A BASS EQUALITY FOR GORENSTEIN INJECTIVE DIMENSION OF MODULES FINITE OVER HOMOMORPHISMS

LARS WINTHER CHRISTENSEN AND DEJUN WU

ABSTRACT. Let $R \to S$ be a local ring homomorphism and N a finitely generated S-module. We prove that if the Gorenstein injective dimension of N over R is finite, then it equals the depth of R.

INTRODUCTION

The homological theory of modules over commutative noetherian rings comes out particularly elegant for finitely generated modules. One way to relax this finiteness condition—without sacrificing elegance—is to settle for finite generation over some noetherian, but otherwise arbitrary, extension ring. This theme has been systematically explored for at least fifteen years. As part of that effort, this short paper answers an open question in Gorenstein homological algebra.

In this paper a ring means a commutative noetherian ring. Let R and S be local rings with unique maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. A ring homomorphism,

$$\varphi\colon R \longrightarrow S\,,$$

is called *local* if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ holds. Given such a homomorphism, every S-module is an R-module via φ; a finitely generated S-module is, when considered as an R-module, said to be *finite over* φ. It is evident from the condition $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ that $\mathfrak{m}N \neq N$ holds for every module $N \neq 0$ that is finite over φ. This is an extension of Nakayama's lemma for finitely generated modules, and the theme that modules finite over φ behave much like finitely generated R-modules was systematically explored by Avramov, Iyengar, and Miller [3]. The first theorem in their study is the *Bass Equality,* $\operatorname{id}_R N = \operatorname{depth} R$, which holds if N is finite over φ and of finite injective dimension over R. We extend this result with

Theorem. *Let $\varphi\colon R \to S$ be a local ring homomorphism and $N \neq 0$ a module finite over φ. If N has finite Gorenstein injective dimension over R, then one has*

$$\operatorname{Gid}_R N \;=\; \operatorname{depth} R\,.$$

The statement here is a special case of Theorem 2.3; it provides a positive answer to Question 6.2 in the survey [5] by Christensen, Foxby, and Holm. Motivation for this question comes, beyond the Bass Equality [3, Thm. 2.1] cited above, from

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Date: 30 April 2019.

2010 *Mathematics Subject Classification.* 13D05.

Key words and phrases. Gorenstein injective dimension, module finite over homomorphism.

L.W.C. was partly supported by Simons Foundation collaboration grant 428308; D.W. was partly supported by NSF of China grants 11761047 and 11861043. The paper was written during D.W.'s year-long visit to Texas Tech University; the hospitality of the TTU Department of Mathematics and Statistics is acknowledged with gratitude.

the similar equality for finitely generated R-modules of finite Gorenstein injective dimension, see Khatami, Tousi, and Yassemi [12, Cor. 2.5], and from the the *Auslander–Bridger Equality,* $\operatorname{Gfd}_R N = \operatorname{depth} R - \operatorname{depth}_R N$, which by work of Christensen and Iyengar [8] holds if N is finite over φ and of finite Gorenstein flat dimension over R.

1. Preliminaries

The proof of the main result uses derived functors on derived categories. Our notation is standard, and to not overload this short paper we refer the reader to the appendix in [4] for unexplained notation.

Let R be a ring, by an *R-complex* we mean a complex of R-modules. The derived category over R is denoted $\mathsf{D}(R)$. We say that a complex X has *bounded homology* if $\mathrm{H}_i(X) = 0$ holds for $|i| \gg 0$. To capture the homological extent of a complex, set

$$\inf X \;=\; \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \quad \text{and} \quad \sup X \;=\; \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}\,.$$

We write $\operatorname{Gid}_R X$ and $\operatorname{Gfd}_R X$ for the Gorenstein injective dimension and Gorenstein flat dimension of an R-complex. For a complex with $\mathrm{H}(X) = 0$ it is standard to set $\inf X = \infty$, $\sup X = -\infty$, and $\operatorname{Gid}_R X = -\infty = \operatorname{Gfd}_R X$.

