Navier-Stokes equations: the normalization map, statistical solutions and fluid dynamics

Ciprian Foias, Luan Hoang*, Basil Nicolaenko

*School of Mathematics, University of Minnesota
www.math.umn.edu/∼lthoang/
lthoang@math.umn.edu

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Outline

Introduction

Physical quantities in studies of Fluid Dynamics:

- the kinetic energy/mass

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x},$$

- the dissipation rate of energy/mass

$$\mathcal{F}(t) = \int_\Omega |\boldsymbol{\omega}(\mathbf{x},t)|^2 d\mathbf{x}, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

- and the helicity/mass

$$\mathcal{H}(t) = \int_\Omega \mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},t) d\mathbf{x}.$$

Their ensemble averages $\langle \mathcal{E}(t)\rangle$, $\langle \mathcal{F}(t)\rangle$ and $\langle \mathcal{H}(t)\rangle$.

Turbulence!

GOAL: Use the analysis of Navier–Stokes equations to understand the above quantities.

Navier-Stokes Equations

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u = -\nabla p - \nabla \phi, \\ \mathrm{div}\ u = 0, \\ \mathbf{u}(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
ϕ is the potential of the body force,
u^0 is the initial velocity.

Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued L-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.
We define

$$\begin{aligned} H &= \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V &= \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \\ \mathcal{D}_A &= V \cap H^2(\Omega)^3. \end{aligned}$$

Norm on H: $|u| = \|u\|_H = \|u\|_{L^2(\Omega)}$,
Norm on V: $\|u\| = \|u\|_V = \|\nabla u\|_{L^2(\Omega)} = \|\nabla \times u\|_{L^2(\Omega)}$,
Norm on $\mathcal{D}_A$: $\|\Delta u\|_{L^2(\Omega)}$.

The Stokes operator: $Au = -\Delta u \text{ for all } u \in \mathcal{D}_A$.
Spectrum of A: $\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}$.

If $N \in \sigma(A)$, denote by $R_N H$ the eigenspace of A corresponding to N.
Otherwise, $R_N H = \{0\}$.

The bilinear mapping:

$$B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}_A.$$

P_L is the Leray projection from $L^2(\Omega)$ onto H.

The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \;\; t > 0,$$

$$u(0) = u^0.$$

Leray-Hopf weak solutions

Recall that for each $u_0 \in H$, there exists a Leray-Hopf weak solution $u(t)$ of the Navier–Stokes equations with $u(0) = u_0$. This weak solution satisfies

$$u \in C([0,\infty), H_{weak}) \cap L^\infty((0,\infty), H) \cap L^2((0,\infty), V).$$

Additionally, let $\mathcal{G} = \mathcal{G}(u(\cdot))$ be the set of $t_0 \geq 0$ such that

$$\lim_{\tau \searrow 0} \|u(t_0+\tau) - u(t_0)\|_H = 0,$$

then $0 \in \mathcal{G}$, the Lebesgue measure of $[0,\infty) \setminus \mathcal{G}$ is zero and for any $t_0 \in \mathcal{G}$

$$\|u(t)\|_H^2 + 2\int_{t_0}^t \|u(s)\|_V^2 ds \leq \|u(t_0)\|_H^2, \quad t \geq t_0.$$

Denote by Σ the set of the Leray-Hopf weak solutions of the Navier–Stokes equations on $[0,\infty)$. Hence $\Sigma \subset C([0,\infty), H_{weak})$.

Statistical solutions

We denote by $\mathcal{T}$ the class of test functionals

$$\Phi(u) = \phi(\langle u, g_1\rangle, \langle u, g_2\rangle, \ldots, \langle u, g_k\rangle), \quad u \in H,$$

for some $k > 0$, where ϕ is a C^1 function on $\mathbb{R}^k$ with compact support and $g_1, g_2, \ldots, g_k$ are in V.

