INJECTIVE MODULES UNDER FAITHFULLY FLAT RING EXTENSIONS

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Abstract. Let R be a commutative ring and S be an R-algebra. It is well-known that if N is an injective R-module, then $\operatorname{Hom}_R(S,N)$ is an injective S-module. The converse is not true, not even if R is a commutative noetherian local ring and S is its completion, but it is close: It is a special case of our main theorem that in this setting, an R-module N with $\operatorname{Ext}_R^{\geqslant 0}(S,N)=0$ is injective if $\operatorname{Hom}_R(S,N)$ is an injective S-module.

Introduction

Faithfully flat ring extensions play a important role in commutative algebra: Any polynomial ring extension and any completion of a noetherian local ring is a faithfully flat extension. The topic of this paper is transfer of homological properties of modules along such extensions.

In this section, R is a commutative ring and S is a commutative R-algebra. It is well-known that if F is a flat R-module, then $S\otimes_R F$ is a flat S-module, and the converse is true if S is faithfully flat over R. If I is an injective R-module, then $S\otimes_R I$ need not be injective over S, but it is standard that $\operatorname{Hom}_R(S,I)$ is an injective S-module. Here the converse is not true, not even if S is faithfully flat over R: Let $(R,\mathfrak{m})$ be a regular local ring with $\mathfrak{m}$-adic completion $S\neq R$. The module $\operatorname{Hom}_R(S,R)$ is then zero—see e.g. Aldrich, Enoch, and Lopez-Ramos [1]—and hence an injective S-module, but R is not an injective R-module, as the assumption $S\neq R$ ensures that R is not artinian. In this paper, we get close to a converse with the following result.

Main Theorem. *Let R be noetherian and S be faithfully flat as an R-module; assume that every flat R-module has finite projective dimension. Let N be an R-module; if $\operatorname{Hom}_R(S,N)$ is an injective S-module and $\operatorname{Ext}_R^n(S,N)=0$ holds for all $n>0$, then N is injective.*

The result stated above follows from Theorem 1.7. The assumption of finite projective dimension of flat modules is satisfied by a wide selection of rings, including rings of finite Krull dimension and rings of cardinality at most $\aleph_n$ for some natural number n; see Gruson, Jensen et. al. [8, prop. 6], [10, thm. II.(3.2.6)], and [7, thm. 7.10]. The projective dimension of a direct sum of modules is the supremum of the projective dimensions of the summands. A direct sum of flat modules is flat, so

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the assumption implies that there is an upper bound d for the projective dimension of a flat module. Notice also that the condition of $\operatorname{Ext}_R^n(S,N)$ vanishing is finite in the sense that vanishing is trivial for n greater than the projective dimension of S.

* * *

The project we report on here is part of Köksal's dissertation work. While the question that started the project—when does injectivity of $\operatorname{Hom}_R(S,N)$ imply injectivity of N?—is natural, it was a result of Christensen and Sather-Wagstaff [5] that suggested that a non-trivial answer might be attainable. The main result in [5] is essentially the equivalent of our Main Theorem for the relative homological dimension known as *Gorenstein injective dimension*. That the result was obtained for the relative dimension before the absolute is already unusual; it is normally the absolute case that serves as a blueprint for the relative. In the end, our proof of the Main Theorem bears little resemblance with the arguments in [5], and we do not readily see how to employ our arguments in the setting of that paper.

1. Injective modules

In the balance of this paper, R is a commutative noetherian ring and S is a flat R-algebra. By an S-module we always mean a left S-module. For convenience, we recall a few basic facts that will be used throughout without further mention.

1.1. A tensor product of flat R-modules is a flat R-module. For every flat R-module F and every injective R-module I, the R-module $\operatorname{Hom}_R(F,I)$ is injective.

For every flat R-module F, the S-module $S \otimes_R F$ is flat, and every flat S-module is flat as an R-module. For every injective R-module I, the S-module $\operatorname{Hom}_R(S,I)$ is injective, and every injective S-module is injective as an R-module.

An R-module C is called *cotorsion* if one has $\operatorname{Ext}_R^1(F,C) = 0$ (equivalently, $\operatorname{Ext}_R^{>0}(F,C) = 0$) for every flat R-module F. It follows by Hom-tensor adjointness that $\operatorname{Hom}_R(F,C)$ is cotorsion whenever C is cotorsion and F is flat.

1.2. Under the sharpened assumption that S is faithfully flat, the exact sequence

$$0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0 \tag{1.2.1}$$

is pure. Another way to say this is that (1.2.1) is an exact sequence of flat R-modules; see [9, Theorems (4.74) and (4.85)].

