

THE IMPROVED NEW INTERSECTION THEOREM REVISITED

LARS WINTHER CHRISTENSEN AND LUIGI FERRARO

ABSTRACT. We prove a generalized version of Evans and Griffith's Improved New Intersection Theorem: Let I be an ideal in a local ring R . If a finite free R -complex, concentrated in nonnegative degrees, has I -torsion homology in positive degrees, and the homology in degree 0 has an I -torsion minimal generator, then the length of the complex is at least $\dim R - \dim R/I$. This improves the bound $\text{ht } I$ obtained by Avramov, Iyengar, and Neeman in 2018.

INTRODUCTION

In its various forms, the The New Intersection Theorem is concerned with the length of a finite free complex, that is, a complex

$$F: 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

of finitely generated free modules, over a local ring (R, \mathfrak{m}) . The classic version, due to Peskine and Szpiro, [14] asserts that if $H(F)$ is non-zero and each homology module $H_i(F)$ is of finite length, then $n \geq \dim R$ holds. The statement known as the Improved New Intersection Theorem was first established within the proof of Evans and Griffith's Syzygy Theorem [6]. Hochster states it in [11] as follows:

- (1) If the homology modules $H_i(F)$ for $i > 0$ have finite length and a nonzero minimal generator of $H_0(F)$ generates a submodule of finite length, then $n \geq \dim R$ holds.

A slightly stronger statement was obtained by Iyengar [12, Theorem 3.1]:

- (2) If the modules $H_i(F)$ for $i > 0$ are of finite length and an ideal I annihilates a nonzero minimal generator of $H_0(F)$, then $n \geq \dim R - \dim R/I$ holds.

We notice that under the assumptions in (1), some power of \mathfrak{m} annihilates $H_i(F)$ for all $i > 0$ as well as a minimal generator of $H_0(F)$. The original statements in [6, 12, 14] were made for equicharacteristic rings. The New Intersection Theorem was proved in mixed characteristics by Roberts [15] and, through the work of André [2], the remaining statements are now also known to hold for all local rings.

The original Improved New Intersection Theorem was generalized by Avramov, Iyengar, and Neeman [3] as follows:

- (3) If an ideal I annihilates the homology modules $H_i(F)$ for $i > 0$ as well as a nonzero minimal generator of $H_0(M)$, then $n \geq \text{ht } I$ holds.

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One always has $\dim R - \dim R/I \geq \text{ht } I$, so the bound in (3) is weaker than the bound in (2), but so are the assumptions on $H(F)$. The main result of this paper is a common generalization of these last two statements:

- (4) If an ideal I annihilates the homology modules $H_i(F)$ for $i > 0$ as well as a nonzero minimal generator of $H_0(F)$, then $n \geq \dim R - \dim R/I$ holds.

If the homology modules $H_i(F)$ for $i > 0$ are I -torsion, then they are all annihilated by some fixed power I^n and one has $\dim R/I^n = \dim R/I$, so (4) is equivalent to the statement made in the abstract. Finally, we notice that for an \mathfrak{m} -primary ideal I the statements (2)–(4) reduce to the original statement (1).

The work of André mentioned above proved the existence of big Cohen-Macaulay modules over any local ring, and it had already been established that the existence of such modules was sufficient to prove the Improved New Intersection Theorem, see Hochster [10] and Iyengar [12]. The proof of our main result, which is Theorem 2.2, is inspired by a more recent proof of (2) by Iyengar, Ma, Schwede, and Walker [13]. Our twist comes down to controlling the depth of derived \mathfrak{m} -complete complexes.

1. DERIVED COMPLETE COMPLEXES

Throughout the paper, R is a commutative noetherian local ring with unique maximal ideal \mathfrak{m} and residue field k . For a finitely generated R -module M and a prime ideal \mathfrak{p} , Bass' [4, Lemma (3.1)] yields the inequality $\text{depth}_R M \leq \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}$. The main result of this section, which is key to our proof of Theorem 2.2, is that the same inequality holds for derived \mathfrak{m} -complete R -complexes.

