ONE-SIDED GORENSTEIN RINGS

LARS WINTHER CHRISTENSEN, SERGIO ESTRADA, LI LIANG, PEDER THOMPSON, AND JUNPENG WANG

Abstract. Distinctive characteristics of Iwanaga–Gorenstein rings are typically understood through their intrinsic symmetry. We show that several of those that pertain to the Gorenstein global dimensions carry over to the one-sided situation, even without the noetherian hypothesis. Our results yield new relations among homological invariants related to the Gorenstein property, not only Gorenstein global dimensions but also the suprema of projective/injective dimensions of injective/projective modules and finitistic dimensions.

Introduction

One of the most basic homological invariants attached to a ring is the left global dimension, that is, the supremum of the projective, or equivalently injective, dimensions of all left modules. For noetherian rings the left and right global dimensions agree, but in general they may differ, and they need not even be simultaneously finite; see Jategaonkar [38]. There are though, beyond the noetherian realm, rings for which finiteness of the left global dimension implies finiteness of the right global dimension. Two results in this direction were achieved by Jensen [39] and Osofsky [44], and one of the several diverse aims of this paper is to establish their counterparts in Gorenstein homological algebra.

A Gorenstein analogue of rings of finite left global dimension are those of finite left Gorenstein global dimension; Beligiannis [7] calls such rings left Gorenstein, and shows that, indeed, for a noetherian ring the left and right Gorenstein global dimensions agree. In other words, a noetherian ring of finite left Gorenstein global dimension is an Iwanaga–Gorenstein ring, that is, a noetherian ring of finite self-injective dimension on both sides. A recurring theme of the paper—illustrated by Theorems 1.18, 2.6, 2.14, 3.2, and 5.15—is that several phenomena associated with Iwanaga–Gorenstein rings carry over to the one-sided situation even without the noetherian hypothesis.

Let A be a unital associative ring. In this paper an A-module means a left A-module; right A-modules are modules over the opposite ring A°. The Gorenstein

Date: 25 August 2023.

2020 *Mathematics Subject Classification.* 16E10, 16E65.

Key words and phrases. Finitistic dimensions, Gorenstein global dimension, Gorenstein flat-cotorsion global dimension, Gorenstein weak global dimension, Iwanaga–Gorenstein ring, silp, spli.

L.W.C. was partly supported by Simons Foundation collaboration grant 962956. S.E. was partly supported by grant PID2020-113206GB-I00 funded by MCIN/AEI/10.13039/ 501100011033 and by grant 22004/PI/22 funded by Fundación Séneca. L.L. was partly supported by NSF of China grant 12271230. Part of the work was done during a visit by P.T. to the University of Murcia, the institution's support and hospitality is much appreciated. J.W. was partly supported by NSF of China grant 12061061.

global dimension of A is defined in terms of the Gorenstein projective dimension:

$$\operatorname{Ggldim}(A) = \sup\{\operatorname{Gpd}_A M \mid M \text{ is an } A\text{-module}\}\,.$$

As in the absolute case, one can equivalently compute $\operatorname{Ggldim}(A)$ based on the Gorenstein injective dimension; see Bennis and Mahdou [10]. Analogous to the weak global dimension, defined in terms of the flat dimension, the Gorenstein weak global dimension is defined in terms of the Gorenstein flat dimension:

$$\operatorname{Gwgldim}(A) = \sup\{\operatorname{Gfd}_A M \mid M \text{ is an } A\text{-module}\}\,.$$

Unlike the elementary fact that the weak global dimension of A is at most the global dimension of A, the corresponding inequality in Gorenstein homological algebra,

$$\operatorname{Gwgldim}(A) \le \operatorname{Ggldim}(A)\,,$$

was only known to hold under extra assumptions on the ring; here we obtain it in Theorem 5.13. Just as the global and weak global dimensions, by a result of Auslander [3], agree for left noetherian rings and, trivially of course, for left perfect rings, we observe in Corollaries 4.6 and 5.14 that the equality

$$\operatorname{Gwgldim}(A) = \operatorname{Ggldim}(A)$$

holds if A is left noetherian or left perfect.

While the Gorenstein weak global dimension is symmetric, see for example [17], the Gorenstein global dimension is not. But the fact from [7] that the invariants $\operatorname{Ggldim}(A)$ and $\operatorname{Ggldim}(A^\circ)$ are simultaneously finite if A is noetherian is in Corollary 5.16 extended to $\aleph_0$-noetherian rings and in Theorem 5.18 to rings of cardinality $\aleph_n$ for some integer $n \ge 0$. Indeed, as in the absolute cases [39, 44] we show that the invariants $\operatorname{Ggldim}(A)$ and $\operatorname{Ggldim}(A^\circ)$ are simultaneously finite and provide bounds on their difference. Namely, if A is $\aleph_0$-noetherian, then one has:

$$|\operatorname{Ggldim}(A) - \operatorname{Ggldim}(A^\circ)| \le 1\,,$$

and if A has cardinality $\aleph_n$ for some integer $n \ge 0$, then one has:

$$|\operatorname{Ggldim}(A) - \operatorname{Ggldim}(A^\circ)| \le n+1\,.$$

Recall that the invariant $\operatorname{spli}(A)$ is $\sup\{\operatorname{pd}_A I \mid I \text{ is an injective } A\text{-module}\}$ while $\operatorname{silp}(A)$ is defined dually. The finitistic projective dimension $\operatorname{FPD}(A)$ is the supremum of projective dimensions of all A-modules of finite projective dimension. The finitistic injective dimension, $\operatorname{FID}(A)$, and the finitistic flat dimension, $\operatorname{FFD}(A)$, are defined similarly. It is known from work of Beligiannis and Reiten [8] that $\operatorname{Ggldim}(A)$ is the maximum of $\operatorname{spli}(A)$ and $\operatorname{silp}(A)$ and that the inequalities $\operatorname{FPD}(A) \le \operatorname{silp}(A)$ and $\operatorname{FID}(A) \le \operatorname{spli}(A)$ hold. Thus the equalities,

$$\max\{\operatorname{spli}(A), \operatorname{FPD}(A)\} = \operatorname{Ggldim}(A) = \max\{\operatorname{silp}(A), \operatorname{FID}(A)\}\,,$$

which follow from Theorem 1.18, constitute an improvement: The question whether finiteness of $\operatorname{silp}(A)$ or $\operatorname{spli}(A)$ implies finiteness of $\operatorname{Ggldim}(A)$ is for an Artin algebra A equivalent to the Gorenstein Symmetry Conjecture, see [8, Note after VII.2.7].

Another strain of results in the paper simplify computations of the Gorenstein global dimensions. Under the assumption that A is left noetherian, Corollary 3.5, Proposition 3.6, and Corollary 4.6 combine to yield

$$\operatorname{Ggldim}(A) = \sup\{\operatorname{Gid}_A M \mid M \text{ is a cotorsion } A\text{-module}\}\,.$$

As recorded in Corollary 3.7 this means that if A is noetherian, then it is Iwanaga–Gorenstein if and only if every cotorsion A-module has finite Gorenstein injective dimension.

Under the assumption that A is left coherent, we also provide alternative ways to compute the Gorenstein weak global dimension. In Theorem 4.5 we establish a parallel in Gorenstein homological algebra to a result of Stenström [46] on the weak global dimension of left coherent rings:

$$\begin{aligned}\operatorname{Gwgldim}(A) &= \sup\{\operatorname{Gfp-id}_A M \mid M \text{ is an } A\text{-module}\} \\ &= \sup\{\operatorname{Gpd}_A M \mid M \text{ is a finitely presented } A\text{-module}\}\,.\end{aligned}$$

Here $\operatorname{Gfp-id}_A$ denotes a Gorenstein dimension defined based on fp-injective modules; see Definition 4.1. The proof of the equalities above proceeds in two steps: (1) The suprema are shown to equal the global invariant based on the Gorenstein flat-cotorsion dimension from [15]; (2) this invariant is shown to agree with the Gorenstein weak global dimension for left coherent rings. In fact, this new global invariant lurks in the background of the proofs of all the results discussed hitherto.

* * *

On the organization of the paper: The results we have advertised above all derive from four separate specializations of one underlying theory. It deals with the abstract notions of Gorenstein modules associated to a cotorsion pair as first defined in [16], and the associated relative homological dimensions developed in [14, 15].

In Sections 1 and 2 we work in the setting of a hereditary cotorsion pair (U,V) in $\mathsf{Mod}(A)$, which is generated by a set and exhibits periodicity, see Definition 1.1. The motivation comes from the cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ and an associated homological invariant, the Gorenstein flat-cotorsion dimension we mentioned above.

The main results of the paper are derived in Sections 3–5 by specializing the results of the first two sections to the cotorsion pairs

$$(\mathsf{Prj}(A),\mathsf{Mod}(A))\,,\quad (\mathsf{Mod}(A),\mathsf{Inj}(A))\,,\quad (\mathsf{Flat}(A),\mathsf{Cot}(A))\,,$$

and, for left coherent rings, to the pair $({}^{\perp}\mathsf{FpInj}(A),\mathsf{FpInj}(A))$. The last specialization produces the result inspired by Stenström that was mentioned above, but the majority of the results come from the specialization to $(\mathsf{Flat}(A),\mathsf{Cot}(A))$, which also contributes to the general theory of the Gorenstein flat-cotorsion dimension.

1. Relative homological dimensions under periodicity

Throughout the paper, A denotes an associative unital ring. By an A-module we mean a left A-module; right A-modules are modules over the opposite ring A°. The category of A-modules is denoted $\mathsf{Mod}(A)$. Chain complexes of A-modules, or A-complexes, are indexed homologically. That is, the degree n component of an A-complex M is denoted M_n, and we denote the degree n cycle, cokernel, and homology modules by $\mathrm{Z}_n(M)$, $\mathrm{C}_n(M)$, and $\mathrm{H}_n(M)$, respectively.

For a class C of A-modules we denote by $\mathsf{C}^{\perp}$ the class of A-modules M with $\operatorname{Ext}_A^1(C,M)=0$ for all $C\in\mathsf{C}$; similarly, the class ${}^{\perp}\mathsf{C}$ consists of the A-modules M with $\operatorname{Ext}_A^1(M,C)=0$ for all $C\in\mathsf{C}$. Recall that a pair (U,V) of classes of A-modules constitutes a *cotorsion pair* if $\mathsf{U}={}^{\perp}\mathsf{V}$ and $\mathsf{U}^{\perp}=\mathsf{V}$ hold. A cotorsion pair (U,V) is *complete* if every A-module M has a special U-precover and a special

V-preenvelope, i.e. there are exact sequences

$$0 \longrightarrow V \longrightarrow U \longrightarrow M \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow M \longrightarrow V' \longrightarrow U' \longrightarrow 0$$

with U, U' in U and V, V' in V. A cotorsion pair (U,V) is *generated by a set* if there is a set S of A-modules with $\mathsf{S}^{\perp} = \mathsf{V}$. By a result of Eklof and Trlifaj [24, Thm. 10], a cotorsion pair that is generated by a set is complete. A cotorsion pair (U,V) is *hereditary* if $\operatorname{Ext}^i_A(U,V) = 0$ holds for all U in U, all V in V, and all integers $i \geq 1$. Equivalently, (U,V) is hereditary if and only if the class U is resolving and if and only if V is coresolving.

Throughout this section, (U,V) is a hereditary cotorsion pair in $\mathsf{Mod}(A)$, and we assume that it is generated by a set. As in [14] we say that an A-complex V is V-*acyclic* if it is acyclic and $\mathrm{Z}_n(V)$ belongs to V for all $n \in \mathbb{Z}$, and an A-complex U of modules in U is *semi*-U if $\operatorname{Hom}_A(U,V)$ is acyclic for every V-acyclic complex V. Similarly one defines U-acyclic complexes and semi-V complexes.

1.1 Definition. We say that (U,V) is *right periodic* if every acyclic complex of modules from V is V-acyclic and *weakly right periodic* if every acyclic complex of injective A-modules is V-acyclic. Dually, we say that (U,V) is *left periodic* if every acyclic complex of modules from U is U-acyclic and *weakly left periodic* if every acyclic complex of projective A-modules is U-acyclic.

Notice that if (U,V) is right/left periodic, then it is weakly right/left periodic. The terminology in the definition above comes from the fact that (U,V) is (weakly) right periodic if and only if in any exact sequence of A-modules of the form

$$0 \longrightarrow M \longrightarrow V \longrightarrow M \longrightarrow 0$$

with V in V (or injective) the module M belongs to V. The (weakly) left periodic property can be described similarly. See for example Bazzoni, Cortés Izurdiaga, and Estrada [5, Prop. 2.4]. Examples of left/right periodic cotorsion pairs come up at the end of this section and in the opening paragraphs of Sections 3 and 4.

Every A-complex M admits a *semi*-U *replacement*, that is, a semi-U complex U that is isomorphic to M in the derived category over A. Every A-complex also admits a similarly defined *semi*-V *replacement.* As a matter of fact, one can even get resolutions; that is, quasi-isomorphisms $U \xrightarrow{\simeq} M$ and $M \xrightarrow{\simeq} V$ in the category of A-complexes where U and V are semi-U and semi-V complexes. See for example Gillespie [33] and Yang and Liu [49, Thm. 3.5]. For an A-module M one can construct a semi-U replacement by taking successive special U-precovers and a semi-V replacement by taking successive special V-preenvelopes in which case one gets resolutions $U \xrightarrow{\simeq} M$ and $M \xrightarrow{\simeq} V$ in the category of A-complexes. An A-complex that is both semi-U and semi-V is called *semi*-U-V. A *semi*-U-V *replacement* of an A-complex M is a semi-U-V complex that is isomorphic to M in the derived category over A. See [14, Sec. 4].

