Non-linear Problems in Fluid Dynamics

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Outline

PART I. 3-D Navier-Stokes Equations

IA. **Theory of normal forms.**

IB. **Global regularity in thin domains.**

Introduction

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u = -\nabla p + f, \\ \mathrm{div}\ u = 0, \\ \mathbf{u}(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
f is the body force,
u^0 is the initial velocity.

Existence and Uniqueness

Denote by N the outward normal vector to the boundary. Define

$$H = \{u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot N = 0 \text{ on } \partial\Omega\},$$

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

[Leray 1933, 1934] Suppose $f = f(x) \in L^2(\Omega)$.

• If $u_0 \in H$, then there exists a weak solution on $[0, \infty)$:

$$u \in C([0,\infty); H_{weak}) \cap L^\infty(0,\infty; H) \cap L^2(0,\infty; V).$$

Question 1: Is this weak solution unique?

• If $u_0 \in V$, then there exists a unique strong solution on $[0, T)$ for some $T > 0$: $u \in C([0,\infty); V) \cap L^\infty(0,\infty; V) \cap L^2(0,\infty; H^2(\Omega))$.

Question 2: Can it be $T = \infty$?

In the 2D case, Questions 1 and 2 have affirmative answers.

In the 3D case, still open!

Small data results: If $\|u_0\|_{H^1(\Omega)}$ and $\|f\|_{L^2(\Omega)}$ are small then the strong solution exists for all $t > 0$.

- $f = -\nabla\phi$ is a potential body force.
- Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3.$$

- Zero average condition

$$\int_\Omega u(x)dx = 0,$$

- Throughout $L = 2\pi$ and $\nu = 1$.
- Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued L-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition. We define

$$\begin{aligned} H &= \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V &= \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \\ D_A &= \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3. \end{aligned}$$

- Norm on H: $|u| = \|u\|_{L^2(\Omega)}$, on V: $\|u\| = |\nabla u|$, on D_A: $|\Delta u|$.

Functional form of NSE

- The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in D_A.$$

- The bilinear mapping:

$$B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in D_A,$$

where P_L is the Leray projection from $L^2(\Omega)$ onto H.
- Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution $u(t)$ is regular for all $t > 0$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \;\; t > 0,$$

$$u(0) = u^0 \in \mathcal{R},$$

where the equation holds in D_A for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into V.

Poincaré–Dulac theory for ODE

Consider an ODE in $\mathbb{R}^n$ of in the formal series form:

$$\frac{dx}{dt} + Ax + \Phi^{[2]}(x) + \Phi^{[3]}(x) + \ldots = 0, \; x \in \mathbb{R}^n,$$

- A is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$
- each $\Phi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$

There exists a formal series $y = x + \sum_{d=1}^{\infty} \Psi^{[d]}(x)$, where $\Psi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$, which transforms the above ODE into an equation

$$\frac{dy}{dt} + Ay + \Theta^{[2]}(y) + \Theta^{[3]}(y) + \ldots = 0, \; y \in \mathbb{R}^n,$$

where all monomials of each $\Theta^{[d]}$ are resonant.

Resonance. Matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\xi_1, \ldots, \xi_n$. For each $x \in \mathbb{R}^n$, let x_i be its coordinate with respect to ξ_i. A monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \xi_k$ is called resonant if

$$\lambda_k = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \ldots + \alpha_n\lambda_n.$$

Poincaré–Dulac normal form for NSE

Functional form of NSE:

$$\frac{du}{dt} + Au + B(u,u) = 0.$$

A differential equation in an infinite dimensional space E

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \Phi^{[d]}(\xi) = 0 \qquad (\star)$$

is a Poincaré–Dulac normal form for the NSE if

1. Each $\Phi^{[d]} \in \mathcal{H}^{[d]}(E)$, the space of homogeneous polynomials of order d, and $\Phi^{[d]}(\xi) = \sum_{k=1}^{\infty} \Phi_k^{[d]}(\xi)$, where all $\Phi_k^{[d]} \in \mathcal{H}^{[d]}(E)$ are resonant monomials,
2. Equation $(\star)$ is obtained from NSE by a formal change of variable $u = \sum_{d=1}^{\infty} \Psi^{[d]}(\xi)$ where $\Psi^{[d]} \in \mathcal{H}^{[d]}(E)$.

Asymptotic expansion - Normalization map

For $u_0 \in \mathcal{R}$, the solution $u(t)$ has an asymptotic expansion: [Foias-Saut]

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + ...,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in t of degree at most $(j-1)$ and with values are trigonometric polynomials. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$,

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.
Let $W(u^0) = \xi_1 \oplus \xi_2 \oplus \cdots$, where $\xi_j = R_j q_j(0)$, for $j = 1, 2, 3...$ Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Fréchet space

$$S_A = R_1 H \oplus R_2 H \oplus \cdots .$$

Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then q_j's are the unique polynomial solutions to the following equations

$$q_j' + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where β_j's are defined by

$$\beta_1 = 0 \text{ and for } j > 1,\ \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$'s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n\geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} (\frac{d}{dt})^n (I - R_j)\beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik\cdot x}$, for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik\cdot x} \in \mathcal{V}$.

Normal form

• The S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^\infty = (W_n(u(t)))_{n=1}^\infty = W(u(t))$ satisfies the following system of differential equations (normal form in S_A):

$$\frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0,$$
$$\frac{d\xi_j(t)}{dt} + A\xi_j(t) + \sum_{k+l=j} R_j B(\mathcal{P}_k(\xi(t)), \mathcal{P}_l(\xi(t))) = 0, \ \ n > 1,$$

where $P_j(\xi) = q_j(0,\xi)$ for $\xi \in S_A$.

• For $d \geq 1$, let $\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^\infty \mathcal{P}_j^{[d]}(\xi) = \sum_{j=d}^\infty q_j^{[d]}(0,\xi)$.

For $d \geq 2$, let $\mathcal{B}^{[d]}(\xi) = \sum_{j=1}^\infty \ \mathcal{B}_j^{[d]}(\xi)$

$$= \sum_{j=1}^\infty \ \sum_{k+l=j} \ \sum_{m+n=d} \ R_j B\big(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)\big).$$

Rewrite in the power series form:

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^\infty \mathcal{B}^{[d]}(\xi) = 0.$$

Main result

Let E^∞ be the Fréchet space $C^\infty(\mathbb{R}^3,\mathbb{R}^3)\cap V$.

