FIVE THEOREMS ON GORENSTEIN GLOBAL DIMENSIONS

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Abstract. We expand on two existing characterizations of rings of Gorenstein (weak) global dimension zero and give two new characterizations of rings of finite Gorenstein (weak) global dimension. We also include the answer to a question of Y. Xiang on Gorenstein weak global dimension of group rings.

Introduction

In this paper, A denotes a unital associative ring. By an A-module we mean a left A-module; right A-modules are considered as modules over the opposite ring A°.

Two classic facts in algebra are that A has finite weak global dimension if and only if so does A°, and that A has global dimension zero if and only if so does A°. Corresponding results are known to hold in the Gorenstein setting: In [7] we observe that the Gorenstein weak global dimension is symmetric, and in [4] Bennis and Mahdou show a ring has Gorenstein global dimension zero if and only if it is Quasi-Frobenius, which is a left–right symmetric property.

Gorenstein flat-cotorsion modules, defined in [6], and the associated dimension developed in [5] turn out to be useful for refining our understanding of Gorenstein global dimensions. Two of our main results, Theorems 2.2 and 3.2, characterize rings of Gorenstein (weak) global dimension zero. In the first, we show that IF rings are precisely those rings where all modules have Gorenstein flat-cotorsion dimension at most zero. In the second, we show that Quasi-Frobenius rings are those rings where all modules are Gorenstein flat-cotorsion. Further, we show in Theorems 4.5 and 4.9 that under assumptions of finite finitistic dimensions, rings of finite Gorenstein (weak) global dimension are precisely the rings over which strongly cotorsion modules and Gorenstein injective modules coincide.

An important ingredient in the proofs of these results are generalizations of two results due to Holm [18]: We prove in [7, Theorem 2.1] that the flat, Gorenstein flat, and Gorenstein flat-cotorsion dimensions agree for modules of finite injective dimension, and in Theorem 1.2 we show that the injective and Gorenstein injective dimensions agree for modules of finite flat dimension. In the final section, Theorem 5.2 provides an answer to a question of Y. Xiang [27] on the Gorenstein global dimension of group rings.

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The notation employed in the paper is standard: we write pd, id, and fd for the projective, injective, and flat dimensions of modules and complexes. The corresponding Gorenstein dimensions are denoted Gpd, Gid, and Gfd. The Gorenstein flat-cotorsion dimension from [5] gets abbreviated Gfcd. We write $\operatorname{FPD}(A)$ and $\operatorname{FFD}(A)$ for the finitistic projective and finitistic flat dimension of A. The invariant $\operatorname{sfli}(A)$ is defined as $\sup\{\operatorname{fd}_A M \mid M \text{ is an injective } A\text{-module}\}$. The Gorenstein global dimension is defined as

$$\operatorname{Ggldim}(A) = \sup\{\operatorname{Gpd}_A M \mid M \text{ is an } A\text{-module}\}\,,$$

and it equals $\sup\{\operatorname{Gid}_A M \mid M \text{ is an } A\text{-module}\}$. The Gorenstein weak global dimension is defined similarly,

$$\operatorname{Gwgldim}(A) = \sup\{\operatorname{Gfd}_A M \mid M \text{ is an } A\text{-module}\}\,.$$

In dealing with A-complexes we use homological notation. For an A-complex M and $n \in \mathbb{Z}$ set $\mathrm{Z}_n(M) = \operatorname{Ker} \partial_n^M$ and $\mathrm{C}_n(M) = \operatorname{Coker} \partial_{n+1}^M$ so that they are, respectively, submodules and quotient modules of the module in degree n.

1. Gorenstein injective dimension of flat modules

Recall that an A-module M is *cotorsion* if $\operatorname{Ext}_A^1(F, M) = 0$ holds for every flat A-module F, and M is *strongly cotorsion* if $\operatorname{Ext}_A^1(L, M) = 0$ holds for every A-module L of finite flat dimension; see Xu [28, Definition 5.4.1].

By convention the homological supremum of an acyclic complex is $-\infty$, so the next result says, in particular, that every cycle module in an acyclic complex of strongly cotorsion modules is strongly cotorsion.

1.1 Proposition. *Let M be a complex of strongly cotorsion A-modules. For every $n \geqslant \sup\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\}$ the module $\mathrm{C}_n(M)$ is strongly cotorsion.*

Proof. If M is acyclic, then $\mathrm{C}_n(M)$ for every $n \in \mathbb{Z}$ is a cokernel in an acyclic complex of strongly cotorsion modules; we first reduce the general case to this special case. Set $s = \sup\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\}$. Splicing a shifted injective resolution of the module $\mathrm{C}_s(M)$ with the acyclic complex $\cdots \to M_{s+1} \to M_s \to \mathrm{C}_s(M) \to 0$ one gets an acyclic complex I with $\mathrm{C}_n(I) = \mathrm{C}_n(M)$ for $n \geqslant s$. It now suffices to show that the modules $\mathrm{C}_i(I)$ are strongly cotorsion.

