

**GORENSTEIN DIMENSIONS OF UNBOUNDED COMPLEXES  
AND CHANGE OF BASE (WITH AN APPENDIX  
BY DRISS BENNIS)**

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ABSTRACT. For a commutative ring  $R$  and a faithfully flat  $R$ -algebra  $S$  we prove, under mild extra assumptions, that an  $R$ -module  $M$  is Gorenstein flat if and only if the left  $S$ -module  $S \otimes_R M$  is Gorenstein flat, and that an  $R$ -module  $N$  is Gorenstein injective if and only if it is cotorsion and the left  $S$ -module  $\text{Hom}_R(S, N)$  is Gorenstein injective. We apply these results to the study of Gorenstein homological dimensions of unbounded complexes. In particular, we prove two theorems on stability of these dimensions under faithfully flat (co-)base change.

1. INTRODUCTION

Auslander and Bridger's [2] notion of G-dimension for finitely generated modules over noetherian rings was generalized and dualized by Enochs and collaborators, who introduced the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of modules over associative rings. In the treatment by Christensen, Frankild, and Holm [6], these invariants were considered for complexes with bounded homology. It is, however, possible to define Gorenstein projective dimension for unbounded complexes: This was done by Veliche [21], and the dual case of Gorenstein injective dimension was treated by Asadollahi and Salarian [1].

Inspired by [21], we propose a definition of Gorenstein flat dimension for unbounded complexes; it coincides with the one introduced by Iacob [15] whenever the latter is defined. Our main results are two theorems on stability of Gorenstein homological dimensions of complexes under faithfully flat change of base.

**1.1 Theorem.** *Let  $R$  be a commutative coherent ring and let  $S$  be a faithfully flat  $R$ -algebra that is left GF-closed. For every  $R$ -complex  $M$  there is an equality*

$$\text{Gfd}_R M = \text{Gfd}_S(S \otimes_R^{\mathbf{L}} M).$$

*In particular, an  $R$ -module  $M$  is Gorenstein flat if and only if the  $S$ -module  $S \otimes_R M$  is Gorenstein flat.*

This result is part of Theorem 6.7; the condition that  $S$  is left GF-closed is discussed in 5.10; it is satisfied if  $S$  is right coherent.

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*Date:* 12 May 2016.

*2010 Mathematics Subject Classification.* 13D05; 13D02.

*Key words and phrases.* Gorenstein injective dimension, faithfully flat co-base change, Gorenstein flat dimension, faithfully flat base change.

This research was partly supported by NSA grant H98230-14-0140 (L.W.C.), by NSFC grant 11301240 and SRF for ROCS, SEM (L.L.). Part of the work was done in the summer of 2015 when L.W.C. and L.L. visited Nanjing University supported by NSFC grant 11371187 (PI N.Q. Ding).

**1.2 Theorem.** *Let  $R$  be a commutative noetherian ring with  $\text{splf } R < \infty$ , i.e. every flat  $R$ -module has finite projective dimension, and let  $S$  be a faithfully flat  $R$ -algebra. For every  $R$ -complex  $N$  with  $H^i(N) = 0$  for all  $i \gg 0$  there is an equality*

$$\text{Gid}_R N = \text{Gid}_S \mathbf{R}\text{Hom}_R(S, N).$$

*In particular, an  $R$ -module  $N$  is Gorenstein injective if and only if the  $S$ -module  $\text{Hom}_R(S, N)$  is Gorenstein injective and  $\text{Ext}_R^i(S, N) = 0$  holds for all  $i > 0$ .*

This result is part of Theorem 4.12. The condition  $\text{splf } R < \infty$  is discussed in 4.9; it is satisfied if  $R$  has finite Krull dimension or cardinality at most  $\aleph_n$  for some integer  $n \geq 0$ . We notice that the condition implies vanishing of  $\text{Ext}_R^i(S, N)$  for  $i \gg 0$ .

A recent result of Štovíček [20] implies that every Gorenstein injective module is cotorsion. It allows us to prove the next result, which appears as Theorem 4.5.

**1.3 Theorem.** *Let  $R$  be commutative noetherian and let  $S$  be a faithfully flat  $R$ -algebra. An  $R$ -module  $N$  is Gorenstein injective if and only if it is cotorsion and the  $S$ -module  $\text{Hom}_R(S, N)$  is Gorenstein injective.*

This result compares to the statement about modules in Theorem 1.2.

By significantly relaxing the conditions on the rings, the results of this paper improve results obtained by Christensen and Holm [7], by Christensen and Sather-Wagstaff [11], and by Liu and Ren [22]. Details pertaining to Theorems 1.1 and 1.2 are given in Remarks 4.13 and 6.8; the trend is that the rings in [11, 22] are assumed to be commutative noetherian and, more often than not, of finite Krull dimension.

The paper is organized as follows: In section 2 we set the notation and recall some background material. Sections 3–4 focus on the Gorenstein injective dimension, and Sections 5–6 deal with the Gorenstein flat dimension. Section 7 has some closing remarks and, finally, an appendix by Bennis answers a question raised in an earlier version of this paper.

## 2. COMPLEXES

Let  $R$  be a ring with identity. We consider only unitary  $R$ -modules, and we employ the convention that  $R$  acts on the left. That is, an  $R$ -module is a left  $R$ -module, and right  $R$ -modules are treated as modules over the opposite ring, denoted  $R^\circ$ .

**2.1 Complexes.** Complexes of  $R$ -modules,  $R$ -complexes for short, is our object of study. Let  $M$  be an  $R$ -complex. With homological grading,  $M$  has the form

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots ;$$

one switches to cohomological grading by setting  $M^i = M_{-i}$  and  $\partial_M^i = \partial_{-i}^M$  to get

$$\cdots \longrightarrow M^{i-1} \xrightarrow{\partial_M^{i-1}} M^i \xrightarrow{\partial_M^i} M^{i+1} \longrightarrow \cdots .$$

For  $n \in \mathbb{Z}$  the symbol  $M_{\geq n}$  denotes the quotient complex of  $M$  with  $(M_{\geq n})_i = M_i$  for  $i \geq n$  and  $(M_{\geq n})_i = 0$  for  $i < n$ .

The subcomplexes  $B(M)$  and  $Z(M)$  of boundaries and cycles, the quotient complex  $C(M)$  of cokernels, and the subquotient complex  $H(M)$  of homology all have

zero differentials; their modules are given by

$$\begin{aligned} B_i(M) &= \text{Im } \partial_{i+1}^M &= & \text{Im } \partial_M^{-i-1} = B^{-i}(M) \\ Z_i(M) &= \text{Ker } \partial_i^M &= & \text{Ker } \partial_M^{-i} = Z^{-i}(M) \\ C_i(M) &= M_i / B_i(M) &= & M^{-i} / B^{-i}(M) = C^{-i}(M) \\ H_i(M) &= Z_i(M) / B_i(M) = Z^{-i}(M) / B^{-i}(M) = H^{-i}(M) \end{aligned}$$

The complex  $M$  is called *acyclic* if  $H(M) = 0$ , i.e.  $H_i(M) = 0$  holds for all  $i \in \mathbb{Z}$ .

**2.2 Morphisms.** A morphism  $\alpha: M \rightarrow N$  of  $R$ -complexes is a family of  $R$ -linear maps  $\{\alpha_i: M_i \rightarrow N_i\}_{i \in \mathbb{Z}}$  that commute with the differentials. That is, for all  $i \in \mathbb{Z}$  one has  $\alpha_{i-1} \partial_i^M = \partial_i^N \alpha_i$ . A morphism  $\alpha: M \rightarrow N$  maps boundaries to boundaries and cycles to cycles, so it induces a morphism  $H(\alpha): H(M) \rightarrow H(N)$ . If the induced morphism  $H(\alpha)$  is bijective, then  $\alpha$  is called a *quasi-isomorphism*. Such morphisms are marked by the symbol ‘ $\simeq$ ’.

**2.3 Resolutions.** An  $R$ -complex  $I$  is called *semi-injective* if each module  $I^i$  is injective, and the functor  $\text{Hom}_R(-, I)$  preserves quasi-isomorphisms. Dually, an  $R$ -complex  $P$  is called *semi-projective* if each module  $P_i$  is projective, and the functor  $\text{Hom}_R(P, -)$  preserves quasi-isomorphisms. An  $R$ -complex  $F$  is called *semi-flat* if each module  $F_i$  is flat, and the functor  $- \otimes_R F$  preserves quasi-isomorphisms. Every semi-projective complex is semi-flat. For the following facts see Avramov and Foxby [3].

- Every  $R$ -complex  $M$  has a semi-projective resolution in the following sense: There is a quasi-isomorphism  $\pi: P \rightarrow M$ , where  $P$  is a semi-projective complex; moreover,  $\pi$  can be chosen surjective.
- Every  $R$ -complex  $M$  has a semi-injective resolution in the following sense: There is a quasi-isomorphism  $\iota: M \rightarrow I$ , where  $I$  is semi-injective; moreover,  $\iota$  can be chosen injective.

**2.4 The derived category.** In the derived category over  $R$ , the objects are  $R$ -complexes, and the morphisms are equivalence classes of diagrams  $\bullet \xleftarrow{\simeq} \bullet \longrightarrow \bullet$  of morphisms of  $R$ -complexes. The isomorphisms in the derived category are classes represented by diagrams with two quasi-isomorphisms; and they are also marked by the symbol ‘ $\simeq$ ’.

The derived tensor product,  $- \otimes_R^{\mathbf{L}} -$ , and the derived Hom,  $\mathbf{R}\text{Hom}_R(-, -)$ , are functors on the derived category; their values on given  $R$ -complexes are computed by means of the resolutions described above. As for modules one sets

$$\text{Ext}_R^i(M, N) = H^i(\mathbf{R}\text{Hom}_R(M, N))$$

for  $R$ -complexes  $M$  and  $N$  and  $i \in \mathbb{Z}$ .

### 3. GORENSTEIN INJECTIVE MODULES AND COTORSION

For definitions and standard results on Gorenstein homological dimensions our references are Holm’s [14] and the monograph [12] by Enochs and Jenda. We start by recalling the definition [12, def. 10.1.1] of a Gorenstein injective module.

