The Navier-Stokes equations: asymptotic expansions for solutions and their associated Lagrangian trajectories

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Outline

1. Introduction

Lagrangian and Eulerian descriptions.

We study the long-time dynamics of the incompressible, viscous fluid flows in the three-dimensional space.

A. Lagrangian description: trajectory $x(t) = x(t, x_0) \in \mathbb{R}^3$ with initial fluid particle (or material point) $x(0, x_0) = x_0$.

- Recent work on short-time properties mostly for inviscid fluids: N. Besse and U. Frisch (2017), G. Camliyurt and I. Kukavica (2018), P. Constantin, I. Kukavica, and V. Vicol (2016), P. Constantin and J. La. (2019), P. Constantin, V. Vicol, and J. Wu. (2015), M. Hernandez (2019).
- 2D dynamics (topological equivalence): T. Ma and S. Wang (book: 2005).
- Solutions have better regularity.
- Issues with viscosity.
- Long-time dynamics is little known.

B. Eulerian description: velocity field $u(x,t)$ and pressure $p(x,t)$, where $x \in \mathbb{R}^3$ is the independent spatial variable representing each fixed position in the fluid.

- Simpler PDEs especially for viscous fluids: Navier–Stokes equations. They have been studied extensively.
- Global weak solutions exist.
- Many results on long-time dynamics, still much is not known.

C. Relation:

$$x'(t) = u(x(t), t).$$

The solutions $x(t)$ of this system are called the Lagrangian trajectories.

D. Our approach:

- Solve for $u(x,t)$ from Navier–Stokes equations. Then study $x(t)$ from the ODE.
- It works sometimes.

2. The Navier–Stokes equations and others

- The Navier–Stokes equations
- Rotating fluids
- Navier–Stokes–Boussinesq system

The Navier–Stokes equations

The Eulerian description turns out to be simpler for deriving the set of equations that govern the fluid flows. They are called the Navier–Stokes equations (NSE),

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, and the unknowns are the velocity $u(x, t)$ and pressure $p(x, t)$.
Initial condition $u(x, 0) = u_0(x)$, where u_0 is a given initial vector field.

Settings

Dirichlet boundary condition (DBC). Let Ω be an bounded, open, connected set in $\mathbb{R}^3$ with C^∞ boundary, $\Omega^* = \bar{\Omega}$.
The boundary condition $u = 0$ on $\partial\Omega \times (0, \infty)$.
Spatial periodicity condition (SPC). Fix a vector $\mathbf{L} = (L_1, L_2, L_3) \in (0, \infty)^3$. We consider $u(\cdot, t)$ and $p(\cdot, t)$ to be $\mathbf{L}$-periodic for $t > 0$.
Here, a function g defined on $\mathbb{R}^3$ is called $\mathbf{L}$-periodic if

$$g(x + L_i e_i) = g(x) \text{ for } i = 1, 2, 3 \text{ and all } x \in \mathbb{R}^3.$$

Define domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ in this case, $\Omega^* = \mathbb{R}^3$.
A function g is said to have zero average over Ω if

$$\int_\Omega g(x)\mathrm{d}x = 0.$$

Notation 1

- $H^m = W^{m,2}$, for $m \in \mathbb{N}$, denotes the standard Sobolev space.
- In the (DBC) case, let $\mathcal{V}$ be the set of divergence-free vector fields in $C_c^\infty(\Omega)^3$.
- In the (SPC) case, let $\mathcal{V}$ be the set of **L**-periodic trigonometric polynomial vector fields on $\mathbb{R}^3$ which are divergence-free and have zero average over Ω.
- In both cases, define space H (respectively, V) to be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ (respectively, $\mathbb{H}^1(\Omega)$).
- The Leray projection $\mathbb{P}$ is the orthogonal projection from $\mathbb{L}^2(\Omega)$ to H.
- The Stokes operator is $A = -\mathbb{P}\Delta$ defined on $V \cap \mathbb{H}^2(\Omega)$.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^{N} f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

Foias–Saut asymptotic expansions

Functional form of NSE:

$$u_t + Au + B(u,u) = 0, \quad u(0) = u_0.$$

Let initial data $u_0 \in H$.