Let L be a module with $\operatorname{fd}_A L \leqslant f$. There is an exact sequence,

$$0 \longrightarrow F_f \longrightarrow \cdots \longrightarrow F_0 \longrightarrow L \longrightarrow 0\,,$$

with each F_i a flat A-module. Given a cotorsion module C, dimension shifting along this sequence yields $\operatorname{Ext}_A^{f+1}(L, C) \cong \operatorname{Ext}_A^1(F_f, C) = 0$. By a result of Bazzoni, Cortés-Izurdiaga, and Estrada [2, Theorem 1.3] each module $\mathrm{C}_i(I)$ is cotorsion, so for every $i \in \mathbb{Z}$ one has $\operatorname{Ext}_A^{f+1}(L, \mathrm{C}_i(I)) = 0$. Now, for every $i \in \mathbb{Z}$ there is an exact sequence,

$$0 \longrightarrow \mathrm{C}_{i+f}(I) \longrightarrow I_{i+f-1} \longrightarrow \cdots \longrightarrow I_i \longrightarrow \mathrm{C}_i(I) \longrightarrow 0\,.$$

Dimension shifting yields $\operatorname{Ext}_A^1(L, \mathrm{C}_i(I)) \cong \operatorname{Ext}_A^{f+1}(L, \mathrm{C}_{i+f}(I)) = 0$, whence $\mathrm{C}_i(I)$ is strongly cotorsion. $\square$

Recall that an acyclic complex I of injective A-modules is called *totally acyclic* if $\operatorname{Hom}_A(E, I)$ is acyclic for every injective A-module E. Recall further that an A-module G is called *Gorenstein injective* if there exists a totally acyclic complex

I of injective A-modules with $\mathrm{Z}_0(I) \cong G$. It follows from Proposition 1.1 that a Gorenstein injective A-module is strongly cotorsion.

Now, if a flat A-module F is Gorenstein injective, then one gets from the defining totally acyclic complex an exact sequence $0 \to G \to I \to F \to 0$, with I injective and G Gorenstein injective. Since $\mathrm{Ext}_A^1(F,G) = 0$ holds this sequence splits, which means that F is a summand of I and hence injective. This argument can be souped up to yield the next theorem, which is dual to [7, Theorem 1.1] and subsumes [18, Theorem 2.1]

Recall, for example from [8, Proposition 3.6], that the Gorenstein injective dimension of an A-complex M can be defined as the least integer n such that (1) $\mathrm{H}_v(M) = 0$ holds for all $v < -n$ and (2) there exists a semi-injective A-complex I, isomorphic to M in the derived category, such that the cycle submodule $\mathrm{Z}_{-n}(I)$ is Gorenstein injective.

1.2 Theorem. *Let M be an A-complex with bounded homology. If $\mathrm{fd}_A M$ is finite, then $\mathrm{Gid}_A M = \mathrm{id}_A M$ holds.*

Proof. The equality $\mathrm{Gid}_A M = \mathrm{id}_A M$ holds trivially if M is acyclic, so assume that M is not acyclic and, without loss of generality, assume further that $\mathrm{fd}_A M = 0$ holds. Set $u = \min\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\}$ and let F be a bounded complex of flat A-modules with $F_v = 0$ for $v > 0$ and $v < u$, such that F and M are isomorphic in the derived category of A. As one has $\mathrm{Gid}_A F \leqslant \mathrm{id}_A F$ it suffices to show that $\mathrm{Gid}_A F = n$ implies $\mathrm{id}_A F \leqslant n$. Let $F \to I$ be a semi-injective resolution with $I_v = 0$ for $v > 0$ and C be its mapping cone. As $I_v = 0$ holds for $v > 0$ one has $C_1 = F_0$ and $C_v = 0$ for $v > 1$. As $F_v = 0$ holds for $v < u$ one has

$$\mathrm{Z}_v(C) = \mathrm{Z}_v(I) \quad \text{for} \quad v \leqslant u\,. \tag{1}$$

