Long-Time Influence of Small Perturbations.

Mark Freidlin

Distinguished Professor of Mathematics
University of Maryland

Texas Tech University
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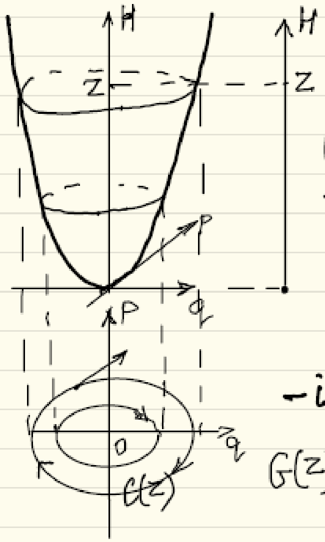
1. Oscillator with One Degree of Freedom

$$\ddot{q}(t) = -F'(q(t)), \quad q(0)=q, \quad \dot{q}(0)=p.$$

$$\begin{cases} \dot{p}(t) = F'(q(t)) \\ \dot{q}(t) = p(t) \end{cases} \qquad H(p,q) = \frac{p^2}{2} + F(q)$$

$$x = (p,q) \in R^2, \; H(x), \; \bar{\nabla} H(x) = \left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$$

$$\boxed{\dot{X}_t = \bar{\nabla} H(X_t), \quad X_0 = (p,q).}$$



$$H(X_t) \overset{t}{\equiv} H(X_0)$$

$C(z) = \{x \in R^2 : H(x) = z\}$, $G(z)$

$T(z) = \oint\limits_{C(z)} \frac{dl}{|\nabla H(x)|}$ – period on $C(z)$

$m_z(x) = \left(T(z)|\nabla H(x)|\right)^{-1}$, $x \in C(z)$

– invariant probability measure

$G(z)$ – domain in R^2 bounded by $C(z)$.

2. Perturbations

$$\dot{X}_t = \overline{\nabla} H(X_t), \quad X_0 = x; \quad \ddot{q}(t) = -F'(q(t))$$

$$\boxed{\dot{\tilde{X}}^{\varepsilon}_t = \overline{\nabla} H(\tilde{X}^{\varepsilon}_t) + \varepsilon \beta(\tilde{X}^{\varepsilon}_t), \quad \tilde{X}^{\varepsilon}_0 = x.}$$

$$\ddot{q}^{\varepsilon}(t) = -F_1'\left(q^{\varepsilon}(t), q^{\varepsilon}_{t-\varepsilon}\right), \quad q^{\varepsilon}(0) = q, \quad \dot{q}^{\varepsilon}(0) = p$$

$$\ddot{q}^{\varepsilon}(t) = -F_1'\left(q^{\varepsilon}(t), \xi_{t/\varepsilon}\right)$$

$$\ddot{q}^{\varepsilon}(t) = -F'\left(q^{\varepsilon}(t)\right), \quad q^{\varepsilon}(0) = q + \delta_1, \quad \dot{q}^{\varepsilon}(0) = p + \delta_2.$$

Stochastic Perturbations

$\xi(\omega)$ – random variable, $\xi_t(\omega)$ – stochastic process.

W_t – Wiener stochastic process, $\dot{W}_t$

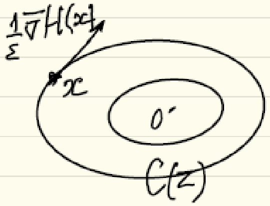
$$\boxed{\dot{\tilde{X}}^{\varepsilon,\delta}_t = \overline{\nabla} H(\tilde{X}^{\varepsilon,\delta}_t) + \varepsilon \beta(\tilde{X}^{\varepsilon,\delta}_t) + \sqrt{\varepsilon\delta}\,\sigma(\tilde{X}^{\varepsilon,\delta}_t) * \dot{W}_t}$$

$$\boxed{\tilde{X}^{\varepsilon,\delta}_{t/\varepsilon} := X^{\varepsilon,\delta}_t} \quad \dot{X}^{\varepsilon,\delta}_t = \frac{1}{\varepsilon} \overline{\nabla} H(X^{\varepsilon,\delta}_t) + \beta(X^{\varepsilon,\delta}_t) + \sqrt{\delta}\,\sigma(X^{\varepsilon,\delta}_t) * \dot{W}_t$$

$$\mathcal{L}^{\varepsilon,\delta} u(x) := \frac{1}{\varepsilon} \overline{\nabla} H(x) \cdot \nabla u + \beta(x) \cdot \nabla u + \frac{\delta}{2} \operatorname{div}(a(x) \nabla u), \quad a(x) = \sigma(x)\sigma^*(x).$$