- Then there exists a Leray–Hopf weak solution $u(x,t)$ for $t \in [0,\infty)$.
- This solution becomes regular $u \in C^\infty(\Omega^* \times [T,\infty))$ for some $T > 0$.
- Foias–Saut (1987) proved that the solution $u(x,t)$ has an asymptotic expansion,

  $$u(\cdot,t) \sim \sum_{n=1}^{\infty} q_n(\cdot,t) e^{-\mu_n t} \text{ in } \mathbb{H}^m(\Omega),$$

  for any $m \in \mathbb{N}$, where $q_j(\cdot,t)$'s are polynomials in t with values in $\mathcal{X} \subset C^\infty(\Omega^*)^3$.

In fact, $q_1(x,t)$ is independent of t, hence we write

$$q_1(x,t) = q_1(x) \in \mathcal{X}.$$

Other NSE results

For (SPC), H.–Martinez (2017) prove that the Foias–Saut expansion holds in Gevrey spaces

$$G_{\alpha,\sigma} = \{u \in H : |u|_{\alpha,\sigma} := \|A^\alpha e^{\sigma A^{1/2}} u\|_{L^2} < \infty\},$$

for any $\alpha, \sigma > 0$.
With non-potential force H.–Martinez (2018)

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^\infty f_n(t) e^{-\gamma_n t}.$$

Cao–H (2020)

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^\infty \chi_n t^{-\gamma_n}.$$

Then

$$u(t) \sim \sum_{n=1}^\infty \xi_n t^{-\mu_n}.$$

Rotating fluids (with the periodicity boundary condition)

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \Omega e_3 \times u = -\nabla p, \quad \nabla\cdot u = 0.$$

Theorem (H.–Titi 2021)

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t} \textit{in all } G_{\alpha,\sigma}, \quad \alpha,\sigma \ge 0,$$

where, with zero average condition,

$$q_n(t) = \sum_{finite} t^m \cos(\omega t) X + \sum_{finite} t^m \sin(\omega t) X,$$

or, without zero average condition,

$$q_n(t) = \sum_{finite} t^m \cos(\cos(\omega t)) X + \sum_{finite} t^m \cos(\sin(\omega t)) X$$
$$+ \sum_{finite} t^m \sin(\cos(\omega t)) X + \sum_{finite} t^m \sin(\sin(\omega t)) X.$$

Ideas

Functional form

$$\frac{du}{dt} + Au + B(u,u) + \Omega Su = 0$$

with $u \in H$, $Ju = e_3 \times u$, $S = \mathcal{P}J\mathcal{P}$.

- Poincaré waves $e^{-\Omega tS}w$, for $w \in H$ and $t \in \mathbb{R}$.
- Set $v(t) = e^{\Omega tS}u(t)$, then

$$\frac{dv}{dt} + Av + B_\Omega(t,v,v) = 0, \quad t > 0,$$

with $B_\Omega(t,u,v) = B(\Omega t, u, v)$,

$$B(t,u,v) = e^{tS}B(e^{-tS}u, e^{-tS}v), \text{ for all } u, v \in \mathcal{D}(A).$$

Let $\check{\mathbf{k}} = (\check{k}_1, \check{k}_2, \check{k}_3) \stackrel{\text{def}}{=\!=} 2\pi(k_1/L_1, k_2/L_2, k_3/L_3)$, and, in case $\mathbf{k} \neq \mathbf{0}$,

$$\widetilde{\mathbf{k}} = (\widetilde{k}_1, \widetilde{k}_2, \widetilde{k}_3) \stackrel{\text{def}}{=\!=} \check{\mathbf{k}}/|\check{\mathbf{k}}|,$$

