

Long-time asymptotic expansions for viscous incompressible fluid flows

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1. Introduction

Lagrangian and Eulerian descriptions.

We study the long-time dynamics of the incompressible, viscous fluid flows in the three-dimensional space.

A. Lagrangian description: trajectory $x(t) = x(t, x_0) \in \mathbb{R}^3$ with initial fluid particle (or material point) $x(0, x_0) = x_0$.

- Recent work on short-time properties mostly for inviscid fluids: N. Besse and U. Frisch (2017), G. Camliyurt and I. Kukavica (2018), P. Constantin, I. Kukavica, and V. Vicol (2016), P. Constantin and J. La. (2019), P. Constantin, V. Vicol, and J. Wu. (2015), M. Hernandez (2019).
- 2D dynamics (topological equivalence): T. Ma and S. Wang (book: 2005).
- Solutions have better regularity.
- Issues with viscosity.
- Long-time dynamics is little known.

B. Eulerian description: velocity field $u(x, t)$ and pressure $p(x, t)$, where $x \in \mathbb{R}^3$ is the independent spatial variable representing each fixed position in the fluid.

- Simpler PDEs especially for viscous fluids: Navier–Stokes equations. They have been studied extensively.
- Global weak solutions exist.
- Many results on long-time dynamics, still much is not known.

C. Relation:

$$x'(t) = u(x(t), t).$$

The solutions $x(t)$ of this system are called the Lagrangian trajectories.

D. Our approach:

- Solve for $u(x, t)$ from Navier–Stokes equations. Then study $x(t)$ from the ODE.
- It works sometimes.

The background of the slide is a reproduction of the Japanese woodblock print 'The Great Wave off Kanagawa' by Katsushika Hokusai. The image shows a massive, curling blue wave with white foam, threatening three small yellow boats filled with people. In the distance, the snow-capped Mount Fuji is visible under a pale, overcast sky. A vertical title slip is present in the upper left corner of the print.

2. The Navier–Stokes equations and others

The Navier–Stokes equations

The Eulerian description turns out to be simpler for deriving the set of equations that govern the fluid flows. They are called the Navier–Stokes equations (NSE),

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, and the unknowns are the velocity $u(x, t)$ and pressure $p(x, t)$.

Initial condition $u(x, 0) = u_0(x)$, where u_0 is a given initial vector field.

Dirichlet boundary condition (DBC). Let Ω be an bounded, open, connected set in \mathbb{R}^3 with C^∞ boundary.

The boundary condition $u = 0$ on $\partial\Omega \times (0, \infty)$.

Spatial periodicity condition (SPC). Fix a vector $\mathbf{L} = (L_1, L_2, L_3) \in (0, \infty)^3$. We consider $u(\cdot, t)$ and $p(\cdot, t)$ to be \mathbf{L} -periodic for $t > 0$.

Here, a function g defined on \mathbb{R}^3 is called \mathbf{L} -periodic if

$$g(x + L_i e_i) = g(x) \text{ for } i = 1, 2, 3 \text{ and all } x \in \mathbb{R}^3.$$

Define domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ in this case.

A function g is said to have zero average over Ω if

$$\int_{\Omega} g(x) dx = 0.$$

- $H^m = W^{m,2}$, for $m \in \mathbb{N}$, denotes the standard Sobolev space.
- In the (DBC) case, let \mathcal{V} be the set of divergence-free vector fields in $C_c^\infty(\Omega)^3$.
Define \mathcal{X} to be the set of functions in $\bigcap_{m=1}^\infty H^m(\Omega)^3$ that are divergence-free and vanish on the boundary $\partial\Omega$, and denote $\Omega^* = \bar{\Omega}$.
- In the (SPC) case, let \mathcal{V} be the set of \mathbf{L} -periodic trigonometric polynomial vector fields on \mathbb{R}^3 which are divergence-free and **have zero average over Ω** .
Define $\mathcal{X} = \mathcal{V}$, and denote $\Omega^* = \mathbb{R}^3$.
- In both cases, define space H (respectively, V) to be the closure of \mathcal{V} in $\mathbb{L}^2(\Omega)$ (respectively, $\mathbb{H}^1(\Omega)$).
The Leray projection \mathbb{P} is the orthogonal projection from $\mathbb{L}^2(\Omega)$ to H .
The Stokes operator is $A = -\mathbb{P}\Delta$ defined on $V \cap \mathbb{H}^2(\Omega)$.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \rightarrow X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X -valued polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \quad \text{as } t \rightarrow \infty,$$