Assume that $\mathrm{Gid}_A F = n$ holds; one then has $-n \leqslant u$. Let G be a Gorenstein injective A-module. From the defining totally acyclic complex of injective A-modules one gets an exact sequence,

$$0 \longrightarrow H \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{-n} \longrightarrow G \longrightarrow 0\,,$$

with H a Gorenstein injective A-module. Dimension shifting along this exact sequence yields

$$\mathrm{Ext}_A^1(\mathrm{Z}_{-(n+1)}(C), G) \;\cong\; \mathrm{Ext}_A^{n+2}(\mathrm{Z}_{-(n+1)}(C), H)\,. \tag{2}$$

Since H is Gorenstein injective, and hence cotorsion by Proposition 1.1, and the modules C_i are direct sums of flat modules and injective modules, dimension shifting along the exact sequence

$$0 \longrightarrow F_0 \longrightarrow C_0 \longrightarrow \cdots \longrightarrow C_{-n} \longrightarrow \mathrm{Z}_{-(n+1)}(C) \longrightarrow 0$$

yields

$$\mathrm{Ext}_A^{n+2}(\mathrm{Z}_{-(n+1)}(C), H) \;\cong\; \mathrm{Ext}_A^1(F_0, H) \;=\; 0\,. \tag{3}$$

Combining (1)–(3) one gets $\mathrm{Ext}_A^1(\mathrm{Z}_{-(n+1)}(I), G) = 0$ for every Gorenstein injective A-module G. In particular $\mathrm{Ext}_A^1(\mathrm{Z}_{-(n+1)}(I), \mathrm{Z}_{-n}(I)) = 0$ holds, so the exact sequence $0 \to \mathrm{Z}_{-n}(I) \to I_{-n} \to \mathrm{Z}_{-(n+1)}(I) \to 0$ splits, which means that $\mathrm{Z}_{-n}(I)$ is injective and $\mathrm{id}_A F \leqslant n$ holds as desired. □

2. IF rings

Following Colby [9] a ring over which every injective module is flat is called a left IF ring. If a ring A and its opposite ring A° are both left IF, then A is called IF. Bennis [3] characterized IF rings in terms of the Gorenstein weak global dimension, which at that point was not known to be symmetric. In the commutative case, where this distinction is irrelevant, the characterization was also obtained by Mahdou, Tamekkante, and Yassemi [22]. Here we use the Gorenstein flat-cotorsion dimension to characterize left-IF rings; Bennis' characterization [3, Proposition 2.14] is recovered as the equivalence of (i) and (iii) in Theorem 2.2.

Recall that an acyclic complex F of flat-cotorsion A-modules is called *totally acyclic* if $\operatorname{Hom}_A(F, C)$ is acyclic for every flat-cotorsion A-module C. Recall further that an A-module G is called *Gorenstein flat-cotorsion* if there exists a totally acyclic complex F of flat-cotorsion A-modules with $\mathrm{C}_0(F) \cong G$. It follows from [2, Theorem 1.3] that a Gorenstein flat-cotorsion A-module is cotorsion.

Recall from [5, Definition 4.1], that the Gorenstein flat-cotorsion dimension of an A-complex M can be defined as the least integer n such that (1) $\mathrm{H}_v(M) = 0$ holds for all $v > n$ and (2) there exists a semi-flat-cotorsion A-complex F—i.e. a semi-flat complex consisting of flat–cotorsion modules—isomorphic to M in the derived category, such that the cokernel $\mathrm{C}_n(F)$ is Gorenstein flat-cotorsion.

2.1 Proposition. *The following conditions are equivalent.*

(i) $\operatorname{Gfcd}_A M \leqslant 0$ *holds for every A-module M.*

(ii) *An A-module is flat-cotorsion if and only if it is injective.*

(iii) *An A-module is cotorsion if and only if it is Gorenstein flat-cotorsion.*

(iv) *An A-module is cotorsion if and only if it is Gorenstein injective.*

(v) *A is a left IF ring and an A-module is Gorenstein flat-cotorsion if and only if it is Gorenstein injective.*

Proof. $(i) \Longrightarrow (ii)$: If F is flat-cotorsion, then $\operatorname{Ext}^1_A(M, F) = 0$ holds for every A-module M by [5, Theorem 4.5], which implies that F is injective. Let I be injective; [7, Theorem 1.1] yields $\operatorname{fd}_A I = \operatorname{Gfcd}_A I \leqslant 0$, so I is flat-cotorsion.

$(ii) \Longrightarrow (iii)$: Every Gorenstein flat-cotorsion module is cotorsion. To prove the converse, let C be a cotorsion A-module. It is a cycle submodule in an acyclic complex of flat-cotorsion A-modules: Indeed, take an injective resolution to the right and a flat-cotorsion resolution to the left. By assumption, every injective module is flat-cotorsion, so this is an acyclic complex of flat-cotorsion modules. Moreover, every flat-cotorsion module is injective, so it is in fact a totally acyclic complex of flat-cotorsion modules; in particular, C is Gorenstein flat-cotorsion.