We recall the main results from a paper by Christensen, Frankild, and Holm [6].

1.1 The Bass category. Let R be a ring with a dualizing complex D. An R-complex X with bounded homology has finite Gorenstein injective dimension if and only if it belongs to the *Bass category* $\mathsf{B}(R)$; that is, if and only if the complex $\mathbf{R}\mathrm{Hom}_R(D, X)$ has bounded homology, and the canonical morphism

$$\beta_D^X \colon D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, X) \;\longrightarrow\; X$$

is an isomorphism in $\mathsf{D}(R)$.

1.2 The Auslander category. Let R be a ring with a dualizing complex D. An R-complex X with bounded homology has finite Gorenstein flat dimension if and only if it belongs to the *Auslander category* $\mathsf{A}(R)$; that is, if and only if the complex $D \otimes_R^{\mathbf{L}} X$ has bounded homology, and the canonical morphism

$$\alpha_D^X \colon X \;\longrightarrow\; \mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^{\mathbf{L}} X)$$

is an isomorphism in $\mathsf{D}(R)$.

The next two lemmas slightly improve standard results [4, Lem. (3.2.9)].

1.3 Lemma. *Let $Q \to R$ be a ring homomorphism. Assume that R has a dualizing complex and let I be an injective Q-module. An R-complex X belongs to $\mathsf{A}(R)$ only if $\mathrm{Hom}_Q(X, I)$ belongs to $\mathsf{B}(R)$, and the converse holds if I is faithfully injective.*

Proof. Let D be a dualizing complex for R. By adjointness there is an isomorphism

$$(*) \qquad \mathrm{Hom}_Q(D \otimes_R^{\mathbf{L}} X, I) \simeq \mathbf{R}\mathrm{Hom}_R(D, \mathrm{Hom}_Q(X, I))$$

in $\mathsf{D}(R)$. It accounts for the horizontal isomorphism in the commutative diagram

$$(\square) \qquad \begin{array}{ccc} \mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^{\mathbf{L}} X), I) & \xrightarrow{\mathrm{Hom}(\alpha_D^X, I)} & \mathrm{Hom}_Q(X, I) \\ \simeq \big\uparrow & & \big\uparrow \beta_D^{\mathrm{Hom}(X,I)} \\ D \otimes_R^{\mathbf{L}} \mathrm{Hom}_Q(D \otimes_R^{\mathbf{L}} X, I) & \xrightarrow[\simeq]{} & D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, \mathrm{Hom}_Q(X, I)) \end{array}$$

and the vertical isomorphism is Hom evaluation; see Christensen and Holm [7, Prop. 2.2(ii)]. If X belongs to $\mathsf{A}(R)$, then $\operatorname{Hom}_Q(X, I)$ has bounded homology by injectivity of I, the complex $\mathbf{R}\operatorname{Hom}_R(D, \operatorname{Hom}_Q(X, I))$ has bounded homology by $(*)$, and $\beta_D^{\operatorname{Hom}(X,I)}$ is an isomorphism by $(\square)$; that is, $\operatorname{Hom}_Q(X, I)$ belongs to $\mathsf{B}(R)$. Conversely, if I is faithfully injective and $\operatorname{Hom}_Q(X, I)$ belongs to $\mathsf{B}(R)$, then X has bounded homology, it follows from $(*)$ that the complex $D \otimes_R^{\mathbf{L}} X$ is has bounded homology and from $(\square)$ that α_D^X is an isomorphism; that is, X belongs to $\mathsf{A}(R)$. $\square$

1.4 Lemma. *Let $Q \to R$ be a ring homomorphism. Assume that R has a dualizing complex and let I be an injective Q-module. An R-complex X belongs to $\mathsf{B}(R)$ only if $\operatorname{Hom}_Q(X, I)$ belongs to $\mathsf{A}(R)$, and the converse holds if I is faithfully injective.*

Proof. Similar to the proof of Lemma 1.3. $\square$

Another key result on Auslander and Bass categories comes from the paper of Avramov and Foxby [1] in which the categories were introduced.

