THE SINGULARITY CATEGORY OF AN EXACT CATEGORY APPLIED TO CHARACTERIZE GORENSTEIN SCHEMES

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Abstract. We construct a non-affine analogue of the singularity category of a Gorenstein local ring. With this Buchweitz's classic equivalence of three triangulated categories over a Gorenstein local ring has been generalized to schemes, a project started by Murfet and Salarian more than ten years ago. Our construction originates in a framework we develop for singularity categories of exact categories. As an application of this framework in the non-commutative setting, we characterize rings of finite finitistic dimension.

Introduction

The singularity category of a commutative noetherian ring A is the Verdier quotient of the finite bounded derived category of A by the subcategory of perfect A-complexes. A motivation for this paper comes from work of Buchweitz [7]: In contemporary terminology, he proved that the singularity category of a Gorenstein local ring A is equivalent to the stable category of finitely generated Gorenstein projective A-modules and to the homotopy category of totally acyclic complexes of finitely generated projective A-modules; in symbols

$$(\diamond) \qquad \mathsf{D}_{\mathrm{sg}}(A) \;\simeq\; \mathsf{StGor}_{\mathsf{prj}}(A) \;\simeq\; \mathsf{K}_{\mathrm{tac}}(\mathsf{prj}(A))\,.$$

A perfect non-affine analogue may be beyond reach: Work of Orlov [32] shows that even under the assumption that a scheme has enough locally free sheaves—a prerequisite for talking about Gorenstein projective coherent sheaves—these sheaves do not form a Frobenius category; i.e. they do not provide an analogue of $\mathsf{StGor}_{\mathsf{prj}}(A)$.

Murfet and Salarian [30] initiated the quest for non-affine analogues of the categories in $(\diamond)$. Noticing that the structure sheaf of a scheme is flat, but not necessarily projective, their idea is to replace projective objects by flat objects. In this process, though, one must give up finite generation. Fortunately, there are "big" versions of all three categories in $(\diamond)$—obtained by dropping the assumptions of finite generation—and they are also equivalent. Indeed, the big singularity category of a Gorenstein ring A is by work of Beligiannis [4, Thm. 6.9] equivalent to

Date: 5 May 2022.

2020 *Mathematics Subject Classification.* Primary 14F08. Secondary 16E65; 18G20.

Key words and phrases. Cotorsion pair; defect category; finitistic dimension; global dimension; Gorenstein scheme; singularity category.

L.W.C. was partly supported by Simons Foundation grant 428308. N.D. was supported by NSFC grant 11771202. S.E. was partly supported by grant PID2020-113206GB-I00 funded by MCIN/AEI/10.13039/501100011033. J.H. was partly supported by NSFC grant 11771212. H.L. was partly supported by NSFC grant 11626179. Part of the work was done when J.H. visited Universidad de Murcia in February 2020; the hospitality of this institution is acknowledged with gratitude.

the stable category $\mathsf{StGor}_{\mathsf{Prj}}(A)$ of Gorenstein projective A-modules, and the latter category is for every ring A equivalent to the homotopy category $\mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(A))$ of totally acyclic complexes of projective A-modules, see for example [14, Cor. 3.9]. Thus, these are equivalences one can aim to replicate in the non-affine setting.

As an analogue of the category $\mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(A))$ for a semi-separated noetherian scheme X, Murfet and Salarian identified the category

$$\mathsf{D}_{\mathrm{F\text{-}tac}}(\mathsf{Flat}(X)) := \frac{\mathsf{K}_{\mathrm{F\text{-}tac}}(\mathsf{Flat}(X))}{\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(X))}\,,$$

i.e. the Verdier quotient of the homotopy category of **F**-totally acyclic complexes of flat quasi-coherent sheaves on X by its subcategory of pure-acyclic complexes. Indeed, for a commutative noetherian ring A of finite Krull dimension and the scheme $X = \operatorname{Spec}(A)$, the categories $\mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(A))$ and $\mathsf{D}_{\mathrm{F\text{-}tac}}(\mathsf{Flat}(X))$ are equivalent by [30, Lem. 4.22]. An analogue for schemes of the stable category was identified in [13], based on a change of focus from flat objects to flat-cotorsion objects: For a semi-separated noetherian scheme X, the category $\mathsf{D}_{\mathrm{F\text{-}tac}}(\mathsf{Flat}(X))$ is equivalent to the homotopy category $\mathsf{K}_{\mathrm{tac}}(\mathsf{FlatCot}(X))$ of totally acyclic complexes of flat-cotorsion sheaves on X, and that category is equivalent to the stable category $\mathsf{StGor}_{\mathsf{FlatCot}}(X)$ of Gorenstein flat-cotorsion sheaves on X. The primary goal of this paper is to identify an analogue of the singularity category in the non-affine setting and to show that it is equivalent to the categories $\mathsf{K}_{\mathrm{tac}}(\mathsf{FlatCot}(X))$ and $\mathsf{StGor}_{\mathsf{FlatCot}}(X)$ for a Gorenstein scheme X of finite Krull dimension.

