

A positivity conjecture related to the Riemann zeta function

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A postcard from Quebec City



- Hugues Bellemare
- Yves Langlois

The functions f_k

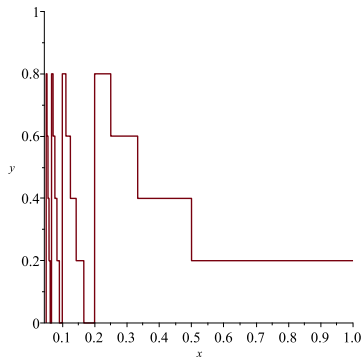
For each integer $k \geq 2$, define $f_k : (0, 1) \rightarrow \mathbb{R}$ by

$$f_k(x) := \frac{1}{k} \left[\frac{1}{x} \right] - \left[\frac{1}{kx} \right].$$

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Graph of $f_5(x)$

A least-squares problem

For $g : (0, 1) \rightarrow \mathbb{C}$, we write

$$\|g\| := \left(\int_0^1 |g(x)|^2 dx \right)^{1/2}.$$

Set

$$d_n := \min_{\lambda_2, \dots, \lambda_n} \left\| 1 - \sum_{k=2}^n \lambda_k f_k \right\|.$$

Does $d_n \rightarrow 0$ as $n \rightarrow \infty$?

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Theorem (Nyman 1950, Báez-Duarte 2003)

$$d_n \rightarrow 0 \iff RH$$

The Riemann hypothesis

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$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = s \int_0^1 \left[\frac{1}{x} \right] x^{s-1} dx = \frac{s}{s-1} - s \int_0^1 \left\{ \frac{1}{x} \right\} x^{s-1} dx.$$

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Riemann hypothesis

$\zeta(s) \neq 0$ for all s with $\operatorname{Re} s > 1/2$.

Proof that $d_n \rightarrow 0 \Rightarrow RH$

A simple calculation shows that, if $\operatorname{Re} s > 0$, then

$$\int_0^1 f_k(x) x^{s-1} dx = \frac{\zeta(s)}{s} \left(\frac{1}{k} - \frac{1}{k^s} \right).$$

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Suppose that $\zeta(s_0) = 0$, where $\operatorname{Re} s_0 > 1/2$. Then

$$\int_0^1 f_k(x) x^{s_0-1} dx = 0 \quad (k = 2, 3, \dots).$$

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Thus, for all $\lambda_2, \dots, \lambda_n$,

$$\int_0^1 \left(1 - \sum_{k=2}^n \lambda_k f_k(x) \right) x^{s_0-1} dx = \int_0^1 x^{s_0-1} dx = \frac{1}{s_0}.$$

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But also, by Cauchy-Schwarz,

$$\left| \int_0^1 \left(1 - \sum_{k=2}^n \lambda_k f_k(x) \right) x^{s_0-1} dx \right| \leq \left\| 1 - \sum_{k=2}^n \lambda_k f_k \right\| \|x^{s_0-1}\|.$$

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Hence $d_n \geq 1/|s_0| \|x^{s_0-1}\|$ for all n . In particular $d_n \not\rightarrow 0$. □

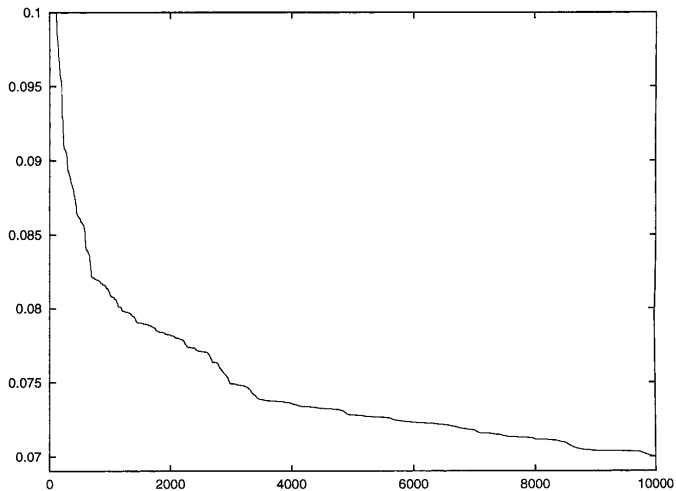


FIG 1. La distance d_n pour n de 2 à 10000.