We first handle some aspects of the relative homological dimension with respect to the class U, primarily in the case where (U,V) is weakly right periodic, and then from Lemma 1.13 we shift focus to V and the weakly left periodic property.

1.2 Definition. Let $M \neq 0$ be an A-module. The U-*projective dimension* of M, denoted $\mathsf{U}\text{-pd}_A M$, is the least integer n such that there exists a semi-U resolution $U \xrightarrow{\simeq} M$ with $U_i = 0$ for $i > n$. If no such resolution exists, then one sets

$\mathsf{U}\text{-pd}_A M = \infty$; further, one sets $\mathsf{U}\text{-pd}_A 0 = -\infty$. We say that a module M has *finite* U-projective dimension if $\mathsf{U}\text{-pd}_A M < \infty$ holds.

We denote by $\mathsf{Prj}(A)$ the class of projective A-modules. For the cotorsion pair $(\mathsf{U}, \mathsf{V}) = (\mathsf{Prj}(A), \mathsf{Mod}(A))$ we of course use the standard notation pd_A for the U-projective dimension. The equivalence of some of the assertions in the next result are standard; we include an entire proof for completeness.

1.3 Proposition. *Let M be an A-module and $n \geq 0$ an integer. The following conditions are equivalent.*

(*i*) $\mathsf{U}\text{-pd}_A M \leq n$.

(*ii*) *There exists a semi-U-V replacement W of M with $W_i = 0$ for $i > n$.*

(*iii*) *For every semi-U-V replacement W of M, the cokernel $\mathrm{C}_n(W)$ is in $\mathsf{U} \cap \mathsf{V}$.*

(*iv*) *For every semi-U replacement U of M, the cokernel $\mathrm{C}_n(U)$ is in U.*

(*v*) *For every projective resolution $P \xrightarrow{\simeq} M$, the cokernel $\mathrm{C}_n(P)$ is in U.*

(*vi*) *For every V in V, one has $\mathrm{Ext}_A^{n+1}(M, V) = 0$.*

Moreover, there is an equality,

$$\mathsf{U}\text{-pd}_A M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_A^i(M, V) \neq 0 \text{ for some module } V \in \mathsf{V}\}\,,$$

and if two out of three modules in a short exact sequence have finite U-projective dimension, then so has the third.

Proof. (*i*)$\Longrightarrow$(*ii*): Let $U \xrightarrow{\simeq} M$ be a semi-U resolution with $U_i = 0$ for $i > n$. By [14, Thm. A.8] there is an exact sequence $0 \to U \to W \to U' \to 0$ of A-complexes such that U' is U-acyclic and W is semi-U-V with $W_i = 0$ for $i \gg 0$. Acyclicity of U' yields an exact sequence $0 \to \mathrm{C}_n(U) \to \mathrm{C}_n(W) \to \mathrm{C}_n(U') \to 0$. As the outer terms belong to U, so does $\mathrm{C}_n(W)$. As (U, V) is hereditary and $W_i = 0$ holds for $i \gg 0$ it follows that that $\mathrm{C}_i(W)$ is in fact in $\mathsf{U} \cap \mathsf{V}$ for $i \geq n$. It now follows that the exact sequence $0 \to \mathrm{C}_{n+1}(W) \to W_n \to \mathrm{C}_n(W) \to 0$ splits and, therefore, we may replace W by the homotopic semi-U-V complex $0 \to \mathrm{C}_n(W) \to W_{n-1} \to \cdots$, which is the desired semi-U-V replacement of M.

(*ii*)$\Longrightarrow$(*iii*): Given any other semi-U-V replacement W' of M, [14, Prop. A.7] says that $\mathrm{C}_n(W')$ is a direct summand of $\mathrm{C}_n(W) \oplus X = W_n \oplus X$ where X is a module in $\mathsf{U} \cap \mathsf{V}$. Thus $\mathrm{C}_n(W')$ belongs to $\mathsf{U} \cap \mathsf{V}$.

(*iii*)$\Longrightarrow$(*iv*): Let U be any semi-U replacement of M. Again by [14, Thm. A.8] there is an exact sequence of A-complexes $0 \to U \to W \to U' \to 0$ where W is semi-U-V and U' is U-acyclic, and again acyclicity of U' yields an exact sequence $0 \to \mathrm{C}_n(U) \to \mathrm{C}_n(W) \to \mathrm{C}_n(U') \to 0$. The middle and right-hand modules belong to U, and thus the resolving property of U implies that $\mathrm{C}_n(U)$ belongs to U.

(*iv*)$\Longrightarrow$(*v*): This is clear since projective resolutions yield semi-U replacements.

(*v*)$\Longrightarrow$(*vi*): Let V be in V, and take a projective resolution $P \xrightarrow{\simeq} M$. Dimension shifting yields $\mathrm{Ext}_A^{n+1}(M, V) \cong \mathrm{Ext}_A^1(\mathrm{C}_n(P), V) = 0$, since $\mathrm{C}_n(P)$ belongs to U.

(*vi*)$\Longrightarrow$(*i*): Construct a semi-U resolution $U \xrightarrow{\simeq} M$ from special U-precovers. For V in V, one has $0 = \mathrm{Ext}_A^{n+1}(M, V) \cong \mathrm{Ext}_A^1(\mathrm{C}_n(U), V)$, thus $\mathrm{C}_n(U)$ is in U.

The equality and the last assertion are immediate from the equivalence of conditions (*i*) and (*vi*). □

1.4 Definition. Let $n \geq 0$ be an integer. Denote by U_n the class of A-modules of U-projective dimension at most n.

1.5 Fact. Let $n \geq 0$ be an integer. The pair $(\mathsf{U}_n, \mathsf{U}_n^{\perp})$ is by work of Cortés Izurdiaga, Estrada, and Guil Asensio [37, Thm. 2.2, Cor. 2.3 and 2.7] a hereditary cotorsion pair generated by a set; in particular, it is complete.

The next proposition shows that (weakly) right periodic cotorsion pairs come in families.

1.6 Proposition. *Let $n \geq 0$ be an integer. If (U, V) is (weakly) right periodic, then $(\mathsf{U}_n, \mathsf{U}_n^{\perp})$ is (weakly) right periodic. In particular, if (U, V) is weakly right periodic, then every Gorenstein injective A-module belongs to $\mathsf{U}_n^{\perp}$.*

Proof. First consider the case where (U, V) is weakly right periodic. Let I be an acyclic complex of injective A-modules and L a module in U_n. For any module V in V, note that Proposition 1.3 yields $\operatorname{Ext}_A^{n+1}(L, V) = 0$. Now, for every $i \in \mathbb{Z}$ there is an exact sequence

$$0 \longrightarrow \mathrm{Z}_{i+n}(I) \longrightarrow I_{i+n} \longrightarrow \cdots \longrightarrow I_{i+1} \longrightarrow \mathrm{Z}_i(I) \longrightarrow 0\,,$$

so dimension shifting yields $\operatorname{Ext}_A^1(L, \mathrm{Z}_i(I)) \cong \operatorname{Ext}_A^{n+1}(L, \mathrm{Z}_{i+n}(I)) = 0$, as $\mathrm{Z}_{i+n}(I)$ belongs to V by hypothesis. Thus $\mathrm{Z}_i(I)$ belongs to $\mathsf{U}_n^{\perp}$. The in particular statement follows simply because every Gorenstein injective A-module is a cycle submodule in an acyclic complex of injective A-modules.

The case where (U, V) is right periodic is handled in the same way, only one takes I to be an acyclic complex of modules from $\mathsf{U}_n^{\perp}$. □

1.7 Definition. Imitating the finitistic projective dimension, set

$$\mathsf{U}\text{-}\mathrm{FPD}(A) = \sup\{\mathsf{U}\text{-}\mathrm{pd}_A M \mid M \text{ is an } A\text{-module with } \mathsf{U}\text{-}\mathrm{pd}_A M < \infty\}\,.$$

Further, imitating the invariant $\mathrm{spli}(A)$, set

$$\mathsf{U}\text{-}\mathrm{spli}(A) = \sup\{\mathsf{U}\text{-}\mathrm{pd}_A I \mid I \text{ is an injective } A\text{-module}\}\,.$$

The next lemma is a more general version of Emmanouil's [26, Lem. 5.2].

1.8 Lemma. *Assume that $\mathsf{U}\text{-}\mathrm{spli}(A) = n < \infty$ holds and let M be an A-module. For every projective resolution $P \xrightarrow{\simeq} M$ there is an acyclic complex U of modules in U with $U_i = P_i$ for $i \geq n$.*

Proof. For the class U of flat A-modules, the statement is proved in [26, Lem. 5.2]. A straightforward generalization of the argument provided there yields the claim asserted here. □

In the next statement, and in the rest of the paper, $\mathsf{Inj}(A)$ and $\mathsf{GInj}(A)$ denote the classes of injective and Gorenstein injective A-modules. The purpose of the statement is to tie in the abstract notion of modules of finite U-projective dimension with the better known notions of injective and Gorenstein injective modules.

1.9 Proposition. *Let $n \geq 0$ be an integer and assume that (U, V) is weakly right periodic. The following conditions are equivalent.*

(*i*) $\max\{\mathsf{U}\text{-}\mathrm{spli}(A), \mathsf{U}\text{-}\mathrm{FPD}(A)\} \leq n$.
(*ii*) $\mathsf{U}\text{-}\mathrm{spli}(A) = \mathsf{U}\text{-}\mathrm{FPD}(A) \leq n$.
(*iii*) $\mathsf{U}_n \cap \mathsf{U}_n^{\perp} = \mathsf{Inj}(A)$.
(*iv*) $\mathsf{U}_n^{\perp} = \mathsf{GInj}(A)$.

Proof. $(i) \Longrightarrow (ii)$: Assuming that both U-spli(A) and U-FPD(A) are finite, the inequality $\mathsf{U}\text{-spli}(A) \leq \mathsf{U}\text{-FPD}(A)$ is evident. For the opposite inequality, set $s = \mathsf{U}\text{-spli}(A)$ and consider a module M of finite U-projective dimension. It follows from Lemma 1.8 that the s^{th} cokernel module in a projective resolution of M appears as a cokernel module in an acyclic complex of modules from U. Finiteness of U-FPD(A) implies that this module belongs to U: Indeed, it is for every $m \geq 0$ an m^{th} syzygy of a module of finite U-projective dimension.

$(ii) \Longrightarrow (iii)$: The inclusion "$\supseteq$" holds as injective A-modules by assumption have U-projective dimension at most n. For the opposite inclusion, let M be a module in the intersection $\mathsf{U}_n \cap \mathsf{U}_n^{\perp}$. Consider an exact sequence $0 \to M \to I \to N \to 0$ of A-modules with I injective. Since I and M have finite U-projective dimension, so has N; see Proposition 1.3. It follows from the assumption $\mathsf{U}\text{-FPD}(A) \leq n$ that N is in U_n, and so one has $\operatorname{Ext}_A^1(N, M) = 0$. Thus the sequence $0 \to M \to I \to N \to 0$ is split exact, whence M is injective.

$(iii) \Longrightarrow (iv)$: Per Proposition 1.6 one has $\mathsf{U}_n^{\perp} \supseteq \mathsf{GInj}(A)$. For the other containment, let M belong to $\mathsf{U}_n^{\perp}$. By Fact 1.5, $(\mathsf{U}_n, \mathsf{U}_n^{\perp})$ is a complete cotorsion pair, so there is an exact sequence $0 \to M' \to U \to M \to 0$ with $U \in \mathsf{U}_n$ and $M' \in \mathsf{U}_n^{\perp}$. Thus U also belongs to $\mathsf{U}_n^{\perp}$, and hence it is injective by hypothesis. Repeating this process and taking an injective resolution of M yields an acyclic complex of injective modules with M as a cycle module. Any such complex I is totally acyclic: Indeed, the cycle modules in such a complex belong to the class V as (U, V) is weakly right periodic, and by assumption injective modules belong to U_n, so for an injective module E one has $\operatorname{Ext}_A^1(E, \mathrm{Z}_i(I)) \cong \operatorname{Ext}_A^{n+1}(E, \mathrm{Z}_{i+n}(I)) = 0$ by Proposition 1.3.

$(iv) \Longrightarrow (i)$: As $(\mathsf{U}_n, \mathsf{GInj}(A))$ per the assumption and Fact 1.5 is a cotorsion pair, one has

$$(*) \qquad {}^{\perp}\mathsf{GInj}(A) = {}^{\perp}(\mathsf{U}_n^{\perp}) = \mathsf{U}_n \,.$$

As injective modules belong to ${}^{\perp}\mathsf{GInj}(A)$ one has $\mathsf{U}\text{-spli}(A) \leq n$. To see that $\mathsf{U}\text{-FPD}(A) \leq n$ holds, let M be an A-module of finite U-projective dimension. It follows from Proposition 1.6 that $\operatorname{Ext}_A^1(M, G) = 0$ holds for every $G \in \mathsf{GInj}(A)$, whence M is in U_n by $(*)$. $\square$

1.10 Proposition. *Assume that* (U, V) *is weakly right periodic and let* V *be a module in* V*. If* $\mathsf{U}\text{-pd}_A V$ *is finite, then the following inequality holds:*

$$\operatorname{id}_A V \leq \max\{\mathsf{U}\text{-spli}(A), \mathsf{U}\text{-FPD}(A)\}\,.$$

Proof. We may assume that $\max\{\mathsf{U}\text{-spli}(A), \mathsf{U}\text{-FPD}(A)\} = n$ holds for some $n \geq 0$. Consider an exact sequence of A-modules,

$$0 \longrightarrow V \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-n+1} \longrightarrow G \longrightarrow 0$$

where each module I_i is injective. For every module M of finite U-projective dimension $\operatorname{Ext}_A^{n+1}(M, V) = 0$ holds by Proposition 1.3, and dimension shifting yields $\operatorname{Ext}_A^1(M, G) \cong \operatorname{Ext}_A^{n+1}(M, V) = 0$, so that G belongs to $\mathsf{U}_n^{\perp}$. Since the modules $V, I_0, \ldots, I_{-n+1}$ have finite U-projective dimension, so does G, see Proposition 1.3. Thus G belongs to $\mathsf{U}_n \cap \mathsf{U}_n^{\perp}$ and is, therefore, injective by Proposition 1.9. $\square$

Dual to the U-projective dimension one defines the V*-injective dimension* of an A-module M; the notation is $\mathsf{V}\text{-id}_A M$ and for an integer $n \geq 0$ the class of A-modules of V-injective dimension at most n is denoted by V_n. For the cotorsion pair $(\mathsf{U},\mathsf{V}) = (\mathsf{Mod}(A),\mathsf{Inj}(A))$ we of course use the standard notation id_A, as already done in Proposition 1.10, for the V-injective dimension.

