A bundling problem: Herd instinct versus individual feeling

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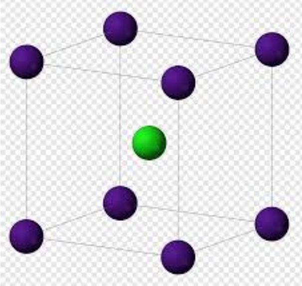
Steady-State Heat Distribution in Space

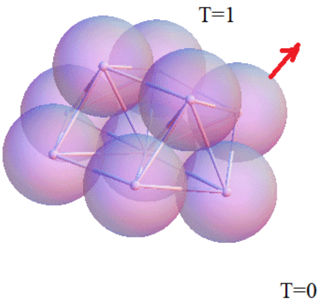
Consider a compact set E in $\mathbb{R}^3$ such that its exterior $\Omega(E) = \mathbb{R}^3 \setminus E$ is an unbounded domain regular for the Dirichlet's problem for harmonic functions. We assume further that E consists of a finite number of connected components $E_1, \ldots, E_n$. We mostly work with n equal balls.



Temperature of $\Omega(E)$

Let $T(x)$ be a steady-state distribution of heat on $\Omega(E)$ with $T = 1$ on E and $T(x) \to 0$ as $x \to \infty$.
Then T is harmonic in $\Omega(E)$:

$$\Delta T = 0 \quad \text{on } \Omega(E).$$



Newtonian Capacity

In fact,

$$T(x) = 1 - g_{\Omega(E)}(x, \infty) = \frac{cap(E)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } x \to \infty, \quad (1)$$

where $g_{\Omega(E)}$ is Green's function of $\Omega(E)$ having singularity at ∞ and $cap(E)$ is the Newtonian capacity of E.

The Newtonian capacity can be defined by the following minimization problem:

$$\text{cap}(E) = \frac{1}{4\pi} \inf_{u \in \mathcal{A}(E)} \int_{\mathbb{R}^3} |\nabla u(x)|^2 \, dV, \quad (2)$$

where $\mathcal{A}(E)$ denote the class of functions u, continuous in $\mathbb{R}^3$ and Lipschitz on compact subsets of $\Omega(E)$, and such that $0 \le u \le 1$ in $\mathbb{R}^3$, $u = 1$ on E, and $\lim_{|x| \to \infty} u(x) = 0$.

Unique Minimizer

As is well known, the function $T(x)$ is the unique minimizer of the Dirichlet integral in (2). Thus,

$$\text{cap}(E) = \frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla T(x)|^2 \, dV. \tag{3}$$

Heat flux

We assume that the boundaries $S_k = \partial E_k$ are sufficiently smooth. Let D be a domain on $S = \cup_{k=1}^{n} S_k$. The "heat flux" (or "loss of heat") from D is given by

$$Q(D|E) = \iint_D \nabla T \cdot \vec{n} \, dS, \tag{4}$$

where $\vec{n}$ is the inward normal to S. Let $Q_k = Q(S_k|E)$ and let $Q = \sum_{k=1}^{n} Q_k$ be the heat flux from S. Then $Q(S|E)$ - the heat flux from the whole surface or total heat flux.

Heat flux and Newtonian capacity:

$$Q(S|E) = 4\pi \mathrm{cap}(E). \tag{5}$$

General Questions

Our goal here is to discuss the following:

(1) How the fluxes Q_k change if components E_k move closer to each other.

(2) How the total flux $Q(S|E)$ or, equivalently, Newtonian capacity changes when ball components of E move in space.

(3) What are minimizing/maximizing configurations of balls in simple cases.

A bundling problem: A little of History

This study was initiated by M. L. Glasser and S. G. Davison, who considered the case of two disjoint balls B_1 and B_2 of unit radius in $\mathbb{R}^3$ at the distance $2a > 2$ between their centers.

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A BUNDLING PROBLEM*

M. L. GLASSER AND S. G. DAVISON†

It is a primeval observation of most warm blooded creatures that by huddling together they can keep each other warm. However, most of the work on this phenomenon appears to be experimental. In this note, we wish to present a calculation which applies, e.g., to the case of armadillos.

