Solutions.

MATH2360. SECTION 002. FALL 2010.

MIDTERM EXAMINATION 2

Name: ....

Signature: ....Date: ....

READ AND FOLLOW THESE INSTRUCTIONS

This booklet contains 4 pages, including this cover page. Check to see if any are missing. PRINT all the requested information above, and sign your name. Put your initials on the top of every page, in case the pages become separated. Books and notes **are not permissible.** Calculators are allowed. Do your work in the blank spaces and back of pages of this booklet.

There are 3 work-out problems making a total score of 100 points. Students should show all the work in order to receive full credits. Unsupported answers will receive little credit.

AFTER YOU FINISH THE EXAM, turn in the whole booklet.

Problem 1 (20)	Problem 2 (40)	Problem 3 (40)	Total Score (100)

1. (20 points) Solve the following system of linear equations by using Cramer's Rule:

$$2x + y - z = -3$$
$$x - 2y + 3z = 14$$
$$5x + y + z = 6$$

Augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 1 & -2 & 3 & 14 \\ 5 & 1 & 1 & 6 \end{array}\right)$$

let $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 5 & 1 & 1 \end{pmatrix} \Rightarrow \det A = 2\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 5 & 1 \end{vmatrix}$

$= 2(-5) - 1(-14) - 1(11) =$
$= -10 + 14 - 11 = -7 \neq 0.$

Then

$A_1 = \begin{pmatrix} \boxed{\begin{matrix} -3 \\ 14 \\ 6 \end{matrix}} & \begin{matrix} 1 \\ -2 \\ 1 \end{matrix} & \begin{matrix} -1 \\ 3 \\ 1 \end{matrix} \end{pmatrix}$ $\quad \det A_1 = -7$

$A_2 = \begin{pmatrix} \begin{matrix} 2 \\ 1 \\ 5 \end{matrix} & \boxed{\begin{matrix} -3 \\ 14 \\ 6 \end{matrix}} & \begin{matrix} -1 \\ 3 \\ 1 \end{matrix} \end{pmatrix}$ $\quad \det A_2 = 14$

$A_3 = \begin{pmatrix} \begin{matrix} 2 \\ 1 \\ 5 \end{matrix} & \begin{matrix} 1 \\ -2 \\ 1 \end{matrix} & \boxed{\begin{matrix} -3 \\ 14 \\ 6 \end{matrix}} \end{pmatrix}$ $\quad \det A_3 = -21$

Solution:

$$x = \frac{\det A_1}{\det A} = \frac{-7}{-7} = 1$$

$$y = \frac{\det A_2}{\det A} = \frac{14}{-7} = -2$$

$$z = \frac{\det A_3}{\det A} = \frac{-21}{-7} = 3$$

2. (40 points) Let $\mathbf{v}_1 = (2,-1,0)^T$, $\mathbf{v}_2 = (-1,2,1)^T$, $\mathbf{v}_3 = (4,-5,-2)^T$ in $\mathbb{R}^3$.

(a) (20 points) Let a, b, c be real numbers. Find a necessary and sufficient condition on a, b, c so that the vector $(a,b,c)^T$ belongs to $Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(b) (10 points) Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not span the whole space $\mathbb{R}^3$.

(c) (10 points) Find a set of linearly independent vectors that spans the same subspace $Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in Span(V_1\ V_2\ V_3) \iff \exists\, x_1, x_2, x_3 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} = x_1 V_1 + x_2 V_2 + x_3 V_3$$

$$\iff \underbrace{\begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{(*)} \text{ has a solution } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Consider $(*)$

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & a \\ -1 & 2 & -5 & b \\ 0 & 1 & -2 & c \end{array}\right)$$

New (2) = old (1)
New (1) = − old (2)

$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & -b \\ 2 & -1 & 4 & a \\ 0 & 1 & -2 & c \end{array}\right)$$

$$\xrightarrow{(2)\to(2)-2(1)} \left(\begin{array}{ccc|c} 1 & -2 & 5 & -b \\ 0 & 3 & -6 & a+2b \\ 0 & 1 & -2 & c \end{array}\right) \xrightarrow{(2)\to(2)-3(3)} \left(\begin{array}{ccc|c} 1 & -2 & 5 & -b \\ 0 & 0 & 0 & \boxed{a+2b-3c} \\ 0 & 1 & -2 & c \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 5 & -b \\ 0 & 1 & -2 & c \\ 0 & 0 & 0 & \boxed{a+2b-3c} \end{array}\right)$$

(a) $(*)$ has a solution $\iff a+2b-3c=0$

Hence

$$Span(V_1\ V_2\ V_3) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a+2b-3c=0 \right\} = \left\{ \begin{pmatrix} -2b+3c \\ b \\ c \end{pmatrix} : b, c \in \mathbb{R} \right\}$$

$$= \left\{ b\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) answer: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

(b) $a=1, b=1, c=0$
$\Rightarrow a+2b-3c \neq 0$
$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin Span(V_1, V_2, V_3)$
$\Rightarrow$ Span is NOT $\mathbb{R}^3$

3. (40 points) Determine whether the following vectors are linearly independent.

(a) $(2,0,0,0)^T$, $(5,-1,1,2)^T$, $(1,3,-2,0)^T$, $(3,2,-1,2)^T$.

Space $\mathbb{R}^4$. 4 vectors $\longrightarrow$ use determinant

$$\det\begin{pmatrix} 2 & 5 & 1 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & 0 & 2 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$= 2\left[2\cdot(-3+4)+2(2-3)\right]$$

$$= 0$$

$\Rightarrow$ they are linearly dependent.

(b) $p_1(x) = x^2 + 2x - 1$, $p_2(x) = 2x^2 - 3$, $p_3(x) = x + 5$.

Suppose $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$\Rightarrow$ $c_1(x^2+2x-1) + c_2(2x^2-3) + c_3(x+5) = 0$

$(c_1+2c_2)x^2 + (2c_1 + c_3)x + (-c_1 - 3c_2 + 5c_3) = 0$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 + c_3 = 0 \\ -c_1 - 3c_2 + 5c_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & -3 & 5 & 0 \end{array}\right]$$

$$\det(\ \ ) = 1(3) - 2(11) \neq 0$$

$\Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow p_1, p_2, p_3$ are linearly Independent

LATER: use basis B_0 and "change of basis". to come to conclusion faster.