The V-injective dimension has properties dual to those in Proposition 1.3:

1.11 Proposition. *Let M be an A-module and $n \geq 0$ an integer. The following conditions are equivalent.*

(*i*) $\mathsf{V}\text{-}\mathrm{id}_A\, M \leq n$.

(*ii*) *There exists a semi-U-V replacement W of M with $W_{-i} = 0$ for $i > n$.*

(*iii*) *For every semi-U-V replacement W of M, the kernel $\mathrm{Z}_{-n}(W)$ is in $\mathsf{U} \cap \mathsf{V}$.*

(*iv*) *For every semi-V replacement V of M, the kernel $\mathrm{Z}_{-n}(V)$ is in V.*

(*v*) *For every injective resolution $M \xrightarrow{\simeq} I$, the kernel $\mathrm{Z}_{-n}(I)$ is in V.*

(*vi*) *For every U in U, one has $\mathrm{Ext}_A^{n+1}(U,M) = 0$.*

Moreover, there is an equality,

$$\mathsf{V}\text{-}\mathrm{id}_A\, M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_A^i(U,M) \neq 0 \text{ for some module } U \in \mathsf{U}\}\,,$$

and if two out of three modules in a short exact sequence have finite V-injective dimension, then so has the third.

Proof. Dual to the proof of Proposition 1.3. □

1.12 Definition. Imitating the finitistic injective dimension, set

$$\mathsf{V}\text{-}\mathrm{FID}(A) = \sup\{\mathsf{V}\text{-}\mathrm{id}_A\, M \mid M \text{ is an } A\text{-module with } \mathsf{V}\text{-}\mathrm{id}_A\, M < \infty\}\,.$$

Further, imitating the invariant $\mathrm{silp}(A)$, set

$$\mathsf{V}\text{-}\mathrm{silp}(A) = \sup\{\mathsf{V}\text{-}\mathrm{id}_A\, P \mid P \text{ is a projective } A\text{-module}\}\,.$$

To investigate the invariants introduced right above, we start by establishing a counterpart to Fact 1.5. The proof stands out from the proofs of 1.11 and 1.14–1.17 by not being simply dual to the proof of Fact 1.5. Our argument below is modeled on the proof of [34, Thm. 8.7].

1.13 Lemma. *Let $n \geq 0$ be an integer. The pair $({}^{\perp}\mathsf{V}_n, \mathsf{V}_n)$ is a complete hereditary cotorsion pair generated by a set.*

Proof. Recall that (U,V) is generated by a set S. For each $S \in \mathsf{S}$, fix a projective resolution $P_S \xrightarrow{\simeq} S$. We argue that the set

$$\mathsf{S}' = \{\mathrm{C}_n(P_S) \mid S \in \mathsf{S}\}$$

generates $({}^{\perp}\mathsf{V}_n, \mathsf{V}_n)$, that is, $\mathsf{S}'^{\perp} = \mathsf{V}_n$. Let M be an A-module and fix an injective resolution $M \xrightarrow{\simeq} I$. Now $M \in \mathsf{S}'^{\perp}$ means $\mathrm{Ext}_A^1(\mathrm{C}_n(P_S),M) = 0$ for every $S \in \mathsf{S}$. However, one has

$$\mathrm{Ext}_A^1(\mathrm{C}_n(P_S),M) \cong \mathrm{Ext}_A^{n+1}(S,M) \cong \mathrm{Ext}_A^1(S,\mathrm{Z}_{-n}(I))\,,$$

which vanishes if and only if $\mathrm{Z}_{-n}(I)$ belongs to V, that is, if and only if M belongs to V_n, see Proposition 1.11. This shows that $({}^{\perp}\mathsf{V}_n, \mathsf{V}_n)$ is a cotorsion pair generated by S'. To see that it is hereditary, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence with $M', M \in \mathsf{V}_n$. Given injective resolutions $M' \xrightarrow{\simeq} I'$ and $M'' \xrightarrow{\simeq} I''$, the Horseshoe Lemma yields an injective resolution $M \xrightarrow{\simeq} I$, such that the sequence $0 \to I' \to I \to I'' \to 0$ is exact. Thus one obtains an exact sequence $0 \to \mathrm{Z}_{-n}(I') \to \mathrm{Z}_{-n}(I) \to \mathrm{Z}_{-n}(I'') \to 0$. As $\mathrm{Z}_{-n}(I')$ and $\mathrm{Z}_{-n}(I)$ belong to V, so does $\mathrm{Z}_{-n}(I'')$ by the coresolving property of V. Thus M'' belongs to V_n; that is, V_n is coresolving and it follows that the pair $({}^\perp\mathsf{V}_n, \mathsf{V}_n)$ is hereditary. Finally, a hereditary cotorsion pair generated by a set is complete. □

1.14 Proposition. *Let $n \geq 0$ be an integer. If (U, V) is (weakly) left periodic, then $({}^\perp\mathsf{V}_n, \mathsf{V}_n)$ is (weakly) left periodic. In particular, if (U, V) is weakly left periodic, then every Gorenstein projective A-module belongs to ${}^\perp\mathsf{V}_n$.*

Proof. Dual to the proof of Proposition 1.6. □

1.15 Lemma. *Assume that $\mathsf{V}\text{-silp}(A) = n < \infty$ holds and let M be an A-module. For every injective resolution $M \xrightarrow{\simeq} I$ there is an acyclic complex V of modules in V with $V_{-i} = I_{-i}$ for $i \geq n$.*

Proof. The argument in the proof of [26, Lem. 5.2] generalizes and dualizes to prove the claim, see also the proof of Lemma 1.8. □

In the next statement, and in the rest of the paper, $\mathsf{GPrj}(A)$ denotes the class of Gorenstein projective A-modules.

1.16 Proposition. *Let $n \geq 0$ be an integer and assume that (U, V) is weakly left periodic. The following conditions are equivalent.*

(*i*) $\max\{\mathsf{V}\text{-silp}(A), \mathsf{V}\text{-FID}(A)\} \leq n$.
(*ii*) $\mathsf{V}\text{-silp}(A) = \mathsf{V}\text{-FID}(A) \leq n$.
(*iii*) ${}^\perp\mathsf{V}_n \cap \mathsf{V}_n = \mathsf{Prj}(A)$.
(*iv*) ${}^\perp\mathsf{V}_n = \mathsf{GPrj}(A)$.

Proof. Dual to the proof of Proposition 1.9. □

1.17 Proposition. *Assume that (U, V) is weakly left periodic and let U be a module in U. If $\mathsf{V}\text{-id}_A U$ is finite, then the following inequality holds:*

$$\mathrm{pd}_A U \leq \max\{\mathsf{V}\text{-silp}(A), \mathsf{V}\text{-FID}(A)\}\,.$$

Proof. Dual to the proof of Proposition 1.10. □

We close this section with an application of the theory developed up to now, more specifically, we apply Propositions 1.10 and 1.17 to the absolute cotorsion pairs $(\mathsf{Prj}(A), \mathsf{Mod}(A))$ and $(\mathsf{Mod}(A), \mathsf{Inj}(A))$. It is trivial that the former is right periodic and the latter is left periodic. For the former, the relevant special case of Fact 1.5 is an earlier result of Aldrich, Enochs, Jenda, and Oyonarte [1, Thm. 4.2].

It is known, e.g. from Beligiannis and Reiten [8, Prop. VII.1.3 and Thm. VII.2.2] or Emmanouil [26, Thm. 4.1], that $\mathrm{Ggldim}(A)$ is finite if and only if the invariants $\mathrm{spli}(A)$ and $\mathrm{silp}(A)$ both are finite. As the inequalities $\mathrm{FPD}(A) \leq \mathrm{silp}(A)$ and $\mathrm{FID}(A) \leq \mathrm{spli}(A)$ always hold, also by [8, Prop. VII.1.3], the next theorem is an improvement on this characterization of rings of finite Gorenstein global dimension. To parse the full statement recall the invariant

$$\mathrm{sfli}(A) = \sup\{\mathrm{fd}_A I \mid I \text{ is an injective } A\text{-module}\}\,.$$

Here, of course, fd_A is the standard notation for the flat dimension.

1.18 Theorem. *Let $n \geq 0$ be an integer. The following conditions are equivalent.*

(*i*) $\mathrm{Ggldim}(A) \leq n$.

(*ii*) $\max\{\mathrm{spli}(A), \mathrm{FPD}(A)\} \leq n$.

(*iii*) $\max\{\mathrm{silp}(A), \mathrm{FID}(A)\} \leq n$.

(*iv*) $\max\{\mathrm{sfli}(A), \mathrm{FPD}(A)\} \leq n$.

Proof. It is known from [8, Prop. VII.1.3 and Thm. VII.2.2] and [26, Thm. 4.1] that (*i*) implies both (*ii*) and (*iii*). It follows from Proposition 1.10 applied to the cotorsion pair $(\mathsf{Prj}(A), \mathsf{Mod}(A))$ that under the assumptions in (*ii*) also $\mathrm{silp}(A) \leq n$ holds, which by the references above means that $\mathrm{Ggldim}(A)$ is at most n. Similarly, it follows from Proposition 1.17 applied to the cotorsion pair $(\mathsf{Mod}(A), \mathsf{Inj}(A))$ that under the assumptions in (*iii*) also $\mathrm{spli}(A) \leq n$ holds, and then $\mathrm{Ggldim}(A) \leq n$ holds by the references above. This shows the equivalence of (*i*)–(*iii*).

Condition (*ii*) implies (*iv*) simply because $\mathrm{sfli}(A) \leq \mathrm{spli}(A)$ holds. Conversely, notice that $\mathrm{FPD}(A) < \infty$ implies $\mathrm{spli}(A) < \infty$ by results of Jensen [40, Prop. 6] and Emmanouil and Talelli [28, Prop. 2.2]. Thus one has $\mathrm{spli}(A) \leq \mathrm{FPD}(A)$. □

1.19 Remark. One could add to the theorem above another equivalent condition,

(*v*) $\max\{\mathrm{silf}(A), \mathrm{FID}(A)\} \leq n$,

but it coincides with (*iii*) as $\mathrm{silf}(A) = \mathrm{silp}(A)$ holds by [28, Prop. 2.1].

2. Relative Gorenstein global dimensions

As in the previous section, (U, V) is a hereditary cotorsion pair in $\mathsf{Mod}(A)$ generated by a set, in particular, it is complete.

Recall from [16, Def. 2.1] that an A-module M is called *right* U-*Gorenstein* if one has $M = \mathrm{C}_0(T)$ for some right U-totally acyclic complex T. That is to say, T is an acyclic complex of modules in U with cycle modules in V, such that $\mathrm{Hom}_A(T, W)$ is acyclic for every module W in $\mathsf{U} \cap \mathsf{V}$. Denote by $\mathsf{RGor}_{\mathsf{U}}$ the category of right U-Gorenstein A-modules. We remind the reader that the $\mathsf{RGor}_{\mathsf{U}}$-projective dimension of an A-module $M \neq 0$, denoted $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A M$, was defined in [14, Def. 4.5] as the least integer $n \geq 0$ such that there is a semi-U-V replacement W of M with $\mathrm{C}_n(W)$ in $\mathsf{RGor}_{\mathsf{U}}$. If no such integer exists, then the $\mathsf{RGor}_{\mathsf{U}}$-projective dimension of M is infinite, and one sets $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A 0 = -\infty$. We say that a module M has *finite* $\mathsf{RGor}_{\mathsf{U}}$-projective dimension if $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A M < \infty$ holds.

For $(\mathsf{U}, \mathsf{V}) = (\mathsf{Prj}(A), \mathsf{Mod}(A))$ the $\mathsf{RGor}_{\mathsf{U}}$-projective dimension is the standard Gorenstein projective dimension and, of course, we use the standard notation Gpd_A.

2.1 Lemma. *Let M be an A-module. There is an inequality,*

$$\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A M \leq \mathsf{U}\text{-pd}_A M\,,$$

and equality holds if $\mathsf{U}\text{-pd}_A M$ is finite.

Proof. We may assume that M is nonzero. Assume that $\mathsf{U}\text{-pd}_A M \leq n$ holds for an integer n. There is by Proposition 1.3 a semi-U-V replacement $W \simeq M$ with $W_i = 0$ for $i > n$. As $\mathrm{C}_n(W) = W_n$ belongs to $\mathsf{U} \cap \mathsf{V}$ and hence to $\mathsf{RGor}_{\mathsf{U}}$, see [16, Exa. 2.2], one has $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A M \leq n$.