We work mostly in the derived category $\mathsf{D}(R)$ whose objects are complexes of R-modules. The next paragraph fixes the necessary terminology and notation.

1.3. Complexes are indexed homologically, so that the ith differential of a complex M is written $\partial_i^M : M_i \to M_{i-1}$. A complex M is called *bounded above* if $M_v = 0$ holds for all $v \gg 0$, *bounded below* if $M_v = 0$ holds for all $v \ll 0$, and *bounded* if it is bounded above and below. Brutal *truncations* of a complex M are denoted $M_{\leqslant n}$ and $M_{\geqslant n}$, and good truncations are denoted $M_{\subset n}$ and $M_{\supset n}$; cf. Weibel [11, 1.2.7].

A complex M is *acyclic* if one has $\mathrm{H}(M) = 0$, equivalently $M \cong 0$ in $\mathsf{D}(R)$. Finally, $\mathbf{R}\operatorname{Hom}_R(-,-)$ denotes the right derived homomorphism functor, and $- \otimes_R^{\mathbf{L}} -$ denotes the left derived tensor product functor.

The proof of Theorem 1.7 passes through a couple of reductions; the first one is facilitated by the next lemma.

1.4 Lemma. *Let N be an R-module of finite injective dimension. If S is faithfully flat, $\operatorname{Hom}_R(S,N)$ is an injective R-module, and $\operatorname{Ext}^n_R(S,N)=0$ holds for all $n>0$, then N is injective.*

Proof. Let i be the injective dimension of N. There exists then an R-module T such that $\operatorname{Ext}^i_R(T,N)\neq 0$. Let E be an injective envelope of T. The exact sequence $0\to T\to E\to X\to 0$ induces an exact sequence of cohomology modules:

$$\cdots\longrightarrow \operatorname{Ext}^i_R(E,N)\longrightarrow \operatorname{Ext}^i_R(T,N)\longrightarrow \operatorname{Ext}^{i+1}_R(X,N)\longrightarrow\cdots.$$

Since $\operatorname{Ext}^{i+1}_R(X,N)=0$ while $\operatorname{Ext}^i_R(T,N)\neq 0$, we conclude that also $\operatorname{Ext}^i_R(E,N)$ is non-zero. Now apply the functor $-\otimes_R E$ to the pure exact sequence (1.2.1) to get the following exact sequence of R-modules

$$0\longrightarrow E\longrightarrow S\otimes_R E\longrightarrow S/R\otimes_R E\longrightarrow 0\,.$$

As E is injective the sequence splits, whence E is a direct summand of the module $S\otimes_R E$. This implies $\operatorname{Ext}^i_R(S\otimes_R E,N)\neq 0$. On the other hand, for every $n>0$ one has

$$\begin{aligned}\operatorname{Ext}^n_R(S\otimes_R E,N)&\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(S\otimes^{\mathbf{L}}_R E,N))\\&\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(E,\mathbf{R}\operatorname{Hom}_R(S,N)))\\&\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(E,\operatorname{Hom}_R(S,N)))\\&\cong \operatorname{Ext}^n_R(E,\operatorname{Hom}_R(S,N))\,,\end{aligned}$$

where the first isomorphism uses that S is flat, the second is Hom-tensor adjointness in the derived category, and the third follows by the vanishing of $\operatorname{Ext}^{>0}_R(S,N)$. As $\operatorname{Hom}_R(S,N)$ is injective, this forces $i=0$; that is, N is injective. ☐

1.5. Let $\operatorname{Spec} R$ be the set of prime ideals in R; for $\mathfrak{p}\in\operatorname{Spec} R$ set $\kappa(\mathfrak{p})=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. To an R-complex X one associates two subsets of $\operatorname{Spec} R$. The (small) *support*, as introduced by Foxby [6], is the set $\operatorname{supp}_R X=\{\mathfrak{p}\in\operatorname{Spec} R\mid \operatorname{H}(\kappa(\mathfrak{p})\otimes^{\mathbf{L}}_R X)\neq 0\}$, and the *cosupport*, as introduced by Benson, Iyengar and Krause [4], is the set $\operatorname{cosupp}_R X=\{\mathfrak{p}\in\operatorname{Spec} R\mid \operatorname{H}(\mathbf{R}\operatorname{Hom}_R(\kappa(\mathfrak{p}),X))\neq 0\}$. A complex X is acyclic if and only if $\operatorname{supp}_R X$ is empty if and only if $\operatorname{cosupp}_R X$ is empty; see [6, (proof of) lem.2.6] and [4, thm. 4.13]. The derived category $\mathsf{D}(R)$ is stratified by R in the sense of [3], see 4.4 *ibid.*, so [4, thm. 9.5] yields for R-complexes X and Y:

$$\operatorname{cosupp}_R \mathbf{R}\operatorname{Hom}_R(Y,X)=\operatorname{supp}_R Y\cap\operatorname{cosupp}_R X\,.$$

