BRAUER–THRALL FOR TOTALLY REFLEXIVE MODULES

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Abstract. Let R be a commutative noetherian local ring that is not Gorenstein. It is known that the category of totally reflexive modules over R is representation infinite, provided that it contains a non-free module. The main goal of this paper is to understand how complex the category of totally reflexive modules can be in this situation.

Local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ are commonly regarded as the structurally simplest rings to admit diverse categorical and homological characteristics. For such rings we obtain conclusive results about the category of totally reflexive modules, modeled on the Brauer–Thrall conjectures. Starting from a non-free cyclic totally reflexive module, we construct a family of indecomposable totally reflexive R-modules that contains, for every $n \in \mathbb{N}$, a module that is minimally generated by n elements. Moreover, if the residue field $R/\mathfrak{m}$ is algebraically closed, then we construct for every $n \in \mathbb{N}$ an infinite family of indecomposable and pairwise non-isomorphic totally reflexive R-modules, each of which is minimally generated by n elements. The modules in both families have periodic minimal free resolutions of period at most 2.

Contents

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1. Introduction and synopsis of the main results

The representation theoretic properties of a local ring bear pertinent information about its singularity type. A notable illustration of this tenet is due to Herzog [10] and to Buchweitz, Greuel, and Schreyer [3]. They show that a complete local Gorenstein algebra is a simple hypersurface singularity if its category of maximal Cohen–Macaulay modules is representation finite. A module category is called *representation finite* if it comprises only finitely many indecomposable modules up to isomorphism. Typical examples of maximal Cohen–Macaulay modules over a Cohen–Macaulay local ring are high syzygies of finitely generated modules.

Over a Gorenstein local ring, all maximal Cohen–Macaulay modules arise as high syzygies, but over an arbitrary Cohen–Macaulay local ring they may not. Totally reflexive modules are infinite syzygies with special duality properties; the precise definition is given below. One reason to study these modules—in fact, the one discovered most recently—is that they afford a characterization of simple hypersurface singularities among all complete local algebras, i.e. without any *a priori* assumption of Gorensteinness. This extension of the result from [3, 10] is obtained in [5]. It is consonant with the intuition that the structure of high syzygies is shaped predominantly by the ring, and the same intuition guides this work.

The key result in [5] asserts that if a local ring is not Gorenstein and the category of totally reflexive modules contains a non-free module, then it is representation infinite. The main goal of this paper is to determine how complex the category of totally reflexive modules is when it is representation infinite. Our results suggest that it is often quite complex; Theorems (1.1) and (1.4) below are modeled on the Brauer–Thrall conjectures.

For a finite dimensional algebra A, the first Brauer–Thrall conjecture asserts that if the category of A-modules of finite length is representation infinite, then there exist indecomposable A-modules of arbitrarily large length. The second conjecture asserts that if the underlying field is infinite, and there exist indecomposable A-modules of arbitrarily large length, then there exist infinitely many integers d such that there are infinitely many indecomposable A-modules of length d. The first conjecture was proved by Roĭter (1968); the second conjecture has been verified, for example, for algebras over algebraically closed fields by Bautista (1985) and Bongartz (1985).

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In this section, R is a commutative noetherian local ring. The central questions addressed in the paper are: Assuming that the category of totally reflexive R-modules is representation infinite and given a non-free totally reflexive R-module, how does one construct an infinite family of pairwise non-isomorphic totally reflexive R-modules? And, can one control the size of the modules in the family in accordance with the Brauer–Thrall conjectures?

A finitely generated R-module M is called *totally reflexive* if there exists an infinite sequence of finitely generated free R-modules

$$F\colon \quad \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots,$$

such that M is isomorphic to the module $\operatorname{Coker}(F_1 \to F_0)$, and such that both F and the dual sequence $\operatorname{Hom}_R(F,R)$ are exact. These modules were first studied by Auslander and Bridger [1], who proved that R is Gorenstein if and only if every R-module has a totally reflexive syzygy. Over a Gorenstein ring, the totally reflexive modules are precisely the maximal Cohen–Macaulay modules, and these have been studied extensively. In the rest of this section we assume that R is not Gorenstein.

Every syzygy of an indecomposable totally reflexive R-module is, itself, indecomposable and totally reflexive; a proof of this folklore result is included in Section 2. Thus if one were given a totally reflexive module whose minimal free resolution is non-periodic, then the syzygies would form the desired infinite family; though one cannot exercise any control over the size of the modules in the family.

In practice, however, the totally reflexive modules that one typically spots have periodic free resolutions. To illustrate this point, consider the $\mathbb{Q}$-algebra

$$A = \mathbb{Q}[s,t,u,v]/(s^2,t^2,u^2,v^2,uv,2su+tu,sv+tv).$$

It has some easily recognizable totally reflexive modules—$A/(s)$ and $A/(s+u)$ for example—whose minimal free resolutions are periodic of period at most 2. It also has indecomposable totally reflexive modules with non-periodic free resolutions. However, such modules are significantly harder to recognize. In fact, when Gasharov and Peeva [8] did so, it allowed them to disprove a conjecture of Eisenbud.

The algebra A above has Hilbert series $1+4\tau+3\tau^2$. In particular, A is a local ring, and the third power of its maximal ideal is zero; informally we refer to such rings as *short.* For these rings, [14] gives a quantitative measure of how challenging it can be to recognize totally reflexive modules with non-periodic resolutions.

Short local rings are the structurally simplest rings that accommodate a wide range of homological behavior, and [8] and [14] are but two affirmations that such rings are excellent grounds for investigating homological questions in local algebra.

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For the rest of this section, assume that R is short and let $\mathfrak{m}$ be the maximal ideal of R. Note that $\mathfrak{m}^3$ is zero and set $e = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. A reader so inclined is welcome to think of a standard graded algebra with Hilbert series $1+e\tau+h_2\tau^2$.

The families of totally reflexive modules constructed in this paper start from cyclic ones. Over a short local ring, such modules are generated by elements with cyclic annihilators; Henriques and Şega [9] call these elements *exact zero divisors.* The ubiquity of exact zero divisors in short local algebras is a long-standing empirical fact. Its theoretical underpinnings are found in works of Conca [7] and Hochster and Laksov [11]; we extend them in Section 8.

The main results of this paper are Theorems (1.1)–(1.4). It is known from work of Yoshino [19] that the length of a totally reflexive R-module is a multiple of e. In Section 4 we prove the existence of indecomposable totally reflexive R-modules of every possible length:

(1.1) **Theorem (Brauer–Thrall I).** *If there is an exact zero divisor in R, then there exists a family $\{M_n\}_{n\in\mathbb{N}}$ of indecomposable totally reflexive R-modules with $\operatorname{length}_R M_n = ne$ for every n. Moreover, the minimal free resolution of each module M_n is periodic of period at most 2.*

Our proof of this result is constructive in the sense that we exhibit presentation matrices for the modules M_n; they are all upper triangular square matrices with exact zero divisors on the diagonal. Yet, the strong converse contained in Theorem (1.2) came as a surprise to us. It illustrates the point that the structure of the ring is revealed in high syzygies.

(1.2) **Theorem.** *If there exists a totally reflexive R-module without free summands, which is presented by a matrix that has a column or a row with only one non-zero entry, then that entry is an exact zero divisor in R.*

These two results—the latter of which is distilled from Theorem (5.3)—show that existence of totally reflexive modules of any size is related to the existence of exact zero divisors. One direction, however, is not unconditional, and in Section 9 we show that non-free totally reflexive R-modules may also exist in the absence of exact zero divisors in R. If this phenomenon appears peculiar, some consolation may be found in the next theorem, which is proved in Section 8. For algebraically closed fields, it can be deduced from results in [7, 11, 19].

(1.3) **Theorem.** *Let k be an infinite field, and let R be a generic standard graded k-algebra which (1) has Hilbert series $1 + h_1\tau + h_2\tau^2$, (2) is not Gorenstein, and (3) admits a non-free totally reflexive module. Then R has an exact zero divisor.*

If the residue field $R/\mathfrak{m}$ is infinite, and there is an exact zero divisor in R, then there are infinitely many different ones; this is made precise in Theorem (7.6). Together with a couple of other results from Section 7 this theorem yields:

(1.4) **Theorem (Brauer–Thrall II).** *If there is an exact zero divisor in R, and the residue field $\mathsf{k} = R/\mathfrak{m}$ is algebraically closed, then there exists for each $n \in \mathbb{N}$ a family $\{M_n^\lambda\}_{\lambda\in\mathsf{k}}$ of indecomposable and pairwise non-isomorphic totally reflexive R-modules with $\operatorname{length}_R M_n^\lambda = ne$ for every λ. Moreover, the minimal free resolution of each module M_n^λ is periodic of period at most 2.*

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The families of modules in Theorems (1.1) and (1.4) come from a construction that can provide such families over a general local ring, contingent on the existence of minimal generators of the maximal ideal with certain relations among them. This construction is presented in Section 2 and analyzed in Sections 3 and 6. To establish the Brauer–Thrall theorems, we prove in Sections 4 and 7 that the necessary elements and relations are available in short local rings with exact zero divisors. The existence of exact zero divisors is addressed in Sections 5, 8, and 9. In Section 10 we give another construction of infinite families of totally reflexive modules. It applies to certain rings of positive dimension, and it does not depend on the existence of exact zero divisors.

2. Totally acyclic complexes and exact zero divisors

In this paper, R denotes a commutative noetherian ring. Complexes of R-modules are graded homologically. A complex

$$F: \quad \cdots \longrightarrow F_{i+1} \xrightarrow{\partial^F_{i+1}} F_i \xrightarrow{\partial^F_i} F_{i-1} \longrightarrow \cdots$$

of finitely generated free R-modules is called *acyclic* if every cycle is a boundary; that is, the equality $\operatorname{Ker}\partial^F_i = \operatorname{Im}\partial^F_{i+1}$ holds for every $i \in \mathbb{Z}$. If both F and the dual complex $\operatorname{Hom}_R(F, R)$ are acyclic, then F is called *totally acyclic*. Thus an R-module is totally reflexive if and only if it is the cokernel of a differential in a totally acyclic complex.

The annihilator of an ideal $\mathfrak{a}$ in R is written $(0 : \mathfrak{a})$. For principal ideals $\mathfrak{a} = (a)$ we use the simplified notation $(0 : a)$.

Recall from [9] the notion of an exact zero divisor: a non-invertible element $x \neq 0$ in R is called an *exact zero divisor* if one of the following equivalent conditions holds.

(i) There is an isomorphism $(0 : x) \cong R/(x)$.

(ii) There exists an element w in R such that $(0 : x) = (w)$ and $(0 : w) = (x)$.

(iii) There exists an element w in R such that

$$\cdots \longrightarrow R \xrightarrow{w} R \xrightarrow{x} R \xrightarrow{w} R \longrightarrow \cdots \tag{2.0.1}$$

is an acyclic complex.

For every element w as above, one says that w and x form an *exact pair* of zero divisors in R. If R is local, then w is unique up to multiplication by a unit in R.

(2.1) **Remark.** For a non-unit $x \neq 0$ the conditions (i)–(iii) above are equivalent to

(iv) There exist elements w and y in R such that the sequence

$$R \xrightarrow{w} R \xrightarrow{x} R \xrightarrow{y} R$$

is exact.

Indeed, exactness of this sequence implies that there are equalities $(0 : y) = (x)$ and $(0 : x) = (w)$. Thus there is an obvious inclusion $(y) \subseteq (w)$ and, therefore, an inclusion $(0 : w) \subseteq (0 : y) = (x)$. As x annihilates w, this forces the equality $(x) = (0 : w)$. Thus (iv) implies (ii) and, clearly, (iv) follows from (iii).

Starting from the next section, we shall assume that R is local. We use the notation $(R, \mathfrak{m})$ to fix $\mathfrak{m}$ as the unique maximal ideal of R and the notation $(R, \mathfrak{m}, \mathsf{k})$ to also fix the residue field $\mathsf{k} = R/\mathfrak{m}$.

A complex F of free modules over a local ring $(R, \mathfrak{m})$ is called *minimal* if one has $\operatorname{Im} \partial_i^F \subseteq \mathfrak{m}F_{i-1}$ for all $i \in \mathbb{Z}$. Let M be a finitely generated R-module and let F be a minimal free resolution of M; it is unique up to isomorphism. The ith *Betti number*, $\beta_i^R(M)$, is the rank of the free module F_i, and the ith *syzygy* of M is the module $\operatorname{Coker} \partial_{i+1}^F$.

(2.2) **Remark.** The complex (2.0.1) is isomorphic to its own dual, so if it is acyclic, then it is totally acyclic. Thus if w and x form an exact pair of zero divisors in R, then the modules $(w) \cong R/(x)$ and $(x) \cong R/(w)$ are totally reflexive. Moreover, it follows from condition (iv) that a totally acyclic complex of free modules, in which four consecutive modules have rank 1, has the form (2.0.1). This means, in particular, that over a local ring R, any totally reflexive module M with $\beta_i^R(M) = 1$ for $0 \leqslant i \leqslant 3$ has the form $M \cong R/(x)$, where x is an exact zero divisor. For modules over short local rings we prove a stronger statement in Theorem (5.3).

The next lemma is folklore; it is proved for Gorenstein rings in [10].

(2.3) **Lemma.** *Let R be local and let T be a minimal totally acyclic complex of finitely generated free R-modules. The following conditions are equivalent:*

(i) *The module* $\operatorname{Coker} \partial_i^T$ *is indecomposable for some* $i \in \mathbb{Z}$.

(ii) *The module* $\operatorname{Coker} \partial_i^T$ *is indecomposable for every* $i \in \mathbb{Z}$.

In particular, every syzygy of an indecomposable totally reflexive R-module is indecomposable and totally reflexive.

Proof. For every $i \in \mathbb{Z}$ set $M_i = \operatorname{Coker} \partial_i^T$. By definition, the modules M_i and $\operatorname{Hom}_R(M_i, R)$ are totally reflexive and one has $M_i \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M_i, R), R)$

for every $i \in \mathbb{Z}$. Assume that for some integer j, the module M_j is indecomposable. For every $i < j$, the module M_j is a syzygy of M_i, and every summand of M_i has infinite projective dimension, as T is minimal and totally acyclic. Thus M_i is indecomposable. Further, if there were a non-trivial decomposition $M_i \cong K \oplus N$ for some $i > j$, then the dual module $\mathrm{Hom}_R(M_i, R)$ would decompose as $\mathrm{Hom}_R(K, R) \oplus \mathrm{Hom}_R(N, R)$, where both summands would be non-zero R-modules of infinite projective dimension. However, $\mathrm{Hom}_R(M_j, R)$ is a syzygy of $\mathrm{Hom}_R(M_i, R)$, so $\mathrm{Hom}_R(M_j, R)$ would then have a non-trivial decomposition and so would $M_j \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M_j, R), R)$; a contradiction. □

(2.4) **Lemma.** *Let T and F be complexes of finitely generated free R-modules. If T is totally acyclic, and the modules F_i are zero for $i \ll 0$, then the complex $T \otimes_R F$ is totally acyclic.*

Proof. The complex $T \otimes_R F$ is acyclic by [4, lem. 2.13]. Adjointness of Hom and tensor yields the isomorphism $\mathrm{Hom}_R(T \otimes_R F, R) \cong \mathrm{Hom}_R(F, \mathrm{Hom}_R(T, R))$. As the complex $\mathrm{Hom}_R(T, R)$ is acyclic, so is $\mathrm{Hom}_R(F, \mathrm{Hom}_R(T, R))$ by [4, lem. 2.4]. □

(2.5) **Definition.** For $n \in \mathbb{N}$ and elements y and z in R with $yz = 0$, let $L^n(y, z)$ be the complex defined as follows

$$L^n(y,z)_i = \begin{cases} R & \text{for } 0 \geqslant i \geqslant -n+1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad \partial_i^{L^n(y,z)} = \begin{cases} y & \text{for } i \text{ even} \\ -z & \text{for } i \text{ odd.} \end{cases}$$

If w and x form an exact pair of zero divisors in R, and T is the complex (2.0.1), then the differentials of the complex $T \otimes_R L^n(y, z)$ have a particularly simple form; see Remark (2.7). In Sections 3 and 6 we study modules whose presentation matrices have this form.

(2.6) **Construction.** Let $n \in \mathbb{N}$, let I_n be the $n \times n$ identity matrix, and let r_i denote its ith row. Consider the $n \times n$ matrices I_n^o, I_n^e, J_n^o, and J_n^e defined by specifying their rows as follows

$$(\mathrm{I}_n^o)_i = \begin{cases} r_i & i \text{ odd} \\ 0 & i \text{ even} \end{cases} \quad \text{and} \quad (\mathrm{I}_n^e)_i = \begin{cases} 0 & i \text{ odd} \\ r_i & i \text{ even} \end{cases}$$

$$(\mathrm{J}_n^o)_i = \begin{cases} r_{i+1} & i \text{ odd} \\ 0 & i \text{ even} \end{cases} \quad \text{and} \quad (\mathrm{J}_n^e)_i = \begin{cases} 0 & i \text{ odd} \\ r_{i+1} & i \text{ even} \end{cases}$$

with the convention $r_{n+1} = 0$. The equality $\mathrm{I}_n^o + \mathrm{I}_n^e = \mathrm{I}_n$ is clear, and the matrix $\mathrm{J}_n = \mathrm{J}_n^o + \mathrm{J}_n^e$ is the $n \times n$ nilpotent Jordan block with eigenvalue zero.

For elements w, x, y, and z in R let $M_n(w, x, y, z)$ be the R-module with presentation matrix $\Theta_n(w, x, y, z) = w\mathrm{I}_n^o + x\mathrm{I}_n^e + y\mathrm{J}_n^o + z\mathrm{J}_n^e$; it is an upper triangular $n \times n$ matrix with w and x alternating on the diagonal, and with y and z alternating on the superdiagonal:

$$\Theta_n(w,x,y,z) = \begin{pmatrix} w & y & 0 & 0 & 0 & \cdots \\ 0 & x & z & 0 & 0 & \cdots \\ 0 & 0 & w & y & 0 & \cdots \\ 0 & 0 & 0 & x & z & \\ 0 & 0 & 0 & 0 & w & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

For $n = 1$ the matrix has only one entry, namely w. For $n = 2$, the matrix does not depend on z, so we set $M_2(w, x, y) = M_2(w, x, y, z)$ for every $z \in R$.