**3.1.** A complex  $U$  of injective  $R$ -modules is called *totally acyclic* if it is acyclic and the complex  $\text{Hom}_R(I, U)$  is acyclic for every injective  $R$ -module  $I$ .

An  $R$ -module  $G$  is called *Gorenstein injective* (for short, G-injective) if there exists a totally acyclic complex  $U$  of injective  $R$ -modules with  $Z^0(U) \cong G$ .

**3.2.** Every  $R$ -module has an injective resolution, so to prove that a module  $N$  is Gorenstein injective it suffices to verify the following:

- (1)  $\text{Ext}_R^i(I, N) = 0$  holds for all  $i > 0$  and every injective  $R$ -module  $I$ .
- (2)  $N$  has a proper left injective resolution. That is, there exists an acyclic complex of  $R$ -modules  $U^+ = \cdots \rightarrow U^{-2} \rightarrow U^{-1} \rightarrow N \rightarrow 0$  with each  $U^i$  injective, such that  $\text{Hom}_R(I, U^+)$  is acyclic for every injective  $R$ -module  $I$ .

**3.3.** An  $R$ -module  $X$  is called *cotorsion* if one has  $\text{Ext}_R^1(F, X) = 0$  (equivalently,  $\text{Ext}_R^{>0}(F, X) = 0$ ) for every flat  $R$ -module  $F$ .

Šťovíček [20, cor. 5,9] shows that for every module  $N$  that has a left injective resolution, one has  $\text{Ext}_R^i(F, N) = 0$  for all  $i > 0$  and every  $R$ -module  $F$  of finite flat dimension. In particular, every Gorenstein injective module is cotorsion.

This provides for the following improvement of [7, Ascent table II.(h)]

**3.4 Lemma.** *Let  $R \rightarrow S$  be a ring homomorphism such that  $\text{fd}_R S$  and  $\text{fd}_{R^\circ} S$  are finite. For a Gorenstein injective  $R$ -module  $G$ , the  $S$ -module  $\text{Hom}_R(S, G)$  is Gorenstein injective.*

**Proof.** Let  $U$  be a totally acyclic complex of injective  $R$ -modules with  $Z^0(U) \cong G$ . The complex  $\text{Hom}_R(S, U)$  consists of injective  $S$ -modules, and it is acyclic as  $\text{fd}_R S$  is finite; see 3.3. One has  $Z^0(\text{Hom}_R(S, U)) \cong \text{Hom}_R(S, G)$ , so it remains to show that the complex

$$\text{Hom}_S(J, \text{Hom}_R(S, U)) \cong \text{Hom}_R(J, U)$$

is acyclic for every injective  $S$ -module  $J$ . As  $\text{fd}_{R^\circ} S$  is finite, an injective  $S$ -module has finite injective dimension over  $R$ ; see [3, cor. 4.2]. Acyclicity of  $\text{Hom}_R(J, U)$  now follows from [6, lem. 2.2].  $\square$

**3.5.** Let  $N$  be an  $R$ -complex. A *complete injective resolution* of  $N$  is a diagram

$$N \xrightarrow{\iota} I \xrightarrow{v} U$$

where  $\iota$  is a semi-injective resolution,  $v^i$  is bijective for all  $i \gg 0$ , and  $U$  is a totally acyclic complex of injective  $R$ -modules.

The *Gorenstein injective dimension* of an  $R$ -complex  $N$  is defined [1, def. 2.2] as

$$\text{Gid}_R N = \inf \left\{ g \in \mathbb{Z} \left| \begin{array}{l} N \xrightarrow{\iota} I \xrightarrow{v} U \\ v^i: I^i \rightarrow U^i \text{ bijective for all } i \geq g \end{array} \right. \right\}.$$

The next result is dual to (parts of) [21, thm. 3.4], and the proof is omitted.<sup>1</sup>

<sup>1</sup> The statement [1, thm. 2.3] is also meant to be the dual of [21, thm. 3.4], but it has an unfortunate typographical error, reproduced in [22, lem. 1]: In parts (ii) and (iii) the kernel has been replaced by the cokernel in the same degree.

**3.6 Proposition.** *Let  $N$  be an  $R$ -complex and  $g$  be an integer. The following conditions are equivalent.*

- (i)  $\text{Gid}_R N \leq g$ .
- (ii)  $H^i(N) = 0$  holds for all  $i > g$ , and there exists a semi-injective resolution  $N \xrightarrow{\cong} I$  such that the module  $Z^g(I)$  is Gorenstein injective.
- (iii)  $H^i(N) = 0$  holds for all  $i > g$ , and for every semi-injective resolution  $N \xrightarrow{\cong} I$  the module  $Z^g(I)$  is Gorenstein injective.  $\square$

The next result improves [22, cor. 9] by removing assumptions that  $R$  and  $S$  should be commutative noetherian with  $\dim R < \infty$ .

**3.7 Proposition.** *Let  $R \rightarrow S$  be a ring homomorphism such that  $\text{fd}_R S$  and  $\text{fd}_{R^e} S$  are finite. For every  $R$ -complex  $N$  one has  $\text{Gid}_S \mathbf{R}\text{Hom}_R(S, N) \leq \text{Gid}_R N$ .*

**Proof.** We may assume that  $H(N)$  is non-zero, otherwise there is nothing to prove. Let  $N \xrightarrow{\cong} I$  be a semi-injective resolution; the  $S$ -complex  $\text{Hom}_R(S, I)$  is semi-injective and isomorphic to  $\mathbf{R}\text{Hom}_R(S, N)$  in the derived category. If  $\text{Gid}_R N$  is finite, say  $g$ , then by Proposition 3.6 one has  $H^i(I) = 0$  for  $i > g$  and  $Z = Z^g(I)$  is a  $G$ -injective  $R$ -module. Now 3.3 yields  $H^{g+n}(\text{Hom}_R(S, I)) = \text{Ext}_R^n(S, Z) = 0$  for all  $n > 0$ . Moreover, the  $S$ -module  $Z^g(\text{Hom}_R(S, I)) \cong \text{Hom}_R(S, Z^g(I))$  is  $G$ -injective by Lemma 3.4, so  $\text{Gid}_S \mathbf{R}\text{Hom}_R(S, N) \leq g$  holds by Proposition 3.6.  $\square$

#### 4. FAITHFULLY FLAT CO-BASE CHANGE

Throughout this section,  $R$  is a commutative ring and  $S$  is an  $R$ -algebra. We are primarily concerned with the following setup.

**4.1.** Let  $S$  be a faithfully flat  $R$ -algebra; there is then a pure exact sequence

$$(4.1.1) \quad 0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0;$$

that is,  $S/R$  is a flat  $R$ -module.

Let  $I$  be an injective  $R$ -module. The induced sequences

$$\begin{aligned} 0 \longrightarrow I \longrightarrow S \otimes_R I \longrightarrow S/R \otimes_R I \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow \text{Hom}_R(S/R, I) \longrightarrow \text{Hom}_R(S, I) \longrightarrow I \longrightarrow 0 \end{aligned}$$

are split exact so, as an  $R$ -module,  $I$  is a direct summand of  $S \otimes_R I$  and of  $\text{Hom}_R(S, I)$ .

**4.2 Lemma.** *Let  $S$  be a faithfully flat  $R$ -algebra and let  $N$  be an  $R$ -module. If  $\text{Ext}_R^i(S, N) = 0$  holds for all  $i > 0$ , then the following conditions are equivalent.*

- (i)  $\text{Ext}_R^i(I, N) = 0$  holds for all  $i > 0$  and every injective  $R$ -module  $I$ .
- (ii)  $\text{Ext}_S^i(J, \text{Hom}_R(S, N)) = 0$  holds for all  $i > 0$  and every injective  $S$ -module  $J$ .

**Proof.** Let  $X$  be an  $S$ -module; for every  $i > 0$  there are isomorphisms

$$\begin{aligned} \text{Ext}_S^i(X, \text{Hom}_R(S, N)) &= H^i(\mathbf{R}\text{Hom}_S(X, \text{Hom}_R(S, N))) \\ &\cong H^i(\mathbf{R}\text{Hom}_S(X, \mathbf{R}\text{Hom}_R(S, N))) \\ &\cong H^i(\mathbf{R}\text{Hom}_R(S \otimes_S^{\mathbf{L}} X, N)) \\ &\cong H^i(\mathbf{R}\text{Hom}_R(X, N)) \\ &= \text{Ext}_R^i(X, N); \end{aligned}$$

the first isomorphism follows by the vanishing of  $\text{Ext}_R^{>0}(S, N)$ , and the second is Hom-tensor adjointness in the derived category. Under the assumptions, every injective  $S$ -module is injective as an  $R$ -module, so it is evident from the computation that (i) implies (ii). For the converse, let  $I$  be an injective  $R$ -module and recall from 4.1 that it is a direct summand of the injective  $S$ -module  $\text{Hom}_R(S, I)$ .  $\square$

For later application, we recall a fact about cotorsion modules.

**4.3.** Let  $X$  be a cotorsion  $R$ -module. For every flat  $R$ -module  $F$  it follows by Hom-tensor adjointness that  $\text{Hom}_R(F, X)$  is cotorsion.

We recall the notion of an injective precover, also known as an injective right approximation.

**4.4.** Let  $M$  be an  $R$ -module. A homomorphism  $\varphi: E \rightarrow M$  is an injective *precover* of  $M$ , if  $E$  is an injective  $R$ -module and every homomorphism from an injective  $R$ -module to  $M$  factors through  $\varphi$ . Every  $R$ -module has an injective precover if and only if  $R$  is noetherian; see [12, thm. 2.5.1].

In the proof of the next theorem, the noetherian hypothesis on  $R$  is used to ensure the existence of injective precovers.