If S is faithfully flat over R then, evidently, one has $\operatorname{supp}_R S=\operatorname{Spec} R$. In this case an R-complex X is acyclic if $\mathbf{R}\operatorname{Hom}_R(S,X)$ is acyclic.

1.6 Lemma. *Let I be an acyclic complex of injective R-modules. Assume that S is faithfully flat and of finite projective dimension over R. If $\operatorname{Hom}_R(S,I)$ is acyclic and $\operatorname{Hom}_R(S,\operatorname{Ker}\partial^I_n)$ is an injective R-module for every $n\in\mathbb{Z}$, then $\operatorname{Hom}_R(M,I)$ is acyclic for every R-module M.*

Proof. Let M be an R-module; in view of 1.5 it is sufficient to show that the complex $\mathbf{R}\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,I))$ is acyclic. Set $d=\operatorname{pd}_R S$ and let $\pi\colon P\to S$ be a projective resolution with $P_i=0$ for all $i>d$. To see that the homology $\operatorname{H}(\mathbf{R}\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,I)))\cong\operatorname{H}(\operatorname{Hom}_R(P,\operatorname{Hom}_R(M,I)))$ is zero, note first that there is an isomorphism

$$\operatorname{Hom}_R(P,\operatorname{Hom}_R(M,I))\;\cong\;\operatorname{Hom}_R(M,\operatorname{Hom}_R(P,I))\,.$$

Fix $m \in \mathbb{Z}$; the truncated complex $J = I_{\leqslant m+d+1}$ is a bounded above complex of injective R-modules, and so is $\operatorname{Hom}_R(P, J)$. It follows that the induced morphism $\operatorname{Hom}_R(\pi, J)$ is a homotopy equivalence; see [11, lem. 10.4.6]. This explains the first isomorphism in the next display. The second isomorphism, like the equality, is immediate from the definition of Hom. The complex $H = \operatorname{Hom}_R(S, I_{\subset m+d+1})$ is acyclic, as $\operatorname{Hom}_R(S, I)$ is acyclic by assumption and $\operatorname{Hom}_R(S, -)$ is left exact. By assumption $\operatorname{Hom}_R(S, \operatorname{Ker} \partial^I_{m+d+1})$ is injective, so H is a complex of injective modules; it is also bounded above, so it splits. It follows that $\operatorname{Hom}_R(M, H)$ is acyclic.

$$\begin{aligned} \mathrm{H}_m(\operatorname{Hom}_R(M, \operatorname{Hom}_R(P, I))) &= \mathrm{H}_m(\operatorname{Hom}_R(M, \operatorname{Hom}_R(P, I_{\leqslant m+d+1}))) \\ &\cong \mathrm{H}_m(\operatorname{Hom}_R(M, \operatorname{Hom}_R(S, I_{\leqslant m+d+1}))) \\ &\cong \mathrm{H}_m(\operatorname{Hom}_R(M, \operatorname{Hom}_R(S, I_{\subset m+d+1}))) \\ &= 0\,. \end{aligned}$$

□

1.7 Theorem. *Let R be a commutative noetherian ring over which every flat module has finite projective dimension. Let N be an R-module and S be a faithfully flat R-algebra; the following conditions are equivalent.*

(*i*) *N is injective.*

(*ii*) $\operatorname{Hom}_R(S, N)$ *is an injective R-module and* $\operatorname{Ext}^n_R(S, N) = 0$ *holds for all $n > 0$.*

(*iii*) $\operatorname{Hom}_R(S, N)$ *is an injective S-module and* $\operatorname{Ext}^n_R(S, N) = 0$ *holds for all $n > 0$.*