To see that equality holds, notice that if $\mathrm{C}_m(W)$ belongs to $\mathsf{RGor}_{\mathsf{U}}$ for some $m < n$, then one has an exact sequence,

$$0 \longrightarrow W_n \longrightarrow \cdots \longrightarrow W_m \longrightarrow \mathrm{C}_m(W) \longrightarrow 0\,,$$

which breaks into split short exact sequences, as $\mathrm{Ext}^i_A(\mathrm{C}_m(W), W_n) = 0$ holds for all $i \geq 1$ by [14, Rmk. 4.9]. Thus also $\mathrm{C}_m(W)$ belongs to U. □

2.2 Theorem. *Let M be an A-module. If $\operatorname{id}_A M$ is finite, then one has*

$$\mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M = \mathsf{U}\text{-}\operatorname{pd}_A M\,.$$

The second paragraph below follows the proof of [17, Thm. 2.1] quite closely. We have included the full argument as we will later direct the reader to dualize it.

Proof. We may assume that M is nonzero. Set $n = \operatorname{id}_A M$ and let $M \xrightarrow{\simeq} I$ be an injective resolution with $I_{-i} = 0$ for $i > n$. As one has $\mathsf{V}\text{-}\operatorname{id}_A M \leq \operatorname{id}_A M = n$ hold, there is by Proposition 1.11 a semi-U-V replacement $W \simeq I$ with $W_{-i} = 0$ for $i > n$. By Avramov and Foxby [4, 1.4.I] there is a quasi-isomorphism $W \xrightarrow{\simeq} I$ in the category of A-complexes.

Fix $m \geq \mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M$; the module $\mathrm{C}_m(W)$ belongs to $\mathsf{RGor}_{\mathsf{U}}$ by [14, Lem. 4.7]. We argue next that $\operatorname{Ext}^1_A(G, \mathrm{C}_m(W)) = 0$ holds for every A-module G in $\mathsf{RGor}_{\mathsf{U}}$. Fix such a module, by definition there is an exact sequence of A-modules,

$$0 \longrightarrow G \longrightarrow U_0 \longrightarrow \cdots \longrightarrow U_{-m-n+1} \longrightarrow G' \longrightarrow 0\,,$$

with each module U_i in $\mathsf{U} \cap \mathsf{V}$ and G' in $\mathsf{RGor}_{\mathsf{U}}$. As $\mathrm{C}_m(W)$ belongs to V, dimension shifting yields:

$$(*) \qquad \operatorname{Ext}^1_A(G, \mathrm{C}_m(W)) \cong \operatorname{Ext}^{m+n+1}_A(G', \mathrm{C}_m(W))\,.$$

Let C be the mapping cone of the quasi-isomorphism $W \xrightarrow{\simeq} I$; one has $C_i = 0$ for $i < -n$, and the complex consists of direct sums of modules that are in $\mathsf{U} \cap \mathsf{V}$ or injective. Moreover, one has $C_{-n} = I_{-n}$ as $W_i = 0$ for $i < -n$. Further, as $I_v = 0$ holds for $v > 0$ and one has $m \geq 0$, there is an isomorphism $\mathrm{C}_{m+1}(C) \cong \mathrm{C}_m(W)$. Thus dimension shifting along the exact sequence

$$0 \longrightarrow \mathrm{C}_{m+1}(C) \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_{-n+1} \longrightarrow I_{-n} \longrightarrow 0$$

yields

$$(\dagger) \qquad \operatorname{Ext}^{m+n+1}_A(G', \mathrm{C}_m(W)) \cong \operatorname{Ext}^1_A(G', I_{-n}) = 0\,.$$

Combining $(*)$ and $(\dagger)$ one gets $\operatorname{Ext}^1_A(G, \mathrm{C}_m(W)) = 0$. With $g = \mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M$ and $m = g + 1$ and $G = \mathrm{C}_g(W)$ one now has $\operatorname{Ext}^1_A(\mathrm{C}_g(W), \mathrm{C}_{g+1}(W)) = 0$. This means that the exact sequence $0 \to \mathrm{C}_{g+1}(W) \to W_g \to \mathrm{C}_g(W) \to 0$ splits, whence $\mathrm{C}_g(W)$ is in $\mathsf{U} \cap \mathsf{V}$. Thus one has $\mathsf{U}\text{-}\operatorname{pd}_A M \leq g$, and the opposite inequality holds by Lemma 2.1. □

2.3 Proposition. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. With $g' = \mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M'$, $g = \mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M$, and $g'' = \mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M''$ there are inequalities,*

$$g' \leq \max\{g, g'' - 1\}\,, \quad g \leq \max\{g', g''\}\,, \quad \text{and} \quad g'' \leq \max\{g' + 1, g\}\,.$$

In particular, if two of the modules have finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension, then so has the third.

Proof. It follows from [14, Prop. 3.8 and Thm. 4.8] that if two of the modules have finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension, then so does the third. The inequalities are now immediate from [14, Rmk. 4.9]. □

2.4 Definition. Imitating the Gorenstein global dimension, set

$$\mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{gldim}(A) = \sup\{\mathsf{RGor}_{\mathsf{U}}\text{-}\operatorname{pd}_A M \mid M \text{ is an } A\text{-module}\}\,.$$

The next result shows that the global dimension defined above can be computed simpler in terms of $\mathsf{RGor}_{\mathsf{U}}$-projective dimensions of modules in the subcategory V, see also Remark 2.17.

2.5 Proposition. *The following conditions are equivalent.*

(i) $\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) < \infty$.

(ii) *Every A-module has finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension.*

(iii) *Every A-module in V has finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension.*

Moreover, there is an equality:

$$\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) = \sup\{\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, M \mid M \text{ is an } A\text{-module in } \mathsf{V}\}\,.$$

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are trivial.

$(ii) \Longrightarrow (i)$: Assume that every A-module has finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension. If one has $\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) = \infty$, then for every $n \in \mathbb{N}$ there is an A-module M_n with $n \le \mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, M_n < \infty$. The A-module $M = \coprod_{n\in\mathbb{N}} M_n$ has finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension, say, k. Now [14, Rmk. 4.9] yields

$$\prod_{n\in\mathbb{N}} \operatorname{Ext}_A^{k+1}(M_n, W) \cong \operatorname{Ext}_A^{k+1}(M, W) = 0$$

for every module W in $\mathsf{U} \cap \mathsf{V}$. Now another application of [14, Rmk. 4.9] yields $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, M_n \le k$ for every $n \in \mathbb{N}$, which is absurd.

$(iii) \Longrightarrow (ii)$: Assume that every module in V has finite $\mathsf{RGor}_{\mathsf{U}}$-projective dimension. For every A-module M, there is an exact sequence

$$0 \longrightarrow M \longrightarrow V \longrightarrow U \longrightarrow 0$$

of A-modules with V in V and U in U. As $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, U \le 0$ holds by Lemma 2.1 it is immediate from Proposition 2.3 that $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, M \le \mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, V$ holds. This also proves the asserted equality. □

To parse the next result, recall the notion of right periodicity from Definition 1.1. For the absolute cotorsion pair $(\mathsf{Prj}(A), \mathsf{Mod}(A))$ the fact that (i) below implies (ii) is known from [8, Thm. VII.2.2], but for $n > 0$ the converse appears to be new.

2.6 Theorem. *Assume that (U, V) is weakly right periodic and let $n \ge 0$ be an integer. The following conditions are equivalent.*

(i) $\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) \le n$.

(ii) $\mathsf{U}_n^{\perp} = \mathsf{GInj}(A)$.

Moreover, when these conditions are satisfied, there is an equality:

$$\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) = \mathsf{U}\text{-FPD}(A)\,.$$

Proof. Assume that (i) holds. We first argue that one has $\mathsf{U}\text{-FPD}(A) \le n$. For an A-module M with $\mathsf{U}\text{-pd}_A\, M$ finite, Lemma 2.1 yields

$$\mathsf{U}\text{-pd}_A\, M = \mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, M \le n\,.$$

Next we argue that $\mathsf{U}\text{-spli}(A) \le n$ holds. For an injective A-module I, one has per Theorem 2.2 the equality $\mathsf{U}\text{-pd}_A\, I = \mathsf{RGor}_{\mathsf{U}}\text{-pd}_A\, I \le n$. Now $\mathsf{U}_n^{\perp} = \mathsf{GInj}(A)$ holds by Proposition 1.9 as (U, V) is weakly right periodic.

Now assume that (*ii*) holds. Proposition 1.9 yields $\mathsf{U}\text{-spli}(A) = \mathsf{U}\text{-FPD}(A) \leq n$. By Proposition 2.5 it suffices to show that every module in V has $\mathsf{RGor}_{\mathsf{U}}(A)$-projective dimension at most n. Let V be a module in V. As V is coresolving, there is an exact sequence $0 \to V \to I \to V' \to 0$ with I injective and V' in V. Taking successive special U-precovers one gets exact sequences,

$$(\sharp) \qquad \cdots \xrightarrow{\partial} W_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow V \longrightarrow 0$$

and

$$(\sharp') \qquad \cdots \xrightarrow{\partial'} W'_n \longrightarrow W'_{n-1} \longrightarrow \cdots \longrightarrow W'_0 \longrightarrow V' \longrightarrow 0\,,$$

and the Horseshoe Lemma [30, Lem. 8.2.1] yields the exact sequence,

$$\cdots \xrightarrow{\eth} W_n \oplus W'_n \longrightarrow W_{n-1} \oplus W'_{n-1} \longrightarrow W_0 \oplus W'_0 \longrightarrow I \longrightarrow 0\,.$$

With $G = \operatorname{Coker}(\partial)$, $F_{-1} = \operatorname{Coker}(\eth)$, and $G' = \operatorname{Coker}(\partial')$ one gets an exact sequence $0 \to G \to F_{-1} \to G' \to 0$. By construction G and G' belong to V and, therefore, so does F_{-1}. It follows from the already established inequality $\mathsf{U}\text{-spli}(A) \leq n$ and Proposition 1.3 that F_{-1} belongs to U. Repeating this process, one establishes an exact sequence

$$(*) \qquad 0 \longrightarrow G \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

where the modules F_i are in $\mathsf{U} \cap \mathsf{V}$. Taking successive special U-precovers yields another exact sequence

$$(**) \qquad \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0$$

of A-modules with each F_i in $\mathsf{U} \cap \mathsf{V}$. Splicing the sequences $(*)$ and $(**)$, one gets an acyclic complex F of modules in $\mathsf{U} \cap \mathsf{V}$, where each cycle module $\mathrm{Z}_i(F)$ belongs to V by construction, and $G = \mathrm{C}_0(F)$. As every module in $\mathsf{U} \cap \mathsf{V}$ by Proposition 1.10 has finite injective dimension, at most n, the complex $\operatorname{Hom}_A(F, W)$ is acyclic for every module W in $\mathsf{U} \cap \mathsf{V}$. Thus, F is a right U-totally acyclic complex and G is in $\mathsf{RGor}_{\mathsf{U}}(A)$, so from $(\sharp)$ one gets $\mathsf{RGor}_{\mathsf{U}}\text{-pd}_A V \leq n$. □

Applied to the absolute cotorsion pair $(\mathsf{Prj}(A), \mathsf{Mod}(A))$, the equivalence of (*i*) and (*iii*) in the next theorem recovers the characterization of left Gorenstein rings from [8, Cor. VII.2.6].

2.7 Theorem. *Assume that* (U, V) *is weakly right periodic and let* $n \geq 0$ *be an integer. The following conditions are equivalent.*

(*i*) $\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) \leq n$.

(*ii*) $\sup\{\operatorname{Gid}_A M \mid M \in \mathsf{V}\} \leq n$.

(*iii*) $\mathsf{U}_n \cap \mathsf{V} = \mathsf{Inj}(A)_n \cap \mathsf{V}$.

(*iv*) $\mathsf{U}\text{-spli}(A) = \sup\{\operatorname{id}_A M \mid M \in \mathsf{U} \cap \mathsf{V}\} \leq n$.

Proof. (*i*)$\Longrightarrow$(*ii*): It follows from Theorem 2.6 and Fact 1.5 that $(\mathsf{U}_n, \mathsf{GInj}(A))$ is a complete cotorsion pair. Let V be a module in V. There is an exact sequence,

$$0 \longrightarrow G \longrightarrow U \longrightarrow V \longrightarrow 0\,,$$

of A-modules with $G \in \mathsf{GInj}(A)$ and $U \in \mathsf{U}_n$. As (U, V) is weakly right periodic, every Gorenstein injective A-module belongs to V, so the module U is in V

and $\operatorname{id}_A U \leq \max\{\mathsf{U}\text{-spli}(A), \mathsf{U}\text{-FPD}(A)\}$ holds by Proposition 1.10. From Proposition 1.9 and Theorem 2.6 one now gets $\mathsf{U}\text{-spli}(A) = \mathsf{U}\text{-FPD}(A) \leq n$. Consequently, one has $\operatorname{Gid}_A V \leq n$ by Holm [36, Thm. 2.22 and 2.25].

$(ii) \Longrightarrow (iii)$: Let $M \in \mathsf{U}_n \cap \mathsf{V}$. By assumption, $\operatorname{Gid}_A M \leq n$ holds, and so there is an exact sequence $0 \to G \to E \to M \to 0$ with $\operatorname{id}_A E \leq n$ and G Gorenstein injective; see Christensen, Frankild, and Holm [19, Lem. 2.18]. The sequence splits as G belongs to $\mathsf{U}_n^{\perp}$, see Proposition 1.6, thus $\operatorname{id}_A M \leq n$. For the opposite containment, suppose that M belongs to V and $\operatorname{id}_A M \leq n$ holds. Let $V \in \mathsf{V}$; as $\operatorname{Gid}_A V \leq n$ holds, [36, Thm. 2.22] yields $\operatorname{Ext}_A^{n+1}(M, V) = 0$, that is, $\mathsf{U}\text{-pd}_A M \leq n$ holds by Proposition 1.3.