We achieve this goal as an application of a more general theory that we develop for singularity categories associated to a cotorsion pair. Before giving an outline of this theory, we remark that in the case of flat-cotorsion sheaves on X, the singularity category takes the form

$$\frac{\mathsf{D}_{\mathrm{b}}(\mathsf{Cot}(X))}{\langle\text{bounded complexes of flat-cotorsion sheaves}\rangle}\,.$$

We notice in Example 1.10 that this singularity category detects when a noetherian semi-separated scheme of finite Krull dimension is regular. Further, for a Gorenstein scheme of finite Krull dimension the category is compactly generated and closely related to Orlov's singularity category, see Remark 4.3.

Here is the outline: Let A be an abelian category. To an additive full subcategory E of A that is closed under extensions and direct summands, we associate in Section 1 two singularity categories relative to the projective and injective objects in E. If E is part of a complete hereditary cotorsion pair, then there are natural functors from categories of Gorenstein objects into these singularity categories. This allows us in Section 2—inspired by works of Bergh, Jorgensen, and Oppermann [6] and Iyengar and Krause [20]—to define associated defect categories. Given a complete hereditary cotorsion pair (U,V) in A, we build on the theory from [14] to develop Gorenstein dimensions for objects in U and V. Assuming that A is Grothendieck—as is the case for the categories of modules over a ring and quasi-coherent sheaves on a semi-separated noetherian scheme—these dimensions extend through work of Gillespie to invariants on the derived category of A; that's the topic of Section 3. In Section 4 we arrive at equivalences of categories akin to those in ($\diamond$)—but now for a Gorenstein scheme. In Section 5 we characterize rings of finite finitistic dimension in terms of the singularity and defect categories developed in the first sections.

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Throughout the paper, A denotes an abelian category; hom-sets of objects in A are denoted $\hom_{\mathsf{A}}$, and that notation is also used for the induced functor from $\mathsf{A}^{\mathrm{op}} \times \mathsf{A}$ to abelian groups. By a *subcategory* of A we mean a full subcategory that is closed under isomorphisms. Let S be a subcategory of A. The category of chain complexes of objects from S, or S-complexes, is denoted $\mathsf{C}(\mathsf{S})$ and $\mathsf{K}(\mathsf{S})$ is the associated homotopy category; the latter is triangulated if S is additive. A complex X in $\mathsf{C}(\mathsf{S})$ is said to be *bounded below* if $X_n = 0$ holds for $n \ll 0$, *bounded above* if $X_n = 0$ holds for $n \gg 0$, and *bounded* if $X_n = 0$ holds for $|n| \gg 0$. The full subcategories of bounded below, bounded above, and bounded complexes are denoted $\mathsf{C}_+(\mathsf{S})$, $\mathsf{C}_-(\mathsf{S})$, and $\mathsf{C}_{\mathrm{b}}(\mathsf{S})$, respectively. The essential images of these subcategories in the homotopy category $\mathsf{K}(\mathsf{S})$ are triangulated subcategories denoted $\mathsf{K}_+(\mathsf{S})$, $\mathsf{K}_-(\mathsf{S})$, and $\mathsf{K}_{\mathrm{b}}(\mathsf{S})$, respectively.

Given a complex $X \in \mathsf{C}(\mathsf{A})$ with differential ∂^X we write $\mathrm{B}_n(X)$ and $\mathrm{Z}_n(X)$ for the subobjects of boundaries and cycles in X_n. The cokernel of ∂^X_{n+1} is the quotient object $\mathrm{C}_n(X) := X_n/\mathrm{B}_n(X)$, and the homology of X in degree n is the subquotient object $\mathrm{H}_n(X) := \mathrm{Z}_n(X)/\mathrm{B}_n(X)$. Given a second A-complex Y, we write $\mathrm{Hom}_{\mathsf{A}}(X,Y)$ for the total hom-complex in $\mathsf{C}(\mathsf{A})$.

1. Singularity categories of exact categories

Let $\mathsf{E} \subseteq \mathsf{A}$ be an additive subcategory closed under extensions and direct summands; it is an idempotent complete exact category with the exact structure induced from A, see Bühler [8, Rmk. 6.2 and Lem. 10.20]. The setup developed in this first section could be done in the generality of idempotent complete exact categories. We don't need that for the applications we pursue in this paper, but we structure the arguments in such a way as to make the generalizations straightforward for the initiated reader.