**4.5 Theorem.** *Let  $R$  be commutative noetherian and let  $S$  be a faithfully flat  $R$ -algebra. An  $R$ -module  $N$  is Gorenstein injective if and only if it is cotorsion and the  $S$ -module  $\text{Hom}_R(S, N)$  is Gorenstein injective.*

**Proof.** The “only if” part of the statement follows from 3.3 and Lemma 3.4. For the converse, note that Lemma 4.2 yields

$$(1) \quad \text{Ext}_R^{>0}(I, N) = 0 \text{ holds for every injective } R\text{-module } I.$$

To prove that  $N$  is G-injective, it is now sufficient to show that  $N$  has a proper left injective resolution; see 3.2.

As  $N$  is cotorsion, application of  $\text{Hom}_R(-, N)$  to (4.1.1) yields an exact sequence  $0 \rightarrow \text{Hom}_R(S/R, N) \rightarrow \text{Hom}_R(S, N) \rightarrow \text{Hom}_R(R, N) \rightarrow 0$ . There is also an exact sequence of  $S$ -modules  $0 \rightarrow G \rightarrow U \rightarrow \text{Hom}_R(S, N) \rightarrow 0$  where  $U$  is injective and  $G$  is G-injective. The composite  $U \rightarrow \text{Hom}_R(S, N) \rightarrow N$  is surjective. Since  $R$  is noetherian, there exists an injective precover  $\varphi: E \rightarrow N$ . The module  $U$  is injective over  $R$ , so a surjective homomorphism factors through  $\varphi$ , whence  $\varphi$  is surjective. Now consider the commutative diagram

$$(2) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \text{Hom}_R(S/R, N) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & U & \longrightarrow & \text{Hom}_R(S, N) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & E & \xrightarrow{\varphi} & N \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

To construct a proper left injective resolution of  $N$ , it suffices to show that  $K$  has the same properties as  $N$ , including (1); that is,  $K$  is cotorsion,  $\mathrm{Hom}_R(S, K)$  is G-injective over  $S$ , and  $\mathrm{Ext}_R^{>0}(I, K) = 0$  holds for every injective  $R$ -module  $I$ .

Injective  $S$ -modules are injective over  $R$ , so by 3.3 the G-injective  $S$ -module  $G$  is cotorsion over  $R$ . Further,  $\mathrm{Hom}_R(S/R, N)$  is cotorsion as  $S/R$  is flat; see 4.3. Let  $F$  be a flat  $R$ -module; application of  $\mathrm{Hom}_R(F, -)$  to (2) yields a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(F, \mathrm{Hom}_R(S, N)) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R(F, N) & \longrightarrow & \mathrm{Ext}_R^1(F, K) \longrightarrow 0 \\ \downarrow & & \\ 0 & & \end{array}$$

from which one concludes  $\mathrm{Ext}_R^1(F, K) = 0$ . That is,  $K$  is cotorsion.

Let  $I$  be an injective  $R$ -module. One has  $\mathrm{Ext}_R^{>0}(I, E) = 0$  and  $\mathrm{Ext}_R^{>0}(I, N) = 0$  by (1), so the exact sequence in cohomology associated to  $0 \rightarrow K \rightarrow E \rightarrow N \rightarrow 0$ , yields  $\mathrm{Ext}_R^{>1}(I, K) = 0$ . Finally,  $\mathrm{Hom}_R(I, -)$  leaves the sequence exact, because  $E \rightarrow N$  is an injective precover, and  $\mathrm{Ext}_R^1(I, K) = 0$  follows.

The exact sequence  $0 \rightarrow \mathrm{Hom}_R(S, K) \rightarrow \mathrm{Hom}_R(S, E) \rightarrow \mathrm{Hom}_R(S, N) \rightarrow 0$  shows that  $\mathrm{Hom}_R(S, K)$  has finite Gorenstein injective dimension over  $S$ , at most 1. Lemma 4.2 yields  $\mathrm{Ext}_R^{>0}(J, \mathrm{Hom}_R(S, K)) = 0$  for every injective  $S$ -module  $J$ , so  $\mathrm{Hom}_R(S, K)$  is G-injective by [14, thm. 2.22].  $\square$

**4.6 Lemma.** *Let  $R$  be commutative noetherian. Let  $0 \rightarrow L \rightarrow Q \rightarrow C \rightarrow 0$  be a pure exact sequence of  $R$ -modules with  $L \neq 0$  free and  $Q$  faithfully flat. For an  $R$ -module  $N$  with  $\mathrm{Ext}_R^i(Q, N) = 0$  for all  $i > 0$  the next conditions are equivalent.*

- (i)  $\mathrm{Ext}_R^i(I, N) = 0$  holds for all  $i > 0$  and every injective  $R$ -module  $I$ .
  - (ii)  $\mathrm{Ext}_R^i(I, \mathrm{Hom}_R(Q, N)) = 0$  holds for all  $i > 0$  and every injective  $R$ -module  $I$ .
- Moreover, if  $N$  is cotorsion, then these conditions are equivalent to
- (iii)  $\mathrm{Ext}_R^i(I, \mathrm{Hom}_R(F, N)) = 0$  holds for all  $i > 0$ , every injective  $R$ -module  $I$ , and every flat  $R$ -module  $F$ .

**Proof.** Let  $F$  be a flat  $R$ -module with  $\mathrm{Ext}_R^{>0}(F, N) = 0$ . For each integer  $i > 0$  and  $R$ -module  $X$  one has

$$\begin{aligned} \mathrm{Ext}_R^i(X, \mathrm{Hom}_R(F, N)) &= \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(X, \mathrm{Hom}_R(F, N))) \\ &\cong \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(F, N))) \\ &\cong \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(F \otimes_R^{\mathbf{L}} X, N)) \\ &\cong \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(F \otimes_R X, N)) \\ &= \mathrm{Ext}_R^i(F \otimes_R X, N); \end{aligned}$$

the first isomorphism holds by the assumption on  $F$ , the second by Hom-tensor adjointness in the derived category, and the third by flatness of  $F$ .

Let  $I$  be an injective  $R$ -module. Since  $R$  is noetherian, the module  $F \otimes_R I$  is injective, so it is immediate from the computation above that (i) implies (ii) and, if  $N$  is cotorsion, also the stronger statement (iii).

To show that (ii) implies (i), let  $I$  be an injective  $R$ -module and consider the exact sequence

$$0 \rightarrow L \otimes_R I \rightarrow Q \otimes_R I \rightarrow C \otimes_R I \rightarrow 0.$$

As  $R$  is noetherian, the module  $L \otimes_R I$  is injective; thus the sequence splits, whence  $I$  is isomorphic to a direct summand of  $Q \otimes_R I$ . The computation above yields  $\text{Ext}_R^{>0}(Q \otimes_R I, N) = 0$ , and  $\text{Ext}_R^{>0}(I, N) = 0$  follows as Ext functors are additive.  $\square$

The next result improves [7, Ascent table I.(d)]. Part (iii) applies, in particular, to the setting where  $Q$  is a faithfully flat  $R$ -algebra; cf. (4.1.1).

**4.7 Proposition.** *Let  $R$  be commutative noetherian. For an  $R$ -module  $N$  the following conditions are equivalent.*

- (i)  $N$  is Gorenstein injective.
- (ii)  $\text{Hom}_R(F, N)$  is Gorenstein injective for every flat  $R$ -module  $F$ .
- (iii)  $N$  is cotorsion and  $\text{Hom}_R(Q, N)$  is Gorenstein injective for some faithfully flat  $R$ -module  $Q$  that contains a non-zero free  $R$ -module as a pure submodule.

**Proof.** Assume first that  $N$  is G-injective and let  $U$  be a totally acyclic complex of injective modules with  $Z^0(U) \cong N$ . Let  $F$  be a flat module. G-injective modules are cotorsion, see 3.3, so the complex  $\text{Hom}_R(F, U)$  is acyclic. Moreover, it is a complex of injective modules, and for every injective module  $I$  the complex

$$\text{Hom}_R(I, \text{Hom}_R(F, U)) \cong \text{Hom}_R(F \otimes_R I, U)$$

is acyclic, as  $F \otimes_R I$  is injective by the assumption that  $R$  is noetherian. This proves the implication (i)  $\implies$  (ii).

That (ii) implies (iii) is trivial, just take  $F = R = Q$  and recall 3.3. To prove that (iii) implies (i), let  $Q$  be a faithfully flat  $R$ -module with a free pure submodule  $L \neq 0$ , and assume that  $\text{Hom}_R(Q, N)$  is G-injective. In particular,  $\text{Ext}_R^{>0}(I, \text{Hom}_R(Q, N)) = 0$  holds for every injective  $R$ -module  $I$ , so by Lemma 4.6

$$(1) \quad \text{Ext}_R^{>0}(I, N) = 0 \text{ holds for every injective } R\text{-module } I.$$

To prove that  $N$  is G-injective, it is now sufficient to show that it has a proper left injective resolution; see 3.2. To this end, consider the pure exact sequence

$$(2) \quad 0 \longrightarrow L \longrightarrow Q \longrightarrow C \longrightarrow 0,$$

and notice that the module  $C$  is flat as  $Q$  is flat. Since  $\text{Hom}_R(Q, N)$  is G-injective, there is an exact sequence of  $R$ -modules  $0 \rightarrow G \rightarrow E \rightarrow \text{Hom}_R(Q, N) \rightarrow 0$ , where  $E$  is injective and  $G$  is G-injective. As  $N$  is cotorsion, applying  $\text{Hom}_R(-, N)$  to (2)



yields another exact sequence, and the two meet in the commutative diagram,

$$(3) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \text{Hom}_R(C, N) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & \text{Hom}_R(Q, N) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & \text{Hom}_R(L, N) \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0. \end{array}$$

The first step towards construction of a proper left injective resolution of  $N$  is to show that  $H$  has the same properties as  $N$ , including (1).

*Claim 1:* The module  $H$  is cotorsion,  $\text{Hom}_R(Q, H)$  is  $G$ -injective, and one has  $\text{Ext}_R^{>0}(I, H) = 0$  for every injective  $R$ -module  $I$ .