A family $\{\mu_t, t \geq 0\}$ of Borel probability measures on H is called a statistical solution of the Navier–Stokes equations with the initial data μ_0 if

- the initial kinetic energy $\int_H \|u\|_H^2 d\mu_0(u)$ is finite;
- the function $t \mapsto \int_H \varphi(u) d\mu_t(u)$ is measurable for every bounded and continuous function φ on H;
- the function $t \mapsto \int_H \|u\|_H^2 d\mu_t(u)$ belongs to $L^\infty_{loc}([0, \infty))$;

- the function $t \mapsto \int_H \|u\|_V^2 d\mu_t(u)$ belongs to $L^1_{loc}([0,\infty))$;
- μ_t satisfies the Liouville equation

$$\int_H \Phi(u) d\mu_t(u) = \int_H \Phi(u) d\mu_0(u) - \int_0^t \int_H \langle Au + B(u,u), \Phi'(u)\rangle d\mu_s(u) ds,$$

  for all $t \geq 0$ and $\Phi \in \mathcal{T}$;
- the following energy inequality holds

$$\int_H \|u\|_H^2 d\mu_t(u) \ + \ 2 \int_0^t \int_H \|u\|_V^2 d\mu_s(u) ds \ \leq \ \int_H \|u\|_H^2 d\mu_0(u).$$

Vishik-Fursikov Statistical solutions

Definition

A statistical solution $\{\mu_t, t \geq 0\}$ is called a Vishik-Fursikov (VF) statistical solution if there is a Borel probability measure $\hat{\mu}$, called the Vishik-Fursikov (VF) measure, on the space $C([0,\infty), H_{weak})$, such that

- $\hat{\mu}(\Sigma) = 1$;
- for each $t \geq 0$, μ_t is the projection measure $Pr_t\hat{\mu}$ on H, i.e.

$$\int_H \Phi(u) d\mu_t(u) = \int_\Sigma \Phi(v(t)) d\hat{\mu}(v(\cdot)),$$

for all $\Phi \in C(H_{weak})$.

The existence of the statistical solutions

We summarize the results by Foias and Vishik-Fursikov.

Theorem

Let m be a Borel probability measure on H such that $\int_H \|u\|_H^2 dm(u)$ *is finite. Then there exists a Vishik-Fursikov statistical solution* $\{\mu_t, t \geq 0\}$ *with* $\mu_0 = m$.

Note: The Vishik-Fursikov statistical solutions and measures are not necessarily unique.

Asymptotic expansion of regular solutions (Foias-Saut)

Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution is regular for all times $t > 0$.

For $u^0 \in \mathcal{R}$, the solution $u(t) = u(t, u^0)$ has the asymptotic expansion of

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in t of degree at most $(j-1)$ and with values in $\mathcal{V}$.
One has for $N \in \mathbb{N}$ and $m \in \mathbb{N}$ that

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.

Normalization map

Let

$$W(u^0) = \Big(W_1(u^0), W_2(u^0), W_3(u^0), \cdots \Big),$$

where $W_j(u^0) = R_j q_j(0)$, for $j = 1, 2, 3 \ldots$

Here we focus on the the first component

$$W_1(u^0) = \lim_{t\to\infty} e^t u(t) = \lim_{t\to\infty} e^t R_1 u(t).$$

Lemma

Let $u(\cdot) \in \Sigma$ and $t_0 \geq 0$ such that $u(t_0) \in \mathcal{R}$. Then

$$e^t W_1(u(t)) = e^{t_0} W_1(u(t_0)), \quad t \geq t_0.$$

Properties of the normalization map

Definition

Let $u(\cdot) \in \Sigma$. We define

$$W_1(u(\cdot)) = e^{t_0} W_1(u(t_0)),$$

where $t_0 \geq 0$ such that $u(t_0) \in \mathcal{R}$.

One also has

$$W_1(u(\cdot)) = \lim_{t\to\infty} e^t u(t) = \lim_{t\to\infty} e^t R_1 u(t).$$

where the limits are taken in either H or V.
Note: If $u_0 = u(0) \in \mathcal{R}$, then $t_0 = 0$ and $W_1(u(\cdot)) = W_1(u_0)$. Thus $W_1(u(\cdot))$ is an extension of $W_1(u_0), u_0 \in \mathcal{R}$.

Properties of the normalization map (cont.)

