Theory of Partial Differential Equations in Sobolev spaces: Perspectives and Developments

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Outline

- Introduction: PDE in Sobolev spaces
  - (a) Theory for Laplace equation (Calderón-Zygmund)
  - (b) Some extension: known results for equations with uniformly elliptic and bounded coefficients
- Equations with singular-degenerate coefficients: motivations, problems/questions
- Some (simplified) results for equations with singular or degenerate coefficients
- Ideas in the proofs and remarks

Linear second order elliptic/parabolic equations

- Non-divergence form equations

$$u_t - a_{ij}(t,x)D_{ij}u(t,x) = f(t,x)$$

- Divergence form equations

$$u_t - D_i[a_{ij}(t,x)D_j u] = f(t,x)$$

- Here, $u(t,x)$ be an unknown physical/biological quantity, $f(t,x)$ is a given "external force", and $a_{ij}(t,x)$. Moreover D_i is the spatial partial derivative in i^{th}-direction:

$$D_i u = u_{x_i}, \quad D_{ij} = u_{x_i x_j}.$$

- Question: If $f \in L_p((0,T)\times\Omega)$ with some $p \in (1,\infty)$ and some spatial domain $\Omega \subset \mathbb{R}^d$, i.e.

$$||f||_{L_p((0,T)\times\Omega))} = \left(\int_0^T \int_\Omega |f(t,x)|^p dxdt\right)^{1/p} < \infty$$

can we control u, Du, D^2u, and u_t in L_p?

Laplace equation

- We consider the Laplace equation

$$-\Delta u(x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^d$$

where

$$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_d x_d}$$

- For $p \in (1, \infty)$, is there $N = N(d, p) > 0$

$$\int_{\mathbb{R}^d} |D^2 u(x)|^p dx \leq N \int_{\mathbb{R}^d} |f(x)|^p dx$$

for every smooth, compactly supported solution u?

- Note that $D^2 u$ is the Hessian matrix of u:

$$D^2 u = (D_{ij} u)_{i,j=1}^d \quad \text{mean while} \quad \Delta u = \text{trace} D^2 u,$$

where $D_i u = u_{x_i}$ and $D_{ij} u = u_{x_i x_j}$.

- **Note**: For an arbitrary matrix M, knowing the trace does not mean we know the matrix.

Laplace equation: energy estimate ($p = 2$)

- Is it true that
$$\int_{\mathbb{R}^d} |D^2u(x)|^2 dx \le N \int_{\mathbb{R}^d} |f(x)|^2 dx \quad \text{when} \quad -\Delta u(x) = f(x)?$$
- By squaring the equations, we obtain
$$\sum_{i,k=1}^{d} \int_{\mathbb{R}^d} u_{x_i x_i}(x) u_{x_k x_k}(x) dx = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$
- Note that by using the integration by parts, we obtain
$$\int_{\mathbb{R}^d} u_{x_i x_i} u_{x_k x_k} dx = \int_{\mathbb{R}^d} u_{x_i x_k} u_{x_i x_k} dx = \int_{\mathbb{R}^d} |u_{x_i x_k}(x)|^2 dx.$$
- Therefore,
$$\int_{\mathbb{R}^d} |D^2u(x)|^2 dx = \int_{\mathbb{R}^d} |f(x)|^2 dx.$$
- Question: What about $p \neq 2$?

Calderón-Zygmund theory (for Laplace equation)

Theorem (Calderón-Zygmund (1950-1960))

If $u \in C_0^\infty(\mathbb{R}^d)$ is a solution of

$$-\Delta u(x) = f(x), \quad x \in \mathbb{R}^d$$

with $f \in L_p(\mathbb{R}^d)$ for $p \in (1, \infty)$, then

$$\int_{\mathbb{R}^d} |D^2 u(x)|^p dx \le N(d,p) \int_{\mathbb{R}^d} |f(x)|^p dx.$$

Proof.