To simplify the situation as much as possible without losing sight of the essentials, we shall consider sleeping armadillos as uniform spherical heat sources and the earth in which they imbed themselves as homogeneous and infinitely extended. Thus we consider two spherical constant temperature heat sources, of the same (unit) radius, at temperature $T = T_0$ above that of an infinite medium of uniform thermal diffusivity, in which they are embedded. We shall determine, in the steady state, the amount of heat given off by either of them as a function of the distance $2a$ between their centers.

Illustrations

Armadillos are cute and friendly animal living in Lubbock and around.

Infants prefer to stay with their mom.

Armadillos like company, especially in cold nights.

When armadillos sleep at night they huddle together and each of them folds in a spherical shape.

Why they pack in groups in cold nights?

Possible reasons:

1. Safety first.
2. Social life.
3. Smell attraction.
4. Baby-mother relations.
5. Keep it warm !

Any questions about armadillos?

M. L. Glasser and S. G. Davison suggested that it was keeping them warm what keep them closer to each other. At least for two armadillos of equal size and of spherical shape.

M. L. Glasser and S. G. Davison found an explicit expression for the loss of heat in this case, assuming that the radius is 1 and the distance between the centers is $2a \geq 2$:

$$Q(a) = 8\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{U_k(a)}, \qquad (6)$$

U_k - Chebyshev polynomial of 2-nd kind.

Glasser-Davison Problems

As these authors noticed, even with a nice explicit expression at hand, a mathematical proof of the monotonicity of $Q(a)$ was problematic. Thus, they raised two questions.

Problem (Quote from [GD])

"It was with some surprise that we were unable to find a mathematical proof of the monotonic increase of $Q(a)$ for $a > 1$ which is an interesting open problem".

Problem (Quote from [GD])

"The numerical calculation supports the empirical result that the greatest warming effect occurs for tangency. Since $Q(1)/Q(\infty) = \log 2$, the output is reduced by nearly a third. The extension of this calculation to three (or more) spheres should be valuable in the study of Armadillo colonies".

Glasser and Davison stop short of proving monotonicity of (6) as a function of *a*. That such a proof is possible was shown by R. A. Todor, who proved the following.

Proposition (R.A. Todor, 2004)

The following inequality holds true:

$$\frac{d}{da}Q(S_1|E(1,a)) = 2\pi\sum_{n=1}^{\infty}(-1)^{n+1}\frac{U_n'(a/2)}{U_n(a/2)^2} > 0 \quad \textit{for all } a > 2. \tag{7}$$

In fact, earlier, in 2003, A. Eremenko noticed that a more general result, for two balls of different radii and even in some situations for any number of components E_k of arbitrary shape, follows from the well-known contraction principle for the energy integral

$$I(\mu) = \int_{E\times E} \frac{d\mu(x)\,d\mu(y)}{|x-y|}, \tag{8}$$

of a measure μ supported on E such that $\mu(E) = 1$. Energy integrals provide the following alternative way to define capacity:

$$\operatorname{cap}(E) = \inf_{\mu} \left(1 \Big/ \int_{E\times E} \frac{d\mu(x)\,d\mu(y)}{|x-y|}\right), \tag{9}$$

where the infimum is taken over all probability measures μ having support on E.

Interpretation in terms of animal behavior: A. Eremenko wrote: "…it is its own loss of heat that an individual animal feels, and the behavior we discuss is probably driven by individual feelings rather then some abstract "community goal"." Therefore, "it seems more interesting from the point of view of animal behavior, and more challenging mathematically, to find under what conditions one can assert that as the animals come closer together, the rate of heat loss decreases for each individual animal". To emphasize Eremenko's questions, we state those in two problems.

Problem (Eremenko's problem 1)

Show, as Eremenko suggested, that, in general, the individual rates of loss of heat might not be monotone under contractions.

Eremenko also suggested that in simple situations monotonicity of the individual losses of heat could occur. In this relation, he proposed the following specific questions.

Problem (Eremenko's problem 2)

Suppose that E consists of

(a) *two balls of unequal radii, or*

(b) *three balls of equal radii.*

Is it true that if we move such balls closer (such that all pairwise distances between their centers decrease) the rate of heat loss for each ball will decrease?