Graph of d_n^{-2} against $\log n$ (Baéz-Duarte et al, 2000)

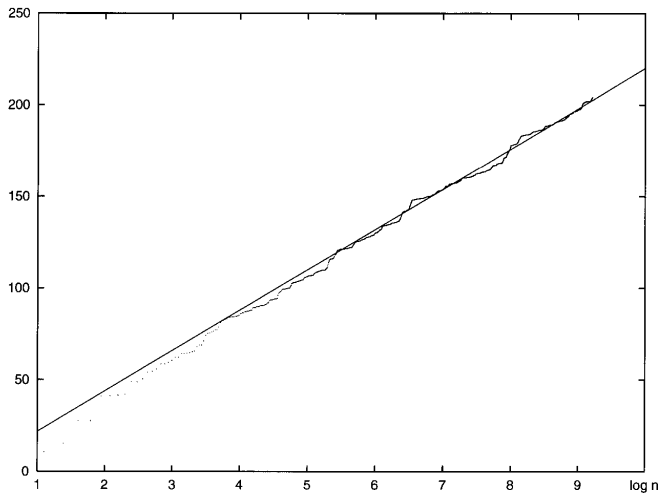


FIG. 2. Comparaison de d_n^{-2} et $21,99357 \log n$.

Behavior of the sequence (d_n)

Numerical calculations suggest that $d_n \approx 0.21/\sqrt{\log n}$.

Conjecture (Báez-Duarte, Balazard, Landreau, Saias (2000))

$$d_n \sim C_0/\sqrt{\log n}$$

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Theorem (Báez-Duarte, Balazard, Landreau, Saias (2000))

There exists $C > 0$ such that $d_n \geq C/\sqrt{\log n}$ for all n .

How to compute d_n ?

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Gram–Schmidt: There is a unique sequence of functions e_2, e_3, \dots on $(0, 1)$ such that:

- $\langle e_j, e_k \rangle = \delta_{jk} \quad \forall j, k,$
- $\text{span}\{e_2, \dots, e_n\} = \text{span}\{f_2, \dots, f_n\} \quad \forall n,$
- $\langle e_k, f_k \rangle > 0 \quad \forall k.$

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Distance formula

$$d_n^2 = 1 - \sum_{k=2}^n |\langle 1, e_k \rangle|^2.$$

How to compute $\langle 1, e_k \rangle$ in practice?

Four steps:

- Compute $\langle 1, f_k \rangle$. This is easy since

$$\langle 1, f_k \rangle = \lim_{s \rightarrow 1} \int_0^1 f_k(x) x^{s-1} dx = \lim_{s \rightarrow 1} \frac{\zeta(s)}{s} \left(\frac{1}{k} - \frac{1}{k^s} \right) = \frac{\log k}{k}.$$

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- Deduce $\langle e_j, f_k \rangle$ (see next slide).

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- Compute $\langle f_j, f_k \rangle$ (see next slide).
- Deduce $\langle e_j, f_k \rangle$ (see next slide).
- Compute $\langle 1, e_j \rangle$ by solving the triangular linear system

$$\sum_{j=2}^n \langle 1, e_j \rangle \langle e_j, f_k \rangle = \langle 1, f_k \rangle \quad (k = 2, \dots, n).$$

The scalar products $\langle f_j, f_k \rangle$ and $\langle e_j, f_k \rangle$

To compute $\langle f_j, f_k \rangle$, we can use

Theorem (Vasyunin, 1996)

Let $j, k \geq 2$, and set $\omega := \exp(2\pi i/jk)$. Then

$$\langle f_j, f_k \rangle = \frac{1}{jk} \sum_{q=1}^{jk-1} \sum_{r=1}^{jk-1} \left\{ \frac{q}{j} \right\} \left\{ \frac{q}{k} \right\} \omega^{-qr} (\omega^{-r} - 1) \log(1 - \omega^r),$$

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And to obtain $\langle e_j, f_k \rangle$ we use

Proposition

Let $P_{jk} := \langle f_j, f_k \rangle$ and $L_{kj} := \langle f_k, e_j \rangle$. Then L is a lower-triangular matrix and $P = LL^t$ (Cholesky decomposition).