$(iii) \Longrightarrow (iv)$: Set $s = \mathsf{U}\text{-spli}(A)$ and $t = \sup\{\operatorname{id}_A M \mid M \in \mathsf{U} \cap \mathsf{V}\}$. As every injective module is in V, the assumption immediately yields $s \leq n$. Similarly, t is finite, at most n. Now there is an injective A-module I with $\mathsf{U}\text{-pd}_A I = s$, so taking successive special U-precovers one gets an exact sequence,

$$0 \longrightarrow U_s \longrightarrow U_{s-1} \longrightarrow \cdots \longrightarrow U_0 \longrightarrow I \longrightarrow 0\,,$$

of A-modules with each module U_i in $\mathsf{U} \cap \mathsf{V}$. One has $\operatorname{Ext}_A^s(I, U_s) \neq 0$ and, therefore, $\operatorname{id}_A U_s \geq s$, so one has $t \geq s$. On the other hand, there is a module M in $\mathsf{U} \cap \mathsf{V}$ with $\operatorname{id}_A M = t$. So there exists an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots \longrightarrow I_{-t} \longrightarrow 0$$

of A-modules with each I_i injective. One has $\operatorname{Ext}_A^t(I_{-t}, M) \neq 0$ and, therefore, $\mathsf{U}\text{-pd}_A I_{-t} \geq t$, so one has $s \geq t$.

$(iv) \Longrightarrow (i)$: Let M be an A-module of finite U-projective dimension. If M is in U, then $\mathsf{U}\text{-pd}_A M \leq n$ holds trivially. Now assume that $d = \mathsf{U}\text{-pd}_A M$ is positive and consider the exact sequence,

$$0 \longrightarrow U_d \longrightarrow U_{d-1} \longrightarrow \cdots \longrightarrow U_0 \longrightarrow M \longrightarrow 0\,,$$

obtained by taking special U-precovers. For $i > 0$ the module U_i is in $\mathsf{U} \cap \mathsf{V}$. One has $\operatorname{Ext}_A^d(M, U_d) \neq 0$ and since, by assumption, $\operatorname{id}_A U_d \leq n$ holds, one gets $d \leq n$. Thus $\mathsf{U}\text{-FPD}(A) \leq n$ holds, and now Proposition 1.9 and Theorem 2.6 yield the desired inequality. □

Applied to the absolute cotorsion pair $(\mathsf{Prj}(A), \mathsf{Mod}(A))$ the next corollary recovers Bennis and Mahdou's result [10, Thm. 1.1] that the Gorenstein global dimension can be computed based on the Gorenstein injective dimension.

2.8 Corollary. *Assume that* (U, V) *is weakly right periodic. The* $\mathsf{RGor}_{\mathsf{U}}$*-global dimension is finite if and only if every module in* V *has finite Gorenstein injective dimension, and there is an equality:*

$$\mathsf{RGor}_{\mathsf{U}}\text{-gldim}(A) = \sup\{\operatorname{Gid}_A M \mid M \in \mathsf{V}\}\,.$$

Proof. As V is closed under products, the assertions follow from Theorem 2.7. □

For the remainder of the section, we shift focus to the class V, recalling the notion of left V-Gorenstein modules from [16, Def. 2.1] and the associated homological dimension $\mathsf{LGor}_{\mathsf{V}}\text{-id}_A$ from [14, Def. 4.5]. For $(\mathsf{U}, \mathsf{V}) = (\mathsf{Mod}(A), \mathsf{Inj}(A))$ the $\mathsf{LGor}_{\mathsf{V}}$-injective dimension is the standard Gorenstein injective dimension, and we use the standard notation Gid_A, as already done in Theorem 2.7 and Corollary 2.8.

2.9 Lemma. *Let M be an A-module. There is an inequality,*

$$\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M \leq \mathsf{V}\text{-}\mathrm{id}_A\, M\ ,$$

and equality holds if $\mathsf{V}\text{-}\mathrm{id}_A\, M$ is finite.

Proof. Dual to the proof of Lemma 2.1. □

2.10 Theorem. *Let M be an A-module. If $\mathrm{pd}_A\, M$ is finite, then one has*

$$\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M = \mathsf{V}\text{-}\mathrm{id}_A\, M\ .$$

Proof. Dual to the proof of Theorem 2.2. □

2.11 Proposition. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. With $g' = \mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M'$, $g = \mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M$, and $g'' = \mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M''$ there are inequalities,*

$$g' \leq \max\{g, g''+1\}\,, \quad g \leq \max\{g', g''\}\,, \quad \text{and} \quad g'' \leq \max\{g'-1, g\}\,.$$

In particular, if two of the modules have finite $\mathsf{LGor}_{\mathsf{V}}$-injective dimension, then so has the third.

Proof. Dual to the proof of Proposition 2.3. □

2.12 Definition. Set

$$\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) = \sup\{\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M \mid M \text{ is an } A\text{-module}\}\,.$$

For $(\mathsf{U},\mathsf{V}) = (\mathsf{Mod}(A), \mathsf{Inj}(A))$ this is by [10, Thm. 1.1] the invariant $\mathrm{Ggldim}(A)$.

2.13 Proposition. *The following conditions are equivalent.*

(*i*) $\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) < \infty$.

(*ii*) *Every A-module has finite $\mathsf{LGor}_{\mathsf{V}}$-injective dimension.*

(*iii*) *Every A-module in U has finite $\mathsf{LGor}_{\mathsf{V}}$-injective dimension.*

Moreover, there is an equality:

$$\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) = \sup\{\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{id}_A\, M \mid M \text{ is an } A\text{-module in } \mathsf{U}\}\,.$$

Proof. Dual to the proof of Proposition 2.5. □

2.14 Theorem. *Assume that (U,V) is weakly left periodic and let $n \geq 0$ be an integer. The following conditions are equivalent.*

(*i*) $\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) \leq n$.

(*ii*) ${}^{\perp}\mathsf{V}_n = \mathsf{GPrj}(A)$.

Moreover, when these conditions are satisfied, there is an equality:

$$\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) = \mathsf{V}\text{-}\mathrm{FID}(A)\,.$$

Proof. Dual to the proof of Theorem 2.6. □

2.15 Theorem. *Assume that (U,V) is weakly left periodic and let $n \geq 0$ be an integer. The following conditions are equivalent.*

(*i*) $\mathsf{LGor}_{\mathsf{V}}\text{-}\mathrm{gldim}(A) \leq n$.

(*ii*) $\sup\{\mathrm{Gpd}_A\, M \mid M \in \mathsf{U}\} \leq n$.

(*iii*) $\mathsf{U} \cap \mathsf{V}_n = \mathsf{U} \cap \mathsf{Prj}(A)_n$.

(*iv*) $\mathsf{V}\text{-}\mathrm{silp}(A) = \sup\{\mathrm{pd}_A\, M \mid M \in \mathsf{U} \cap \mathsf{V}\} \leq n$.

Proof. Dual to the proof of Theorem 2.7. □

2.16 Corollary. *Assume that* (U,V) *is weakly left periodic. The* $\mathsf{LGor}_{\mathsf{V}}$*-global dimension is finite if and only if every module in* U *has finite Gorenstein projective dimension, and there is an equality:*

$$\mathsf{LGor}_{\mathsf{V}}\text{-gldim}(A) = \sup\{\operatorname{Gpd}_A M \mid M \in \mathsf{U}\}\,.$$

Proof. Dual to the proof of Corollary 2.8. □

We denote by $\mathsf{Flat}(A)$ the class of flat A-modules and by $\mathsf{Cot}(A)$ the class of cotorsion A-modules; they form a cotorsion pair.

2.17 Remark. The equalities in Proposition 2.5 and Corollary 2.8, and dually in Proposition 2.13 and Corollary 2.16, are the counterparts in Gorenstein homological algebra of the following equalities:

$$\begin{aligned}\mathsf{U}\text{-gldim}(A) &= \sup\{\mathsf{U}\text{-pd}_A V \mid V \in \mathsf{V}\} = \sup\{\operatorname{id}_A V \mid V \in \mathsf{V}\}\,,\\ \mathsf{V}\text{-gldim}(A) &= \sup\{\mathsf{V}\text{-id}_A U \mid U \in \mathsf{U}\} = \sup\{\operatorname{pd}_A U \mid U \in \mathsf{U}\}\,.\end{aligned}$$

Here, of course, U-gldim(A) and V-gldim(A) are defined as the suprema of $\mathsf{U}\text{-pd}_A$ and $\mathsf{V}\text{-id}_A$ over all A-modules. The displayed equalities can be shown as in Mao and Ding [42, Thm. 19.2.5(1)], where they consider the equalities in the second line for the cotorsion pair $(\mathsf{U},\mathsf{V}) = (\mathsf{Flat}(A),\mathsf{Cot}(A))$.

3. Gorenstein flat-cotorsion global dimension

In this section we apply the theory developed in the previous two sections to the cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ in order to study the global invariant based on the Gorenstein flat-cotorsion dimension from [15]. We relate it to the Gorenstein global dimension and obtain new characterizations of Iwanaga–Gorenstein rings and Ding–Chen rings. The results in this section find further applications in Section 5.

The cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ is hereditary and generated by a set; see Bican, El Bashir, and Enochs [11, Prop. 2]. The relevant special case of Fact 1.5 was noted earlier by Hügel, Herbera, and Trlifaj [2, Sec. 1.C]. The cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ is right periodic by [5, Thm. 1.3].

In this setting, $\mathsf{U} = \mathsf{Flat}(A)$, the invariants $\mathsf{U}\text{-pd}_A$, U-FPD(A), and U-spli(A) are the flat dimension fd_A, the finitistic flat dimension FFD(A), and sfli(A). Modules in $\mathsf{RGor}_{\mathsf{U}}(A)$ are called Gorenstein flat-cotorsion, see [15, Rmk. 4.8]. The $\mathsf{RGor}_{\mathsf{U}}$-projective dimension is known as the Gorenstein flat-cotorsion dimension and written Gfcd_A. For the corresponding global dimension we naturally introduce the symbol Gfc-gldim(A).

3.1 Fact. It follows from [15, Thm. 5.7] that there is an inequality,

$$\text{Gfc-gldim}(A) \le \operatorname{Gwgldim}(A)\,,$$

and equality holds if Gwgldim(A) is finite.

For a large class of rings the Gorenstein flat-cotorsion global dimension agrees with the Gorenstein weak global dimension, see Section 5. As such, the next theorem can be considered an improvement of [26, Thm. 5.3], see also [17, Thm. 2.4]. Indeed, Gwgldim(A) is finite if and only if sfli(A) and sfli(A°) are finite, but the next theorem yields

$$\text{Gfc-gldim}(A) = \max\{\operatorname{sfli}(A), \operatorname{FFD}(A)\}\,,$$

and one always has $\mathrm{FFD}(A) \leq \mathrm{sfli}(A^{\circ})$ by [28, Prop. 2.4].

3.2 Theorem. *Let $n \geq 0$ be an integer. The following conditions are equivalent.*

(i) $\text{Gfc-gldim}(A) \leq n$.

(ii) $\max\{\mathrm{sfli}(A), \mathrm{FFD}(A)\} \leq n$.

(iii) $\sup\{\mathrm{Gid}_A M \mid M \text{ is cotorsion}\} \leq n$.

(iv) $\mathsf{Flat}(A)_n \cap \mathsf{Cot}(A) = \mathsf{Inj}(A)_n \cap \mathsf{Cot}(A)$.

(v) $\mathrm{sfli}(A) = \sup\{\mathrm{id}_A M \mid M \text{ is flat-cotorsion}\} \leq n$.

(vi) $\mathsf{Flat}(A)_n^{\perp} = \mathsf{GInj}(A)$.

Moreover, when these conditions are satisfied there are equalities:

$$\text{Gfc-gldim}(A) = \mathrm{FFD}(A) = \mathrm{sfli}(A)\ .$$

Proof. Apply Proposition 1.9 and Theorems 2.6 and 2.7 to $(\mathsf{Flat}(A), \mathsf{Cot}(A))$. □

The next corollary provides the converse to [18, Prop. 4.2] that was anticipated in [18, Quest. 4.11]. Recall that an A-module M is said to be *strongly cotorsion* if $\mathrm{Ext}_A^1(L, M) = 0$ for every A-module L of finite flat dimension.

3.3 Corollary. *The following conditions are equivalent.*

(i) $\text{Gfc-gldim}(A) < \infty$.

(ii) $\mathrm{FFD}(A) < \infty$ *and every strongly cotorsion A-module is Gorenstein injective.*

Proof. With $n = \mathrm{FFD}(A)$ the class of strongly cotorsion modules is $\mathsf{Flat}(A)_n^{\perp}$ and the assertion follows from Theorem 3.2. □

As discussed in [18, Sec. 4] there has been interest in the question of when Gorenstein injective modules coincide with strongly cotorsion modules.

3.4 Remark. Assume that $\mathrm{FFD}(A) = n$ is finite and let $\mathsf{SCot}(A)$ be the class of strongly cotorsion A-modules. The pair $(\mathsf{Flat}(A)_n, \mathsf{SCot}(A))$ is per Fact 1.5, or [2, Sec. 1.C], a hereditary cotorsion pair generated by a set, and it is right periodic by Proposition 1.6. Applied to this cotorsion pair, Corollary 2.8 yields

$$\mathsf{RGor}_{\mathsf{Flat}(A)_n}\text{-gldim}(A) = \sup\{\mathrm{Gid}_A M \mid M \text{ is a strongly cotorsion } A\text{-module}\}\ .$$

This measures how far the strongly cotorsion A-modules are from being Gorenstein injective: It is zero precisely when the strongly cotorsion and Gorenstein injective modules coincide, or alternatively when $\text{Gfc-gldim}(A)$ is finite; see Corollary 3.3.

