Navier-Stokes equations in thin domains with Navier friction boundary conditions

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Outline

Introduction

Navier-Stokes equations for fluid dynamics:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u = -\nabla p + f, \\ \mathrm{div}\ u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
$f(t)$ is the body force,
u_0 is the given initial data.

Navier friction boundary conditions

On the boundary $\partial\Omega$:

$$u \cdot N = 0,$$

$$\nu[D(u)N]_{\tan} + \gamma u = 0,$$

- N is the unit outward normal vector
- $\gamma \geq 0$ denotes the friction coefficients
- $[\,\cdot\,]_{\tan}$ denotes the tangential part
- $$D(u) = \frac{1}{2}\Big(\nabla u + (\nabla u)^*\Big).$$

Remarks

- $\nu = 0$, $\gamma = 0$: Boundary condition for inviscid fluids
- $\gamma = \infty$: Dirichlet condition.
- $\gamma = 0$: Navier boundary conditions (without friction) [Iftimie-Raugel-Sell](with flat bottom), [H.-Sell].
- If the boundary is flat, say, part of $x_3 = const$, then the conditions become the Robin conditions (see [Hu])

$$u_3 = 0, \quad u_1 + \gamma \partial_3 u_1 = u_2 + \gamma \partial_3 u_2 = 0.$$

- Compressible fluids on half planes with Navier friction boundary conditions [Hoff].
  Assume $\nu = 1$.

Thin domains

$$\Omega = \Omega_\varepsilon = \{(x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2,\ h_0^\varepsilon(x_1, x_2) < x_3 < h_1^\varepsilon(x_1, x_2)\},$$

where $\varepsilon \in (0, 1]$,

$$h_0^\varepsilon = \varepsilon g_0, \quad h_1^\varepsilon = \varepsilon g_1,$$

and g_0, g_1 are given C^3 functions defined on $\mathbb{T}^2$,

$$g = g_1 - g_0 \geq c_0 > 0.$$

The boundary is $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the bottom and Γ_1 is the top.

Boundary conditions on thin domains

The velocity u satisfies the Navier friction boundary conditions on Γ_1 and Γ_0 with friction coefficients $\gamma_1 = \gamma_1^\varepsilon$ and $\gamma_0 = \gamma_0^\varepsilon$, respectively.

Assumption

There is $\delta \in [0,1]$, such that for i=0,1,

$$0 < \liminf_{\varepsilon \to 0} \frac{\gamma_i^\varepsilon}{\varepsilon^\delta} \le \limsup_{\varepsilon \to 0} \frac{\gamma_i^\varepsilon}{\varepsilon^\delta} < \infty.$$

Notation

Leray-Helmholtz decomposition

$$L^2(\Omega_\varepsilon)^3 = H \oplus H^\perp$$

where

- $H = \{u \in L^2(\Omega_\varepsilon)^3 : \nabla \cdot u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot N = 0 \text{ on } \Gamma\}$,
- $H^\perp = \{\nabla\phi : \phi \in H^1(\Omega_\varepsilon)\}$.

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Let V be the closure in $H^1(\Omega_\varepsilon, \mathbb{R}^3)$ of $u \in C^\infty(\overline{\Omega_\varepsilon}, \mathbb{R}^3) \cap H$ that satisfies the friction boundary conditions.

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Averaging operator:

$$M_0\phi(x') = \frac{1}{\varepsilon g}\int_{h_0}^{h_1} \phi(x', x_3)dx_3, \quad \widehat{M}u = (M_0 u_1, M_0 u_2, 0).$$

Main result

Theorem (Global strong solutions)

Let $\delta \in [2/3, 1]$*. There are* $\varepsilon_0 > 0$ *and* $\kappa > 0$ *such that if* $\varepsilon \in (0, \varepsilon_0]$ *and* $u_0 \in V$ *and* $f \in L^\infty(L^2)$ *satisfy*

$$m_{u,0} = \|\widehat{M} u_0\|_{L^2}^2, \quad m_{u,1} = \varepsilon \|u_0\|_{H^1}^2,$$
$$m_{f,0} = \|\widehat{M} f\|_{L^\infty L^2}^2, \quad m_{f,1} = \varepsilon \|f\|_{L^\infty L^2}^2,$$

are smaller than κ*, then the regular solution exists for all* $t \geq 0$*:*

$$u \in C([0,\infty), H^1(\Omega_\varepsilon)) \cap L^2_{loc}([0,\infty), H^2(\Omega_\varepsilon)).$$

Remark: The condition on u_0 is acceptable.