Theorem (Foias-**H.**-Saut 2011)

The formal power series change of variable

$$u=\xi+\sum_{d=2}^{\infty}\mathcal{P}^{[d]}(\xi),$$

where $\xi\in E^\infty=C^\infty(\mathbb{R}^3,\mathbb{R}^3)\cap V$, *reduces the NSE to a Poincaré–Dulac normal form*

$$\frac{d}{dt}\xi+A\xi+\sum_{d=2}^{\infty}\mathcal{B}^{[d]}(\xi)=0.$$

- Along the way, $\mathcal{P}^{[d]}(\xi)$, $\mathcal{B}^{[d]}(\xi)$ are proved to converge in appropriate Sobolev spaces (depending on d).
- The change of variables, in fact, is the formal inverse of the normalization map W.

Explicit change of variable

$$u \sim \sum_{j=1}^{\infty} q_j(0,\xi) = \sum_{j=1}^{\infty}\sum_{d=1}^{j} q_j^{[d]}(0,\xi) = \sum_{d=1}^{\infty}\sum_{j=d}^{\infty} q_j^{[d]}(0,\xi) = \sum_{d=1}^{\infty} \mathcal{P}^{[d]}(\xi).$$

This has the formal inverse

$$\xi = \tilde{\mathcal{P}}(u) \stackrel{\text{def}}{=\!=} u + \sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) = \sum_{d=1}^{\infty} \tilde{\mathcal{P}}^{[d]}(u).$$

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} Q^{[d]}(\xi) = 0.$$

Explicitly, $Q^{[1]}(\xi) = A\xi$, and for $d \geq 2$,

$$Q^{[d]}(\xi) = \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{\substack{2\leq k,l\leq d-1\\ k+l=d+1}} D\mathcal{P}^{[k]}(\xi)(Q^{[l]}(\xi)) + H_A^{(d)}\mathcal{P}^{[d]}(\xi),$$

where $H_A^{(d)}\mathcal{P}^{[d]}(\xi) = A\mathcal{P}^{[d]}(\xi) - D\mathcal{P}^{[d]}(\xi)A\xi$ (Poincaré homology oper.).
Proved $Q^{[d]}(\xi) = \mathcal{B}^{[d]}(\xi)$ for all $\xi \in E^\infty$ and $d \geq 2$.

Well-posedness in Banach spaces

Constructed normed spaces. Let $(\tilde{\kappa}_n)_{n=2}^{\infty}$ be a fixed sequence of real numbers in the interval $(0,1]$ satisfying

$$\lim_{n\to\infty} (\tilde{\kappa}_n)^{1/2^n} = 0.$$

We define the sequence of positive weights $(\rho_n)_{n=1}^{\infty}$ by

$$\rho_1 = 1, \quad \rho_n = \tilde{\kappa}_n \gamma_n \rho_{n-1}^2,\ n > 1,$$

where $\gamma_n \in (0,1]$ are known and decrease to zero faster than n^{-n}.
For $\bar{u} = (u_n)_{n=1}^{\infty} \in V^{\infty}$, let

$$\|\bar{u}\|_\star = \sum_{n=1}^{\infty} \rho_n \|\nabla u_n\|_{L^2(\Omega)},$$

Define $V^\star = \{\, \bar{u} \in V^{\infty} : \|\bar{u}\|_\star < \infty \,\}, \quad S_A^\star = S_A \cap V^\star.$
Clearly $V^\star$ and $S_A^\star$ are Banach spaces.

Well-posedness of the normal form

[Foias-**H.**-Olson-Ziane 2006,2009]
We summarize our results in the commutative diagram

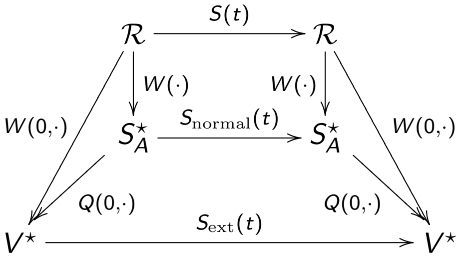

Figure: Commutative diagram

where all mappings are continuous.

Part IB. Global regularity in thin domains

Fast Rotation. Chemin, Babin-Mahalov-Nicolaenko, ...

Thin domains.

- Damped hyperbolic equations in thin domains: Hale-Raugel 1992
- NSE on thin domains: Raugel-Sell 1993, 1994
- Spherical domains: Temam-Ziane 1996
- Later: Avrin, Chueshov, Hu, Iftimie, Kukavica, Rekalo, Raugel, Sell, Ziane, **H.**, ...

Navier friction boundary conditions.

$$u \cdot N = 0, \quad \nu[D(u)N]_{\tan} + \gamma u = 0,$$

- $\gamma = \infty$: Dirichlet condition.
- $\gamma = 0$: Navier boundary conditions (without friction/free slip). One-layer domain: flat bottom [Iftimie-Raugel-Sell 2005]
- If the boundary is flat, say, $x_3 = const$, then $u_3 = 0, \quad \gamma u_1 + \nu\partial_3 u_1 = \gamma u_2 + \nu\partial_3 u_2 = 0$. See [Hu 2007].

Single-layer domains

Theorem (H.-Sell 2010, H. 2011)

Consider NSE in $\Omega = \mathbb{T}^2 \times (h_0(\varepsilon), h_1(\varepsilon))$ *with Navier boundary conditions on the top and bottom boundaries.*
If $\|u_0\|_V,\ \|f\|_H = o(\varepsilon^{-1/2})$ *as* $\varepsilon \to 0$*, then the strong solution exists for all time when* ε *is sufficiently small.*

Note: We also prove the existence of the global attractor and estimate its size.

Two-layer domains

$$\Omega_\varepsilon^+ = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_0(x_1, x_2) < x_3 < h_+(x_1, x_2)\},$$

$$\Omega_\varepsilon^- = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_-(x_1, x_2) < x_3 < h_0(x_1, x_2)\},$$

where $h_0 = \varepsilon g_0, \quad h_+ = \varepsilon(g_0 + g_+)$, and $h_- = \varepsilon(g_0 - g_-)$.