For $u = \sum' \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}} \in H$, we have

$$e^{tS} u = \sum' E_{\mathbf{k}}(\widetilde{k}_3 t)\widehat{\mathbf{u}}_{\mathbf{k}} e^{i\check{\mathbf{k}}\cdot\mathbf{x}}, \quad \text{where } E_{\mathbf{k}}(t) = \cos(t) I_3 + \sin(t) J_{\mathbf{k}},$$

with $J_{\mathbf{k}}\mathbf{z} = \widetilde{\mathbf{k}} \times \mathbf{z}$,

$$J_{\mathbf{k}} = \begin{pmatrix} 0 & -\widetilde{k}_3 & \widetilde{k}_2 \\ \widetilde{k}_3 & 0 & -\widetilde{k}_1 \\ -\widetilde{k}_2 & \widetilde{k}_1 & 0 \end{pmatrix}.$$

Properties

$$(E_{\mathbf{k}}(t))^* = E_{\mathbf{k}}(-t),$$

$$(e^{tS})^* = e^{-tS} \quad (\text{on } H).$$

$$|e^{tS} u|_{\alpha,\sigma} = |u|_{\alpha,\sigma}, \text{ for all } \alpha, \sigma \geq 0 \text{ and } u \in \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}).$$

Then we can follow Foias-Saut's proof.

Navier–Stokes–Boussinesq system

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - \alpha(\theta-\theta_*)\mathbf{g} \quad \text{on } \Omega\times(0,\infty),\\
&\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - \kappa_0\Delta\theta = 0 \quad \text{on } \Omega\times(0,\infty),\\
&\operatorname{div}\mathbf{u} = 0 \quad \text{on } \Omega\times(0,\infty),
\end{aligned} \tag{2.1}$$

where $\Omega := [0,L_1]\times[0,L_2]\times[0,h]\subset\mathbb{R}^3$, $\theta_*\in\mathbb{R}$ is a given reference temperature.

Boundary conditions:

$$\begin{aligned}
&\mathbf{u},\theta \quad \text{are } L_j\text{-periodic in } x_j \text{ for } j=1,2,\\
&\theta\big|_{x_3=0} = \theta\big|_{x_3=h} = \Theta_0 = const.\\
&u^3\big|_{x_3=0,h} = 0, \quad \frac{\partial u^1}{\partial x_3}\Big|_{x_3=0,h} = \frac{\partial u^2}{\partial x_3}\Big|_{x_3=0,h} = 0.
\end{aligned} \tag{2.2}$$

By shifting the temperature, we can assume $\Theta_0 = 0$.
Functional spaces. Consider

- θ odd in x_3
- u^1, u^2 even in x_3
- u^3 odd in x_3.

Write solutions in Fourier series ($L_3 = 2h$).
Define Sobolev and Gevrey spaces using the Fourier series:

$$|\mathbf{v}|_{\alpha,\sigma} = \left(\sum_{\mathbf{k}\neq 0} |\check{\mathbf{k}}|^{4\alpha} e^{2\sigma|\check{\mathbf{k}}|} |\hat{\mathbf{v}}(\mathbf{k})|^2 \right)^{1/2}.$$

Then we re-write the system as

$$u_t + Au + B(u,u) = \mathcal{L}\theta,$$
$$\theta_t + \tilde{A}\theta + \tilde{B}(u,\theta) = 0.$$

Theorem (Biswas-H.-Martinez 2021)

Any Leray-Hopf solution $(u(t), \theta(t))$ decays exponentially, as $t \to \infty$, in any Gevrey norms.

Asymptotic expansion

Theorem (Biswas-H.-Martinez 2021)

Any Leray-Hopf solution $(u(t), \theta(t))$ admits an asymptotic expansion, as $t \to \infty$, in any Gevrey spaces

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t}, \quad \theta_n(t) \sim \sum_{n=1}^{\infty} p_n(t) e^{-\mu_n t},$$

where $q_n(t), p_n(t)$ are polynomials in t, with values being trigonometric polynomials in $\mathbf{x}$. Moreover,

$$q_n' + (A - \mu_n) q_n + \sum_{\mu_k + \mu_l = \mu_n} B(q_k, q_l) = \mathcal{L} p_n, \tag{*}$$

$$p_n' + (\tilde{A} - \mu_n) p_n + \sum_{\mu_k + \mu_l = \mu_n} B(u_k, q_l) = 0. \tag{**}$$

In fact, at each step, p_n is determined first, and the q_n.