Note that if $(R, \mathfrak{m})$ is local and w, x, y, and z are non-zero elements in $\mathfrak{m}$, then $M_n(w, x, y, z)$ is minimally generated by n elements.

(2.7) **Remark.** Assume that w and x form an exact pair of zero divisors in R, and let T be the complex (2.0.1), positioned such that ∂_1^T is multiplication by w. Let $n \in \mathbb{N}$ and let y and z be elements in R that satisfy $yz = 0$. It follows from Lemma (2.4) that the complex $T \otimes_R L^n(y, z)$ is totally acyclic. It is elementary to verify that the differential $\partial_i^{T \otimes_R L^n(y,z)}$ is given by the matrix $\Theta_n(w, x, y, z)$ for i odd and by $\Theta_n(x, w, -y, -z)$ for i even, cf. Construction (2.6). In particular, the module $M_n(w, x, y, z)$ is totally reflexive.

3. Families of indecomposable modules of different size

With appropriately chosen ring elements as input, Construction (2.6) yields the infinite families of modules in the Brauer–Thrall theorems advertised in Section 1. In this section we begin to analyze the requirements on the input.

(3.1) **Theorem.** *Let $(R, \mathfrak{m})$ be a local ring and assume that w and x are elements in $\mathfrak{m} \backslash \mathfrak{m}^2$, that form an exact pair of zero divisors. Assume further that y and z are elements in $\mathfrak{m} \backslash \mathfrak{m}^2$ with $yz = 0$ and that one of the following conditions holds:*

(a) *The elements w, x, and y are linearly independent modulo $\mathfrak{m}^2$.*

(b) *One has $w \in (x) + \mathfrak{m}^2$ and $y, z \notin (x) + \mathfrak{m}^2$.*

For every $n \in \mathbb{N}$, the R-module $M_n(w, x, y, z)$ is indecomposable, totally reflexive, and non-free. Moreover, $M_n(w, x, y, z)$ has constant Betti numbers, equal to n, and its minimal free resolution is periodic of period at most 2.

The proof of (3.1)—which takes up the balance of the section—employs an auxiliary result of independent interest, Proposition (3.2) below; its proof is deferred to the end of the section.

(3.2) **Proposition.** *Let $(R, \mathfrak{m})$ be a local ring, let n be a positive integer, and let w, x, y, and z be elements in $\mathfrak{m} \backslash \mathfrak{m}^2$.*

(a) *Assume that w, x, and y are linearly independent modulo $\mathfrak{m}^2$.*
  - *If n is even, then $M_n(w, x, y, z)$ is indecomposable.*
  - *If n is odd, then $M_n(w, x, y, z)$ or $M_n(x, w, y, z)$ is indecomposable.*

(b) *If $y \notin (w, x) + \mathfrak{m}^2$ and $z \notin (x) + \mathfrak{m}^2$ hold, then $M_n(w, x, y, z)$ is indecomposable.*

Proof of Theorem (3.1). Let n be a positive integer; in view of Remark (2.7), all we need to show is that the R-module $M_n(w, x, y, z)$ is indecomposable.

(a): Assume that w, x, and y are linearly independent modulo $\mathfrak{m}^2$. By Remark (2.7) the module $M_n(w, x, y, z)$ is the first syzygy of $M_n(x, w, -y, -z)$. If the module $M_n(x, w, y, z)$ is indecomposable, then so is the isomorphic module $M_n(x, w, -y, -z)$, and it follows from Lemma (2.3) that $M_n(w, x, y, z)$ is indecomposable as well. Thus by Proposition (3.2)(a) the module $M_n(w, x, y, z)$ is indecomposable.

(b): Under the assumptions $w \in (x) + \mathfrak{m}^2$ and $y, z \notin (x) + \mathfrak{m}^2$, the conditions in Proposition (3.2)(b) are met, so the R-module $M_n(w, x, y, z)$ is indecomposable. $\square$

Proof of Proposition (3.2). Let n be a positive integer and let w, x, y, and z be elements in $\mathfrak{m}\backslash\mathfrak{m}^2$. It is convenient to work with a presentation matrix $\Phi_n(w,x,y,z)$ for $M = M_n(w,x,y,z)$ that one obtains as follows. Set $p = \lceil \frac{n}{2} \rceil$ and let Π be the $n \times n$ matrix obtained from I_n by permuting its rows according to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots & n \\ 1 & p+1 & 2 & p+2 & 3 & p+3 & \dots & \delta p + p \end{pmatrix},$$

with $\delta = 0$ if n is odd and $\delta = 1$ if n is even. Set $\Phi_n(w,x,y,z) = \Pi\Theta_n(w,x,y,z)\Pi^{-1}$. If n is even, then Φ_n is the block matrix

$$\Phi_n(w,x,y,z) = \begin{pmatrix} w\mathrm{I}_p & y\mathrm{I}_p \\ z\mathrm{J}_p & x\mathrm{I}_p \end{pmatrix},$$

and if n is odd, then $\Phi_n(w,x,y,z)$ is the matrix obtained from $\Phi_{n+1}(w,x,y,z)$ by deleting the last row and the last column.

To verify that M is indecomposable, assume that $\varepsilon \in \mathrm{Hom}_R(M,M)$ is idempotent and not the identity map 1_M. The goal is to show that ε is the zero map. The only idempotent automorphism of M is 1_M, so ε is not an isomorphism. Thus ε is not surjective, as M is noetherian. Set $\Phi = \Phi_n(w,x,y,z)$ and consider the commutative diagram with exact rows

$$\begin{tikzcd} R^n \arrow[r, "\Phi"] \arrow[d, "\mathrm{B}"] & R^n \arrow[r] \arrow[d, "\mathrm{A}"] & M \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ R^n \arrow[r, "\Phi"] & R^n \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \tag{1}$$

obtained by lifting ε. Let $\bar{\mathrm{A}} = (a_{ij})$ and $\bar{\mathrm{B}}$ be the $n \times n$ matrices obtained from A and B by reducing their entries modulo $\mathfrak{m}$.

To prove that ε is the zero map, it suffices to show that the matrix $\bar{\mathrm{A}}$ is nilpotent. Indeed, if $\bar{\mathrm{A}}$ is nilpotent, then so is the map $\bar{\varepsilon}\colon M/\mathfrak{m}M \to M/\mathfrak{m}M$. As $\bar{\varepsilon}$ is also idempotent, it is the zero map. Now it follows from Nakayama's lemma that the map $1_M - \varepsilon$ is surjective and hence an isomorphism. As $1_M - \varepsilon$ is idempotent, it follows that $1_M - \varepsilon$ is the identity map 1_M; that is, $\varepsilon = 0$.

Claim. If the matrix $\bar{\mathrm{A}}$ is upper triangular with identical entries on the diagonal, i.e. $a_{11} = a_{22} = \cdots = a_{nn}$, then it is nilpotent.

Proof. Since ε is not surjective, the matrix A does not represent a surjective map and, by Nakayama's lemma, neither does $\bar{\mathrm{A}}$. Therefore, the diagonal entries of $\bar{\mathrm{A}}$ cannot all be non-zero, whence they are all zero, and $\bar{\mathrm{A}}$ is nilpotent.

Denote by $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ the images of w, x, y, and z in $\mathfrak{m}/\mathfrak{m}^2$, and let V be the k-subspace of $\mathfrak{m}/\mathfrak{m}^2$ spanned by $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$. Consider the following possibilities:

(I) The elements $\tilde{w}$, $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ form a basis for V.
(II) The elements $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ form a basis for V, and $\mathsf{k}\tilde{w} = \mathsf{k}\tilde{x}$ holds.
(III) The elements $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ form a basis for V.
(IV) The elements $\tilde{x}$ and $\tilde{y}$ form a basis for V, and one has $\mathsf{k}\tilde{w} = \mathsf{k}\tilde{x}$ and $\tilde{z} \notin \mathsf{k}\tilde{x}$.

Under the assumptions on w, x, and y in part (a), one of the conditions (I) or (III) holds. Under the assumptions in part (b), one of the conditions (I)–(IV) holds. Indeed, if V has dimension 4, then (I) holds. If that is not the case, then the dimension of V is 2 or 3. In case $\dim_{\mathsf{k}} V = 2$, the elements $\tilde{x}$ and $\tilde{y}$ form a basis for V, whereas $\tilde{w}$ and $\tilde{x}$ cannot be linearly independent; thus (IV) holds. In case

$\dim_{\mathsf{k}} V = 3$, condition (II) or (III) holds, depending on whether or not the equality $\mathsf{k}\tilde{w} = \mathsf{k}\tilde{x}$ holds.

The rest of the proof is split in two, according to the parity of n. To prove that $M_n(w,x,y,z)$ is indecomposable, it suffices to prove that the matrix $\bar{\mathrm{A}}$ is nilpotent. This is how we proceed under each of the conditions (I), (II), and (IV), and under condition (III) when n is even. When n is odd, and condition (III) holds, we show that one of the modules $M_n(w,x,y,z)$ and $M_n(x,w,y,z)$ is indecomposable.

Case 1: n is even. Let $\widetilde{\Phi}$ denote the matrix obtained from Φ by reducing the entries modulo $\mathfrak{m}^2$. Write $\bar{\mathrm{A}}$ and $\bar{\mathrm{B}}$ as block matrices

$$\bar{\mathrm{A}} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix} \qquad \text{and} \qquad \bar{\mathrm{B}} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix},$$

where A_{ij} and B_{ij} are $p \times p$ matrices with entries from k. By (1), the equality $\mathrm{A}\Phi = \Phi\mathrm{B}$ holds; it implies an equality of block matrices

$$\begin{pmatrix} \tilde{w}\mathrm{A}_{11} + \tilde{z}\mathrm{A}_{12}\mathrm{J}_p & \tilde{y}\mathrm{A}_{11} + \tilde{x}\mathrm{A}_{12} \\ \tilde{w}\mathrm{A}_{21} + \tilde{z}\mathrm{A}_{22}\mathrm{J}_p & \tilde{y}\mathrm{A}_{21} + \tilde{x}\mathrm{A}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{w}\mathrm{B}_{11} + \tilde{y}\mathrm{B}_{21} & \tilde{w}\mathrm{B}_{12} + \tilde{y}\mathrm{B}_{22} \\ \tilde{z}\mathrm{J}_p\mathrm{B}_{11} + \tilde{x}\mathrm{B}_{21} & \tilde{z}\mathrm{J}_p\mathrm{B}_{12} + \tilde{x}\mathrm{B}_{22} \end{pmatrix}. \tag{2}$$

Assume first that condition (I) or (III) holds, so that the elements $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are linearly independent. Then the equality of the upper right blocks, $\tilde{y}\mathrm{A}_{11} + \tilde{x}\mathrm{A}_{12} = \tilde{w}\mathrm{B}_{12} + \tilde{y}\mathrm{B}_{22}$, yields

$$\mathrm{A}_{11} = \mathrm{B}_{22} \qquad \text{and} \qquad \mathrm{A}_{12} = 0 = \mathrm{B}_{12}.$$

From the blocks on the diagonals, one now gets

$$\mathrm{A}_{11} = \mathrm{B}_{11}, \qquad \mathrm{A}_{21} = 0 = \mathrm{B}_{21}, \qquad \text{and} \qquad \mathrm{A}_{22} = \mathrm{B}_{22}.$$

Thus the matrix $\bar{\mathrm{A}}$ has the form $\left(\begin{smallmatrix} \mathrm{A}_{11} & 0 \\ 0 & \mathrm{A}_{11} \end{smallmatrix}\right)$. Finally, the equality of the lower left blocks yields $\mathrm{A}_{11}\mathrm{J}_p = \mathrm{J}_p\mathrm{A}_{11}$. Since J_p is non-derogatory, this implies that A_{11} belongs to the algebra $\mathsf{k}[\mathrm{J}_p]$ of polynomials in J_p. That is, there are elements $c_0, \ldots, c_{p-1}$ in k such that $\mathrm{A}_{11} = c_0\mathrm{I}_p + c_1\mathrm{J}_p + \cdots + c_{p-1}\mathrm{J}_p^{p-1}$; see [13, thm. 3.2.4.2]. In particular, the matrices A_{11} and, therefore, $\bar{\mathrm{A}}$ are upper-triangular with identical entries on the diagonal. By Claim, $\bar{\mathrm{A}}$ is nilpotent as desired.

Under either condition (II) or (IV), the elements $\tilde{x}$ and $\tilde{y}$ are linearly independent, $\tilde{z}$ is not in $\mathsf{k}\tilde{x}$, and the equality $\mathsf{k}\tilde{w} = \mathsf{k}\tilde{x}$ holds. In particular, there is an element $t \neq 0$ in k such that $t\tilde{w} = \tilde{x}$. From the off-diagonal blocks in (2) one obtains the following relations:

$$\begin{aligned} \mathrm{A}_{11} &= \mathrm{B}_{22} \\ t\mathrm{A}_{12} &= \mathrm{B}_{12} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{A}_{22}\mathrm{J}_p &= \mathrm{J}_p\mathrm{B}_{11} \\ \mathrm{A}_{21} &= t\mathrm{B}_{21}. \end{aligned} \tag{3}$$

If (II) holds, then the blocks on the diagonals in (2) yield

$$\mathrm{A}_{11} = \mathrm{B}_{11}, \qquad \mathrm{A}_{22} = \mathrm{B}_{22}, \qquad \text{and} \qquad \mathrm{A}_{21} = 0.$$

Thus the matrix $\bar{\mathrm{A}}$ has the form $\left(\begin{smallmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ 0 & \mathrm{A}_{11} \end{smallmatrix}\right)$, and the equality $\mathrm{A}_{11}\mathrm{J}_p = \mathrm{J}_p\mathrm{A}_{11}$ holds. As above, it follows that the matrices A_{11} and, therefore, $\bar{\mathrm{A}}$ are upper-triangular with identical entries on the diagonal. That is, $\bar{\mathrm{A}}$ is nilpotent by Claim.

Now assume that (IV) holds. There exist elements r and $s \neq 0$ in k such that $\tilde{z} = r\tilde{x} + s\tilde{y}$. Comparison of the blocks on the diagonals in (2) now yields

$$\begin{aligned} \mathrm{A}_{11} + rt\mathrm{A}_{12}\mathrm{J}_p &= \mathrm{B}_{11} \\ s\mathrm{A}_{12}\mathrm{J}_p &= \mathrm{B}_{21} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{A}_{22} &= r\mathrm{J}_p\mathrm{B}_{12} + \mathrm{B}_{22} \\ \mathrm{A}_{21} &= s\mathrm{J}_p\mathrm{B}_{12}. \end{aligned} \tag{4}$$

Combine these equalities with those from (3) to get

$$\bar{\mathrm{A}} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ st\mathrm{J}_p\mathrm{A}_{12} & \mathrm{A}_{11} + rt\mathrm{J}_p\mathrm{A}_{12} \end{pmatrix}.$$

It follows from the equalities

$$\mathrm{J}_p\mathrm{A}_{12} = t^{-1}\mathrm{J}_p\mathrm{B}_{12} = (st)^{-1}\mathrm{A}_{21} = s^{-1}\mathrm{B}_{21} = \mathrm{A}_{12}\mathrm{J}_p,$$

derived from (3) and (4), that the matrix A_{12} commutes with J_p; hence it belongs to $\mathsf{k}[\mathrm{J}_p]$. Similarly, the chain of equalities

$$\begin{aligned} \mathrm{J}_p\mathrm{A}_{11} = \mathrm{J}_p\mathrm{B}_{11} - rt\mathrm{J}_p\mathrm{A}_{12}\mathrm{J}_p &= \mathrm{J}_p\mathrm{B}_{11} - r\mathrm{J}_p\mathrm{B}_{12}\mathrm{J}_p \\ &= \mathrm{A}_{22}\mathrm{J}_p - r\mathrm{J}_p\mathrm{B}_{12}\mathrm{J}_p = \mathrm{B}_{22}\mathrm{J}_p = \mathrm{A}_{11}\mathrm{J}_p \end{aligned}$$

shows that A_{11} is in $\mathsf{k}[\mathrm{J}_p]$. Thus all four blocks in $\bar{\mathrm{A}}$ belong to $\mathsf{k}[\mathrm{J}_p]$. For notational bliss, identify $\mathsf{k}[\mathrm{J}_p]$ with the ring $S = \mathsf{k}[\chi]/(\chi^p)$, where χ corresponds to $t\mathrm{J}_p$. With this identification, $\bar{\mathrm{A}}$ takes the form of a 2×2 matrix with entries in S:

$$\begin{pmatrix} f & g \\ sg\chi & f + rg\chi \end{pmatrix}.$$

As $\bar{\mathrm{A}}$ is not invertible, the determinant $f^2 + frg\chi - sg^2\chi$ belongs to the maximal ideal (χ) of S. It follows that f is in (χ), whence one has $\bar{\mathrm{A}}^{2p} = 0$ as desired.