- $g_+(x_1, x_2), g_+(x_1, x_2) \geq c_0 > 0$.
- Top Γ_+, bottom Γ_-, and interface Γ_0.
- Let $\Omega_\varepsilon = [\Omega_\varepsilon{}^+, \Omega_\varepsilon{}^-]$, $\partial\Omega_\varepsilon = [\partial\Omega_\varepsilon{}^+, \partial\Omega_\varepsilon{}^-]$, $\Gamma = [\Gamma_+, \Gamma_-]$.
- Consider the Navier–Stokes equations in Ω_ε

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u = -\nabla p + f, \\ \mathrm{div}\ u = 0, \\ u(x, 0) = u^0(x), \end{cases}$$

where $u = [u_+, u_-]$, $p = [p_+, p_-]$, $f = [f_+, f_-]$, $\nu = [\nu_+, \nu_-]$, $u^0(x) = [u_+^0, u_-^0]$.

Boundary conditions

Let N_+, N_- be the outward normal vector to the boundary of ${\Omega_\varepsilon}^+, {\Omega_\varepsilon}^-$.

- The slip boundary condition on $\partial\Omega_\varepsilon$ is

$$u_+ \cdot N_+ = 0 \text{ on } \Gamma_+, \Gamma_0; \qquad u_- \cdot N_- = 0 \text{ on } \Gamma_-, \Gamma_0.$$

- The Navier friction conditions on the top and bottom boundaries are

$$[\nu_+(Du_+)\ N_+]_{\tan} + \gamma_+ u_+ = 0 \quad \text{ on } \Gamma_+,$$

$$[\nu_-(Du_-)\ N_-]_{\tan} + \gamma_- u_- = 0 \quad \text{ on } \Gamma_-.$$

- The interface boundary condition on Γ_0 is

$$[\nu_+(Du_+)N_+]_{\tan} + \gamma_0(u_+ - u_-) = 0,$$

$$[\nu_-(Du_-)N_-]_{\tan} + \gamma_0(u_- - u_+) = 0.$$

- Assumptions on coefficients: with $2/3 \le \delta \le 1,$

$$C^{-1}\varepsilon^\delta \le \gamma_-, \gamma_+ \le C\varepsilon^\delta, \quad C^{-1}\varepsilon \le \gamma_0 \le C\varepsilon.$$

Main results

Stokes operator A. Some appropriate averaging operator: $\overline{M}$.

Theorem (**H.** 2013)

There are positive numbers ε_* *and* κ *such that if* $\varepsilon < \varepsilon_*$ *and* $u_0 \in V$, $f \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ *satisfy that*

$$\|\overline{M}u_0\|_{L^2}^2, \quad \varepsilon\|A^{\frac{1}{2}}u_0\|_{L^2}^2, \quad \varepsilon^{1-\delta}\|\overline{M}Pf\|_{L^\infty L^2}^2, \quad \varepsilon\|Pf\|_{L^\infty L^2}^2 \le \kappa,$$

then the strong solution $u(t)$ *of NSE exists for all* $t \ge 0$. *Moreover,*

$$\|u(t)\|_{L^2}^2 \le C\kappa, \quad \|A^{\frac{1}{2}}u(t)\|_{L^2}^2 \le C\varepsilon^{-1}\kappa, \quad t \ge 0,$$

where $C > 0$ *is independent of* ε, u_0, f, *and*

$$\|A^{1/2}u\|_{L^2}^2 = 2\int_\Omega \nu|Du|^2dx + 2\int_\Gamma \gamma|u|^2d\sigma + 2\gamma_0\int_{\Gamma_0}|u_+ - u_-|^2d\sigma.$$

Remark: The condition on u_0 is acceptable.

Key estimates

Proposition

If $\varepsilon > 0$ is sufficiently small, then for all $u \in D_A$,

$$\|Au + \Delta u\|_{L^2} \le C\varepsilon\|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2} + C\varepsilon^{\delta-1}\|u\|_{L^2},$$

$$C\|Au\|_{L^2} \le \|u\|_{H^2} \le C'\|Au\|_{L^2}.$$

Proposition

Given $\alpha > 0$, there is $C_\alpha > 0$ such that for any $\varepsilon \in (0,1]$ and $u \in D_A$, we have

$$\begin{aligned} |\langle u \cdot \nabla u, Au\rangle| \le \alpha\|u\|_{H^2}^2 + C\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\|Au\|_{L^2}^2 \\ + C_\alpha\Big(\|u\|_{L^2}^2 + \big[\varepsilon\|A^{1/2}u\|_{L^2}^2\|u\|_{L^2}^2\big]^{1/3}\Big)(\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2). \end{aligned}$$

where $C > 0$ is independent of ε and α.

PART II. Forchheimer flows in porous media

IIA. **Single-phase slihgtly compressible fluids.**

IIB. **Two-phase incompressible fluids.**

Darcy's and Forchheimer's flows

Fluid flows in porous media with velocity u and pressure p:

- Darcy's Law:

$$\alpha u = -\nabla p,$$

- the Forchheimer "two term" law

$$\alpha u + \beta |u|\, u = -\nabla p,$$

- the Forchheimer "three term" law

$$\mathcal{A} u + \mathcal{B}\, |u|\, u + \mathcal{C} |u|^2\, u = -\nabla p.$$

- the Forchheimer "power" law

$$a u + c^n |u|^{n-1}\, u = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Generalized Forchheimer equations

[Aulisa-Bloshanskaya-**H.**-Ibragimov 2009]
Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow u = -K(|\nabla p|)\nabla p,$$

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

Class $FP(N, \vec{\alpha})$. Let $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$,

$$FP(N, \vec{\alpha}) = \Big\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \Big\},$$

where $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$. Notation: $\alpha_N = \deg(g)$,
$\vec{a} = (a_0, a_1, \ldots, a_N)$, $a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1)$, $b = \frac{\alpha_N}{\alpha_N + 2} \in (0, 1)$.