Key idea of proof

Prove by Induction. Induction step $n = N$.

- Let $\rho_N(t) = e^{\mu_{N+1}t}(\theta(t) - \sum_{n=1}^N p_n(t)e^{-\mu_n t})$. Then

$$\rho_N' + (\tilde{A} - \mu_{N+1})\rho_N + \sum_{\mu_k+\mu_l=\mu_{N+1}} \tilde{B}(q_k, p_l) = \mathcal{O}(e^{-\delta t}).$$

Then approximate $\rho_N(t)$ by polynomial solution $p_{N+1}(t)$ of

$$p_{N+1}' + (\tilde{A} - \mu_{N+1})p_{N+1} + \sum_{\mu_k+\mu_l=\mu_{N+1}} \tilde{B}(q_k, p_l) = 0,$$

with

$$|\rho_N(t) - p_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\varepsilon t}).$$

- Let $w_N(t) = e^{\mu_{N+1}t}(u(t) - \sum_{n=1}^N q_n(t)e^{-\mu_n t})$. Then

$$w_N' + (A - \mu_{N+1})w_N + \sum_{\mu_k+\mu_l=\mu_{N+1}} B(q_k, p_l) = \mathcal{L}p_{N+1} + \mathcal{O}(e^{-\delta' t}).$$

Approximate $w_N(t)$ by polynomial solution $q_{N+1}(t)$ so that

$$|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(e^{-\varepsilon' t}).$$

3. Lagrangian trajectories

Notation 2

- In the (DBC) case, let $\mathcal{V}$ be the set of divergence-free vector fields in $C_c^\infty(\Omega)^3$.
  Define $\mathcal{X}$ to be the set of functions in $\bigcap_{m=1}^\infty H^m(\Omega)^3$ that are divergence-free and vanish on the boundary $\partial\Omega$, and denote $\Omega^* = \bar{\Omega}$.
- In the (SPC) case, let $\mathcal{V}$ be the set of $\mathbf{L}$-periodic trigonometric polynomial vector fields on $\mathbb{R}^3$ which are divergence-free and have zero average over Ω.
  Define $\mathcal{X} = \mathcal{V}$, and denote $\Omega^* = \mathbb{R}^3$.

Assumption

Fix a Leray–Hopf weak solution $u(x,t)$ with large time regularity

$$u \in C^\infty(\Omega^* \times [T,\infty)).$$

Fix a Lagrangian trajectory $x(t)$.

- $x(t) \in C^1([T,\infty),\Omega)$ in the (DBC) case, or
- $x(t) \in C^1([T,\infty),\mathbb{R}^3)$ in the (SPC) case.

Foias–Saut expansion

$$u(\cdot,t) \sim \sum_{n=1}^{\infty} q_n(\cdot,t)e^{-\mu_n t} \text{ in } \mathbb{H}^m(\Omega),$$

where μ_n increases strictly to infinity, and $\mathcal{S} = \{\mu_n\}$ preserves the addition.
For $m = 2$, we have

$$\left\| u(\cdot,t) - \sum_{n=1}^{N} q_n(\cdot,t)e^{-\mu_n t} \right\|_{H^2(\Omega)^3} = \mathcal{O}(e^{-(\mu_N+\delta_N)t}),$$

for any $N \in \mathbb{N}$, and some $\delta_N > 0$.
By Morrey's embedding theorem, it follows that

$$\sup_{x\in\Omega^*} \left| u(x,t) - \sum_{n=1}^{N} q_n(x,t)e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}).$$