Let $u(\cdot) \in \Sigma$. Then

- $$\|W_1(u(\cdot))\|_H \leq e^t \|u(t)\|_H, \quad t \in \mathcal{G}(u(\cdot)).$$

  In particular,

  $$\|W_1(u(\cdot))\|_H \leq \|u(0)\|_H.$$

- For $t \geq 0$,

  $$\|W_1(u(\cdot))\|_H^2 \leq \frac{1}{T}\int_t^{t+T} e^{2\tau}\|u(\tau)\|_H^2 d\tau \leq \|u(0)\|_H^2,$$

  $$\int_t^{t+T} \|u(\tau)\|_V^2 d\tau \leq \frac{e^{-2t}}{2}\|u(0)\|_H^2 - \frac{e^{-2(t+T)}}{2}\|W_1(u(\cdot))\|_H^2.$$

- $$\|R_1 u(0) - W_1(u(\cdot))\|_H \leq C_1 \|u(0)\|_H^2.$$

Asymptotic behavior of the mean flows.

Proposition

$$\lim_{t\to\infty} e^{2t} \int_H \|u\|_H^2 d\mu_t(u) = \int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)).$$

Theorem

We have for any $T > 0$ that

$$\lim_{t\to\infty} \frac{e^{2t}}{T} \int_t^{t+T} \int_H \|u\|_H^2 d\mu_s(u) ds = \frac{1-e^{-2T}}{2T} \int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)),$$

Theorem (continued)

$$\lim_{t\to\infty} \frac{e^{2t}}{T} \int_t^{t+T} \int_H \|u\|_V^2 d\mu_s(u) ds$$

$$= \frac{1-e^{-2T}}{2T} \int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)),$$

and

$$\lim_{t\to\infty} \frac{e^{2t}}{T} \int_t^{t+T} \int_H \mathcal{H}(u) d\mu_s(u) ds$$

$$= \frac{1-e^{-2T}}{2T} \int_\Sigma \mathcal{H}(W_1(u(\cdot))) d\hat{\mu}(u(\cdot)).$$

Proposition

For any $T > 0$ *and* $t \geq 0$,

$$e^{-2(t+T)} \int_{\Sigma} \|W_1 u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) \leq \frac{1}{T} \int_t^{t+T} \int_H \|u\|_H^2 d\mu_\tau(u) d\tau$$
$$\leq e^{-2t} \int_H \|u\|_H^2 d\mu_0(u),$$

and

$$e^{-2(t+T)} \int_{\Sigma} \|W_1 u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) \leq \frac{1}{T} \int_t^{t+T} \int_H \|u\|_V^2 d\mu_\tau(u) d\tau$$
$$\leq \frac{e^{-2t}}{2T} \int_H \|u\|_H^2 d\mu_0(u) - \frac{e^{-2(t+T)}}{2T} \int_{\Sigma} \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)).$$

Beltrami flows

A C^1 vector field $\mathbf{u}(\mathbf{x})$ in $\mathbb{R}^3$ is a said to be Beltrami if

$$\nabla \times \mathbf{u}(\mathbf{x}) = \alpha(\mathbf{x})\mathbf{u}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \text{ some } \alpha(\mathbf{x}) \in \mathbb{R}.$$

If $u = \mathbf{u}(\cdot)$ is an eigenfunction of the curl operator $\mathfrak{C}$, then it is Beltrami with $\alpha \equiv \pm\sqrt{n}$, for some $n \in \sigma(A)$.

Proposition

Let $u \in R_n H \setminus \{0\}$ where $n \in \sigma(A)$. Then u is Beltrami if and only if u is an eigenfunction of the curl operator, i.e., $\mathfrak{C}u = \sqrt{n}u$ or $\mathfrak{C}u = -\sqrt{n}u$.

Notation: $R_n^{\pm} H$ is the eigenspace of the curl operator corresponding to $\pm\sqrt{n}$.

Asymptotic Beltrami flows

Definition

We say that a time dependent vector field $\mathbf{u}(\mathbf{x}, t)$ is asymptotically Beltrami if there are $\alpha(\mathbf{x}, t) \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t) - \alpha(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t)}{|\mathbf{u}(\mathbf{x}, t)|} = 0, \text{ a.e. on } \mathbb{R}^3.$$

Let $u(\cdot) \in \Sigma$ such that there is $t_0 \geq 0$, $u(t_0) \in \mathcal{R} \setminus \{0\}$. Denote

$$n_* = n_*(u(\cdot)) = \lim_{t\to\infty} \frac{\|u(t)\|_V^2}{\|u(t)\|_H^2}.$$

Define

$$W_*(u(\cdot)) = e^{n_* t_0} W_{n_*}(u(t_0)) = \lim_{t\to\infty} e^{n_* t} u(t),$$

where the limit is taken in either H or V.