Given a subcategory $\mathsf{S} \subseteq \mathsf{E}$, a complex X in $\mathsf{K}(\mathsf{S})$ is called E-*acyclic* if

$$0 \longrightarrow \mathrm{Z}_n(X) \longrightarrow X_n \longrightarrow \mathrm{Z}_{n-1}(X) \longrightarrow 0 \tag{1.0.1}$$

is an exact sequence in E for every $n \in \mathbb{Z}$. The full subcategory of such complexes is denoted $\mathsf{K}_{\mathsf{E}\text{-ac}}(\mathsf{S})$; it is isomorphism closed, see Keller [23, 4.1], and is indeed triangulated, see Neeman [31, Lem. 1.1]. The symbol $\mathsf{K}_{\mathsf{E}\text{-ac}}(\mathsf{S})$ is used with subscripts $+$, $-$, and b to indicate boundedness: $\mathsf{K}_{\mathsf{E}\text{-ac,b}}(\mathsf{S}) := \mathsf{K}_{\mathsf{E}\text{-ac}}(\mathsf{S}) \cap \mathsf{K}_{\mathrm{b}}(\mathsf{S})$ etc. An A-acyclic complex is simply referred to as *acyclic*, and the corresponding category is denoted $\mathsf{K}_{\mathrm{ac}}(\mathsf{S})$. A morphism in $\mathsf{K}(\mathsf{S})$ is called an E-*quasi-isomorphism* if its mapping cone is E-acyclic and simply a *quasi-isomorphism* if its mapping cone is acyclic.

The *derived category* of E is defined to be the Verdier quotient

$$\mathsf{D}(\mathsf{E}) := \mathsf{K}(\mathsf{E})/\mathsf{K}_{\mathsf{E}\text{-ac}}(\mathsf{E})\,;$$

see [31] and Keller [24]. In the case where $\mathsf{E} = \mathsf{A}$ is the module category of a ring, this construction yields the usual derived category. The bounded versions $\mathsf{D}_+(\mathsf{E})$, $\mathsf{D}_-(\mathsf{E})$, and $\mathsf{D}_{\mathrm{b}}(\mathsf{E})$ are defined analogously: $\mathsf{D}_+(\mathsf{E}) := \mathsf{K}_+(\mathsf{E})/\mathsf{K}_{\mathsf{E}\text{-ac},+}(\mathsf{E})$ etc.

A complex X in $\mathsf{K}(\mathsf{S})$ is called *eventually* E-*acyclic* if the sequences (1.0.1) are exact in E for all $|n| \gg 0$. The full subcategory of such complexes is denoted $\mathsf{K}_{(\mathsf{E}\text{-ac})}(\mathsf{S})$; this notation is also used with subscripts to indicate boundedness.

Recall that for an A-complex X and an integer n, the *hard truncation below of X at n* is the complex $X_{\geqslant n} := \cdots \to X_{n+1} \to X_n \to 0$, and the *soft truncation below of X at n* is the complex $X_{\supseteq n} := \cdots \to X_{n+1} \to \mathrm{Z}_n(X) \to 0$. Hard and soft truncations above are defined similarly: $X_{\leqslant n} := 0 \to X_n \to X_{n-1} \to \cdots$ and $X_{\subseteq n} := 0 \to \mathrm{C}_n(X) \to X_{n-1} \to \cdots$.

1.1 Proposition. *Let E be an additive subcategory of A that is closed under extensions and direct summands; let S be an additive subcategory of E.*

(a) *The category $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{S})$ is a triangulated subcategory of $\mathsf{K}_{+}(\mathsf{S})$.*
(b) *The category $\mathsf{K}_{(\mathsf{E}\text{-ac}),-}(\mathsf{S})$ is a triangulated subcategory of $\mathsf{K}_{-}(\mathsf{S})$.*

Proof. We prove (a); the proof of (b) is similar. That $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{S})$ is additive and closed under shifts is evident; it remains to show that it is closed under isomorphisms and cones.

Let $\alpha\colon X \to Y$ be an isomorphism in $\mathsf{K}_{+}(\mathsf{S})$ with $X \in \mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{S})$. The complex $\mathrm{Cone}(\alpha)$ is contractible and hence E-acyclic as E is closed under direct summands. For every $n \in \mathbb{Z}$ the short exact sequence

$$0 \longrightarrow Y_n \longrightarrow \mathrm{Cone}(\alpha)_n \longrightarrow (\Sigma X)_n \longrightarrow 0 \tag{1}$$

yields a commutative diagram

$$\begin{array}{ccccc} \mathrm{Cone}(\alpha)_n & \longrightarrow & X_{n-1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Z}_{n-1}(\mathrm{Cone}(\alpha)) & \longrightarrow & \mathrm{Z}_{n-2}(X)\,. & & \end{array} \tag{2}$$