In particular, letting $N = 1$, we infer

$$\sup_{x\in\Omega^*} |u(x,t)| \le \sup_{x\in\Omega^*} |q_1(x)| e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\delta_1)t}) = \mathcal{O}(e^{-\mu_1 t}).$$

Therefore, there is $C_0 > 0$ such that

$$\sup_{x\in\Omega^*} |u(x,t)| \le C_0 e^{-\mu_1 t} \text{ for all } t \ge T.$$

Taking $x = x(t)$ gives

$$\left| u(x(t),t) - \sum_{n=1}^{N} q_n(x(t),t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}),$$

$$|u(x(t),t)| \le C_0 e^{-\mu_1 t} \text{ for all } t \ge T.$$

Convergence of the Lagrangian trajectories

$$x'(t) = u(x(t), t).$$

Proposition (H. 2020)

The limit $x_* \overset{\text{def}}{=\!=} \lim_{t\to\infty} x(t)$ *exists and belongs to* Ω^*, *and*

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu_1 t}).$$

Proof. For $t \ge T$, we have $x(t) = x(T) + \int_T^t u(x(\tau),\tau)\mathrm{d}\tau$.
Since $|u(x(t), t)| \le Ce^{-\mu_1 t}$ for $t \ge T$,

$$x_* = \lim_{t\to\infty} x(t) = x(T) + \int_T^\infty u(x(\tau),\tau)\mathrm{d}\tau \text{ which exists in } \mathbb{R}^3.$$

Obviously, $x_* \in \Omega^*$. Error estimate:

$$|x(t) - x_*| = \Big| \int_t^\infty u(x(\tau),\tau)\mathrm{d}\tau \Big| \le \int_t^\infty C_0 e^{-\mu_1\tau}\mathrm{d}\tau = C_0\mu_1^{-1}e^{-\mu_1 t}.$$

□

Consideration I. (SPC) or $x_* \in \Omega$ for (DBC).

Foias–Saut expansion: $u(x,t) \sim \sum q_n(x,t)e^{-\mu_n t}$. Write

$$q_n(x,t) = \sum_{k=0}^{d_n} t^k q_{n,k}(x), \text{ where } d_n \geq 0, \text{ and } q_{n,k} \in \mathcal{X}.$$

The Taylor expansion: for any $s \geq 0$,

$$q_{n,k}(x) = \sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x),$$

where $D_x^m q_{n,k}$ denotes the m-th order derivative of $q_{n,k}$ (m-linear mapping), and $g_{n,k,s} \in C(\Omega^*)^3$ satisfying

$$g_{n,k,s}(x) = \mathcal{O}(|x - x_*|^{s+1}) \text{ as } x \to x_*.$$

Then

$$q_n(x,t) = \sum_{k=0}^{d_n} t^k \Big[\sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x) \Big].$$

Rewrite

$$q_n(x,t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*,t)(x-x_*)^{(m)} + \sum_{k=0}^{d_n} t^k g_{n,k,s}(x),$$

where

$$\mathcal{Q}_{n,m}(x_*,t) = \sum_{k=0}^{d_n} \frac{t^k}{m!} D_x^m q_{n,k}(x_*) = \frac{1}{m!} D_x^m q_n(x_*,t).$$

In particular,

$$\mathcal{Q}_{n,0}(x_*,t) = q_n(x_*,t), \quad \mathcal{Q}_{n,1}(x_*,t) = D_x q_n(x_*,t),$$
$$\mathcal{Q}_{n,2}(x_*,t) = \frac{1}{2} D_x^2 q_n(x_*,t).$$

Note that $\mathcal{Q}_{n,m}(x_*,t)$ is a polynomial in t valued in the space of m-linear mappings from $(\mathbb{R}^3)^m$ to $\mathbb{R}^3$.