3. Classical Averaging Principle

$$\dot{X}_t^\varepsilon = \frac{1}{\varepsilon}\overline{\nabla} H(X_t^\varepsilon) + \beta(X_t^\varepsilon).$$ Fast-Slow components



$$H(X_{t+\Delta}^\varepsilon) - H(X_t^\varepsilon) = \cancel{\frac{1}{\varepsilon}\int_t^{t+\Delta} \nabla H(X_s^\varepsilon)\cdot\overline{\nabla} H(X_s^\varepsilon)\,ds} + \int_t^{t+\Delta} \nabla H(X_s^\varepsilon)\cdot\beta(X_s^\varepsilon)\,ds$$

$$\int_t^{t+\Delta} \nabla H(X_s^\varepsilon)\cdot\beta(X_s^\varepsilon)\,ds = \Delta\left(\oint_{C(H(X_t^\varepsilon))} \frac{\nabla H(y)\cdot\beta(y)}{T(H(X_s^\varepsilon))}\frac{dl}{|\nabla H(y)|} + \rho_\varepsilon\right) + \tilde{\rho}_\Delta$$

$$\frac{1}{T(z)}\oint_{C(z)} \frac{\nabla H(y)\cdot\beta(y)\,dl}{|\nabla H(y)|} = \frac{1}{T(z)}\int_{G(z)} \operatorname{div}\beta(x)\,dx = \frac{1}{T(z)}\overline{\beta}(z)$$

$$H(X_t^\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{} Z_t;$$

$$\boxed{\dot{Z}_t = \frac{1}{T(Z_t)}\overline{\beta}(Z_t),\quad Z_0 = H(x)}\ \checkmark$$

Long-time behavior of X_t^ε for $\varepsilon \ll 1$ can be described by Z_t and a distribution on $C(Z_t)$.

The point 0 (minimum of $H(x)$) is inaccessable in finite time.

4. Stochastic perturbations. One well.

$$\dot{X}_t^{\varepsilon,\delta} = \frac{1}{\varepsilon}\bar{\nabla} H(X_t^{\varepsilon,\delta}) + \beta(X_t^{\varepsilon,\delta}) + \sqrt{\delta}\,\sigma(X_t^{\varepsilon,\delta}) * \dot{W}_t, \quad X_0^{\varepsilon,\delta} = x.$$

$H(X_t^{\varepsilon,\delta}) \to Z_t^\delta$ as $\varepsilon \downarrow 0$.

$$\boxed{\begin{gathered} \dot{Z}_t^\delta = \frac{\bar{B}(Z_t^\delta)}{T(Z_t^\delta)} + \sqrt{\frac{\delta}{T(Z_t^\delta)}}\,\bar{\sigma}(Z_t^\delta) * \dot{W}_t, \quad Z_0^\delta = H(x) \\ Z_t^\delta \sim L^\delta u(z) = \frac{\delta}{2T(z)} \frac{d}{dz}\left(\bar{a}(z)\frac{du}{dz}\right) + \frac{1}{T(z)}\bar{\beta}(z)\frac{du}{dz} \end{gathered}}$$

$$\bar{a}(z) = \int_{G(z)} \operatorname{div}(a(x)\nabla H(x))\,dx, \quad \bar{\sigma}(z) = \sqrt{\bar{a}(z)}, \quad \bar{\beta}(z) = \int_{G(z)} \operatorname{div}\beta(x)\,dx$$

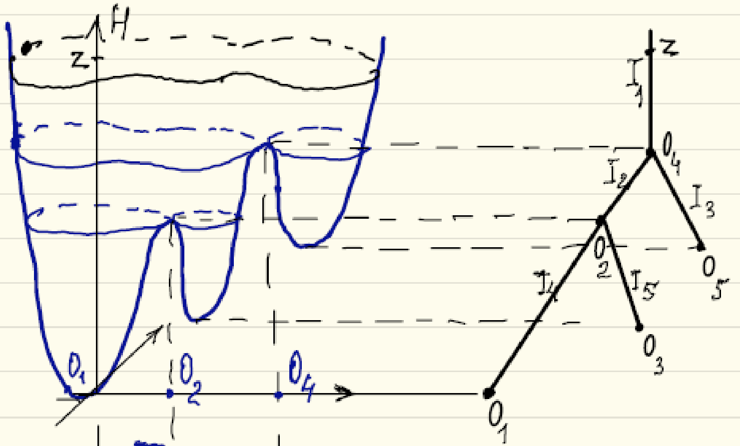
$$T(z) = \oint_{C(z)} \frac{dl}{|\nabla H(x)|}.$$

Z_t^δ – diffusion process on $\Gamma = [0, \infty)$.
Point 0 is inaccessible for Z_t^δ.