1.1. We use homological notation, i.e. lower indexing, for R -complexes. For an R -complex M , the homological supremum and infimum are

$$\sup M = \sup\{n \in \mathbb{Z} \mid H_n(M) \neq 0\} \quad \text{and} \quad \inf M = \inf\{n \in \mathbb{Z} \mid H_n(M) \neq 0\}.$$

1.2. Let I be an ideal in R . As is standard we denote the right derived I -torsion functor by $\text{R}\Gamma_I$ and the left derived I -completion functor by $\text{L}\Lambda^I$. They are adjoint functors, see Alonso Tarrío, Jeremías Lopez, and Lipman [1, Theorem (0.3)], and an R -complex M is called derived I -torsion or derived I -complete if it is isomorphic in the derived category to $\text{R}\Gamma_I(M)$ or $\text{L}\Lambda^I(M)$, respectively. For an R -complex M the vanishing of local (co)homology, i.e. $H(\text{R}\Gamma_I M)$ and $H(\text{L}\Lambda^I M)$ detects, or if one wishes defines, the depth and width invariants relative to I :

$$(1.2.1) \quad \text{depth}_R(I, M) = -\sup \text{RHom}_R(R/I, M) = -\sup \text{R}\Gamma_I(M)$$

$$(1.2.2) \quad \text{width}_R(I, M) = \inf(R/I \otimes_R^L M) = \inf \text{L}\Lambda^I(M);$$

see Foxby and Iyengar [9, Theorem 2.1 and Theorem 4.1]. From these equalities and standard inequalities regarding homological suprema and infima, see Foxby [7, Lemma 2.1], one gets:

$$(1.2.3) \quad \text{depth}(I, M) \geq -\sup M \text{ with equality if and only if } \Gamma_I(H_{\sup M}(M)) \neq 0.$$

$$(1.2.4) \quad \text{width}(I, M) \geq \inf M \text{ with equality if and only if } \Lambda^I(H_{\inf M}(M)) \neq 0.$$

Vanishing of local cohomology supported at the maximal ideal also detects the dimension of a finitely generated R -module:

$$(1.2.5) \quad \dim_R M = -\inf \text{R}\Gamma_{\mathfrak{m}}(M).$$

The next lemma is folklore—Foxby and Iyengar allude to it in the text preceding [9, Proposition 2.2]—but we didn't find a reference to cite.

Lemma 1.3. *Let I be an ideal in R and M and N be R -complexes. If M is derived I -torsion with $H(M)$ nonzero and bounded below, then the next inequalities hold*

$$(a) \quad \inf(M \otimes_R^{\mathbf{L}} N) \geq \inf M + \text{width}_R(I, N) .$$

$$(b) \quad -\sup \text{RHom}_R(M, N) \geq \inf M + \text{depth}_R(I, N) .$$

Proof. To prove the inequality (a), we first observe that there are isomorphisms in the derived category as follows:

$$\begin{aligned} M \otimes_R^{\mathbf{L}} N &\simeq \text{R}\Gamma_I M \otimes_R^{\mathbf{L}} N \\ &\simeq M \otimes_R^{\mathbf{L}} \text{R}\Gamma_I N \\ &\simeq M \otimes_R^{\mathbf{L}} \text{R}\Gamma_I \mathbf{L}\Lambda^I N \\ &\simeq \text{R}\Gamma_I M \otimes_R^{\mathbf{L}} \mathbf{L}\Lambda^I N \\ &\simeq M \otimes_R^{\mathbf{L}} \mathbf{L}\Lambda^I N . \end{aligned}$$

Indeed, the first and last isomorphisms hold as M is derived I -torsion, the second and fourth isomorphism follow from [1, (2.1)], and the third isomorphism follows from [1, Corollary (5.1.1)]. This justifies the first equality in the following chain of (in)equalities

$$\begin{aligned} \inf(M \otimes_R^{\mathbf{L}} N) &= \inf(M \otimes_R^{\mathbf{L}} \mathbf{L}\Lambda^I N) \\ &\geq \inf M + \inf \mathbf{L}\Lambda^I N \\ &= \inf M + \text{width}_R(I, N) ; \end{aligned}$$

here the inequality holds by [7, Lemma 2.1] and (1.2.2) yields the last equality.