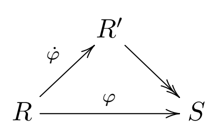
1.5 Regular homomorphisms. Let R be local with maximal ideal $\mathfrak{m}$ and $R \to R'$ be a flat local homomorphism such that the closed fiber $R'/\mathfrak{m}R'$ is regular; such a homomorphism is called *regular*. If R has a dualizing complex, then R' has a dualizing complex, see [1, (2.11)], and by [1, Cor. (7.9)] the next assertions hold:

(a) An R'-complex belongs to $\mathsf{A}(R')$ if and only if it belongs to $\mathsf{A}(R)$.

(b) An R'-complex belongs to $\mathsf{B}(R')$ if and only if it belongs to $\mathsf{B}(R)$.

2. The main result

We start by proving the main result in a special case, and then we reduce the general case to the special. Let R be a local ring with maximal ideal $\mathfrak{m}$. A local ring homomorphism $\varphi\colon R \to S$ is said to have a *regular factorization* if there is a commutative diagram of local ring homomorphisms



where $\dot{\varphi}$ is flat and the closed fiber $R'/\mathfrak{m}R'$ is regular.

2.1 Lemma. *Let $\varphi\colon R \to S$ be a local ring homomorphism and N an S-complex with bounded and degreewise finitely generated homology. Assume that R has a dualizing complex and φ has a regular factorization. If N has finite Gorenstein injective dimension over R, then one has*

$$\operatorname{Gid}_R N \;=\; \operatorname{depth} R - \inf N\,.$$

Proof. Let D be a dualizing complex for R and $R \to R' \to S$ a regular factorization of φ. If $\mathrm{H}(N) = 0$ the claim is trivial under the conventions from Section 1, so we may assume that $\mathrm{H}(N)$ is nonzero. By 1.1 and 1.5 the complex N belongs to $\mathsf{B}(R')$, so $g := \operatorname{Gid}_{R'} N$ is finite. By [6, Thm. 6.3] one has

$$g \;=\; \operatorname{depth} R' - \inf N\,.$$

Let D' be a dualizing complex for R', cf. 1.5, and assume without loss of generality that it is normalized in the sense of [1]. For every R'-complex X with bounded and degreewise finitely generated homology one then has

$$(\dagger) \qquad \operatorname{depth}_{R'} X \;=\; \inf \mathbf{R}\mathrm{Hom}_{R'}(X, D')$$

and

$$(\dagger\dagger) \qquad X \;\simeq\; \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{R'}(X, D'), D') \quad \text{in} \quad \mathsf{D}(R')\,;$$

see [1, Lem. (1.5.3), (2.6), and (2.7)]. Moreover, $\operatorname{Gfd}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(N, D') = g$ holds by [6, Cor. 6.4]. Set $n = -\dim R - \inf N$; by [8, Thm. 3.1] there is a distinguished triangle in $\mathsf{D}(R')$ of complexes with bounded and degreewise finitely generated homology,

$$(\triangle) \qquad \mathbf{R}\mathrm{Hom}_{R'}(N, D') \longrightarrow P \longrightarrow H \longrightarrow \Sigma\mathbf{R}\mathrm{Hom}_{R'}(N, D')\,,$$

where

$$(*) \qquad \operatorname{pd}_{R'} P \;=\; g \qquad \text{and} \qquad \sup H \;\le\; \operatorname{Gfd}_{R'} H \;\le\; n\,.$$

By the Auslander–Buchsbaum formula and [8, Thm. 4.1] one has

$$\operatorname{depth} R' - \operatorname{depth}_{R'} P \;=\; g \;=\; \operatorname{depth} R' - \operatorname{depth}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(N, D')\,.$$