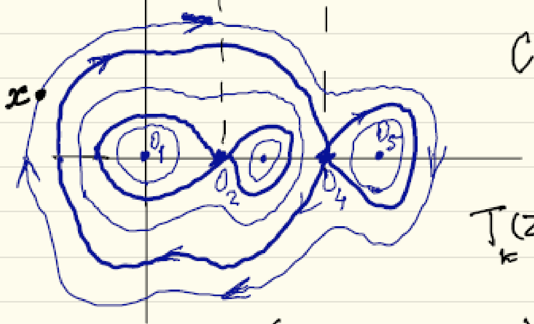
In the case of one well, the classical averaging principle (extended to stoch. perturbations) describes the long-time evolution of the perturbed system.

Remark: If σ is independent of x, the stochastic integral is defined in the unique way.

5. Multi-well Hamiltonians.



$\Gamma = \{ 0_1, 0_2, 0_3, 0_4, 0_5;$
$I_1, I_2, I_3, I_4, I_5 \}$



$$C(z) = \{x \in R^2 : H(x) = z\} = \bigcup_{k=1}^{n(z)} C_k(z)$$

$G_k(z)$ - domain bounded by $C_k(z)$

$$T_k(z) = \oint_{C_k(z)} \frac{dl}{|\nabla H(x)|} \text{ - period on } C_k(z).$$

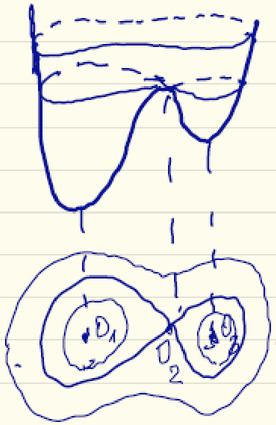
$m_{z,k}(x) = \left(T_k(z)\,|\nabla H(x)|\right)^{-1}$ - density of invariant measure on $C_k(z)$, $x \in C_k(z)$

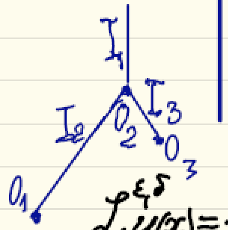
(z,k) - Global coordinate system on Γ.

map $Y: R^2 \longrightarrow \Gamma$: $Y(x) = (H(x), k(x))$.

$$\boxed{\dot{X}_t = \bar{\nabla} H(X_t),\ X_0 = x} \qquad Y(X_t) \equiv Y(x) = (H(x), k(x)).$$

6. Perturbations of multi-well potential





$$\boxed{\dot{X}_t^{\varepsilon\delta} = \frac{1}{\varepsilon}\bar{\nabla} H(X_t^{\varepsilon,\delta}) + \beta(X_t^{\varepsilon,\delta}) + \sqrt{\delta}\,\sigma \cdot \dot{W}_t}$$

$$L^{\varepsilon,\delta} u(x) = \frac{1}{\varepsilon}\bar{\nabla} H \cdot \nabla u + \beta \cdot \nabla u + \frac{\delta}{2} \operatorname{div}(a(x)\nabla u),$$

$$a(x) = \sigma(x)\,\sigma^*(x).$$

$Y_t^\varepsilon = Y(X_t^{\varepsilon\delta})$ – slow component

Fast component $\approx$ motion along X_t.

Inside an edge I_k: $\quad Y_t^\varepsilon \to Z_k(t)$

$$\dot{Z}_k^\delta(t) \approx \frac{1}{T_k(Z_k(t))}\bar{\beta}_k(Z_k^\delta(t)) + \sqrt{\frac{\delta}{T_k(Z_t^\delta)}}\,\bar{\sigma}_k(Z_t^\delta) * \dot{\tilde{W}}_t$$

$$\bar{\beta}_k(z) = \int_{G_k(z)} \operatorname{div}\beta(x)\,dx, \quad \bar{\sigma}_k(z) = \sqrt{\bar{a}(z)}, \quad \bar{a}(z) = \int_{G_k(z)} \operatorname{div}(a(x)\nabla H)\,dx$$

$$Z_k^\delta(t) \sim L_k^\delta u(z) = \frac{\bar{\beta}_k(z)}{T_k(z)}\frac{du}{dz} + \frac{\delta}{2T_k(z)} \operatorname{div}(\bar{a}_k(x)\nabla u(x)).$$

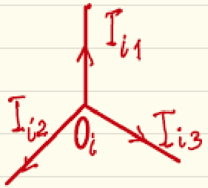
But interior vertices (O_2 in the Figure) are accessable in finite time.

Therefore the classical averaging principle should be supplemented by a discription what the trajectory is doing after hitting the vertex.