$(iii) \Longrightarrow (i)$: Let M be an A-module. There exists a semi-flat-cotorsion complex F isomorphic to M in the derived category; see Nakamura and Thompson [23, Theorem A.6]. Since the cokernel $\mathrm{C}_0(F)$ has a left resolution by cotorsion modules, it is itself cotorsion by [5, Lemma 5.6]. Thus $\mathrm{C}_0(F)$ is Gorenstein flat-cotorsion by assumption. By the definition of Gorenstein flat-cotorsion dimension it follows that $\operatorname{Gfcd}_A M \leqslant 0$ holds.

$(ii) \Longrightarrow (iv)$: By Proposition 1.1 every Gorenstein injective A-module is cotorsion. Now, let C be cotorsion. A left resolution of C constructed by taking flat covers consists of flat-cotorsion modules. Splice this resolution together with a right injective resolution of C. As flat-cotorsion modules are injective, this produces an acyclic complex of injective modules. In particular, the complex has cotorsion cycles, see [2, Theorem 1.3]. As injective modules are flat, this complex is $\operatorname{Hom}_A(I,-)$-exact for every injective A-module I. Thus C is Gorenstein injective.

$(iv) \Longrightarrow (ii)$: Let I be an injective A-module. For every cotorsion module C, one has $\operatorname{Ext}^1_A(I,C) = 0$ because C is also Gorenstein injective. Thus I is flat-cotorsion. Conversely, let F be flat-cotorsion. By assumption, F must then be Gorenstein injective, so F is injective by Theorem 1.2.

$(ii) \Longrightarrow (v)$: As injective A-modules are flat, A is a left IF ring. Further, as flat-cotorsion modules are precisely the injective modules, Gorenstein flat-cotorsion modules and Gorenstein injective modules coincide.

$(v) \Longrightarrow (ii)$: Every injective module is flat and hence flat-cotorsion. A flat-cotorsion module F is Gorenstein flat-cotorsion and hence Gorenstein injective, so Theorem 1.2 yields $\operatorname{id}_A F = \operatorname{Gid}_A F = 0$. □

2.2 Theorem. *The following conditions are equivalent.*

- (i) $\operatorname{Gwgldim}(A) = 0$.
- (ii) $\operatorname{Gfcd}_A M \leqslant 0$ *holds for every A-module M and* $\operatorname{Gfcd}_{A^\circ} M \leqslant 0$ *holds for every A°-module M.*
- (iii) *A is an IF ring.*

Proof. The Gorenstein weak global dimension is symmetric, see [7, Corollary 2.5], so (i) implies (ii) in view of [5, Theorem 5.7]. Further (ii) implies (iii) by Proposition 2.1. Finally, to see that (iii) implies (i) note that since every injective A°-module is flat, every acyclic complex of flat A-modules is $\mathbf{F}$-totally acyclic, i.e. it remains exact when tensored by an injective A°-module. Let M be an A-module. It suffices to build an acyclic complex F of flat A-modules such that $\mathrm{Z}_0(F) = M$. The left half is built by taking successive flat covers, whereas the right half is obtained by taking an injective resolution of M and observing that it is a complex of flat modules by assumption. □

2.3 Remark. While condition (i) in Theorem 2.2 can be interpreted as saying that all A- and A°-modules are Gorenstein flat, (ii) does not say that all A- and A°-modules are Gorenstein flat-cotorsion. In fact, an A-module M with $\operatorname{Gfcd}_A M = 0$ is Gorenstein flat-cotorsion if and only if it is cotorsion; see [5, Remark 4.6].

Colby [9, Proposition 5] shows that a ring A is von Neumann regular if and only if it is left IF with $\operatorname{wgldim}(A) < \infty$. Here is the Gorenstein analog of this result:

2.4 Corollary. *The following conditions are equivalent.*

- (i) *A is an IF ring.*
- (ii) *A is a left or right IF ring with* $\operatorname{Gwgldim}(A) < \infty$.

Proof. By Theorem 2.2 and [7, Corollary 2.5] it suffices to show that a left IF ring A of finite Gorenstein weak global dimension has $\operatorname{Gwgldim}(A) = 0$. Let A be such a ring and M an A-module. An injective resolution of M is a right resolution by flat A-modules, so for every n the module M is a flat syzygy of its n^{th} injective cosyzygy. Thus M is Gorenstein flat. □

2.5 Remark. There exist right IF rings which are not IF, see [9, Example 2]. Per Corollary 2.4 such rings must be of infinite Gorenstein weak global dimension.