Case 2: n is odd. Set $q = p - 1$, where $p = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. The presentation matrix Φ takes the form

$$\Phi = \begin{pmatrix} w\mathrm{I}_p & y\mathrm{H} \\ z\mathrm{K} & x\mathrm{I}_q \end{pmatrix},$$

where H and K are the following block matrices $\mathrm{H} = \begin{pmatrix} \mathrm{I}_q \\ 0_{1\times q} \end{pmatrix}$ and $\mathrm{K} = \begin{pmatrix} 0_{q\times 1} & \mathrm{I}_q \end{pmatrix}$. Notice that there are equalities

$$\mathrm{HX} = \begin{pmatrix} \mathrm{X} \\ 0_{1\times m} \end{pmatrix} \quad \text{and} \quad \mathrm{X}'\mathrm{K} = \begin{pmatrix} 0_{m'\times 1} & \mathrm{X}' \end{pmatrix} \tag{5}$$

for every $q \times m$ matrix X and every $m' \times q$ matrix X′. Furthermore, it is straightforward to verify the equalities

$$\mathrm{HK} = \mathrm{J}_p \quad \text{and} \quad \mathrm{KH} = \mathrm{J}_q. \tag{6}$$

As in Case 1, write

$$\bar{\mathrm{A}} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix} \quad \text{and} \quad \bar{\mathrm{B}} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix},$$

where, now, A_{ij} and B_{ij} are matrices of size $m_i \times m_j$, for $m_1 = p$ and $m_2 = q$. With $\widetilde{\Phi}$ as defined in Case 1, the relation $\bar{\mathrm{A}}\widetilde{\Phi} = \widetilde{\Phi}\bar{\mathrm{B}}$, derived from (1), yields:

$$\begin{pmatrix} \tilde{w}\mathrm{A}_{11} + \tilde{z}\mathrm{A}_{12}\mathrm{K} & \tilde{y}\mathrm{A}_{11}\mathrm{H} + \tilde{x}\mathrm{A}_{12} \\ \tilde{w}\mathrm{A}_{21} + \tilde{z}\mathrm{A}_{22}\mathrm{K} & \tilde{y}\mathrm{A}_{21}\mathrm{H} + \tilde{x}\mathrm{A}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{w}\mathrm{B}_{11} + \tilde{y}\mathrm{HB}_{21} & \tilde{w}\mathrm{B}_{12} + \tilde{y}\mathrm{HB}_{22} \\ \tilde{z}\mathrm{KB}_{11} + \tilde{x}\mathrm{B}_{21} & \tilde{z}\mathrm{KB}_{12} + \tilde{x}\mathrm{B}_{22} \end{pmatrix}. \tag{7}$$

Assume first that condition (I) or (III) holds, so that the elements $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are linearly independent. From the equality of the upper right blocks in (7) one gets

$$\mathrm{A}_{11}\mathrm{H} = \mathrm{HB}_{22} \quad \text{and} \quad \mathrm{A}_{12} = 0 = \mathrm{B}_{12}. \tag{8}$$

In view of (5), comparison of the blocks on the diagonals now yields

$$\begin{matrix} \mathrm{A}_{11} = \mathrm{B}_{11} \\ \mathrm{B}_{21} = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathrm{A}_{21}\mathrm{H} = 0 \\ \mathrm{A}_{22} = \mathrm{B}_{22}. \end{matrix} \tag{9}$$

From the first equality in (8) and the last equality in (9) one gets in view of (5)

$$\mathrm{A}_{11} = \begin{pmatrix} \mathrm{A}_{22} & * \\ 0_{1\times q} & * \end{pmatrix}; \tag{10}$$

here and in the following the symbol '$*$' in a matrix denotes an unspecified block of appropriate size. To glean information from the equality of the lower left blocks in (7), assume first that $\tilde{w}$ and $\tilde{z}$ are linearly independent. Then one has

$$\mathrm{A}_{21} = 0 \qquad \text{and} \qquad \mathrm{A}_{22}\mathrm{K} = \mathrm{K}\mathrm{B}_{11}. \tag{11}$$

Combine this with (10) and the second equality in (8) to see that the matrix $\bar{\mathrm{A}}$ has the form

$$\bar{\mathrm{A}} = \left(\begin{array}{cc|c} \mathrm{A}_{22} & * & \\ 0_{1\times q} & * & 0_{p\times q} \\ \hline \multicolumn{2}{c|}{0_{q\times p}} & \mathrm{A}_{22} \end{array}\right). \tag{12}$$

The equalities

$$\mathrm{A}_{11}\mathrm{J}_p = \mathrm{A}_{11}\mathrm{H}\mathrm{K} = \mathrm{H}\mathrm{B}_{22}\mathrm{K} = \mathrm{H}\mathrm{A}_{22}\mathrm{K} = \mathrm{H}\mathrm{K}\mathrm{B}_{11} = \mathrm{J}_p\mathrm{A}_{11}$$

and

$$\mathrm{A}_{22}\mathrm{J}_q = \mathrm{A}_{22}\mathrm{K}\mathrm{H} = \mathrm{K}\mathrm{B}_{11}\mathrm{H} = \mathrm{K}\mathrm{A}_{11}\mathrm{H} = \mathrm{K}\mathrm{H}\mathrm{B}_{22} = \mathrm{J}_q\mathrm{A}_{22},$$

derived from (6), (8), (9), and (11), show that A_{11} is in $\mathsf{k}[\mathrm{J}_p]$ and A_{22} is in $\mathsf{k}[\mathrm{J}_q]$. It follows that $\bar{\mathrm{A}}$ is upper triangular with identical entries on the diagonal. Thus $\bar{\mathrm{A}}$ is nilpotent, and $M_n(w,x,y,z)$ is indecomposable. If, on the other hand, $\tilde{z}$ and $\tilde{w}$ are linearly dependent, then $\tilde{z}$ and $\tilde{x}$ are linearly independent, as $\tilde{w}$ and $\tilde{x}$ are linearly independent by assumption. It follows from what we have just shown that $M_n(x,w,y,z)$ is indecomposable.

Under either condition (II) or (IV), the elements $\tilde{x}$ and $\tilde{y}$ are linearly independent, $\tilde{z}$ is not in $\mathsf{k}\tilde{x}$, and the equality $\mathsf{k}\tilde{w} = \mathsf{k}\tilde{x}$ holds. In particular, there is an element $t \neq 0$ in k such that $t\tilde{w} = \tilde{x}$. Compare the off-diagonal blocks in (7) to get

$$\begin{aligned} t\mathrm{A}_{12} &= \mathrm{B}_{12} \\ \mathrm{A}_{11}\mathrm{H} &= \mathrm{H}\mathrm{B}_{22} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{A}_{21} &= t\mathrm{B}_{21} \\ \mathrm{A}_{22}\mathrm{K} &= \mathrm{K}\mathrm{B}_{11}. \end{aligned} \tag{13}$$

If (II) holds, then a comparison of the blocks on the diagonals in (7) combined with (5) implies

$$\mathrm{A}_{11} = \mathrm{B}_{11}, \quad \mathrm{A}_{12} = 0 = \mathrm{B}_{21}, \quad \text{and} \quad \mathrm{A}_{22} = \mathrm{B}_{22}. \tag{14}$$

It follows from (13) and (14) that the matrix A_{21} is zero. In view of the equality $\mathrm{A}_{11}\mathrm{H} = \mathrm{H}\mathrm{B}_{22}$ from (13), it now follows that $\bar{\mathrm{A}}$ has the form given in (12). Using the equalities in (13) and (14), one can repeat the arguments above to see that A_{11} is in $\mathsf{k}[\mathrm{J}_p]$ and A_{22} is in $\mathsf{k}[\mathrm{J}_q]$, and continue to conclude that $\bar{\mathrm{A}}$ is nilpotent.

Finally, assume that (IV) holds. There exist elements r and $s \neq 0$ in k such that $\tilde{z} = r\tilde{x} + s\tilde{y}$. Comparison of the blocks on the diagonals in (7) yields

$$\begin{aligned} \mathrm{A}_{11} + rt\mathrm{A}_{12}\mathrm{K} &= \mathrm{B}_{11} \\ s\mathrm{A}_{12}\mathrm{K} &= \mathrm{H}\mathrm{B}_{21} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{A}_{21}\mathrm{H} &= s\mathrm{K}\mathrm{B}_{12} \\ \mathrm{A}_{22} &= \mathrm{B}_{22} + r\mathrm{K}\mathrm{B}_{12}. \end{aligned} \tag{15}$$

The equality $s\mathrm{A}_{12}\mathrm{K} = t^{-1}\mathrm{H}\mathrm{A}_{21}$, obtained from (13) and (15), shows, in view of (5), that the matrices A_{12} and A_{21} have the following form:

$$\mathrm{A}_{12} = \begin{pmatrix} * \\ \mathrm{E} \end{pmatrix} \qquad \text{and} \qquad \mathrm{A}_{21} = \begin{pmatrix} 0_{q\times 1} & \Gamma \end{pmatrix},$$

where E and Γ are $q \times q$ matrices, and the last row of E is zero. From the equalities in (6), (13), and (15) one gets

$$\mathrm{A}_{21}\mathrm{J}_p = s\mathrm{K}\mathrm{B}_{12}\mathrm{K} = st\mathrm{K}\mathrm{A}_{12}\mathrm{K} = t\mathrm{K}\mathrm{H}\mathrm{B}_{21} = \mathrm{J}_q\mathrm{A}_{21}.$$

From here it is straightforward to verify that Γ commutes with J_q; i.e. Γ belongs to $\mathsf{k}[\mathrm{J}_q]$. Similarly, from the equalities

$$\mathrm{A}_{12}\mathrm{J}_q = s^{-1}\mathrm{H}\mathrm{B}_{21}\mathrm{H} = (st)^{-1}\mathrm{H}\mathrm{A}_{21}\mathrm{H} = t^{-1}\mathrm{H}\mathrm{K}\mathrm{B}_{12} = \mathrm{J}_p\mathrm{A}_{12},$$

it follows that E belongs to $\mathsf{k}[\mathrm{J}_q]$. Since the last row in E is zero, all entries on the diagonal of E are zero, and the matrix is nilpotent. The first equality in (15) can now be written as

$$\mathrm{A}_{11} = \mathrm{B}_{11} - \begin{pmatrix} 0 & * \\ 0_{q\times 1} & rt\mathrm{E} \end{pmatrix}.$$

Combine this with the last equality in (13) to get

$$\bar{\mathrm{A}} = \left(\begin{array}{cc|c} a_{11} & * & * \\ 0_{q\times 1} & \mathrm{A}_{22} - rt\mathrm{E} & \mathrm{E} \\ \hline 0_{q\times 1} & \Gamma & \mathrm{A}_{22} \end{array}\right).$$

The equalities

$$\begin{aligned} \mathrm{A}_{11}\mathrm{J}_p = \mathrm{H}\mathrm{B}_{22}\mathrm{K} = \mathrm{H}(\mathrm{A}_{22} - r\mathrm{K}\mathrm{B}_{12})\mathrm{K} &= \mathrm{H}\mathrm{A}_{22}\mathrm{K} - r\mathrm{H}\mathrm{K}\mathrm{B}_{12}\mathrm{K} \\ &= \mathrm{H}\mathrm{K}\mathrm{B}_{11} - rt\mathrm{H}\mathrm{K}\mathrm{A}_{12}\mathrm{K} = \mathrm{J}_p\mathrm{A}_{11} \end{aligned}$$

show that A_{11} is in $\mathsf{k}[\mathrm{J}_p]$, and a similar chain of equalities shows that A_{22} is in $\mathsf{k}[\mathrm{J}_q]$. It follows that all entries on the diagonal of $\bar{\mathrm{A}}$ are identical. Let Δ be the matrix obtained by deleting the first row and first column in $\bar{\mathrm{A}}$ and write it in block form

$$\Delta = \begin{pmatrix} \mathrm{A}_{22} - rt\mathrm{E} & \mathrm{E} \\ \Gamma & \mathrm{A}_{22} \end{pmatrix}.$$

As $\bar{\mathrm{A}}$ is not invertible, one has $0 = \det \bar{\mathrm{A}} = a_{11}(\det \Delta)$. If a_{11} is non-zero, then Δ has determinant 0; in particular, it is not invertible. Considered as a 2×2 matrix over the artinian local ring $\mathsf{k}[\mathrm{J}_q]$, its determinant $(\mathrm{A}_{22})^2 - rt\mathrm{E}\mathrm{A}_{22} - \Gamma\mathrm{E}$ belongs to the maximal ideal (J_q). As E is nilpotent, it belongs to (J_q) and hence so does A_{22}; this contradicts the assumption that a_{11} is non-zero. If the diagonal entry a_{11} is 0, then the matrix A_{22} is nilpotent, which implies that Δ is nilpotent and, finally, that $\bar{\mathrm{A}}$ is nilpotent. □

4. Brauer–Thrall I over short local rings with exact zero divisors

Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring. The *embedding dimension* of R, denoted $\operatorname{emb.dim} R$, is the minimal number of generators of $\mathfrak{m}$, i.e. the dimension of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. The *Hilbert series* of R is the power series $\mathrm{H}_R(\tau) = \sum_{i=0}^{\infty} \dim_{\mathsf{k}}(\mathfrak{m}^i/\mathfrak{m}^{i+1})\tau^i$.

In the rest of this section $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^3 = 0$. The main result, Theorem (4.4), together with (4.1.1) and Theorem (3.1), establishes Theorem (1.1). Towards the proof of Theorem (4.4), we first recapitulate a few facts about totally reflexive modules and exact zero divisors.

If R is Gorenstein, then every R-module is totally reflexive; see [1, thm. (4.13) and (4.20)]. If R is not Gorenstein, then existence of a non-free totally reflexive R-module forces certain relations among invariants of R. The facts in (4.1) are proved by Yoshino [19]; see also [6] for the non-graded case.

(4.1) **Totally reflexive modules.** Assume that R is not Gorenstein and set $e = \operatorname{emb.dim} R$. If M is a totally reflexive R-module without free summands and minimally generated by n elements, then the equalities

$$\text{(4.1.1)} \qquad \operatorname{length}_R M = ne \quad \text{and} \quad \beta_i^R(M) = n \ \text{ for all } i \geqslant 0$$

hold. Moreover, $\mathfrak{m}^2$ is non-zero and the following hold:

$$\text{(4.1.2)} \qquad (0:\mathfrak{m}) = \mathfrak{m}^2, \quad \dim_{\mathsf{k}} \mathfrak{m}^2 = e-1, \quad \text{and} \quad \operatorname{length} R = 2e.$$

In particular, e is at least 3, and the Hilbert series of R is $1 + e\tau + (e-1)\tau^2$.

Let k be a field. For the Gorenstein ring $R = \mathsf{k}[x]/(x^3)$, two of the relations in (4.1.2) fail. This ring also has an exact zero divisor, x^2, in the square of the maximal ideal; for rings with embedding dimension 2 or higher this cannot happen.

(4.2) **Exact zero divisors.** Set $e = \operatorname{emb.dim} R$ and assume $e \geqslant 2$. Suppose that w and x form an exact pair of zero divisors in R. As $\mathfrak{m}^2$ is contained in the annihilator of $\mathfrak{m}$, there is an inclusion $\mathfrak{m}^2 \subseteq (x)$, which has to be strict, as $(w) = (0:x)$ is strictly contained in $\mathfrak{m}$. Thus x is a minimal generator of $\mathfrak{m}$ with

$$x\mathfrak{m} = \mathfrak{m}^2.$$

By symmetry, w is a minimal generator of $\mathfrak{m}$ with $w\mathfrak{m} = \mathfrak{m}^2$. Let $\{v_1, \ldots, v_{e-1}, w\}$ be a minimal set of generators for $\mathfrak{m}$; then the elements $xv_1, \ldots, xv_{e-1}$ generate $\mathfrak{m}^2$, and it is elementary to verify that they form a basis for $\mathfrak{m}^2$ as a k-vector space. It follows that the relations in (4.1.2) hold and, in addition, there is an equality

$$\operatorname{length}_R(x) = e.$$

Note that the socle $(0:\mathfrak{m})$ of R has dimension $e-1$ over k, so R is Gorenstein if and only if $e = 2$.

(4.3) **Lemma.** *Assume that $(R, \mathfrak{m})$ has Hilbert series $1 + e\tau + (e-1)\tau^2$ with $e \geqslant 2$. For every element $x \in \mathfrak{m} \backslash \mathfrak{m}^2$ the following hold:*

(a) *The ideal (x) in R has length at most e.*
(b) *There exists an element $w \in \mathfrak{m} \backslash \mathfrak{m}^2$ that annihilates x, and if w generates $(0:x)$, then w and x form an exact pair of zero divisors in R.*
(c) *If the equalities $wx = 0$ and $w\mathfrak{m} = \mathfrak{m}^2 = x\mathfrak{m}$ hold, then w and x form an exact pair of zero divisors in R.*

Proof. (a): By assumption, the length of $\mathfrak{m}$ is $2e-1$. As x and $e-1$ other elements form a minimal set of generators for $\mathfrak{m}$, the inequality $2e - 1 \geqslant \operatorname{length}_R(x) + e - 1$ holds; whence $\operatorname{length}_R(x)$ is at most e.

(b): Additivity of length on short exact sequences yields

$$\operatorname{length}_R(0:x) = \operatorname{length} R - \operatorname{length}_R(x) \geqslant e,$$

so $\mathfrak{m}^2$ is properly contained in $(0:x)$; choose an element w in $(0:x) \backslash \mathfrak{m}^2$. There is an inclusion $(x) \subseteq (0:w)$, and if the equality $(w) = (0:x)$ holds, then two length counts yield

$$\operatorname{length}_R(0:w) = \operatorname{length} R - \operatorname{length}_R(w) = \operatorname{length}_R(x).$$

Thus $(x) = (0:w)$ holds; hence w and x form an exact pair of zero divisors in R.

(c): As both ideals (w) and (x) strictly contain $\mathfrak{m}^2$, the equalities $\operatorname{length}_R(x) = e = \operatorname{length}_R(w)$ hold in view of part (a). As w and x annihilate each other, simple length counts show that they form an exact pair of zero divisors in R. □

(4.4) **Theorem.** *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$ and* emb.dim $R \geqslant 3$. *Assume that w and x form an exact pair of zero divisors in R. For every element y in $\mathfrak{m}\backslash(w,x)$ there exists an element $z \in \mathfrak{m}\backslash\mathfrak{m}^2$ such that the R-modules $M_n(w,x,y,z)$ are indecomposable and totally reflexive for all $n \in \mathbb{N}$.*

Proof. By (4.2) the equalities in (4.1.2) hold for R. Let y be an element in $\mathfrak{m}\backslash(w,x)$. By Lemma (4.3)(b) the ideal $(0:y)$ contains an element z of $\mathfrak{m}\backslash\mathfrak{m}^2$. Since y is not contained in the ideal $(w,x) = (w,x)+\mathfrak{m}^2$, the element z is not in $(x) = (x)+\mathfrak{m}^2$. The desired conclusion now follows from Theorem (3.1). □

The key to the theorem is that existence of an exact zero divisor in R implies the existence of additional elements such that the conditions in Theorem (3.1) are satisfied. This phenomenon does not extend to rings with $\mathfrak{m}^4 = 0$.

(4.5) **Example.** Let F be a field and set $S = \mathsf{F}[x,y,z]/(x^2, y^2z, yz^2, y^3, z^3)$; it is a standard graded F-algebra with Hilbert series $1+3\tau+5\tau^2+3\tau^3$, and x is an exact zero divisor in S. Set $\mathfrak{n} = (x,y,z)S$ and let v be an element in $\mathfrak{n}\backslash((x)+\mathfrak{n}^2)$. A straightforward calculation shows that the annihilator $(0:v)$ is contained in $\mathfrak{n}^2$.

5. Exact zero divisors from totally reflexive modules

Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$. If R is not Gorenstein, then a cyclic totally reflexive R-module is either free or generated by an exact zero divisor. Indeed, if it is not free, then by (4.1.1) it has constant Betti numbers, equal to 1, so by Remark (2.2) it is generated by an exact zero divisor in R. The next results improve on this elementary observation; in particular, Corollary (5.4) should be compared to Remark (2.2).