7. Behavior near the vertices.

Exterior vertices are inaccessible

Interior vertex O_i:



When Y_t

$\alpha_{i1} = \alpha_{i2} + \alpha_{i3}$; $\quad \alpha_{ij} = \int_{G_{ij}} \mathrm{div}(a(x)\nabla H(x))\,dx, \quad = 2,3.$

The limiting process $Y_t = (Z_t, k_t)$ on Γ is defined in the unique way by the operators L_k and gluing conditions at the interior vertices O_i:

$$\alpha_{i1} D_1 v(O_i) = \alpha_{i2} D_2 v(O_i) + \alpha_{i3} D_3 v(O_i)$$

where $D_{ij} v(O_i)$ is the first derivative of a function v on Γ along I_{ij} at the vertex O_i.

Function $v(z,k)$ should be continuous on Γ as well as the function $L_k^\delta v(z,k)$.

The operators L_k^δ together with the gluing conditions at interior vertices allow to calculate main term of many interesting characteristics of $X_t^{\varepsilon\delta}$ for $0 < \varepsilon \ll 1$.

8. Modified averaging principle for deterministic perturbations. Stochasticity of systems close to Hamiltonian systems.

$$\dot X_t = \bar\nabla H(X_t), \quad X_0 = x \in R^2.$$

$$\dot{\tilde X}^{\varepsilon}_t = \bar\nabla H(\tilde X^\varepsilon_t) + \varepsilon\beta(\tilde X^\varepsilon_t), \quad X^\varepsilon_t = \tilde X^\varepsilon_{t/\varepsilon},$$

$$\dot X^\varepsilon_t = \frac{1}{\varepsilon}\bar\nabla H(X^\varepsilon_t) + \beta(X^\varepsilon_t)$$

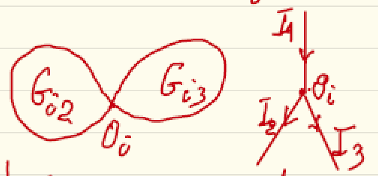
Let $\operatorname{div}\beta(x) < 0$, $H(x)$ has saddle points.

$$\dot X^{\varepsilon,\delta}_t = \frac{1}{\varepsilon}\bar\nabla H(X^{\varepsilon,\delta}_t) + \beta(X^{\varepsilon,\delta}_t) + \sqrt{\delta}\,\sigma(X^{\varepsilon,\delta}_t)\dot W_t$$

First, take $\varepsilon \downarrow 0$: $Y(X^{\varepsilon,\delta}_t) \to Y^\delta_t \sim (L^\delta_k$, gluing conditions$)$

Then take $\delta \downarrow 0$: $Y^\delta_t \to Y_t = (Z_k(t), k(t))$ deterministic inside the edges (L^0_k); inside I_k, $\dot Z_k = \dfrac{B_k(Z_k)}{T_k(Z_k)}$.

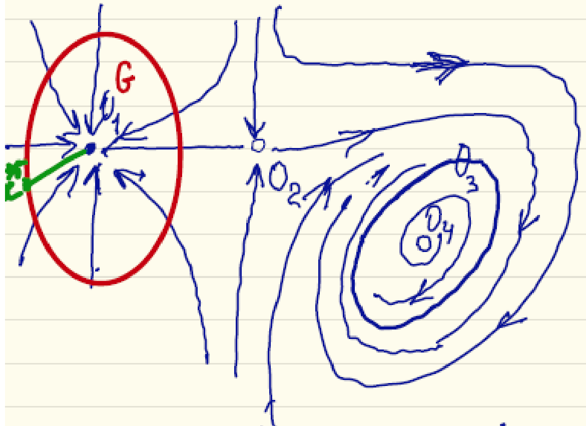
But when Y_t comes to an interior vertex O_i it goes to one of the wells related to O_i with probabilities proportional to

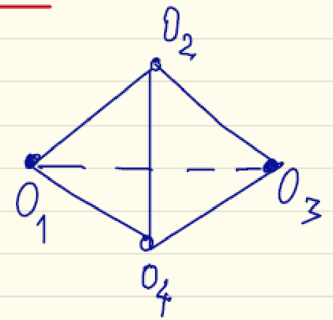


$\left|\int_{G_{ij}} \operatorname{div}\beta(x)\,dx\right|$, $j \in \{2,3\}$.

This stochasticity is independent of the type of regularization.

9. Perturbations of systems with asymptotically stable regimes. Large deviations.





$$X_0 = X_0^\varepsilon = x \in R^n$$

$$\dot{X}_t = b(X_t), \quad \dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \sqrt{\varepsilon}\, \sigma(X_t^\varepsilon)\, \dot{W}_t$$

For each fixed T: $\max\limits_{0 \le t \le T} |X_t^\varepsilon - X_t| \to 0$ in probability.