Theorem

Let $u(\cdot) \in \Sigma$ *such that* $u(t_0) \in \mathcal{R} \setminus \{0\}$, *for some* $t_0 > 0$. *The following are equivalent*

1. $u(t)$ *is asymptotically Beltrami.*
2. *There is a subsequence* $t_k \nearrow \infty$ *and* $\alpha(\mathbf{x}, t_k) \in \mathbb{R}$ *such that*

   $$\lim_{k\to\infty} \frac{\nabla \times \mathbf{u}(\mathbf{x}, t_k) - \alpha(\mathbf{x}, t_k)\mathbf{u}(\mathbf{x}, t_k)}{|\mathbf{u}(\mathbf{x}, t_k)|} = 0,$$

   a.e. on $\mathbb{R}^3$.
3. $W_*(u(\cdot))$ *is a Beltrami vector filed.*
4. *For* $n_* = n_*(u(\cdot))$,

   $$\lim_{t\to\infty} \frac{\|\mathfrak{C}u(t) - \varepsilon\sqrt{n_*}u(t)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\Omega)}} = 0,$$

   where $\varepsilon = 1$ *or* -1.

Definition

Let $\hat{\mu}$ be a Vishik-Fursikov measure on Σ. We say that the $\hat{\mu}$ is asymptotically Beltrami if almost surely every solution $u(\cdot)$ in Σ is asymptotic Beltrami; more precisely,

$$\hat{\mu}\Big(\{u(\cdot) \in \Sigma : u(\cdot) \text{ is asymptotically Beltrami}\}\Big) = 1.$$

The necessary condition for $\hat{\mu}$ to be asymptotically Beltrami is that

$$\hat{\mu}\Big(\{u(\cdot) \in \Sigma : |R_1^+ W_1(u(\cdot))|\ |R_1^- W_1(u(\cdot))| = 0\}\Big) = 1,$$

equivalently,

$$\int_\Sigma |R_1^+ W_1(u(\cdot))|\ |R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) = 0, \quad \text{or}$$

$$\int_\Sigma |R_1^+ W_1(u(\cdot))| + |R_1^- W_1(u(\cdot))| - |W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) = 0.$$

Proposition

If $\hat{\mu}$ is a VF measure with initial data μ satisfying

$$\int_H \Big[|R_1^+ u| + |R_1^- u| - |R_1 u| \Big] d\mu(u) > 3c_5 \int_H |u|^2 d\mu(u),$$

then $\hat{\mu}$ is not asymptotically Beltrami.

Theorem

There exists a VF measure $\hat{\mu}$ with initial Gaussian probability measure such that $\hat{\mu}$ is not asymptotically Beltrami.

Fast decaying mean flows

We study the case when

$$\int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) = 0.$$

Denote

$$\mathcal{M}_1 = \{u \in \mathcal{R} : W_1(u) = 0\}.$$

Let

$$\mathcal{N}_1 = \{u_0 \in H : \exists u(\cdot) \in \Sigma, u(0) = u_0, u(t) \in \mathcal{M}_1, t \geq t_0(u(\cdot))\},$$

$$\Sigma_1 = \{u(\cdot) \in \Sigma : u(t) \in \mathcal{M}_1, t \geq t_0, \text{ some } t_0 \text{ depending on } u(\cdot)\}.$$

Proposition

$\int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) = 0$ *if and only if* $\hat{\mu}(\Sigma_1) = 1$,
equivalently,
$\int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) > 0$ *if and only if* $\hat{\mu}(\Sigma_1) < 1$,

Corollary

If $\int_\Sigma \|W_1(u(\cdot))\|_H^2 d\hat{\mu}(u(\cdot)) = 0$ *then* $\mu_0(\mathcal{N}_1) = 1$.

Generic properties

Let $\hat{\mu}$ and $\tilde{\mu}$ be two Borel measures on Σ. We define $d_1(\hat{\mu}, \tilde{\mu})$ by the total variation of the measure $\hat{\mu} - \tilde{\mu}$, that is,

$$d_1(\hat{\mu}, \tilde{\mu}) = \sup \Big\{ \sum_{j=1}^{N} |\hat{\mu}(E_j) - \tilde{\mu}(E_j)| \Big\},$$

where the supremum is taken over all Borel partitions $\{E_1, E_2, \ldots, E_N\}$, $N \in \mathbb{N}$, of Σ.