A Green's formula

[Solonnikov-Ščădilov]

$$\int_\Omega \Delta u \cdot v \, dx = \int_\Omega [-2(Du : Dv) + (\nabla \cdot u)(\nabla \cdot v)] \, dx + \int_{\partial\Omega} \{2((Du)N) \cdot v - (\nabla \cdot u)(v \cdot N)\} \, d\sigma.$$

If u is divergence-free and satisfies the Navier friction boundary conditions, v is tangential to the boundary then

$$-\int_{\Omega_\varepsilon} \Delta u \cdot v \, dx = 2 \int_{\Omega_\varepsilon} (Du : Dv) \, dx + 2\gamma_0 \int_{\Gamma_0} u \cdot v \, d\sigma + 2\gamma_1 \int_{\Gamma_1} u \cdot v \, d\sigma.$$

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The right hand side is denoted by $E(u, v)$.

Uniform Korn inequality

Is $E(\cdot,\cdot)$ bounded and coercive in $H^1(\Omega_\varepsilon)$?
We need Korn's inequality: $\|u\|^2_{H^1(\Omega_\varepsilon)} \le C_\varepsilon E(u,u)$.

Lemma

There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0]$, $u \in H^1(\Omega_\varepsilon) \cap H_0^\perp$ and u is tangential to the boundary of Ω_ε, one has

$$\|u\|^2_{L^2} \le C\varepsilon^{1-\delta} E(u,u),$$

$$C\|u\|^2_{H^1} \le E(u,u) \le C'(\|\nabla u\|^2_{L^2} + \varepsilon^{1-\delta}\|u\|^2_{L^2}),$$

where C, C' are positive constants independent of ε.

Stokes operator

Let P denotes the (Leray) projection on H. Then the Stokes operator is:

$$Au = -P\Delta u, \quad u \in D_A,$$

$D_A = \{u \in H^2(\Omega_\varepsilon)^3 \cap V : \text{ u satisfies the Navier friction boundary conditions}\}$

For $u \in D_A, v \in V$, one has

$$\langle Au, v \rangle = E(u, v).$$

Navier-Stokes equations:

$$\frac{du}{dt} + Au + B(u, u) = Pf,$$

where $B(u, v) = P(u \cdot \nabla u)$.

Inequalities

For $\varepsilon \in (0, \varepsilon_0]$, one has the following:

- If $u \in V = D_{A^{\frac{1}{2}}}$ then

$$\|u\|_{L^2} \leq C\varepsilon^{(1-\delta)/2}\|A^{\frac{1}{2}}u\|_{L^2}, \quad \|u\|_{H^1} \leq C\|A^{\frac{1}{2}}u\|_{L^2},$$

$$\|A^{\frac{1}{2}}u\|_{L^2} \leq C(\|\nabla u\|_{L^2} + \varepsilon^{(\delta-1)/2}\|u\|_{L^2}).$$

- If $u \in D_A$ then

$$\|A^{\frac{1}{2}}u\|_{L^2} \leq C\varepsilon^{(1-\delta)/2}\|Au\|_{L^2}, \quad \|u\|_{L^2} \leq C\varepsilon^{1-\delta}\|Au\|_{L^2}.$$

Interpreting the boundary conditions

Lemma

Let τ be a tangential vector field on the boundary. If u satisfies the Navier friction boundary conditions then one has on Γ that

$$\frac{\partial u}{\partial \tau} \cdot N = -u \cdot \frac{\partial N}{\partial \tau},$$
$$\frac{\partial u}{\partial N} \cdot \tau = u \cdot \left\{ \frac{\partial N}{\partial \tau} - 2\gamma\tau \right\}.$$

One also has [Chueshov-Raugel-Rekalo]

$$N \times (\nabla \times u) = 2N \times \{N \times ((\nabla N)^* u) - \gamma N \times u\}.$$

Our case: $|\nabla N| \sim \varepsilon$ and $\gamma \sim \varepsilon^\delta$.

Linear and Non-linear Estimates

Proposition

If $\varepsilon \in (0, \varepsilon_0]$ and $u \in D_A$, then

$$\|Au + \Delta u\|_{L^2} \le C_1 \varepsilon^{\delta} \|\nabla u\|_{L^2} + C_1 \varepsilon^{\delta - 1} \|u\|_{L^2},$$

$$C_2 \|Au\|_{L^2} \le \|u\|_{H^2} \le C_3 \|Au\|_{L^2}.$$

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Proposition

There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $\alpha > 0$ and $u \in D_A$, one has

$$|\langle u \cdot \nabla u, Au \rangle| \le \{\alpha + C\varepsilon^{1/2} \|A^{\frac{1}{2}} u\|_{L^2}\} \|Au\|_{L^2}^2 + C_\alpha \varepsilon^{2\delta} \|u\|_{L^2}^2 \|A^{\frac{1}{2}} u\|_{L^2}^4 + C_\alpha \varepsilon^{-1} \varepsilon^{\delta - 2/3} \|u\|_{L^2}^{2/3} \|A^{\frac{1}{2}} u\|_{L^2}^2 + C_\alpha \varepsilon^{-1} \|u\|_{L^2}^2 \|A^{\frac{1}{2}} u\|_{L^2}^2.$$

where the positive number C_α depends on α but not on ε.