for some $\varepsilon_N > 0$.

Foias–Saut asymptotic expansions

Functional form of NSE:

$$u_t + Au + B(u, u) = 0, \quad u(0) = u_0.$$

Let initial data $u_0 \in H$.

- Then there exists a Leray–Hopf weak solution $u(x, t)$ for $t \in [0, \infty)$.
- This solution becomes regular $u \in C^\infty(\Omega^* \times [T, \infty))$ for some $T > 0$.
- Foias–Saut (1987) proved that the solution $u(x, t)$ has an asymptotic expansion,

$$u(\cdot, t) \sim \sum_{n=1}^{\infty} q_n(\cdot, t) e^{-\mu_n t} \text{ in } \mathbb{H}^m(\Omega),$$

for any $m \in \mathbb{N}$, where $q_j(\cdot, t)$'s are polynomials in t with values in $\mathcal{X} \subset C^\infty(\Omega^*)^3$.

In fact, $q_1(x, t)$ is independent of t , hence we write

$$q_1(x, t) = q_1(x) \in \mathcal{X}.$$

Exponential decaying rates

- Denote the spectrum of the Stokes operator A by $\{\Lambda_k : k \in \mathbb{N}\}$, where Λ_k 's are positive, strictly increasing to infinity.
- Let \mathcal{S} be the additive semigroup generated by $\nu\Lambda_k$'s, that is,

$$\mathcal{S} = \left\{ \nu \sum_{j=1}^N \Lambda_{k_j} : N, k_1, \dots, k_N \in \mathbb{N} \right\}.$$

- We arrange the set \mathcal{S} as a sequence $(\mu_n)_{n=1}^{\infty}$ of positive, strictly increasing numbers. Clearly,

$$\lim_{n \rightarrow \infty} \mu_n = \infty,$$

$$\mu_n + \mu_k \in \mathcal{S} \quad \forall n, k \in \mathbb{N}.$$

Other NSE results

For (SPC), H.–Martinez (2017) prove that the Foias–Saut expansion holds in Gevrey spaces

$$G_{\alpha,\sigma} = \{u \in H : \|A^\alpha e^{\sigma A^{1/2}} u\|_{L^2} < \infty\},$$

for any $\alpha, \sigma > 0$.

With non-potential force H.–Martinez (2018)

$$u_t + Au + B(u, u) = f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\gamma_n t}.$$

Cao–H (2020)

$$u_t + Au + B(u, u) = f(t) \sim \sum_{n=1}^{\infty} \chi_n t^{-\gamma_n}.$$

Then

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n e^{-\mu_n}.$$

$$u_t - \nu \Delta u + (u \cdot \nabla)u + Re_3 \times u = -\nabla p.$$

H.-Titi (2020):

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t},$$

where, with zero average condition,

$$q_n(t) = \sum_{finite} t^m \cos(\omega t) X + \sum_{finite} t^m \sin(\omega t) X,$$

or, without zero average condition,

$$\begin{aligned} q_n(t) = & \sum_{finite} t^m \cos(\cos(\omega t)) X + \sum_{finite} t^m \cos(\sin(\omega t)) X \\ & + \sum_{finite} t^m \sin(\cos(\omega t)) X + \sum_{finite} t^m \sin(\sin(\omega t)) X. \end{aligned}$$

- Dissipative wave equations: Shi (2000)
- Navier–Stokes–Boussinesq system: Biswas–H.–Martinez (in preparation)

Back to NSE. From Foias–Saut expansions:

- Non-linear spectral manifolds characterizing rates of decay for the solutions: Foias–Saut papers in 1980s and 1990s.
- Normal form and normalization map: Foias–H.–Olson–Ziane (2006,2009).
- Foias–Saut normalization gives a Poincaré–Dulac normal form: Foias–H.–Saut (2011).