- Write
$$u_{x_i x_j}(x) = \int_{\mathbb{R}^d} K_{ij}(x-y) f(y) dy$$
with some singular kernel K_{ij}.
- Use their developed "theory of singular integral operators". □

Modern approaches

- Krylov (~2003): Based on Fefferman-Stein theorem for sharp function

$$\|f^{\#}\|_{L_p(\mathbb{R}^d)} \sim \|f\|_{L_p(\mathbb{R}^d)}$$

where

$$f^{\#}(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - \bar{f}_{B_r(x)}| dy$$

with $\bar{f}_{B_r(x)}$ the average of f in the ball $B_r(x)$.

Use the PDE to control $(D^2u)^{\#}$

- Caffarelli-Peral (CPAM - 2003): Based on level set estimates

$$\int_{\mathbb{R}^d} |D^2u(x)|^p dx = N(n,p) \int_0^\infty \lambda^{p-1} \Big|\{x : |D^2u(x)| > \lambda\}\Big|^p d\lambda$$

Use the PDE to control $\Big|\{x : |D^2u(x)| > \lambda\}\Big|$

- Both approaches work for linear, nonlinear and fully nonlinear equations.

Main steps in Krylov's approach (oscillation estimates)

- Let us consider the harmonic function

$$-\Delta u = 0 \quad \text{in} \quad B_2(x_0).$$

- We know (1st PDE course)

$$\|D^k u\|_{L_\infty(B_1(x_0))} \le N(d,k) \fint_{B_2(x_0)} |u(x)|dx.$$

- Then, by mean value theorem (Cal I)

$$\fint_{B_1(x_0)} |D^2u(y) - \overline{D^2u}_{B_1(x_0)}|dy \le \|D^3u\|_{L_\infty(B_1(x_0))}$$

- Therefore, we can control the oscillation of D^2u in $B_1(x_0)$ by

$$\fint_{B_1(x_0)} |D^2u(y) - \overline{D^2u}_{B_1(x_0)}|dy \le N(d) \fint_{B_2(x_0)} |u(x)|dx.$$

- Then (with a little bit more work) we use Fefferman-Stein theorem to derive the L_p-estimate of D^2u.

Equations with variable coefficients

- CZ theory has been extended equations in non-divergence and divergence form (elliptic, parabolic, linear, nonlinear)

$$
\begin{aligned}
(ND) \qquad & -a_{ij}(x)D_{ij}u(x) + c(x)u(x) = f(x) \\
(D) \qquad & -D_i[a_{ij}(x)D_j u(x)] + c(x)u(x) = f(x)
\end{aligned}
$$

- The coefficients matrix a_{ij} is bounded, and uniformly elliptic: there is $\nu \in (0,1)$ such that

$$\nu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \quad \text{and} \quad |a_{ij}(x)| \leq \nu^{-1}$$

  for all x and for all $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$.
- (a_{ij}) is sufficiently smooth: $a_{ij} \in VMO$ is sufficient (Sarason's class of functions).
- Refs: Di Fazio-Ragusa (1991); Maugeri-Palagachev-Softova (2000-book); Krylov (2003-book); Caffarelli-Peral (2003); Acerbi-Mingione (2007); Byun-Wang (2012,...), Hoang-Nguyen-P. (2015),...

A simple example (for the perturbation technique)

- Consider

$$-a_{ij}(x)D_{ij}u = f \quad \text{in} \quad \mathbb{R}^d.$$

- Assume (for simplification) that

$$|a_{ij}(x) - \delta_{ij}| \leq \epsilon \quad \text{for all} \quad x.$$

- We write (freezing the coefficients)

$$-\Delta u = g, \quad g := [a_{ij}(x) - \delta_{ij}]D_{ij}u + f$$

- Then,

$$||D^2 u||_{L_p(\mathbb{R}^d)} \leq N\Big[\epsilon ||D^2 u||_{L_p(\mathbb{R}^d)} + ||f||_{L_p(\mathbb{R}^d)}\Big].$$