Corollary

Suppose $\delta \in [2/3, 1]$, *then there exists* $\varepsilon_* \in (0, 1]$ *such that for any* $\varepsilon < \varepsilon_*$ *and* $u \in D_A$, *one has*

$$|\langle u \cdot \nabla u, Au\rangle| \leq \Big\{\frac{1}{4} + d_1 \varepsilon^{1/2} \|A^{\frac{1}{2}} u\|_{L^2}\Big\} \|Au\|_{L^2}^2 + d_2 \Big\{\|u\|_{L^2}^2 \|A^{\frac{1}{2}} u\|_{L^2}^2\Big\} \|A^{\frac{1}{2}} u\|_{L^2}^2 + d_3 \Big\{1 + \|u\|_{L^2}^2\Big\} \varepsilon^{-1} \|A^{\frac{1}{2}} u\|_{L^2}^2.$$

where positive constants d_1, d_2 *and* d_3 *are independent of* ε.

Key identity

Lemma

Let $u \in D_A$ and $\Phi \in H^1(\Omega_\varepsilon)^3$. One has

$$\int_{\Omega_\varepsilon} (\nabla \times (\nabla \times u)) \cdot \Phi dx = \int_{\Omega_\varepsilon} (\nabla \times \Phi) \cdot (\nabla \times u + G(u)) dx - \int_{\Omega_\varepsilon} \Phi \cdot (\nabla \times G(u)) dx.$$

where

$$|G(u)| \le C\varepsilon^\delta |u|, \quad |\nabla G(u)| \le C\varepsilon^\delta |\nabla u| + C\varepsilon^{\delta-1}|u|.$$

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- Linear estimate: $\Phi = Au + \Delta u$, $\nabla \times \Phi = 0$.
- Non-linear estimate: $\Phi = u \times (\nabla \times u)$.

Estimate of $\|\nabla^2 u\|_{L^2}$

Lemma

There is $\varepsilon_0 \in (0,1]$ such that if $\varepsilon < \varepsilon_0$ and $u \in H^2(\Omega_\varepsilon)^3$ satisfies the Navier friction boundary conditions, then

$$\|\nabla^2 u\|_{L^2} \leq C\|\Delta u\|_{L^2} + C\|u\|_{H^1}.$$

Remarks on the proof. Integration by parts

$$\int_{\Omega_\varepsilon} |\nabla^2 u|^2 dx = \int_{\Omega_\varepsilon} |\Delta u|^2 dx + \int_\Gamma \left(\frac{1}{2} \frac{\partial |\nabla u|^2}{\partial N} - \frac{\partial u}{\partial N} \cdot \Delta u \right) d\sigma.$$

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- Remove the second derivatives in the boundary integrals
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- Remove the second derivatives in the boundary integrals
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- The role of the positivity of the friction coefficients

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- Remove the second derivatives in the boundary integrals
- Appropriate order for ε
- The role of the positivity of the friction coefficients

$$\|\nabla^2 u\|_{L^2}^2 \le \|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2.$$

Estimate of the non-linear term

Write $u = v + w$ where

$$v = Mu = (\widehat{M}u, \widehat{M}u \cdot \psi), \quad \psi(x) = \frac{1}{\varepsilon g}\left\{(x_3 - h_0)\nabla_2 h_1 + (h_1 - x_3)\nabla_2 h_0\right\}.$$

Then v is divergence free and tangential to the boundary.
Important properties:

- v is a 2D-like vector field.
- w satisfies "good" inequalites:

$$\|v\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}, \quad \|\nabla v\|_{L^4} \le C\varepsilon^{-1/4}\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}$$

$$\|w\|_{L^2} \le C\varepsilon\|\nabla w\|_{L^2}, \quad \|\nabla w\|_{L^2} \le C\varepsilon\|u\|_{H^2} + C\varepsilon^{\delta}\|u\|_{L^2},$$

$$\|w\|_{L^\infty} \le C\varepsilon^{1/2}\|u\|_{H^2} + C\varepsilon^{\delta/2}\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}.$$

Then write

$$\langle (u\cdot\nabla)u, Au\rangle = \langle (w\cdot\nabla)u, Au\rangle + \langle (v\cdot\nabla)u, Au + \Delta u\rangle - \langle (v\cdot\nabla)u, \Delta u\rangle.$$

Strong global solutions

- Do not need $u = (v, w)$ and equations for each v and w
- Non-linear estimate and Uniform Gronwall's inequality

Steps:

- Estimates for $\|u(t)\|_{L^2}$ and $\int_{t-1}^{t} \|A^{\frac{1}{2}} u(s)\|_{L^2}^2 ds$
- Estimates for $\|A^{\frac{1}{2}} u(t)\|_{L^2}^2$ and the "right" ε^{-1} size.