To prove the inequality (b), we first observe that the following chain of isomorphisms in the derived category holds:

$$\begin{aligned} \text{RHom}_R(M, N) &\simeq \text{RHom}_R(\text{R}\Gamma_I M, N) \\ &\simeq \text{RHom}_R(M, \mathbf{L}\Lambda^I N) \\ &\simeq \text{RHom}_R(M, \mathbf{L}\Lambda^I \text{R}\Gamma_I N) \\ &\simeq \text{RHom}_R(\text{R}\Gamma_I M, \text{R}\Gamma_I N) \\ &\simeq \text{RHom}_R(M, \text{R}\Gamma_I N) . \end{aligned}$$

Indeed, the first and last isomorphisms hold as M is derived I -torsion, the second and fourth isomorphisms follows from the fact that $\text{R}\Gamma_I$ and $\mathbf{L}\Lambda^I$ are adjoint functors, see [1, Theorem (0.3)], and the third isomorphism follows from [1, Corollary (5.1.1)]. This explains the first equality in the following chain of (in)equalities:

$$\begin{aligned} -\sup \text{RHom}_R(M, N) &= -\sup \text{RHom}_R(M, \text{R}\Gamma_I N) \\ &\geq \inf M - \sup \text{R}\Gamma_I N \\ &= \inf M + \text{depth}_R(I, N) ; \end{aligned}$$

here the inequality holds by [7, Lemma 2.1], and (1.2.2) yields the last equality. \square

Theorem 1.4. *Let M be a derived \mathfrak{m} -complete R -complex. For every prime ideal \mathfrak{p} in R there is an inequality*

$$\text{depth}_R M \leq \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} .$$

Proof. The claim follows from the following chain of (in)equalities

$$\begin{aligned}
\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &\geq \text{depth}_R(\mathfrak{p}, M) \\
&= -\sup \text{RHom}_R(R/\mathfrak{p}, M) \\
&= -\sup \text{RHom}_R(R/\mathfrak{p}, \mathbf{L}\Lambda^{\mathfrak{m}} M) \\
&= -\sup \text{RHom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}(R/\mathfrak{p}), M) \\
&\geq \inf \mathbf{R}\Gamma_{\mathfrak{m}}(R/\mathfrak{p}) + \text{depth}_R M \\
&= \text{depth}_R M - \dim R/\mathfrak{p},
\end{aligned}$$

where the first inequality holds by [9, Proposition 2.10], the first equality is part of (1.2.1), the second equality follows from the hypothesis that M is derived \mathfrak{m} -complete, the third equality holds as $\mathbf{L}\Lambda$ and $\mathbf{R}\Gamma$ are adjoint functors, the last inequality holds by Lemma 1.3, and (1.2.5) yields the last equality. \square

Corollary 1.5. *Let M be a derived \mathfrak{m} -complete R -complex. For every ideal I in R there is an inequality*

$$\text{depth}_R M \leq \text{depth}_R(I, M) + \dim R/I.$$

Proof. From [9, Proposition 2.10] one gets

$$\text{depth}_R(I, M) = \inf\{\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in V(I)\}.$$

Therefore, $\text{depth}_R(I, M) = \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ holds for some $\mathfrak{p} \in V(I)$, and now the asserted inequality follows from Theorem 1.4 as $\dim R/\mathfrak{p} \leq \dim R/I$ holds. \square

Notice that for a prime ideal I the inequality in Corollary 1.5 may be stronger than the inequality in Theorem 1.4.

2. AN IMPROVED NEW INTERSECTION THEOREM

We now get to the main result of the paper.

2.1. We recall from [13] that an R -complex of maximal depth is a complex M satisfying the following three conditions:

- (1) $H(M)$ is bounded;
- (2) The canonical map $H_0(M) \rightarrow H_0(k \otimes_R^{\mathbf{L}} M)$ is nonzero;
- (3) $\text{depth}_R M = \dim R$.