A. With analytic nonlinear terms, no forcing.

$$y' + Ay = F(y).$$

- Normal forms: Poincaré, Dulac, Lyapunov (first method), Bruno.
- Power geometry: Bruno (1960s–present).
- Foias–Saut approach: Minea (1998).

B. Non-smooth nonlinearity.

- Cao–H.–Kieu (2020). Example:

$$F(y) \sim \sum |y|^{\alpha_n} P_n(y).$$

C. With forcing.

$$y' + Ay = F(y) + f(t).$$

- Cao–H. (2020).

$$f(t) = \sum t^{-\mu}, (\ln t)^r, (\ln \ln t)^r (\ln \ln \ln t)^r, \dots$$

The background of the slide is a visualization of a turbulent flow field, likely a wake or a shear layer, showing complex, swirling structures. A semi-transparent blue box with rounded corners is overlaid on the upper portion of the image, containing the section title.

3. Lagrangian trajectories

Assumption

Fix a Leray–Hopf weak solution $u(x, t)$ with large time regularity

$$u \in C^\infty(\Omega^* \times [T, \infty)).$$

Fix a Lagrangian trajectory $x(t)$.

- $x(t) \in C^1([T, \infty), \Omega)$ in the (DBC) case, or
- $x(t) \in C^1([T, \infty), \mathbb{R}^3)$ in the (SPC) case.

Foias–Saut expansion

$$u(\cdot, t) \sim \sum_{n=1}^{\infty} q_n(\cdot, t) e^{-\mu_n t} \text{ in } \mathbb{H}^m(\Omega),$$

For $m = 2$, we have

$$\left\| u(\cdot, t) - \sum_{n=1}^N q_n(\cdot, t) e^{-\mu_n t} \right\|_{H^2(\Omega)^3} = \mathcal{O}(e^{-(\mu_N + \delta_N)t}),$$

for any $N \in \mathbb{N}$, and some $\delta_N > 0$.

By Morrey's embedding theorem, it follows that

$$\sup_{x \in \Omega^*} \left| u(x, t) - \sum_{n=1}^N q_n(x, t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N + \delta_N)t}).$$

In particular, letting $N = 1$, we infer

$$\sup_{x \in \Omega^*} |u(x, t)| \leq \sup_{x \in \Omega^*} |q_1(x)| e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1 + \delta_1)t}) = \mathcal{O}(e^{-\mu_1 t}).$$

Therefore, there is $C_0 > 0$ such that

$$\sup_{x \in \Omega^*} |u(x, t)| \leq C_0 e^{-\mu_1 t} \text{ for all } t \geq T.$$

Taking $x = x(t)$ gives

$$\left| u(x(t), t) - \sum_{n=1}^N q_n(x(t), t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N + \delta_N)t}),$$

$$|u(x(t), t)| \leq C_0 e^{-\mu_1 t} \text{ for all } t \geq T.$$

Convergence of the Lagrangian trajectories

$$x'(t) = u(x(t), t).$$

Proposition (H. 2020)

The limit $x_* \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} x(t)$ exists and belongs to Ω^* , and

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu_1 t}).$$

Proof. For $t \geq T$, we have $x(t) = x(T) + \int_T^t u(x(\tau), \tau) d\tau$.
Since $|u(x(t), t)| \leq Ce^{-\mu_1 t}$ for $t \geq T$,

$$x_* = \lim_{t \rightarrow \infty} x(t) = x(T) + \int_T^\infty u(x(\tau), \tau) d\tau \text{ which exists in } \mathbb{R}^3.$$