Contraction

First we give a version of a well known theorem of Landkof.

Proposition

Let E and $\widehat{E}$ be compact sets in $\mathbb{R}^3$ having sufficiently smooth boundaries S and $\widehat{S}$ respectively, and let φ be one-to one map from E onto $\widehat{E}$, such that for $x, x' \in E$

$$|\varphi(x) - \varphi(x')| \leq k|x - x'|, \tag{10}$$

where $k > 0$ is a constant. Then we have

$$Q(\widehat{S}|\widehat{E}) \leq kQ(S|E). \tag{11}$$

If there are D_1 and D_2 on S of positive surface area such that (10) holds with the sign of strict inequality for $x \in D_1$, $x' \in D_2$, then (11) holds with the sign of strict inequality.

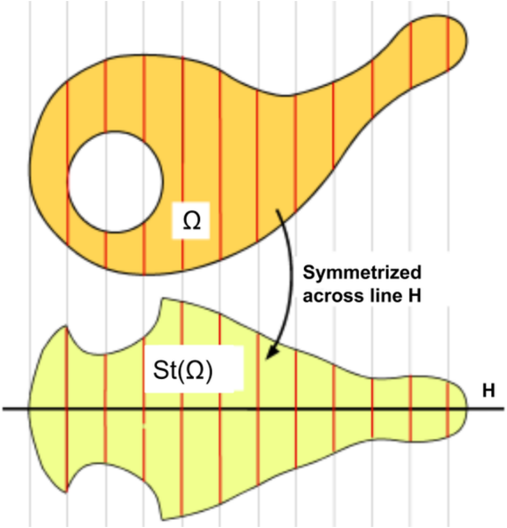
Steiner symmetrization

Figure: Jakob Steiner (1796–1863)

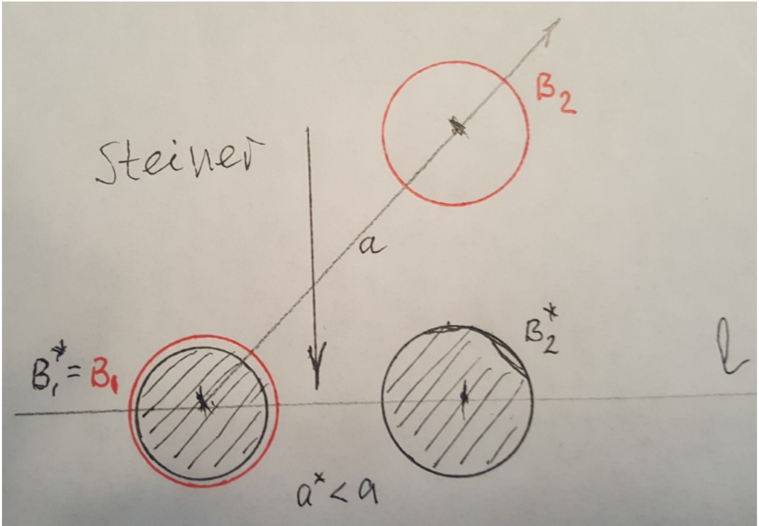
"Calculating replaces thinking while geometry stimulates it".

Jakob Steiner, a self made Swiss farmer's son and contemporary of Gauss was the foremost "synthetic geometer." He hated the use of algebra and analysis and distrusted figures. He proposed several arguments to prove that the circle is the largest figure with given boundary length. Besides symmetrization, his four-hinge method has great intuitive appeal, but is limited to two dimensions. He published several proofs trying to avoid analysis and the calculus of variations.