The obvious example of a complex of maximal depth is a big Cohen-Macaulay module, and such modules exist over every local ring. The interest in complexes derives from the fact that homological conjectures—The Canonical Element Conjecture to be specific—in the presence of a dualizing complex implies the existence of complexes of maximal depth with degreewise finitely generated homology, see [13, Remarks 4.7 and 4.15].

Theorem 2.2. *Let I be an ideal in R and*

$$F: 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

a finite free R -complex with $H_0(F) \neq 0$. If $H_i(F)$ is I -torsion for $i > 0$ and a minimal generator of $H_0(F)$ is I -torsion, then $n \geq \dim R - \dim R/I$ holds.

Proof. Let M be a derived \mathfrak{m} -complete complex of maximal depth; such a complex exists by [13, Lemma 3.4]. Let s be the integer $\sup(F \otimes_R M)$ and notice from [13, Lemma 3.1] that one has that $s \geq 0$.

Let \mathfrak{p} be in $\text{Ass}_R H_s(F \otimes_R M)$. It follows that $H(F \otimes_R M)_{\mathfrak{p}}$ is nonzero and, hence, $H(F)_{\mathfrak{p}}$ and $H(M)_{\mathfrak{p}}$ are nonzero as well. We have the following chain of (in)equalities

$$\begin{aligned}
 \text{proj. dim}_{R_{\mathfrak{p}}} F_{\mathfrak{p}} &= \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} (F \otimes_R M)_{\mathfrak{p}} \\
 &= \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + s \\
 (2.2.1) \quad &\geq \text{depth}_R M - \dim R/\mathfrak{p} + s \\
 &= \dim R - \dim R/\mathfrak{p} + s,
 \end{aligned}$$

where the first equality is the Auslander-Buchsbaum equality, see [9, Theorem 2.4], the second equality follows from (1.2.3), the inequality holds by Theorem 1.4, and the last equality holds as M is a complex of maximal depth.

Assume first that $s \geq 1$ holds. In this case it suffices to show that I is contained in \mathfrak{p} as one then has,

$$n \geq \text{proj. dim}_R F \geq \text{proj. dim}_{R_{\mathfrak{p}}} F_{\mathfrak{p}} \geq \dim R - \dim R/\mathfrak{p} + s > \dim R - \dim R/I.$$

To see that \mathfrak{p} contains I , assume towards a contradiction that $I \not\subseteq \mathfrak{p}$. It follows that $F_{\mathfrak{p}}$ is isomorphic to $H_0(F)_{\mathfrak{p}}$ in the derived category, as $H_i(F)$ is I -torsion for $i \geq 1$ and, therefore, $\sup F_{\mathfrak{p}} = 0$. One now has the following chain of (in)equalities

$$\begin{aligned}
 \text{depth } R_{\mathfrak{p}} &= \text{depth } F_{\mathfrak{p}} + \text{proj. dim } F_{\mathfrak{p}} \\
 &\geq \text{proj. dim } F_{\mathfrak{p}} \\
 &\geq \dim R - \dim R/\mathfrak{p} + s \\
 &\geq \dim R_{\mathfrak{p}} + s,
 \end{aligned}$$

which is absurd as s is positive. The equality in the display above is the Auslander-Buchsbaum equality, see [9, Theorem 2.4], the first inequality is trivial, the second follows from (2.2.1), and the last inequality is standard.