Proof. It is well-known that (*i*) implies (*iii*) implies (*ii*), so we need to show that (*i*) follows from (*ii*). Let $N \to E$ be an injective resolution, then $\operatorname{Hom}_R(S, E)$ is a complex of injective R-modules. By assumption $\mathrm{H}_n(\operatorname{Hom}_R(S, E))$ is zero for $n < 0$, so $\operatorname{Hom}_R(S, E)$ is an injective resolution of the module $\operatorname{Hom}_R(S, N)$, which is injective by assumption. It follows that the co-syzygies

$$\operatorname{Ker} \partial_n^{\operatorname{Hom}_R(S,E)} = \operatorname{Hom}_R(S, \operatorname{Ker} \partial^E_n)$$

are injective for all $n \leqslant 0$. As remarked in the Introduction, there is an upper bound d for the projective dimension of a flat R-module. Set $K = \operatorname{Ker} \partial^E_{-d}$; by Lemma 1.4 it is sufficient to show that K is injective. The complex $J = \Sigma^d(E_{\leqslant -d})$ is an injective resolution of K, so we need to show that $\operatorname{Ext}^1_R(M, K) = \mathrm{H}_{-1}(\operatorname{Hom}_R(M, J))$ is zero for every R-module M.

For every flat R-module F and all $i > 0$ one has $\operatorname{Ext}^i_R(F, K) \cong \operatorname{Ext}^{i+d}_R(F, N) = 0$ by dimension shifting; that is, K is cotorsion. For every $i > 0$ the i-fold tensor product $(S/R)^{\otimes i}$ is a flat R-module, and we set $(S/R)^{\otimes 0} = R$. Let η denote the pure exact sequence (1.2.1); splicing together the exact sequences of flat modules $\eta \otimes_R (S/R)^{\otimes i}$ for $i \geqslant 0$ one gets an acyclic complex

$$G = 0 \to R \to S \to S \otimes_R S/R \to S \otimes_R (S/R)^{\otimes 2} \to \cdots \to S \otimes_R (S/R)^{\otimes i} \to \cdots$$

concentrated in non-positive degrees. As K is cotorsion, the functor $\operatorname{Hom}_R(-, K)$ leaves each sequence $\eta \otimes_R (S/R)^{\otimes i}$ exact, so the complex $\operatorname{Hom}_R(G, K)$ is acyclic. For every $n > 0$, the R-module

$$\operatorname{Hom}_R(G, K)_n = \operatorname{Hom}_R(S \otimes_R (S/R)^{\otimes n-1}, K) \cong \operatorname{Hom}_R((S/R)^{\otimes n-1}, \operatorname{Hom}_R(S, K))$$

is injective; indeed, $\operatorname{Hom}_R(S, K)$ is injective and $(S/R)^{\otimes n-1}$ is flat. Moreover, one has $\operatorname{Hom}_R(G, K)_0 \cong K$, so the complexes $\operatorname{Hom}_R(G, K)_{\geqslant 1}$ and J splice together to yield an acyclic complex I of injective R-modules.

We argue that Lemma 1.6 applies to I. For $n < 0$ one has $\mathrm{H}_n(\mathrm{Hom}_R(S,I)) = \mathrm{H}_n(\mathrm{Hom}_R(S,J)) = \mathrm{H}_{n-d}(\mathrm{Hom}_R(S,E)) = 0$, and the module $\mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^I_n) = \mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^E_{n-d})$ is injective. For $n \geqslant 0$ one has

$$\mathrm{Ker}\,\partial^I_n = \mathrm{Hom}_R(\mathrm{Im}\,\partial^G_{-n},K) = \mathrm{Hom}_R((S/R)^{\otimes n},K)\ .$$

Since K is cotorsion and $(S/R)^{\otimes n}$ is flat, the module $\mathrm{Ker}\,\partial^I_n$ is cotorsion. The truncated complex $I_{\leqslant n+1}$ is an injective resolution of the module $\mathrm{Ker}\,\partial^I_{n+1}$, so for all $n \geqslant 0$ one has $\mathrm{H}_n(\mathrm{Hom}_R(S,I)) = \mathrm{Ext}^1_R(S,\mathrm{Ker}\,\partial^I_{n+1}) = 0$. Furthermore, the R-module $\mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^I_n) \cong \mathrm{Hom}_R((S/R)^{\otimes n},\mathrm{Hom}_R(S,K))$ is injective.

Now it follows from Lemma 1.6 that $\mathrm{Hom}_R(M,I)$ is acyclic for every R-module M; in particular, one has $\mathrm{H}_{-1}(\mathrm{Hom}_R(M,J)) = \mathrm{H}_{-1}(\mathrm{Hom}_R(M,I)) = 0$. □

2. Injective dimension

To draw the immediate consequences of our theorem, we need some terminology.