Steiner symmetrization



Steiner symmetrization



Symmetrization

Proposition

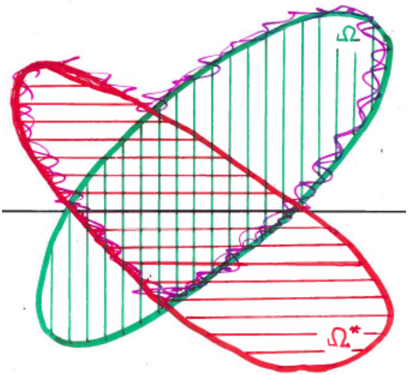
Let E^* be a Steiner symmetrization or a cap-symmetrization of a compact set E in $\mathbb{R}^n$. Then

$$Q(S^*) \leq Q(S) \quad \text{or equivalently} \quad cap\,(E^*) \leq cap\,(E). \qquad (12)$$

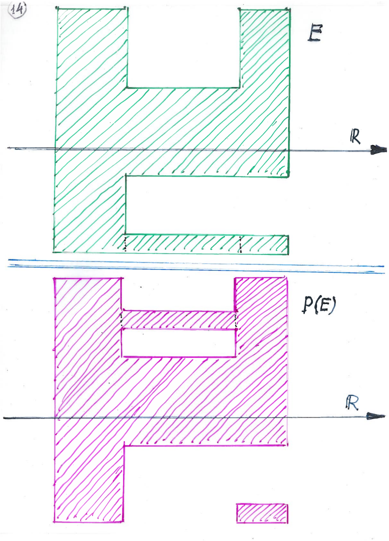
Furthermore equality occurs in (13) if and only if E^* coincides with E up to the corresponding symmetry.

Polarization

Polarization was introduced by V. Wolontis in 1972, then two different approaches to polarization were developed in papers by V. Dubinin and in my papers.



Polarization



Polarization

Proposition

Let E_p be a polarization of a compact set E in $\mathbb{R}^n$. Then

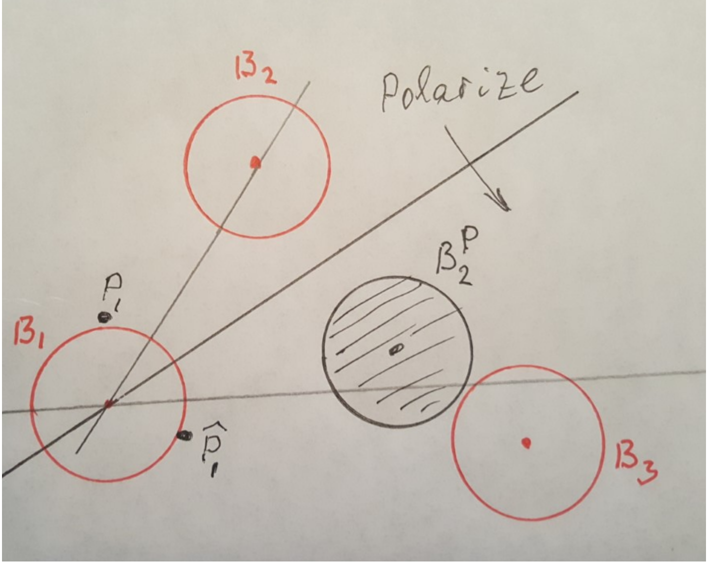
$$Q(S_p) \leq Q(S) \quad \text{or equivalently} \quad cap\,(E_p) \leq cap\,(E). \qquad (13)$$

Furthermore equality occurs in (13) if and only if E_p coincides with E up to reflection with respect the plane of polarization.

Interestingly enough, ANY ONE !!! of the above transformations / propositions can be used to prove the following.

Corollary

The individual rate of loss of heat by each ball $B_1(a)$ and $B_2(a)$ is a strictly increasing function on $a > 0$.


B_2
Polarize
B_2^P
P_1
B_1
$\hat{P}_1$
B_3

Approximation: Heat flux from a fixed region on the boundary of varying compact sets

We need this to construct counterexamples concerning possible extensions to more general cases.

Proposition

Let D be a C^2 boundary component of S or a C^2 domain on S. Let K_j be a sequence of admissible compact sets such that $K_j \subset K_{j+1}$ for all $j \in \mathbb{N}$, $K = \cup_{j=1}^{\infty} K_j$, and $D \subset \partial\Omega(K_j)$ for all $j \in \mathbb{N}$. Then for any domain G such that $\overline{G} \subset D$,

$$Q(G|K_j) \to Q(G|K) \quad \text{as } j \to \infty. \tag{14}$$

Kangaroo bag effect

First, we answer to Eremenko's problem 1. Monotonicity property of individual losses of heat fails, in general, when two equal balls are replaced by two continua of different shapes.