For all $n \gg 0$ the vertical morphisms in (2) are epimorphisms, hence so is the lower horizontal morphism. Thus the vertical morphisms in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Z}_n(\mathrm{Cone}(\alpha)) & \longrightarrow & \mathrm{Cone}(\alpha)_n & \longrightarrow & \mathrm{Z}_{n-1}(\mathrm{Cone}(\alpha)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Z}_{n-1}(X) & \longrightarrow & X_{n-1} & \longrightarrow & \mathrm{Z}_{n-2}(X) & \longrightarrow & 0 \end{array}$$

are epimorphisms. As the cycle functor is left exact, (1) and the Snake Lemma yield the exact sequence

$$0 \longrightarrow \mathrm{Z}_n(Y) \longrightarrow Y_n \longrightarrow \mathrm{Z}_{n-1}(Y) \longrightarrow 0\,.$$

In (2) the upper horizontal and right-hand vertical morphisms are epimorphisms with kernels in E, namely Y_n and $\mathrm{Z}_{n-1}(X)$. As E is an exact category closed under direct summands, [8, Prop. 7.6] shows that the lower horizontal morphism in (2) is an epimorphism with kernel in E. By left exactness of the cycle functor, it now follows from (1) that $\mathrm{Z}_{n-1}(Y)$ is in E; thus Y belongs to $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{S})$.

Finally, if $\alpha\colon X \to Y$ is a morphism in $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{S})$, then for $n \gg 0$ the complexes $X_{\supseteq n}$ and $Y_{\supseteq n}$ are E-acyclic and α induces a map α' between them. In high degrees, the cones of α and α' agree, and the cone of α' is E-acyclic; see [8, Lem. 10.3]. $\square$

We denote by $\mathsf{Prj}(\mathsf{E})$ and $\mathsf{Inj}(\mathsf{E})$ the subcategories of projective and injective objects in E.

1.2 Proposition. *Let E be an additive subcategory of A that is closed under extensions and direct summands.*

(a) *If* E *has enough projectives, then there are triangulated equivalences of triangulated categories*

$$\mathsf{K}_+(\mathsf{Prj}(\mathsf{E})) \simeq \mathsf{D}_+(\mathsf{E}) \qquad \textit{and} \qquad \mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E})) \simeq \mathsf{D}_{\mathrm{b}}(\mathsf{E})\,.$$

(b) *If* E *has enough injectives, then there are triangulated equivalences of triangulated categories*

$$\mathsf{K}_-(\mathsf{Inj}(\mathsf{E})) \simeq \mathsf{D}_-(\mathsf{E}) \qquad \textit{and} \qquad \mathsf{K}_{(\mathsf{E}\text{-ac}),-}(\mathsf{Inj}(\mathsf{E})) \simeq \mathsf{D}_{\mathrm{b}}(\mathsf{E})\,.$$

Proof. We prove part (a); the proof of part (b) is dual. For every complex X in $\mathsf{K}_+(\mathsf{E})$ there is a distinguished triangle in $\mathsf{K}_+(\mathsf{E})$,

$$p(X) \longrightarrow X \longrightarrow a(X) \longrightarrow \Sigma p(X)\,,$$

with $p(X)$ in $\mathsf{K}_+(\mathsf{Prj}(\mathsf{E}))$ and $a(X)$ an E-acyclic complex; moreover the inclusion of E-acyclic complexes into $\mathsf{K}_+(\mathsf{E})$ has a left adjoint. For a proof (of the dual result) see [23, Lem. 4.1]. It is now standard, see Krause [26, Prop. 4.9.1 and the proof of Prop. 4.13.1], that the functor $G\colon \mathsf{K}_+(\mathsf{Prj}(\mathsf{E})) \to \mathsf{K}_+(\mathsf{E}) \to \mathsf{D}_+(\mathsf{E})$ yields the equivalence $\mathsf{K}_+(\mathsf{Prj}(\mathsf{E})) \simeq \mathsf{D}_+(\mathsf{E})$; see also [24, Exa. 12.2].

By Proposition 1.1 the triangulated subcategory $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))$ is equivalent to its essential image under G in $\mathsf{D}_+(\mathsf{E})$. For X in $\mathsf{K}_{\mathrm{b}}(\mathsf{E})$ the complex $p(X)$ belongs to $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))$, and a complex in $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))$ is E-quasi-isomorphic in $\mathsf{K}_+(\mathsf{E})$ to a complex in $\mathsf{K}_{\mathrm{b}}(\mathsf{E})$. As $\mathsf{D}_{\mathrm{b}}(\mathsf{E})$ is equivalent to the subcategory of $\mathsf{D}(\mathsf{E})$ generated by $\mathsf{K}_{\mathrm{b}}(\mathsf{E})$, see [24, Lem. 11.7], the restriction of G to $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))$ induces the second equivalence. □

As $\mathsf{K}_{\mathrm{b}}(\mathsf{Prj}(\mathsf{E}))$ evidently is a thick subcategory of $\mathsf{K}_+(\mathsf{Prj}(\mathsf{E}))$, it follows that for a category E as in Proposition 1.2 with enough projectives, the subcategory $\mathsf{K}_{\mathrm{b}}(\mathsf{Prj}(\mathsf{E}))$ is equivalent to a thick subcategory of $\mathsf{D}_{\mathrm{b}}(\mathsf{E})$. Similarly, if E has enough injectives, then $\mathsf{K}_{\mathrm{b}}(\mathsf{Inj}(\mathsf{E}))$ is equivalent to a thick subcategory of $\mathsf{D}_{\mathrm{b}}(\mathsf{E})$.