- If ϵ is sufficiently small, we obtain

$$||D^2 u||_{L_p(\mathbb{R}^d)} \leq N||f||_{L_p(\mathbb{R}^d)}.$$

Equations with singular/degenerate coefficients

- Denote $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, \infty)$ the upper half space. We write $x = (x', x_d) \in \mathbb{R}^d_+$ where
$$x' \in \mathbb{R}^{d-1} \quad \text{and} \quad x_d \in \mathbb{R}_+ = (0, \infty).$$
- We study the following class of equations:
$$(D) \qquad x_d^\alpha (u_t + u) - D_i[x_d^\alpha a_{ij}(t,x) D_j u] = x_d^\alpha f(t,x)$$
or
$$(ND) \qquad u_t + u - a_{ij}(t,x) D_{ij} u(t,x) + \frac{\alpha}{x_d} a_{dj}(t,x) D_j u = f(t,x)$$
- Boundary condition on $\partial \mathbb{R}^d_+ = \{x_d = 0\}$:
$$\text{Conormal (zero flux)}: \quad \lim_{x_d \to 0^+} x_d^\alpha a_{dj}(t,x) D_j u(t,x) = 0 \quad \text{or}$$
$$\text{Dirichlet}: \quad u(t, x', 0) = 0$$
- Here, $\alpha \in \mathbb{R}$ is given constant; (a_{ij}) is bounded, and uniformly elliptic.

Equations with singular/degenerate coefficients

- Recall $x = (x', x_d) \in \mathbb{R}^d_+$, $t \in \mathbb{R}$ and we consider

$$(D) \qquad x_d^\alpha(u_t + u) - D_i[x_d^\alpha a_{ij}(t,x) D_j u] = x_d^\alpha f(t,x)$$

- When $\alpha > 0$, the coefficients are degenerate. Meanwhile, when $\alpha < 0$ the coefficients are singular (on $\{x_d = 0\} = \partial\mathbb{R}^d_+$).
- Motivation: Geometric PDE, Calculus of Variations, Probability theory, Mathematical finance, Math Biology, Non-local PDE.
- Question: L_p-theory for the PDE (what is the right functional space setting for the PDE?).
- Issue: The coefficients $x_d^\alpha a_{ij}(t,x)$ may not bounded, not uniformly elliptic, and not sufficiently smooth (even not locally integrable as $\alpha \leq -1$).

Energy estimate (the first try)

- We consider a solution $u \in C_0^\infty$ of the PDE

$$x_d^\alpha(u_t + u) - D_i[x_d^\alpha a_{ij}(t,x)D_j u] = x_d^\alpha f(t,x) \quad x \in \mathbb{R}^d_+, \quad t \in \mathbb{R}$$

  with either the Dirichlet or the conormal boundary condition on $\{x_d = 0\}$.

- Energy estimate (integration by parts, and Cauchy-Schwartz inequality):

$$\int_{\mathbb{R}^{d+1}_+} |Du(t,x)|^2 x_d^\alpha dxdt + \int_{\mathbb{R}^{d+1}_+} |u(t,x)|^2 x_d^\alpha dxdt$$
$$\leq N \int_{\mathbb{R}^{d+1}_+} |f(t,x)|^2 x_d^\alpha dxdt.$$

- Common thought: Upgrade this estimate: replacing 2 by p for $p \in (1,\infty)$?

A check point with ODEs

- Conormal boundary condition (zero flux):

$$(x^{\alpha} u')' = x^{\alpha-1}, \quad x \in (0,1) \quad \text{with} \quad \lim_{x \to 0^+} x^{\alpha} u'(x) = 0.$$

  Integrate the ODE, we obtain $u(x) = Cx$. Therefore,

$$\int_0^1 |u'(x)|^p x^{\alpha} dx < \infty \quad \text{if and only if} \quad \alpha > -1.$$

- Dirichlet boundary condition: For $\alpha \in (0,1)$, we have $u(x) = x^{1-\alpha}$ is a solution of

$$(x^{\alpha} u')' = 0, \quad x \in (0,1) \quad \text{with} \quad u(0) = 0.$$

  However,

$$\int_0^1 |u'(x)|^p x^{\alpha} dx = C \int_0^1 x^{(1-p)\alpha} dx < \infty$$

  if and only if $p < \frac{1}{\alpha} + 1$.