Historical remarks

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

A. Single-phase flows.

- 1990's Numerical study
- L^2-theory (for slightly compressible flows): Aulisa-Bloshanskaya-**H.**-Ibragimov (2009), **H.**-Ibragimov: Dirichlet B.C. (2011), **H.**-Ibragimov Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012), Inhomogeneous media Celik-**H.**(in preparation).
- L^α-theory: **H.**-Ibragimov-Kieu-Sobol (2012-preprint)
- L^∞-theory: **H.**-Kieu-Phan (2013-preprint), **H.**-Kieu-Phan (in preparation), **H.**-Celik (in preparation).

B. Multi-phase flows.

- One-dimensional case: **H.**-Ibragimov-Kieu (2013).
- Multi-dimensional case: **H.**-Ibragimov-Kieu (preprint).

Note: there are more works on Forchheimer flows (2-terms or 3 terms).

Part IIA. Single-phase slightly compressible fluids

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Then

$$\frac{dp}{dt} = \kappa \nabla \cdot \Big(K(|\nabla|)\nabla p \Big) + K(|\nabla p|)|\nabla p|^2.$$

Since $\kappa \gg 1$, we neglect the last terms, after scaling the time variable:

$$\frac{dp}{dt} = \nabla \cdot \Big(K(|\nabla|)\nabla p \Big).$$

Flux condition on the boundary:

$$-K(|\nabla p|)\nabla p \cdot \vec{\nu} = u \cdot \vec{\nu} = \psi(x,t), \quad x \in \Gamma,\ t > 0,$$

where $\vec{\nu}$ is the outward normal vector on Γ and the flux $\psi(x,t)$ is known.

Degeneracy - The Degree Condition

Lemma

Let $g(s, \vec{a})$ be in class FP($N, \vec{\alpha}$). One has for any $\xi \geq 0$ that

$$\frac{C_1(\vec{a})}{(1+\xi)^a} \leq K(\xi, \vec{a}) \leq \frac{C_2(\vec{a})}{(1+\xi)^a},$$

$$C_3(\vec{a})(\xi^{2-a} - 1) \leq K(\xi, \vec{a})\xi^2 \leq C_2(\vec{a})\xi^{2-a}.$$

Degree Condition (DC)

$$\deg(g) \leq \frac{4}{n-2} \iff 2 \leq (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Under the (DC), the Sobolev space $W^{1,2-a}(U) \hookrightarrow L^2(U)$.

Strict Degree Condition (SDC)

$$\deg(g) < \frac{4}{n-2} \iff 2 < (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

IBVP - Shifted solution

The initial data

$$p(x,0) = p_0(x) \text{ is given.}$$

Let $\bar{p}(x,t) = p(x,t) - \frac{1}{|U|}\int_U p(x,t)dx$. Then

$$\int_U \bar{p}(x,t)dx = 0 \quad \text{for all } t \geq 0,$$

and

$$\bar{p}(x,t) = p(x,t) - \frac{1}{|U|}\int_U p_0(x)dx + \frac{1}{|U|}\int_0^t \int_\Gamma \psi(x,\tau)d\sigma d\tau.$$

L^2-estimates

Let $f(t) = \|\psi(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty}^{\frac{2-a}{1-a}}$ and $\tilde{f}(t) = \|\psi_t(t)\|_{L^\infty}^2 + \|\psi_t(t)\|_{L^\infty}^{\frac{2-a}{1-a}}$.
Let $M_f(t)$ be a continuous, increasing majorant of $f(t)$ on $[0,\infty)$.
Let $A = \limsup_{t\to\infty} f(t)$ and $\beta = \limsup_{t\to\infty}[f'(t)]^-$.

Theorem (H.-Ibragimov 2012)

Assume (DC), then

$$\|\bar{p}(t)\|_{L^2}^2 \le \|\bar{p}_0\|_{L^2}^2 + C(1 + M_f(t)^{\frac{2}{2-a}}) \quad \text{for all } t \ge 0.$$

Moreover, if $A < \infty$ then

$$\limsup_{t\to\infty} \|\bar{p}(t)\|_{L^2}^2 \le C(A + A^{\frac{2}{2-a}}),$$

and if $\beta < \infty$ then there is $T > 0$ such that

$$\|\bar{p}(t)\|_{L^2}^2 \le C\left(1 + \beta^{\frac{1}{1-a}} + f(t)^{\frac{2}{2-a}}\right) \quad \text{for all } t > T.$$

Notation

We use the following notation:

$$m_1(t) = 1 + \|\bar{p}_0\|_{L^2}^2 + M_f(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau)d\tau,$$

$$m_2(t) = 1 + A^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau)d\tau,$$

$$m_3(t) = 1 + \beta^{\frac{1}{1-a}} + \sup_{[t-1,t]} f^{\frac{2}{2-a}} + \int_{t-1}^{t} \tilde{f}(\tau)d\tau,$$

$$\mathcal{A}_1 = A + A^{\frac{2}{2-a}} + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{f}(\tau)d\tau.$$

Below,

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})ds \quad \text{for } \xi \geq 0,$$

$$J_H[u](t) = \int_U H(|\nabla u(x,t)|)dx \sim \int_U K(|\nabla p|)|\nabla p|^2 dx \sim \int_U |\nabla p|^{2-a} dx.$$

Estimates for derivatives

Theorem (H.-Ibragimov 2012)

Assume (DC).

(i) *For all* $t \geq 1$, *one has*

$$J_H[p](t),\ \|\bar{p}_t(t)\|_{L^2}^2 \leq C m_1(t) \quad \textit{for all } t \geq 1.$$

(iii) *If* $A < \infty$ *then there is* $T > 1$ *such that*

$$J_H[p](t),\ \|\bar{p}_t(t)\|_{L^2}^2 \leq C m_2(t) \quad \textit{for all } t > T,$$

$$\limsup_{t\to\infty} J_H[p](t),\ \limsup_{t\to\infty} \|\bar{p}_t(t)\|_{L^2}^2 \leq C\mathcal{A}_1.$$

(iv) *If* $\beta < \infty$ *then there is* $T > 1$ *such that*

$$J_H[p](t),\ \|\bar{p}_t(t)\|_{L^2}^2 \leq C m_3(t) \quad \textit{for all } t > T.$$

L^∞-estimate - I. De Giorgi technique.

