Berry phase and the phase of the determinant

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Plan

1. Role of phase in QM
2. Adiabatic process
   - Adiabatic theorem of Born and Fock, 1928'
3. Berry phase (1983')
   - Geometric interpretation due to B. Simon 1983'
4. Determinant of the Shrödinger operator
5. Main theorem in finite dim (B., 2012)
6. Infinite dimensional case (to appear)

In QM the state of a system is described by $\psi \in \mathcal{H}$ – element of a complex Hilbert space $\mathcal{H}$. It satisfies the Schrödinger equation

$$\boxed{i \frac{d\psi(t)}{dt} = H(t)\,\psi(t), \qquad \hbar = 1}$$

Ex For a particle one a line

$\mathcal{H} = L^2(\mathbb{R}) \Rightarrow \psi = \psi(t,x),\ \psi(t,\cdot) \in L^2(\mathbb{R})$

$|\psi(t,x)|$ – density of probability to find ψ near x

Phase $\psi(t,x)$ – often disregarded

However, if we study interaction between 2 particles, then difference of phases is important for interference, etc.

How does the phase $\mathrm{Ph}(\psi(t))$ depend on t?

① H - independent of t

Then for eigenstate $H\psi(0) = E\psi(0)$

$$\boxed{\psi(t) = e^{-iEt}\psi(0)}$$

② If $H(t)$ it is harder.

Adiabatic process when $H(t)$ changes very slowly. Mathematically introduce small $\varepsilon > 0$

$$i\frac{d}{dt}\psi_\varepsilon(t) = H(\varepsilon t)\psi_\varepsilon(t) \Leftrightarrow i\varepsilon\frac{d}{dt}\psi_\varepsilon = H(t)\psi_\varepsilon(t)$$

$\varepsilon \to 0$

Remark $\varepsilon \neq \hbar$ in applications

Adiabatic theorem (Max Born, Vladimir Fock, 1928, Kato 50'): In the simplest form:

Suppose $E(t)$ - an isolated eigenvalue of $H(t)$.

Th Suppose $H(0)\psi_m(0) = E(0)\psi_m(0)$

Then $\psi_m(t)$ is an "almost eigenvector for all t, i.e. $\exists\ \varphi_\varepsilon(t)$ s.t. $H(t)\varphi_\varepsilon(t) = E(t)\varphi_\varepsilon(t)$

and $\psi_\varepsilon(t) = \varphi_\varepsilon(t) + o(1)$, $\varepsilon \to 0$

Suppose $H(t)=H(t+2\pi)$ - periodic

Then $\varphi_\varepsilon(2\pi)=e^{i\alpha_\varepsilon}\varphi_\varepsilon(0)$

$$\Rightarrow \boxed{\psi_\varepsilon(2\pi)=e^{i\alpha_\varepsilon}\psi_\varepsilon(0)+o(1)}$$

α_ε - ? If $H(t)=\text{const}$

$$\alpha_\varepsilon=-tE/\varepsilon$$

One expects $\alpha_\varepsilon=-\frac{1}{\varepsilon}\int_0^{2\pi}E(t)\,dt$

$$\boxed{\text{Th (M. Berry, 83')}\quad \alpha_\varepsilon=-\frac{1}{\varepsilon}\int_0^{2\pi}E(t)\,dt+\gamma_E}$$

γ_E - Berry phase at energy level E

In particular, if $E(t)\equiv 0$ then

$$\psi_\varepsilon(2\pi)=e^{i\gamma_0}\psi_\varepsilon(0)+o(1)$$

Geometric enterpritation of Berry phase

(B. Simon, 83')

$$\begin{array}{l} S^1\times\mathcal{H} \\ \downarrow \\ S^1 \end{array}$$

$S^1\times\mathcal{H}$ - view as a trivial v. bundle of Hilbert spaces

S^1 - $t\in[0,2\pi]$

$L_{E(t)}$ - eigenspace $<\mathcal{H}$

$$\begin{array}{l} L_E\subset S^1\times\mathcal{H} \\ \qquad\quad\downarrow \\ \qquad\quad S^1 \end{array}$$

- Line bundle over S^1

Then $\exists$ a natural connection on L_E

(*) $\nabla\varphi(t)=P_t\frac{d}{dt}\varphi(t)$, $P_t:\mathcal{H}\to L_E$ orth. projection