The next corollary is subsumed by Theorem 5.5, but we record it here for use in Corollary 3.7, Corollary 3.9, and Theorem 4.5.

3.5 Corollary. *If A is left or right coherent, then there is an equality:*

$$\text{Gfc-gldim}(A) = \mathrm{Gwgldim}(A)\ .$$

Proof. By Fact 3.1 one has the inequality $\text{Gfc-gldim}(A) \leq \mathrm{Gwgldim}(A)$ with equality if $\mathrm{Gwgldim}(A)$ is finite. It follows from Corollary 3.3 and [18, Thm. 4.9] that the two global dimensions are simultaneously finite over a left or right coherent ring, whence the displayed equality holds. □

3.6 Proposition. *The following conditions are equivalent.*

(i) $\text{Gfc-gldim}(A) < \infty$.

(*ii*) *Every cotorsion A-module has finite Gorenstein flat-cotorsion dimension.*

(*iii*) *Every cotorsion A-module has finite Gorenstein injective dimension.*

Further, there are equalities,

$$\begin{aligned}\text{Gfc-gldim}(A) &= \sup\{\text{Gfcd}_A M \mid M \text{ is a cotorsion } A\text{-module}\}\\ &= \sup\{\text{Gid}_A M \mid M \text{ is a cotorsion } A\text{-module}\}\,.\end{aligned}$$

Proof. The equivalence of (*i*)–(*iii*) as well as the equalities follow from Proposition 2.5 and Corollary 2.8 applied to the cotorsion pair $(\mathsf{Flat}(A), \mathsf{Cot}(A))$. □

The equalities in Proposition 3.6 are the correspondents in Gorenstein homological algebra to the fact that the weak global dimension is the supremum of flat dimensions of cotorsion modules and also the supremum of injective dimensions of cotorsion modules, see Remark 2.17.

3.7 Corollary. *If A is noetherian, then it is Iwanaga–Gorenstein if (and only if) every cotorsion A-module has finite Gorenstein injective dimension.*

Proof. Combine Corollary 3.5 and Proposition 3.6 with [17, Rmk. 3.11]. □

Recall that A is called *Ding–Chen* if it is coherent with finite self-fp-injective dimension on both sides. Symmetry of the Gorenstein weak global dimension has the following immediate consequence:

3.8 Proposition. *If A is coherent, then it is Ding–Chen if and only if* $\text{Gwgldim}(A)$ *is finite.*

Proof. As A is coherent, it follows from work of Ding and Chen [23, Thm. 7] combined with [17, Cor. 2.5] that A is Ding–Chen if and only if the Gorenstein weak global dimension is finite. □

One can get an even broader statement than Corollary 3.7:

3.9 Corollary. *If A is coherent, then the following conditions are equivalent.*

(*i*) *A is Ding–Chen.*

(*ii*) *Every cotorsion A-module has finite Gorenstein injective dimension.*

(*iii*) *Every A-module has finite Gorenstein flat dimension.*

Proof. In view of [17, Thm. 2.4] one gets the equivalence of the three conditions by combining Corollary 3.5 with Propositions 3.6 and 3.8. □

To parse the next result recall the invariant

$$\text{splf}(A) = \sup\{\text{pd}_A F \mid F \text{ is a flat } A\text{-module}\}\,.$$

If A is right coherent, the next result reduces per Corollary 3.5 to [17, Cor. 3.5].

3.10 Corollary. *There are inequalities:*

$$\text{Gfc-gldim}(A) \le \text{Ggldim}(A) \le \text{Gfc-gldim}(A) + \text{splf}(A)\,.$$

Moreover, the following conditions are equivalent.

(*i*) $\text{Gfc-gldim}(A)$ *and* $\text{splf}(A)$ *are finite.*

(*ii*) $\text{Ggldim}(A)$ *is finite.*

Proof. The left-hand inequality holds by [17, Thm. 3.3]; it is also a consequence of Theorems 1.18 and 3.2. To prove the right-hand inequality one can assume that $g = \text{Gfc-gldim}(A)$ and $s = \text{splf}(A)$ are finite. Let M be an A-module and $M \xrightarrow{\simeq} I$ an injective resolution. By dimension shifting or Remark 2.17 the module $\mathrm{Z}_{-s}(I)$ is cotorsion, so by Proposition 3.6 one has $\mathrm{Gid}_A M \leq \mathrm{Gid}_A \mathrm{Z}_{-s}(I) + s \leq g + s$.

It is clear from the second inequality that (i) implies (ii). For the converse it follows from the first inequality that $\text{Gfc-gldim}(A)$ is finite and one has $\text{splf}(A) \leq \text{FPD}(A) = \text{Ggldim}(A)$ by [40, Prop. 6] and [36, Thm. 2.28]. □

4. A Gorenstein analogue of a result due to Stenström

Recall that an A-module I is called *fp-injective* if $\mathrm{Ext}^1_A(F, I) = 0$ holds for every finitely presented A-module F. Let $\mathsf{FpInj}(A)$ be the class of fp-injective A-modules. In a pure homological sense, fp-injective modules are dual to flat modules as every exact sequence $0 \to I \to M \to N \to 0$ with I fp-injective is pure.

4.1 Definition. For simplicity we call $\mathsf{LGor}_{\mathsf{FpInj}(A)}$-modules *Gorenstein fp-injective* A-modules. The associated dimensions $\mathsf{LGor}_{\mathsf{FpInj}(A)}\text{-id}_A$ and $\mathsf{LGor}_{\mathsf{FpInj}(A)}\text{-gldim}(A)$ are similarly simplified to Gfp-id_A and $\text{Gfpinj-gldim}(A)$.

It is standard that $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ is a complete cotorsion pair generated by the set of isomorphism classes of finitely presented modules. It is hereditary if and only if A is left coherent; see for example [46, Thm. 3.2] for the non-trivial "only if" part of the statement.

4.2 Fact. Let A be left coherent. Every acyclic complex of A-modules in ${}^{\perp}\mathsf{FpInj}(A)$ has cycle submodules in ${}^{\perp}\mathsf{FpInj}(A)$, that is, $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ is left periodic; see Šaroch and Šťovíček [47, Exa. 4.3].

4.3 Remark. An A-module is Ding injective if it is a cycle module in an acyclic complex of injective modules that remains exact upon application of $\mathrm{Hom}_A(I, -)$ for every fp-injective A-module I; these modules were called Gorenstein fp-injective by Mao and Ding [43]. If A is left noetherian, then Ding injective A-modules and Gorenstein fp-injective A-modules are simply Gorenstein injective A-modules. In general, Ding injective modules and Gorenstein fp-injective modules can differ.

For $\mathsf{V} = \mathsf{FpInj}(A)$ we use the notation fp-id_A for $\mathsf{V}\text{-id}_A$ and, similarly, $\text{fp-silp}(A)$ for $\mathsf{V}\text{-silp}(A)$ and $\text{fp-FID}(A)$ for $\mathsf{V}\text{-FID}(A)$.

4.4 Lemma. *If A is left coherent, then there are equalities,*

$$\text{sfli}(A^{\circ}) = \text{fp-silp}(A) \quad \textit{and} \quad \text{FFD}(A^{\circ}) = \text{fp-FID}(A)\,.$$

Proof. Since A is left coherent, one has $\text{sfli}(A^{\circ}) = \text{fp-id}_A A = \text{fp-silp}(A)$ by Ding and Chen [22, Thm. 3.8]. On the other hand, for every A°-module M and A-module N, the equalities

$$\mathrm{fd}_{A^{\circ}} M = \text{fp-id}_A \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \quad \text{and} \quad \text{fp-id}_A N = \mathrm{fd}_{A^{\circ}} \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$$

hold by Fieldhouse [32, Thm. 2.1 and 2.2], whence $\text{FFD}(A^{\circ}) = \text{fp-FID}(A)$. □

The next result is an analogue in Gorenstein homological algebra to a result of Stenström [46, Thm. 3.3]. The equality of the first and last quantities can alternatively be deduced from results of Bennis [9, Thm. 3.3] and Li, Yan, and Zhang [41, Thm. 2.10], but the remaining equalities are new.

4.5 Theorem. *If A is left coherent, then there are equalities,*

$$\begin{aligned}\operatorname{Gwgldim}(A) &= \text{Gfpinj-gldim}(A)\\ &= \sup\{\text{Gfp-id}_A\, M \mid M \textit{ is in } {}^{\perp}\mathsf{FpInj}(A)\}\\ &= \sup\{\operatorname{Gpd}_A M \mid M \textit{ is a finitely presented } A\textit{-module}\}\,.\end{aligned}$$

Proof. In view of the equalities in Lemma 4.4 it follows from Theorem 3.2, Proposition 1.16, and Theorem 2.14 that the quantities Gfc-gldim(A°) and Gfpinj-gldim(A) are simultaneously finite whence one has

$$\operatorname{Gwgldim}(A) = \text{Gfc-gldim}(A^\circ) = \operatorname{FFD}(A^\circ) = \text{fp-FID}(A) = \text{Gfpinj-gldim}(A)$$

by the same references combined with [17, Cor. 2.5] and Corollary 3.5. This proves the first of the asserted equalities, and the second holds by Proposition 2.13.

To prove the third equality, notice first that Corollary 2.16 yields

$$\text{Gfpinj-gldim}(A) = \sup\{\operatorname{Gpd}_A M \mid M \text{ is in } {}^{\perp}\mathsf{FpInj}(A)\}\,,$$

and since all finitely presented A-modules are in ${}^{\perp}\mathsf{FpInj}(A)$, the right-hand quantity is at least $\sup\{\operatorname{Gpd}_A M \mid M$ is a finitely presented A-module$\}$. Assume now that this last quantity is at most n and let S be a set of representatives of isomorphism classes of finitely presented A-modules. Let L be the class of A-modules of Gorenstein projective dimension at most n. By assumption, $\mathsf{S} \subseteq \mathsf{L}$ and, therefore, $\operatorname{Filt}(\mathsf{S}) \subseteq \operatorname{Filt}(\mathsf{L})$. Here, for a class C of A-modules, $\operatorname{Filt}(\mathsf{C})$ is the class of direct transfinite extensions of modules in C. By Enochs, Iacob, and Jenda [29, Thm. 3.4] one has $\operatorname{Filt}(\mathsf{L}) \subseteq \mathsf{L}$, so all modules in $\operatorname{Filt}(\mathsf{S})$ have Gorenstein projective dimension at most n. Since S is a set that contains A, [34, Cor. 6.14] yields that ${}^{\perp}(\mathsf{S}^{\perp}) = \operatorname{add}(\operatorname{Filt}(\mathsf{S}))$. It is standard that L is closed under direct summands. Since $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ is generated by the set S, one has ${}^{\perp}\mathsf{FpInj}(A) = {}^{\perp}(\mathsf{S}^{\perp})$, so all modules in ${}^{\perp}\mathsf{FpInj}(A)$ have Gorenstein projective dimension at most n. □

The next corollary is an equivalent in Gorenstein homological algebra to [3, Thm. 4]. As noticed in [41, Cor. 2.7] one could alternatively derive it by combining a result of Bouchiba [12, Thm. 7] with symmetry of the Gorenstein weak global dimension [17, Cor. 2.5]. As also noticed in [41] this symmetry could alternatively be obtained by combining [12, Thm. 6] with [47, Cor. 4.12].

4.6 Corollary. *If A is left noetherian, then the following equality holds:*

$$\operatorname{Gwgldim}(A) = \operatorname{Ggldim}(A)\,.$$

Proof. As A is left noetherian, fp-injective modules are simply injective modules and one has $\text{Gfp-id}_A\, M = \operatorname{Gid}_A M$ for every A-module M. Now invoke the first equality in Theorem 4.5. □

4.7 Corollary. *If A is coherent, then it is Ding–Chen if and only if every A-module has finite Gorenstein fp-injective dimension.*

Proof. Combine Proposition 3.8 with Proposition 2.13 and Theorem 4.5. □

We observe that the left periodicity of $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ provides for a nicer description of Gorenstein fp-injective modules.

4.8 Proposition. *Let A be left coherent. An A-module G is Gorenstein fp-injective if and only if there exists an acyclic complex T of modules from the class $\mathsf{W} = {}^{\perp}\mathsf{FpInj}(A) \cap \mathsf{FpInj}(A)$ with $G = \mathrm{Z}_0(T)$ such that $\mathrm{Hom}_A(W, T)$ is acyclic for every module W in W.*

Proof. The assertion follows from [16, Prop. 1.3(L)] since the left periodic property of $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ takes care of condition (3). □

4.9 Remark. If A is not left coherent, then the cotorsion pair $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ is not hereditary. To obtain a cotorsion pair that is hereditary, generated by a set, and agrees with $({}^{\perp}\mathsf{FpInj}(A), \mathsf{FpInj}(A))$ if A is left coherent, one can replace the fp-injective modules by the smaller class of strongly fp-injective modules. This yields a cotorsion pair which is weakly left periodic by Emmanouil and Kaperonis [27, Rmk. 4.10(ii)], and we can thus apply the results of Sections 1 and 2 to this cotorsion pair. The question of left periodicity of this cotorsion pair has been recently considered by Bazzoni, Hrbek, and Positselski [6].