It is warmer in the bag!

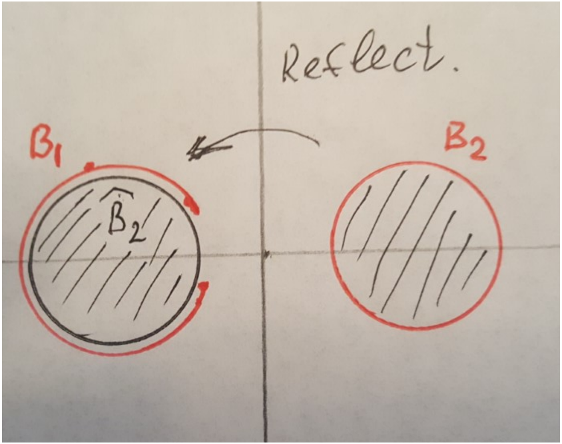
Kangaroo bag effect

For $\varepsilon \geq 0$ sufficiently small, let

$$E_1(\varepsilon) = \{x \in \mathbb{S} : \ x_1 \leq 1 - \varepsilon\} \quad \text{and} \quad E_2(\varepsilon) = \mathbb{B}_{1-\varepsilon}(2\mathbf{i}). \qquad (15)$$

Let $E_1^*(\varepsilon) = E_1(\varepsilon)$, $E_2^*(\varepsilon) = \mathbb{B}_{1-\varepsilon}$, $E(\varepsilon) = E_1(\varepsilon) \cup E_2(\varepsilon)$, $E^*(\varepsilon) = E_1^*(\varepsilon) \cup E_2^*(\varepsilon)$, and let $\Omega(\varepsilon) = \Omega(E(\varepsilon))$, $\Omega^*(\varepsilon) = \Omega(E^*(\varepsilon))$.
In case $\varepsilon = 0$, $E(\varepsilon)$ is a union of two touching balls of unit radius and $E^*(0)$ coincides with $\overline{\mathbb{B}}$. In both cases the loss of heat by each ball is known.

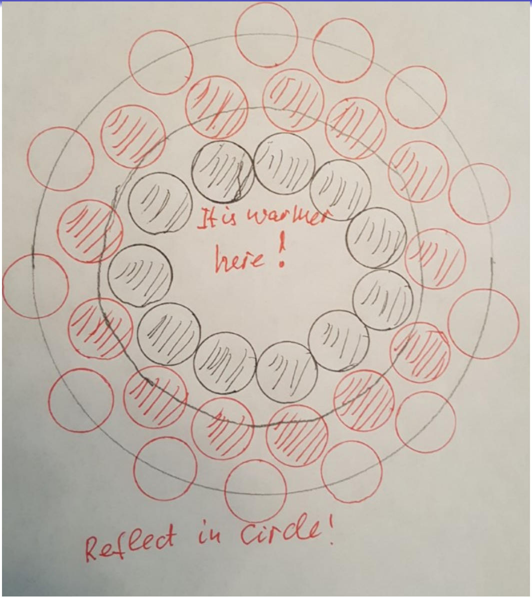
Kangaroo bag effect



Herd effect

We replace the arcs $E_1(\varepsilon)$ and $E_2(\varepsilon)$ in the previous example by two chains of small balls approximating these arcs. Polarizing configuration as above, we find that some of the balls in the exterior chain lose more heat than they lose in the original configuration.

Heard effect



Herd effect

Problem

a) *Given n balls of equal radii, find an arrangement with the minimal loss of heat (with minimal capacity).*

b) *Prove that the limiting positions (as $n \to \infty$) of the centers of the exterior balls form a spherical shape (circular shape in $\mathbb{R}^2$).*

c) *Prove that the limiting positions (as $n \to \infty$) of the centers of all disks form a hexagonal lattice in $\mathbb{R}^2$.*