Th (B. Simon, 84) $e^{i\gamma_E}=\mathrm{Hol}_\nabla$

A slight generalization

Suppose $0 \notin sp(H(t))$ for all t

$\mathcal{H} = \mathcal{H}_{<0}(t) \oplus \mathcal{H}_{>0}(t)$ - spectral decomposition

$\mathcal{H} \times S^1 = F^+ \oplus F^-$

$F_t^+ = \mathcal{H}_{<0}(t), \quad F_t^- = \mathcal{H}_{>0}(t)$

Then (*) defines a natural connections $\nabla^{F^\pm}$ on $F^\pm$

Assume $\dim F^- = k < \infty$

Def $\gamma_{<0}$: $e^{i\lambda_{<0}} = \det Hol_{\nabla^{F^-}}$

Remarks

1. If $sp(H(t)) \cap (-\infty, 0) = \{E_1(t), \dots E_k(t)\}$ - isolated eigenvalues

Then $\gamma_{<0} = \gamma_{E_1} + \dots + \gamma_{E_k}$

2. $H(t)$ defines $H_k(t) : \Lambda^k \mathcal{H} \to \Lambda^k \mathcal{H}$ (Fock space)

Then $\lambda_{<0}$ - Berry phase of the ground state.

Goal: Give an alternative formula for $\gamma_{<0}$ related to path integrals, etc.

Before doing it

Wick rotation $\quad t \to it$

What does it do to the Berry phase?

$$\psi_\varepsilon(2\pi) = e^{i\alpha_\varepsilon}\psi_\varepsilon(0) + o(1)$$

$$\alpha_\varepsilon = -\frac{1}{\varepsilon}\int_0^{2\pi} E(t)\,dt + \gamma_E$$

"dynamical phase" large $\sim \frac{1}{\varepsilon}$

Berry phase fixed indep. of ε

After Wick rotation

$$\tilde{\alpha}_m = -\frac{i}{\varepsilon}\int_0^{2\pi} E(t)\,dt + \gamma_E + o(1)$$

γ_E – only phase which survive

Problem (often overlooked) After rotation the Schrödinger eq. is

$$-\varepsilon\frac{d}{dt}\psi_\varepsilon = H\psi_\varepsilon$$

Adiabatic theorem does not hold for this eq.

For usual eq. $\psi_\varepsilon(t) = \varphi_\varepsilon(t) + o(1)$

$\varepsilon \sin\frac{1}{\varepsilon}$

Main theorem

Set $D_\varepsilon = -i\varepsilon\frac{d}{dt} - iH(t)$

imaginary time Schrödinger operator

Suppose $\dim \mathcal{H} = N < \infty$

$\dim F^\pm = N^\pm$ so that $N = N^+ \oplus N^-$

Then modulo $2\pi\mathbb{Z}$

$$\operatorname{Im}\log\det{}_\pm D_\varepsilon = N^\mp\pi + \gamma_{<0} + o(1)$$

Path integrals $\int e^{-\frac{1}{\hbar}\langle D_\varepsilon\psi,\psi\rangle}\,d\psi$

Remarks ① Formula was known in many examples but without ν^T term

② All the proofs I've seen are not correct (even by physics standards) because they use the Adiabatic th. for twisted Schrödinger

③ $\mathcal{D}_\varepsilon = -i\varepsilon\frac{d}{dt} - iH(t)$

elliptic diff. operator of order 1

Its spectrum $\to \infty$ (e.g. 1, 2, 3, ...)

What is $\det \mathcal{D}_\varepsilon$ -?

Need to regularize

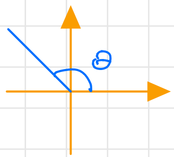
ζ-function regularization

E_i - eigenvalues of $\mathcal{D}_\varepsilon$

$\zeta_{\mathcal{D},\theta}(s) := \sum E_i^{-s}$

Subtelty: what is E^{-s} -?, $E \in \mathbb{C}$

Need a spectral cut



$\zeta'(s) = -\sum \log\lambda_i \, \lambda_i^{-s}$

$\zeta'(0) =$ " $-\sum \log\lambda_i$ " $= -\log(\lambda_1\lambda_2 \dots)$ "