Obviously, $x_* \in \Omega^*$. Error estimate:

$$|x(t) - x_*| = \left| \int_t^\infty u(x(\tau), \tau) d\tau \right| \leq \int_t^\infty C_0 e^{-\mu_1 \tau} d\tau = C_0 \mu_1^{-1} e^{-\mu_1 t}.$$



Consideration I. (SPC) or $x_* \in \Omega$ for (DBC).

Foias–Saut expansion: $u(x, t) \sim \sum q_n(x, t)e^{-\mu_n t}$. Write

$$q_n(x, t) = \sum_{k=0}^{d_n} t^k q_{n,k}(x), \text{ where } d_n \geq 0, \text{ and } q_{n,k} \in \mathcal{X}.$$

The Taylor expansion: for any $s \geq 0$,

$$q_{n,k}(x) = \sum_{m=0}^s \frac{1}{m!} D_x^m q_{n,k}(x_*) (x - x_*)^{(m)} + g_{n,k,s}(x),$$

where $D_x^m q_{n,k}$ denotes the m -th order derivative of $q_{n,k}$ (m -linear mapping), and $g_{n,k,s} \in C(\Omega^*)^3$ satisfying

$$g_{n,k,s}(x) = \mathcal{O}(|x - x_*|^{s+1}) \text{ as } x \rightarrow x_*.$$

Then

$$q_n(x, t) = \sum_{k=0}^{d_n} t^k \left[\sum_{m=0}^s \frac{1}{m!} D_x^m q_{n,k}(x_*) (x - x_*)^{(m)} + g_{n,k,s}(x) \right].$$

Rewrite

$$q_n(x, t) = \sum_{m=0}^s Q_{n,m}(x_*, t)(x - x_*)^{(m)} + \sum_{k=0}^{d_n} t^k g_{n,k,s}(x),$$

where

$$Q_{n,m}(x_*, t) = \sum_{k=0}^{d_n} \frac{t^k}{m!} D_x^m q_{n,k}(x_*) = \frac{1}{m!} D_x^m q_n(x_*, t).$$

In particular,

$$Q_{n,0}(x_*, t) = q_n(x_*, t), \quad Q_{n,1}(x_*, t) = D_x q_n(x_*, t),$$
$$Q_{n,2}(x_*, t) = \frac{1}{2} D_x^2 q_n(x_*, t).$$

Note that $Q_{n,m}(x_*, t)$ is a polynomial in t valued in the space of m -linear mappings from $(\mathbb{R}^3)^m$ to \mathbb{R}^3 .

Above, $x(t) \rightarrow x_*$ as $t \rightarrow \infty$. Denote $z(t) = x(t) - x_*$. Then

$$|z(t)| = \mathcal{O}(e^{-\mu_1 t}).$$

We have

$$q_n(x(t), t) = \sum_{m=0}^s \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \sum_{k=0}^{d_n} t^k \mathcal{O}(|z(t)|^{s+1}),$$

thus

$$q_n(x(t), t) = \sum_{m=0}^s \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \mathcal{O}(e^{-(\mu_1(s+1)-\delta)t}) \quad \forall \delta > 0.$$

Heuristic arguments

Assume $z(t) \sim \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t}$.

$$z'(t) = x'(t) = u(x(t), t) = u(x(t), t) \sim \sum_{k=1}^{\infty} q_k(x(t), t) e^{-\mu_k t},$$

$$\sum_{n=1}^{\infty} (\zeta_n'(t) - \mu_n \zeta_n(t)) e^{-\mu_n t} \sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} Q_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} Q_{k,m}(x_*, t) \left(\sum_{j_1} \zeta_{j_1}(t) e^{-\mu_{j_1} t}, \dots, \sum_{j_m} \zeta_{j_m}(t) e^{-\mu_{j_m} t} \right) e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_1, j_2, \dots, j_m} Q_{k,m}(x_*, t) (\zeta_{j_1}(t), \dots, \zeta_{j_m}(t)) e^{-(\mu_{j_1} + \dots + \mu_{j_m})t} e^{-\mu_k t}.$$