3. Quasi-Frobenius rings

The difference between a module being Gorenstein flat-cotorsion and having Gorenstein flat-cotorsion dimension zero was already commented on in Remark 2.3. The former quality is the stronger one, and the main result of this section is that Gorenstein flat-cotorsionness of all modules characterizes rings of Gorenstein global dimension zero. First we characterize left perfect rings in terms of Gorenstein flat-cotorsion modules.

3.1 Proposition. *The following conditions are equivalent.*

(i) *Every Gorenstein flat A-module is Gorenstein projective.*
(ii) *Every flat A-module is Gorenstein projective.*
(iii) *Every Gorenstein projective A-module is Gorenstein flat-cotorsion.*
(iv) *Every projective A-module is cotorsion.*
(v) *A is left perfect.*

Moreover, when these conditions are satisfied, an A-module is Gorenstein projective if and only if it is Gorenstein flat-cotorsion.

Proof. Notice first that if A is left perfect, then every A-module is cotorsion and the definitions of Gorenstein projective A-modules and Gorenstein flat-cotorsion A-modules coincide. This justifies the last assertion as well as the implication $(v) \implies (iii)$. The implications $(i) \implies (ii)$ and $(iii) \implies (iv)$ are trivial.

$(ii) \Longrightarrow (iv)$: As projective A-modules are right Ext-orthogonal to Gorenstein projective modules, it follows that each projective A-module is right Ext-orthogonal to flat A-modules and hence cotorsion.

$(iv) \Longrightarrow (v)$: The free A-module $A^{(\mathbb{N})}$ is in particular cotorsion, whence A is left perfect by Guil Asensio and Herzog [16, Corollary 20].

$(v) \Longrightarrow (i)$: Every A-module is cotorsion, so per [5, Theorem 5.2] every Gorenstein flat A-module is Gorenstein flat-cotorsion, which over a left perfect ring means Gorenstein projective. $\square$

The equivalence of (i), (i'), and (iii) in the next result was proved by Bennis and Mahdou in [4, Proposition 2.6]. Recall that a Quasi-Frobenius ring is one where the projective modules are injective. Over such a ring, the projective and injective modules coincide, and the same is true for the opposite ring.

3.2 Theorem. *The following conditions are equivalent.*

(i) $\operatorname{Ggldim}(A) = 0$.
(i') $\operatorname{Ggldim}(A^\circ) = 0$.
(ii) *Every A-module is Gorenstein flat-cotorsion.*
(ii') *Every A°-module is Gorenstein flat-cotorsion.*
(iii) *A is Quasi-Frobenius.*

Proof. As the Quasi-Frobenius property is left–right symmetric, it suffices to show the equivalence of the unprimed conditions.

$(i) \Longrightarrow (ii)$: By assumption every A-module is Gorenstein projective, so it follows from Proposition 3.1 that every A-module is Gorenstein flat-cotorsion.

(ii)$\Longrightarrow$*(iii)*: Let P be a projective A-module. As every module is Gorenstein flat-cotorsion, P is flat-cotorsion and $\operatorname{Ext}_A^1(M,P)=0$ holds for every A-module M, so P is injective.

(iii)$\Longrightarrow$*(i)*: A Quasi-Frobenius ring is noetherian and self-injective on both sides, so it is in particular an Iwanaga-Gorenstein ring. Thus for every A-module M one has $\operatorname{Gpd}_A M \leqslant \operatorname{id}_A A = 0$, see for example Holm [17, Theorem 2.28] and Bass [1, Proposition 4.3]. □

4. Strongly cotorsion modules

As noticed it follows from Proposition 1.1 that every Gorenstein injective module is strongly cotorsion. A question considered in the literature is: For which rings do Gorenstein injective modules and strongly cotorsion modules coincide?

Yoshizawa [29, Theorem 2.7] shows that over (commutative) Gorenstein complete local rings, Gorenstein injective modules and strongly cotorsion modules are the same. Huang [19, Theorem 3.10] proves that the same statement holds for every Iwanaga-Gorenstein ring A such that the injective envelope of A is flat, and Iacob [20, Theorem 10] proves it for all Iwanaga-Gorenstein rings; cf. [12, Proposition 9.1.2]. The best result to date can be pieced together from work of Gillespie:

4.1 Remark. A ring is *Ding-Chen* if it is coherent and has finite self FP-injective dimension on both sides. Over a Ding-Chen ring, the classes of Gorenstein injective and strongly cotorsion modules coincide: Indeed, let A be a Ding-Chen ring. Gillespie shows in [14, Theorem 4.2 and Corollary 4.5] that the modules that are right Ext_A^1-orthogonal to the class of A-modules with finite flat dimension—that is, the strongly cotorsion A-modules—coincide with the so-called Ding injective A-modules. Further, Gillespie shows [15, Theorem 1.1] that Ding injective A-modules are the same as Gorenstein injective A-modules.