(5.1) **Lemma.** *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$ and let*

$$F: \quad F_2 \longrightarrow F_1 \xrightarrow{\psi} F_0 \xrightarrow{\varphi} F_{-1}$$

be an exact sequence of finitely generated free R-modules, where the homomorphisms are represented by matrices with entries in $\mathfrak{m}$. Let Ψ be any matrix that represents ψ. For every row Ψ_r of Ψ the following hold:

(a) *The ideal $\mathfrak{r}$, generated by the entries of Ψ_r, contains $\mathfrak{m}^2$.*

(b) *If $\dim_{\mathsf{k}} \mathfrak{m}^2$ is at least 2 and $\operatorname{Hom}_R(F,R)$ is exact, then Ψ_r has an entry from $\mathfrak{m}\backslash\mathfrak{m}^2$, the entries in Ψ_r from $\mathfrak{m}\backslash\mathfrak{m}^2$ generate $\mathfrak{r}$, and $\mathfrak{m}\mathfrak{r} = \mathfrak{m}^2$ holds.*

Proof. Let Ψ and Φ be the matrices for ψ and ϕ with respect to bases $\mathcal{B}_1$, $\mathcal{B}_0$, and $\mathcal{B}_{-1}$ for F_1, F_0, and F_{-1}. For every $p \geqslant 1$, let $\{e_1,\dots,e_p\}$ be the standard basis for R^p. The matrix Φ is of size $l\times m$, and Ψ is of size $m \times n$, where l, m and n denote the ranks of F_{-1}, F_0, and F_1, respectively. We make the identifications $F_{-1} = R^l$, $F_0 = R^m$, and $F_1 = R^n$, by letting $\mathcal{B}_{-1}$, $\mathcal{B}_0$, and $\mathcal{B}_1$ correspond to the standard bases. The map ψ is now left multiplication by the matrix Ψ, and ϕ is left multiplication by Φ. For every $x \in \mathfrak{m}^2$ and every basis element e_i in R^m one has $\varphi(xe_i) = 0$; indeed, Φ has entries in $\mathfrak{m}$, the entries of xe_i are in $\mathfrak{m}^2$, and $\mathfrak{m}^3 = 0$ holds by assumption. By exactness of F, the element xe_i is in the image of ψ, and (a) follows.

(b): Fix $q \in \{1,\dots,m\}$ and let Ψ_q be the qth row of $\Psi = (x_{ij})$; we start by proving the following:

Claim. Every entry from $\mathfrak{m}^2$ in Ψ_q is contained in the ideal generated by the other entries in Ψ_q.

Proof. Assume, towards a contradiction, that some entry from $\mathfrak{m}^2$ in Ψ_q is not in the ideal generated by the other entries. After a permutation of the columns of Ψ, one can assume that the entry x_{q1} is in $\mathfrak{m}^2$ but not in the ideal $(x_{q2},\dots,x_{qn})$. Since the element $x_{q1}e_q$ belongs to $\operatorname{Ker}\varphi = \operatorname{Im}\psi$, there exist elements a_i in R such that $\psi(\sum_{i=1}^n a_i e_i) = x_{q1}e_q$. In particular, one has $\sum_{i=1}^n a_i x_{qi} = x_{q1}$, whence it follows that a_1 is invertible. The $n \times n$ matrix

$$\mathrm{A} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ a_n & 0 & \dots & 1 \end{pmatrix}$$

is invertible, as it has determinant a_1. The first column of the matrix $\Psi\mathrm{A}$, which is the first row of the transposed matrix $(\Psi\mathrm{A})^{\mathrm{T}}$, has only one non-zero entry, namely x_{q1}. As $\Psi\mathrm{A}$ represents ψ, the matrix $(\Psi\mathrm{A})^{\mathrm{T}}$ represents the dual homomorphism $\operatorname{Hom}_R(\psi, R)$. By assumption the sequence $\operatorname{Hom}_R(F, R)$ is exact, so it follows from part (a) that the element x_{q1} spans $\mathfrak{m}^2$. This contradicts the assumption that $\mathfrak{m}^2$ is a k-vector space of dimension at least 2. This finishes the proof of Claim.

Suppose, for the moment, that every entry of Ψ_r is in $\mathfrak{m}^2$. Performing column operations on Ψ results in a matrix that also represents ψ, so by Claim one can assume that Ψ_r is the zero row, which contradicts part (a). Thus Ψ_r has an entry from $\mathfrak{m}\backslash\mathfrak{m}^2$. After a permutation of the columns of Ψ, one may assume that the entries $x_{r1},\dots,x_{rt}$ are in $\mathfrak{m}\backslash\mathfrak{m}^2$ while $x_{r(t+1)},\dots,x_{rn}$ are in $\mathfrak{m}^2$, where t is in $\{1,\dots,n\}$. Claim shows that—after column operations that do not alter the first t columns—one can assume that the entries $x_{r(t+1)},\dots,x_{rn}$ are zero. Thus the entries $x_{r1},\dots,x_{rt}$ from $\mathfrak{m}\backslash\mathfrak{m}^2$ generate the ideal $\mathfrak{r}$.

Finally, after another permutation of the columns of Ψ, one can assume that $\{x_{r1},\dots,x_{rs}\}$ is maximal among the subsets of $\{x_{r1},\dots,x_{rt}\}$ with respect to the property that its elements are linearly independent modulo $\mathfrak{m}^2$. Now use column operations to ensure that the elements $x_{r(s+1)},\dots,x_{rn}$ are in $\mathfrak{m}^2$. As above, it follows that $x_{r1},\dots,x_{rs}$ generate $\mathfrak{r}$. To verify the last equality in (b), note first the obvious inclusion $\mathfrak{m}\mathfrak{r} \subseteq \mathfrak{m}^2$. For the reverse inclusion, let $x \in \mathfrak{m}^2$ and write $x = x_{r1}b_1 + \cdots + x_{rs}b_s$ with $b_i \in R$. If some b_i were a unit, then the linear independence of the elements x_{ri} modulo $\mathfrak{m}^2$ would be contradicted. Thus each b_i is in $\mathfrak{m}$, and the proof is complete. □

The condition $\dim_{\mathsf{k}} \mathfrak{m}^2 \geqslant 2$ in part (b) of the lemma cannot be relaxed:

(5.2) **Example.** Let k be a field; the local ring $R = \mathsf{k}[x,y]/(x^2, xy, y^3)$ has Hilbert series $1 + 2\tau + \tau^2$. The sequence

$$R^2 \xrightarrow{(x\ y)} R \xrightarrow{x} R \xrightarrow{\binom{x}{y}} R^2$$

is exact and remains exact after dualization, but the product $(x,y)x$ is zero.

(5.3) **Theorem.** *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$ and $e = \operatorname{emb.dim} R \geqslant 3$. Let x be an element of $\mathfrak{m}\backslash\mathfrak{m}^2$; the following conditions are equivalent.*

(i) *The element x is an exact zero divisor in R.*

(ii) *The Hilbert series of R is $1+e\tau+(e-1)\tau^2$, and there exists an exact sequence of finitely generated free R-modules*

$$F: \quad F_3 \longrightarrow F_2 \longrightarrow F_1 \xrightarrow{\psi} F_0 \longrightarrow F_{-1}$$

such that $\operatorname{Hom}_R(F,R)$ is exact, the homomorphisms are represented by matrices with entries in $\mathfrak{m}$, and ψ is represented by a matrix in which some row has x as an entry and no other entry from $\mathfrak{m}\backslash\mathfrak{m}^2$.

Proof. If x is an exact zero divisor in R, then the complex (2.0.1) supplies the desired exact sequence, and R has Hilbert series $1 + e\tau + (e-1)\tau^2$; see (4.2).

To prove the converse, let $\Psi = (x_{ij})$ be a matrix of size $m \times n$ that represents ψ and assume, without loss of generality, that the last row of Ψ has exactly one entry $x = x_{mq}$ from $\mathfrak{m}\backslash\mathfrak{m}^2$. By Lemma (5.1)(b) there is an equality $x\mathfrak{m} = \mathfrak{m}^2$ and, therefore, the length of (x) is e by Lemma (4.3)(a). As R has length $2e$, additivity of length on short exact sequences yields $\operatorname{length}_R(0:x) = e$.

Let (w_{ij}) be an $n \times p$ matrix that represents the homomorphism $F_2 \to F_1$. The matrix equality $(x_{ij})(w_{ij}) = 0$ yields $xw_{qj} = 0$ for $j \in \{1,\dots,p\}$; it follows that the ideal $\mathfrak{r} = (w_{q1},\dots,w_{qp})$ is contained in $(0:x)$. By Lemma (5.1)(b) some entry $w = w_{ql}$ is in $\mathfrak{m}\backslash\mathfrak{m}^2$, and there are inclusions

$$(w) + \mathfrak{m}^2 \subseteq \mathfrak{r} \subseteq (0:x). \tag{1}$$

Now the inequalities

$$e \leqslant \operatorname{length}_R((w)+\mathfrak{m}^2) \leqslant \operatorname{length}_R(0:x) = e$$

imply that equalities hold throughout (1); in particular, $(w) + \mathfrak{m}^2 = \mathfrak{r}$ holds. This equality and Lemma (5.1)(b) yield $w\mathfrak{m} = \mathfrak{m}\mathfrak{r} = \mathfrak{m}^2$; hence w and x form an exact pair of zero divisors by Lemma (4.3)(c). □

(5.4) **Corollary.** *Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$. If R is not Gorenstein, then the following conditions are equivalent:*

(i) *There is an exact zero divisor in R.*

(ii) *For every $n \in \mathbb{N}$ there is an indecomposable totally reflexive R-module that is presented by an upper triangular $n \times n$ matrix with entries in $\mathfrak{m}$.*

(iii) *There is a totally reflexive R-module without free summands that is presented by a matrix with entries in $\mathfrak{m}$ and a row/column with only one entry in $\mathfrak{m}\backslash\mathfrak{m}^2$.*

Proof. Set $e = \operatorname{emb.dim} R$; it is at least 2 as R is not Gorenstein. If (i) holds, then e is at least 3, see (4.2), so (ii) follows from Theorem (4.4). It is clear from Lemma (5.1) that (iii) follows from (ii). To prove that (iii) implies (i), let Ψ be a presentation matrix for a totally reflexive R-module without free summands and assume—possibly after replacing Ψ with its transpose, which also presents a totally reflexive module—that some row of Ψ has only one entry in $\mathfrak{m}\backslash\mathfrak{m}^2$. By (4.1) the Hilbert series of R is $1 + e\tau + (e-1)\tau^2$, and e is at least 3, so it follows from Theorem (5.3) that there is an exact zero divisor in R. □

The corollary manifests a strong relation between the existence of exact zero divisors in R and existence of totally reflexive R-modules of any size. A qualitatively different relation is studied later; see Remark (8.7). These relations notwithstanding, totally reflexive modules may exist in the absence of exact zero divisors. In Section 9 we exhibit a local ring $(R, \mathfrak{m})$, which has no exact zero divisors, and a totally reflexive R-module that is presented by a 2×2 matrix with all four entries from $\mathfrak{m} \backslash \mathfrak{m}^2$. Thus the condition on the entries of the matrix in (5.4)(*iii*) is sharp.

6. Families of non-isomorphic modules of the same size

In this section we continue the analysis of Construction (2.6).

(6.1) **Definition.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring. Given a subset $\mathcal{K} \subseteq \mathsf{k}$, a subset of R that contains exactly one lift of every element in $\mathcal{K}$ is called a *lift* of $\mathcal{K}$ in R.

(6.2) **Theorem.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring and let $\mathcal{L}$ be a lift of k in R. Let w, x, y, y', and z be elements in $\mathfrak{m} \backslash \mathfrak{m}^2$ and let n be an integer. Assume that w and x form an exact pair of zero divisors in R and that one of the following holds.*

(a) *$n = 2$ and the elements w, x, y, and y' are linearly independent modulo $\mathfrak{m}^2$;*
(b) *$n = 2$, the elements x, y, and y' are linearly independent modulo $\mathfrak{m}^2$, and the element w belongs to $(x) + \mathfrak{m}^2$;*
(c) *$n \geqslant 3$, the elements w, x, y, and y' are linearly independent modulo $\mathfrak{m}^2$, and the following hold: $z \notin (w) + \mathfrak{m}^2$, $z \notin (x) + \mathfrak{m}^2$, and $(y, y') \subseteq (0 : z)$; or*
(d) *$n \geqslant 3$, the elements x, y, and y' are linearly independent modulo $\mathfrak{m}^2$, and the following hold: $w \in (x) + \mathfrak{m}^2$, $z \notin (x) + \mathfrak{m}^2$, and $(y, y') \subseteq (0 : z)$.*

Then the modules in the family $\{M_n(w, x, \lambda y + y', z)\}_{\lambda \in \mathcal{L}}$ are indecomposable, totally reflexive, and pairwise non-isomorphic.

The proof of (6.2) takes up the balance of this section; here is the cornerstone:

(6.3) **Proposition.** *Let $(R, \mathfrak{m})$ be a local ring and let w, x, y, y', and z be elements in $\mathfrak{m} \backslash \mathfrak{m}^2$. Assume that the following hold:*

$$z \notin (w) + \mathfrak{m}^2, \quad z \notin (x) + \mathfrak{m}^2, \quad y \notin (w, x) + \mathfrak{m}^2, \quad \text{and} \quad y' \notin (w, y) + \mathfrak{m}^2.$$

If θ and λ are elements in R with $\theta - \lambda \notin \mathfrak{m}$, and $n \geqslant 3$ is an integer, then the R-modules $M_n(w, x, \theta y + y', z)$ and $M_n(w, x, \lambda y + y', z)$ are non-isomorphic.

The next example shows that the condition $n \geqslant 3$ in (6.3) cannot be relaxed.

(6.4) **Example.** Let $(R, \mathfrak{m})$ be a local ring with $\operatorname{emb.dim} R \geqslant 3$ and assume that 2 is a unit in R. Let w, x, and y be linearly independent modulo $\mathfrak{m}^2$, and set $y' = y - 2^{-1}x$. The equality

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} w & y' \\ 0 & x \end{pmatrix} = \begin{pmatrix} w & y' - 2y \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

shows that the R-modules $M_2(w, x, 0y + y')$ and $M_2(w, x, -2y + y')$ are isomorphic.

To get a statement similar to Proposition (6.3) for 2-generated modules, it suffices to assume that y' is outside the span of w, x, and y modulo $\mathfrak{m}^2$.

(6.5) **Proposition.** *Let $(R,\mathfrak{m})$ be a local ring and let w, x, y, and y' be elements in $\mathfrak{m}\backslash\mathfrak{m}^2$. Assume that one of the following conditions holds:*

(a) *The elements w, x, y, and y' are linearly independent modulo $\mathfrak{m}^2$.*

(b) *The elements x, y, and y' are linearly independent modulo $\mathfrak{m}^2$, and the element w belongs to $(x)+\mathfrak{m}^2$.*

If θ and λ are elements in R with $\theta-\lambda\notin\mathfrak{m}$, then the R-modules $M_2(w,x,\theta y+y')$ and $M_2(w,x,\lambda y+y')$ are non-isomorphic.

Proof. Let θ and λ be in R. Assume that (a) holds and that the R-modules $M_2(w,x,\theta y+y')$ and $M_2(w,x,\lambda y+y')$ are isomorphic. It follows that there exist matrices A and B in $\mathrm{GL}_2(R)$ such that the equality

$$\mathrm{A}(w\mathrm{I}^o+x\mathrm{I}^e+(\theta y+y')\mathrm{J}^o)=(w\mathrm{I}^o+x\mathrm{I}^e+(\lambda y+y')\mathrm{J}^o)\mathrm{B}$$

holds; here the matrices $\mathrm{I}^o=\mathrm{I}^o_2$, $\mathrm{I}^e=\mathrm{I}^e_2$, and $\mathrm{J}^o=\mathrm{J}^o_2$ are as defined in Construction (2.6). The goal is to prove that $\theta-\lambda$ is in $\mathfrak{m}$. After reduction modulo $\mathfrak{m}$, the equality above yields, in particular,

$$\bar{\mathrm{A}}\mathrm{J}^o-\mathrm{J}^o\bar{\mathrm{B}}=0=\bar{\theta}\bar{\mathrm{A}}\mathrm{J}^o-\bar{\lambda}\mathrm{J}^o\bar{\mathrm{B}},$$

and, therefore, $(\bar{\theta}-\bar{\lambda})\mathrm{J}^o\bar{\mathrm{B}}=0$. As the matrix $\bar{\mathrm{B}}$ is invertible and J^o is non-zero, this implies that $\theta-\lambda$ is in $\mathfrak{m}$ as desired.

If (b) holds, the desired conclusion is proved under Case 1 in the next proof. □

Proof of (6.3). Let θ and λ be elements in R and assume that $M_n(w,x,\theta y+y',z)$ and $M_n(w,x,\lambda y+y',z)$ are isomorphic as R-modules. It follows that there exist matrices A and B in $\mathrm{GL}_n(R)$ such that the equality

$$\mathrm{A}(w\mathrm{I}^o+x\mathrm{I}^e+(\theta y+y')\mathrm{J}^o+z\mathrm{J}^e)=(w\mathrm{I}^o+x\mathrm{I}^e+(\lambda y+y')\mathrm{J}^o+z\mathrm{J}^e)\mathrm{B} \tag{1}$$

holds; here the matrices $\mathrm{I}^o=\mathrm{I}^o_n$, $\mathrm{I}^e=\mathrm{I}^e_n$, $\mathrm{J}^o=\mathrm{J}^o_n$, and $\mathrm{J}^e=\mathrm{J}^e_n$ are as defined in Construction (2.6). The goal is to prove that $\theta-\lambda$ is in $\mathfrak{m}$.