2.1. An R-complex I is *semi-injective* if it is a complex of injective R-modules and the functor $\mathrm{Hom}_R(-,I)$ preserves acyclicity. A *semi-injective resolution* of an R-complex N is a semi-injective complex I that is isomorphic to N in $\mathsf{D}(R)$. If N is a module, then an injective resolution of N is a semi-injective resolution in this sense. The *injective dimension* of an R-complex N is denoted $\mathrm{id}_R\,N$ and defined as

$$\mathrm{id}_R\,N = \inf\left\{ i\in\mathbb{Z} \;\middle|\; \begin{array}{c}\text{There is a semi-injective resolution}\\ I \text{ of } N \text{ with } I_n = 0 \text{ for all } n < -i\end{array}\right\}\ ;$$

see [2, 2.4.I], where “DG-injective” is the same as “semi-injective”.

2.2 Theorem. *Let R be a commutative noetherian ring over which every flat module has finite projective dimension, and let S be a flat R-algebra. For every R-complex N there are inequalities*

$$\mathrm{id}_R\,N \geqslant \mathrm{id}_S\,\mathbf{R}\mathrm{Hom}_R(S,N) \geqslant \mathrm{id}_R\,\mathbf{R}\mathrm{Hom}_R(S,N)\ ,$$

and equalities hold if S is faithfully flat.

Proof. Let N be an R-complex and let I be a semi-injective resolution of N. In $\mathsf{D}(S)$ there is an isomorphism $\mathbf{R}\mathrm{Hom}_R(S,N) \cong \mathrm{Hom}_R(S,I)$. It follows by Hom-tensor adjointness that $\mathrm{Hom}_R(S,I)$ is a semi-injective S-complex, whence the left-hand inequality holds. As S is flat over R, Hom-tensor adjointness also shows that every semi-injective S-complex is semi-injective over R. In particular, any semi-injective resolution of $\mathbf{R}\mathrm{Hom}_R(S,N)$ over S is a semi-injective resolution over R, and the second inequality follows.

Assume now that S is faithfully flat and that $\mathrm{id}_R\,\mathbf{R}\mathrm{Hom}_R(S,N) \leqslant i$ holds for some integer i. Let I be a semi-injective resolution of N; our first step is to prove that the R-module $K = \mathrm{Ker}\,\partial^I_{-i}$ is injective. As $\mathrm{Hom}_R(S,-)$ is left exact one has

$$\mathrm{Ker}\,\partial^{\mathrm{Hom}_R(S,I)}_{-i} \;\cong\; \mathrm{Hom}_R(S,K)\ .$$

In $\mathsf{D}(R)$ there is an isomorphism $\mathrm{Hom}_R(S,I) \cong \mathbf{R}\mathrm{Hom}_R(S,N)$, and by previous arguments the R-complex $\mathrm{Hom}_R(S,I)$ is semi-injective. It now follows from [2, 2.4.I] that the R-module $\mathrm{Hom}_R(S,K)$ is injective, and the truncated complex $\mathrm{Hom}_R(S,I)_{\supset -i} = \mathrm{Hom}_R(S,I_{\supset -i})$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(S,N)$ in $\mathsf{D}(R)$. In particular, one has

$$\mathrm{Ext}^n_R(S,K) = \mathrm{H}_{-n}(\mathrm{Hom}_R(S,\Sigma^i(I_{\leqslant -i}))) = \mathrm{H}_{-i-n}(\mathbf{R}\mathrm{Hom}_R(S,N)) = 0$$

for all $n > 0$, so K is injective by Theorem 1.7.

To conclude that N has injective dimension at most i, it is now sufficient to show that $\mathrm{H}_n(N) = 0$ holds for all $n < -i$; see [2, 2.4.I]. Let X be the cokernel of the embedding $\iota\colon I_{\supset -i} \to I$; the sequence $0 \to I_{\supset -i} \to I \to X \to 0$ is a degree-wise split exact sequence of complexes of injective modules. In the induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(S, I_{\supset -d}) \longrightarrow \mathrm{Hom}_R(S, I) \longrightarrow \mathrm{Hom}_R(S, X) \longrightarrow 0\ ,$$

the embedding is a homology isomorphism, so $\mathrm{Hom}_R(S,X)$ is acyclic. As X is a bounded above complex of injective modules, it is semi-injective. That is, the complex $\mathbf{R}\mathrm{Hom}_R(S,X)$ is acyclic, and then it follows that X is acyclic; see 1.5. Thus ι is a quasi-isomorphism, whence one has $\mathrm{H}_n(N) = \mathrm{H}_n(I) = 0$ for all $n < -i$. □

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