Then

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n} Q_{k,m}(x_*, t) (\zeta_{j_1}(t), \dots, \zeta_{j_m}(t)).$$

Theorem (H. 2020)

Under Consideration I, there exist polynomials $\zeta_n : \mathbb{R} \rightarrow \mathbb{R}^3$, for $n \geq 0$, such that solution $x(t)$ has an asymptotic expansion,

$$x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3,$$

where each ζ_n , for $n \geq 1$, is the unique polynomial solution of the following differential equation

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \dots, \zeta_{j_m}(t)).$$

for all $t \in \mathbb{R}$.

Remarks on $\zeta_n(t)$

- Find **polynomial solution** $\zeta_n(t)$ of

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \dots, \zeta_{j_m}(t)).$$

- Equation for $\zeta_n(t)$ is linear. The RHS comes from previous steps.
- The RHS sum is finitely many. In fact, for each $n \geq 1$, and integers $M \geq \mu_n/\mu_1 - 1$, $K \geq n$, $J \geq n - 1$, one has

$$\sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n} = \sum_{m=0}^M \sum_{k=1}^K \sum_{\substack{j_1, \dots, j_m=1, \\ \mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n}}^J$$

- Examples

$$\zeta_1'(t) - \mu_1 \zeta_1(t) = q_1(x_*),$$

$$\zeta_2'(t) - \mu_2 \zeta_2(t) = D_x q_1(x_*) \zeta_1(t) + q_2(x_*, t),$$

$$\zeta_3'(t) - \mu_3 \zeta_3(t) = \frac{1}{2} D_x^2 q_1(x_*) (\zeta_1(t), \zeta_1(t)) + D_x q_2(x_*, t) \zeta_1(t) + q_3(x_*, t)$$

Proof I. First step of induction.

By induction. First step. We have

$$\begin{aligned}z'(t) &= x'(t) = u(x(t), t) = q_1(x(t))e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1 + \delta_1)t}) \\ &= [q_1(x_*) + \mathcal{O}(e^{-\mu_1 t/2})]e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1 + \delta_1)t}) \\ &= q_1(x_*)e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1 + \varepsilon_1)t}).\end{aligned}$$

Let $w_0(t) = e^{\mu_1 t} z(t)$. Then

$$w_0'(t) - \mu_1 w_0(t) = q_1(x_*) + \mathcal{O}(e^{-\varepsilon_1 t}).$$

Need an Approximation Lemma (see below): there is polynomial $\zeta_1(t)$ such that

$$|w_0(t) - \zeta_1(t)| = \mathcal{O}(e^{-\varepsilon_1 t}).$$

Multiplying by $e^{-\mu_1 t}$ gives

$$|z(t) - e^{-\mu_1 t} \zeta_1(t)| = \mathcal{O}(e^{-(\mu_1 + \varepsilon_1)t}).$$

Proof II. Approximation lemma

Let $(X, \|\cdot\|_X)$ be a Banach space. Let $p : \mathbb{R} \rightarrow X$ be a polynomial, and $\|g(t)\|_X \leq Me^{-\delta t}$ for $t \geq t_*$, for some $M, \delta > 0$.