The matrix $P_{jk} := \langle f_j, f_k \rangle$ for $n = 9$

	2	3	4	5	6	7	8	9
2	0.1733	0.1063	0.1184	0.0918	0.0931	0.0784	0.0778	0.0683
3	0.1063	0.1770	0.1220	0.1118	0.1178	0.0976	0.0908	0.0914
4	0.1184	0.1220	0.1618	0.1194	0.1103	0.1023	0.1060	0.0912
5	0.0918	0.1118	0.1194	0.1456	0.1125	0.1019	0.0956	0.0918
6	0.0931	0.1178	0.1103	0.1125	0.1313	0.1049	0.0957	0.0909
7	0.0784	0.0976	0.1023	0.1019	0.1049	0.1192	0.0976	0.0889
8	0.0778	0.0908	0.1060	0.0956	0.0957	0.0976	0.1089	0.0910
9	0.0683	0.0914	0.0912	0.0918	0.0909	0.0889	0.0910	0.1002

The matrix $L_{kj} := \langle f_k, e_j \rangle$ for $n = 9$

	2	3	4	5	6	7	8	9
2	0.4163	0	0	0	0	0	0	0
3	0.2554	0.3343	0	0	0	0	0	0
4	0.2845	0.1475	0.2430	0	0	0	0	0
5	0.2205	0.1659	0.1325	0.2277	0	0	0	0
6	0.2237	0.1814	0.0819	0.0976	0.1792	0	0	0
7	0.1883	0.1480	0.1107	0.0929	0.0991	0.1764	0	0
8	0.1868	0.1288	0.1395	0.0638	0.0721	0.0841	0.1471	0
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The first few values of d_n

n	d_n
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Let's go back one slide ...

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The below-diagonal entries are positive. Should we expect this?

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Here is what we have been able to establish.

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Numerical computation

$$\langle e_j, f_k \rangle > 0 \quad \text{for } 2 \leq j \leq k \leq 50000.$$

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Here is what we have been able to establish.

Numerical computation

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This leads us to formulate:

Conjecture

$$\langle e_j, f_k \rangle > 0 \quad \text{for all } j, k \text{ with } 2 \leq j \leq k.$$

Reformulation of the conjecture

It is possible to reformulate the conjecture purely in terms of the original functions (f_k), thanks to the following proposition.

Proposition

$$\langle e_j, f_k \rangle > 0 \iff \begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & \langle f_2, f_k \rangle \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & \langle f_3, f_k \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & \langle f_j, f_k \rangle \end{vmatrix} > 0.$$

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Corollary

The conjecture is false if we exchange f_2 and f_6 .

Other formulas for $\langle f_j, f_k \rangle$

Plancherel's formula

For all $j, k \geq 2$,

$$\langle f_j, f_k \rangle = \frac{1}{2\pi jk} \int_{-\infty}^{\infty} (j^{\frac{1}{2}-it} - 1)(k^{\frac{1}{2}+it} - 1) \frac{|\zeta(\frac{1}{2} + it)|^2}{\frac{1}{4} + t^2} dt.$$

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Asymptotic formula (Báez-Duarte, Balazard, Landreau, Saias '05)

For each $j \geq 2$,

$$\lim_{k \rightarrow \infty} \frac{\langle f_j, f_k \rangle}{(\log k)/2k} = \frac{j-1}{j}.$$

Using this, we deduce:

Another positivity result

Theorem

Let $j \geq 2$. If

$$\begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & 1/2 \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & 2/3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & (j-1)/j \end{vmatrix} > 0,$$

then there exists $k_0(j)$ such that $\langle e_j, f_k \rangle > 0$ for all $k \geq k_0(j)$.

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then there exists $k_0(j)$ such that $\langle e_j, f_k \rangle > 0$ for all $k \geq k_0(j)$.

The hypothesis holds for $2 \leq j \leq 100$ (numerical check). Hence

Corollary

$\langle e_j, f_k \rangle > 0$ for $2 \leq j \leq 100$ and all sufficiently large k .

Questions:

- Is the conjecture true?
- If yes, does it imply RH?
- Perhaps it is a consequence of RH?

All these questions remain open. . .

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