Eremenko's problem for three equal balls

Theorem

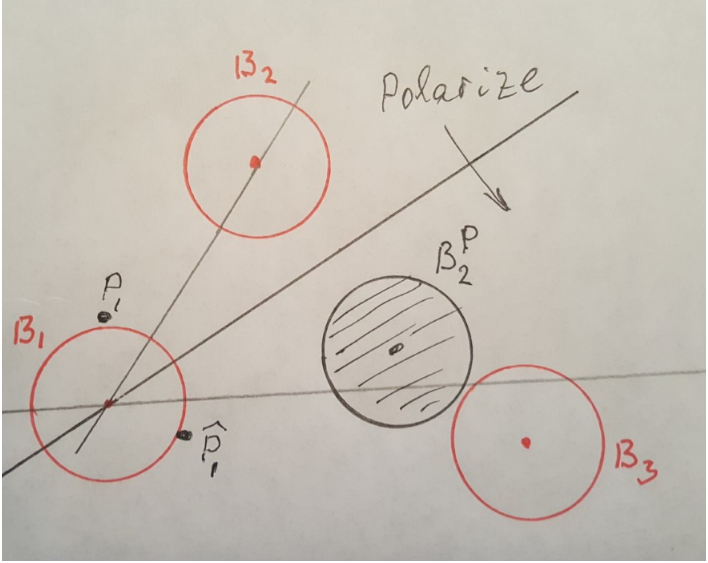
Let E_p be polarization of a compact set $E \subset \mathbb{R}^3$ into the half-space $\{x_3 < 0\}$. Let $T(x)$ and $T_p(x)$ be steady-state distributions of heat in $\Omega(E)$ and $\Omega(E_p)$ respectively. If $x = (x_1, x_2, x_3) \in \Omega(E_p) \setminus \{x_3 < 0\}$ and $\overline{x} = (x_1, x_2, -x_3)$ then

$$T_p(x) \leq \min\{T(x), T(\overline{x})\}, \tag{16}$$

$$T_p(x) + T_p(\overline{x}) \leq T(x) + T(\overline{x}). \tag{17}$$

Furthermore, if equality occurs in (16) or (17) then E_p coincides with E up to reflection in the plane $\{x_3 = 0\}$.

Proof. Since $T(x) = 1 - g_{\Omega(E)}(x, \infty)$ and $T_p(x) = 1 - g_{\Omega(E_p)}(x, \infty)$, the required result follows from Theorem 1 in [Sol96]. □



B_2
Polarize
B_2^P
P_1
B_1
$\hat{P}_1$
B_3

Theorem

Under the assumptions of Theorem 7, assume in addition that E_0 is a compact subset of E, which is symmetric with respect to $\{x_3 = 0\}$. Then $E_0 \subset E_p$ and

$$Q(E_0|E) \leq Q(E_0|E_p). \tag{18}$$

Furthermore, if equality occurs in (18) then E_p coincides with E up to reflection in the plane $\{x_3 = 0\}$.

Proof. Proof follows from the previous theorem. □

Negative answer to Eremenko’s question for 3 balls.

Theorem

The loss of heat by the central body $E_0 = \overline{\mathbb{D}}$ is an increasing function of θ.

Proof. Follows from the previous theorem. □

Applying Theorem 9 to configuration consisting of three equal balls two of which are at equal distance from the third one, we obtain the following.

Corollary

The loss of heat by the central ball $E_0 = \overline{\mathbb{D}}$ is a strictly increasing function of θ while the loss of heat by each of two other balls is a strictly decreasing function of θ.

This gives a negative answer to a question on the loss of heat under motion which decrease pairwise distances between the centers of three balls raised by Alex Eremenko.

Corollary

There is a motion strictly decreasing distances between centers of three equal balls, which strictly increase the loss of heat by the central ball.

Eremenko's problem for two unequal balls

The answer to this Eremenko's question is again negative! This unexpected result was found by P. Ivanishvili in 2016.

Theorem (P. Ivanishvili, 2016)

For very fixed r, $0 < r < r_0$, there is $a(r) > 1 + r$ such that $Q(S_1|E(r,a))$ strictly decreases on the interval $1 + r \leq a \leq a(r)$. Furthermore, for every r, fixed in the interval $(0, r_0)$, $Q(S_1|E(r,a))$ is not a monotonic function of a.