Proposition (**H.**-Kieu-Phan 2013)

Assume (SDC).

(i) *Then*

$$\sup_{[0,T]} \|\bar{p}\|_{L^\infty} \le C\Big\{\|\bar{p}_0\|_{L^\infty} + (1+T)^{\frac{2}{2-a}}\big(\sup_{[0,T]}\|\psi\|_{L^\infty}+1\big)^{\frac{1}{1-a}}\Big\}.$$

(ii) *For any* $T_0 \ge 0,\ T>0,\ \delta\in(0,1]$ *and* $\theta\in(0,1)$,

$$\sup_{[T_0+\theta T, T_0+T]} \|\bar{p}\|_{L^\infty} \le C\Big\{\sqrt{\mathcal{E}} + (T+1)^{\frac{n}{4-(n+2)a}}\Big(1+\frac{1}{\delta^a\theta T}\Big)^{\frac{n+2}{4-(n+2)a}} \cdot \Big(\|\bar{p}\|_{L^2(U\times(T_0,T_0+T))} + \|\bar{p}\|_{L^2(U\times(T_0,T_0+T))}^{\frac{4}{4-(n+2)a}}\Big)\Big\},$$

$$\mathcal{E} = \mathcal{E}_{T_0,T} \overset{\text{def}}{=\!=} \delta^{2-a} + T\delta^{-a}\sup_{[T_0,T_0+T]}\|\psi\|_{L^\infty}^2 + \delta^{-\frac{a(2-a)}{1-a}}\sup_{[T_0,T_0+T]}\|\psi\|_{L^\infty}^{\frac{2-a}{1-a}}.$$

Theorem (H.-Kieu-Phan 2013)

Assume (SDC). (i) *For* $t > 0$,

$$\|\bar p(t)\|_{L^\infty} \le C\Big(1 + \|\bar p_0\|_{L^\infty} + \|\bar p_0\|_{L^2}^{\mu_4(2-a)} + M_f(t)^{\mu_4}\Big)$$

$$\|\bar p(t)\|_{L^\infty} \le C\Big(1 + t^{-\frac{1}{\mu_2(2-a)}}\Big)\Big\{1 + \|\bar p_0\|_{L^2}^{\mu_4(2-a)} + M_f(t)^{\mu_4}\Big\}.$$

(ii) *If* $A < \infty$ *then* $\limsup_{t\to\infty} \|\bar p(t)\|_{L^\infty} \le C\big(1 + A^{\mu_4}\big)$.
(iii) *If* $\beta < \infty$ *then there is* $T > 0$ *such that*

$$\|\bar p(t)\|_{L^\infty} \le C\Big\{1 + \beta^{\frac{\mu_4(2-a)}{2(1-a)}} + \sup_{[t-2,t]} \|\psi\|_{L^\infty}^{\frac{\mu_4(2-a)}{1-a}}\Big\} \quad \text{for all } t > T.$$

(iv) *If* $\lim_{t\to\infty} \|\psi(t)\|_{L^\infty} = 0$ *then* $\lim_{t\to\infty} \|\bar p(t)\|_{L^\infty} = 0$.

Gradient estimate

Theorem (**H.**-Kieu-Phan 2013)

Assume (SDC). For $s \geq 2$, $U' \Subset U$ and $T > 1$,

$$\sup_{[0,T]} \int_{U'} |\nabla p(x,t)|^s dx \leq C L_4(s) \Big(1 + M_f(T)\Big)^{\mu_4(s-2)} \Big\{1 + \int_0^T f(t)dt\Big\},$$

where $L_4(s) = \Big(1 + \|\bar{p}_0\|_{L^\infty} + \|\bar{p}_0\|_{L^2}^{\mu_4(2-a)}\Big)^{s-2} \Big\{1 + \int_U |\nabla p_0(x)|^s dx\Big\}.$

Lemma (Ladyzhenskaya-Uraltseva-type embedding theorem)

For each $s \geq 1$, smooth cut-off function $\zeta(x) \in C_c^\infty(U)$,

$$\int_U K(|\nabla w|)|\nabla w|^{2s+2}\zeta^2 dx \leq C \sup_{\mathrm{supp}\zeta} |w|^2 \Big\{ \int_U K(|\nabla w|)|\nabla w|^{2s-2}|\nabla^2 w|^2 \zeta^2 dx + \int_U K(|\nabla w|)|\nabla w|^{2s}|\nabla \zeta|^2 dx \Big\}.$$

Dependence on the boundary data

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions having fluxes ψ_1 and ψ_2.
Let $\Psi = \psi_1 - \psi_2$, $P = p_1 - p_2$, and $\bar{P} = P - |U|^{-1} \int_U P dx$.
Notation. We define for $i = 1, 2$,

$$f_i(t) = \|\psi_i(t)\|_{L^\infty}^2 + \|\psi_i(t)\|_{L^\infty}^{\frac{2-a}{1-a}}, \ \tilde{f}_i(t) = \|\psi_{it}(t)\|_{L^\infty}^2 + \|\psi_{it}(t)\|_{L^\infty}^{\frac{2-a}{1-a}}.$$

For $i = 1, 2$, we assume $f_i(t), \tilde{f}_i(t) \in C([0,\infty))$ and when needed $f_i(t) \in C^1((0,\infty))$; let

$$A_i = \limsup_{t\to\infty} f_i(t) \text{ and } \beta_i = \limsup_{t\to\infty} [f_i'(t)]^-.$$

Set $\bar{A} = A_1 + A_2$, $\bar{\beta} = \beta_1 + \beta_2$.
Let $F(t) = f_1(t) + f_2(t)$, $M_F(t) = M_{f_1}(t) + M_{f_2}(t)$, $\tilde{F}(t) = \tilde{f}_1(t) + \tilde{f}_2(t)$.

$W(t) = 1 + M_F(t) + \int_{t-1}^t \tilde{F}(\tau) d\tau$ in general case, or
$W(t) = 1 + \bar{\beta} + F(t) + F(t-1) + \int_{t-1}^t \tilde{F}(\tau) d\tau$ in case $\bar{\beta} < \infty$.