Combined with (†) and (††) these equalities yield

$$(**) \qquad \operatorname{depth}_{R'} P \;=\; \inf N\,.$$

Applying the functor $\mathbf{R}\mathrm{Hom}_{R'}(-, D')$ to ($\triangle$) one gets via (††) the triangle

$$(\triangledown) \qquad \Sigma^{-1} N \longrightarrow \mathbf{R}\mathrm{Hom}_{R'}(H, D') \longrightarrow \mathbf{R}\mathrm{Hom}_{R'}(P, D') \longrightarrow N\,.$$

One has $\operatorname{id}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(P, D') \;=\; g$; see [6, Cor. 6.4]. As $\dot{\varphi}$ is flat, the complex $\mathbf{R}\mathrm{Hom}_{R'}(P, D')$ has finite injective dimension over R. By [3, Cor. 8.2.2] and [14, Thm. 4.4] one has $\operatorname{id}_R \mathbf{R}\mathrm{Hom}_{R'}(P, D') = \operatorname{depth} R - \inf \mathbf{R}\mathrm{Hom}_{R'}(P, D')$, which by (†) and (∗∗) can be rewritten as

$$(***) \qquad \operatorname{id}_R \mathbf{R}\mathrm{Hom}_{R'}(P, D') \;=\; \operatorname{depth} R - \inf N\,.$$

The complex $\mathbf{R}\mathrm{Hom}_{R'}(H, D')$ has finite Gorenstein injective dimension over R' by [6, Cor. 6.4] and hence over R; see 1.5. The first inequality in the next computation holds by [6, Thm. 3.3]; the equality follows from (†); the second inequality holds by the definition of depth; the final inequality follows from (∗).

$$\begin{aligned}
\operatorname{Gid}_R \mathbf{R}\mathrm{Hom}_{R'}(H, D') &\le \dim R - \inf \mathbf{R}\mathrm{Hom}_{R'}(H, D') \\
&= \dim R - \operatorname{depth}_{R'} H \\
&\le \dim R + \sup H \\
&\le -\inf N\,.
\end{aligned}$$

For every injective R-module I and every $i \le \inf N$ one gets from ($\triangledown$) an exact sequence in homology

$$\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(I, \mathbf{R}\mathrm{Hom}_{R'}(P, D'))) \longrightarrow \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_R(I, N)) \longrightarrow 0\,.$$

Thus, per [6, Thm. 3.3] and (∗∗∗) one has

$$\operatorname{Gid}_R N \le \operatorname{id}_R \mathbf{R}\mathrm{Hom}_{R'}(P, D') = \operatorname{depth} R - \inf N\,.$$

The opposite inequality $\operatorname{Gid}_R N \ge \operatorname{depth} R - \inf N$ holds by [6, Thm. 6.3]. □

As is standard, we denote by $\widehat{R}$ and $\widehat{S}$ the completions of R and S in the topologies induced by their maximal ideals. The homomorphism $\varphi\colon R \to S$ extends to a homomorphism of complete local rings; that is, there is a commutative diagram of local ring homomorphisms

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\hat{\varphi}} & \widehat{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

In particular, every $\widehat{S}$-complex is an $\widehat{R}$-complex.

In the special case $R = S$, and φ the identity, the next result was proved by Christensen, Frankild, and Iyengar; see Foxby and Frankild [10, Thm. 3.6].

2.2 Lemma. *Let $\varphi\colon R \to S$ be a local ring homomorphism and N an S-complex with bounded and degreewise finitely generated homology. If N has finite Gorenstein injective dimension over R, then $N \otimes_S \widehat{S}$ has finite Gorenstein injective dimension over $\widehat{R}$.*

Proof. Let K^S be the Koszul complex on a minimal set of generators for $\mathfrak{n}$, the maximal ideal of S. Since the S-complex $\mathrm{H}(K)$ has degreewise finite length, one has $\widehat{S} \otimes_S K^S \simeq K^S$ in $\mathsf{D}(S)$. Under the flat map $S \to \widehat{S}$ the minimal generators of $\mathfrak{n}$ extend to a minimal set of generators for the maximal ideal $\hat{\mathfrak{n}}$ of $\widehat{S}$, so $\widehat{S} \otimes_S K^S$ is the Koszul complex $K^{\widehat{S}}$ on a minimal set of generators for $\hat{\mathfrak{n}}$. Thus one has

$$(\diamond) \qquad K^S \simeq K^{\widehat{S}}$$

in $\mathsf{D}(S)$, and we simply denote this complex K.

