Forchheimer Equations in Porous Media - Part III

Luan Hoang, Akif Ibragimov

Department of Mathematics and Statistics, Texas Tech University
http://www.math.umn.edu/∼lhoang/
`luan.hoang@ttu.edu`

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Outline

Introduction

- Darcy's Law:

$$\alpha u = -\nabla p,$$

- the "two term" law

$$\alpha u + \beta |u|\, u = -\nabla p,$$

- the "power" law

$$c^n |u|^{n-1}\, u + au = -\nabla p,$$

- the "three term" law

$$\mathcal{A}u + \mathcal{B}\,|u|\,u + \mathcal{C}|u|^2\,u = -\nabla p.$$

Here α, β, c, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

General Forchheimer equations

Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived non-linear Darcy equations from Forchheimer equations.

Equations of Fluids

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho u),$$

For slightly compressible fluid it takes

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp}\frac{dp}{dt} = -\rho\nabla \cdot u - \frac{d\rho}{dp} u \cdot \nabla p,$$

$$\frac{dp}{dt} = -\kappa\nabla \cdot u - u \cdot \nabla p.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa\nabla \cdot u\,.$$

Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

$$\frac{dp}{dt} = \nabla \cdot \left(K(|\nabla p|) \, \nabla p \right).$$

Consider the equation on a bounded domain U in $\mathbb{R}^3$. The boundary of U consists of two connected components: exterior boundary Γ_2 and interior (accessible) boundary Γ_1.

- Dirichlet condition on Γ_1: $p(x,t) = \psi(x,t)$ which is known for $x \in \Gamma_i$.
- On Γ_2:

$$u \cdot N = 0 \Leftrightarrow \frac{\partial p}{\partial N} = 0.$$

Class $FP(N, \vec{\alpha})$

We introduce a class of "Forchheimer polynomials"

Definition

A function $g(s)$ is said to be of class $FP(N, \vec{\alpha})$ if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} = \sum_{j=0}^{N} a_j s^{\alpha_j},$$

where $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$, and $a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0$. Notation $\alpha_N = \deg(g)$.

Let $N \geq 1$ and $\vec{\alpha}$ be fixed. Let

$$R(N) = \{\vec{a} = (a_0, a_1, \ldots, a_N) \subset \mathbb{R}^{N+1} : \, a_0, a_N > 0, a_1, \ldots, a_{N-1} \geq 0\}.$$

Let $g = g(s, \vec{a})$ be in $\mathrm{FP}(N, \vec{\alpha})$ with $\vec{a} \in R(N)$. We denote

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0,1), \quad b = \frac{a}{2-a} = \frac{\alpha_N}{\alpha_N + 2} \in (0,1),$$

$$\chi(\vec{a}) = \max\left\{a_0, a_1, \ldots, a_N, \frac{1}{a_0}, \frac{1}{a_N}\right\} \geq 1.$$

Many estimates below will have constants depending on $\chi(\vec{a})$.

Lemma

Let $g(s, \vec{a})$ be in class FP($N, \vec{\alpha}$). One has for any $\xi \geq 0$ that

$$\frac{C_0^{-1}\chi(\vec{a})^{-1-a}}{(1+\xi)^a} \leq K(\xi, \vec{a}) \leq \frac{C_0\chi(\vec{a})^{1+a}}{(1+\xi)^a},$$

and for any $m \geq 1$, $\delta > 0$ that

$$C_0^{-1}\chi(\vec{a})^{-1-a}\frac{\delta^a}{(1+\delta)^a}(\xi^{m-a} - \delta^{m-a}) \leq K(\xi, \vec{a})\xi^m \leq C_0\chi(\vec{a})^{1+a}\xi^{m-a},$$

where $C_0 = C_0(N, \alpha_N)$ depends on N, α_N only.
In particular, when $m = 2$, $\delta = 1$, one has

$$2^{-a}C_0^{-1}\chi(\vec{a})^{-1-a}(\xi^{2-a} - 1) \leq K(\xi, \vec{a})\xi^2 \leq C_0\chi(\vec{a})^{1+a}\xi^{2-a}.$$

The Monotonicity

Proposition

(i) For any $y, y' \in \mathbb{R}^n$, one has

$$\big(K(|y|,\vec{a})y - K(|y'|,\vec{a})y'\big)\cdot\big(y-y'\big) \geq aK(|y|\vee|y'|,\vec{a})|y-y'|^2.$$

(ii) For any functions p_1 and p_2 one has

$$\int_U \big(K(|\nabla p_1|,\vec{a})\nabla p_1 - K(|\nabla p_2|,\vec{a})\nabla p_2\big)\cdot\big(\nabla p_1 - \nabla p_2\big)dx$$

$$\geq a\left(\int_U K(|\nabla p_1|\vee|\nabla p_2|,\vec{a})|\nabla p_1 - \nabla p_2|^2 dx\right)$$

$$\geq C_5\left(\int_U |\nabla p_1 - \nabla p_2|^{2-a}dx\right)^{\frac{2}{2-a}}\big(1+\|\nabla p_1\|_{L^{2-a}(U)}\vee\|\nabla p_2\|_{L^{2-a}(U)}\big)^{-a},$$

where $C_5 = C_5(N, \deg(g), \chi(\vec{a}))$.