4.2 Proposition. *If the quantity*

$$\sup\{\operatorname{Gfcd}_A M \mid M \text{ is an } A\text{-module}\}$$

is finite, then $\operatorname{FFD}(A)$ *is finite and an A-module is strongly cotorsion if and only if it is Gorenstein injective.*

Proof. Set $n = \sup\{\operatorname{Gfcd}_A M \mid M \text{ is an } A\text{-module}\}$ and assume that it is finite. This assumption implies that both $\operatorname{FFD}(A)$ and $\operatorname{sfli}(A)$ are finite: Indeed [5, Theorem 5.12] yields $\operatorname{FFD}(A) \leqslant n$, and [7, Theorem 2.1] yields $\operatorname{sfli}(A) \leqslant n$.

As already noticed, every Gorenstein injective A-module is strongly cotorsion. Now let M be a strongly cotorsion A-module. Since $\operatorname{sfli}(A)$ is finite, it follows from Proposition 1.1 that every acyclic complex of injectives is totally acyclic. Therefore, it suffices to show that M is the homomorphic image of an injective A-module with strongly cotorsion kernel. Since $\operatorname{FFD}(A)$ is finite, results of Trlifaj [25, Lemma 1.5(3) and Theorem 1.14] apply to yield a short exact sequence,

$$0 \longrightarrow K \longrightarrow E \longrightarrow M \longrightarrow 0\,,$$

with K a strongly cotorsion A-module and E an A-module of finite flat dimension. By the assumption on M it follows that also E is strongly cotorsion. To complete the proof we show that E is an injective A-module. Consider an exact sequence,

$$0 \longrightarrow E \longrightarrow I \longrightarrow C \longrightarrow 0\,,$$

with I an injective A-module. Since E and I have finite flat dimension, the A-module C has finite flat dimension. But then, since E is a strongly cotorsion A-module, the sequence splits and thus E is injective. $\square$

A noetherian ring is Ding-Chen if and only if it is Iwanaga-Gorenstein. Thus for noetherian rings, the next result is equivalent to the statement in Remark 4.1.

4.3 Corollary. *If* $\operatorname{Ggldim}(A)$ *is finite, then an A-module is strongly cotorsion if and only if it is Gorenstein injective.*

Proof. The inequality $\sup\{\operatorname{Gfcd}_A M \mid M \text{ is an } A\text{-module}\} \leqslant \operatorname{Ggldim}(A)$ holds by [7, Theorem 3.3], so the assertion is a special case of Proposition 4.2. $\square$

This gives new examples of rings, including non-coherent rings, with the property that strongly cotorsion modules and Gorenstein injective modules coincide.

4.4 Example. Over any ring of finite global dimension the strongly cotorsion modules, injective modules, and Gorenstein injective modules coincide. A result of Enochs, Estrada, and Iacob [11, Theorem 3.2] yields less trivial examples: the ring of dual numbers over any ring of finite Gorenstein global dimension. A concrete example of a non-coherent ring of finite global dimension is provided by Estrada, Iacob, and Yeomans [13, Section 4(1)].

Yoshizawa [29] in fact shows that among (commutative) Cohen-Macaulay complete local rings, those that are Gorenstein are characterized by the property that strongly cotorsion modules and Gorenstein injective modules are the same. More generally, Iacob [20] shows that this property characterizes Iwanaga-Gorenstein rings among noetherian rings A with $\operatorname{FPD}(A) < \infty$ and $\operatorname{FPD}(A^\circ) < \infty$. In view of Proposition 1.1, the characterizing property is really that strongly cotorsion modules are Gorenstein injective. Further, a noetherian ring A is Iwanaga-Gorenstein if and only if $\operatorname{Ggldim}(A)$ is finite, so the next result can be interpreted as removing the noetherian assumption from [20, Theorem 10]:

4.5 Theorem. *The following conditions are equivalent.*

(i) $\operatorname{Ggldim}(A) < \infty$.