Above, $x(t) \to x_*$ as $t \to \infty$. Denote $z(t) = x(t) - x_*$. Then

$$|z(t)| = \mathcal{O}(e^{-\mu_1 t}).$$

We have

$$q_n(x(t), t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \sum_{k=0}^{d_n} t^k \mathcal{O}(|z(t)|^{s+1}),$$

thus

$$q_n(x(t), t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \mathcal{O}(e^{-(\mu_1(s+1)-\delta)t}) \quad \forall \delta > 0.$$

Heuristic arguments

Assume $z(t) \sim \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t}$.

$$z'(t) = x'(t) = u(x(t), t) = u(x(t), t) \sim \sum_{k=1}^{\infty} q_k(x(t), t) e^{-\mu_k t},$$

$$\sum_{n=1}^{\infty} (\zeta_n'(t) - \mu_n \zeta_n(t)) e^{-\mu_n t} \sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{Q}_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{Q}_{k,m}(x_*, t) (\sum_{j_1} \zeta_{j_1}(t) e^{-\mu_{j_1} t}, \ldots, \sum_{j_m} \zeta_{j_m}(t) e^{-\mu_{j_m} t}) e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_1, j_2, \ldots, \mu_{j_m}} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)) e^{-(\mu_{j_1} + \ldots + \mu_{j_m}) t} e^{-\mu_k t}.$$

Then

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

Theorem (H. 2020)

Under Consideration I, there exist polynomials $\zeta_n : \mathbb{R} \to \mathbb{R}^3$, for $n \geq 0$, such that solution $x(t)$ has an asymptotic expansion,

$$x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3,$$

where each ζ_n, for $n \geq 1$, is the unique polynomial solution of the following differential equation

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

for all $t \in \mathbb{R}$.

Remarks on $\zeta_n(t)$

- Find polynomial solution $\zeta_n(t)$ of

$$\zeta_n'(t) - \mu_n\zeta_n(t) = \sum_{\mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n} \mathcal{Q}_{k,m}(x_*,t)(\zeta_{j_1}(t),\ldots,\zeta_{j_m}(t)).$$

- Equation for $\zeta_n(t)$ is linear. The RHS comes from previous steps.
- The RHS sum is finitely many. In fact, for each $n\ge 1$, and integers $M\ge \mu_n/\mu_1 - 1$, $K\ge n$, $J\ge n-1$, one has

$$\sum_{\mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n} = \sum_{m=0}^{M}\sum_{k=1}^{K}\sum_{\substack{j_1,\ldots,j_m=1,\\ \mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n}}^{J}$$

- Examples

$$\zeta_1'(t) - \mu_1\zeta_1(t) = q_1(x_*),$$
$$\zeta_2'(t) - \mu_2\zeta_2(t) = D_x q_1(x_*)\zeta_1(t) + q_2(x_*,t),$$
$$\zeta_3'(t) - \mu_3\zeta_3(t) = \frac{1}{2}D_x^2 q_1(x_*)(\zeta_1(t),\zeta_1(t)) + D_x q_2(x_*,t)\zeta_1(t) + q_3(x_*,t$$

Proof I. First step of induction.

By induction. First step. We have

$$\begin{aligned} z'(t) = x'(t) = u(x(t),t) &= q_1(x(t))e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\delta_1)t}) \\ &= [q_1(x_*) + \mathcal{O}(e^{-\mu_1 t/2})]e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\delta_1)t}) \\ &= q_1(x_*)e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\varepsilon_1)t}). \end{aligned}$$

Let $w_0(t) = e^{\mu_1 t} z(t)$. Then

$$w_0'(t) - \mu_1 w_0(t) = q_1(x_*) + \mathcal{O}(e^{-\varepsilon_1 t}).$$

Need an Approximation Lemma (see below): there is polynomial $\zeta_1(t)$ such that

$$|w_0(t) - \zeta_1(t)| = \mathcal{O}(e^{-\varepsilon_1 t}).$$

Multiplying by $e^{-\mu_1 t}$ gives

$$|z(t) - e^{-\mu_1 t}\zeta_1(t)| = \mathcal{O}(e^{-(\mu_1+\varepsilon_1)t}).$$

Proof II. Approximation lemma

Let $(X, \|\cdot\|_X)$ be a Banach space. Let $p : \mathbb{R} \to X$ be a polynomial, and $\|g(t)\|_X \le Me^{-\delta t}$ for $t \ge t_*$, for some $M, \delta > 0$.
Let $\gamma > 0$. Suppose that $y : [t_*, \infty) \to X$ solves