Poincaré–Sobolev inequality

Let $f(x)$ and $\xi(x)$ be two functions on U with $f(x)$ vanishing on Γ_1 and $\xi(x) \geq 0$. Then

$$\Big(\int_U |f(x)|^{(2-a)^*}dx\Big)^{\frac{2}{(2-a)^*}} \leq C_6\Big(\int_U K(\xi(x),\vec{a})|\nabla f(x)|^2 dx\Big) \times \Big(1+\int_U H(\xi(x),\vec{a})dx\Big)^{\frac{a}{2-a}},$$

where $(2-a)^* = n(2-a)/(n-(2-a))$ and $C_6 = C_6(N, \deg(g), \chi(\vec{a}), U)$. Subsequently, when $\deg(g) \leq 4/(n-2)$ one has

$$\int_U |f(x)|^2 dx \leq C_7\Big(\int_U K(\xi(x),\vec{a})|\nabla f(x)|^2 dx\Big)\Big(1+\int_U H(\xi(x),\vec{a})dx\Big)^{\frac{a}{2-a}},$$

where $C_7 = C_7(N, \deg(g), \chi(\vec{a}), U)$.

Degree Condition (DC)

$$\deg(g) \le \frac{4}{n-2} \iff 2 \le (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Extension of the boundary data

Let $\Psi(x,t)$, $x \in U, t \in [0,\infty)$ be an extension of $\psi(x,t)$ from Γ_1 to U. For instance, one can use the following harmonic extension Ψ of ψ:

$$\Delta\Psi = 0 \quad \text{on} \quad U, \quad \Psi\Big|_{\Gamma_1} = \psi \quad \text{and} \quad \Psi\Big|_{\Gamma_2} = 0.$$

We denote such Ψ by $\mathcal{H}(\psi)$. Then the Sobolev norms of $\mathcal{H}(\psi)$ on U can be bounded by the Sobolev or Besov norms of ψ on Γ_1. In particular, we have

$$\Big\|\frac{\partial^k}{\partial t^k}\mathcal{H}(\psi)\Big\|_{W^{r,2}(U)} \le C(k,r)\Big\|\frac{\partial^k}{\partial t^k}\psi\Big\|_{W^{r,2}(\Gamma_1)},$$

for $k = 0,1,2$, $r = 0,1$.

Shift of solutions

Shifted solution: Let $\overline{p} = p - \Psi$, then $\overline{p}$ satisfies

$$\frac{\partial \overline{p}}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p) - \Psi_t \quad \text{on} \quad U \times (0, \infty),$$

$$\overline{p} = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty).$$

Define:

$$H(\xi, \vec{a}) = \int_0^{\xi^2} K(\sqrt{s}, \vec{a}) ds.$$

We denote $H(x, t) = H[p](x, t) = H(|\nabla p(x, t)|, \vec{a})$.
Constants in estimates below depend on $\chi(\vec{a})$.

A priori Estimates - I(a)

Lemma

One has

$$\frac{1}{2}\frac{d}{dt}\int_U \overline{p}^2(x,t)dx \le -C\int_U H(x,t)dx + CG_1(t),$$

where

$$G_1(t) = \int_U |\nabla\Psi(x,t)|^2 dx + \Big(\int_U |\Psi_t(x,t)|^{r_0} dx\Big)^{\frac{2-a}{r_0(1-a)}} + \Big(\int_U |\Psi_t(x,t)|^{r_0} dx\Big)^{\frac{1}{r_0}}$$

with r_0 denoting the conjugate exponent of $(2-a)^*$, thus explicitly having the value

$$r_0 = \frac{n(2-a)}{(2-a)(n+1)-n} = \frac{n(2+\alpha_N)}{n+2+\alpha_N}.$$

A priori Estimates - I(b)

Corollary

One has for $t \geq 0$ that

$$\int_U \overline{p}^2(x,t)dx \leq \int_U \overline{p}^2(x,0)dx + C\Lambda_1(t),$$

where

$$\Lambda_1(t) = \int_0^t G_1(\tau)d\tau.$$

In the case $\deg(g) \leq 4/(n-2)$ *one has*

$$\int_U \overline{p}^2(x,t)dx \leq \int_U \overline{p}^2(x,0)dx + C\Lambda_2(t),$$

where $\Lambda_2(t) = (1 + Env(G_1)(t))^{\frac{2}{2-a}}$, *with* $Env(G_1)(t)$ *being a continuous, increasing envelop of the function* $G_1(t)$ *on* $[0,\infty)$.