But, if $a(x) = \sigma(x)\sigma^*(x)$ is non-degenerate, X_t^ε sooner or later will deviate of X_t on a distance of order 1.

$$P\left\{\max_{0 \le t \le T} |X_t^\varepsilon - X_t| < \delta\right\} \approx \exp\left\{-\frac{1}{\varepsilon} S_{0T}(\varphi)\right\}; \quad 0 < \varepsilon, \delta \ll 1.$$

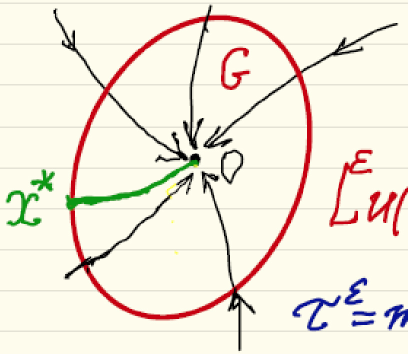
$S_{0T}(\varphi)$ - action functional.

$$\boxed{V(x,y) = \inf\left\{S_{0T}(\varphi) : \varphi \in C_{0T},\ \varphi_0 = x,\ \varphi_T = y,\ T \ge 0\right\}}$$

Exit problem.

Hierarchy of cycles. Metastability.

10. Exit Problem



$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \sqrt{\varepsilon}\,\sigma(X_t^\varepsilon)\dot{W}_t, \quad X_0^\varepsilon = x$$

$$a(x) = \sigma(x)\sigma^*(x), \quad x \in R^n$$

$$\overset{\varepsilon}{L}u(x) = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1}^{n} b_i(x)\frac{\partial u}{\partial x_i}$$

$$\tau^\varepsilon = \min\{t : X_t^\varepsilon \in \partial G\}, \quad X_0^\varepsilon = x \in G.$$

What are $X_{\tau^\varepsilon}^\varepsilon$ – exit position on ∂G,

Asymptotics of τ^ε and of $E_x\tau^\varepsilon$ as $\varepsilon \to 0$?

Quasi-potential: $V(x) = V(0,x)$

$$V_0 = \min_{x \in \partial G} V(x) = V(x^*), \quad x^* \in \partial G.$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln \tau^\varepsilon = V_0, \quad \lim_{\varepsilon \downarrow 0} E_x \tau^\varepsilon = V_0$$

If x^* is a unique point of ∂G where $V(x) = V_0$, then $X_{\tau^\varepsilon}^\varepsilon \to x^*$ in probability when $\varepsilon \downarrow 0$.

Dirichlet problem: $Lu^\varepsilon(x) = 0$ in G, $u^\varepsilon(x) = \psi(x)$.

Then $\lim_{\varepsilon \downarrow 0} u^\varepsilon(x) = \psi(X^*)$

$$u^\varepsilon(x) = E_x \psi(X_{\tau^\varepsilon}^\varepsilon) \to \psi(X^*) \text{ as } \varepsilon \downarrow 0.$$

11. Hierarchy of cycles. Metastability.

Consider a system $\dot{X}_t = b(X_t)$ in $\mathbb{R}^n$ with N asymptotically stable regimes. Denote them $1, 2, \ldots, N$.

Assume that each point $x \in \mathbb{R}^n$, besides the points of separatrix surfaces is attracted to one of these stable attractors.

$V_{ij} =$

$= V(i, j) =$

$\inf\{S_{0T}(\varphi) : \varphi_0 = i, \varphi_T = j, T \geq 0\}$

17 asymptotically stable regimes:

6 cycles of rank 1,

3 cycles of rank 2,

2 cycles of rank 3,

1 cycle of rank 4.

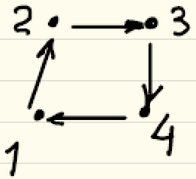
Operator N (Next):

$N(i) = j$ such that

$V_{ij} = \min_{k:\, k \neq i,\, 1 \leq k \leq N} V_{ik}.$

12. Hierarchy of cycles (generic case).

1-cycle



Rotation rate $\tau(C) = \max_{i \in C} V_{i N(i)} = V_{i^* N(i^*)}$;

$i^* = i^*(C)$ - main state of C;

invariant distribution on C: $m^\varepsilon_C(k)$, $k \in C$,

$\tilde{m}^\varepsilon_C(k) = A(\varepsilon) \exp\{\frac{1}{\varepsilon} V_{k,N(k)}\}$, $\sum_{k \in C} m^\varepsilon_C(k) = 1$.

invariant distribution rate:

$m^\varepsilon_C(k) = V_{k,N(k)} - \max_{i \in C} V_{i,N(i)}$

Transition probability from $i \in C$ to $j \notin C$

rate

$$\min_{i \in C} [m_C(i) - V_{ij}] = \rho_{C,j}$$

Operator N (next) for 1-cycles: $N(C_1) = C_2$

if C_2 contains k such that

$$\min_{j \notin C_1} \rho_{C,j} = \rho_{C,k}$$

Exit time (from C) rate $e(C) = \min_{k \in C} \rho_{C,k}$

cycles of rank 2, ...