Case 1: w is in $(x)+\mathfrak{m}^2$. Under this assumption, one can write $w=rx+\delta$, where $\delta\in\mathfrak{m}^2$, and rewrite (1) as

$$\begin{aligned}&x(r(\mathrm{A}\mathrm{I}^o-\mathrm{I}^o\mathrm{B})+\mathrm{A}\mathrm{I}^e-\mathrm{I}^e\mathrm{B})+y(\theta\mathrm{A}\mathrm{J}^o-\lambda\mathrm{J}^o\mathrm{B})\\&\qquad+y'(\mathrm{A}\mathrm{J}^o-\mathrm{J}^o\mathrm{B})+z(\mathrm{A}\mathrm{J}^e-\mathrm{J}^e\mathrm{B})=\Delta,\end{aligned} \tag{2}$$

where Δ is a matrix with entries in $\mathfrak{m}^2$. The assumptions on w, x, y, and y' imply $y\notin(x)+\mathfrak{m}^2$ and $y'\notin(x,y)+\mathfrak{m}^2$, so the elements x, y, and y' are linearly independent modulo $\mathfrak{m}^2$. There exist elements v_i such that $v_1,\dots,v_{e-3},x,y,y'$ form a minimal set of generators for $\mathfrak{m}$. Write

$$z=sx+ty+uy'+\sum_{i=1}^{e-3}d_iv_i,$$

substitute this expression into (2), and reduce modulo $\mathfrak{m}$ to get

$$0=\bar{r}(\bar{\mathrm{A}}\mathrm{I}^o-\mathrm{I}^o\bar{\mathrm{B}})+\bar{\mathrm{A}}\mathrm{I}^e-\mathrm{I}^e\bar{\mathrm{B}}+\bar{s}(\bar{\mathrm{A}}\mathrm{J}^e-\mathrm{J}^e\bar{\mathrm{B}}), \tag{3}$$

$$0=\bar{\theta}\bar{\mathrm{A}}\mathrm{J}^o-\bar{\lambda}\mathrm{J}^o\bar{\mathrm{B}}+\bar{t}(\bar{\mathrm{A}}\mathrm{J}^e-\mathrm{J}^e\bar{\mathrm{B}}), \tag{4}$$

$$0=\bar{\mathrm{A}}\mathrm{J}^o-\mathrm{J}^o\bar{\mathrm{B}}+\bar{u}(\bar{\mathrm{A}}\mathrm{J}^e-\mathrm{J}^e\bar{\mathrm{B}}),\ \text{and} \tag{5}$$

$$0=\bar{d_i}(\bar{\mathrm{A}}\mathrm{J}^e-\mathrm{J}^e\bar{\mathrm{B}}),\ \text{for } i\in\{1,\dots,e-3\}. \tag{6}$$

The arguments that follow use the relations that (3)–(6) induce between the entries of the matrices $\bar{\mathrm{A}} = (a_{ij})$ and $\bar{\mathrm{B}} = (b_{ij})$. With the convention $a_{hl} = 0 = b_{hl}$ for $h, l \in \{0, n+1\}$, it is elementary to verify that the following systems of equalities hold for i and j in $\{1, \dots, n\}$ and elements f and g in $R/\mathfrak{m}$:

$$(\bar{\mathrm{A}}\mathrm{I}^o - \mathrm{I}^o\bar{\mathrm{B}})_{ij} = \begin{cases} a_{ij} - b_{ij} & i \text{ odd}, j \text{ odd} \\ -b_{ij} & i \text{ odd}, j \text{ even} \\ a_{ij} & i \text{ even}, j \text{ odd} \\ 0 & i \text{ even}, j \text{ even} \end{cases} \tag{7}$$

$$(\bar{\mathrm{A}}\mathrm{I}^e - \mathrm{I}^e\bar{\mathrm{B}})_{ij} = \begin{cases} 0 & i \text{ odd}, j \text{ odd} \\ a_{ij} & i \text{ odd}, j \text{ even} \\ -b_{ij} & i \text{ even}, j \text{ odd} \\ a_{ij} - b_{ij} & i \text{ even}, j \text{ even} \end{cases} \tag{8}$$

$$(f\bar{\mathrm{A}}\mathrm{J}^o - g\mathrm{J}^o\bar{\mathrm{B}})_{ij} = \begin{cases} gb_{(i+1)j} & i \text{ odd}, j \text{ odd} \\ fa_{i(j-1)} - gb_{(i+1)j} & i \text{ odd}, j \text{ even} \\ 0 & i \text{ even}, j \text{ odd} \\ fa_{i(j-1)} & i \text{ even}, j \text{ even} \end{cases} \tag{9}$$

$$(\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}})_{ij} = \begin{cases} a_{i(j-1)} & i \text{ odd}, j \text{ odd} \\ 0 & i \text{ odd}, j \text{ even} \\ a_{i(j-1)} - b_{(i+1)j} & i \text{ even}, j \text{ odd} \\ -b_{(i+1)j} & i \text{ even}, j \text{ even} \end{cases} \tag{10}$$

Each of the equalities (3)–(6) induces four subsystems, which are referred to by subscripts 'oo', 'oe', 'eo', and 'ee', where 'oo' stands for 'i odd and j odd' etc. For example, $(4)_{\mathrm{oe}}$ refers to the equalities $0 = \bar{\theta}a_{i(j-1)} - \bar{\lambda}b_{(i+1)j}$, for i odd and j even.

Let $n \geqslant 2$; the goal is to prove the equality $\bar{\theta} = \bar{\lambda}$, as that implies $\theta - \lambda \in \mathfrak{m}$. First assume $\bar{d_i} \neq 0$ for some $i \in \{1, \dots, e-3\}$, then (6) yields $\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}} = 0$. From (4) and (5) one then gets

$$\bar{\mathrm{A}}\mathrm{J}^o - \mathrm{J}^o\bar{\mathrm{B}} = 0 = \bar{\theta}\bar{\mathrm{A}}\mathrm{J}^o - \bar{\lambda}\mathrm{J}^o\bar{\mathrm{B}}$$

and thus $(\bar{\theta} - \bar{\lambda})\mathrm{J}^o\bar{\mathrm{B}} = 0$. As $\bar{\mathrm{B}}$ is invertible and J^o is non-zero, this yields the desired equality $\bar{\theta} = \bar{\lambda}$. Henceforth we assume $\bar{d_i} = 0$ for all $i \in \{1, \dots, e-3\}$.

By assumption, z is not in $(x) + \mathfrak{m}^2$, so $\bar{u}$ or $\bar{t}$ is non-zero. In case $\bar{u}$ is non-zero, $(5)_{\mathrm{oo}}$ and $(5)_{\mathrm{eo}}$ yield $b_{h1} = 0$ for $1 < h \leqslant n$. If $\bar{t}$ is non-zero, then $(4)_{\mathrm{oo}}$ and $(4)_{\mathrm{eo}}$ yield the same conclusion. The matrix $\bar{\mathrm{B}}$ is invertible, so each of its columns contains a non-zero element. It follows that b_{11} is non-zero, and by $(3)_{\mathrm{oo}}$ one has $a_{11} = b_{11}$. From $(4)_{\mathrm{oe}}$ and $(5)_{\mathrm{oe}}$ one gets $a_{11} = b_{22}$ and $(\bar{\theta} - \bar{\lambda})a_{11} = 0$, whence the equality $\bar{\theta} = \bar{\lambda}$ holds.

For $n = 2$ the arguments above establish the assertion in Proposition (6.5) under assumption (b) *ibid.*

Case 2: w is not in $(x) + \mathfrak{m}^2$. It follows from the assumption $y \notin (w, x) + \mathfrak{m}^2$ that w, x, and y are linearly independent modulo $\mathfrak{m}^2$, so there exist elements v_i such that $v_1, \dots, v_{e-3}, w, x, y$ form a minimal set of generators for $\mathfrak{m}$. Write

$$y' = pw + qx + ry + \sum_{i=1}^{e-3} c_i v_i \qquad \text{and} \qquad z = sw + tx + uy + \sum_{i=1}^{e-3} d_i v_i.$$

Substitute these expressions into (1) and reduce modulo $\mathfrak{m}$ to get

$$0 = \bar{\mathrm{A}}\mathrm{I}^o - \mathrm{I}^o\bar{\mathrm{B}} + \bar{p}(\bar{\mathrm{A}}\mathrm{J}^o - \mathrm{J}^o\bar{\mathrm{B}}) + \bar{s}(\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}}), \tag{11}$$

$$0 = \bar{\mathrm{A}}\mathrm{I}^e - \mathrm{I}^e\bar{\mathrm{B}} + \bar{q}(\bar{\mathrm{A}}\mathrm{J}^o - \mathrm{J}^o\bar{\mathrm{B}}) + \bar{t}(\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}}), \tag{12}$$

$$0 = \bar{\theta}\bar{\mathrm{A}}\mathrm{J}^o - \bar{\lambda}\mathrm{J}^o\bar{\mathrm{B}} + \bar{r}(\bar{\mathrm{A}}\mathrm{J}^o - \mathrm{J}^o\bar{\mathrm{B}}) + \bar{u}(\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}}), \text{ and} \tag{13}$$

$$0 = \bar{c}_i(\bar{\mathrm{A}}\mathrm{J}^o - \mathrm{J}^o\bar{\mathrm{B}}) + \bar{d}_i(\bar{\mathrm{A}}\mathrm{J}^e - \mathrm{J}^e\bar{\mathrm{B}}), \text{ for } i \in \{1, \dots, e-3\}. \tag{14}$$

As in Case 1, set $\bar{\mathrm{A}} = (a_{ij})$ and $\bar{\mathrm{B}} = (b_{ij})$ so that the equalities (7)–(10) hold. The subscripts 'oo', 'oe', 'eo', and 'ee' are used, as in Case 1, to denote the subsystems induced by (11)–(14).

Let $n \geqslant 3$; the goal is, again, to prove the equality $\bar{\theta} = \bar{\lambda}$. In the following, h and l are integers in $\{1, \dots, n\}$. First, notice that if $\bar{c}_m \neq 0$ for some $m \in \{1, \dots, e-3\}$, then $(14)_{\mathrm{oe}}$ implies $a_{11} = b_{22}$ and, in turn, $(13)_{\mathrm{oe}}$ yields $(\bar{\theta} - \bar{\lambda})a_{11} = 0$. To conclude $\bar{\theta} = \bar{\lambda}$, it must be verified that a_{11} is non-zero. To this end, assume first $\bar{d}_m = 0$; from $(14)_{\mathrm{ee}}$ and $(14)_{\mathrm{oo}}$ one immediately gets

$$a_{h1} = 0 = b_{h1} \quad \text{for } h \text{ even.} \tag{15}$$

As z is in $\mathfrak{m}\backslash\mathfrak{m}^2$, one of the coefficients $\bar{d}_1, \dots, \bar{d}_{e-3}, \bar{s}, \bar{t}, \bar{u}$ is non-zero. If $\bar{u}$ or one of $\bar{d}_1, \dots, \bar{d}_{e-3}$ is non-zero, then $(13)_{\mathrm{eo}}$ or $(14)_{\mathrm{eo}}$ yields $b_{h1} = 0$ for $h > 1$ odd. If $\bar{s}$ or $\bar{t}$ is non-zero, then the same conclusion follows from (15) combined with $(11)_{\mathrm{eo}}$ or with $(12)_{\mathrm{eo}}$. Now that $b_{h1} = 0$ holds for all $h \geqslant 2$, it follows that b_{11} is non-zero. Finally, $(11)_{\mathrm{oo}}$ yields $a_{11} = b_{11}$, so the entry a_{11} is not zero, as desired. Now assume $\bar{d}_m \neq 0$, then $(14)_{\mathrm{oo}}$ and $(14)_{\mathrm{eo}}$ immediately give $b_{h1} = 0$ for $h > 1$. As above, we conclude that b_{11} is non-zero, whence $a_{11} \neq 0$ by $(11)_{\mathrm{oo}}$. This concludes the argument under the assumption that one of the coefficients $\bar{c}_i$ is non-zero. Henceforth we assume $\bar{c}_i = 0$ for all $i \in \{1, \dots, e-3\}$; it follows that $\bar{q}$ is non-zero, as $y' \notin (w, y) + \mathfrak{m}^2$ by assumption.

If $\bar{d}_i \neq 0$ for some $i \in \{1, \dots, e-3\}$, then $(14)_{\mathrm{oo}}$ yields $a_{12} = 0$, as n is at least 3. From $(12)_{\mathrm{oe}}$ one gets $a_{11} = b_{22}$, and then $(13)_{\mathrm{oe}}$ implies $(\bar{\theta} - \bar{\lambda})a_{11} = 0$. To see that a_{11} is non-zero, notice that $(14)_{\mathrm{eo}}$ yields $b_{h1} = 0$ for $h > 1$ odd, while $(12)_{\mathrm{oo}}$ yields $b_{h1} = 0$ for h even. It follows that b_{11} is non-zero, and then $a_{11} \neq 0$, by $(11)_{\mathrm{oo}}$. Thus the desired equality $\bar{\theta} = \bar{\lambda}$ holds. This concludes the argument under the assumption that one of the coefficients $\bar{d}_i$ is non-zero. Henceforth we assume $\bar{d}_i = 0$ for all $i \in \{1, \dots, e-3\}$.

To finish the argument, we deal separately with the cases $\bar{u} \neq 0$ and $\bar{u} = 0$. Assume first $\bar{u} \neq 0$. By assumption one has $n \geqslant 3$, so $(13)_{\mathrm{eo}}$ yields $a_{22} = b_{33}$, and then it follows from $(12)_{\mathrm{eo}}$ that b_{23} is zero. From $(13)_{\mathrm{oo}}$ one gets $a_{12} = 0$, and then the equality $a_{11} = b_{22}$ follows from $(12)_{\mathrm{oe}}$. In turn, $(13)_{\mathrm{oe}}$ implies $(\bar{\theta} - \bar{\lambda})a_{11} = 0$. To see that a_{11} is non-zero, notice that there are equalities $b_{h1} = 0$ for $h > 1$ odd by $(13)_{\mathrm{eo}}$ and $b_{h1} = 0$ for h even by $(12)_{\mathrm{oo}}$. It follows that b_{11} is non-zero, and the equality $a_{11} = b_{11}$ holds by $(11)_{\mathrm{oo}}$. As above we conclude $\bar{\theta} = \bar{\lambda}$.

Finally, assume $\bar{u} = 0$; the assumptions $z \notin (w) + \mathfrak{m}^2$ and $z \notin (x) + \mathfrak{m}^2$ imply that $\bar{s}$ and $\bar{t}$ are both non-zero. From $(13)_{\mathrm{ee}}$ and $(13)_{\mathrm{oo}}$ one gets:

$$(\bar{\theta} + \bar{r})a_{h1} = 0 = (\bar{\lambda} + \bar{r})b_{h1} \quad \text{for } h \text{ even.} \tag{16}$$

First assume that $\bar{\lambda} + \bar{r}$ is zero. If $\bar{\theta} + \bar{r}$ is zero, then the desired equality $\bar{\theta} = \bar{\lambda}$ holds. If $\bar{\theta} + \bar{r}$ is non-zero, then (16) gives $a_{h1} = 0$ for h even. Moreover, $(13)_{\mathrm{oe}}$ implies $(\bar{\theta} + \bar{r})a_{h1} = (\bar{\lambda} + \bar{r})b_{(h+1)2} = 0$, so $a_{h1} = 0$ for h odd as well, which is absurd as $\bar{\mathrm{A}}$ is invertible. Now assume that $\bar{\lambda} + \bar{r}$ is non-zero. From (16) one gets

$b_{h1} = 0$ for h even and, in turn, $(12)_{\mathrm{eo}}$ yields $b_{h1} = 0$ for $h > 1$ odd. It follows that b_{11} is non-zero, and then one has $a_{11} \neq 0$ by $(11)_{\mathrm{oo}}$. By assumption, n is at least 3, so $(13)_{\mathrm{oo}}$ gives $(\bar{\lambda} + \bar{r})b_{23} = 0$, which implies $b_{23} = 0$. Now $(12)_{\mathrm{oo}}$ yields $a_{12} = 0$, and then $(12)_{\mathrm{oe}}$ implies $a_{11} = b_{22}$. From $(13)_{\mathrm{oe}}$ one gets $(\bar{\theta} - \bar{\lambda})a_{11}$, whence the equality $\bar{\theta} = \bar{\lambda}$ holds. $\square$

Proof of (6.2). Under the assumptions in part (a) or (b), it is immediate from Theorem (3.1) that the module $M_2(w, x, \lambda y + y')$ is indecomposable and totally reflexive for every $\lambda \in R$. It follows from Proposition (6.5) that the modules in the family $\{M_2(w, x, \lambda y + y')\}_{\lambda \in \mathcal{L}}$ are pairwise non-isomorphic.

Fix $n \geqslant 3$. Proposition (6.3) shows that, under the assumptions in part (c) or (d), the modules in the family $\{M_n(w, x, \lambda y + y', z)\}_{\lambda \in \mathcal{L}}$ are pairwise non-isomorphic. Moreover, for every $\lambda \in R$, one has $(\lambda y + y')z = 0$, so it follows from Theorem (3.1) the module $M_n(w, z, \lambda y + y', z)$ is totally reflexive and indecomposable. $\square$

7. Brauer–Thrall II over short local rings with exact zero divisors

In this section $(R, \mathfrak{m}, \mathsf{k})$ is a local ring with $\mathfrak{m}^3 = 0$. Together, (4.1.1) and the theorems (3.1), (7.4), (7.6), and (7.8) establish Theorem (1.4).

(7.1) **Remark.** Assume that R has embedding dimension 2. If there is an exact zero divisor in R, then the Hilbert series of R is $1 + 2\tau + \tau^2$, and the equality $(0 : \mathfrak{m}) = \mathfrak{m}^2$ holds; see (4.2). Therefore, R is Gorenstein, and the equality $x\mathfrak{m} = \mathfrak{m}^2$ holds for all $x \in \mathfrak{m} \backslash \mathfrak{m}^2$; in particular, every element in $\mathfrak{m} \backslash \mathfrak{m}^2$ is an exact zero divisor by Lemma (4.3)(c). On the other hand, if R is Gorenstein, then one has $\mathrm{H}_R(\tau) = 1 + 2\tau + \tau^2$ and $(0 : \mathfrak{m}) = \mathfrak{m}^2$. It is now elementary to verify that the equalities $\mathrm{length}_R(x) = 2 = \mathrm{length}_R(0 : x)$ hold for every $x \in \mathfrak{m} \backslash \mathfrak{m}^2$, so every such element is an exact zero divisor in R by Lemma (4.3)(c). It follows from work of Serre [18, prop. 5] that R is Gorenstein if and only if it is complete intersection; thus the following conditions are equivalent:

(*i*) There is an exact zero divisor in R.
(*ii*) Every element in $\mathfrak{m} \backslash \mathfrak{m}^2$ is an exact zero divisor.
(*iii*) R is complete intersection.