A priori Estimates - I: Proofs

- No Degree Condition: minimal information
- With Degree Condition: Sobolev-Poincare inequality and non-linear differential inequality (weaker than the usual Gronwall's)

Non-linear differential inequality

Lemma

Suppose

$$y' \le -Ay^{\alpha} + f(t),$$

for all $t > 0$*, with* $A, \alpha > 0$ *and* $y(t), f(t) \ge 0$*.*
Let $F(t)$ *be a continuous, increasing envelop of* $f(t)$ *on* $[0, \infty)$*. Then one has*

$$y(t) \le y(0) + A^{-1/\alpha} F(t)^{1/\alpha}, \quad \forall t \ge 0.$$

A priori Estimates - II(a)

Lemma

For any $\varepsilon > 0$, one has

$$\frac{d}{dt}\int_U H(x,t)dx + \int_U \overline{p}_t^2(x,t)dx \le \varepsilon \int_U H(x,t)dx + C_\varepsilon G_2(t),$$

where C_ε is positive and

$$G_2(t) = \int_U |\nabla \Psi_t(x,t)|^2 dx + \int_U |\Psi_t(x,t)|^2 dx.$$

Consequently, one has

$$\frac{d}{dt}\left[\int_U H(x,t) + \overline{p}^2(x,t)dx\right] + \int_U \overline{p}_t^2(x,t)dx \le -C\int_U H(x,t)dx + CG_3(t),$$

where $G_3(t) = G_1(t) + G_2(t)$.

A priori Estimates - II(b)

Corollary

(i) Given $\delta > 0$*, there is* $C_\delta > 0$ *such that for all* $t \geq 0$ *one has*

$$\int_U H(x,t)dx \leq e^{\delta t}\int_U H(x,0)dx + C_\delta \int_0^t e^{\delta(t-\tau)} G_2(\tau)d\tau.$$

(ii) For $t \geq 0$*, one has*

$$\int_U H(x,t) + \bar{p}^2(x,t)dx + \boxed{\int_0^t \int_U \bar{p}_t^2(x,\tau)dxd\tau}$$
$$\leq \int_U H(x,0) + \bar{p}^2(x,0)dx + C\int_0^t G_3(\tau)d\tau.$$

A priori Estimates - II(c)

Corollary

For $t \geq 0$, one has

$$\int_U H(x,t)dx \leq e^{-C_1 t}\int_U H(x,0)dx + C\int_U \overline{p}^2(x,0)dx + C\int_0^t e^{-C_1(t-\tau)}\Big(\Lambda_1(\tau) + G_3(\tau)\Big)d\tau.$$

In case $\deg(g) \leq 4/(n-2)$, *one has for $t \geq 0$ that*

$$\int_U H(x,t)dx \leq e^{-C_1 t}\int_U H(x,0)dx + C\int_U \overline{p}^2(x,0)dx + C\int_0^t e^{-C_1(t-\tau)}\Big(\Lambda_2(\tau) + G_3(\tau)\Big)d\tau.$$

A priori Estimates - II(d)

Lemma

Suppose $\deg(g) \le \frac{4}{n-2}$. *Then*

$$\int_U \overline{p}^2(x,t)dx \le Ch(t)\Big(\int_U H(x,t)dx + \int_U |\nabla\Psi(x,t)|^2 dx\Big),$$

$$\Big(\int_U \overline{p}^2(x,t)dx\Big)^{\frac{2-a}{2}} \le C\Big(1+\int_U H(x,t)dx\Big) + C\int_U |\nabla\Psi(x,t)|^2 dx,$$

where

$$h(t) = \Big(1+\int_U H(x,t)dx\Big)^{\frac{a}{2-a}}.$$

A priori Estimates - II(e)

Proposition

Suppose $\deg(g) \le \frac{4}{n-2}$. One has the following two estimates

$$\int_U H(x,t) + \bar{p}^2(x,t)dx \le e^{-C_1\int_0^t h^{-1}(\tau)d\tau}\Big(\int_U H(x,0) + \bar{p}^2(x,0)dx\Big) + C\int_0^t e^{-C_1\int_\tau^t h^{-1}(\theta)d\theta} G_3(\tau)d\tau,$$

and

$$\int_U H(x,t)+\bar{p}^2(x,t)dx \le \int_U H(x,0)+\bar{p}^2(x,0)dx + C\big(1+Env(G_3)(t)\big)^{\frac{2}{2-a}}.$$

A priori Estimates - III(a)

Let $q = p_t$, $\overline{q} = \overline{p}_t$.