Theorem 1: Conormal boundary condition (zero flux)

Theorem (Dong-P. (2020))

Let $\alpha \in (-1, \infty)$, $T > 0$, *and* $\Omega_T = (-\infty, T) \times \mathbb{R}^d_+$. *Assume that* $f : \Omega_T \to \mathbb{R}$

$$\|f\|_{L_p(\Omega_T,\mu)} := \left(\int_{\Omega_T} |f(t,x)|^p x_d^\alpha dxdt\right)^{1/p} < \infty, \quad p \in (1, \infty).$$

There exists a unique weak solution u of

$$\begin{cases} x_d^\alpha(u_t + u) - D_i[x_d^\alpha a_{ij}(x_d) D_j u] = x_d^\alpha f(t,x) \quad \text{in} \quad \Omega_T \\ \lim\limits_{x_d \to 0^+} x_d^\alpha a_{dj}(x_d) D_j u(t,x) = 0. \end{cases}$$

Moreover,

$$\|Du\|_{L_p(\Omega_T,\mu)} + \|u\|_{L_p(\Omega_T,\mu)} \leq N \|f\|_{L_p(\Omega_T,\mu)}$$

for $N = N(d, p, \nu, \alpha)$ *and* $d\mu(x,t) = x_d^\alpha dxdt$.

Some remarks on Theorem 1

- The condition $\alpha > -1$ is optimal for the kind of estimates: Example from the ODE and also the measure $\mu(dz) = x_d^{\alpha} dxt$ is finite.

- We only state a simplified version.

- Local boundary estimates, estimates in mixed-norms, weighted mixed-norms are obtained. These are important in case we have anisotropic data.

Dirichlet boundary value problem

- Let us recall that for the Laplace equation: $-\Delta u = f$, we use

$$\int_{\mathbb{R}^d} |D^2 u|^2 dx \leq N \int_{\mathbb{R}^d} |f(x)|^2 dx$$

to derive

$$\int_{\mathbb{R}^d} |D^2 u|^p dx \leq N(d,p) \int_{\mathbb{R}^d} |f(x)|^p dx.$$

- For problem $-D_i[x_d^\alpha a_{ij}(x_d) D_j u] = x_d^\alpha f(t,x)$ with $u(t,x',0) = 0$, we know (the energy estimate)

$$\int_{\Omega_T} |Du|^2 x_d^\alpha dxdt \leq N \int_{\Omega_T} |f(t,x)|^2 x_d^\alpha dxdt.$$

The ODE check point tells us that (when p is large) it is not correct to control

$$\int_{\Omega_T} |Du|^p x_d^\alpha dxdt.$$

What is the right functional setting for the PDE?

Homogeneous equations (Harmonic; α-Harmonic?)

- Recall that for Harmonic function $-\Delta u = 0$, we have

$$\|D^k u\|_{L_\infty(B_1)} \leq N \fint_{B_2} |u(x)|dx, \quad k = 0, 1, 2, \ldots, .$$

- Now, for the equation

$$\begin{cases} -D_i[x_d^\alpha a_{ij}(x_d)D_j u] & = & 0 & \text{in} \quad B_2^+ \\ u(x', 0) & = & 0 & x' \in B_1' \end{cases}$$

  what is the corresponding estimate?