Proposition (P. Ivanishvili, 2016)

Let $0 < r < 1$, $a > 1 + r$ and let $d = a - 1 - r$. Then the following holds:

$$Q(S_1|E(r,a)) = -\frac{4\pi}{1+r}\left(\gamma r + r\psi\left(\frac{r}{1+r}\right)\right.$$
$$\left. + \frac{d}{6(1+r)^2}\left(2(1+r^3)\left(\gamma + \psi\left(\frac{r}{1+r}\right)\right) + r + r^2 + 2r\right.\right. \tag{19}$$

as $d \to 0$. In the above notation γ is the Euler's constant and ψ is the digamma function.

Paata Ivanisvili’s graph

4

The answer is negative.

Assume that K_0 and K_1 are two disjoint balls. If K_0 is at least twice as big as K_1, then K_0 should keep some nonzero distance in order to minimize his heat loss while K_1 should try to be as close as possible to K_0.

To be more precise let r_0 and r_1 denote the radii of the balls K_0 and K_1. Let

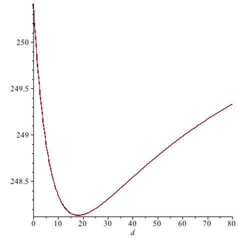
$$Q_j(d) = \int_{\partial K_j} \frac{\partial u}{\partial n} ds \quad \text{for} \quad j = 0, 1$$

denote the heat loss as a function of distance between the balls. Then $Q_0(d)$ is not monotonically increasing provided that $\frac{r_0}{r_1} > \ell$ where $\ell \approx 1.95$ is the positive solution of the equation

$$2(1+x^3)\left(\gamma + \psi\left(\frac{1}{1+x}\right)\right) + x^2 + x + 2(x^2 - x)\psi'\left(\frac{1}{1+x}\right) = 0.$$

In the above notation γ is the Euler--Mascheroni constant, and ψ is the digamma function. I have uploaded a preprint on arXiv about this problem.

In the following figure the graph of $Q_0(d)$ is given when $r_0 = 20$, $r_1 = 1$, $0 \le d \le 80$ and the temperature of the balls equals 1.



Actually numerical computations show that if the sizes of the balls are comparable, namely, $\frac{1}{\ell} \le \frac{r_0}{r_1} \le \ell$ then the individual heat loss Q_j, $j = 0, 1$ is monotonically increasing, however, I was not able to prove this.

Also based on numerical computations if $\frac{r_0}{r_1} > \ell$ then the nonzero distance that the big ball has to keep approximately equal to r_0. The heat loss of the small ball is always monotonically increasing.

Numerical computations performed by Ivanishvili suggest that the following properties hold.

Conjecture:

(a) Prove that for every fixed r, $0 < r < 1$, $Q(S_r(a)|E(r,a)$ is a strictly increasing function of a, $a \geq 1 + r$.

(b) Prove that for every fixed r, $r_0 < r < 1$, $Q(S_1|E(r,a)$ is a strictly increasing function of a, $a \geq 1 + r$.

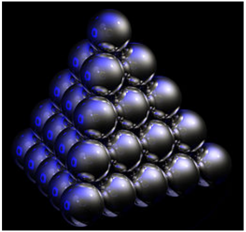
(c) Prove that for every fixed r, $0 < r < r_0$, there is $a(r)$, $1 + r < a(r) < \infty$, such that $Q(S_1|E(r,a)$ is a strictly decreasing function on the interval $1 + r < a < a(r)$ and a strictly increasing function on the interval $(a(r) < a < \infty$.

n-ball problems

In this section, we work with n equal balls in $\mathbb{R}^n$. Our main goal here is to draw attention to several challenging problems.

Problem

Given $n \geq 2$ equal balls, arrange them in $\mathbb{R}^3$ in such a way that the Newtonian capacity or, equivalently, the total loos of heat of their union is the minimal possible.



A configuration of balls is called *liner* if their centers lie on the same line.

Problem

Prove that the linear connected configuration of n equal balls has the maximal Newtonian capacity or, equivalently, the maximal total loos of heat among all connected configurations of n equal balls.

Glasser-Davison tangency problem

The following theorem gives a partial answer to the question on tangency raised by Glasser and Davison as it was stated in Problem 2.