Lemma

One has for $t > 0$ that

$$\frac{d}{dt}\int_U \overline{q}^2 dx \le -C\int_U K(|\nabla p|)|\nabla \overline{q}|^2 dx + C\int_U |\nabla \Psi_t|^2 dx + \int_U |\overline{q}\Psi_{tt}| dx.$$

Note: May happen

$$\limsup_{t\to 0}\int_U \overline{p}_t^2(x,t)dx = \infty.$$

A priori Estimates - III(b)

Proposition

One has for $t \geq t_0 > 0$ that

$$\int_U \bar{p}_t^2(x,t)dx \leq t_0^{-1}\Big[\int_U H(x,0) + \bar{p}^2(x,0)dx + C\int_0^{t_0} G_3(\tau)d\tau\Big] + C\int_0^t \Big\{\int_U |\nabla\Psi_t(x,\tau)|^2 dx + h(\tau)\Big(\int_U |\Psi_{tt}(x,\tau)|^{r_0} dx\Big)^{\frac{2}{r_0}}\Big\} d\tau.$$

If $\deg(g) \leq \frac{4}{n-2}$, then for $t \geq t_0 > 0$ one has

$$\int_U \bar{p}_t^2(x,t)dx \leq t_0^{-1} e^{-C_1\int_{t_0}^t \frac{1}{h(\tau)}d\tau}\Big[\int_U H(x,0)+\bar{p}^2(x,0)dx+C\int_0^{t_0} G_3(\tau)d\tau + C\int_0^t e^{-C_1\int_\tau^t \frac{1}{h(\theta)}d\theta}\Big(\int_U |\nabla\Psi_t(x,\tau)|^2 dx + h(\tau)\int_U |\Psi_{tt}(x,\tau)|^2 dx\Big)d\tau.$$

A priori Estimates - IV(a)

Proposition

Suppose $\deg(g) \le 4/(n-2)$.
(i) For any $t \ge t_0 > 0$ one has

$$\int_U H(x,t) + \overline{p}_t^2(x,t) + \overline{p}^2(x,t)dx \le (1+t_0^{-1})e^{-C_1\int_{t_0}^t h^{-1}(\tau)d\tau}\Big[\int_U H(x,0) + \overline{p}^2(x,0)dx + C\int_0^{t_0} G_3(\tau)d\tau\Big] + C\int_0^t e^{-C_1\int_\tau^t h^{-1}(\theta)d\theta} G_4(\tau)d\tau,$$

where $G_4(t) = G_3(t) + \int_U \Psi_{tt}^2(x,t)dx$.

A priori Estimates - IV(b)

Proposition (continued)

(ii) Assume, in addition, that

$$M_0 \overset{\text{def}}{=\!=} \int_0^\infty e^{-C_1(t-\tau)}\Big(\Lambda_2(t) + G_3(t)\Big)d\tau < \infty.$$

Then there is $d_0 > 0$ depending on the initial data of the solution and the value M_0 above so that

$$\int_U H(x,t) + \overline{p}_t^2(x,t) + \overline{p}^2(x,t)dx \le (1 + t_0^{-1})e^{-d_0(t-t_0)}\Big[\int_U H(x,0) + \overline{p}^2(x,0)dx + C\int_0^{t_0} G_3(\tau)d\tau\Big] + C\int_0^t e^{-d_0(t-\tau)}G_4(\tau)d\tau,$$

for all $t \ge t_0 > 0$.

A priori Estimates - IV(c): No condition on $h(t)$

Proposition

Suppose $\deg(g) \le \frac{4}{n-2}$. One has for $t \ge t_0 > 0$ that

$$\int_U H(x,t) + \overline{p}_t^2(x,t) + \overline{p}^2(x,t)dx$$
$$\le (1 + t_0^{-1}) \int_U H(x,0) + \overline{p}^2(x,0)dx + C\big(1 + Env(G_4)(t)\big)^{\frac{2}{2-a}}.$$

A priori Estimates - V(a): Uniform bounds

Theorem

Suppose $\deg(g) \le 4/(n-2)$*. Assume that*

$$\sup_{[0,\infty)} \|\psi(\cdot,t)\|_{W^{1,2}(\Gamma_1)},\ \sup_{[0,\infty)} \|\psi_t(\cdot,t)\|_{W^{1,2}(\Gamma_1)} < \infty.$$

Then

$$\sup_{[0,\infty)} \int_U H(x,t) + p^2(x,t)dx \le 4\int_U H(x,0) + p^2(x,0)dx + C\Big(1+$$

$$\sup_{[0,\infty)} \|\psi(\cdot,t)\|^2_{W^{1,2}(\Gamma_1)} + \sup_{[0,\infty)} \|\psi_t(\cdot,t)\|^{\frac{2-a}{1-a}}_{L^2(\Gamma_1)} + \sup_{[0,\infty)} \|\psi_t(\cdot,t)\|^2_{W^{1,2}(\Gamma_1)}\Big)^{\frac{2}{2-a}}.$$

A priori Estimates - V(b): Uniform bounds

Theorem (continued)