- In the above, B_ρ is the ball in $\mathbb{R}^d$ of radius $\rho > 0$, B_ρ^+ is the upper half ball:

$$B_\rho^+ = B_\rho \cap \{x_d > 0\}$$

  Moreover, B_ρ' is the ball of radius ρ in $\mathbb{R}^{d-1}$

Theorem 2: Dirichlet boundary condition

Theorem (Dong-P. (2020))

Let $\alpha \in (-\infty, 1)$, $T > 0$, $\Omega_T = (-\infty, T) \times \mathbb{R}^d_+$, and $\mu_1(dz) = x_d^{-\alpha} dxdt$. Assume $f : \Omega_T \to \mathbb{R}$ such that

$$\|x_d^\alpha f\|_{L_p(\Omega_T,\mu_1)} = \left(\int_{\Omega_T} |x_d^\alpha f(t,x)|^p x_d^{-\alpha} dxdt\right)^{1/p} < \infty, \quad p \in (1,\infty).$$

There exists a unique weak solution u of

$$x_d^\alpha (u_t + u) - D_i[x_d^\alpha a_{ij}(x_d) D_j u] = x_d^\alpha f(t,x) \quad \text{in} \quad \Omega_T$$

with $u(t,x',0) = 0$ for $t \in (-\infty, T)$, $x' \in \mathbb{R}^{d-1}$. Moreover,

$$\|x_d^\alpha Du\|_{L_p(\Omega_T,\mu_1)} + \|x_d^\alpha u\|_{L_p(\Omega_T,\mu_1)} \le N \|x_d^\alpha f\|_{L_p(\Omega_T,\mu_1)}$$

for $N = N(d,p,\nu,\alpha)$.

Some remarks on Theorem 2

- The condition $\alpha < 1$ is optimal for the kind of estimates: Example from the ODE and also the measure $\mu_1(dz) = x_d^{-\alpha} dxt$ is locally finite.
- We only state a simplified version.
- Estimates in mixed-norms, weighted mixed-norms (anisotropic data), local boundary estimates are obtained.
- Similar results for elliptic equations are also proved.
- The method in our approach also works for system of equations.

Ideas in the proof: an observation

- The energy estimate

$$\int_{\Omega_T} |Du|^2 x_d^{\alpha} dxdt \leq N \int_{\Omega_T} |f(t,x)|^2 x_d^{\alpha} dxdt.$$

is equivalent to

$$\int_{\Omega_T} |x_d^{\alpha} Du|^2 x_d^{-\alpha} dxdt \leq N \int_{\Omega_T} |x_d^{\alpha} f(t,x)|^2 x_d^{-\alpha} dxdt.$$

(note that $1 = 2 - 1$)

- The theorem proves

$$\int_{\Omega_T} |x_d^{\alpha} Du|^p x_d^{-\alpha} dxdt \leq N \int_{\Omega_T} |x_d^{\alpha} f(t,x)|^p x_d^{-\alpha} dxdt.$$

Ideas in the proof: Important regularity estimates

Proposition

For a solution u of

$$x_d^\alpha u_t - D_i[x_d^\alpha a_{ij}(x_d) D_j u] = 0 \quad in \quad Q_2^+ \quad with \quad u(t, x', 0) = 0.$$

Then,

$$x_d^\alpha |u(t,x)| \leq N x_d \left(\fint_{Q_2^+} |x_d^\alpha u(\hat{t}, \hat{x})|^2 \hat{x}_d^{-\alpha} d\hat{x} d\hat{t} \right)^{1/2},$$

$$x_d^\alpha |Du(t,x)| \leq N \left(\fint_{Q_2^+} |\hat{x}_d^\alpha u(\hat{t}, \hat{x})|^2 \hat{x}_d^{-\alpha} d\hat{x} d\hat{t} \right)^{1/2},$$

for a.e. $(t, x) \in Q_1^+$.

Proof: Energy estimates, Sobolev embedding, and an iteration technique.