(ii) $\operatorname{FPD}(A) < \infty$ *and every strongly cotorsion A-module is Gorenstein injective.*

Proof. The implication $(i) \Longrightarrow (ii)$ holds by Corollary 4.3 and [17, Theorem 2.28] which shows that the assumption on A implies that $\operatorname{FPD}(A)$ is finite. For $(ii) \Longrightarrow (i)$, set $n = \operatorname{FPD}(A)$. By a result of Jensen [21, Proposition 6], every flat A-module, and hence every module of finite flat dimension, has projective dimension at most n. It suffices to show that $\operatorname{Gid}_A N \leqslant n$ holds for every A-module N. Let N be an A-module and consider an exact sequence,

$$0 \longrightarrow N \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_{-n+1} \longrightarrow D \longrightarrow 0\,,$$

with each E_i injective. For every A-module M with $\operatorname{fd}_A M \leqslant n$, dimension shifting along this sequence yields $\operatorname{Ext}^1_A(M,D) \cong \operatorname{Ext}^{n+1}_A(M,N) = 0$. Therefore, D is a strongly cotorsion A-module. By assumption D is thus Gorenstein injective, whence $\operatorname{Ggldim}(A)$ is finite. $\square$

For ease of comparison to the literature, we include:

4.6 Corollary. *If* $\operatorname{FPD}(A)$ *is finite, then the following conditions are equivalent.*

(i) $\operatorname{Ggldim}(A) < \infty$.

(ii) *An A-module is strongly cotorsion if and only if it is Gorenstein injective.*

4.7 Remark. In the literature it is often remarked that Ding-Chen rings have properties that generalize those of Iwanaga-Gorenstein rings to a non-noetherian setting; see for example the abstract of [14]. Ding and Chen [10, Theorem 7] show that a Ding-Chen ring has finite Gorenstein weak global dimension. In light of Remark 4.1, however, Theorem 4.5 says that a Ding-Chen ring A has finite Gorenstein global dimension if and only if $\operatorname{FPD}(A) < \infty$ holds. It follows from [21, Proposition 6] that a von Neumann regular ring A of infinite global dimension is a a Ding-Chen ring with $\operatorname{FPD}(A) = \infty$. By a result of Pierce [24, Corollary 5.2], a free boolean ring with $\aleph_\omega$ generators, for an infinite cardinal ω, is a von Neumann regular ring of infinite global dimension. We remark that this observation was essentially already made by Wang [26, Example 3.3].

Here is another consequence of Proposition 4.2; without assumptions on the ring it does not readily compare to Corollary 4.3.

4.8 Corollary. *If* $\operatorname{Gwgldim}(A)$ *is finite, then an A-module is strongly cotorsion if and only if it is Gorenstein injective.*

Proof. The inequality $\sup\{\operatorname{Gfcd}_A M \mid M \text{ is an } A\text{-module}\} \leqslant \operatorname{Gwgldim}(A)$ holds by [5, Theorem 5.7], so the assertion is a special case of Proposition 4.2. □

4.9 Theorem. *If A is left or right coherent, then the next conditions are equivalent.*

(i) $\operatorname{Gwgldim}(A) < \infty$.

(ii) $\operatorname{FFD}(A) < \infty$ *and every strongly cotorsion A-module is Gorenstein injective.*

Proof. Condition (i) implies (ii) by Corollary 4.8 and [17, Theorem 3.24] as the Gorenstein weak global dimension is symmetric by [7, Corollary 2.5]. For the converse, assume that A is left coherent and set $n = \operatorname{FFD}(A)$. Let M be an A°-module with a flat resolution F, and let $N = \mathrm{C}_n(F)$ be the n^{th} syzygy in this resolution. By [17, Theorem 3.6] the module N is Gorenstein flat if and only if the dual $N^+ = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective A-module. Let L be an A-module of finite flat dimension. The modules F_i^+ are injective, so dimension shifting along

$$0 \longrightarrow M^+ \longrightarrow (F_0)^+ \longrightarrow \cdots \longrightarrow (F_{n-1})^+ \longrightarrow N^+ \longrightarrow 0$$

yields $\operatorname{Ext}_A^1(L, N^+) \cong \operatorname{Ext}_A^{n+1}(L, M^+)$. As $\operatorname{fd}_A L \leqslant n$ and M^+ is cotorsion, dimension shifting along a flat resolution of L yields $\operatorname{Ext}_A^{n+1}(L, M^+) = 0$. Thus N^+ is strongly cotorsion and hence Gorenstein injective. This shows that every A°-module has Gorenstein flat dimension at most n, and $\operatorname{Gwgldim}(A) = \operatorname{Gwgldim}(A^\circ)$ holds by [7, Corollary 2.5]. If A is instead right coherent, then the same argument applies with A and A° interchanged. □

Again for ease of comparison to the literature, we include:

4.10 Corollary. *If A is left or right coherent and* $\operatorname{FFD}(A)$ *is finite, then the following conditions are equivalent.*

(i) $\operatorname{Gwgldim}(A) < \infty$.