13. Metastable state (for a given x and λ)

Initial point $x \in R^n$ is attracted to an asymptotically stable $i(x)$.

$$i[x) = C^{(0)}_{i_0}(x) \in C^{(1)}_{i_1}(x) \in \dots \in C^{(k)}_{i_k}(x) \in C^{(k+1)}_{i_{k+1}}(x) \in \dots \in C^{(n)} = C$$

$$0 < e(C^{(1)}_{i_1}) < \dots < e(C^{(k)}_{i_k}) < e(C^{(k+1)}_{i_{k+1}}) < \dots < \infty.$$

If $e_k < \lambda < e_{k+1}$, the process has enough time to come from x to $C^{(k+1)}_{i_{k+1}}(x)$ and not enough time to leave it.

Then the trajectory spends almost all time near main state $i^*(C^{(k+1)}_{i_{k+1}}(x)) = I(x,\lambda)$

The state (distribution) $I(x,\lambda)$ is called the **metastable distribution** for given initial point x and time scale $e^{\frac{\lambda}{\varepsilon}} \in \left(\exp\left\{\frac{e_k}{\varepsilon}\right\}, \exp\left\{\frac{e_{k+1}}{\varepsilon}\right\}\right)$.

PDE formulation: $\frac{\partial u^\varepsilon(t,x)}{\partial t} = L^\varepsilon u^\varepsilon$, $u^\varepsilon(0,x) = g(x)$.

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(e^{\frac{\lambda}{\varepsilon}}, x) = g(I(x,\lambda)).$$

14. General point of view – motion on the simplex of invariant probability measures of the non-perturbed system.

(i) Finite number of ergodic measures.

(ii) Exit problem and stoped process

(iii) Hamiltonian systems with one degree of freedom: graphs give a parametrization of ergodic probability measures.

Landau – Lifshitz equation

$$\boxed{\dot{X}_t = \nabla F(X_t) \times \nabla G(X_t), \; X_0 = x \in R^3, \; \lim_{|x| \to \infty} F(x) = \infty.}$$

$F(X_t) \equiv F(x)$, $G(X_t) \equiv G(x)$, X_t preserves the volume.

$S_F(z) = \{x \in R^3 : F(x) = z\}$, $S_G(z) = \{x \in R^3 : G(x) = z\}$.

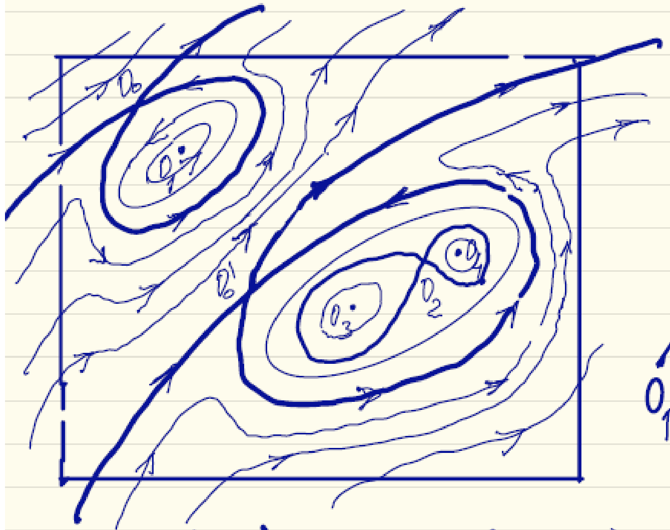
$S_F(z)$ – compact smooth orientable manifold.

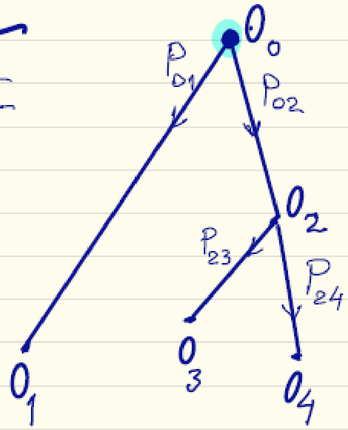
Let $\varkappa$ be the genus of $S_F(z)$.

$m_z(x)$ – invariant density on $S_F(z)$: $m_z(x) = \dfrac{const}{|\nabla F(x)|}$.

If $\varkappa = 0$, the set of invariant measures on $S_F(z)$ can be described similarly to the set of invariant measures of a Hamiltonian system with one degree of freedom.