$$y'(t) - \gamma y(t) = p(t) + g(t) \quad \text{for } t > t_*,$$

and satisfies

$$\lim_{t\to\infty} (e^{-\gamma t}\|y(t)\|_X) = 0.$$

Then there exists a unique polynomial $q : \mathbb{R} \to X$ such that

$$\|y(t) - q(t)\|_X \le \frac{M}{\gamma + \delta} e^{-\delta t} \quad \text{for all } t \ge t_*.$$

More precisely, $q(t)$ is the unique polynomial solution of

$$q'(t) - \gamma q(t) = p(t) \quad \text{for } t \in \mathbb{R},$$

and can be explicitly defined by

$$q(t) = -\int_t^\infty e^{\gamma(t-\tau)} p(\tau) \mathrm{d}\tau.$$

Proof III. Sketch of the induction step

Let $z_N(t) = \sum_{n=1}^{N} \zeta_n(t) e^{-\mu_n t}$ and $\tilde{z}_N(t) = z(t) - z_N(t)$.
Denote $\tilde{J}_n = \sum_{\mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t))$.
Induction hypothesis:

$$\zeta_n' - \mu_n \zeta_n = \tilde{J}_n \text{ for } (1 \le n \le N) \text{ and } |\tilde{z}_N(t)| = \mathcal{O}(e^{-(\mu_N+\varepsilon_N)t}).$$

Define $w_N(t) = e^{\mu_{N+1} t} \tilde{z}_N(t)$.

$$w_N' = \mu_{N+1} w_N + e^{\mu_{N+1} t} \Big(z' - \sum_{n=1}^{N} e^{-\mu_n t} (\zeta_n' - \mu_n \zeta_n) \Big).$$

Approximate $z'(t) = u(x(t), t)$ same as in heuristic arguments. Need to control the errors.

Let $s_{N+1} \in \mathbb{N}$: $s_{N+1} \ge \mu_{N+1}/\mu_1 - 1$.
Calculations give

$$z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \mathcal{Q}_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t} + \mathcal{O}(e^{-(\mu_{N+1}+\widehat{\delta}_{N+1})t}),$$

with

$$z^{(m)} = (z, z, \dots, z) \quad (m \text{ times.})$$

Write $z(t) = \sum_{j=1}^{N} \zeta_j(t) e^{-\mu_j t} + \mathcal{O}(e^{-(\mu_{N+1}+\widehat{\delta}_{N+1})t})$. Then

$$\begin{aligned} z'(t) = &\sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \sum_{j_1,\dots,j_m=1}^{N} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}, \dots, \zeta_{j_m}) e^{-(\mu_k+\mu_{j_1}+\dots+\mu_{j_m})t} \\ &+ \sum_{k=1}^{N+1} \sum_{m=1}^{s_{N+1}} \mathcal{O}(e^{-(\mu_N+\varepsilon_N/2)t})) e^{-\mu_k t} + \mathcal{O}(e^{-(\mu_{N+1}+\widehat{\delta}_{N+1})t}). \end{aligned}$$

• Observation $\mu_N + \mu_k \in \mathcal{S}$ and is greater than μ_N. Then $\mu_N + \mu_k \geq \mu_{N+1}$.
• Also, $\mu_k + \mu_{j_1} + \ldots + \mu_{j_m} \in \mathcal{S}$, then

$$\mu_k + \mu_{j_1} + \ldots + \mu_{j_m} = \mu_n \text{ for some } n \in \mathbb{N}.$$

Split the first sum on the RHS: $n \leq N+1$ and $n \geq N+2$. We obtain

$$z'(t) = \sum_{n=1}^{N+1} J_n(t) e^{-\mu_n t} + \mathcal{O}(e^{-(\mu_{N+1}+\varepsilon_{N+1})t}),$$

$$J_n(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \sum_{\substack{j_1,\ldots,j_m=1, \\ \mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n}}^{N} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