1.3 Definition. Let E be an additive subcategory of A that is closed under extensions and direct summands.

(a) Assume that E has enough projectives. The Verdier quotient

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{E}) := \mathsf{D}_{\mathrm{b}}(\mathsf{E})/\mathsf{K}_{\mathrm{b}}(\mathsf{Prj}(\mathsf{E}))$$

is called the *projective singularity category of* E or $\mathsf{Prj}(\mathsf{E})$-*singularity category.*

(b) Assume that E has enough injectives. The Verdier quotient

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{E}) := \mathsf{D}_{\mathrm{b}}(\mathsf{E})/\mathsf{K}_{\mathrm{b}}(\mathsf{Inj}(\mathsf{E}))$$

is called the *injective singularity category of* E or $\mathsf{Inj}(\mathsf{E})$-*singularity category.*

With an eye towards module categories, the next result justifies the term "singularity category." Resolutions of objects in exact categories are defined in the usual way, see [8, Sec. 12].

1.4 Theorem. *Let* E *be an additive subcategory of* A *that is closed under extensions and direct summands.*

(a) *Assume that* E *has enough projectives. The category* $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{E})$ *vanishes if and only if every object in* E *has a bounded resolution by objects from* $\mathsf{Prj}(\mathsf{E})$.

(b) *Assume that* E *has enough injectives. The category* $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{E})$ *vanishes if and only if every object in* E *has a bounded coresolution by objects from* $\mathsf{Inj}(\mathsf{E})$.

Proof. We prove (a); the proof of (b) is dual. The category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{E})$ is per Proposition 1.2 equivalent to $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))/\mathsf{K}_{\mathrm{b}}(\mathsf{Prj}(\mathsf{E}))$. The "only if" statement is clear. For the "if" statement let P be a complex in $\mathsf{K}_{(\mathsf{E}\text{-ac}),+}(\mathsf{Prj}(\mathsf{E}))$. For $n \gg 0$ the objects $\mathrm{Z}_n(P)$ belong to E, and the sequences $0 \to \mathrm{Z}_{n+1}(P) \to P_{n+1} \to \mathrm{Z}_n(P) \to 0$ are exact. From the Comparison Theorem for projective resolutions [8, Thm. 12.4] it follows that P is isomorphic to a complex in $\mathsf{K}_{\mathrm{b}}(\mathsf{Prj}(\mathsf{E}))$. □

Given a ring A we write $\mathsf{Mod}(A)$ for the abelian category of A-modules.

1.5 Example. Let A be a ring. The category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Mod}(A))$ vanishes if and only if every A-module has finite projective dimension; that is, if and only if A has finite global dimension. In particular, for a commutative noetherian local ring A the category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Mod}(A))$ provides a measure of how singular—i.e. how far from being regular—A is. It thus serves the same purpose as the classic singularity category $\mathsf{D}_{\mathrm{sg}}(A)$ recalled in the introduction.

1.6 Remark. A variety of relative singularity categories are considered in the literature. They are in certain aspects more general and/or more restrictive than our notions. Closest to ours is, perhaps, the category considered by Chen [11]: In the case $\mathsf{E} = \mathsf{A}$ his category $D_{\mathsf{Prj}(\mathsf{A})}(\mathsf{A})$ agrees with $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{A})$.

Let (U,V) be a cotorsion pair in A; the next result shows that the setup developed above applies to U and V, and in the balance of the paper, we work in that setting. A cotorsion pair (U,V) is called *complete* if for each object $M \in \mathsf{A}$ there are exact sequences $0 \to V \to U \to M \to 0$ and $0 \to M \to V' \to U' \to 0$ with $U, U' \in \mathsf{U}$ and $V, V' \in \mathsf{V}$. Such sequences are known as special U-precovers and special V-envelopes, respectively. A cotorsion pair is called *hereditary* if $\mathrm{Ext}^i_{\mathsf{A}}(U,V) = 0$ holds for all objects $U \in \mathsf{U}$ and $V \in \mathsf{V}$ and all $i > 0$.

1.7 Proposition. *Let* (U,V) *be a complete cotorsion pair in* A.

(a) *The category* V *has enough projectives, and one has* $\mathsf{Prj}(\mathsf{V}) = \mathsf{U} \cap \mathsf{V}$.
(b) *The category* U *has enough injectives, and one has* $\mathsf{Inj}(\mathsf{U}) = \mathsf{U} \cap \mathsf{V}$.