15. Landau-Lifshitz equ. with $\varkappa > 0$ $(\varkappa = 1)$.

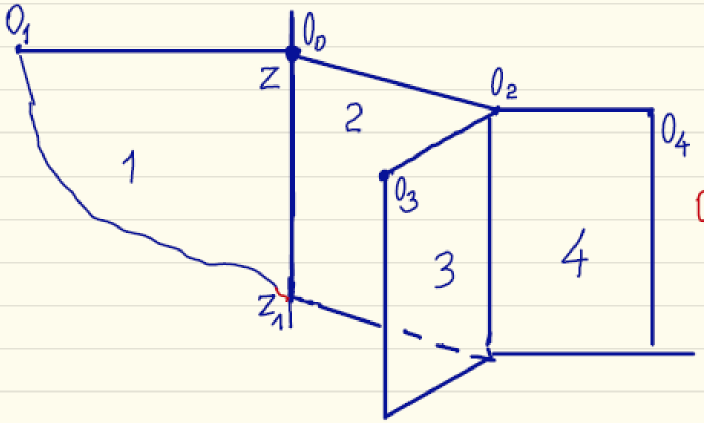




ergodic probability measures

$S_F(z)$ - Torus $(\varkappa = 1)$.

The whole set of ergodic probability measures can be parametrized by an open book space.



Perturbations

Pozitive time at O_0 - binding.

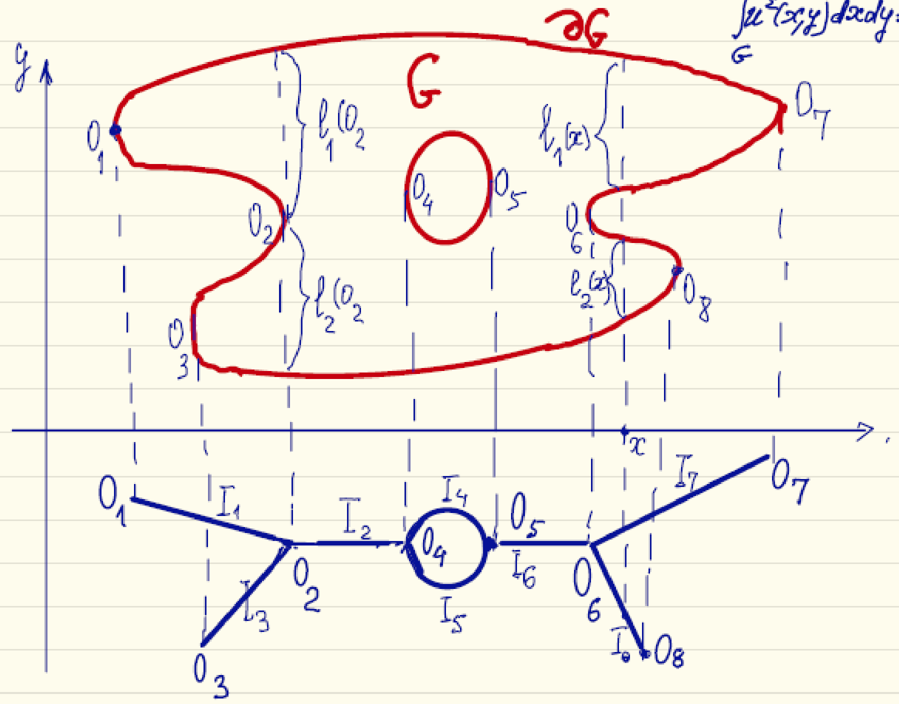
Four pages Open ~~Book~~.

16. The Neumann problem.

$$L^\varepsilon u^\varepsilon(x) = \frac{1}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{1}{2\varepsilon}\frac{\partial^2 u}{\partial y^2} = f(x,y), (x,y) \in G,$$

$$\frac{\partial u^\varepsilon(x,y)}{\partial n_\varepsilon(x,y)} = 0, (x,y) \in \partial G, \quad \int_G f(x,y)\,dx\,dy = 0,$$

$$\int_G u^\varepsilon(x,y)\,dx\,dy = 0.$$



$$n_\varepsilon(x,y) = (\varepsilon n_1(x,y), n_2(x,y)), \quad (n_1, n_2)\text{-interior normal}$$

$$\dot X^\varepsilon = \sqrt{\varepsilon}\,\dot W_t^1 + \varepsilon n_1(X_t^\varepsilon, Y_t^\varepsilon)\dot\varphi_t^\varepsilon, \quad \dot Y^\varepsilon = \dot W_t^2 + n_2(X_t^\varepsilon, Y_t^\varepsilon)\dot\varphi_t^\varepsilon,$$

$$u^\varepsilon(x,y) = -\int_0^\infty E_{x,y} f(X^\varepsilon_{s/\varepsilon}, Y^\varepsilon_{s/\varepsilon})\,ds, \qquad x,y \in G, \quad \varphi_0^\varepsilon = 0.$$

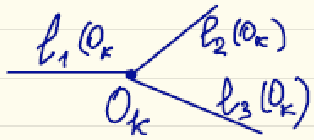
17. Neumann's problem.