Theorem (**H.**-Ibragimov 2012)

Assume (DC). (i) *If* $\bar{A} < \infty$ *and* $\int_1^\infty \left(1 + \int_{\tau-1}^{\tau} \tilde{F}(s)ds\right)^{-b} d\tau = \infty$ *then*

$$\limsup_{t\to\infty} \|\bar{P}(t)\|_{L^2}^2 \le C \limsup_{t\to\infty} \left\{ \|\Psi(t)\|_{L^\infty}^2 \left(1 + \bar{A} + \int_{t-1}^{t} \tilde{F}(\tau)d\tau\right)^{2b} \right\}.$$

(ii) *If* $\bar{A} = \infty$ *and* $\int_1^\infty W^{-b}(t)dt = \infty$ *then*

$$\limsup_{t\to\infty} \|\bar{P}(t)\|_{L^2}^2 \le C \limsup_{t\to\infty} \left\{ \|\Psi(t)\|_{L^\infty}^2 W^{2b}(t) \right\}.$$

Theorem (**H.**-Kieu-Phan 2013)

Assume (SDC). For $T > 0$*, we have*

$$\sup_{[0,T]} \|\bar{P}\|_{L^\infty(U')} \le 2\|\bar{P}(0)\|_{L^\infty}$$
$$+ CL_{11}M_{2,T}\left(\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty} + \left[\|\bar{P}(0)\|_{L^2} + \sup_{[0,T]} \|\Psi(t)\|_{L^\infty}\right]^{\frac{\gamma_1}{\gamma_1+1}}\right)$$

Continuous dependence for pressure gradient

Theorem (**H.**-Ibragimov 2012)

Assume (DC).

(i) *If* $\bar{A} < \infty$ *and* $\int_{t-1}^{t} \tilde{F}(\tau)d\tau$ *is uniformly bounded on* $[1, \infty)$, *then*

$$\limsup_{t\to\infty} \|\nabla P(t)\|_{L^{2-a}}^2 \le CM_4^{2b+1/2} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty} + CM_4^{2b} \limsup_{t\to\infty} \|\Psi(t)\|_{L^\infty}^2,$$

where $M_4 = 1 + \bar{A} + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{F}(\tau)d\tau$.

(ii) *If* $\bar{A} = \infty$ *and* $\lim_{t\to\infty} W'(t)W^{b-1}(t) = 0$ *and* $\int_1^\infty W^{-b}(\tau)d\tau = \infty$, *then*

$$\limsup_{t\to\infty} \|\nabla P(t)\|_{L^{2-a}}^2 \le C \limsup_{t\to\infty} \Big(W^{2b+1/2}(t)\|\Psi(t)\|_{L^\infty}\Big) + C \limsup_{t\to\infty} \Big(W^{2b}(t)\|\Psi(t)\|_{L^\infty}^2\Big).$$

Dependence on the Forchheimer polynomials

Recall

$$g(|u|)u = -\nabla p,$$

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N}.$$

• Let $N > 0$ and $\vec{\alpha} = (0, \alpha_1, \ldots, \alpha_N)$ be fixed. Denote

$$R(N) = \{\vec{a} = (a_0, a_1, \ldots, a_N) : a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\}.$$

• Let D be a compact set in $R(N)$.
• Let $g_1(s) = g(s, \vec{a}^{(1)})$ and $g_2(s) = g(s, \vec{a}^{(2)})$ be two functions of class $\mathrm{FP}(N, \vec{\alpha})$, where $\vec{a}^{(1)}$ and $\vec{a}^{(2)}$ belong to D.
• Let $p_k = p_k(x, t; \vec{a}^{(k)})$ be two solutions with the same flux $\psi(x, t)$.
• Let $P = p_1 - p_2$, $\bar{P} = \bar{p}_1 - \bar{p}_2$.

Theorem (**H.**-Ibragimov 2012)

Assume (DC). If $f(t)$ and $\tilde{f}(t)$ are bounded, then

$$\limsup_{t\to\infty} \|\bar{P}(t)\|_{L^2}^2 \le C_* M_8^{b+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|,$$

$$\limsup_{t\to\infty} \|\nabla \bar{P}(t)\|_{L^{2-a}}^2 \le C_* M_8^{3b/2+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{1/2} + C_* M_8^{b+1} |\vec{a}^{(1)} - \vec{a}^{(2)}|,$$

where $M_8 = 1 + A + \limsup_{t\to\infty} \int_{t-1}^{t} \tilde{f}(\tau) d\tau$.

Theorem (**H.**-Kieu-Phan 2013)

Assume (SDC). Let $\mathcal{A} = \|\bar{P}(0)\|_{L^2} + |\vec{a}^{(1)} - \vec{a}^{(2)}|^{1/2}$.

$$\sup_{[0,T]} \|\bar{P}\|_{L^\infty(U')} \le 2\|\bar{P}(0)\|_{L^\infty} + CM_{6,T}\big(\mathcal{A} + \mathcal{A}^{\frac{\gamma_1}{\gamma_1+1}}\big), \quad T > 0,$$

$$\sup_{[2,\infty)} \|\bar{P}\|_{L^\infty(U')} \le C\Upsilon_8(\mathcal{A} + \mathcal{A}^{\frac{\gamma_1}{\gamma_1+1}}),$$

$$\limsup_{t\to\infty} \|\bar{P}(t)\|_{L^\infty(U')} \le C\Upsilon_9\Big(|\vec{a}^{(1)} - \vec{a}^{(2)}| + |\vec{a}^{(1)} - \vec{a}^{(2)}|^{\frac{\gamma_1}{\gamma_1+1}}\Big)^{1/2}.$$