Proof. We prove (a); the proof of (b) is dual. Let $M \in \mathsf{V}$; since (U,V) is a complete cotorsion pair in A, there is an exact sequence

$$0 \longrightarrow V \longrightarrow U \longrightarrow M \longrightarrow 0$$

with $V \in \mathsf{V}$ and $U \in \mathsf{U}$. As V is closed under extensions, the object U is in $\mathsf{U} \cap \mathsf{V}$. It is evident that objects in $\mathsf{U} \cap \mathsf{V}$ are projective in V, and given any projective object M in V the exact sequence above shows that it is a quotient object, and hence a direct summand, of an object in $\mathsf{U} \cap \mathsf{V}$. Thus one has $\mathsf{Prj}(\mathsf{V}) = \mathsf{U} \cap \mathsf{V}$. □

1.8 Remark. Let (U,V) be a complete cotorsion pair. It follows in view of Theorem 1.4 that the projective singularity category of V vanishes if and only if every object in A has a bounded resolution by objects from U. Indeed, every object in A has a resolution by objects from U in which the first syzygy belongs to V, and $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V})$ vanishes if and only if this object has a bounded resolution by objects from $\mathsf{U} \cap \mathsf{V}$. Similarly, vanishing of the category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})$ means that every object in A has a bounded coresolution by objects from V.

As U contains $\mathsf{Prj}(\mathsf{A})$ the singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{U})$ is defined if A has enough projectives, but we are not aware of any interpretation of it in this generality.

For the absolute cotorsion pair $(\mathsf{Prj}(\mathsf{A}), \mathsf{A})$ in a category with enough projectives, the singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Prj}(\mathsf{A}))$, of course, vanishes. The category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{V})$ is defined if A has enough injectives and in general it appears equally intractable.

1.9 Example. Let A be a ring and consider the cotorsion pair $(\mathsf{Flat}(A), \mathsf{Cot}(A))$, which is complete.

The singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(A))$ vanishes if and only if every A-module has finite flat dimension; that is, if and only if A has finite weak global dimension.

The singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{Flat}(A))$ vanishes if and only if every A-module has finite cotorsion dimension; by a result of Mao and Ding [28, Thm. 19.2.5] this is equivalent to A being n-perfect—every flat module has projective dimension at most n—for some n.

If A has finite global dimension, then both $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(A))$ and $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{Flat}(A))$ vanish, and the converse holds by [28, Thm. 19.2.14].

1.10 Example. Let X be a semi-separated noetherian scheme of finite Krull dimension. The flat sheaves and cotorsion sheaves on X form a complete cotorsion pair $(\mathsf{Flat}(X), \mathsf{Cot}(X))$ in the category of quasi-coherent sheaves on X; see [13, Rmk. 2.4]. Now the singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))$ vanishes if and only if every sheaf has finite flat dimension. This is equivalent to $\mathcal{O}_{X,x}$ being a regular local ring for every $x \in X$; that is $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))$ vanishes if and only if X is a regular scheme.

2. Gorenstein defect categories

In this section, (U, V) is a complete cotorsion pair in A. The additive categories U and V are closed under extensions and direct summands, and by Proposition 1.7 the category V has enough projectives, and U has enough injectives. Thus the singularity categories $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V})$ and $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})$ are defined in this context. In the case of a complete hereditary cotorsion pair we give a more concrete interpretation of these singularity categories in terms of the notions of U- and V-Gorenstein objects.

We recall from [14] that an acyclic complex T of objects from U is called *right* U*-totally acyclic* if and only if the cycle objects $\mathrm{Z}_n(T)$ belong to V and the complex $\hom_{\mathsf{A}}(T, W)$ is acyclic for every $W \in \mathsf{U} \cap \mathsf{V}$. As V is extension closed, the objects in a U-totally acyclic complex in fact belong to $\mathsf{U} \cap \mathsf{V}$; the symbol $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U} \cap \mathsf{V})$ denotes the homotopy category of such complexes. An object is called *right* U*-Gorenstein* if it equals $\mathrm{Z}_0(T)$ for some right U-totally acyclic complex T. The subcategory of right U-Gorenstein objects in A is denoted $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$; it is by [14, Thm. 2.11] a Frobenius category, and the associated stable category is denoted $\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A})$. Dually one defines *left* V*-totally acyclic* complexes, *left* V*-Gorenstein* objects, and associated categories $\mathsf{K}^{\mathrm{L}}_{\mathsf{V}\text{-tac}}(\mathsf{U} \cap \mathsf{V})$, $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$, and $\mathsf{StLGor}_{\mathsf{V}}(\mathsf{A})$.