Let $\mathfrak{M}$ be the set of all VF measures and define the following metric in $\mathfrak{M}$:

$$d(\hat{\mu}, \tilde{\mu}) = d_1(\hat{\mu}, \tilde{\mu}) + \int_\Sigma |u(0)|^2 d|\hat{\mu} - \tilde{\mu}|(u(\cdot)),$$

where $|\hat{\mu} - \tilde{\mu}|$ is the total variation measure of the signed measure $(\hat{\mu} - \tilde{\mu})$.

Proposition

The metric space $(\mathfrak{M}, d)$ is complete.

A property $P(\hat{\mu})$ of a VF measure $\hat{\mu}$ is called *generic* if the set of all VF measures $\hat{\mu}$ enjoying the property $P(\hat{\mu})$ contains an intersection of dense open sets in $\mathfrak{M}$.

Theorem

The set $\mathfrak{M}_E$ of all $\hat{\mu} \in \mathfrak{M}$ such that

$$\int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)) > 0 \tag{1}$$

holds is open and dense in $\mathfrak{M}$. Subsequently, (1) is generic.

Denote by $\mathfrak{M}_H$ the set of all $\hat{\mu} \in \mathfrak{M}$ such that

$$\int_\Sigma \mathcal{H}(W_1(u(\cdot)))d\hat{\mu}(u(\cdot)) \neq 0. \tag{2}$$

holds.

Theorem

The set $\mathfrak{M}_H$ is open and dense in $\mathfrak{M}$. Subsequently, (2) is generic.

The genericity of the VF measures which are asymptotically Beltrami: let

$$\mathfrak{M}_B = \{\hat{\mu} \in \mathfrak{M} : \hat{\mu} \text{ is asymptotically Beltrami}\},$$

$$\mathfrak{N}_B = \left\{\hat{\mu} \in \mathfrak{M} : \int_\Sigma |R_1^+ W_1(u(\cdot))|\, |R_1^- W_1(u(\cdot))| d\hat{\mu}(u(\cdot)) > 0\right\}.$$

Note: $\mathfrak{N}_B \subset \mathfrak{M} \setminus \mathfrak{M}_B$.

Theorem

The set $\mathfrak{N}_B$ is open and dense in $\mathfrak{M}$. Consequently, the property "$\hat{\mu}$ is not asymptotically Beltrami" for a VF measure $\hat{\mu}$ is generic.

Idea

Let $\hat{\mu}, \hat{m} \in \mathfrak{M}$, $\varepsilon \in (0,1)$ and $\tilde{\mu} = (1-\varepsilon)\hat{\mu} + \varepsilon\hat{m}$. Then $\tilde{\mu} \in \mathfrak{M}$ and

$$d(\tilde{\mu}, \hat{\mu}) \le 2\varepsilon + \varepsilon\left\{\int_\Sigma |u(0)|^2 d\hat{\mu}(u(\cdot)) + \int_\Sigma |u(0)|^2 d\hat{m}(u(\cdot))\right\}.$$

Applications to decaying turbulence

Kolmogorov's empirical theory of turbulence: let

$$U^2 = \frac{1}{L^3}\langle\int_{[0,L]^3} |\mathbf{u}(\mathbf{x},t)|^2 dx\rangle \text{ and } \epsilon = \frac{\nu}{L^3}\langle\int_{[0,L]^3} |\nabla\times\mathbf{u}(\mathbf{x},t)|^2 dx\rangle,$$

where $\langle\ \cdot\ \rangle$ denotes an ensemble average. These two quantities are connected by

$$U^2 \sim \int_{k_i}^{k_d} S(k)dk, \quad \epsilon \sim \nu\int_{k_i}^{k_d} k^2 S(k)dk,$$

where $S(k)$ is the energy spectrum and $[k_i, k_d]$ is so called the "inertial range" of the turbulent flows.