In contrast, if R has embedding dimension at least 3, and k is algebraically closed, then there exist elements in $\mathfrak{m} \backslash \mathfrak{m}^2$ that are not exact zero divisors. This fact follows from Lemma (7.3), and it is essential for our proof of Theorem (7.4).

(7.2) **Remark.** Assume that R has Hilbert series $1 + e\tau + f\tau^2$, and let x be an element in $\mathfrak{m}$. The equality $x\mathfrak{m} = \mathfrak{m}^2$ holds if and only if the k-linear map from $\mathfrak{m}/\mathfrak{m}^2$ to $\mathfrak{m}^2$ given by multiplication by x is surjective. Let Ξ_x be a matrix that represents this map; it is an $f \times e$ matrix with entries in k, so the equality $x\mathfrak{m} = \mathfrak{m}^2$ holds if and only if Ξ_x has rank f.

Assume that the (in)equalities $f = e - 1 \geqslant 1$ hold. If w and x are elements in R with $wx = 0$, then they form an exact pair of zero divisors if and only if both matrices Ξ_x and Ξ_w have a non-zero maximal minor; cf. Lemma (4.3)(c).

(7.3) **Lemma.** *Assume that $(R, \mathfrak{m}, \mathsf{k})$ has Hilbert series $1 + e\tau + f\tau^2$ and that k is algebraically closed; set $n = e - f + 1$. If one has $2 \leqslant f \leqslant e - 1$ and $v_0, \ldots, v_n \in \mathfrak{m}$ are linearly independent modulo $\mathfrak{m}^2$, then there exist $r_0, \ldots, r_n \in R$, at least one of which is invertible, such that the ideal $(\sum_{h=0}^{n} r_h v_h)\mathfrak{m}$ is properly contained in $\mathfrak{m}^2$.*

Proof. Let $\{x_1,\dots,x_e\}$ be a minimal set of generators for $\mathfrak{m}$ and let $\{u_1,\dots,u_f\}$ be a basis for the k-vector space $\mathfrak{m}^2$. For $h \in \{0,\dots,n\}$ and $j \in \{1,\dots,e\}$ write

$$v_h x_j = \sum_{i=1}^{f} \xi_{hij} u_i \,,$$

where the elements ξ_{hij} are in k. In the following, $\bar{r}$ denotes the image in k of the element $r \in R$. For $r_0,\dots,r_n$ in R and $v = \sum_{h=0}^{n} r_h v_h$ the equality $v\mathfrak{m} = \mathfrak{m}^2$ holds if and only if the $f \times e$ matrix

$$\Xi_v = \left(\sum_{h=0}^{n} \bar{r}_h \xi_{hij} \right)_{ij}$$

has rank f; see Remark (7.2). Let X be the matrix obtained from Ξ_v by replacing $\bar{r}_0,\dots,\bar{r}_n$ with indeterminates $\chi_0,\dots,\chi_n$. The non-zero entries in X are then homogeneous linear forms in $\chi_0,\dots,\chi_n$, and in the polynomial algebra $\mathsf{k}[\chi_0,\dots,\chi_n]$ the ideal $I_f(X)$, generated by the maximal minors of X, has height at most n; see [2, thm. (2.1)]. As $I_f(\mathrm{X})$ is generated by homogeneous polynomials, it follows from Hilbert's Nullstellensatz that there exists a point $(\rho_0,\dots,\rho_n)$ in $\mathsf{k}^{n+1}\backslash\{0\}$ such that all maximal minors of the matrix $(\sum_{h=0}^{n} \rho_h \xi_{hij})_{ij}$ vanish. Let $r_0,\dots,r_n$ be lifts of $\rho_0,\dots,\rho_n$ in R; then at least one of them is not in $\mathfrak{m}$, and the ideal $(\sum_{h=0}^{n} r_h v_h)\mathfrak{m}$ is properly contained in $\mathfrak{m}^2$. □

(7.4) **Theorem.** *Let $(R,\mathfrak{m},\mathsf{k})$ be a local ring with $\mathfrak{m}^3 = 0$, emb.dim $R \geqslant 3$, and k algebraically closed. If w and x form an exact pair of zero divisors in R, then there exist elements y, y', and z in $\mathfrak{m}\backslash\mathfrak{m}^2$, such that for every lift $\mathcal{L}$ of $\mathsf{k}\backslash\{0\}$ in R and for every integer $n \geqslant 3$ the modules in the family $\{M_n(w,x,\lambda y + y',z)\}_{\lambda\in\mathcal{L}}$ are indecomposable, totally reflexive, and pairwise non-isomorphic.*

Proof. Assume that w and x form an exact pair of zero divisors in R. By (4.2) the Hilbert series of R is $1 + e\tau + (e-1)\tau^2$, so it follows from Lemma (7.3) that there exists an element $z \in \mathfrak{m}\backslash\mathfrak{m}^2$ such that $z\mathfrak{m}$ is properly contained in $\mathfrak{m}^2$. In particular, one has $\operatorname{length}_R(z) < e$ and, therefore, $\operatorname{length}_R(0:z) > e$ by additivity of length on short exact sequences. It follows that there exist two elements, call them y and y', in $(0:z)$ that are linearly independent modulo $\mathfrak{m}^2$. The inequality $\operatorname{length}_R(z) < e$ implies that z is not in $(w) = (w) + \mathfrak{m}^2$ and not in $(x) = (x) + \mathfrak{m}^2$. If y and y' were both in $(w,x) = (w,x) + \mathfrak{m}^2$, then x would be in (y,y'), which is impossible as $z \notin (w)$. Without loss of generality, assume $y \notin (w,x) + \mathfrak{m}^2$. If y' were in $(w,y) = (w,y) + \mathfrak{m}^2$, then w would be in (y,y'), which is also impossible. Now let $\mathcal{L}$ be a lift of $\mathsf{k}\backslash\{0\}$ in R and let $n \geqslant 3$ be an integer. It follows from Proposition (6.3) that the modules in the family $\{M_n(w,x,\lambda y + y',z)\}_{\lambda\in\mathcal{L}}$ are pairwise non-isomorphic.

If w is in $(x) = (x) + \mathfrak{m}^2$, then it follows from Theorem (3.1) that the modules in the family $\{M_n(w,x,\lambda y + y',z)\}_{\lambda\in\mathcal{L}}$ are indecomposable and totally reflexive. Indeed, for every $\lambda \in R$ the element $\lambda y + y'$ annihilates z, whence it is not in $(x) = (x) + \mathfrak{m}^2$.

If w is not in $(x) = (x) + \mathfrak{m}^2$, then it follows from the assumption $y \notin (w,x) + \mathfrak{m}^2$ that w, x, and y are linearly independent modulo $\mathfrak{m}^2$. There exist elements v_i such that $v_1, \dots, v_{e-3}, w, x, y$ form a minimal set of generators for $\mathfrak{m}$. Write

$$y' = pw + qx + ry + \sum_{i=1}^{e-3} c_i v_i.$$

As $yz = 0$ holds and the elements y and y' are linearly independent modulo $\mathfrak{m}^2$, we may assume $r = 0$. For $\lambda \in \mathcal{L}$, the elements w, x, and $\lambda y + y'$ are linearly independent. Indeed, if there is a relation

$$sw + tx + u(\lambda y + y') = (s + up)w + (t + uq)x + u\lambda y + u \sum_{i=1}^{e-3} c_i v_i \in \mathfrak{m}^2,$$

then u is in $\mathfrak{m}$ as $\lambda \notin \mathfrak{m}$, and then s and t are in $\mathfrak{m}$ as w and x are linearly independent modulo $\mathfrak{m}^2$. Now it follows from Theorem (3.1) that the modules in the family $\{M_n(w, x, \lambda y + y', z)\}_{\lambda \in \mathcal{L}}$ are indecomposable and totally reflexive. $\square$

(7.5) **Remark.** Assume that R has embedding dimension 3 and that $\mathfrak{m}$ is minimally generated by elements v, w, and x, where w and x form an exact pair of zero divisors. Let y and y' be elements in $\mathfrak{m}$ and let $\mathcal{L}$ be a subset of R. It is elementary to verify that each module in the family $\{M_2(w, x, \lambda y + y')\}_{\lambda \in \mathcal{L}}$ is isomorphic to either $R/(w) \oplus R/(x)$ or to the indecomposable R-module $M_2(w, x, v)$. Thus the requirement $n \geqslant 3$ in Theorem (7.4) cannot be relaxed.

We now proceed to deal with 1- and 2-generated modules.

(7.6) **Theorem.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring with $\mathfrak{m}^3 = 0$, $\operatorname{emb.dim} R \geqslant 2$, and k infinite. If there is an exact zero divisor in R, then the set*

$$\mathcal{N}_1 = \{\, R/(x) \mid x \text{ is an exact zero divisor in } R \,\}$$

of totally reflexive R-modules contains a subset $\mathcal{M}_1$ of cardinality $\operatorname{card}(\mathsf{k})$, such that the modules in $\mathcal{M}_1$ are pairwise non-isomorphic.

Proof. Assume that there is an exact zero divisor in R; then R has Hilbert series $1 + e\tau + (e-1)\tau^2$; see (4.2). Let $\{x_1, \dots, x_e\}$ be a minimal set of generators for $\mathfrak{m}$ and let $\{u_1, \dots, u_{e-1}\}$ be a basis for $\mathfrak{m}^2$. For h and j in $\{1, \dots, e\}$ write

$$x_h x_j = \sum_{i=1}^{e-1} \xi_{hij} u_i,$$

where the elements ξ_{hij} are in k. For $r_1, \dots, r_e$ in R let $\bar{r}_h$ denote the image of r_h in k, and set $x = r_1 x_1 + \cdots + r_e x_e$. The equality $x\mathfrak{m} = \mathfrak{m}^2$ holds if and only if the $(e-1) \times e$ matrix

$$\Xi_x = \left(\sum_{h=1}^{e} \bar{r}_h \xi_{hij} \right)_{ij}$$

has rank $e-1$; see Remark (7.2). For $j \in \{1, \dots, e\}$ let $\mu_j(x)$ be the maximal minor of Ξ_x obtained by omitting the jth column. Notice that each minor $\mu_j(x)$ is a homogeneous polynomial expression in the elements $\bar{r}_1, \dots, \bar{r}_e$. The column vector $(\mu_1(x) \;\; -\mu_2(x) \;\; \dots \;\; (-1)^{e-1}\mu_e(x))^{\mathrm{T}}$ is in the null-space of the matrix Ξ_x, so the element

$$w = \mu_1(x)x_1 - \mu_2(x)x_2 + \cdots + (-1)^{e-1}\mu_e(x)x_e$$

annihilates x. Let $\nu_1(w), \dots, \nu_e(w)$ be the maximal minors of the matrix Ξ_w; they are homogeneous polynomial expressions in $\mu_1(x), \dots, \mu_e(x)$ and, therefore, in $\bar{r}_1, \dots, \bar{r}_e$. By Remark (7.2) the elements x and w form an exact pair of zero divisors if and only if both matrices Ξ_x and Ξ_w have rank $e-1$.

For $j \in \{1, \dots, e\}$ define μ_j and ν_j to be the homogeneous polynomials in $\mathsf{k}[\chi_1, \dots, \chi_e]$ obtained from $\mu_j(x)$ and $\nu_j(w)$ by replacing the elements $\bar{r}_h$ by the indeterminates χ_h, for $h \in \{1, \dots, e\}$. In the projective space $\mathbb{P}_{\mathsf{k}}^{e-1}$, the complement $\mathcal{E}$ of the intersection of vanishing sets $\mathcal{Z}(\nu_1) \cap \cdots \cap \mathcal{Z}(\nu_e)$ is an open set. No point in $\mathcal{E}$ is in the intersection $\mathcal{Z}(\mu_1) \cap \cdots \cap \mathcal{Z}(\mu_e)$, as each polynomial ν_j is a polynomial in $\mu_1, \dots, \mu_e$. Therefore, each point $(\rho_1 : \cdots : \rho_e)$ in $\mathcal{E}$ corresponds to an exact zero divisor as follows. Let $r_1, \dots, r_e$ be lifts of $\rho_1, \dots, \rho_e$ in R; then the element $x = r_1x_1 + \cdots + r_ex_e$ is an exact zero divisor. It is clear that two distinct points in $\mathcal{E}$ correspond to non-isomorphic modules in $\mathcal{N}_1$. Take as $\mathcal{M}_1$ any subset of $\mathcal{N}_1$ such that the elements of $\mathcal{M}_1$ are in one-to-one correspondence with the points in $\mathcal{E}$. By assumption, the set $\mathcal{E}$ is non-empty, and, if k is infinite, then $\mathcal{E}$ has the same cardinality as k. □

(7.7) **Remark.** The modules in $\mathcal{M}_1$ are in one-to-one correspondence with the points of a non-empty Zariski open set in $\mathbb{P}_{\mathsf{k}}^{e-1}$. See also Remark (8.8).

(7.8) **Theorem.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring with $\mathfrak{m}^3 = 0$, emb.dim $R \geqslant 3$, and k infinite. If there is an exact zero divisor in R, then the set*

$$\mathcal{N}_2 = \left\{ M_2(w, x, y) \;\middle|\; \begin{array}{c} w,\ x, \text{ and } y \text{ are elements in } R, \text{ such that} \\ w \text{ and } x \text{ form an exact pair of zero divisors} \end{array} \right\}$$

of totally reflexive R-modules contains a subset $\mathcal{M}_2$ of cardinality card(k), *such that the modules in $\mathcal{M}_2$ are indecomposable and pairwise non-isomorphic.*

Proof. Assume that there is an exact zero divisor in R and let $\mathcal{M}_1$ be the set of cyclic totally reflexive R-modules afforded by Theorem (7.6); the cardinality of $\mathcal{M}_1$ is card(k). From $\mathcal{M}_1$ one can construct another set of the same cardinality, whose elements are exact pairs of zero divisors, such that for any two of them, say w, x and w', x', one has $(x) \ncong (x')$ and $(w) \ncong (x')$. Given two such pairs, choose elements y and y' in $\mathfrak{m} \backslash \mathfrak{m}^2$ such that $y \notin (w, x)$ and $y' \notin (w', x')$. By Theorem (4.4), the modules $M_2(w, x, y)$ and $M_2(w', x', y')$ are indecomposable and totally reflexive.

Suppose that the R-modules $M_2(w, x, y)$ and $M_2(w', x', y')$ are isomorphic; then there exist matrices $\mathrm{A} = (a_{ij})$ and $\mathrm{B} = (b_{ij})$ in $\mathrm{GL}_2(R)$ such that the equality

$$\mathrm{A} \begin{pmatrix} w & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} w' & y' \\ 0 & x' \end{pmatrix} \mathrm{B}$$

holds. In particular, there are equalities

$$a_{21}w = b_{21}x' \qquad \text{and} \qquad a_{21}y + a_{22}x = b_{22}x'.$$

The first one shows that the entries a_{21} and b_{21} are elements in $\mathfrak{m}$, and then it follows from the second one that a_{22} and b_{22} are in $\mathfrak{m}$. Thus A and B each have a row with entries in $\mathfrak{m}$, which contradicts the assumption that they are invertible. □

8. Existence of exact zero divisors

In previous sections we constructed families of totally reflexive modules starting from an exact pair of zero divisors. Now we address the question of existence of exact zero divisors; in particular, we prove Theorem (1.3); see Remark (8.7).

A local ring $(R,\mathfrak{m})$ with $\mathfrak{m}^3 = 0$ and embedding dimension 1 is, by Cohen's Structure Theorem, isomorphic to $D/(d^2)$ or $D/(d^3)$, where $(D,(d))$ is a discrete valuation domain. In either case, d is an exact zero divisor. In the following we focus on rings of embedding dimension at least 2.

(8.1) **Remark.** Let $(R,\mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$. It is elementary to verify that elements in $\mathfrak{m}$ annihilate each other if and only if their images in the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ annihilate each other. Thus an element $x \in \mathfrak{m}$ is an exact zero divisor in R if and only if $\bar{x} \in \mathfrak{m}/\mathfrak{m}^2$ is an exact zero divisor in $\mathrm{gr}_{\mathfrak{m}}(R)$.

Let $(R,\mathfrak{m},\mathsf{k})$ be a standard graded k-algebra with $\mathfrak{m}^3 = 0$ and embedding dimension $e \geqslant 2$. Assume that there is an exact zero divisor in R; by (4.2) one has $\mathrm{H}_R(\tau) = 1 + e\tau + (e-1)\tau^2$. If e is 2, then it follows from [17, Hilfssatz 7] that R is complete intersection with Poincaré series

$$\sum_{i=0}^{\infty} \beta_i^R(\mathsf{k})\tau^i = \frac{1}{1-2\tau+\tau^2} = \frac{1}{\mathrm{H}_R(-\tau)};$$

hence R is Koszul by a result of Löfwall [15, thm. 1.2]. If e is at least 3, then R is not Gorenstein and, therefore, R is Koszul by [6, thm. A]. It is known that Koszul algebras are quadratic, and the goal of this section is to prove that if k is infinite, then a generic quadratic standard graded k-algebra with Hilbert series $1 + e\tau + (e-1)\tau^2$ has an exact zero divisor.

Recall that R being quadratic means it is isomorphic to $\mathsf{k}[x_1,\ldots,x_e]/\mathfrak{q}$, where $\mathfrak{q}$ is an ideal generated by homogeneous quadratic forms. The ideal $\mathfrak{q}$ corresponds to a subspace V of the k-vector space W spanned by $\{\, x_ix_j \mid 1 \leqslant i \leqslant j \leqslant e \,\}$. The dimension of W is $m = \frac{e(e+1)}{2}$, and since R has Hilbert series $1+e\tau+(e-1)\tau^2$, the ideal $\mathfrak{q}$ is minimally generated by $n = \frac{e^2-e+2}{2}$ quadratic forms; that is, $\dim_{\mathsf{k}} V = n$. In this way, R corresponds to a point in the Grassmannian $\mathrm{Grass}_{\mathsf{k}}(n,m)$.