L^2-Estimates for u

Poincaré-like inequalities: $\|(I-\widehat{M})u\|_{L^2} \le C\varepsilon\|u\|_{H^1}$.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|A^{\frac{1}{2}}u\|_{L^2}^2 \le |\langle u, Pf\rangle| \le |\langle \hat{M}u, \hat{M}Pf\rangle| + |\langle (I-\hat{M})u, (I-\hat{M})Pf\rangle|$$
$$\le \frac{1}{2}\|A^{\frac{1}{2}}u\|_{L^2}^2 + \|\widehat{M}Pf\|_{L^\infty L^2}^2 + \varepsilon^2\|Pf\|_{L^\infty L^2}^2.$$

Hence Gronwall's inequality yields

$$\|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2 e^{-c_1 t} + \|\widehat{M}Pf\|_{L^\infty L^2}^2 + \varepsilon^2\|Pf\|_{L^\infty L^2}^2.$$

$$\|u_0\|_{L^2}^2 = \|\widehat{M}u_0\|_{L^2}^2 + \|(I-\widehat{M})u_0\|_{L^2}^2 \le \|\widehat{M}u_0\|_{L^2}^2 + C\varepsilon^2\|u_0\|_{H^1}^2 \le C\kappa.$$

$$\|u(t)\|_{L^2}^2, \int_t^{t+1} \|A^{\frac{1}{2}}u\|_{L^2}^2 ds \le C\kappa.$$

H^1-Estimate for u

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \le \left\{\frac{1}{4} + d_1\varepsilon^{1/2}\|A^{\frac{1}{2}}u\|_{L^2}\right\}\|Au\|_{L^2}^2$$
$$+d_2\left\{\|u\|_{L^2}^2\|A^{\frac{1}{2}}u\|_{L^2}^2\right\}\|A^{\frac{1}{2}}u\|_{L^2}^2 + d_3\left\{1+\|u\|_{L^2}^2\right\}\varepsilon^{-1}\|A^{\frac{1}{2}}u\|_{L^2}^2 + \|Au\|_{L^2}\|f\|_{L^\infty L^2}.$$

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_{L^2}^2 + (1 - 2d_1\varepsilon^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|_{L^2})\|Au\|_{L^2}^2$$
$$\le g\|A^{\frac{1}{2}}u\|_{L^2}^2 + h,$$

where

$$g = 2d_2\|u\|_{L^2}^2\|A^{\frac{1}{2}}u\|_{L^2}^2,$$
$$h = 2d_3\left\{1 + \|u\|_{L^2}^2\right\}\varepsilon^{-1}\|A^{\frac{1}{2}}u\|_{L^2}^2 + 2\|f\|_{L^\infty L^2}^2.$$

One has

$$\int_{t-1}^{t} g(s)ds \le C, \quad \int_{t-1}^{t} h(s)ds \le \varepsilon^{-1}k,$$

where $k = k(\kappa)$ is small.

Note $\|A^{\frac{1}{2}}u_0\|_{L^2}^2 \le C(\|u_0\|_{H^1}^2 + \varepsilon^{\delta-1}\|u_0\|_{L^2}) \le k(\kappa)\varepsilon^{-1}$.

As far as $(1 - 2d_1\varepsilon^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|_{L^2}) \ge \frac{1}{2}$, equivalently, $\|A^{\frac{1}{2}}u\|_{L^2}^2 \le d\varepsilon^{-1}$, one

- estimates $\|A^{\frac{1}{2}}u(t)\|_{L^2}^2$ for $t \le 1$ by (usual) Gronwall's inequality,
- uses Uniform Gronwall's inequality for $t \ge 1$ to obtain

$$\|A^{\frac{1}{2}}u(t)\|_{L^2}^2 \le \left(\int_{t-1}^{t} \|A^{\frac{1}{2}}u(s)\|_{L^2}^2 ds + \int_{t-1}^{t} h(s)ds\right) \exp\left(\int_{t-1}^{t} g(s)ds\right),$$

The result is:

$$\|A^{\frac{1}{2}}u(t)\|_{L^2}^2 \le \varepsilon^{-1}k(\kappa).$$

THANK YOU FOR YOUR ATTENTION.