Combine calculations

$$w_N' = \mu_{N+1} w_N + e^{\mu_{N+1} t} \sum_{n=1}^{N} e^{-\mu_n t} \Big\{ J_n - (\zeta_n' - \mu_n \zeta_n) \Big\} + J_{N+1} + \mathcal{O}(e^{-\varepsilon_{N+1} t}).$$

Note $J_n = \tilde{J}_n$. Then

$$w_N' - \mu_{N+1} w_N = J_{N+1} + \mathcal{O}(e^{-\varepsilon_{N+1} t}).$$

Applying Approximation Lemma, one has

$$\Big| w_N(t) - \zeta_{N+1}(t) \Big| = \mathcal{O}(e^{-\varepsilon_{N+1} t}).$$

Multiplying by $e^{-\mu_{N+1} t}$ gives

$$\Big| \tilde{z}_N(t) - \zeta_{N+1}(t) e^{-\mu_{N+1} t} \Big| = \mathcal{O}(e^{-(\mu_{N+1} + \varepsilon_{N+1}) t}).$$

Consideration II. (DBC) with $x_* \in \partial\Omega$

Theorem (H. 2020)

Under Consideration II, one has

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu t}) \text{ for all } \mu > 0.$$

Proof. Recall $|z(t) - \sum_{n=1}^N \zeta_n(t) e^{-\mu_n t}| = \mathcal{O}(e^{-(\mu_n+\varepsilon_n)t})$.
Explicit formula:

$$\zeta_n(t) = -\int_t^\infty e^{\mu_n(t-\tau)} \Big\{ q_n(x_*,\tau) + \sum_{m=1}^{s_n} \sum_{\substack{k,j_1,\ldots,j_m=1,\\ \mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_n}}^{n-1} \mathcal{Q}_{k,m}(x_*,\tau)(\zeta_{j_1}(\tau),\ldots,\zeta_{j_m}(\tau)) \Big\} \mathrm{d}\tau.$$

Note $q_n(x_*,t) = 0$ for all n.
When $n = 1$, one has $\zeta_1(t) = -q_1(x_*)/\mu_1$. Thus, $\zeta_1(t) = 0$.
Recursively, $\zeta_2(t) = 0$, $\zeta_3(t) = 0$, etc. So, $\zeta_n(t) = 0$ for all n. □

(SPC) without the zero average condition

Let $(u(x,t), p(x,t))$ be a **L**-periodic, classical solution the NSE on $\mathbb{R}^3 \times (0,\infty)$.
Let $x(t) \in \mathbb{R}^3$ be a Lagrangian trajectory corresponding to $u(x,t)$.

Theorem (H. 2020)

There exist $x_* \in \mathbb{R}^3$ and polynomials $X_n : \mathbb{R} \to \mathbb{R}^3$, for $n \in \mathbb{N}$, such that

$$x(t) \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t} \ \text{in } \mathbb{R}^3,$$

where $U_0 = (L_1 L_2 L_3)^{-1} \int_\Omega u(x,0)\mathrm{d}x$.

Proof.

Galilean transformation. Set

$$v(X,t) = u(X + U_0 t, t) - U_0 \text{ and } P(X,t) = p(X + U_0 t, t).$$

Then (v, P) is a solution of the NSE, $\mathbf{L}$-periodic, and $v(\cdot, t)$ has zero average.

Let $X(t) = x(t) - U_0 t$. We have

$$X'(t) = x'(t) - U_0 = u(x(t), t) - U_0 = v(x(t) - U_0 t, t) + U_0 - U_0 = v(X(t), t).$$

Applying above result (for zero average solutions) to $v(X,t)$ and $X(t)$ yields

$$X(t) \sim x_* + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$

Consequently, we obtain

$$x(t) = X(t) + U_0 t \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$

$\square$

THANK YOU!