The first step is to notice that $N \otimes_S K$ has finite Gorenstein injective dimension over R. For every element $x \in \mathfrak{n}$ there is an exact sequence of S-complexes,

$$0 \longrightarrow N \longrightarrow \operatorname{Cone} x^N \longrightarrow \Sigma N \longrightarrow 0\,,$$

where x^N is the homothety. Since N and ΣN have finite Gorenstein injective dimension over R, so has $\operatorname{Cone} x^N$; this folklore fact is dual to a result of Veliche [13, Thm. 3.9] for Gorenstein projective dimension. Now, $\operatorname{Cone} x^N$ is isomorphic to $N \otimes_S K(x)$, where $K(x)$ denotes the elementary Koszul complex on x. Since K is a tensor product of such elementary Koszul complexes, it follows that $N \otimes_S K$ has finite Gorenstein injective dimension over R.

Set $M = N \otimes_S K$; it is an $\widehat{S}$-complex via K and, therefore, an $\widehat{R}$-complex. The second step is to prove that M belongs to $\mathsf{B}(\widehat{R})$. The composite $R \xrightarrow{\varphi} S \longrightarrow \widehat{S}$, called the semi-completion of φ, has a regular factorization; see Avramov, Foxby, and Herzog [2, Thm. (1.1)]. Let E' denote the injective hull of the residue field of R' and E denote the injective hull $\mathrm{E}_R(k) \cong \mathrm{E}_{\widehat{R}}(k)$. As $\mathrm{H}(M)$ has degreewise finite length over $\widehat{S}$ and, therefore, over R' one has

$$M \;\simeq\; \operatorname{Hom}_{R'}(\operatorname{Hom}_{R'}(M,E'),E')\,.$$

As $\operatorname{Gid}_R M$ is finite, it follows from [9, Thm. 1.7] that the R'-complex

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(R',M) \;&\simeq\; \mathbf{R}\mathrm{Hom}_R(R',\operatorname{Hom}_{R'}(\operatorname{Hom}_{R'}(M,E'),E')) \\ &\simeq\; \operatorname{Hom}_{R'}(R' \otimes_R \operatorname{Hom}_{R'}(M,E'),E') \end{aligned}$$

has finite Gorenstein injective dimension. The R'-complex $R' \otimes_R \operatorname{Hom}_{R'}(M,E')$ then has finite Gorenstein flat dimension by Lemma 1.3. As R' is faithfully flat over R, it follows from [9, Thm. 1.8] that the complex $\operatorname{Hom}_{R'}(M,E')$ has finite Gorenstein flat dimension over R. By another application of the same result the complex $\widehat{R}\otimes_R \operatorname{Hom}_{R'}(M,E')$ has finite Gorenstein flat dimension over $\widehat{R}$, whence it belongs to $\mathsf{A}(\widehat{R})$. By Lemma 1.3 the dual complex

$$\operatorname{Hom}_{\widehat{R}}(\widehat{R}\otimes_R \operatorname{Hom}_{R'}(M,E'),E) \;\cong\; \operatorname{Hom}_R(\operatorname{Hom}_{R'}(M,E'),E)$$

belongs to $\mathsf{B}(\widehat{R})$. As E is faithfully injective, it follows from Lemma 1.3 that the complex $\operatorname{Hom}_{R'}(M,E')$ belongs to $\mathsf{A}(\widehat{R})$ and hence to $\mathsf{A}(R')$; see 1.5. By Lemma 1.4 the complex M now belongs to $\mathsf{B}(R')$ and hence to $\mathsf{B}(\widehat{R})$.