Part IIB. Two-phase incompressible fluids

For each ith-phase ($i = 1, 2$), saturation $S_i \in [0, 1]$, density $\rho_i \geq 0$, velocity $\mathbf{u_i} \in \mathbb{R}^\mathbf{n}$, and , and pressure $p_i \in \mathbb{R}$. The saturations satisfy

$$S_1 + S_2 = 1.$$

Each phase's velocity obeys the generalized Forchheimer equation. Conservation of mass holds for each of the phases:

$$\partial_t(\phi\rho_i S_i) + \mathrm{div}(\rho_i \mathbf{u}_i) = 0, \quad i = 1, 2.$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, it is reduced to

$$\phi\partial_t S_i + \mathrm{div}\ \mathbf{u}_i = 0, \quad i = 1, 2.$$

Let p_c be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c.$$

Denote $S = S_1$ and $p_c = p_c(S)$. Then

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -f_i(S)\nabla p_i, \quad i = 1, 2,$$

$$\nabla p_1 - \nabla p_2 = p_c'(S)\nabla S.$$

Hence

$$F_2(S)g_2(|\mathbf{u}_2|)\mathbf{u}_2 - F_1(S)g_1(|\mathbf{u}_1|)\mathbf{u}_1 = \nabla S,$$

where

$$F_i(S) = \frac{1}{p_c'(S)f_i(S)}, \quad i = 1, 2.$$

In summary,

$$\begin{aligned}
&0 \le S = S(\mathbf{x}, t) \le 1,\\
&S_t = -\mathrm{div}\ \mathbf{u}_1,\\
&S_t = \mathrm{div}\ \mathbf{u}_2,\\
&\nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1).
\end{aligned}$$

One-dimensional problem

Assumption A.

$$f_1, f_2 \in C([0,1]) \cap C^1((0,1)),$$

$$f_1(0) = 0, \quad f_2(1) = 0,$$

$$f_1'(S) > 0, \quad f_2'(S) < 0 \text{ on } (0,1).$$

Assumption B.

$$p_c' \in C^1((0,1)), \quad p_c'(S) > 0 \text{ on } (0,1).$$

Theorem (H.-Kieu-Ibragimov 2013)

- *There are 16 types of non-constant steady states (based on their monotonicity and asymptotic behavior as $x \to \pm\infty$).*
- *The steady states which are never 0 nor 1 are linearly stable.*

Multi-dimensional problem [**H.**-Kieu-Ibragimov 2013]

Steady states with geometric constraints:

$$\mathbf{u}_1^*(\mathbf{x}) = c_1|\mathbf{x}|^{-n}\mathbf{x}, \quad \mathbf{u}_2^*(\mathbf{x}) = c_2|\mathbf{x}|^{-n}\mathbf{x}, \quad S_*(\mathbf{x}) = S(|\mathbf{x}|),$$

where c_1, c_2 are constants and $S(r)$ is a solution of the following ODE:

$$S' = F(r, S(r)) \quad \text{for } r > r_0, \quad S(r_0) = s_0, \quad 0 < S(r) < 1.$$

where s_0 is always a number in $(0, 1)$ and

$$F(r, S(r)) = G_2(c_2 r^{1-n})F_2(S) - G_1(c_1 r^{1-n})F_1(S).$$

Theorem

There exists a maximal interval of existence $[r_0, R_{\max})$, *where* $R_{\max} \in (r_0, \infty]$, *and a unique solution* $S \in C^1([r_0, R_{\max}); (0,1))$. *Moreover, if* $R_{\max}$ *is finite then either*

$$\lim_{r \to R_{\max}^-} S(r) = 0 \quad \text{or} \lim_{r \to R_{\max}^-} S(r) = 1.$$

Theorem

If solution $S(r)$ *exists in* $[r_0, \infty)$, *then it eventually becomes monotone and, consequently,* $s_\infty = \lim_{r \to \infty} S(r)$ *exists.*
In case $n = 2$ *and* $c_1^2 + c_2^2 > 0$, *let* $s^* = (f_1/f_2)^{-1}\left(\frac{c_1 a_1^0}{c_2 a_2^0}\right)$.

(i) *If* $c_1 \le 0$ *and* $c_2 \ge 0$ *then* $s_\infty = 1$.

(ii) *If* $c_1 \ge 0$ *and* $c_2 \le 0$ *then* $s_\infty = 0$.

(iii) *If* $c_1, c_2 < 0$ *then* $s_\infty = s^*$.

(iv) *If* $c_1, c_2 > 0$ *then* $s_\infty \in \{0, 1, s^*\}$.

Linearized problem

The formal linearized system at the steady state $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ is

$$\sigma_t = -\mathrm{div}\ \mathbf{v}_1, \quad \sigma_t = \mathrm{div}\ \mathbf{v}_2,$$
$$\nabla\sigma = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*)\mathbf{v}_2 + F_2'(S_*)\sigma\mathbf{G}_2(\mathbf{u}_2^*) - \Big(F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{v}_1 + F_1'(S_*)\sigma\mathbf{G}_1(\mathbf{u}_1^*)\Big).$$

Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Then $\mathrm{div}\ \mathbf{v} = 0$. Assume $\mathbf{v} = \mathbf{V}(\mathbf{x}, t)$ is given . Let

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}(\mathbf{x}) = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*) + F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*),$$
$$\underline{\mathbf{A}} = \underline{\mathbf{A}}(\mathbf{x}) = \underline{\mathbf{B}}(\mathbf{x})^{-1}$$
$$\mathbf{b} = \mathbf{b}(\mathbf{x}) = F_2'(S_*)\mathbf{G}_2(\mathbf{u}_2^*) - F_1'(S_*)\mathbf{G}_1(\mathbf{u}_1^*),$$
$$\mathbf{c} = \mathbf{c}(\mathbf{x}, t) = F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{V}(\mathbf{x}, t).$$

Decoupling the linearized system:

$$\sigma_t = \nabla\cdot\Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c}),$$
$$\mathbf{v}_2 = \underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b}) + \underline{\mathbf{A}}\mathbf{c}, \quad \mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2.$$

In Bounded domains

Condition (E1). $F_1, F_2 \in C^7((0,1))$ and $V \in C^6_{\mathbf{x}}(\bar{D})$; $V_t \in C^3_{\mathbf{x}}(\bar{D})$.