If, in addition,

$$\sup_{[0,\infty)} \|\psi_{tt}(\cdot,t)\|_{L^2(\Gamma_1)} < \infty,$$

then for any $t_0 > 0$ one has

$$\sup_{[t_0,\infty)} \int_U H(x,t) + p_t^2(x,t) + p^2(x,t)dx \le C(1+t_0^{-1}) \int_U H(x,0)$$

$$+ p^2(x,0)dx + Ct_0^{-1} \int_{\Gamma_1} |\psi(x,0)|^2 d\sigma + C\Big(1 + \sup_{[0,\infty)} \|\psi(\cdot,t)\|^2_{W^{1,2}(\Gamma_1)}$$

$$+ \sup_{[0,\infty)} \|\psi_t(\cdot,t)\|^{\frac{2-a}{1-a}}_{L^2(\Gamma_1)} + \sup_{[0,\infty)} \|\psi_t(\cdot,t)\|^2_{W^{1,2}(\Gamma_1)} + \sup_{[0,\infty)} \|\psi_{tt}(\cdot,t)\|^2_{L^2(\Gamma_1)}\Big)^{\frac{2}{2-a}}.$$

Dependence on the boundary data

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions of the IBVP with the boundary profiles $\psi_1(x,t)$ and $\psi_2(x,t)$, respectively. Let $\Psi_k(x,t)$ be an extension of $\psi_k(x,t)$, for $k=1,2$.
We denote

$$z(x,t) = p_1(x,t) - p_2(x,t), \quad \Psi(x,t) = \Psi_1(x,t) - \Psi_2(x,t),$$

$$\overline{p}_k = p_k - \Psi_k, \quad k=1,2, \quad \overline{z} = \overline{p}_1 - \overline{p}_2 = z - \Psi.$$

Let $H_k(x,t) = H[p_k](x,t) = H(|\nabla p_k(x,t)|, \vec{a})$ for $k=1,2$.
Recall that $a = \frac{\deg(g)}{\deg(g)+1}$ and $b = \frac{a}{2-a} = \frac{\deg(g)}{\deg(g)+2}$.
We will establish various estimates for $\overline{Z}(t) \stackrel{\text{def}}{=\!=} \int_U \overline{z}^2(x,t)dx$, for $t \geq 0$.

First, we derive a general differential inequality for $\overline{Z}(t)$.

Lemma

One has for all $t > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_U \bar{z}^2 dx \le -C\Big(\int_U |\nabla \bar{z}|^{2-a} dx\Big)^{\frac{2}{2-a}} (1 + \|H_1\|_{L^1} + \|H_2\|_{L^1})^{-b}$$
$$+ C(1 + \|H_1\|_{L^1} + \|H_2\|_{L^1})^{-b} \|\nabla \Psi\|^2_{L^{2-a}}$$
$$+ C(\|H_1\|_{L^1} + \|H_2\|_{L^1})^{1/2} \|\nabla \Psi\|_{L^2}$$
$$+ C\,(1 + \|H_1\|_{L^1} + \|H_2\|_{L^1})^b\, \|\Psi_t\|^2_{L^{r_0}},$$

where r_0 is the conjugate of $(2-a)^*$.

Let $G_j[\Psi_k]$, $k = 1, 2$, $j = 1, 2, 3, 4$, denote the quantity G_j for corresponding solution p_k with boundary data extension Ψ_k. Similarly, let $\Lambda_j[\Psi_k]$, $k = 1, 2$, $j = 1, 2$, denote the corresponding quantity Λ_j defined for Ψ_k.
Let $\overline{m}(t) = \overline{m}_1(t) + \overline{m}_2(t)$, where for $k = 1, 2$,

$$\overline{m}_k(t) = e^{-C_1 t} \int_U H_k(x,0)dx + \int_U \overline{p}_k^2(x,0)dx + \int_0^t e^{-C_1(t-\tau)}\Big(\Lambda_1[\Psi_k](\tau) + G_3[\Psi_k](\tau)\Big)d\tau.$$

Estimate for finite time interval

Theorem

One has for all $t \geq 0$ that

$$\int_U \bar{z}^2(x,t)dx \leq \int_U \bar{z}^2(x,0)dx + C\int_0^t \Big(\|\nabla\Psi(\cdot,\tau)\|_{L^{2-a}}^2 + \overline{m}(\tau)^{1/2}\|\nabla\Psi(\cdot,\tau)\|_{L^2} + (1+\overline{m}(\tau))^b \|\Psi_t(\cdot,\tau)\|_{L^{r_0}}^2 \Big) d\tau.$$

Consequently, for any given $T > 0$,

$$\sup_{[0,T]} \int_U z^2(x,t)dx \leq 4\int_U z^2(x,0)dx + 6\sup_{[0,T]} \|\Psi(\cdot,t)\|_{L^2}^2 + CT\sup_{[0,T]} \|\nabla\Psi(\cdot,t) + CT(1+A_* + D_*(T))^\delta \Big(\sup_{[0,T]} \|\nabla\Psi(\cdot,t)\|_{L^2} + \sup_{[0,T]} \|\Psi_t(\cdot,t)\|_{L^{r_0}}^2 \Big),$$