(ii) *An A-module is strongly cotorsion if and only if it is Gorenstein injective.*

For modules over a right coherent ring, the Gorenstein flat dimension agrees with the Gorenstein flat-cotorsion dimension, see [5, Corollary 5.8]. In view of Theorem 4.9 it is thus natural to ask if Proposition 4.2 has a converse:

4.11 Question. If $\mathrm{FFD}(A)$ is finite and an A-module is strongly cotorsion if and only if it is Gorenstein injective, is $\sup\{\mathrm{Gfcd}_A M \mid M \text{ is an } A\text{-module}\}$ finite?

4.12 Remark. We remark that finiteness of $\sup\{\mathrm{Gfcd}_A M \mid M \text{ is an } A\text{-module}\}$ implies that another two quantities are finite, namely $\mathrm{sfli}(A)$, as already noticed in the proof of Proposition 4.2, and $\sup\{\mathrm{id}_A M \mid M \text{ is a flat-cotorsion } A\text{-module}\}$. If one adds to the assumption $\mathrm{FFD}(A) < \infty$ the assumption that these two quantities are finite, then it seems feasible to prove that $\sup\{\mathrm{Gfcd}_A M \mid M \text{ is an } A\text{-module}\}$ is finite, but at this time we are not convinced that this is the best one can do.

5. Group rings

We take the opportunity to include a note on how the Gorenstein flat-cotorsion dimension behaves along certain ring homomorphisms. This leads to an answer to a question of Y. Xiang [27] on the Gorenstein weak global dimension of group rings.

First make the following observation: Let $A \to B$ be a flat ring homomorphism. If C is a cotorsion B-module, then it is also cotorsion as an A-module. To see this, let F be a flat A-module and note that standard Hom-tensor adjunction yields $\mathrm{Ext}^1_A(F, C) \cong \mathrm{Ext}^1_A(F, \mathrm{Hom}_B(B, C)) \cong \mathrm{Ext}^1_B(B \otimes_A F, C) = 0$.

5.1 Proposition. *Let $A \to B$ be a homomorphism of rings and M a B-module. If B is free as an A-module and $\mathrm{Gfcd}_B M < \infty$, then $\mathrm{Gfcd}_A M \geqslant \mathrm{Gfcd}_B M$.*

Proof. Set $n = \mathrm{Gfcd}_B M$. By [5, Theorem 4.5], one thus has $\mathrm{Ext}^n_B(M, C) \neq 0$ for some flat-cotorsion B-module C. By the earlier remark, C is also flat-cotorsion viewed as an A-module. It now follows that

$$\mathrm{Ext}^n_A(M, C) \cong \mathrm{Ext}^n_A(B \otimes_B M, C) \cong \mathrm{Ext}^n_B(M, \mathrm{Hom}_A(B, C)),$$

which contains $\mathrm{Ext}^n_B(M, C)$ as a direct summand and hence is non-zero. Thus $n \leqslant \mathrm{Gfcd}_A M$. □

As an application, the following result removes the right coherence assumption in [27, Proposition 4.9], thus affirmatively answering [27, Question 4.11].

5.2 Theorem. *Let k be a field, G a group, and H a subgroup of G of finite index. If $\mathrm{Gwgldim}(k[G])$ is finite, then it is equal to $\mathrm{Gwgldim}(k[H])$.*

Proof. The inequality $\mathrm{Gwgldim}(k[H]) \leqslant \mathrm{Gwgldim}(k[G])$ holds by [27, Proposition 4.3(2)]. Assume that $\mathrm{Gwgldim}(k[G]) = n < \infty$. By [27, Proposition 4.2], one has $\mathrm{Gwgldim}(k[G]) = \mathrm{Gfd}_{k[G]}\, k$, which in turn equals $\mathrm{Gfcd}_{k[G]}\, k$ since Gorenstein flat-cotorsion and Gorenstein flat dimensions agree when the latter is finite by [5, Theorem 5.7]. Since $k[G]$ is free as a $k[H]$-module, Proposition 5.1 yields $\mathrm{Gfcd}_{k[G]}\, k \leqslant \mathrm{Gfcd}_{k[H]}\, k$, and so we obtain the other inequality: $\mathrm{Gwgldim}(k[H]) \geqslant \mathrm{Gwgldim}(k[G])$. □

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