*Proof:* Since the module  $G$  is  $G$ -injective, it is also cotorsion, and so is the module  $\text{Hom}_R(C, N)$  as  $C$  is flat; see 3.3 and 4.3. Let  $F$  be a flat  $R$ -module and apply  $\text{Hom}_R(F, -)$  to (3) to get the commutative diagram

$$\begin{array}{ccccc} \text{Hom}_R(F, \text{Hom}_R(Q, N)) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \\ \text{Hom}_R(F, \text{Hom}_R(L, N)) & \longrightarrow & \text{Ext}_R^1(F, H) & \longrightarrow & 0 \\ \downarrow & & & & \\ 0, & & & & \end{array}$$

which shows that  $\text{Ext}_R^1(F, H)$  vanishes, whence  $H$  is cotorsion.

Notice that one has  $\text{Hom}_R(L, N) \cong N^\Lambda$  for some index set  $\Lambda$ . Let  $I$  be an injective module; by (1) one has  $\text{Ext}_R^{>0}(I, N^\Lambda) = 0$ , so the exact sequence in cohomology induced by the last non-zero row of (3) yields  $\text{Ext}_R^{>1}(I, H) = 0$ . Now we argue that also  $\text{Ext}_R^1(I, H)$  vanishes. As  $G$  is  $G$ -injective, one has  $\text{Ext}_R^{>0}(I, G) = 0$ ; by the assumptions on  $N$  and (1), Lemma 4.6 applies to yield  $\text{Ext}_R^{>0}(I, \text{Hom}_R(C, N)) = 0$ . Applying  $\text{Hom}_R(I, -)$  to (3) one thus gets the commutative diagram

$$(4) \quad \begin{array}{ccccc} \text{Hom}_R(I, \text{Hom}_R(Q, N)) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \\ \text{Hom}_R(I, \text{Hom}_R(L, N)) & \longrightarrow & \text{Ext}_R^1(I, H) & \longrightarrow & 0 \\ \downarrow & & & & \\ 0, & & & & \end{array}$$

which yields  $\text{Ext}_R^1(I, H) = 0$ .

It now follows from Lemma 4.6 that  $\text{Ext}_R^{>0}(I, \text{Hom}_R(Q, H)) = 0$  holds for every injective  $R$ -module  $I$ . To prove that the module  $\text{Hom}_R(Q, H)$  is  $G$ -injective it is thus, by [14, thm. 2.22], enough to show that it has finite Gorenstein injective

dimension. However, that is immediate as application of  $\mathrm{Hom}_R(Q, -)$  to last non-zero row of (3) yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, H) \longrightarrow \mathrm{Hom}_R(Q, E) \longrightarrow \mathrm{Hom}_R(Q, N^\Lambda) \longrightarrow 0,$$

where  $\mathrm{Hom}_R(Q, E)$  is injective, and  $\mathrm{Hom}_R(Q, N^\Lambda) \cong \mathrm{Hom}_R(Q, N)^\Lambda$  is G-injective; see [14, thm. 2.6]. This finishes the proof of Claim 1.

In the commutative diagram below, the second non-zero row is the last non-zero row from (3), and  $\pi$  is a canonical projection.

$$(5) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & N^{\Lambda'} & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & E & \longrightarrow & N^\Lambda \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \pi \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & N \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

To construct a proper left injective resolution of  $N$ , it is sufficient to prove that  $K$  has the same properties as  $N$ , including (1).

*Claim 2:* The module  $K$  is cotorsion,  $\mathrm{Hom}_R(Q, K)$  is G-injective, and one has  $\mathrm{Ext}_R^{>0}(I, K) = 0$  for every injective  $R$ -module  $I$ .

*Proof:* Apply the Snake Lemma to the diagram (5) to get the exact sequence  $0 \rightarrow H \rightarrow K \rightarrow N^{\Lambda'} \rightarrow 0$ . By assumption  $N$  is cotorsion, and  $H$  is cotorsion by Claim 1; it follows that  $K$  is cotorsion. Moreover, by (1) and Claim 1 one has  $\mathrm{Ext}_R^{>0}(I, H) = 0 = \mathrm{Ext}_R^{>0}(I, N)$  and hence  $\mathrm{Ext}_R^{>0}(I, K) = 0$  for every injective module  $I$ . The induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, H) \longrightarrow \mathrm{Hom}_R(Q, K) \longrightarrow \mathrm{Hom}_R(Q, N^{\Lambda'}) \longrightarrow 0$$

is exact, as  $H$  is cotorsion. The modules  $\mathrm{Hom}_R(Q, N^{\Lambda'})$  and  $\mathrm{Hom}_R(Q, H)$  are G-injective by assumption and Claim 1, so by [14, thm. 2.6] also  $\mathrm{Hom}_R(Q, K)$  is G-injective. That finishes the proof of Claim 2.  $\square$

**4.8 Corollary.** *Let  $R$  be commutative noetherian. For every flat  $R$ -module  $F$  and for every  $R$ -complex  $N$  there is an inequality  $\mathrm{Gid}_R \mathbf{R}\mathrm{Hom}_R(F, N) \leq \mathrm{Gid}_R N$ .*

**Proof.** We may assume that  $\mathrm{H}(N)$  is non-zero, otherwise there is nothing to prove. Let  $N \xrightarrow{\simeq} I$  be a semi-injective resolution and assume that  $\mathrm{Gid}_R N = g$  is finite. By Proposition 3.6 the module  $Z = Z^g(I)$  is G-injective, and arguing as in the proof of 3.7 one gets  $\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(F, N)) = 0$  for all  $i > g$ . The complex,  $\mathrm{Hom}_R(F, I)$  is semi-injective and isomorphic to  $\mathbf{R}\mathrm{Hom}_R(F, N)$  in the derived category. From Proposition 4.7 it follows that the module  $Z^g(\mathrm{Hom}_R(F, I)) \cong \mathrm{Hom}_R(F, Z^g(I))$  is G-injective, and then Proposition 3.6 yields  $\mathrm{Gid}_R \mathbf{R}\mathrm{Hom}_R(F, N) \leq g$ .  $\square$

**4.9.** For the statement of the next corollary we recall the invariant

$$\mathrm{splf} R = \sup\{\mathrm{proj.dim}_R F \mid F \text{ is a flat } R\text{-module}\}.$$

If  $R$  is noetherian of finite Krull dimension  $d$ , then one has  $\text{splf } R \leq d$  by works of Jensen [16, prop. 6], and Gruson and Raynaud [19, thm. II.(3.2.6)]. If  $R$  has cardinality at most  $\aleph_n$  for some integer  $n$ , then a theorem of Gruson and Jensen [13, thm. 7.10] yields  $\text{splf } R \leq n + 1$ . Osofsky [18, 3.1] gives examples of rings for which the  $\text{splf}$  invariant is infinite.

Since an arbitrary direct sum of flat  $R$ -modules is flat, the invariant  $\text{splf } R$  is finite if and only if every flat  $R$ -module has finite projective dimension.

**4.10 Proposition.** *Let  $R$  be commutative noetherian with  $\text{splf } R < \infty$ . Let  $S$  be a faithfully flat  $R$ -algebra and  $N$  be an  $R$ -module with  $\text{Ext}_R^i(S, N) = 0$  for all  $i > 0$ . If  $\text{Hom}_R(S, N)$  is Gorenstein injective as an  $R$ -module or as an  $S$ -module, then  $N$  is Gorenstein injective.*

**Proof.** Set  $n = \text{splf } R$ . Let  $N \xrightarrow{\sim} E$  be an injective resolution. As every flat  $R$ -module has projective dimension at most  $n$ , it follows by dimension shifting that the cosyzygy  $Z = Z^n(E)$  is cotorsion. Vanishing of  $\text{Ext}_R^{>0}(S, N)$  implies that  $\text{Hom}_R(S, -)$  leaves the sequence  $0 \rightarrow N \rightarrow E^0 \rightarrow \dots \rightarrow E^{n-1} \rightarrow Z \rightarrow 0$  exact.

The class of G-injective modules is injectively resolving; see [14, thm. 2.6]. If  $\text{Hom}_R(S, N)$  is G-injective over  $R$ , it thus follows that  $\text{Hom}_R(S, Z)$  is G-injective over  $R$ . From Proposition 4.7 it follows that  $Z$  is G-injective over  $R$ , so  $N$  has finite Gorenstein injective dimension. Finally, Lemma 4.6 yields  $\text{Ext}_R^{>0}(I, N) = 0$  for every injective  $R$ -module  $I$ , so  $N$  is G-injective by [14, thm. 2.22].

If  $\text{Hom}_R(S, N)$  is G-injective over  $S$ , then it follows as above that  $\text{Hom}_R(S, Z)$  is G-injective over  $S$ . From Theorem 4.5 it follows that  $Z$  is G-injective over  $R$ , so  $N$  has finite Gorenstein injective dimension. Finally, it follows from Lemma 4.2 that  $\text{Ext}_R^{>0}(I, N) = 0$  holds for every injective  $R$ -module  $I$ , so  $N$  is G-injective.  $\square$

**4.11 Lemma.** *Let  $R$  be commutative noetherian and let  $S$  be a faithfully flat  $R$ -algebra. For every  $R$ -complex  $N$  of finite Gorenstein injective dimension one has*

$$\text{Gid}_S \mathbf{RHom}_R(S, N) = \text{Gid}_R N = \text{Gid}_R \mathbf{RHom}_R(S, N).$$

**Proof.** We may assume that  $\mathbf{H}(N)$  is non-zero, otherwise there is nothing to prove. Set  $g = \text{Gid}_R N$ ; Proposition 3.7 and Corollary 4.8 yield  $\text{Gid}_S \mathbf{RHom}_R(S, N) \leq g$  and  $\text{Gid}_R \mathbf{RHom}_R(S, N) \leq g$ . Let  $N \xrightarrow{\sim} I$  be a semi-injective resolution; the complex  $\text{Hom}_R(S, I)$  is isomorphic to  $\mathbf{RHom}_R(S, N)$  in the derived category, and it is semi-injective as an  $R$ -complex and as an  $S$ -complex. By [1, thm. 2.4] there exists an injective  $R$ -module  $E$  with  $\text{Ext}_R^g(E, N) \neq 0$ .