It remains to consider the case $s = 0$. It follows from the finite generation of $H_0(F)$, Nakayama's Lemma, and [13, Lemma 3.1] that each minimal generator of $H_0(F)$ gives rise to a nonzero element in $H_0(F \otimes_R M)$. Thus, by hypothesis, there is an I -torsion element of $H_0(F \otimes_R M)$, i.e. $\Gamma_I(H_0(F \otimes_R M)) \neq 0$ and, therefore,

$$\text{depth}_R(I, F \otimes_R M) = -\sup(F \otimes_R M) = -s = 0$$

by (1.2.3). By [13, (2.2)], the complex $F \otimes_R M$ is derived \mathfrak{m} -complete, therefore, applying Corollary 1.5, one gets

$$\text{depth}_R(F \otimes_R M) \leq \dim R/I.$$

It remains to apply the Auslander-Buchsbaum equality:

$$\text{proj. dim}_R F = \text{depth}_R M - \text{depth}_R(F \otimes_R M) \geq \dim R - \dim R/I. \quad \square$$

2.3. Recall from Foxby [8] that for an R -complex M the small support is the set

$$\text{supp}_R M = \{\mathfrak{p} \in \text{Spec } R \mid H(M \otimes_R^{\mathbb{L}} k(\mathfrak{p})) \neq 0\},$$

where $k(\mathfrak{p})$ denotes the residue field of the local ring $R_{\mathfrak{p}}$.

Remark 2.4. Let M be an R -complex with bounded homology and \mathfrak{p} a prime ideal in $\operatorname{supp}_R M$. There are inequalities,

$$(2.4.1) \quad \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim R_{\mathfrak{p}} - \operatorname{inf} M_{\mathfrak{p}} \leq \dim R - \dim R/\mathfrak{p} - \operatorname{inf} M_{\mathfrak{p}};$$

indeed, the first inequality holds by [8, Corollary 3.9], the second inequality follows from the definition of dimension of complexes, also from [8], and the third is standard. Thus, for a derived \mathfrak{m} -complete R -complex M of maximal depth it follows from Theorem 1.4 and (2.4.1) that the inequalities

$$\dim R - \dim R/\mathfrak{p} \leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim R - \dim R/\mathfrak{p} - \operatorname{inf} M_{\mathfrak{p}},$$

hold for every prime ideal \mathfrak{p} in $\operatorname{supp}_R M$.

Theorem 2.5. *Let M be a derived \mathfrak{m} -complete R -module of maximal depth. For every prime ideal \mathfrak{p} in $\operatorname{supp}_R M$ the equality $\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$ holds and $M_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$ -module of maximal depth.*

Proof. Let \mathfrak{p} be a prime ideal in $\operatorname{supp}_R M$; as \mathfrak{p} in particular is in the support of M , the inequalities (2.4.1) read $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim R - \dim R/\mathfrak{p}$. The inequality from Theorem 1.4 can be rewritten as $\dim R - \dim R/\mathfrak{p} \leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Combining these inequalities one gets the equality $\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$ as well as $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It remains to see that $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p})$ is non-zero. Set $d = \dim R_{\mathfrak{p}}$ and let K be the Koszul complex on a sequence $\mathbf{x} = x_1, \dots, x_d$ of parameters for $R_{\mathfrak{p}}$. By [9, Definitions 2.3 and 4.3] one has $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = d - \sup(K \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$ and $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{inf}(K \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}})$. Therefore, one has

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = d - \sup(K \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) + \operatorname{inf}(K \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) \leq d.$$

This forces $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$, whence $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} k(\mathfrak{p}) \neq 0$ by (1.2.2) as desired. (The inequality displayed above is [18, Corollary 6.1.10], and Strooker credits Bartijn with observing it in his thesis.) \square

Remark 2.6. The \mathfrak{m} -adic completion of a big Cohen-Macaulay module is an example of a derived \mathfrak{m} -complete module of maximal depth. Indeed, such a module is a balanced big Cohen-Macaulay module, see Bruns and Herzog [5, Corollary 8.5.3], and derived \mathfrak{m} -complete, see for example Schenzel and Simon [16, Proposition 2.5.7(a), Example 7.3.2(d), and Theorem 7.5.13(a)]. For such modules the equality in Theorem 2.5 was proved by Sharp [17, Theorem 3.2], who called the set $\operatorname{supp}_R M$ the supersupport of M .

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TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409, U.S.A.

Email address: lars.w.christensen@ttu.edu

Email address: lferraro@ttu.edu