5. Applications to the Gorenstein (weak) global dimension

In settings where the Gorenstein flat-cotorsion global dimension agrees with the Gorenstein weak global dimension, results from the previous sections yield new relations among the Gorenstein global dimensions, see 5.13–5.18. It was established in [15, Cor. 5.8] that the equality $\mathrm{Gfc\text{-}gldim}(A) = \mathrm{Gwgldim}(A)$ holds if A is right coherent, and by Corollary 3.5 it also holds if A is left coherent. In this section, we enlarge the class of rings for which the equality holds in three directions, see Theorems 5.5, 5.8, and 5.11.

For an A-module M we use the abbreviated notation M^+ for the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. The definable closure of A is denoted $\langle A \rangle$. Recall from Cortés Izurdiaga [20] that a ring A is *right weak coherent* if the product of flat A-modules has finite flat dimension.

5.1 Lemma. *If A is right weak coherent, then there are inequalities:*

$$\sup\{\mathrm{fd}_A I^+ \mid I \text{ is an injective } A^\circ\text{-module}\} \leq \sup\{\mathrm{fd}_A M \mid M \in \langle A \rangle\} < \infty\,.$$

Proof. The character module of every injective A°-module belongs to $\langle A \rangle$, see Estrada, Iacob, and Pérez [31, Rmk. 2.12], whence the left-hand inequality holds. To see that the right-hand supremum is finite, notice that the definable closure of A agrees with the definable closure of the class of flat A-modules. By a result of Rothmaler, see Herbera [35, Lem. 2.9], the definable closure of the class of flat A-modules can be constructed by first closing under direct products, then direct limits and, finally, pure submodules. As A is right weak coherent, there is per [20, Prop. 4.1] an integer n such that the product of any family of flat A-modules has flat dimension at most n. The class of modules of flat dimension at most n is closed both under direct limits and pure submodules, so all modules in $\langle A \rangle$ have finite flat dimension, and so the right-hand supremum is finite, at most n. □

The next theorem subsumes [15, Thm. 5.2].

5.2 Theorem. *Let M be an A-module. If M is Gorenstein flat and cotorsion, then it is Gorenstein flat-cotorsion. The converse holds if A is right weak coherent.*

Proof. Per [15, Thm. 5.2] an A-module that is both Gorenstein flat and cotorsion is Gorenstein flat-cotorsion, so it suffices to show that the converse holds under the assumption that A is right weak coherent. Let M be a Gorenstein flat-cotorsion A-module and T a totally acyclic complex of flat-cotorsion A-modules with $M = \mathrm{C}_0(T)$. As M is cotorsion, see e.g. [15, Lem. 3.2], it remains to show that it is Gorenstein flat. Let I be an injective A°-module. The A-module I^+ is cotorsion, and since A is right weak coherent, it follows from Lemma 5.1 that I^+ has finite flat dimension. By a standard dimension shifting argument, the complex $\mathrm{Hom}_A(T, I^+)$ is acyclic. Adjunction now yields

$$\mathrm{Hom}_{\mathbb{Z}}(I \otimes_A T, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_A(T, I^+)\ ,$$

and so $I \otimes_A T$ is acyclic by faithful injectivity of $\mathbb{Q}/\mathbb{Z}$. Thus T is **F**-totally acyclic, and M is Gorenstein flat. □

5.3 Theorem. *Let A be right weak coherent. For every A-complex M one has:*

$$\mathrm{Gfcd}_A M = \mathrm{Gfd}_A M\ .$$

Proof. The inequality $\mathrm{Gfcd}_A M \le \mathrm{Gfd}_A M$ holds by [15, Thm. 5.7], and equality holds trivially if $\mathrm{Gfcd}_A M$ is infinite. Assume now that $\mathrm{Gfcd}_A M = n < \infty$ holds. By definition there exists a semi-flat-cotorsion replacement W of M with $\mathrm{H}_i(W) = 0$ for $i > n$ and $\mathrm{C}_n(W)$ a Gorenstein flat-cotorsion A-module. By Theorem 5.2, the module $\mathrm{C}_n(W)$ is Gorenstein flat, whence $\mathrm{Gfd}_A M \le \mathrm{Gfcd}_A M$ holds by the definition of Gorenstein flat dimension. □

5.4 Remark. While right weak coherence is a natural generalization of right coherence, notice that the previous two theorems remain true under the assumption that the character module of every injective A°-module has finite flat dimension.

5.5 Theorem. *If A is left or right weak coherent, then the next equality holds:*

$$\text{Gfc-gldim}(A) = \mathrm{Gwgldim}(A)\ .$$

Proof. If A is right weak coherent, then the equality follows immediately from Theorem 5.3. Assume now that A is left weak coherent and notice that by Fact 3.1 one has the inequality $\text{Gfc-gldim}(A) \le \mathrm{Gwgldim}(A)$ with equality if $\mathrm{Gwgldim}(A)$ is finite. If $\text{Gfc-gldim}(A)$ is infinite, then the equality trivially holds, so we assume that $\text{Gfc-gldim}(A)$ is finite, and proceed to show that $\mathrm{Gwgldim}(A)$ is finite. It suffices per [17, Cor. 2.5] to show $\mathrm{Gwgldim}(A^\circ)$ is finite.

As A° is right weak coherent, it follows from Lemma 5.1 that there is an integer n with

$$(*) \quad \sup\{\mathrm{fd}_{A^\circ} I^+ \mid I \text{ is an injective } A\text{-module}\} \le n \quad \text{and} \quad \text{Gfc-gldim}(A) \le n\ .$$

Let M be an A°-module. As M^+ is a cotorsion A-module, Proposition 3.6 yields $\mathrm{Gid}_A M^+ \le n$. By [19, Lem. 2.18] there is an exact sequence of A-modules,

$$0 \longrightarrow H \longrightarrow E \longrightarrow M^+ \longrightarrow 0\ ,$$

with $\mathrm{id}_A E \le n$ and H is Gorenstein injective. Taking character modules yields an embedding $M^{++} \to E^+$ and, therefore, an embedding $M \to E^+$ of A°-modules. Since $\mathrm{id}_A E \le n$ holds one has $\mathrm{fd}_{A^\circ} E^+ \le 2n$ per the first inequality in $(*)$. Thus every A°-module can be embedded into an A°-module of flat dimension at most $2n$. This process allows one to build a right resolution of M by A°-modules M_i of flat dimension at most $2n$. Following Cartan and Eilenberg [13, Chap. XVII,

§1] one can construct a projective resolution of this complex in the category of A°-complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1^{(0)} & \longrightarrow & P_1^{(-1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0^{(0)} & \longrightarrow & P_0^{(-1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & M_0 & \longrightarrow & M_{-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This induces an exact sequence

$$0 \longrightarrow \mathrm{C}_{2n}(P) \longrightarrow \mathrm{C}_{2n}(P^{(0)}) \longrightarrow \mathrm{C}_{2n}(P^{(-1)}) \longrightarrow \cdots .$$

As the modules $\mathrm{C}_{2n}(P^{(i)})$ are flat, this means that a syzygy, namely $\mathrm{C}_{2n}(P)$, of M is a cokernel in an acyclic complex of flat A°-modules. The inequality $\text{Gfc-gldim}(A) \le n$ implies $\operatorname{sfli}(A) \le n$ by Theorem 3.2. Thus every acyclic complex of flat A°-modules is $\mathbf{F}$-totally acyclic, whence M has finite Gorenstein flat dimension. □

5.6 Corollary. *If A is left or right weak coherent, then the next equality holds:*

$$\text{Gfc-gldim}(A) = \text{Gfc-gldim}(A^\circ) .$$

Proof. As both A and A° are left or right weak coherent, the equality follows from Theorem 5.5 and [17, Cor. 2.5]. □

One also obtains the following extension of [18, Thm. 4.9]:

5.7 Corollary. *If A is left or right weak coherent, then the following conditions are equivalent.*

(i) $\operatorname{Gwgldim}(A) < \infty$.

(ii) $\operatorname{FFD}(A) < \infty$ *and every strongly cotorsion A-module is Gorenstein injective.*

Proof. This is immediate from Corollary 3.3 in light of Theorem 5.5. □

Per [40, Prop. 6] it follows from next theorem that the equality $\text{Gfc-gldim}(A) = \operatorname{Gwgldim}(A)$ holds if $\operatorname{FPD}(A)$ is finite.

5.8 Theorem. *If $\mathsf{Cot}(A)\text{-id}_A\, A$ is finite, then the following equality holds:*

$$\text{Gfc-gldim}(A) = \operatorname{Gwgldim}(A) .$$

In particular, the equality holds if $\operatorname{splf}(A)$ *is finite.*

Proof. The inequality $\text{Gfc-gldim}(A) \le \operatorname{Gwgldim}(A)$ holds by Fact 3.1. To prove the opposite inequality one can assume that $\text{Gfc-gldim}(A)$ is finite, and by Theorem 5.5 it suffices to show that A is right weak coherent. Fix an integer n with $\mathsf{Cot}(A)\text{-id}_A\, A \le n$ and $\text{Gfc-gldim}(A) \le n$. We first argue that the product of any set of flat-cotorsion A-modules has finite flat dimension. This follows from Theorem 3.2, since Gfc-gldim$(A) \leq n$ implies $\mathsf{Flat}(A)_n \cap \mathsf{Cot}(A) = \mathsf{Inj}(A)_n \cap \mathsf{Cot}(A)$, and since $\mathsf{Inj}(A)_n \cap \mathsf{Cot}(A)$ is closed under products, so is $\mathsf{Flat}(A)_n \cap \mathsf{Cot}(A)$.

Taking special cotorsion preenvelopes yields per Proposition 1.11 a resolution,

$$0 \longrightarrow A \longrightarrow C_0 \longrightarrow \cdots \longrightarrow C_{-n} \longrightarrow 0\,,$$

where the A-modules C_i are flat-cotorsion. For any set X it yields an exact sequence

$$0 \longrightarrow A^X \longrightarrow C_0^X \longrightarrow \cdots \longrightarrow C_{-n}^X \longrightarrow 0\,.$$

As C_i^X has finite flat dimension for each i, so does A^X. Hence by [20, Thm. 4.2], the ring A is right weak coherent. For the last assertion, recall from [42, Thm. 19.2.5(1)] that $\operatorname{splf}(A)$ equals the cotorsion global dimension of A, see also Remark 2.17. □

In comments on an earlier version of this paper, Ioannis Emmanouil made the following observation.

5.9 Remark. If every flat A-module has finite Gorenstein projective dimension, then $\mathsf{Cot}(A)\text{-id}_A\, A$ is finite and Theorem 5.8 applies. Indeed, as flat modules are closed under coproducts, there is an integer n such that $\operatorname{Gpd}_A F \leq n$ holds for every flat A-module F. To see that $\mathsf{Cot}(A)\text{-id}_A\, A \leq n$ holds, it suffices by Proposition 1.11 to show that $\operatorname{Ext}_A^{n+1}(F, A) = 0$ holds for every flat A-module F. Let $P \xrightarrow{\simeq} F$ be a projective resolution. Dimension shifting yields $\operatorname{Ext}_A^{n+1}(F, A) \cong \operatorname{Ext}_A^1(\mathrm{C}_n(P), A)$, which vanishes because $\mathrm{C}_n(P)$ is Gorenstein projective.

5.10 Corollary. *If* $\operatorname{splf}(A)$ *and* $\operatorname{splf}(A^\circ)$ *are both finite, then one has:*

$$\text{Gfc-gldim}(A) = \text{Gfc-gldim}(A^\circ)\,.$$

Proof. The assertion follows in view of [17, Cor. 2.5] from Theorem 5.8. □

Recall that A is called left $\aleph_0$-noetherian if every left ideal is countably generated.

5.11 Theorem. *If A is left $\aleph_0$-noetherian, then the following equality holds:*

$$\text{Gfc-gldim}(A) = \text{Gwgldim}(A)\,.$$

Proof. If $\operatorname{splf}(A)$ is finite, the equality holds by Theorem 5.8. If $\operatorname{splf}(A)$ is infinite, then so is $\operatorname{FPD}(A)$ by [40, Prop. 6]. From [28, Prop. 2.8] it follows that $\operatorname{FFD}(A)$ is infinite, whence Gfc-gldim(A) is infinite by Theorem 3.2, and by Fact 3.1 one has Gfc-gldim$(A) \leq$ Gwgldim(A) . □

5.12 Corollary. *If A is $\aleph_0$-noetherian, then the following equality holds:*

$$\text{Gfc-gldim}(A) = \text{Gfc-gldim}(A^\circ)\,.$$

Proof. The assertion follows in view of [17, Cor. 2.5] from Theorem 5.11. □

In what remains of this section we derive some further consequences of the theorems above in combination with Theorems 1.18 and 3.2.

The next result is also obtained by Wang, Yang, Shao, and Zhang [48, Thm. 3.7], and it can easily be deduced from a result of El Maaouy [25, Thm. 4.16]. Both arguments are based on the PGF-modules introduced in [47]; our proof is different. For commutative rings the inequality is already known to hold from [10, Cor. 1.2], see also [26, Rmk. 5.4(ii)], and for right coherent rings it holds by [17, Cor. 3.5].

5.13 Theorem. *The following inequality holds:*

$$\mathrm{Gwgldim}(A) \le \mathrm{Ggldim}(A)\,.$$

Proof. One can assume that $\mathrm{Ggldim}(A) < \infty$ holds, in which case A is right weak coherent by [20, Exa. 4.4(3)]. Now combine Corollary 3.10 and Theorem 5.5. □

The next corollary applies, in particular, if A is left perfect.