As an  $R$ -module,  $E$  is a direct summand of the injective  $S$ -module  $\text{Hom}_R(S, E)$ , see 4.1, so one gets

$$\begin{aligned} \text{Ext}_S^g(\text{Hom}_R(S, E), \mathbf{RHom}_R(S, N)) &\cong \mathbf{H}^g(\text{Hom}_S(\text{Hom}_R(S, E), \text{Hom}_R(S, I))) \\ &\cong \mathbf{H}^g(\text{Hom}_R(\text{Hom}_R(S, E), I)) \\ &\cong \text{Ext}_R^g(\text{Hom}_R(S, E), N) \neq 0, \end{aligned}$$

and hence  $\text{Gid}_S \mathbf{RHom}_R(S, N) \geq g$  by [1, thm. 2.4]; i.e. equality holds.

As an  $R$ -module,  $E$  is a direct summand of  $S \otimes_R E$ , see 4.1, so one gets

$$\begin{aligned} \text{Ext}_R^g(E, \mathbf{RHom}_R(S, N)) &\cong \mathbf{H}^g(\text{Hom}_R(E, \text{Hom}_R(S, I))) \\ &\cong \mathbf{H}^g(\text{Hom}_R(S \otimes_R E, I)) \\ &\cong \text{Ext}_R^g(S \otimes_R E, N) \neq 0, \end{aligned}$$

and hence  $\text{Gid}_R \mathbf{RHom}_R(S, N) \geq g$  by [1, thm. 2.4]; i.e. equality holds.  $\square$

**4.12 Theorem.** *Let  $R$  be commutative noetherian with  $\text{splf } R < \infty$  and let  $S$  be a faithfully flat  $R$ -algebra. For every  $R$ -complex  $N$  with  $H^i(N) = 0$  for all  $i \gg 0$  there are equalities*

$$\text{Gid}_S \mathbf{R}\text{Hom}_R(S, N) = \text{Gid}_R N = \text{Gid}_R \mathbf{R}\text{Hom}_R(S, N).$$

**Proof.** In view of Lemma 4.11 it is sufficient to prove that  $\text{Gid}_R N$  is finite if  $\mathbf{R}\text{Hom}_R(S, N)$  has finite Gorenstein injective dimension as an  $R$ -complex or as an  $S$ -complex. Assume, without loss of generality, that  $H^i(N) = 0$  holds for  $i > 0$ , and let  $N \xrightarrow{\sim} I$  be a semi-injective resolution. The complex  $\text{Hom}_R(S, I)$  is isomorphic to  $\mathbf{R}\text{Hom}_R(S, N)$  in the derived category, and it is semi-injective as an  $R$ -complex and as an  $S$ -complex.

If  $\text{Gid}_S \mathbf{R}\text{Hom}_R(S, N)$  is finite, then one has  $\text{Gid}_S \mathbf{R}\text{Hom}_R(S, N) \leq g$  for some  $g > 0$ . By Proposition 3.6 the  $S$ -module  $Z^g(\text{Hom}_R(S, I)) \cong \text{Hom}_R(S, Z^g(I))$  is G-injective and one has  $\text{Ext}_R^n(S, Z^g(I)) = H^{g+n}(\text{Hom}_R(S, I)) = 0$  for all  $n > 0$ . Now follows from Proposition 4.10 that  $Z^g(I)$  is G-injective, so  $\text{Gid}_R N \leq g$  holds.

The same argument applies if  $\text{Gid}_R \mathbf{R}\text{Hom}_R(S, N)$  is finite.  $\square$

The equivalent of this theorem in absolute homological algebra is [10, thm. 2.2] about injective dimension of a complex  $N$ . In that statement, the *a priori* assumption of vanishing of  $H^i(N)$  for  $i \gg 0$  is missing, though it is applied in the proof in the same manner as above. However, [10, thm. 2.2] is correct as stated, and an even stronger result is proved in [9, cor. 3.1].

**4.13 Remark.** Theorem 4.5 improves [11, lem. 1.3.(a)] by removing assumptions that  $S$  should be commutative and noetherian with  $\dim S < \infty$ .

The first equality in Theorem 4.12 compares to the equality in [11, thm. 1.7] as follows: There is no assumption of finite Krull dimension of  $R$ , and  $S$  is not assumed to be commutative nor noetherian. The inequality in [11, thm. 1.7] is subsumed by [22, cor. 9] and compares to Proposition 3.7 as discussed there.

In [22, thm. 11] the first equality in Theorem 4.12 is proved without conditions on the homology of  $N$  but under the assumption that  $S$  is finitely generated as an  $R$ -module. Note that this assumption implies that  $R$  is an algebra retract of  $S$ .

## 5. GORENSTEIN FLAT DIMENSION

Recall the definition [12, def. 10.3.1] of a Gorenstein flat module.

**5.1.** A complex  $T$  of flat  $R$ -modules is called *totally acyclic* if it is acyclic and the complex  $I \otimes_R T$  is acyclic for every injective  $R^\circ$ -module  $I$ .

An  $R$ -module  $G$  is called *Gorenstein flat* (for short, G-flat) if there exists a totally acyclic complex  $T$  of flat  $R$ -modules with  $C_0(T) \cong G$ .

**5.2.** Every module has a projective resolution, so to prove that an  $R$ -module  $M$  is Gorenstein flat it suffices to verify the following:

- (1)  $\text{Tor}_i^R(I, M) = 0$  holds for all  $i > 0$  and every injective  $R^\circ$ -module  $I$ .
- (2) There exists an acyclic complex  $T^+ = 0 \rightarrow M \rightarrow T_{-1} \rightarrow T_{-2} \rightarrow \dots$  of  $R$ -modules with each  $T_i$  flat, such that  $I \otimes_R T^+$  is acyclic for every injective  $R^\circ$ -module  $I$ .

The next result is a non-commutative version of [7, Ascent table I.(a)].

**5.3 Lemma.** *Let  $R \rightarrow S$  be a ring homomorphism such that  $\text{fd}_R S$  and  $\text{fd}_{R^\circ} S$  are finite. For a Gorenstein flat  $R$ -module  $G$ , the  $S$ -module  $S \otimes_R G$  is Gorenstein flat.*

**Proof.** Let  $T$  be a totally acyclic complex of flat  $R$ -modules with  $C_0(T) \cong G$ . The complex  $S \otimes_R T$  consists of flat  $S$ -modules, and it is acyclic as  $\text{fd}_{R^\circ} S$  is finite; see [6, lem. 2.3]. One has  $C_0(S \otimes_R T) \cong S \otimes_R G$ , so it suffices to show that the complex  $J \otimes_S (S \otimes_R T) \cong J \otimes_R T$  is acyclic for every injective  $S^\circ$ -module  $J$ . As  $\text{fd}_R S$  is finite, every injective  $S^\circ$ -module has finite injective dimension over  $R^\circ$ ; see [3, cor. 4.2]. Acyclicity of  $J \otimes_R T$  hence follows from [6, lem. 2.3].  $\square$

In standard homological algebra, the study of flat modules and flat dimension is propelled by flat–injective duality: An  $R$ -module  $F$  is flat if and only if the  $R^\circ$ -module  $\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$  is injective. A similar Gorenstein flat–injective duality is only known to hold if  $R$  is right coherent, so that is a convenient setting for studies of Gorenstein flat dimension; see for example [14, thm. 3.14].

As Bennis demonstrates [4, thm. 2.8], one can do a little better and get a well-behaved theory of Gorenstein flat dimension, as long as the class of Gorenstein flat modules is projectively resolving. Rings with that property have become known as left GF-closed; we recall the definition in 5.10.

Iacob [15] defined a notion of Gorenstein flat dimension for unbounded complexes, but only for complexes over left GF-closed rings. Here we give a definition, inspired by Veliche’s [21, def. 3.1]; it applies to complexes over any ring and coincides with Iacob’s for left GF-closed rings.

To get started we recall some terminology from Liang [17].

**5.4.** Let  $M$  be an  $R$ -complex; a *semi-flat replacement* of  $M$  is a semi-flat  $R$ -complex  $F$  such that there is an isomorphism  $F \simeq M$  in the derived category. Every complex has a semi-projective resolution and hence a semi-flat replacement; see 2.3. A *Tate flat resolution* of  $M$  is a pair  $(T, F)$  where  $T$  is a totally acyclic complex of flat  $R$ -modules and  $F \simeq M$  a semi-flat replacement with  $T_{\geq g} \cong F_{\geq g}$  for some  $g \in \mathbb{Z}$ .

**5.5 Definition.** Let  $M$  be an  $R$ -complex. The *Gorenstein flat dimension* of  $M$  is given by

$$\text{Gfd}_R M = \inf\{g \in \mathbb{Z} \mid (T, F) \text{ is a Tate flat resolution of } M \text{ with } T_{\geq g} \cong F_{\geq g}\}.$$

**5.6.** Let  $M$  be an  $R$ -complex; the following facts are evident from the definition.

- (a) One has  $\text{Gfd}_R M < \infty$  if and only if  $M$  admits a Tate flat resolution.
- (b)  $M$  is acyclic if and only if one has  $\text{Gfd}_R M = -\infty$ .
- (c)  $\text{Gfd}_R(\Sigma^n M) = \text{Gfd}_R M + n$  holds for every  $n \in \mathbb{Z}$ .<sup>2</sup>

A Gorenstein flat  $R$ -module  $M \neq 0$  clearly has  $\text{Gfd}_R M = 0$  per Definition 5.5. In Remark 5.13 we compare 5.5 to competing definitions in the literature.

**5.7 Proposition.** *Let  $M$  be an  $R$ -complexes and  $g \in \mathbb{Z}$ . The following conditions are equivalent.*

- (i)  $\text{Gfd}_R M \leq g$ .
- (ii)  $H_i(M) = 0$  holds for all  $i > g$ , and there exists a semi-flat replacement  $F \simeq M$  such that  $C_i(F)$  is Gorenstein flat for every  $i \geq g$ .

<sup>2</sup> Here  $\Sigma^n M$  is the complex with  $(\Sigma^n M)_i = M_{i-n}$  and  $\partial_i^{\Sigma^n M} = (-1)^n \partial_{i-n}^M$  for all  $i \in \mathbb{Z}$ .