Th (Seeley 67') $\zeta(s)$ is meromorphic and regular at 0

Def $\operatorname{Det}_\theta \mathcal{D}_\varepsilon := e^{-\zeta'(0)}$

Lemma Depends only on whether the cut is in lower or upper half-plane $\Rightarrow$ $\mathrm{Det}_{\pm} \mathcal{D}_\varepsilon$

$$\mathrm{Im} \log \det_{\pm} \mathcal{D}_\varepsilon = N^{\mp} \pi + \gamma_{<0} + o(1)$$

Main steps of the proof

① Burghelea - Friedlander - Kappeler formula (91') (special case which we need)

$$\det{}_+ \mathcal{D}_\varepsilon = \det\left(I - T_\varepsilon(2\pi)\right)$$

$$\det{}_- \mathcal{D}_\varepsilon = (-1)^N \exp\left(\frac{1}{\varepsilon}\int_0^{2\pi} H(t)\, dt\right) \cdot \det\left(I - T_\varepsilon(2\pi)\right)$$

where $T_\varepsilon(2\pi)$ - monodromy map

$\mathcal{D}_\varepsilon T_\varepsilon(t) = 0$, $T_\varepsilon(0) = I$

(2) Fact: $\exists\ \mathcal{U}(t): \mathcal{H} \to \mathcal{H}$

such that

- $\mathcal{U}(0) = \mathcal{U}(2\pi) = I$
- $\mathcal{U}(t)\, \mathcal{H}(t)\, \mathcal{U}(t)^{-1} = \underbrace{\begin{pmatrix} H^+(t) & 0 \\ 0 & H^-(t) \end{pmatrix}}_{\tilde{H}(t)}$

$H^+ > 0$, $H^- < 0$

Then

$$\boxed{\gamma_{<0} = i \int_0^{2\pi} \operatorname{Tr} P_0\, \mathcal{U}(t)^{-1}\, \dot{\mathcal{U}}(t)\, P_0 \, dt}$$

and

$$\det{}_{\pm} \mathcal{D}_\varepsilon = \det{}_{\pm} \overbrace{\mathcal{U}^{-1} \mathcal{D}_\varepsilon \mathcal{U}}^{\tilde{\mathcal{D}}_\varepsilon}$$

$$= \det{}_{\pm} \left(-i\varepsilon \frac{d}{dt} - i\tilde{H}(t) - i\varepsilon\, \mathcal{U}^{-1} \dot{\mathcal{U}} \right)$$

(3) Set $A(t) = i\tilde{H}(t) - i\varepsilon\, \mathcal{U}^{-1}(t)\, \dot{\mathcal{U}}(t)$

$\tilde{T}_\varepsilon$ – monodromy $\tilde{\mathcal{D}}_\varepsilon \tilde{T}_\varepsilon = \frac{d}{dt} \tilde{T}_\varepsilon$

Since adiabatic th. is not true

$\tilde{T}_\varepsilon$ is not asympt. block diagonal

Idea Deform $A(t)$ to $i\tilde{H}(t)$

so that the phase does not change (but abs. value will change)

Then apply BFK

Infinite dimensional $\mathcal{H}$

Setting
$$\begin{array}{c} E \\ \downarrow \\ M\times S^1 \\ \downarrow \\ S^1 \end{array}$$
M - compact

- fiber bundle (or, more generally, mapping torus)

Let g_t^M, $t\in S^1$ - family of Riemannian metrics

$$H(t)=\Delta_{g(t)}+V(t)$$

Schrödinger operator (after twist)

$$D_\varepsilon=-i\varepsilon\frac{d}{dt}-iH(t)$$

first order

second order

If is not elliptic but hypoelliptic

$\Rightarrow$ $\det_-$ is defined (not $\det_+$)

If is bounded below $\Rightarrow$

$\Rightarrow$ only fin many eigenvalues <0. $\Rightarrow$ F^- is finite dim

$(F^+\oplus F^-=\mathcal{H}=L^2(M,E))$

Th

$$\operatorname{Im}\log\det{}_-D_h=N\pi+\gamma_{<0}+o(1)$$

Then basically the same proof

but $\nexists$ BFK formula, so

direct estimates

and T_ε-monodromy $\exists$ σ-det

but of <u>trace class</u>