Non-perturbed system: $\dot{X}_t = 0,\ \dot{Y}_t = \dot{W}_t^2 + n_2(X_t, Y_t)\dot{\varphi}_t$

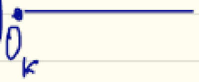
Limiting process on Γ:

$X_i(t)$ on $I_i \subset \Gamma$: $X_i(t) \sim \bar{L}_i v(z) = \frac{1}{2 l_i(x)} \frac{d}{dx}\left(l_i(x)\frac{dv}{dx}\right)$

Conditions at the vertices:

(*) 

$$l_1(O_k) D_1 v(O_k) = l_2(O_k) D_2 v(O_k) + l_3(O_k) D_3 v(O_k)$$

(**) 

$$\lim l_i(x) D_i v(O_k) = 0, \quad \bar{f}_k(x) = \frac{1}{l_k(x)} \int_{l_k(x)} f(x,y)\,dy$$

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(x,y) = v(x, k(x,y))$$

$$L_k v(x,k) = \bar{f}_k(x), \quad x \in l_k(x)$$

$v(x,k)$ satisfies (*) and (**),

$$\sum_{k: I_k \subset \Gamma} \int_{I_k} v(x,k)\, l_k(x)\, dx = 0. \qquad \cup \partial G.$$

18. Homogenization:

$$\dot{X}_t^\varepsilon = b\left(X_t^\varepsilon, \frac{X_t^\varepsilon}{\varepsilon}\right) + \sigma\left(X_t^\varepsilon, \frac{X_t^\varepsilon}{\varepsilon}\right)\dot{W}_t, \quad X_0^\varepsilon = x \in R^n$$

$b(x,y)$, $\sigma(x,y)$ are 1-periodic in y.

$$L^\varepsilon = b\left(x,\frac{x}{\varepsilon}\right)\cdot\nabla + \frac{1}{2}\sum_{i,j}^{n} a_{ij}\left(x,\frac{x}{\varepsilon}\right)\frac{\partial^2}{\partial x_i\,\partial x_j}, \quad a(x,y)=\sigma\sigma^*.$$

$$\boxed{\tilde{X}_t^\varepsilon = X_{\varepsilon^2 t}^\varepsilon, \quad \tilde{Y}_t^\varepsilon = \varepsilon^{-1}X_{\varepsilon^2 t}^\varepsilon}$$

$$\begin{cases} \dot{\tilde{X}}_t^\varepsilon = \varepsilon^2 b(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon) + \varepsilon\sigma(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon)\dot{W}_t \\ \dot{\tilde{Y}}_t^\varepsilon = \varepsilon b(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon) + \sigma(\tilde{X}_t^\varepsilon, \tilde{Y}_t^\varepsilon)\dot{W}_t \end{cases}$$

Non-perturbed system:

$$\dot{\tilde{X}}_t = 0, \quad \dot{\tilde{Y}}_t = \sigma(\tilde{X}_t, \tilde{Y}_t)\dot{W}_t.$$

For each x, $\tilde{Y}_t$ on T^n has one inv. probab. measure $m_x(y)$, $y\in T^n$. Put $\bar{b}(x) = \int_{T^n} b(x,y)\,m_x(y)\,dy$, $\bar{a}(x) = \int_{T^n} a(x,y)\,m_x(y)\,dy$. Then

$$\tilde{X}_{t/\varepsilon^2}^\varepsilon = X_t^\varepsilon \to \bar{X}_t \sim \bar{L}u = \bar{b}(x)\cdot\nabla u + \frac{1}{2}\sum_1^n \bar{a}_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}$$

Solutions of initial-boundary and boundary problems for L^ε converge to solutions of corresponding problems for $\bar{L}$.

19. Reaction-Diffusion Equations

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + f(x, u^\varepsilon)$$

$$u^\varepsilon(x) = g(x).$$

non-perturbed problem:

$$\dot{u}(t,x) = f(x,u), \quad u(0,x) = g(x)$$

considered as a flow in the space of functions $g(x) \to u(t,x)$.

The invariant measures of this flow are δ-functions concentrated on the piece-wise constant functions with values from the set of zeroes of $f(x,u)$.

Under certain conditions one can describe the long-time evolution of these function under perturbation $\frac{\varepsilon}{2}\sum a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$.

20. Perturbations of Markov chains,

Fast oscillating perturbations,
Diffusion approximation,
Small delay, ...