(iii)  $H_i(M) = 0$  holds for all  $i > g$ , and there exists a semi-flat replacement  $F \simeq M$  such that  $C_g(F)$  is Gorenstein flat.

**Proof.** If  $\text{Gfd}_R M \leq g$  holds, then there exists a Tate flat resolution  $(T, F)$  of  $M$  with  $T_{\geq g} \cong F_{\geq g}$ . As  $T$  is acyclic and  $F \simeq M$  one has  $H_i(M) \cong H_i(F) \cong H_i(T) = 0$  for all  $i > g$ , and  $C_i(F) \cong C_i(T)$  is G-flat for every  $i \geq g$ . Thus (i) implies (ii).

Clearly, (ii) implies (iii); to finish the proof assume that  $H_i(M) = 0$  holds for all  $i > g$  and that there exists a semi-flat replacement  $F \simeq M$  such that  $C_g(F)$  is G-flat. As one has  $H(F) \cong H(M)$  it follows that the complex

$$F' = \cdots \longrightarrow F_{g+1} \longrightarrow F_g \longrightarrow C_g(F) \longrightarrow 0$$

is acyclic; i.e. it is an augmented flat resolution of  $C_g(F)$ . For every injective  $R^\circ$ -module  $I$  one has  $\text{Tor}_{>0}^R(I, C_g(F)) = 0$ , so the complex  $I \otimes_R F'$  is acyclic. Moreover, there exists an acyclic complex of  $R$ -modules

$$T' = 0 \longrightarrow C_g(F) \longrightarrow T_{g-1} \longrightarrow T_{g-2} \longrightarrow \cdots$$

with each  $T_i$  flat, such that  $I \otimes_R T'$  is acyclic for every injective  $R^\circ$ -module  $I$ . Now let  $T$  be the complex obtained by splicing together  $F'$  and  $T'$  at  $C_g(F)$ ; that is

$$T = \cdots \longrightarrow F_{g+1} \longrightarrow F_g \longrightarrow T_{g-1} \longrightarrow T_{g-2} \longrightarrow \cdots$$

It is a totally acyclic complex of flat  $R$ -modules, so  $(T, F)$  is a Tate flat resolution of  $M$  with  $T_{\geq g} \cong F_{\geq g}$ , and so one has  $\text{Gfd}_R M \leq g$ .  $\square$

In the next statement ‘fd’ means flat dimension as defined in [3, sec. 2.F].

**5.8 Proposition.** *Let  $M$  be an  $R$ -complex; the following assertions hold.*

- (a) *One has  $\text{Gfd}_R M \leq \text{fd}_R M$ ; equality holds if  $\text{fd}_R M < \infty$ .*
- (b) *One has  $\text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M$ ; equality holds if  $R$  is right coherent.*

**Proof.** (a): For an acyclic  $R$ -complex  $M$  one has  $\text{Gfd}_R M = -\infty = \text{fd}_R M$ , and the inequality is trivial if  $\text{fd}_R M = \infty$ . Now assume that  $H(M) \neq 0$  and  $\text{fd}_R M = g$  holds for some  $g \in \mathbb{Z}$ . It follows from [3, thm. 2.4.F] that  $H_i(M) = 0$  holds for all  $i > g$ , and that there exists a semi-flat replacement  $F \simeq M$  such that  $C_g(F)$  is flat, in particular G-flat. Thus one has  $\text{Gfd}_R M \leq g$  by Proposition 5.7. To prove that equality holds, assume towards a contradiction that  $\text{Gfd}_R M \leq g - 1$  holds. One then has  $H_i(M) = 0$  for all  $i > g - 1$  and there exists a semi-flat replacement  $F \simeq M$  such that  $C_{g-1}(F)$  is G-flat; see Proposition 5.7. Consider the exact sequence

$$0 \longrightarrow C_g(F) \longrightarrow F_{g-1} \longrightarrow C_{g-1}(F) \longrightarrow 0.$$

The module  $C_g(F)$  is flat, so one has  $\text{fd}_R(C_{g-1}(F)) \leq 1$ , and then  $C_{g-1}(F)$  is flat; see Bennis [5, thm. 2.2]. Thus one has  $\text{fd}_R M \leq g - 1$ , which is a contradiction.

(b): For an acyclic  $R$ -complex  $M$  one has  $\text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = -\infty = \text{Gfd}_R M$ , and the inequality is trivial if  $\text{Gfd}_R M = \infty$ . Now assume that  $H(M) \neq 0$  and  $\text{Gfd}_R M = g$  holds for some  $g \in \mathbb{Z}$ . By Proposition 5.7 one has  $H_i(M) = 0$  for all  $i > g$ , and there exists a semi-flat replacement  $F \simeq M$  such that  $C_g(F)$  is G-flat. The  $R^\circ$ -complex  $\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$  is semi-injective and yields a semi-injective resolution  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ ; see [3, 1.4.I]. For all  $i > g$  one has

$$H^i(\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(H_i(M), \mathbb{Q}/\mathbb{Z}) = 0,$$

and by [14, prop. 3.11] the  $R^\circ$ -module  $Z^g(\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(C_g(F), \mathbb{Q}/\mathbb{Z})$  is G-injective, so one gets  $\text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq g$  from Proposition 3.6.

Assume now that  $g = \text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is finite. Let  $F \simeq M$  be a semi-flat replacement; as above it yields a semi-injective resolution

$$\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}).$$

By Proposition 3.6 one has  $\text{Hom}_{\mathbb{Z}}(\text{H}_i(M), \mathbb{Q}/\mathbb{Z}) \cong \text{H}^i(\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = 0$  for all  $i > g$  and the module  $\text{Hom}_{\mathbb{Z}}(\text{C}_g(F), \mathbb{Q}/\mathbb{Z}) \cong \text{Z}^g(\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}))$  is G-injective. It follows that  $\text{H}_i(M) = 0$  holds for all  $i > g$ , and under the assumption that  $R$  is right coherent it follows from [14, prop. 3.11] that  $\text{C}_g(F)$  is G-flat. Thus one has  $\text{Gfd}_R M \leq g$  by Proposition 5.7, and  $\text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M$  holds.  $\square$

The next result is proved by standard applications of the Snake Lemma.

**5.9 Lemma.** *Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be an exact sequence of  $R$ -complexes and let  $n \in \mathbb{Z}$ . The following assertions hold.*

- (a) *If  $\text{H}_n(Z) = 0$ , then the sequence  $0 \rightarrow \text{C}_{n-1}(X) \rightarrow \text{C}_{n-1}(Y) \rightarrow \text{C}_{n-1}(Z) \rightarrow 0$  is exact. Moreover if  $\text{H}_n(Y) = 0$ , then the converse holds.*
- (b) *If  $\text{H}_n(X) = 0$ , then the sequence  $0 \rightarrow \text{Z}_{n+1}(X) \rightarrow \text{Z}_{n+1}(Y) \rightarrow \text{Z}_{n+1}(Z) \rightarrow 0$  is exact. Moreover if  $\text{H}_n(Y) = 0$ , then the converse holds.*  $\square$

**5.10.** Recall from [4] that  $R$  is called left *GF-closed* if the class of Gorenstein flat  $R$ -modules is closed under extensions (equivalently, it is projectively resolving). The class of left GF-closed rings strictly contains the class of right coherent rings.

The next result appears as [15, lem. 3.2] with the additional hypothesis that  $R$  is left GF-closed; that hypothesis is, however, not needed.

**5.11 Lemma.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be an exact sequence of  $R$ -modules. If  $F$  is flat,  $G$  is Gorenstein flat, and  $\text{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z})$  is Gorenstein injective, then  $H$  is Gorenstein flat.*

**Proof.** It is argued in the proof of [15, lem. 3.2] that there is an exact sequence  $0 \rightarrow H \rightarrow X \rightarrow Y \rightarrow 0$  with  $X$  flat and  $Y$  G-flat, so  $H$  is G-flat by [4, lem. 2.4].  $\square$

**5.12 Proposition.** *Let  $M$  be an  $R$ -complex and  $g \in \mathbb{Z}$ . Conditions (i) and (ii) below are equivalent and imply (iii). Moreover, if  $R$  is left GF-closed, then all three conditions are equivalent.*

- (i)  $\text{H}_i(M) = 0$  holds for all  $i > g$ , and for every semi-flat replacement  $F \simeq M$  the cokernel  $\text{C}_g(F)$  is Gorenstein flat.
- (ii)  $\text{H}_i(M) = 0$  holds for all  $i > g$ , and there exists a semi-projective resolution  $P \xrightarrow{\simeq} M$  such that the cokernel  $\text{C}_g(P)$  is Gorenstein flat.
- (iii)  $\text{Gfd}_R M \leq g$ .

**Proof.** Clearly (i) implies (ii), and (ii) implies (iii) by Proposition 5.7.

(ii)  $\implies$  (i): Let  $P \xrightarrow{\simeq} M$  be a semi-projective resolution such that  $\text{C}_g(P)$  is G-flat. Let  $F \simeq M$  be a semi-flat replacement. There is by [3, 1.4.P] a quasi-isomorphism  $\varphi: P \xrightarrow{\simeq} F$ , and after adding to  $P$  a projective precover of  $F$ , we may assume that  $\varphi$  is surjective. One then has an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow F \rightarrow 0$ . The kernel  $K$  is semi-flat and acyclic, so the module  $\text{C}_i(K)$  is flat for every  $i \in \mathbb{Z}$ ; see Christensen and Holm [8, thm. 7.3]. The sequence

$$(1) \quad 0 \longrightarrow \text{C}_g(K) \longrightarrow \text{C}_g(P) \longrightarrow \text{C}_g(F) \longrightarrow 0$$

is exact by Lemma 5.9(a). The module  $C_g(P)$  is G-flat, so to prove that  $C_g(F)$  is G-flat, it is by Lemma 5.11 enough to prove that  $\text{Hom}_{\mathbb{Z}}(C_g(F), \mathbb{Q}/\mathbb{Z})$  is G-injective. By Propositions 5.7 and 5.8(b) one has  $\text{Gid}_{R^\circ} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq g$ . The  $R^\circ$ -complex  $\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$  is semi-injective and yields a semi-injective resolution  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})$ ; see [3, 1.4.I]. It now follows from Proposition 3.6 that the module  $\text{Hom}_{\mathbb{Z}}(C_g(F), \mathbb{Q}/\mathbb{Z}) \cong Z^g(\text{Hom}_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}))$  is G-injective.