Above, $\delta = \max\{1/2, b\}$,

$$A_* = \int_U H_1(x,0)dx + \int_U \overline{p}_1^2(x,0)dx + \int_U H_2(x,0)dx + \int_U \overline{p}_2^2(x,0)dx,$$

$$D_*(T) = \sum_{k=1}^{2} \sup_{[0,T]} \int_0^t e^{-C_1(t-\tau)} \Big(\Lambda_1[\Psi_k](\tau) + G_3[\Psi_k](\tau)\Big) d\tau.$$

Corollary

Suppose for both $k = 1, 2$ and $t \geq 0$ one has

$$\|\nabla \Psi_k(\cdot, t)\|_{L^2}^2 + \|(\Psi_k)_t(\cdot, t)\|_{L^{r_0}}^{\frac{2-a}{1-a}} \leq C(1+t)^{r_1}$$

and

$$\|\nabla (\Psi_k)_t(\cdot, t)\|_{L^2}^2 + \|(\Psi_k)_t(\cdot, t)\|_{L^2}^2 \leq C(1+t)^{r_2},$$

where $r_1, r_2 > 0$. Let $r_3 = 1 + \max\{r_1 + 1, r_2\}$. Then

$$\int_U \bar{z}^2(x,t)dx \leq \int_U \bar{z}^2(x,0)dx$$
$$+ C(1 + A_*) \int_0^t (1+\tau)^{r_3/2} \|\nabla \Psi(\cdot, \tau)\|_{L^2} + (1+\tau)^{r_3 b} \|\Psi_t(\cdot, \tau)\|_{L^r}^2 d\tau,$$

where A_* is defined previously.

Estimate for all time under the Degree Condition

Using appropriate estimates of $\int_U H_k(x,t)dx$, we denote

$$m_k(t) = e^{-C_1 t}\int_U H_k(x,0)dx + \int_U \overline{p}_k^2(x,0)dx \\ + \int_0^t e^{-C_1(t-\tau)}\Big(\Lambda_2[\Psi_k](\tau) + G_3[\Psi_k](\tau)\Big)d\tau, \quad k=1,2.$$

Also, let

$$m(t) = m_1(t) + m_2(t) \text{ and } S(t',t) = \int_{t'}^t (1+m(\tau))^{-b}d\tau.$$

Theorem

Suppose $\deg(g) \le 4/(n-2)$. *Then for all* $t \ge 0$ *one has*

$$\int_U \bar{z}^2(x,t)dx \le e^{-C_1 S(0,t)} \int_U \bar{z}^2(x,0)dx + C \int_0^t e^{-C_1 S(\tau,t)} \Big(\|\nabla\Psi(\cdot,\tau)\|^2_{L^{2-a}} + m(\tau)^{1/2} \|\nabla\Psi(\cdot,\tau)\|_{L^2} + (1+m(\tau))^b \|\Psi_t(\cdot,\tau)\|^2_{L^{r_0}} \Big) d\tau.$$

Corollary

Suppose $\deg(g) \le 4/(n-2)$. *Assume that*

$$\sum_{k=1}^{2}\Big(\sup_{[0,\infty)} \Lambda_2[\Psi_k](t) + \sup_{[0,\infty)} G_3[\Psi_k](t)\Big) < \infty.$$

Then one has

$$\sup_{[0,\infty)} \int_U z^2(x,t)dx \le 4\int_U z^2(x,0)dx + C\sup_{[0,\infty)} \|\Psi(\cdot,t)\|_{L^2}^2$$
$$+ C\sup_{[0,\infty)}\Big(A_*^b\|\nabla\Psi(\cdot,t)\|_{L^{2-a}}^2 + A_*^{b+\frac{1}{2}}\|\nabla\Psi(\cdot,t)\|_{L^2} + A_*^{2b}\|\Psi_t(\cdot,t)\|_{L^{r_0}}^2\Big),$$

where

$$A_* = 1+\sum_{k=1}^{2}\Big(\int_U |\nabla p_k(x,0)|^{2-a} + \overline{p}_k^2(x,0)dx + \sup_{[0,\infty)} \Lambda_2[\Psi_k](t) + \sup_{[0,\infty)} G_3[\Psi_k](t$$

Corollary

Suppose $\deg(g) \le 4/(n-2)$. *Assume that*

$$\lim_{t\to\infty} S(0,t) = \infty,$$

$$\lim_{t\to\infty} (1+m(t))^{b+\frac{1}{2}} \|\nabla\Psi(\cdot,t)\|_{L^2} = 0, \quad \lim_{t\to\infty} (1+m(t))^{b} \|\Psi_t(\cdot,t)\|_{L^{r_0}} = 0.$$

Then $\lim_{t\to\infty} \int_U \bar{z}^2(x,t)dx = 0$.