Let $\gamma > 0$. Suppose that $y : [t_*, \infty) \rightarrow X$ solves

$$y'(t) - \gamma y(t) = p(t) + g(t) \quad \text{for } t > t_*,$$

and satisfies

$$\lim_{t \rightarrow \infty} (e^{-\gamma t} \|y(t)\|_X) = 0.$$

Then there exists a unique polynomial $q : \mathbb{R} \rightarrow X$ such that

$$\|y(t) - q(t)\|_X \leq \frac{M}{\gamma + \delta} e^{-\delta t} \quad \text{for all } t \geq t_*.$$

More precisely, $q(t)$ is the unique polynomial solution of

$$q'(t) - \gamma q(t) = p(t) \quad \text{for } t \in \mathbb{R},$$

and can be explicitly defined by

$$q(t) = - \int_t^\infty e^{\gamma(t-\tau)} p(\tau) d\tau.$$

Proof III. Sketch of the induction step

Let $z_N(t) = \sum_{n=1}^N \zeta_n(t)e^{-\mu_n t}$ and $\tilde{z}_N(t) = z(t) - z_N(t)$.

Denote $\tilde{J}_n = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \dots, \zeta_{j_m}(t))$.

Induction hypothesis:

$$\zeta_n' - \mu_n \zeta_n = \tilde{J}_n \text{ for } (1 \leq n \leq N) \text{ and } |\tilde{z}_N(t)| = \mathcal{O}(e^{-(\mu_N + \varepsilon_N)t}).$$

Define $w_N(t) = e^{\mu_{N+1}t} \tilde{z}_N(t)$.

$$w_N' = \mu_{N+1} w_N + e^{\mu_{N+1}t} \left(z' - \sum_{n=1}^N e^{-\mu_n t} (\zeta_n' - \mu_n \zeta_n) \right).$$

Approximate $z'(t) = u(x(t), t)$ same as in heuristic arguments. Need to control the errors.

Let $s_{N+1} \in \mathbb{N}$: $s_{N+1} \geq \mu_{N+1}/\mu_1 - 1$.

Calculations give

$$z'(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \mathcal{Q}_{k,m}(x_*, t) z(t)^{(m)} e^{-\mu_k t} + \mathcal{O}(e^{-(\mu_{N+1} + \widehat{\delta}_{N+1})t}),$$

with

$$z^{(m)} = (z, z, \dots, z) \quad (m \text{ times.})$$

Write $z(t) = \sum_{j=1}^N \zeta_j(t) e^{-\mu_j t} + \mathcal{O}(e^{-(\mu_{N+1} + \widehat{\delta}_{N+1})t})$. Then

$$\begin{aligned} z'(t) &= \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \sum_{j_1, \dots, j_m=1}^N \mathcal{Q}_{k,m}(x_*, t) (\zeta_{j_1}, \dots, \zeta_{j_m}) e^{-(\mu_k + \mu_{j_1} + \dots + \mu_{j_m})t} \\ &\quad + \sum_{k=1}^{N+1} \sum_{m=1}^{s_{N+1}} \mathcal{O}(e^{-(\mu_N + \varepsilon_N/2)t}) e^{-\mu_k t} + \mathcal{O}(e^{-(\mu_{N+1} + \widehat{\delta}_{N+1})t}). \end{aligned}$$

- Observation $\mu_N + \mu_k \in \mathcal{S}$ and is greater than μ_N . Then $\mu_N + \mu_k \geq \mu_{N+1}$.
- Also, $\mu_k + \mu_{j_1} + \dots + \mu_{j_m} \in \mathcal{S}$, then

$$\mu_k + \mu_{j_1} + \dots + \mu_{j_m} = \mu_n \text{ for some } n \in \mathbb{N}.$$

Split the first sum on the RHS: $n \leq N + 1$ and $n \geq N + 2$. We obtain

$$z'(t) = \sum_{n=1}^{N+1} J_n(t) e^{-\mu_n t} + \mathcal{O}(e^{-(\mu_{N+1} + \varepsilon_{N+1})t}),$$

$$J_n(t) = \sum_{k=1}^{N+1} \sum_{m=0}^{s_{N+1}} \sum_{\substack{j_1, \dots, j_m=1, \\ \mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n}}^N \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \dots, \zeta_{j_m}(t)).$$