Assume $k_i \sim k_0 = \sqrt{\lambda_1} = 2\pi/L$, $k_d \sim (\epsilon/\nu^3)^{1/4}$ and $S(k) \sim \epsilon^{2/3}k^{-5/3}$ (based on the dimensional analysis), we obtain

$$U^2 \sim \epsilon^{2/3}\int_{k_i}^{k_d} k^{-5/3}dk \sim \epsilon^{2/3}k_i^{-2/3} \sim (L\epsilon)^{2/3}.$$

Let $(\mu_t)_{t\geq 0}$ be a VF statistical solution to the Navier–Stokes equations with the VF measure $\hat{\mu}$ and $T > 0$. We define for $t \geq 0$

$$U_t^2 = \lambda_1^{3/2} \frac{1}{T} \int_t^{t+T} \int_H |u|^2 d\mu_\tau(u) d\tau,$$

and

$$\epsilon_t = \nu \lambda_1^{3/2} \frac{1}{T} \int_t^{t+T} \int_H \|u\|^2 d\mu_\tau(u) d\tau,$$

The first component of the normalization map is defined now by

$$W_1(u(\cdot)) = \lim_{t\to\infty} e^{\nu\lambda_1 t} u(t),$$

where the limit is taken in any Sobolev norms.

Let

$$\alpha_0^2 = \lambda_1^{3/2} \int_H |u|^2 d\mu_0(u) \text{ and } \alpha_1^2 = \lambda_1^{3/2} \int_\Sigma |W_1(u(\cdot))|^2 d\hat{\mu}(u(\cdot)).$$

Proposition

One has for each $T > 0$ that

$$\lim_{t\to\infty} e^{2\nu\lambda_1 t} U_t^2 = \frac{1-e^{-2T}}{2T}\alpha_1^2,$$

$$\lim_{t\to\infty} e^{2\nu\lambda_1 t} \epsilon_t = \frac{1-e^{-2T}}{2T}\alpha_1^2.$$

If Kolmogorov's theory applies to U_t^2 and ϵ_t then there are absolute positive constants c_K and C_K such that

$$c_K \le \frac{U_t^2}{(L/2\pi)^{2/3}\epsilon_t^{2/3}} = \frac{\lambda_1^{1/3} U_t^2}{\epsilon_t^{2/3}} \le C_K.$$

Proposition

For $T > 0$ and $t \geq 0$, one has

$$e^{-2\nu\lambda_1(t+T)}\alpha_1^2 \leq \quad U_t^2 \quad \leq e^{-2\nu\lambda_1 t}\alpha_0^2,$$
$$\nu\lambda_1 U_t^2 \leq \quad \epsilon_t \quad \leq \frac{e^{-2\nu\lambda_1 t}}{2T}(\alpha_0^2 - e^{-2\nu\lambda_1 T}\alpha_1^2).$$

Consequently, let $Q = \alpha_1^2/\alpha_0^2$, one hase for $t \geq 0$ that

$$\left\{\frac{2\nu\lambda_1 T}{Q^{-1}e^{2\nu\lambda_1 T} - 1}\right\}^{2/3}\left\{\frac{e^{-2\nu\lambda_1 t}\alpha_1^2}{\lambda_1\nu^2}\right\}^{1/3} \leq \frac{\lambda_1^{1/3}U_t^2}{\epsilon_t^{2/3}} \leq \left\{\frac{e^{-2\nu\lambda_1 t}\alpha_0^2}{\lambda_1\nu^2}\right\}^{1/3}.$$

Corollary

Kolmogorov's universal features may only be valid on the time interval $[t_K, T_K]$ *where*

$$t_K = \frac{1}{2\nu\lambda_1}\left(\log\frac{\alpha_1^2}{\lambda_1\nu^2} - 3\log C_K - 2\log\frac{Q^{-1}e^{2\nu\lambda_1 T} - 1}{2\nu\lambda_1 T}\right),$$

$$T_K = \frac{1}{2\nu\lambda_1}\left(\log\frac{\alpha_0^2}{\lambda_1\nu^2} - 3\log c_K\right).$$

Example

Let $L = 2\pi$ ($\lambda_1 = 1$), $\nu = 1$, $M > 0$, and $\theta \in (0,1)$. There is a VF measure $\hat{\mu}$ with initial Gaussian data such that $\alpha_0^2 \geq M$ and $\theta \leq Q \leq 1$, hence one obtains

$$t_K \geq \frac{1}{2}\left(\log M - 3\log C_K - 2\log\frac{\theta^{-1}e^{2T} - 1}{2T}\right),$$

$$T_K = \frac{1}{2}\{\log(\alpha_0^2) - 3\log c_K\}.$$