Dependence on the Forchheimer polynomials

Let the boundary data $\psi(x,t)$ on Γ_1 be fixed. For each coefficient vector $\vec{a}$, we denote by $p(x,t;\vec{a})$ the solution of

$$p_t = \nabla \cdot (K(|\nabla p|)\nabla p), \quad p|_{\Gamma_1} = \psi, \quad \frac{\partial p}{\partial N}\Big|_{\Gamma_2} = 0,$$

with $K = K(\xi, \vec{a})$.
Let $g_1 = g(s, \vec{a}^{(1)})$ and $g_2 = g(s, \vec{a}^{(2)})$ be two functions of class $\mathrm{FP}(N, \vec{\alpha})$.
Let $p_k = p_k(x,t;\vec{a}^{(k)}))$ for $k = 1, 2$. Let $p = p_1 - p_2$, then

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p_1|, \vec{a}^{(1)})\nabla p_1) - \nabla \cdot (K(|\nabla p_2|, \vec{a}^{(2)})\nabla p_2).$$

Multiplying this equation by p and integrating by parts over the domain yield

$$\frac{1}{2}\frac{d}{dt}\int_U p^2 dx = -\int_U (K(|\nabla p_1|, \vec{a}^{(1)})\nabla p_1 - K(|\nabla p_2|, \vec{a}^{(2)})\nabla p_1)\cdot(\nabla p_1 - \nabla p_2)dx$$

Perturbed Monotonicity

Let $\vec{a}$ and $\vec{a}'$ be two arbitrary vectors. We denote by $\vec{a} \vee \vec{a}'$ and $\vec{a} \wedge \vec{a}'$ the maximum and minimum vectors of the two, respectively, with components

$$(\vec{a} \vee \vec{a}')_j = \max\{a_j, a'_j\} \quad \text{and} \quad (\vec{a} \wedge \vec{a}')_j = \min\{a_j, a'_j\}.$$

Then component-wise one has $\vec{a} \wedge \vec{a}' \leq \vec{a}, \vec{a}' \leq \vec{a} \vee \vec{a}'$.
Define $\chi(\vec{a}, \vec{a}') = \max\{\chi(\vec{a}), \chi(\vec{a}')\}$. Note that

$$\chi(\vec{a} \vee \vec{a}'),\ \chi(\vec{a} \wedge \vec{a}') \leq \chi(\vec{a}, \vec{a}'),$$

$$\chi(t\vec{a} + (1-t)\vec{a}') \leq \chi(\vec{a}, \vec{a}') \quad \forall t \in [0,1].$$

Perturbed Monotonicity

Lemma

Let $g(s,\vec{a})$ and $g(s,\vec{a}')$ belong to class FP($N,\vec{\alpha}$). Then for any y, y' in $\mathbb{R}^n$, one has

$$\begin{aligned}(K(|y|,\vec{a})y - K(|y'|,\vec{a}')y')\cdot(y-y') \geq (1-a)K(|y|\vee|y'|,\vec{a}\vee\vec{a}')|y-y'|^2 \\ - N\chi(\vec{a},\vec{a}')|\vec{a}-\vec{a}'|K(|y|\vee|y'|,\vec{a}\wedge\vec{a}')(|y|\vee|y'|)|y-y'|,\end{aligned}$$

where $a \in (0,1)$.

Let

$$\overline{M}_k(t) = 1 + e^{-C_1 t}\int_U |\nabla p_k(x,0)|^{2-a}dx + \int_U \overline{p}_k^2(x,0)dx + \int_0^t e^{-C_1(t-\tau)}\big(\Lambda_1(\tau) + G_3(\tau)\big)d\tau, \quad k = 1,2,$$

and $\overline{M} = \overline{M}_1 + \overline{M}_2$. One has

$$\int_U K(|\nabla p_k|, \vec{a}^{(1)} \wedge \vec{a}^{(2)})|\nabla p_k|^2 dx \le C + C\int_U |\nabla p_k|^{2-a}dx \le C\overline{M}_k,$$

where C depends on $\chi(\vec{a}^{(1)} \wedge \vec{a}^{(2)})$ and $\chi(\vec{a}^{(k)})$.
Denote $H_k(x,t) = H(|\nabla p_k(x,t)|, \vec{a}^{(k)})$ for $k = 1,2$.

Finite time estimate

First, we obtain the continuous dependence for finite time.

Proposition

For $t \geq 0$ one has

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \leq \int_U |p_1(x,0) - p_2(x,0)|^2 dx + C|\vec{a}^{(1)} - \vec{a}^{(2)}| \int_0^t \overline{M}(\tau) d\tau,$$

where $C > 0$ depends on N, α_N, and $\chi(\vec{a}^{(1)}, \vec{a}^{(2)})$. Consequently, the solution $p(x,t;\vec{a})$ depends continuously (in finite time intervals) on the initial data and the coefficient vector $\vec{a} \in R(N)$.