It is straightforward to verify that the functors described in the next theorem are those induced by the embeddings of V and U into $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ and $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$.

2.1 Theorem. *Let* (U, V) *be a complete cotorsion pair in* A*. There are fully faithful triangulated functors*

$$\mathrm{F}_{\mathrm{Prj}} \colon \mathsf{StRGor}_{\mathsf{U}}(\mathsf{A}) \longrightarrow \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V}) \qquad \text{and} \qquad \mathrm{F}_{\mathrm{Inj}} \colon \mathsf{StLGor}_{\mathsf{V}}(\mathsf{A}) \longrightarrow \mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})\,,$$

where $\mathrm{F}_{\mathrm{Prj}}$ *sends a right* U*-Gorenstein object to the hard truncation below at* 0 *of a defining* U*-totally acyclic complex, and* $\mathrm{F}_{\mathrm{Inj}}$ *is defined similarly.*

Proof. We prove the assertion regarding $\mathrm{F}_{\mathrm{Prj}}$; the one for $\mathrm{F}_{\mathrm{Inj}}$ is proved similarly. The functor $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A}) \to \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{V})$ that maps a right U-Gorenstein object to a defining right U-totally acyclic complex induces a triangulated equivalence $\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A}) \to \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{V})$; this was shown in [14, Thm. 3.8]. The arguments in the proof of [6, Thm. 3.1] show that hard truncation below at 0 yields a fully faithful triangulated functor $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{V}) \to \mathsf{K}_{(\mathsf{U}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})/\mathsf{K}_{\mathrm{b}}(\mathsf{U}\cap\mathsf{V})$. Propositions 1.2 and 1.7 now yield the desired fully faithful triangulated functor $\mathrm{F}_{\mathrm{Prj}}$. $\square$

Theorem 2.1 facilitates the following definition; see also [6].

2.2 Definition. Let (U,V) be a complete cotorsion pair in A. The Verdier quotient of the projective singularity category by the essential image of $\mathrm{F}_{\mathrm{Prj}}$,

$$\mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{V}) := \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V})/\operatorname{im}(\mathrm{F}_{\mathrm{Prj}})\;,$$

is called the *projective Gorenstein defect category of* V or $\mathsf{Prj}(\mathsf{V})$-*Gorenstein defect category*. Similarly, the *injective Gorenstein defect category of* U or $\mathsf{Inj}(\mathsf{U})$-*Gorenstein defect category* is the Verdier quotient

$$\mathsf{D}^{\mathrm{def}}_{\mathrm{Inj}}(\mathsf{U}) := \mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})/\operatorname{im}(\mathrm{F}_{\mathrm{Inj}})\;.$$

It was shown in [14, Lem. 2.10] that the categories $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ are closed under extensions. In the context of a complete hereditary cotorsion pair, we now show that they are closed under direct summands; our proof relies of work of Chen, Liu, and Yang [10, 35]. In the next two proofs we use the notation $\mathcal{G}(\mathsf{U})$ from [35, Def. 3.1] to denote cycle subobjects of acyclic complexes of objects from U that are $\hom_{\mathsf{A}}(-,\mathsf{U}\cap\mathsf{V})$-exact.

2.3 Lemma. *Let* (U,V) *be a complete hereditary cotorsion pair in* A. *The categories* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ *and* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ *are closed under direct summands.*

Proof. It follows from [14, Def. 1.1] and [35, Lem. 3.2] that the class $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is the intersection of V and the class $\mathcal{G}(\mathsf{U})$ defined in [35, Def. 3.1]. (We notice that [35, 3.1–3.3] do not depend on the blanket assumption that the underlying abelian category is bicomplete.) It now follows from [10, Prop. 3.3] that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is closed under direct summands. That $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ is closed under direct summands is proved similarly. $\square$

2.4 Lemma. *Let* (U,V) *be a complete hereditary cotorsion pair in* A.

(a) *Let* $0 \to V' \to V \to V'' \to 0$ *be an exact sequence in* V.
  (1) *If* V'' *is in* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$, *then* V' *is in* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ *if and only if* V *is in* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$.
  (2) *If* V' *and* V *are in* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$, *then* $V'' \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ *if and only if* $\operatorname{Ext}^1_{\mathsf{A}}(V'',W) = 0$ *holds for all* $W \in \mathsf{U}\cap\mathsf{V}$.

(b) *Let* $0 \to U' \to U \to U'' \to 0$ *be an exact sequence in* U.
  (1) *If* U' *is in* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$, *then* U'' *is in* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ *if and only if* U *is in* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$.
  (2) *If* U *and* U'' *are in* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$, *then* U' *is in* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ *if and only if* $\operatorname{Ext}^1_{\mathsf{A}}(W,U') = 0$ *holds for all* $W \in \mathsf{U}\cap\mathsf{V}$.