5.14 Corollary. *If* $\mathrm{FFD}(A) = \mathrm{FPD}(A)$ *holds, then the following equalities hold:*

$$\text{Gfc-gldim}(A) = \mathrm{Gwgldim}(A) = \mathrm{Ggldim}(A)\,.$$

Proof. If $\mathrm{FFD}(A) = \mathrm{FPD}(A)$ holds, then Theorems 1.18 and 3.2 yield

$$\mathrm{Ggldim}(A) = \max\{\mathrm{sfli}(A), \mathrm{FPD}(A)\} = \max\{\mathrm{sfli}(A), \mathrm{FFD}(A)\} = \text{Gfc-gldim}(A)\,.$$

If this quantity is infinite, then Fact 3.1 implies that $\mathrm{Gwgldim}(A)$ is infinite. If the quantity is finite, then Theorem 5.13 implies that $\mathrm{Gwgldim}(A)$ is finite and thus agrees with $\text{Gfc-gldim}(A)$ per Fact 3.1. □

If A is noetherian it is known from [7, Cor. 6.11] that $\mathrm{Ggldim}(A)$ and $\mathrm{Ggldim}(A^\circ)$ are simultaneously finite. In the terminology from [7, Def. 6.8] or [8, Def. VII.2.5] the final two theorems say that the Gorenstein property is symmetric for $\aleph_0$-noetherian rings and rings of cardinality $\aleph_n$ for some $n \ge 0$.

The next inequality is a counterpart in Gorenstein homological algebra to [39, Thm. 1], and the subsequent corollary is a counterpart to [39, Cor. 1].

5.15 Theorem. *If A is left $\aleph_0$-noetherian, then the next inequality holds:*

$$\mathrm{Ggldim}(A) \le \mathrm{Gwgldim}(A) + 1\,.$$

Proof. One can assume that $\mathrm{Gwgldim}(A) = n < \infty$ holds. Thus Fact 3.1 yields $\text{Gfc-gldim}(A) = n$. From Theorem 3.2 it now follows that the equalities $\mathrm{sfli}(A) = \mathrm{FFD}(A) = n$ hold. Since [28, Prop. 2.8] yields $\mathrm{FPD}(A) \le n + 1$, the desired inequality $\mathrm{Ggldim}(A) \le n + 1$ follows from Theorem 1.18. □

5.16 Corollary. *If A is $\aleph_0$-noetherian, then* $\mathrm{Ggldim}(A)$ *is finite if and only if* $\mathrm{Ggldim}(A^\circ)$ *is finite, and in that case the following inequality holds:*

$$|\,\mathrm{Ggldim}(A) - \mathrm{Ggldim}(A^\circ)| \le 1\,.$$

Proof. This follows from Theorem 5.15 via Theorem 5.13 and [17, Cor. 2.5]. □

5.17 Remark. Under the a priori stronger assumption that both $\mathrm{Ggldim}(A)$ and $\mathrm{Ggldim}(A^\circ)$ are finite, the inequality in the previous corollary was noticed in [26, Rmk. 5.4(iii)]. One could alternatively obtain Corollary 5.16 by combining Theorem 5.13 with [28, Thm. 2.9] and a result of Dalezios and Emmanouil [21, Prop. 5.3].

The last assertion in the next result is a counterpart in Gorenstein homological algebra to a result of Osofsky [44, Cor. 1.6]. As a countable ring is $\aleph_0$-noetherian, the case $n = 0$ is a special instance of Corollary 5.16.

5.18 Theorem. *If* $\mathrm{splf}(A)$ *and* $\mathrm{splf}(A^\circ)$ *are both finite, then* $\mathrm{Ggldim}(A)$ *is finite if and only if* $\mathrm{Ggldim}(A^\circ)$ *is finite, and in that case there is an inequality:*

$$|\,\mathrm{Ggldim}(A) - \mathrm{Ggldim}(A^\circ)| \le \max\{\mathrm{splf}(A), \mathrm{splf}(A^\circ)\}\,.$$

In particular, if A has cardinality $\aleph_n$ for an integer $n \ge 0$, then one has:

$$|\,\mathrm{Ggldim}(A) - \mathrm{Ggldim}(A^\circ)| \le n + 1\,.$$

Proof. By Corollary 3.10 and Corollary 5.10 there are inequalities

$$\begin{aligned}\operatorname{Ggldim}(A) &\le \text{Gfc-gldim}(A) + \operatorname{splf}(A)\\ &= \text{Gfc-gldim}(A^\circ) + \operatorname{splf}(A) \le \operatorname{Ggldim}(A^\circ) + \operatorname{splf}(A)\,,\end{aligned}$$

and by symmetry one has $\operatorname{Ggldim}(A^\circ) \le \operatorname{Ggldim}(A) + \operatorname{splf}(A^\circ)$. This proves the first inequality. For the last assertion recall that by a result of Simson [45] a ring of cardinality $\aleph_n$ has $\operatorname{splf}(A) \le n+1$ and $\operatorname{splf}(A^\circ) \le n+1$. □

Acknowledgment

We thank Ioannis Emmanouil for valuable comments on a draft of this paper.

References

[1] S. Tempest Aldrich, Edgar E. Enochs, Overtoun M. G. Jenda, and Luis Oyonarte, *Envelopes and covers by modules of finite injective and projective dimensions*, J. Algebra **242** (2001), no. 2, 447–459. MR1848954

[2] Lidia Angeleri Hügel, Dolors Herbera, and Jan Trlifaj, *Tilting modules and Gorenstein rings*, Forum Math. **18** (2006), no. 2, 211–229. MR2218418

[3] Maurice Auslander, *On the dimension of modules and algebras. III. Global dimension*, Nagoya Math. J. **9** (1955), 67–77. MR0074406

[4] Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155. MR1117631

[5] Silvana Bazzoni, Manuel Cortés-Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, Algebr. Represent. Theory **23** (2020), no. 5, 1861–1883. MR4140057

[6] Silvana Bazzoni, Michal Hrbek, and Leonid Positselski, *FP-pojective periodicity*, J. Pure Appl. Alg., 107497; 30 pp.

[7] Apostolos Beligiannis, *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization*, Comm. Algebra **28** (2000), no. 10, 4547–4596. MR1780017

[8] Apostolos Beligiannis and Idun Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207. MR2327478

[9] Driss Bennis, *A note on Gorenstein flat dimension*, Algebra Colloq. **18** (2011), no. 1, 155–161. MR2753665

[10] Driss Bennis and Najib Mahdou, *Global Gorenstein dimensions*, Proc. Amer. Math. Soc. **138** (2010), no. 2, 461–465. MR2557164

[11] Ladislav Bican, Robert El Bashir, and Edgar E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390. MR1832549

[12] Samir Bouchiba, *On Gorenstein flat dimension*, J. Algebra Appl. **14** (2015), no. 6, 1550096, 18 pp. MR3338092

[13] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999, With an appendix by David A. Buchsbaum, Reprint of the 1956 original. MR1731415

[14] Lars Winther Christensen, Nanqing Ding, Sergio Estrada, Jiangsheng Hu, Huanhuan Li, and Peder Thompson, *The singularity category of an exact category applied to characterize Gorenstein schemes*, Q. J. Math. **74** (2023), no. 1, 1–27. MR4571621

[15] Lars Winther Christensen, Sergio Estrada, Li Liang, Peder Thompson, Dejun Wu, and Gang Yang, *A refinement of Gorenstein flat dimension via the flat–cotorsion theory*, J. Algebra **567** (2021), 346–370. MR4159258

[16] Lars Winther Christensen, Sergio Estrada, and Peder Thompson, *Homotopy categories of totally acyclic complexes with applications to the flat–cotorsion theory*, Categorical, Homological and Combinatorial Methods in Algebra, Contemp. Math., vol. 751, Amer. Math. Soc., Providence, RI, 2020, pp. 99–118. MR4132086

[17] Lars Winther Christensen, Sergio Estrada, and Peder Thompson, *Gorenstein weak global dimension is symmetric*, Math. Nachr. **294** (2021), no. 11, 2121–2128. MR4371287

[18] Lars Winther Christensen, Sergio Estrada, and Peder Thompson, *Five theorems on Gorenstein global dimensions*, Algebra and coding theory, Contemp. Math., vol. 785, Amer. Math. Soc., [Providence], RI, [2023] ©2023, pp. 67–78. MR4586431

[19] Lars Winther Christensen, Anders Frankild, and Henrik Holm, *On Gorenstein projective, injective and flat dimensions—A functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279. MR2236602

[20] Manuel Cortés-Izurdiaga, *Products of flat modules and global dimension relative to $\mathcal{F}$-Mittag-Leffler modules*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4557–4571. MR3544508

[21] Georgios Dalezios and Ioannis Emmanouil, *Homological dimension based on a class of gorenstein flat modules*, C.R. Math. Acad. Sci. Paris, to appear. Preprint arXiv:2208.05692 [math.RA]; 20 pp.

[22] Nan Qing Ding and Jian Long Chen, *The flat dimensions of injective modules*, Manuscripta Math. **78** (1993), no. 2, 165–177. MR1202159

[23] Nanqing Ding and Jianlong Chen, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra **24** (1996), no. 9, 2963–2980. MR1396867

[24] Paul C. Eklof and Jan Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), no. 1, 41–51. MR1798574

[25] Rachid El Maaouy, *Model structures, n-Gorenstein flat modules and PGF dimension*, preprint arXiv:2302.12905 [math.RA]; 22 pp.

[26] Ioannis Emmanouil, *On the finiteness of Gorenstein homological dimensions*, J. Algebra **372** (2012), 376–396. MR2990016

[27] Ioannis Emmanouil and Ilias Kaperonis, *On K-absolutely pure complexes*, preprint (2022) available from http://users.uoa.gr/~emmanoui/research.html.

[28] Ioannis Emmanouil and Olympia Talelli, *On the flat length of injective modules*, J. Lond. Math. Soc. (2) **84** (2011), no. 2, 408–432. MR2835337

[29] Edgar E. Enochs, Alina Iacob, and Overtoun M. G. Jenda, *Closure under transfinite extensions*, Illinois J. Math. **51** (2007), no. 2, 561–569. MR2342674

[30] Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146

[31] Sergio Estrada, Alina Iacob, and Marco A. Pérez, *Model structures and relative Gorenstein flat modules and chain complexes*, Categorical, homological and combinatorial methods in algebra, Contemp. Math., vol. 751, Amer. Math. Soc., [Providence], RI, [2020] ©2020, pp. 135–175. MR4132088

[32] David J. Fieldhouse, *Character modules, dimension and purity*, Glasgow Math. J. **13** (1972), 144–146. MR0313319

[33] James Gillespie, *The flat model structure on* $\mathrm{Ch}(R)$, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. MR2052954

[34] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules. Volume 1*, extended ed., De Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2012, Approximations. MR2985554

[35] Dolors Herbera, *Definable classes and Mittag-Leffler conditions*, Ring theory and its applications, Contemp. Math., vol. 609, Amer. Math. Soc., Providence, RI, 2014, pp. 137–166. MR3204357

[36] Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. MR2038564

[37] M. Cortés Izurdiaga, S. Estrada, and P. A. Guil Asensio, *A model structure approach to the finitistic dimension conjectures*, Math. Nachr. **285** (2012), no. 7, 821–833. MR2924515

[38] Arun Vinayak Jategaonkar, *A counter-example in ring theory and homological algebra*, J. Algebra **12** (1969), 418–440. 0240131

[39] Christian U. Jensen, *On homological dimensions of rings with countably generated ideals*, Math. Scand. **18** (1966), 97–105. MR0207796

[40] Christian U. Jensen, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra **15** (1970), 151–166. MR0260839

[41] Weiqing Li, Liang Yan, and Dongdong Zhang, *Some results on Gorenstein (weak) global dimension of rings*, Comm. Algebra **51** (2023), no. 1, 264–275. MR4525298

[42] Lixin Mao and Nanqing Ding, *The cotorsion dimension of modules and rings*, Abelian groups, rings, modules, and homological algebra, Lect. Notes Pure Appl. Math., vol. 249, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 217–233. MR2229114

[43] Lixin Mao and Nanqing Ding, *Gorenstein FP-injective and Gorenstein flat modules*, J. Algebra Appl. **7** (2008), no. 4, 491–506. MR2442073
[44] Barbara L. Osofsky, *Upper bounds on homological dimensions*, Nagoya Math. J. **32** (1968), 315–322. MR0232805
[45] D. Simson, *A remark on projective dimension of flat modules*, Math. Ann. **209** (1974), 181–182. MR0344292
[46] Bo Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. MR0258888
[47] Jan Šaroch and Jan Šťovíček, *Singular compactness and definability for* Σ*-cotorsion and Gorenstein modules*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 23, 40 pp. MR4076700
[48] Junpeng Wang, Gang Yang, Qingyu Shao, and Xioaxiang Zhang, *On Gorenstein global and Gorenstein weak global dimensions*, Colloq. Math., to appear.
[49] Gang Yang and Zhongkui Liu, *Cotorsion pairs and model structures on* $\mathrm{Ch}(R)$, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 3, 783–797. MR2837480

L.W.C. Texas Tech University, Lubbock, TX 79409, U.S.A.
Email address: lars.w.christensen@ttu.edu
URL: http://www.math.ttu.edu/~lchriste

S.E. Universidad de Murcia, Murcia 30100, Spain
Email address: sestrada@um.es
URL: http://webs.um.es/sestrada

L.L. Lanzhou Jiaotong University, Lanzhou 730070, China
Email address: lliangnju@gmail.com
URL: https://sites.google.com/site/lliangnju

P.T. Mälardalen University, Västerås 72123, Sweden
Email address: peder.thompson@mdu.se
URL: https://sites.google.com/view/pederthompson

J.W. Northwest Normal University, Lanzhou 730070, China
Email address: wangjunpeng1218@163.com