(8.2) **Definition.** Let $e \geqslant 2$ be an integer and set $\mathcal{G}_{\mathsf{k}}(e) = \mathrm{Grass}_{\mathsf{k}}(n,m)$, where $n = \frac{e^2-e+2}{2}$ and $m = \frac{e(e+1)}{2}$. Points in $\mathcal{G}_{\mathsf{k}}(e)$ are in bijective correspondence with k-algebras of embedding dimension e whose defining ideal is minimally generated by n homogeneous quadratic forms. For a point $\pi \in \mathcal{G}_{\mathsf{k}}(e)$ let R_π denote the corresponding k-algebra and let $\mathfrak{M}_\pi$ denote the irrelevant maximal ideal of R_π. Notice that $\mathrm{H}_{R_\pi}(\tau)$ has the form $1+e\tau+(e-1)\tau^2+\sum_{i=3}^{\infty} h_i\tau^i$ for every $\pi \in \mathcal{G}_{\mathsf{k}}(e)$.

Consider the sets

$$\mathcal{E}_{\mathsf{k}}(e) = \{\, \pi \in \mathcal{G}_{\mathsf{k}}(e) \mid \text{there is an exact zero divisor in } R_\pi \,\} \quad \text{and}$$
$$\mathcal{H}_{\mathsf{k}}(e) = \{\, \pi \in \mathcal{G}_{\mathsf{k}}(e) \mid \mathrm{H}_{R_\pi}(\tau) = 1 + e\tau + (e-1)\tau^2 \,\}$$

and recall that a subset of $\mathcal{G}_{\mathsf{k}}(e)$ is called open, if it maps to a Zariski open set under the Plücker embedding $\mathcal{G}_{\mathsf{k}}(e) \hookrightarrow \mathbb{P}_{\mathsf{k}}^N$, where $N = \binom{m}{n} - 1$.

The sets $\mathcal{E}_{\mathsf{k}}(e)$ and $\mathcal{H}_{\mathsf{k}}(e)$ are non-empty:

(8.3) **Example.** Let k be a field and let $e \geqslant 2$ be an integer. The k-algebra

$$R = \frac{\mathsf{k}[x_1,\ldots,x_e]}{(x_1^2) + (x_ix_j \mid 2 \leqslant i \leqslant j \leqslant e)}$$

is local with Hilbert series $1+e\tau+(e-1)\tau^2$. One has $(0 : x_1) = (x_1)$; in particular, x_1 is an exact zero divisor in R.

(8.4) **Theorem.** *For every field* k *and every integer* $e \geqslant 2$ *the sets* $\mathcal{E}_{\mathsf{k}}(e)$ *and* $\mathcal{H}_{\mathsf{k}}(e)$ *are non-empty open subsets of the Grassmannian* $\mathcal{G}_{\mathsf{k}}(e)$.

The fact that $\mathcal{H}_{\mathsf{k}}(e)$ is open and non-empty is a special case of [11, thm. 1]; for convenience a proof is included below.

Recall that a property is said to hold for a generic algebra over an infinite field if there is a non-empty open subset of an appropriate Grassmannian such that every point in that subset corresponds to an algebra that has the property.

(8.5) **Corollary.** *Let* k *be an infinite field and let* $e \geqslant 2$ *be an integer. A generic standard graded* k*-algebra with Hilbert series* $1 + e\tau + (e-1)\tau^2$ *has an exact zero divisor.*

Proof. The assertion follows as the set $\mathcal{E}_{\mathsf{k}}(e) \cap \mathcal{H}_{\mathsf{k}}(e)$ is open by Theorem (8.4) and non-empty by Example (8.3). □

(8.6) **Remark.** Let $(R, \mathfrak{m})$ be a local ring. Following Avramov, Iyengar, and Şega (2008) we call an element x in R with $x^2 = 0$ and $x\mathfrak{m} = \mathfrak{m}^2$ a *Conca generator* of $\mathfrak{m}$. Conca proves in [7, sec. 4] that if k is algebraically closed, then the set

$$\mathcal{C}_{\mathsf{k}}(e) = \{\, \pi \in \mathcal{G}_{\mathsf{k}}(e) \mid \text{there is a Conca generator of } \mathfrak{M}_\pi \,\}$$

is open and non-empty in $\mathcal{G}_{\mathsf{k}}(e)$.

If $\mathfrak{m}^3$ is zero, then it follows from Lemma (4.3)(c) that an element x is a Conca generator of $\mathfrak{m}$ if and only if the equality $(0 : x) = (x)$ holds. In particular, there is an inclusion

$$\mathcal{C}_{\mathsf{k}}(e) \cap \mathcal{H}_{\mathsf{k}}(e) \subseteq \mathcal{E}_{\mathsf{k}}(e) \cap \mathcal{H}_{\mathsf{k}}(e).$$

If k is algebraically closed, then it follows from Conca's result combined with Theorem (8.4) and Example (8.3) that both sets are non-empty and open in $\mathcal{G}_{\mathsf{k}}(e)$; in the next section we show that the inclusion may be strict.

(8.7) **Remark.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local k-algebra with $\mathfrak{m}^3 = 0$, and assume that it is not Gorenstein. If R admits a non-free totally reflexive module, then it has embedding dimension $e \geqslant 3$ and Hilbert series $1 + e\tau + (e-1)\tau^2$; see (4.1). Set

$$\mathcal{T}_{\mathsf{k}}(e) = \{\, \pi \in \mathcal{H}_{\mathsf{k}}(e) \mid R_\pi \text{ admits a non-free totally reflexive module} \,\}.$$

We do not know if this is an open subset of $\mathcal{G}_{\mathsf{k}}(e)$, but it contains the non-empty open set $\mathcal{E}_{\mathsf{k}}(e) \cap \mathcal{H}_{\mathsf{k}}(e)$; hence the assertion in Theorem (1.3).

Proof of (8.4). Let k be a field, let $e \geqslant 2$ be an integer, and let S denote the standard graded polynomial algebra $\mathsf{k}[x_1, \ldots, x_e]$. Set $N = \binom{m}{n} - 1$, where $m = \frac{e(e+1)}{2}$ and $n = \frac{e^2-e+2}{2}$. The Plücker embedding maps a point π in the Grassmannian $\mathcal{G}_{\mathsf{k}}(e)$ to the point $(\mu_0(\pi) : \cdots : \mu_N(\pi))$ in the projective space $\mathbb{P}^N_{\mathsf{k}}$, where $\mu_0(\pi), \ldots, \mu_N(\pi)$ are the maximal minors of any $m \times n$ matrix Π corresponding to π. The columns of such a matrix give the coordinates, in the lexicographically ordered basis $\mathcal{B} = \{\, x_i x_j \mid 1 \leqslant i \leqslant j \leqslant e \,\}$ for S_2, of homogeneous quadratic forms $q_1, \ldots, q_n$. The algebra R_π is the quotient ring $S/(q_1, \ldots, q_n)$.

The sets $\mathcal{E}_{\mathsf{k}}(e)$ and $\mathcal{H}_{\mathsf{k}}(e)$ are non-empty by Example (8.3). Let Z be an $m \times n$ matrix of indeterminates and let $\chi_0, \ldots, \chi_N$ denote the maximal minors of Z. We prove openness of each set $\mathcal{E}_{\mathsf{k}}(e)$ and $\mathcal{H}_{\mathsf{k}}(e)$ in $\mathcal{G}_{\mathsf{k}}(e)$ by proving that the Plücker embedding maps it to the complement in $\mathbb{P}^N_{\mathsf{k}}$ of the vanishing set for a finite collection of homogeneous polynomials in $\mathsf{k}[\chi_0, \ldots, \chi_N]$.

Openness of $\mathcal{E}_{\mathsf{k}}(e)$. Let π be a point in $\mathcal{G}_{\mathsf{k}}(e)$, let Π be a corresponding matrix, and let $\mathfrak{q}$ be the defining ideal for $R = R_\pi$. For a linear form $\ell = a_1x_1 + \cdots + a_ex_e$ in S, let l denote the image of ℓ in R. For $i \in \{1, \ldots, e\}$ let $[x_i\ell]$ denote the column that gives the coordinates of $x_i\ell$ in the basis $\mathcal{B}$. Multiplication by l defines a k-linear map from R_1 to R_2. By assumption, one has $\dim_{\mathsf{k}} R_2 = \dim_{\mathsf{k}} R_1 - 1$, so the equality $lR_1 = R_2$ holds if and only if l is annihilated by a unique, up to scalar multiplication, homogeneous linear form in R. Set

$$\Xi_\ell = \big([x_1\ell] \mid [x_2\ell] \mid \cdots \mid [x_e\ell] \mid \Pi \big);$$

it is an $m \times (m+1)$ matrix with entries in k. The equality $lR_1 = R_2$ holds if and only if the equality $\ell S_1 + \mathfrak{q}_2 = S_2$ holds, and the latter holds if and only if the matrix Ξ_ℓ has maximal rank. For $i \in \{1, \ldots, m+1\}$ let $\nu_i(\ell)$ denote the maximal minor obtained by omitting the ith column of Ξ_ℓ. The columns of Π are linearly independent, so Ξ_ℓ has maximal rank if and only if one of the minors $\nu_1(\ell), \ldots, \nu_e(\ell)$ is non-zero. Notice that each of the minors $\nu_1(\ell), \ldots, \nu_e(\ell)$ is a polynomial expression of degree $e-1$ in the coefficients $a_1, \ldots, a_e$ of ℓ and linear in the Plücker coordinates $\mu_1(\pi), \ldots, \mu_N(\pi)$. The column vector $(\nu_1(\ell)\ -\nu_2(\ell)\ \ldots\ (-1)^m\nu_{m+1}(\ell))^{\mathrm{T}}$ is in the null-space of the matrix Ξ_ℓ, so the element

$$\Big[\Big(\sum_{i=1}^{e}(-1)^{i-1}\nu_i(\ell)x_i\Big)\ell\Big] = \sum_{i=1}^{e}(-1)^{i-1}\nu_i(\ell)[x_i\ell]$$

is in the column space of the matrix Π. Set $\ell' = \sum_{i=1}^{e}(-1)^{i-1}\nu_i(\ell)x_i$; it follows that $\ell'\ell$ belongs to the ideal $\mathfrak{q}$, so one has $l'l = 0$ in R. By the discussion above, l is the unique, up to scalar multiplication, annihilator in R_1 of l' if and only if one of the maximal minors $\nu_1(\ell'), \ldots, \nu_e(\ell')$ of $\Xi_{\ell'}$ is non-zero. Moreover, if one of $\nu_1(\ell'), \ldots, \nu_e(\ell')$ is non-zero, then also one of the minors $\nu_1(\ell), \ldots, \nu_e(\ell)$ is non-zero by the definition of ℓ'. Thus l is an exact zero divisor in R if and only if one of $\nu_1(\ell'), \ldots, \nu_e(\ell')$ is non-zero. Notice that each of these minors is a polynomial expression of degree e in $\mu_1(\pi), \ldots, \mu_N(\pi)$ and degree $(e-1)^2$ in $a_1, \ldots, a_e$.

Let $\zeta_1, \ldots, \zeta_e$ be indeterminates and set $L = \zeta_1x_1 + \cdots + \zeta_ex_e$. For $i \in \{1, \ldots, e\}$ let ν_i denote the maximal minor of the matrix

$$\big([x_1L] \mid [x_2L] \mid \ldots \mid [x_eL] \mid \mathrm{Z} \big),$$

obtained by omitting the ith column. Each minor ν_i is a polynomial of degree $e-1$ in the indeterminates $\zeta_1, \ldots, \zeta_e$ and linear in $\chi_0, \ldots, \chi_N$. For $i \in \{1, \ldots, e\}$ set

$$F_i = \nu_i(\nu_1, -\nu_2, \ldots, (-1)^{e-1}\nu_e);$$

each F_i is a polynomial of degree $(e-1)^2$ in $\zeta_1, \ldots, \zeta_e$ and of degree e in $\chi_0, \ldots, \chi_N$. Consider $F_1, \ldots, F_e$ as polynomials in $\zeta_1, \ldots, \zeta_e$ with coefficients in $\mathsf{k}[\chi_0, \ldots, \chi_N]$, and let $\mathcal{P}$ denote the collection of these coefficients. The algebra R has an exact zero divisor, i.e. the point π belongs to $\mathcal{E}_{\mathsf{k}}(e)$, if and only if one of the polynomials F_i in the algebra $(\mathsf{k}[\chi_0, \ldots, \chi_N])[\zeta_1, \ldots, \zeta_e]$ is non-zero, that is, if and only if the Plücker embedding maps π to a point in the complement of the algebraic variety

$$\bigcap_{P \in \mathcal{P}} \mathcal{Z}(P) \subseteq \mathbb{P}^N_{\mathsf{k}}.$$

Openness of $\mathcal{H}_{\mathsf{k}}(e)$. Let π be a point in $\mathcal{G}_{\mathsf{k}}(e)$, let Π be a corresponding matrix, and let $\mathfrak{q} = (q_1, \ldots, q_n)$ be the defining ideal for R_π. Clearly, π belongs to the subset $\mathcal{H}_{\mathsf{k}}(e)$ if and only if the equality $S_1\mathfrak{q}_2 = S_3$ holds. Set $c = \binom{e+2}{3}$ and take as

k-basis for S_3 the c homogeneous cubic monomials ordered lexicographically. For a homogeneous cubic form $f \in S_3$, let $[f]$ denote the column that gives its coordinates in this basis. The equality $S_1\mathfrak{q}_2 = S_3$ holds if and only if the $c \times ne$ matrix

$$\Xi = \big([x_1q_1] \mid [x_1q_2] \mid \ldots \mid [x_1q_n] \mid [x_2q_1] \mid \ldots \mid [x_eq_n] \big)$$

has maximal rank, i.e. rank c as one has $c \leqslant ne$. Set

$$\mathrm{E} = \big([x_1(x_1^2)] \mid [x_1(x_1x_2)] \mid \ldots \mid [x_1(x_e^2)] \mid [x_2(x_1^2)] \mid \ldots \mid [x_e(x_e^2)] \big);$$

it is a $c \times me$ matrix, and each column of E is identical to a column in the $c \times c$ identity matrix I_c. In particular, E has entries from the set $\{0, 1\}$. Let Δ be the matrix $\Pi^{\oplus e}$, that is, the block matrix with e copies of Π on the diagonal and 0 elsewhere; it is a matrix of size $me \times ne$. It is straightforward to verify the equality $\Xi = \mathrm{E}\Delta$. Set $g = ne - c$ and let C denote the collection of all subsets of $\{1, \ldots, me\}$ that have cardinality g. For $i \in \{1, \ldots, me\}$ let r_i denote the ith row of the identity matrix I_{me}. For each $\gamma = \{t_1, \ldots, t_g\}$ in C set

$$\mathrm{E}_\gamma = \begin{pmatrix} r_{t_1} \\ \vdots \\ r_{t_g} \\ \hline E \end{pmatrix};$$

it is an $ne \times me$ matrix with entries from the set $\{0, 1\}$.

Claim. The matrix Ξ has maximal rank if and only if there exists a $\gamma \in C$ such that $\mathrm{E}_\gamma\Delta$ has non-zero determinant.

Proof. Assume that the $ne \times ne$ matrix $\mathrm{E}_\gamma\Delta$ has non-zero determinant. One can write the determinant as a linear combination with coefficients in $\{-1, 0, 1\}$ of the c-minors of the submatrix $\mathrm{E}\Delta$, so $\Xi = \mathrm{E}\Delta$ has maximal rank. To prove the converse, assume that $\mathrm{E}\Delta$ has maximal rank, i.e. rank c. The matrix Δ has maximal rank, ne, so the rows of Δ span k^{ne}. Therefore, one can choose $\gamma = \{t_1, \ldots, t_g\}$ in C such that rows number $t_1, \ldots, t_g$ in Δ together with the rows of $\mathrm{E}\Delta$ span k^{ne}. That is, the rows of $\mathrm{E}_\gamma\Delta$ span k^{ne}, so the determinant of $\mathrm{E}_\gamma\Delta$ is non-zero.

The determinants $P_\gamma = \det(\mathrm{E}_\gamma \mathrm{Z}^{\oplus e})$, for $\gamma \in C$, yield $\binom{me}{g}$ homogeneous polynomials of degree e in $\mathsf{k}[\chi_0, \ldots, \chi_N]$. Under the Plücker embedding, π is mapped to a point in the complement of the algebraic variety

$$\bigcap_{\gamma \in C} \mathcal{Z}(P_\gamma) \subseteq \mathbb{P}^N_{\mathsf{k}}$$

if and only if $\det(\mathrm{E}_\gamma\Pi^{\oplus e})$ is non-zero for some $\gamma \in C$. By Claim such a γ exists if and only if Ξ has maximal rank, that is, if and only if π belongs to $\mathcal{H}_{\mathsf{k}}(e)$. $\square$

(8.8) **Remark.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local k-algebra with $\mathfrak{m}^3 = 0$. The argument that shows the openness of $\mathcal{E}_{\mathsf{k}}(e)$ yields additional information. Namely, if there is an exact zero divisor in R, then one of the polynomials F_i in the variables $\zeta_1, \ldots, \zeta_e$ is non-zero, and every point in the complement of its vanishing set $\mathcal{Z}(F_i) \subseteq \mathbb{P}^{e-1}_{\mathsf{k}}$ corresponds to an exact zero divisor. Thus if k is infinite, then a generic homogeneous linear form in R is an exact zero divisor.

9. Short local rings without exact zero divisors—An example

Let k be a field; set

$$R = \mathsf{k}[s,t,u,v]/(s^2, sv, t^2, tv, u^2, uv, v^2 - st - su) \quad \text{and} \quad \mathfrak{m} = (s,t,u,v)R.$$

Conca mentions in [7, Example 12] that although R is a standard graded k-algebra with Hilbert series $1 + 4\tau + 3\tau^2$ and $(0 : \mathfrak{m}) = \mathfrak{m}^2$, empirical evidence suggests that there is no element $x \in R$ with $(0 : x) = (x)$; that is, there is no Conca generator of $\mathfrak{m}$. Proposition (9.1) confirms this, and together with Proposition (9.2) it exhibits properties of R that frame the results in the previous sections. In particular, these propositions show that non-free totally reflexive modules may exist even in the absence of exact zero divisors, and that exact zero divisors may exist also in the absence of Conca generators.