Theorem

Let $E^* = \cup_{k=1}^{n} B_k^* \in \mathcal{B}_n$, $n \geq 3$, be a configuration of balls extremal for Problem 5 and let B_k^* has its center at the point c_k^*.

(1) *Suppose that the point c_k^* is one of the vertices of the convex hull $\widehat{C}$ of the set of points $c_1,^*, \ldots, c_n^*$.*
Then there is a ball B_l^*, $l \neq k$, that is tangent to B_k^*.

(2) *Suppose that the points $c_{k_1}^*, \ldots, c_{k_j}^*$, $2 \leq j < n$, belong to an edge L of the convex hull $\widehat{C}$ and there are no other points c_m^* on L.*
Then there is a ball B_l^* centered at $c_l^* \notin L$, which is tangent to at least one of the balls $B_{k_1}^*, \ldots, B_{k_j}^*$.

(3) *Suppose that the points $c_{k_1}^*, \ldots, c_{k_j}^*$, $3 \leq j < n$, belong to a face F of the convex hull $\widehat{C}$ and there are no other points c_m^* on F.*
Then there is a ball B_l^* centered at $c_l^* \notin F$, which is tangent to at least one of the balls $B_{k_1}^*, \ldots, B_{k_j}^*$.

Problem

Tangency problem Is it true that every configuration of $n \geq 5$ balls minimizing the Newtonian capacity over the set $\mathcal{B}_n$ is connected?
If the answer to this question is negative then finding a counterexample would also be an interesting task.

Solution of the *n*-ball problem for $n \leq 4$

Theorem

For each n, $2 \leq n \leq 4$, there is only one, up to a rigid motion in $\mathbb{R}^3$, arrangement of n balls each of radius 1 minimizing the Newtonian capacity of the union of these balls. These minimizing arrangements are the following:

- $n = 2$ *- two touching balls,*
- $n = 3$ *- three balls touching each other with centers at the vertices of an equilateral triangle,*
- $n = 4$ *- four balls touching each other with centers at the vertices of a tetrahedron,*

Two more problems

It is a common knowledge that problems in geometry and its applications to other areas of mathematics, which include many balls, are notoriously difficult. Thus, it is expected that, for $n > 5$, solution of Problem 5 will require extensive numerical calculations with modern computers. However, two more specific questions stated below may be easier to handle.

Problem

Prove or disprove that the configuration of five touching balls of unit radii having centers at the vertices of the triangular bipyramid minimizes the Newtonian capacity over the family $\mathcal{B}_5$.

Problem

Let $\mu(n) = \min\{cap(E) : E \in \mathcal{B}_n\}$. Find asymptotic of $\mu(n)$ as $n \to \infty$.

Chains of of balls

Next, we consider connected configurations of balls. By a chain of n balls we understand a set $E = \cup_{k=1}^{n} B_k \in \mathcal{B}_n$ such that B_k touches B_{k+1} for $k = 1, \ldots, n-1$. Thus, for $k = 2, \ldots, n-1$, B_k has at least two neighbors and the balls B_1 and B_n each has at least one neighbor. By E_n^- we denote a linear chain of balls B_k centered at the points $c_k = (2(k-1), 0, 0)$, $k = 1, \ldots, n$.

Theorem

Let $E = \cup_{k=1}^{n} B_k \in \mathcal{B}_n$ be a chain of $n \geq 3$ balls. Then

$$cap(E) \leq cap(E_n^-) \tag{20}$$

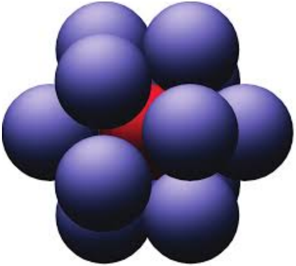
with the sign of equality if and only if E coincides with E_n^- up to a rigid motion.

Individual loss of heat

Our previous results deal with the total loss of heat by n balls. Next, we state a problem on the loss of heat by individual balls.

Problem

Arrange $n \geq 2$ equal balls in $\mathbb{R}^3$ in such a way that the loss of heat by a fixed ball, say by the ball B_1, is the minimal possible.



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THANK YOU!