Thus, the conditions (i) and (ii) are equivalent. We now assume that  $R$  is left GF-closed and verify the implication (iii)  $\implies$  (ii). By Proposition 5.7 one has  $H_i(M) = 0$  for all  $i > g$ , and there exists a semi-flat replacement  $F \simeq M$  with  $C_g(F)$  G-flat. Choose a surjective semi-projective resolution  $P \xrightarrow{\simeq} F$ ; it yields by [3, 1.4.P] a semi-projective resolution  $P \xrightarrow{\simeq} M$ . As above one gets an sequence (1), where now  $C_g(K)$  is flat and  $C_g(F)$  is G-flat; it follows that  $C_g(P)$  is G-flat as  $R$  is left GF-closed.  $\square$

For commentary on the proof, see Remark 5.16 at the end of the section.

**5.13 Remark.** From Proposition 5.12 and [15, thm. 1] it is clear that for complexes over a left GF-closed ring, Definition 5.5 agrees with Iacob's definition [15, def. 3.2]. In this setting, it thus extends the definitions in [14, def. 3.9] and [6, 1.9] of Gorenstein flat dimension for modules and complexes with bounded below homology; see [15, rmk. 3].

Though they potentially differ over rings that are not GF-closed, we use the same symbol, 'Gfd', for our notion and the one from [6, 1.9]. It causes no ambiguity, as the results we prove in the remainder of this paper are valid for either notion. When needed, we make remarks to that effect.

The next result improves [22, thm. 12] by removing assumptions that  $R$  and  $S$  should be commutative noetherian.

**5.14 Proposition.** *Let  $R \rightarrow S$  be a ring homomorphism such that  $\text{fd}_R S$  and  $\text{fd}_{R^\circ} S$  are finite. For every  $R$ -complex  $M$  one has  $\text{Gfd}_S(S \otimes_R^{\mathbf{L}} M) \leq \text{Gfd}_R M$ .*

**Proof.** We may assume that  $H(M)$  is non-zero, otherwise there is nothing to prove. Assume that  $\text{Gfd}_R M = g$  holds for some  $g \in \mathbb{Z}$ . By Proposition 5.7 one has  $H_i(M) = 0$  for all  $i > g$ , and there exists a semi-flat replacement  $F \simeq M$  with  $C_g(F)$  G-flat. The  $S$ -complex  $S \otimes_R F$  is semi-flat and isomorphic to  $S \otimes_R^{\mathbf{L}} M$  in the derived category. For  $n > 0$  one has

$$H_{g+n}(S \otimes_R^{\mathbf{L}} M) \cong H_{g+n}(S \otimes_R F) \cong \text{Tor}_n^R(S, C_g(F)) = 0,$$

where vanishing follows from [6, lem. 2.3] as  $\text{fd}_{R^\circ} S$  is finite. Finally, the module  $C_g(S \otimes_R F) \cong S \otimes_R C_g(F)$  is G-flat by Lemma 5.3, so  $\text{Gfd}_S(S \otimes_R^{\mathbf{L}} M) \leq g$  holds by Proposition 5.7.  $\square$

**5.15 Remark.** For an  $R$ -complex  $M$  with  $H_i(M) = 0$  for  $i \ll 0$  the inequality in Proposition 5.14 is also valid with the definition of Gorenstein flat dimension from [6, 1.9]. Indeed, the argument for [7, (4.4)] also applies when  $R$  is not commutative.

**5.16 Remark.** From the proof of (iii)  $\implies$  (ii) in Proposition 5.12, consider the exact sequence

$$(5.16.1) \quad 0 \longrightarrow C_g(K) \longrightarrow C_g(P) \longrightarrow C_g(F) \longrightarrow 0$$



where the module  $C_g(K)$  is flat and  $C_g(F)$  is Gorenstein flat. Under the assumption that  $R$  is left GF-closed, the module in the middle,  $C_g(P)$ , is Gorenstein flat. Without that assumption, the best we know is to consider the pull-back diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & G & \xlongequal{\quad} & G & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & C_g(K) & \longrightarrow & X & \longrightarrow & T \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & C_g(K) & \longrightarrow & C_g(P) & \longrightarrow & C_g(F) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & 
 \end{array}$$

where  $T$  is flat and  $G$  is Gorenstein flat. It follows that  $X$  is flat, so  $C_g(P)$  has Gorenstein flat dimension at most 1, per Definition 5.5 as well as per [14, def. 3.9].

Notice that the module  $C_g(K)$  in (5.16.1) is not just Gorenstein flat but flat. In an earlier version of this paper we asked if a ring is GF-closed provided that in every exact sequence of  $R$ -modules  $0 \rightarrow F \rightarrow M \rightarrow G \rightarrow 0$  with  $F$  flat and  $G$  Gorenstein flat also  $M$  is Gorenstein flat. A positive answer to this question is provided in the Appendix by Driss Bennis.

## 6. FAITHFULLY FLAT BASE CHANGE

In this section,  $R$  is commutative and  $S$  is an  $R$ -algebra.

**6.1 Lemma.** *Let  $S$  be a faithfully flat  $R$ -algebra. For an  $R$ -module  $M$  the following conditions are equivalent.*

- (i)  $\mathrm{Tor}_i^R(I, M) = 0$  holds for all  $i > 0$  and every injective  $R$ -module  $I$ .
- (ii)  $\mathrm{Tor}_i^S(J, S \otimes_R M) = 0$  holds for all  $i > 0$  and every injective  $S^\circ$ -module  $J$ .

**Proof.** For every  $S^\circ$ -module  $X$  and every  $i > 0$  one has

$$\mathrm{Tor}_i^S(X, S \otimes_R M) \cong \mathrm{H}_i(X \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M)) \cong \mathrm{H}_i(X \otimes_R^{\mathbf{L}} M) \cong \mathrm{Tor}_i^R(X, M).$$

As  $S$  is flat over  $R$ , every injective  $S^\circ$ -module is an injective  $R$ -module, so (i) implies (ii). Let  $I$  be an  $R$ -module and recall from 4.1 that it is a direct summand of the injective  $S^\circ$ -module  $\mathrm{Hom}_R(S, I)$ ; now (i) follows from (ii).  $\square$

For use in the next proof, we recall the notion of a flat preenvelope, also known as a flat left approximation.

**6.2.** Let  $M$  be an  $R$ -module. A homomorphism  $\varphi: M \rightarrow F$  is a flat *preenvelope* of  $M$ , if  $F$  is a flat  $R$ -module and every homomorphism from  $M$  to a flat  $R$ -module factors through  $\varphi$ . Every  $R$ -module has a flat preenvelope if and only if  $R$  is coherent; see [12, thm. 5.4.1].

**6.3 Lemma.** *Let  $R$  be coherent and let  $S$  be a faithfully flat  $R$ -algebra that is left GF-closed. An  $R$ -module  $M$  is Gorenstein flat if and only if the  $S$ -module  $S \otimes_R M$  is Gorenstein flat.*

**Proof.** The “only if” part is a special case of Lemma 5.3. To prove “if”, assume that  $S \otimes_R M$  is a G-flat  $S$ -module. Now  $\mathrm{Tor}_{>0}^S(J, S \otimes_R M) = 0$  holds for every injective  $S^\circ$ -module  $J$ , so by Lemma 6.1 one has

$$(1) \quad \mathrm{Tor}_{>0}^R(I, M) = 0 \text{ for every injective } R\text{-module } I.$$

Thus, 5.2.(1) holds and we proceed to verify 5.2.(2).

Let  $\varphi: M \rightarrow F$  be a flat preenvelope; it exists because  $R$  is coherent. The G-flat module  $S \otimes_R M$  embeds into a flat  $S$ -module, and  $M$  embeds into  $S \otimes_R M$  as  $S$  is faithfully flat; cf. (4.1.1). A flat  $S$ -module is also flat as an  $R$ -module, so  $M$  embeds into a flat  $R$ -module, and it follows that  $\varphi$  is injective. Consider the associated exact sequence

$$\eta = 0 \longrightarrow M \xrightarrow{\varphi} F \longrightarrow C \longrightarrow 0.$$

Let  $I$  be an injective  $R$ -module. The induced sequence  $I \otimes_R \eta$  is exact if and only if  $\mathrm{Hom}_{\mathbb{Z}}(I \otimes_R \eta, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(\eta, \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}))$  is exact. The  $R$ -module  $\mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$  is flat, as  $R$  is coherent, so exactness of  $\mathrm{Hom}_R(\eta, \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z}))$  holds because  $\varphi$  is a flat preenvelope. Notice that exactness of  $I \otimes_R \eta$  and (1) yield

$$(2) \quad \mathrm{Tor}_{>0}^R(I, C) = 0 \text{ for every injective } R\text{-module } I.$$

To construct the complex in 5.2.(2), it is now sufficient to show that  $S \otimes_R C$  is G-flat over  $S$ . Consider the following push-out diagram in the category of  $S$ -modules where  $H$  is flat and  $G$  is G-flat.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S \otimes_R M & \longrightarrow & S \otimes_R F & \longrightarrow & S \otimes_R C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & X & \longrightarrow & S \otimes_R C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & G & \xlongequal{\quad} & G & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Since  $S$  is left GF-closed, the module  $X$  is G-flat. To conclude from Lemma 5.11, applied to the diagram’s second non-zero row, that  $S \otimes_R C$  is G-flat, we need to verify that the  $S^\circ$ -module  $\mathrm{Hom}_{\mathbb{Z}}(S \otimes_R C, \mathbb{Q}/\mathbb{Z})$  is G-injective. Application of  $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$  to the second row shows that the module has finite Gorenstein injective dimension (at most 1). Let  $J$  be an injective  $S^\circ$ -module and  $i > 0$  be an integer. The isomorphism below is Hom-tensor adjointness, and vanishing follows from Lemma 6.1 in view of (2)

$$\mathrm{Ext}_{S^\circ}^i(J, \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R C, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_i^S(J, S \otimes_R C), \mathbb{Q}/\mathbb{Z}) = 0.$$

It follows that  $\mathrm{Hom}_{\mathbb{Z}}(S \otimes_R C, \mathbb{Q}/\mathbb{Z})$  is G-injective; see [14, thm. 2.22].  $\square$

It is straightforward to verify that for a flat  $R$ -module  $F$  and a Gorenstein flat  $R$ -module  $G$ , the module  $F \otimes_R G$  is Gorenstein flat; see [7, Ascent table I.(a)]. The next lemma provides a partial converse; it applies, in particular, to the setting where  $Q$  is a faithfully flat  $R$ -algebra; cf. (4.1.1).