Combine calculations

$$w'_N = \mu_{N+1} w_N + e^{\mu_{N+1}t} \sum_{n=1}^N e^{-\mu_n t} \left\{ J_n - (\zeta'_n - \mu_n \zeta_n) \right\} + J_{N+1} + \mathcal{O}(e^{-\varepsilon_{N+1}t}).$$

Note $J_n = \tilde{J}_n$. Then

$$w'_N - \mu_{N+1} w_N = J_{N+1} + \mathcal{O}(e^{-\varepsilon_{N+1}t}).$$

Applying Approximation Lemma, one has

$$\left| w_N(t) - \zeta_{N+1}(t) \right| = \mathcal{O}(e^{-\varepsilon_{N+1}t}).$$

Multiplying by $e^{-\mu_{N+1}t}$ gives

$$\left| \tilde{z}_N(t) - \zeta_{N+1}(t) e^{-\mu_{N+1}t} \right| = \mathcal{O}(e^{-(\mu_{N+1} + \varepsilon_{N+1})t}).$$

Consideration II. (DBC) with $x_* \in \partial\Omega$

Theorem (H. 2020)

Under Consideration II, one has

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu t}) \text{ for all } \mu > 0.$$

Proof. Recall $|z(t) - \sum_{n=1}^N \zeta_n(t) e^{-\mu_n t}| = \mathcal{O}(e^{-(\mu_n + \varepsilon_n)t})$.

Explicit formula:

$$\begin{aligned} \zeta_n(t) = & - \int_t^\infty e^{\mu_n(t-\tau)} \left\{ q_n(x_*, \tau) \right. \\ & + \sum_{m=1}^{s_n} \sum_{\substack{k, j_1, \dots, j_m=1, \\ \mu_k + \mu_{j_1} + \mu_{j_2} + \dots + \mu_{j_m} = \mu_n}}^{n-1} \mathcal{Q}_{k,m}(x_*, \tau) (\zeta_{j_1}(\tau), \dots, \zeta_{j_m}(\tau)) \left. \right\} d\tau. \end{aligned}$$

Note $q_n(x_*, t) = 0$ for all n .

When $n = 1$, one has $\zeta_1(t) = -q_1(x_*)/\mu_1$. Thus, $\zeta_1(t) = 0$.

Recursively, $\zeta_2(t) = 0$, $\zeta_3(t) = 0$, etc. So, $\zeta_n(t) = 0$ for all n . □

(SPC) without the zero average condition

Let $(u(x, t), p(x, t))$ be a \mathbf{L} -periodic, classical solution the NSE on $\mathbb{R}^3 \times (0, \infty)$.

Let $x(t) \in \mathbb{R}^3$ be a Lagrangian trajectory corresponding to $u(x, t)$.

Theorem (H. 2020)

There exist $x_* \in \mathbb{R}^3$ and polynomials $X_n : \mathbb{R} \rightarrow \mathbb{R}^3$, for $n \in \mathbb{N}$, such that

$$x(t) \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3,$$

where $U_0 = (L_1 L_2 L_3)^{-1} \int_{\Omega} u(x, 0) dx$.

Proof.

Galilean transformation. Set

$$v(X, t) = u(X + U_0 t, t) - U_0 \text{ and } P(X, t) = p(X + U_0 t, t).$$

Then (v, P) is a solution of the NSE, \mathbf{L} -periodic, and $v(\cdot, t)$ has zero average.

Let $X(t) = x(t) - U_0 t$. We have

$$X'(t) = x'(t) - U_0 = u(x(t), t) - U_0 = v(x(t) - U_0 t, t) + U_0 - U_0 = v(X(t), t).$$

Applying above result (for zero average solutions) to $v(X, t)$ and $X(t)$ yields

$$X(t) \sim x_* + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$

Consequently, we obtain

$$x(t) = X(t) + U_0 t \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$





THANK YOU!