Theorem 3: Mixed-norm weighted estimates

Theorem (Dong-P. (2020))

For α, p, T as in Theorem 2 and for $\gamma \in (p\alpha - 1, p-1), q > 1$. Then, there exists a unique weak solution u of

$$x_d^\alpha(u_t + u) - D_i[x_d^\alpha a_{ij}(x_d) D_j u] = x_d^\alpha f(t,x) \quad \text{in} \quad \Omega_T$$

with $u(t, x', 0) = 0$. Moreover,

$$\int_{-\infty}^T \left(\int_{\mathbb{R}^d_+} (|Du|^p + |u|^p) x_d^\gamma dx \right)^{q/p} dt \leq N \int_{-\infty}^T \left(\int_{\mathbb{R}^d_+} |f|^p x_d^\gamma dx \right)^{q/p} dt$$

for $N = N(d, p, q, \nu, \alpha, \gamma)$.

Remark: The range for γ is optimal. Such weighted estimate is necessary in Probability as explained in Kyrlov (*Probab. Theory Related Fields*, 1994).

In Theorem 2: $\gamma = \alpha(p-1)$.

Corollary: Local boundary estimates

Corollary (Dong-P. (2020))

Let α, p, γ as in Theorem 3. If u is a "weak solution" of

$$\begin{cases} x_d^\alpha u_t - D_i[x_d^\alpha a_{ij}(x_d) D_j u] = x_d^\alpha f(t,x) & in \quad Q_2^+ \\ u(t,x',0) = 0 & (t,x') \in (-4,0) \times B_1' \end{cases}$$

Then,

$$\left(\int_{Q_1^+} (|Du|^p + |u|^p) x_d^{\gamma} dxdt \right)^{\frac{1}{p}} \leq N \left(\int_{Q_2^+} |f|^{p_*} x_d^{\gamma} dxdt \right)^{\frac{1}{p_*}} + N \|u\|_{L_1(Q_2^+)}.$$

where $p_* \in (1,p)$ is a number satisfying some weighted Sobolev embedding, and $N = N(d,p,\gamma,\alpha)$.

Notation: B_ρ' is the ball in $\mathbb{R}^{d-1}$, Q_ρ^+ upper-half parabolic cylinder of radius $\rho > 0$: $Q_\rho^+ = (-\rho^2, 0) \times B_\rho^+$ where B_ρ^+ is upper-half ball.

Remarks on related work

- The existence and uniqueness of un-weighted L_p solutions for elliptic equations with smooth coefficients that are degenerate on the boundary domains have been studied in classical work (See book by O. A. Oleĭnik and E. V. Radkevič)
  Method: Barrier function techniques, maximum principle.
- Similar class of equations are also studied recently by Y. Sire, S. Terracini, and S. Vita in which Schauder's estimates are obtained under some smoothness and structural conditions of the coefficients.
  Method: contradiction argument, blow-up method, Liouville theorem.
- The approach only use energy estimates and Sobolev embedding theorem. The approach works for system of equations.
- Our kind of estimates seems to appear for the first time.

Eqns with singular/degerate coefficients

- Fully nonlinear singular-degenerate equations in geometric PDE: F.-H. Lin (*Invent. Math.*, 1989); H. Jian and X.-J. Wang (*Adv. Math.*, 2012).
- Mathematical finance: S. Heston (*Review of Financial Studies*, 1993); P. M. N. Feehan and C. Pop (2014)
- Mathematical biology: book by C. L. Epstein and R. Mazzeo
- Probability: N. V. Krylov (*Probab. Theory Related Fields*, 1994)
- Fractional Laplace equations: L. Caffarelli and L. Sylvestre(Comm. PDE, 2007)
- Porous media: P. Daskalopoulos, R. Hamilton, and K. Lee (*Duke Math. J.*, 2001)

Possible future directions/collaborations

- Theory (in Sobolev spaces): Singular/degenerate quasilinear equations, fully nonlinear equations, Stokes system of equations.
- Applications: Porous media, geometric PDEs, mathematical finance, mathematical biology, obstacle problems, optimal control problems, Hamilton-Jacobi equations.

Thank you for your attention

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- All lectures are in zoom. They will be recorded and posted together with lecture notes.
- This fall: Speaker: Ovidiu Savin (Columbia University, NY). Time 2:50PM (Eastern time) on each Thursday of October 29, November 5, 12, 19.