**6.4 Proposition.** *Let  $R$  be coherent and  $M$  be an  $R$ -module. If  $Q \otimes_R M$  is Gorenstein flat for some faithfully flat  $R$ -module  $Q$  that contains a non-zero free  $R$ -module as a pure submodule, then  $M$  is Gorenstein flat.*

**Proof.** As  $Q$  is faithfully flat, vanishing of  $\mathrm{Tor}_i^R(I, Q \otimes_R M)$  implies vanishing of  $\mathrm{Tor}_i^R(I, M)$ , so one has  $\mathrm{Tor}_{>0}^R(I, M) = 0$  for every injective  $R$ -module  $I$ . Thus 5.2.(1) is satisfied and we proceed to construct the complex in 5.2.(2).

Let  $L \neq 0$  be a free pure submodule of  $Q$ ; application of  $-\otimes_R M$  to the associated pure embedding yields an embedding  $L \otimes_R M \rightarrow Q \otimes_R M$ . By assumption, the module  $Q \otimes_R M$  embeds into a flat  $R$ -module, whence  $L \otimes_R M$ , and therefore  $M$ , embeds into a flat  $R$ -module. Since  $R$  is coherent, there exists a flat preenvelope,  $\varphi: M \rightarrow F$ , and it is necessarily injective. Consider the associated exact sequence  $0 \rightarrow M \rightarrow F \rightarrow C \rightarrow 0$ . As in the proof of Lemma 6.3 one gets

$$(1) \quad \mathrm{Tor}_{>0}^R(I, C) = 0 \text{ for every injective } R\text{-module } I.$$

To construct the complex in 5.2.(2) it is now sufficient to show that  $Q \otimes_R C$  is G-flat. It is immediate from the induced exact sequence

$$0 \longrightarrow Q \otimes_R M \longrightarrow Q \otimes_R F \longrightarrow Q \otimes_R C \longrightarrow 0$$

that  $Q \otimes_R C$  has finite Gorenstein flat dimension. For every injective  $R$ -module  $I$  one gets

$$\mathrm{Tor}_{>0}^R(I, Q \otimes_R C) = 0$$

from (1), so  $Q \otimes_R C$  is G-flat by [14, thm. 3.14].  $\square$

**6.5 Proposition.** *For every flat  $R$ -module  $F$  and for every  $R$ -complex  $M$  there is an inequality*

$$\mathrm{Gfd}_R(F \otimes_R M) \leq \mathrm{Gfd}_R M.$$

**Proof.** We may assume that  $H(M)$  is non-zero, otherwise there is nothing to prove. Assume that  $\mathrm{Gfd}_R M = g$  holds for some  $g \in \mathbb{Z}$ . By Proposition 5.7 one has  $H_i(M) = 0$  for all  $i > g$ , and there exists a semi-flat replacement  $P \simeq M$  with  $C_g(P)$  G-flat. The complex  $F \otimes_R P$  is semi-flat and isomorphic to  $F \otimes_R M$  in the derived category. For  $i > g$  one has  $H_i(F \otimes_R M) \cong F \otimes_R H_i(M) = 0$ , and the module  $C_g(F \otimes_R P) \cong F \otimes_R C_g(P)$  is G-flat by [7, Ascent table I.(a)], so  $\mathrm{Gfd}_R(F \otimes_R M) \leq g$  holds by Proposition 5.7.  $\square$

**6.6 Remark.** For an  $R$ -complex  $M$  with  $H_i(M) = 0$  for  $i \ll 0$  the inequality in Proposition 6.5 is by [7, (4.4)] valid for Gorenstein flat dimension defined in [6, 1.9].

**6.7 Theorem.** *Let  $R$  be commutative coherent, let  $S$  be a faithfully flat  $R$ -algebra, and let  $M$  be an  $R$ -complex. There is an equality*

$$\mathrm{Gfd}_R M = \mathrm{Gfd}_R(S \otimes_R M),$$

and if  $S$  is left GF-closed also an equality

$$\mathrm{Gfd}_S(S \otimes_R M) = \mathrm{Gfd}_R M.$$

**Proof.** We may assume that  $H(M)$  is non-zero, otherwise there is nothing to prove.

Consider the first equality. The inequality “ $\geq$ ” is a special case of Proposition 6.5. For the opposite inequality, assume  $\mathrm{Gfd}_R(S \otimes_R M) = g$  holds for some  $g \in \mathbb{Z}$ . Choose a semi-projective resolution  $P \xrightarrow{\simeq} M$ ; the induced quasi-isomorphism  $S \otimes_R P \rightarrow S \otimes_R M$  is a semi-flat replacement. From Proposition 5.12 it follows

that  $S \otimes_R H_i(M) \cong H_i(S \otimes_R M) = 0$  holds for all  $i > g$  and that the module  $S \otimes_R C_g(P) \cong C_g(S \otimes_R P)$  is  $G$ -flat. Since  $S$  is faithfully flat, it follows that  $H_i(M) = 0$  holds for all  $i > g$ , and by Proposition 6.4 the module  $C_g(P)$  is  $G$ -flat. By Proposition 5.12 one now has  $\text{Gfd}_R M \leq g$ , and this proves the first equality.

A parallel argument applies to establish the second equality; only one has to invoke Proposition 5.14 instead of 6.5 and Lemma 6.3 instead of 6.4.  $\square$

**6.8 Remark.** The first equality in Theorem 6.7 improves the second equality in [11, prop. 1.9] by removing the requirements that  $R$  be semi-local and  $S$  be commutative noetherian. The equality in [11, thm. 1.8] are subsumed by [22, thm. 15], and the second equality in Theorem 6.7 improves [22, thm. 15] by removing the assumption that  $S$  is commutative and by relaxing the conditions on  $R$  and  $S$  from noetherian to coherent/left GF-closed.

## 7. CLOSING REMARKS

Research on Gorenstein homological dimensions continues to be guided by the idea that every result about absolute homological dimensions should have a counterpart in Gorenstein homological algebra. In a curious departure from this principle, the behavior of Gorenstein injective dimension under faithfully flat base change was understood [11, thm. 1.7] years before the absolute case [10, thm. 2.2]. The main results of the present paper have brought the (co-)basechange results for Gorenstein dimensions in closer alignment with the results for absolute homological dimensions by relaxing the conditions on rings and complexes. It would still be interesting to remove the *a priori* assumption of homological boundedness from Theorem 4.12 to align it even closer with [9, cor. 3.1].

## ACKNOWLEDGMENTS

This paper was written during visits by L.L. to Texas Tech University—while F.K. was a student there—and by L.W.C. and L.L. to Nanjing University; we gratefully acknowledge the hospitality of both institutions. In particular, we thank Nanqing Ding, our host at Nanjing, for inviting and supporting us.

## APPENDIX. GF CLOSED RINGS BY DRISS BENNIS

Recall that a ring  $R$  is said to be left GF-closed if in every exact sequence of  $R$ -modules  $0 \rightarrow G \rightarrow M \rightarrow G' \rightarrow 0$  with  $G$  and  $G'$  Gorenstein flat, also the middle module is Gorenstein flat. No example is known of a ring that is not GF-closed.

From [4, lem. 2.5] it is known that the middle module is Gorenstein flat in every exact sequence  $0 \rightarrow G \rightarrow M \rightarrow F \rightarrow 0$  with  $F$  flat and  $G$  Gorenstein flat.

**A.1 Theorem.** *For a ring  $R$  the following conditions are equivalent.*

- (i)  $R$  is left GF-closed.
- (ii) In every exact sequence of  $R$ -modules  $0 \rightarrow F \rightarrow M \rightarrow G \rightarrow 0$  with  $F$  flat and  $G$  Gorenstein flat, the module  $M$  is Gorenstein flat.

**Proof.** The implication (i)  $\implies$  (ii) is trivial. To prove that (ii) implies (i), let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of  $R$ -modules where  $A$  and  $C$  are G-flat. The goal is to prove that  $B$  is G-flat. For every injective  $R^\circ$ -module  $I$  one evidently has  $\text{Tor}_{>0}^R(I, B) = 0$ . Thus, 5.2.(1) holds and we proceed to verify 5.2.(2).

Since  $A$  is G-flat, there is an exact sequence

$$0 \longrightarrow A \longrightarrow F' \longrightarrow G' \longrightarrow 0$$

with  $F'$  flat and  $G'$  G-flat. From the push-out diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & F' & \longrightarrow & X & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & G' & = & G' & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

one gets by (ii) that  $X$  is G-flat, and so there exists an exact sequence

$$0 \longrightarrow X \longrightarrow F \longrightarrow G \longrightarrow 0$$

with  $F$  flat and  $G$  G-flat. Consider the second non-zero row in the push-out diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & G' \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & B' \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & G & = & G \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0. \end{array}$$

The module  $B'$  has the same properties as  $B$ ; indeed  $G$  and  $G'$  are G-flat, and hence one has  $\text{Tor}_{>0}^R(I, B') = 0$  for every injective  $R^\circ$ -module  $I$ . Thus, by repeating the above process one can construct the complex in 5.2.(2).  $\square$

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