To finish the proof we now prove that $N\otimes_S \widehat{S}$ belongs to $\mathsf{B}(\widehat{R})$, cf. 1.1. First notice that by ($\diamond$) and associativity of the tensor product one has

$$(\diamond\diamond) \qquad N\otimes_S K \;\cong\; N\otimes_S(\widehat{S}\otimes_{\widehat{S}} K) \;\cong\; (N\otimes_S \widehat{S})\otimes_{\widehat{S}} K\,.$$

By ($\diamond\diamond$) and an application of tensor evaluation [7, Prop. 2.2(v)] one gets

$$(\ddagger\ddagger) \qquad \begin{aligned}\mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S K) &\simeq \mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,(N\otimes_S \widehat{S})\otimes_{\widehat{S}} K)\\ &\simeq \mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S \widehat{S})\otimes_{\widehat{S}} K\,.\end{aligned}$$

As $N\otimes_S K$ belongs to $\mathsf{B}(\widehat{R})$, the complex $\mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S \widehat{S})\otimes_{\widehat{S}} K$ has bounded homology, so $\mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S \widehat{S})$ has bounded homology. This follows from work of Foxby and Iyengar [11, 1.3]; indeed, as the $\widehat{R}$-complex D and the $\widehat{S}$-complex $N\otimes_S K$ have degreewise finitely generated homology, it follows from [3, Lem. 1.3.2] that the $\widehat{S}$-complex $\mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S \widehat{S})$ has degreewise finitely generated homology. There is a commutative diagram in $\mathsf{D}(\widehat{R})$,

$$\begin{array}{ccc}
D\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S K) & \xrightarrow[\simeq]{\beta_D^{N\otimes K}} & N\otimes_S K\\
\simeq\downarrow & & \downarrow\simeq\\
(D\otimes^{\mathbf{L}}_{\widehat{R}} \mathbf{R}\operatorname{Hom}_{\widehat{R}}(D,N\otimes_S \widehat{S}))\otimes_{\widehat{S}} K & \xrightarrow{\beta_D^{N\otimes\widehat{S}}\otimes K} & (N\otimes_S \widehat{S})\otimes_{\widehat{S}} K
\end{array}$$

where the right-hand vertical isomorphism is ($\diamond\diamond$), and the left-hand vertical isomorphism follows by tensor evaluation [7, Prop. 2.2(v)] and associativity of the tensor product. It follows that $\beta_D^{N\otimes_S\widehat{S}}\otimes_{\widehat{S}} K$ is an isomorphism; that is, the mapping cone

$$\operatorname{Cone}(\beta_D^{N\otimes_S\widehat{S}}\otimes_{\widehat{S}} K) \;\simeq\; (\operatorname{Cone}\beta_D^{N\otimes_S\widehat{S}})\otimes_{\widehat{S}} K$$

is acyclic. As $\beta_D^{N\otimes_S\widehat{S}}$ per [3, Lem. 1.3.2] is a morphism of $\widehat{S}$-complexes with degreewise finitely generated homology, it follows from [11, 1.3] that the complex $\operatorname{Cone}\beta_D^{N\otimes_S\widehat{S}}$ is acyclic, whence $\beta_D^{N\otimes_S\widehat{S}}$ is an isomorphism in $\mathsf{D}(\widehat{R})$, and $N\otimes_S \widehat{S}$ belongs to $\mathsf{B}(\widehat{R})$. □

The main result, which we can now prove, compares to [8, Thm. 4.1 and Cor. 4.8].