(9.1) **Proposition.** *The following hold for the* k*-algebra* R*.*

(a) *There is no element* x *in* R *with* $(0 : x) = (x)$*.*

(b) *If* k *does not have characteristic* 2 *or* 3*, then the elements* $s + t + 2u - v$ *and* $3s + t - 2u + 4v$ *form an exact pair of zero divisors in* R*.*

(c) *Assume that* k *has characteristic* 3*. If* $\vartheta \in \mathsf{k}$ *is not an element of the prime subfield* F_3*, then the element* $(1 - \vartheta)s + \vartheta t + u + v$ *is an exact zero divisor in* R*. If* k *is* F_3*, then there are no exact zero divisors in* R*.*

(d) *If* k *has characteristic* 2*, then there are no exact zero divisors in* R*.*

(9.2) **Proposition.** *The* R*-module presented by the matrix*

$$\Phi = \begin{pmatrix} t & -t + u - v \\ t + u - v & s + u \end{pmatrix}$$

is indecomposable and totally reflexive, its first syzygy is presented by

$$\Psi = \begin{pmatrix} -t + v & 2s + t - u + 2v \\ t + u & s - u + v \end{pmatrix},$$

and its minimal free resolution is periodic of period 2.

Proof of (9.1). For an element $x = \alpha s + \beta t + \gamma u + \delta v$ in $\mathfrak{m}$ we denote the images of α, β, γ, and δ in $R/\mathfrak{m} \cong \mathsf{k}$ by a, b, c, and d. For x and $x' = \alpha' s + \beta' t + \gamma' u + \delta' v$ the product xx' can be written in terms of the basis $\{st, su, tu\}$ for $\mathfrak{m}^2$ as follows:

$$xx' = (ab' + ba' + dd')st + (ac' + ca' + dd')su + (bc' + cb')tu. \tag{1}$$

The product $x\mathfrak{m}$ is generated by the elements $xs = bst + csu$, $xt = ast + ctu$, $xu = asu + btu$, and $xv = dst + dsu$. By Remark (7.2) the equality $x\mathfrak{m} = \mathfrak{m}^2$ holds if and only if the matrix

$$\Xi_x = \begin{pmatrix} b & a & 0 & d \\ c & 0 & a & d \\ 0 & c & b & 0 \end{pmatrix}$$

has a non-zero 3-minor; that is, if and only if one of

$$\begin{aligned} &\mu_1(x) = -2abc, && \mu_2(x) = cd(c - b), \\ &\mu_3(x) = bd(c - b), \quad \text{and} && \mu_4(x) = -ad(c + b) \end{aligned} \tag{2}$$

is non-zero. Thus elements x and x' with $xx' = 0$ form an exact pair of zero divisors if and only if one of the minors $\mu_1(x), \dots, \mu_4(x)$ is non-zero, and one of the minors $\mu_1(x'), \dots, \mu_4(x')$ of the matrix $\Xi_{x'}$ is non-zero.

(a): For the equality $(0 : x) = (x)$ to hold, x must be an element in $\mathfrak{m}\backslash\mathfrak{m}^2$. If $x = \alpha s + \beta t + \gamma u + \delta v$ satisfies $x^2 = 0$, then (1) yields

$$2ab + d^2 = 0, \quad 2ac + d^2 = 0, \quad \text{and} \quad 2bc = 0.$$

It follows that b or c is zero and then that d is zero. Thus each of the minors $\mu_1(x), \ldots, \mu_4(x)$ is zero, whence x does not generate $(0 : x)$.

(b): For the elements $x = s+t+2u-v$ and $x' = 3s+t-2u+4v$, it is immediate from (1) that the product xx' is zero, while (2) yields $\mu_3(x) = -1$ and $\mu_1(x') = 12$. If k does not have characteristic 2 or 3, then $\mu_1(x')$ is non-zero, so x and x' form an exact pair of zero divisors in R.

(c): Assume that k has characteristic 3. If k properly contains F_3, then choose an element $\vartheta \in \mathsf{k}\backslash\mathsf{F}_3$ and set

$$x = (1-\vartheta)s + \vartheta t + u + v.$$

Observe that $\mu_2(x) = 1 - \vartheta$ is non-zero. If ϑ is not a 4th root of unity, set

$$x' = (1+\vartheta)s + t - \vartheta u - (1+\vartheta^2)v,$$

and note that the minor $\mu_2(x') = \vartheta(1+\vartheta)(1+\vartheta^2)$ is non-zero. From (1) one readily gets $xx' = 0$, so x and x' form an exact pair of zero divisors. If ϑ is a 4th root of unity, then one has $\vartheta^2 = -1$. Set

$$x'' = (1-\vartheta)s + t + \vartheta u - v;$$

then the minor $\mu_2(x'') = \vartheta(1-\vartheta)$ is non-zero, and it is again straightforward to verify the equality $xx'' = 0$. Therefore, x and x'' form an exact pair of zero divisors.

Assume now $\mathsf{k} = \mathsf{F}_3$ and assume that the elements $x = \alpha s + \beta t + \gamma u + \delta v$ and $x' = \alpha' s + \beta' t + \gamma' u + \delta' v$ form an exact pair of zero divisors in R. One of the minors $\mu_1(x), \ldots, \mu_4(x)$ is non-zero, and one of the minors $\mu_1(x'), \ldots, \mu_4(x')$ is non-zero, so it follows from (2) that abc or d is non-zero and that $a'b'c'$ or d' is non-zero.

First assume $dd' \neq 0$; we will show that the six elements a, a', b, b', c, c' are non-zero and derive a contradiction. Suppose $b = 0$, then (1) yields $cb' = 0$ and $ab' \neq 0$, which forces $c = 0$. This, however, contradicts the assumption that one of the minors $\mu_1(x), \ldots, \mu_4(x)$ is non-zero, cf. (2). Therefore, b is non-zero. A parallel arguments show that c is non-zero, and by symmetry the elements b' and c' are non-zero. Suppose $a = 0$, then it follows from (1) that a' is non-zero, as $dd' \neq 0$ by assumption. However, (1) also yields

$$a(b' - c') = a'(c - b),$$

so the assumption $a = 0$ forces $c = b$, which contradicts the assumption that one of the minors $\mu_1(x), \ldots, \mu_4(x)$ is non-zero. Thus a is non-zero, and by symmetry also a' is non-zero. Without loss of generality, assume $a = 1 = a'$, then (1) yields

$$b' + b = c' + c \quad \text{and} \quad bc' + cb' = 0. \tag{3}$$

Eliminate b' between these two equalities to get

$$b(c' - c) + c(c' + c) = 0. \tag{4}$$

As c and c' are non-zero elements in F_3, the elements $c' - c$ and $c' + c$ are distinct, and their product is 0. Thus one and only one of them is 0, which contradicts (4) as both b and c are non-zero.

Now assume $dd' = 0$. Without loss of generality, assume that d is zero, then abc is non-zero. It follows from (2) that the elements a', b', and c' cannot all be zero, and then (1) shows that they are all non-zero. Thus all six elements a, a', b, b', c, c' are non-zero, and as above this leads to a contradiction.

(d): Assume that k has characteristic 2 and that the elements $x = \alpha s+\beta t+\gamma u+\delta v$ and $x' = \alpha' s+\beta' t+\gamma' u+\delta' v$ form an exact pair of zero divisors in R. From (2) one gets $d \neq 0$, $d' \neq 0$, $b \neq c$, and $b' \neq c'$. Arguing as in part (c), it is straightforward to verify that the six elements a, a', b, b', c, c' are non-zero. Without loss of generality, assume $a = 1 = a'$. From (1) the following equalities emerge:

$$b' + b + dd' = 0, \quad c' + c + dd' = 0, \quad \text{and} \quad bc' + cb' = 0.$$

The first two equalities yield $b' + b \neq 0$ and $c' + c \neq 0$. Further, elimination of dd' yields $b' = b + c + c'$. Substitute this into the third equality to get

$$b(c' + c) + c(c' + c) = 0.$$

As $c' + c$ is non-zero this implies $b = c$, which is a contradiction. $\square$

Proof of (9.2). It is easy to verify that the products $\Phi\Psi$ and $\Psi\Phi$ are zero, whence

$$F: \quad \cdots \longrightarrow R^2 \xrightarrow{\Phi} R^2 \xrightarrow{\Psi} R^2 \xrightarrow{\Phi} R^2 \longrightarrow \cdots,$$

is a complex. We shall first prove that F is totally acyclic. To see that it is acyclic, one must verify the equalities $\operatorname{Im}\Psi = \operatorname{Ker}\Phi$ and $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$. Assume that the element $\binom{x}{x'}$ is in $\operatorname{Ker}\Phi$. It is straightforward to verify that $\mathfrak{m}^2R^2$ is contained in the image of Ψ, so we may assume that x and x' have the form $x = as+bt+cu+dv$ and $x' = a's+b't+c'u+d'v$, where a, b, c, d and a', b', c', d' are elements in k. Using that $\{st, su, tu\}$ is a basis for $\mathfrak{m}^2$, the assumption $\Phi\binom{x}{x'} = 0$ can be translated into the following system of equations

$$0 = \begin{cases} a - a' - d' \\ a' - d' \\ c + b' - c' \\ a - d + b' \\ a - d + a' + c' \\ b + c + b' \end{cases}$$

From here one derives, in order, the following identities

$$a' = d', \quad a = 2a', \quad d = 2a' + b', \quad c' = -a' + b', \quad c = -a', \quad \text{and} \quad b = a' - b',$$

which immediately yield $\binom{x}{x'} = \Psi\binom{b'}{a'}$. This proves the equality $\operatorname{Im}\Psi = \operatorname{Ker}\Phi$. Similarly, it is easy to check that $\mathfrak{m}^2R^2$ is contained in $\operatorname{Im}\Phi$, and for an element

$$(1) \qquad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} as + bt + cu + dv \\ a's + b't + c'u + d'v \end{pmatrix},$$

where $a, a', \dots, d, d'$ are elements in k, one finds that $\Psi\binom{x}{x'} = 0$ implies $\binom{x}{x'} = \Phi\binom{b'}{a'}$. This proves the equality $\operatorname{Im}\Phi = \operatorname{Ker}\Psi$, so F is acyclic. The differentials in the dual complex $\operatorname{Hom}_R(F, R)$ are represented by the matrices Φ^{T} and Ψ^{T}. One easily checks the inclusions $\mathfrak{m}^2R^2 \subseteq \operatorname{Im}\Phi^{\mathrm{T}}$ and $\mathfrak{m}^2R^2 \subseteq \operatorname{Im}\Psi^{\mathrm{T}}$. Moreover, for an element of the form (1) one finds

$$\Phi^{\mathrm{T}}\binom{x}{x'} = 0 \quad \text{implies} \quad \binom{x}{x'} = \Psi^{\mathrm{T}}\binom{b'}{a'-2b'} \quad \text{and}$$
$$\Psi^{\mathrm{T}}\binom{x}{x'} = 0 \quad \text{implies} \quad \binom{x}{x'} = \Phi^{\mathrm{T}}\binom{-b'}{a'}.$$

This proves that also $\operatorname{Hom}_R(F,R)$ is acyclic, so the module, M, presented by Φ is totally reflexive. Moreover, the first syzygy of M is presented by Ψ, and the minimal free resolution of M is periodic of period 2.

To prove that M is indecomposable, assume that there exist matrices $\mathrm{A} = (a_{ij})$ and $\mathrm{B} = (b_{ij})$ in $\mathrm{GL}_2(R)$, such that $\mathrm{A}\Phi\mathrm{B}$ is a diagonal matrix; i.e. the equalities

$$
\begin{aligned}
0 = (a_{12}b_{22})s + ((a_{11}+a_{12})b_{12} - a_{11}b_{22})t + (a_{12}b_{12} + (a_{11}+a_{12})b_{22})u \\
- (a_{12}b_{12} + a_{11}b_{22})v
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
0 = (a_{22}b_{21})s + ((a_{21}+a_{22})b_{11} - a_{21}b_{21})t + (a_{22}b_{11} + (a_{21}+a_{22})b_{21})u \\
- (a_{22}b_{11} + a_{21}b_{21})v
\end{aligned} \tag{3}
$$

hold. As the matrices A and B are invertible, neither has a row or a column with both entries in $\mathfrak{m}$. Since the elements s, t, u, and v are linearly independent modulo $\mathfrak{m}^2$, it follows from (2) that $a_{12}b_{22}$ is in $\mathfrak{m}$. Assume that a_{12} is in $\mathfrak{m}$, then a_{11} and a_{22} are not in $\mathfrak{m}$. It also follows from (2) that $a_{12}b_{12} + a_{11}b_{22}$ is in $\mathfrak{m}$, which forces the conclusion $b_{22} \in \mathfrak{m}$. However, this implies that b_{21} is not in $\mathfrak{m}$, so $a_{22}b_{21}$ is not in $\mathfrak{m}$ which contradicts (3). A parallel argument shows that also the assumption $b_{22} \in \mathfrak{m}$ leads to a contradiction. Thus M is indecomposable. $\square$

10. Families of non-isomorphic modules of infinite length

Most available proofs of the existence of infinite families of totally reflexive modules are non-constructive. In the previous sections we have presented constructions that apply to local rings with exact zero divisors. In [12] Holm gives a different construction; it applies to rings of positive dimension which have a special kind of exact zero divisors. Here we provide one that does not depend on exact zero divisors.

(10.1) **Construction.** Let $(R,\mathfrak{m})$ be a local ring and let $\varkappa = \{x_1,\dots,x_e\}$ be a minimal set of generators for $\mathfrak{m}$. Let N be a finitely generated R-module and let N_1 be its first syzygy. Let $F \twoheadrightarrow N$ be a projective cover, and consider the element

$$\xi: \quad 0 \longrightarrow N_1 \xrightarrow{\iota} F \longrightarrow N \longrightarrow 0$$

in $\operatorname{Ext}^1_R(N,N_1)$. For $i \in \{1,\dots,e\}$ and $j \in \mathbb{N}$ recall that $x_i^j\xi$ is the second row in the diagram

$$
\begin{tikzcd}
\xi\colon\ 0 \arrow[r] & N_1 \arrow[r, "\iota"] \arrow[d, "x_i^j"] & F \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
x_i^j\xi\colon\ 0 \arrow[r] & N_1 \arrow[r, "\omega^{(i,j)}"] & P^{(i,j)} \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}
$$

where the left-hand square is the pushout of ι along the multiplication map x_i^j. The diagram defines $P^{(i,j)}$ uniquely up to isomorphism of R-modules. Set

$$\mathcal{P}(\varkappa\,; N) = \{\, P^{(i,j)} \mid 1 \leqslant i \leqslant e,\ j \in \mathbb{N} \,\};$$

note that every module in $\mathcal{P}(\varkappa\,;N)$ can be generated by $\beta_0^R(N) + \beta_1^R(N)$ elements.

(10.2) **Lemma.** *Let $(R,\mathfrak{m})$ be a local ring and let N be a finitely generated R-module. If, for some minimal set $\varkappa = \{x_1,\dots,x_e\}$ of generators for $\mathfrak{m}$, the set $\mathcal{P}(\varkappa\,;N)$ contains only finitely many pairwise non-isomorphic modules, then the R-module $\operatorname{Ext}^1_R(N,N_1)$ has finite length.*

Proof. Let $\varkappa = \{x_1, \dots, x_e\}$ be a minimal set of generators for $\mathfrak{m}$, and assume that $\mathcal{P}(\varkappa\,; N)$ contains only finitely many pairwise non-isomorphic modules. Given an index $i \in \{1, \dots, e\}$, there exist positive integers m and n such that the modules $P^{(i,m)}$ and $P^{(i,n+m)}$ are isomorphic. Since $x_i^{n+m}\xi$ equals $x_i^n(x_i^m\xi)$, the module $P^{(i,n+m)}$ comes from the pushout of $\omega^{(i,m)}$ along the multiplication map x_i^n; cf. Construction (10.1). Thus there is an exact sequence

$$0 \longrightarrow N_1 \xrightarrow{\alpha} N_1 \oplus P^{(i,m)} \longrightarrow P^{(i,n+m)} \longrightarrow 0,$$

where $\alpha = \begin{pmatrix} x_i^n & -\omega^{(i,m)} \end{pmatrix}$. It follows from the isomorphism $P^{(i,m)} \cong P^{(i,n+m)}$ and Miyata's theorem [16] that this sequence splits. Hence, it induces a split monomorphism

$$\operatorname{Ext}_R^1(N, N_1) \xrightarrow{\operatorname{Ext}_R^1(N,\alpha)} \operatorname{Ext}_R^1(N, N_1) \oplus \operatorname{Ext}_R^1(N, P^{(i,m)}).$$

Let β be a left-inverse of $\operatorname{Ext}_R^1(N, \alpha)$, set $m_i = m$ and notice that the element

$$x_i^{m_i}\xi = \beta \operatorname{Ext}_R^1(N,\alpha)(x_i^{m_i}\xi) = \beta \begin{pmatrix} x_i^{n+m_i}\xi & 0 \end{pmatrix} = x_i^{n+m_i}\beta \begin{pmatrix} \xi & 0 \end{pmatrix}$$

belongs to $\mathfrak{m}^{n+m_i} \operatorname{Ext}_R^1(N, N_1)$.

For every index $i \in \{1, \dots, e\}$ let m_i be the positive integer obtained above. With $h = m_1 + \cdots + m_e$ there is an inclusion $\mathfrak{m}^h \operatorname{Ext}_R^1(N, N_1) \subseteq \mathfrak{m}^{h+1} \operatorname{Ext}_R^1(N, N_1)$, so Nakayama's lemma yields $\mathfrak{m}^h \operatorname{Ext}_R^1(N, N_1) = 0$. □

(10.3) **Theorem.** *Let $(R, \mathfrak{m})$ be a local ring and let $\varkappa = \{x_1, \dots, x_e\}$ be a minimal set of generators for $\mathfrak{m}$. If there exists a totally reflexive R-module N and a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ such that $N_{\mathfrak{p}}$ is not free over $R_{\mathfrak{p}}$, then the set $\mathcal{P}(\varkappa\,; N)$ contains infinitely many indecomposable and pairwise non-isomorphic totally reflexive R-modules.*

Proof. It follows from the assumptions on N that every module in the set $\mathcal{P}(\varkappa\,; N)$ is totally reflexive and that the R-module $\operatorname{Ext}_R^1(N, N_1)$ has infinite length, as its support contains the prime ideal $\mathfrak{p} \neq \mathfrak{m}$. By (10.2) the set $\mathcal{P}(\varkappa\,; N)$ contains infinitely many pairwise non-isomorphic modules. Every module in $\mathcal{P}(\varkappa\,; N)$ is minimally generated by at most $\beta_0^R(N) + \beta_1^R(N)$ elements; see Construction (10.1). Therefore, every infinite collection of pairwise non-isomorphic modules in $\mathcal{P}(\varkappa\,; N)$ contains infinitely many indecomposable modules. □

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