Theorem

Assume **(E1)** *and* $\Delta_4 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) + \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *is finite. Then the solution* $\sigma(\mathbf{x},t)$ *of the linearized equation satisfies*

$$\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)| \le C\left[e^{-\eta_1 t}\sup_U |\sigma_0(\mathbf{x})| + \Delta_4\right] \quad \text{for all } t > 0.$$

Moreover,

$$\limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)|\right] \le C\Delta_5,$$

where

$$\Delta_5 = \limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma} |g(\mathbf{x},t)|\right].$$

Theorem

Assume **(E1)**, *and* $\Delta_6 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|)$ *and* $\Delta_7 \stackrel{\text{def}}{=\!=} \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *are finite. Then for any* $U' \Subset U$*, there is* $\tilde{M} > 0$ *such that for* $i = 1, 2$, $\mathbf{x} \in U'$ *and* $t > 0$,

$$\sup_{\mathbf{x}\in U'} |\mathbf{v}_i(\mathbf{x},t)| \le \tilde{M}\Big(1 + \frac{1}{\sqrt{t}}\Big)\Big[e^{-\eta_1 t}\sup_U |\sigma_0(\mathbf{x})| + \Delta_6 + \sqrt{\Delta_6} + \Delta_7\Big].$$

Consequently, if

$$\lim_{t\to\infty}\Big\{\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma}|g(\mathbf{x},t)|\Big\} = 0,$$

then for any $\mathbf{x} \in U$,

$$\lim_{t\to\infty}\mathbf{v}_1(\mathbf{x},t) = \lim_{t\to\infty}\mathbf{v}_2(\mathbf{x},t) = 0.$$

Theorem (In unbounded domains)

Let $n \geq 3$. *Assume* **(E1)**, $\Delta_{11} \overset{\text{def}}{=\!=} \sup_D |\nabla \cdot (\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty$ *and*

$$\Delta_{10} \overset{\text{def}}{=\!=} \max\{\sup_U |\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty.$$

(i) *There exists a solution* $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *such that*

$$|\sigma(\mathbf{x},t)| \leq C\big[\Delta_{10} + \Delta_{11}(t+1)\big].$$

(ii) *If* $\lim_{|\mathbf{x}|\to\infty} \sigma_0(\mathbf{x}) = 0$ *and* $\lim_{|\mathbf{x}|\to\infty} \sup_{0\leq t\leq T} |\nabla \cdot (\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0$ *for each* $T > 0$, *then*

$$\lim_{r\to\infty}\Big(\sup_{\{|\mathbf{x}|=r\}\times[0,T]} |\sigma(\mathbf{x},t)|\Big) = 0 \quad \textit{for any } T > 0;$$

there is a continuous, increasing function $r(t) > 0$ *with* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty}\Big(\sup_{\{|\mathbf{x}|\geq r(t)\}} |\sigma(\mathbf{x},t)|\Big) = 0.$$

Publications

1. L. Hoang, T. Kieu, T. Phan, *Properties of generalized Forchheimer flows in porous media,* 64pp, submitted for publication.
2. L. Hoang, A. Ibragimov, T. Kieu, Z. Sobol, *Stability of solutions to generalized Forchheimer equations of any degree,* 50pp, submitted for publication.
3. L. Hoang, *Incompressible fluids in thin domains with Navier friction boundary conditions (II),* J. Math. Fluid Mech., Vol. 15, Issue 2, (2013) 361–395.
4. L. Hoang, A. Ibragimov, T. Kieu, *One-dimensional two-phase generalized Forchheimer flows of incompressible fluids,* J. Math. Anal. Appln., Volume 401, Issue 2, (2013) 921–938.
5. L. Hoang, A. Ibragimov, *Qualitative study of generalized Forchheimer flows with the flux boundary condition,* Adv. Diff. Eq., Vol. 17, 5–6, (2012) 511–556.
6. C. Foias, L. Hoang, J.-C. Saut, *Asymptotic integration of Navier-Stokes equations with potential forces. II. An explicit Poincare-Dulac normal form*, J. of Functional Anal., Vol. 260, Issue 10 (2011), 3007–3035.
7. L. Hoang, A. Ibragimov, *Structural stability of generalized Forchheimer equations for compressible fluids in porous media*, Nonlinearity, Vol. 24, No. 1 (2011) 1–41.
8. L. Hoang, G. R. Sell, *Navier–Stokes equations with Navier boundary conditions for an oceanic model*, J. of Dyn. and Diff. Eq., Vol. 22, No 3 (2010), 563–616.

Publications (continued)

9. L. Hoang, *Incompressible fluids with Navier friction boundary conditions in thin domains (I)*, J. of Math. Fluid Mech., Vol. 12, No. 3 (2010), 435–472.
10. E. Aulisa, L. Bloshanskaya, L. Hoang, A. Ibragimov, *Analysis of generalized Forchheimer equations of compressible fluids in porous media*, J. of Math. Phys. 50 (2009), 103102:1–44.
11. C. Foias, L. Hoang, B. Nicolaenko, *On the helicity in 3D Navier–Stokes equations II: The statistical case*, Communications in Mathematical Physics, Vol. 290, Issue 2 (2009), 679–717.
12. C. Foias, L. Hoang, E. Olson, M. Ziane, *The normal form of the Navier–Stokes equations in suitable normed spaces*, Annal. de L'Inst. H. Poincaré - Analyse Non Linéaire, Vol. 26, Issue 5 (2009), 1635–1673.
13. L. Hoang, *A basic inequality for the Stokes operator related to the Navier boundary condition*, J. of Diff. Eq., Vol. 245, Issue 9 (2008), 2585–2594.
14. C. Foias, L. Hoang, B. Nicolaenko, *On the helicity in 3D Navier–Stokes equations I: The non-statistical case*, Proc. of London Math. Soc., Vol. 94 (2007) 53–90.
15. C. Foias, L. Hoang, E. Olson, M. Ziane, *On the solutions to the normal form of the Navier–Stokes equations*, Indiana Univ. Math. J., Vol. 55, No 2 (2006) 631–686.

Scientific activities (since joining TTU in Fall 2008)

- Conference talks: 8
- TTU talks: 11
- Invited talks in other institutions: University of Chicago (2010), Indiana University (2011), University of Tennessee (2011).
- Co-organizer of special sessions/mini-symposia: 3 in America, 1 in Italy
- Co-organizer of RedRaider Mini-symposia 2009 and 2013
- Co-organizer of Applied Mathematics Seminars (2008–present)
- Reviewer for: Discrete and Continuous Dynamical Systems - Series A, Journal of Mathematical Analysis and Applications, Journal of Dynamics and Differential Equations, Journal of Differential Equations, Electronic Journal of Differential Equations, Nonlinear Analysis Series A: Theory, Methods and Applications

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