2.3 Theorem. *Let $\varphi\colon R \to S$ be a local ring homomorphism and N an S-complex with bounded and degreewise finitely generated homology. If N has finite Gorenstein injective dimension over R, then one has*

$$\begin{aligned}\operatorname{Gid}_R N &= \operatorname{depth} R - \inf N \\ &= -\inf \mathbf{R}\operatorname{Hom}_R(\mathrm{E}_R(k), N) \\ &= \operatorname{Gid}_{\widehat{R}}(N \otimes_S \widehat{S})\,.\end{aligned}$$

Proof. The homomorphism $\widehat{R} \xrightarrow{\hat{\varphi}} \widehat{S}$ has a regular factorization; see [2, Thm. (1.1)]. By Lemma 2.2 the $\widehat{R}$-complex $N \otimes_S \widehat{S}$ has finite Gorenstein injective dimension, and it has bounded and degreewise finite homology over $\widehat{S}$, so Lemma 2.1 yields

$$\operatorname{Gid}_{\widehat{R}}(N \otimes_S \widehat{S}) \;=\; \operatorname{depth} \widehat{R} - \inf\,(N \otimes_S \widehat{S})\,.$$

There are equalities $\operatorname{depth} \widehat{R} = \operatorname{depth} R$ and $\inf\,(N \otimes_S \widehat{S}) = \inf N$; the latter holds by faithful flatness of $\widehat{S}$ over S. Moreover, one has

$$\operatorname{Gid}_R N \;\geq\; \operatorname{depth} R - \inf N \;=\; -\inf \mathbf{R}\operatorname{Hom}_R(\mathrm{E}_R(k), N)$$

by [6, Thm. 6.3] and [7, Cor. 6.5], so it is sufficient to prove that the inequality $\operatorname{Gid}_R N \leq \operatorname{Gid}_{\widehat{R}}(N \otimes_S \widehat{S})$ holds. By [9, Thm. 2.2] one has

$$(\S)\qquad \begin{aligned}\operatorname{Gid}_R N &= \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \quad \text{and} \\ \operatorname{Gid}_{\widehat{R}}(N \otimes_S \widehat{S}) &= \sup\{\operatorname{depth} \widehat{R}_{\mathfrak{q}} - \operatorname{width}_{\widehat{R}_{\mathfrak{q}}}(N \otimes_S \widehat{S})_{\mathfrak{q}} \mid \mathfrak{q} \in \operatorname{Spec} \widehat{R}\}\,.\end{aligned}$$

For $\mathfrak{p} \in \operatorname{Spec} R$ choose $\mathfrak{q} \in \operatorname{Spec} \widehat{R}$ minimal over $\mathfrak{p}\widehat{R}$, the local homomorphism $R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{q}}$ is flat with artinian closed fiber, whence one has $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth} \widehat{R}_{\mathfrak{q}}$; see e.g. [2, Prop. (2.8)]. In the next computation the first and fourth equalities hold by the definition of width, the second holds by faithful flatness of $\widehat{S}$ over S, and the last holds as $R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{q}}$ is a local homomorphism; see Wu and Kong [15, Lem. 3.6].

$$\begin{aligned}\operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} &= \inf\,((R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R N) \\ &= \inf\,((R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R N) \otimes_S \widehat{S} \\ &= \inf\,(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R (N \otimes_S \widehat{S}) \\ &= \operatorname{width}_{R_{\mathfrak{p}}}(N \otimes_S \widehat{S})_{\mathfrak{p}} \\ &= \operatorname{width}_{\widehat{R}_{\mathfrak{q}}}(N \otimes_S \widehat{S})_{\mathfrak{q}}\,.\end{aligned}$$

For every prime ideal $\mathfrak{p}$ in R there is thus a prime ideal $\mathfrak{q}$ in $\widehat{R}$ with

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \operatorname{depth} \widehat{R}_{\mathfrak{q}} - \operatorname{width}_{\widehat{R}_{\mathfrak{q}}}(N \otimes_S \widehat{S})_{\mathfrak{q}}$$

so the desired inequality is immediate from (§). □

Acknowledgment

We thank the anonymous referee for suggestions and comments that helped us improve the exposition.

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Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409, U.S.A.

Email address: lars.w.christensen@ttu.edu

URL: http://www.math.ttu.edu/~lchriste

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China

Email address: wudj@lut.cn