Large time estimate under the Degree Condition

Under the Degree Condition, $\int_U |\nabla p_k(x,t)|^{2-a}dx$ is bounded by $CM_k(t)$ where

$$M_k(t) = 1 + e^{-C_1 t}\int_U |\nabla p_k(x,0)|^{2-a}dx + \int_U \overline{p}_k^2(x,0)dx \\ + \int_0^t e^{-C_1(t-\tau)}\big(\Lambda_2(\tau) + G_3(\tau)\big)d\tau, \quad k = 1,2,$$

and $M(t) = M_1(t) + M_2(t)$.

Proposition

Suppose $\alpha_N \leq 4/(n-2)$. Then for $t \geq 0$,

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \leq e^{-C_1 \int_0^t M(\tau)^{-b}(\tau) d\tau} \int_U |p_1(x,0) - p_2(x,0)|^2 dx$$
$$+ C_2 |\vec{a}^{(1)} - \vec{a}^{(2)}| \int_0^t e^{-C_1 \int_\tau^t M(\theta)^{-b}(\theta) d\theta} M(\tau) d\tau.$$

Assume, in addition, that $M_0 \stackrel{\text{def}}{=\!=} \sup_{[0,\infty)} M(t) < \infty$ then

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \leq e^{-C_1 t / M_0^b} \int_U |p_1(x,0) - p_2(x,0)|^2 dx$$
$$+ C_2 C_1 M_0^{1+b} |\vec{a}^{(1)} - \vec{a}^{(2)}|$$

for all $t \geq 0$, and consequently

$$\limsup_{t \to \infty} \int_U |p_1(x,t) - p_2(x,t)|^2 dx \leq C_2 C_1 M_0^{1+b} |\vec{a}^{(1)} - \vec{a}^{(2)}|.$$

Uniform estimate in the coefficients

Let D be a compact set in $R(N)$. Define

$$\hat{\chi}(D) = \max\{\chi(\vec{a}),\, \vec{a} \in D\}.$$

Note that $\hat{\chi}(D) \in (0,\infty)$ and for any $\vec{a} \in D$, one has

$$\hat{\chi}(D)^{-1} \le \chi(\vec{a})^{-1} \le \chi(\vec{a}) \le \hat{\chi}(D).$$

Let $\vec{a}^{(k)}$ belong to D for $k = 1,2$. Set

$$A_* = 2 + \sum_{k=1}^{2} \left(\int_U |\nabla p_k(x,0)|^{2-a} dx + \int_U \overline{p}_k^2(x,0) dx \right)$$

$$\overline{\lambda}(t) = 1 + \int_0^t e^{-C_1(t-\tau)} \big(\Lambda_1(\tau) + G_3(\tau)\big) d\tau,$$

$$\lambda(t) = 1 + \int_0^t e^{-C_1(t-\tau)} \big(\Lambda_2(\tau) + G_3(\tau)\big) d\tau,$$

where C_1 depends on N, α_N and $\hat{\chi}(D)$.
Then $\overline{M}(t) \le A_* \overline{\lambda}(t)$, $M(t) \le A_* \lambda(t)$.

Theorem

Assume, additionally, that $\alpha_N \le 4/(n-2)$.
(i) One has for $t \ge 0$ *that*

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le e^{-C_1 A_*^{-b} \int_0^t \lambda(\tau)^{-b} d\tau} \int_U |p_1(x,0) - p_2(x,0)|^2 dx$$
$$+ C_2 A_* |\vec{a}^{(1)} - \vec{a}^{(2)}| \int_0^t e^{-C_1 A_*^{-b} \int_\tau^t \lambda(\theta)^{-b} d\theta} \lambda(\tau) d\tau.$$

(ii) Assume, in addition, that $M_0 \stackrel{\text{def}}{=\!=} \sup_{[0,\infty)} \lambda(t) < \infty$. *Then*

$$\int_U |p_1(x,t) - p_2(x,t)|^2 dx \le e^{-C_1 t/(A_* M_0)^b} \int_U |p_1(x,0) - p_2(x,0)|^2 dx$$
$$+ C_2 C_1^{-1} (A_* M_0)^{1+b} |\vec{a}^{(1)} - \vec{a}^{(2)}|.$$

Consequently

$$\limsup_{t\to\infty} \int_U |p_1(x,t) - p_2(x,t)|^2 dx \le C_2 C_1^{-1} (A_* M_0)^{1+b} |\vec{a}^{(1)} - \vec{a}^{(2)}|.$$

Next Semester - Spring 2011

Forchheimer Equations in Porous Media - Part IV

- Boundary Flux condition
- Refined asymptotic estimates: limsup estimates for nonlinear differential inequalities, uniform Gronwall inequality, ...
- Continuous dependence for pressure gradients and velocity

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THANK YOU!