Proof. As noticed in the proof of Lemma 2.3, the class $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is the intersection of V and the class $\mathcal{G}(\mathsf{U})$ defined in [35, Def. 3.1]. The assertions in part (a) now follow from [35, Prop. 3.3]; the proof of part (b) is similar. □

The next definition foreshadows notions of Gorenstein dimensions relative to a cotorsion pair.

2.5 Definition. Let (U,V) be a complete hereditary cotorsion pair in A.

(a) A complex V in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ is called $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-*perfect* if V is isomorphic in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ to a complex X with $X_n \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for all $n \in \mathbb{Z}$ and $X_n = 0$ for $|n| \gg 0$.

(b) A complex U in $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ is called $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-*perfect* if U is isomorphic in $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ to a complex Y with $Y_n \in \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ for all $n \in \mathbb{Z}$ and $Y_n = 0$ for $|n| \gg 0$.

2.6 Theorem. *Let (U,V) be a complete hereditary cotorsion pair in A.*

(a) *For a complex $V \in \mathsf{D}_{\mathrm{b}}(\mathsf{V})$ the following conditions are equivalent:*
  - (i) *V is $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-perfect.*
  - (ii) *For every complex $X \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})$ isomorphic to V in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ one has $\mathrm{Z}_n(X) \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for $n \gg 0$.*
  - (iii) *There exists a complex $X \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})$ isomorphic to V in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ with $\mathrm{Z}_n(X) \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for $n \gg 0$.*
  - (iv) *For every complex $X \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{RGor}_{\mathsf{U}}(\mathsf{A}))$ isomorphic to V in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ one has $\mathrm{Z}_n(X) \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for $n \gg 0$.*

(b) *For a complex $U \in \mathsf{D}_{\mathrm{b}}(\mathsf{U})$ the following conditions are equivalent:*
  - (i) *U is $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-perfect.*
  - (ii) *For every complex $Y \in \mathsf{K}_{(\mathsf{U}\text{-ac}),-}(\mathsf{U}\cap\mathsf{V})$ isomorphic to U in $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ one has $\mathrm{Z}_n(Y) \in \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ for $n \ll 0$.*
  - (iii) *There exists a complex $Y \in \mathsf{K}_{(\mathsf{U}\text{-ac}),-}(\mathsf{U}\cap\mathsf{V})$ isomorphic to U in $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ with $\mathrm{Z}_n(Y) \in \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ for $n \ll 0$.*
  - (iv) *For every complex $Y \in \mathsf{K}_{(\mathsf{U}\text{-ac}),-}(\mathsf{LGor}_{\mathsf{V}}(\mathsf{A}))$ isomorphic to U in $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ one has $\mathrm{Z}_n(Y) \in \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ for $n \ll 0$.*

Proof. We prove (a); the proof of (b) is dual.

(i)$\Longrightarrow$(ii): Let X be a complex in $\mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})$ with $V \cong X$ in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$. As X is isomorphic in $\mathsf{K}(\mathsf{U}\cap\mathsf{V})$ to a bounded below complex, and as $\mathsf{U}\cap\mathsf{V}$ is closed under direct summands, the complex X is isomorphic in $\mathsf{K}(\mathsf{U}\cap\mathsf{V})$ to a soft truncation below, so we may assume that $X_n = 0$ holds for $n \ll 0$. By assumption, there is a complex X' isomorphic to V in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ with $X'_n \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for all $n \in \mathbb{Z}$ and $X'_n = 0$ for $|n| \gg 0$. As $X \cong X'$ in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$, there exists a V-quasi-isomorphism $\alpha\colon X \to X'$; indeed, this follows from the dual statement to Bühler's [9, Lem. 3.3.3], cf. Proposition 1.7. In particular, $\mathrm{Cone}(\alpha)$ is a V-acyclic complex. The category $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is additive, so $\mathrm{Cone}(\alpha)$ is a complex of right U-Gorenstein objects and $\mathrm{Cone}(\alpha)_n = 0$ holds for $n \ll 0$. It now follows from Lemma 2.4(a,1) that $\mathrm{Z}_n(\mathrm{Cone}(\alpha))$ is right U-Gorenstein for all $n \in \mathbb{Z}$. For $n \gg 0$ one has $\mathrm{Z}_n(\mathrm{Cone}(\alpha)) \cong \mathrm{Z}_n(X)$ and thus $\mathrm{Z}_n(X)$ belongs to $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$.

(ii)$\Longrightarrow$(iii): Trivial in view of Proposition 1.2.

(iii)$\Longrightarrow$(iv): Let X be a complex in $\mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{RGor}_{\mathsf{U}}(\mathsf{A}))$ with $V \cong X$ in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$. By assumption, there exists a complex X' in $\mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})$ such that $X \cong$

X' in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ and $\mathrm{Z}_n(X') \in \mathsf{RGor}_\mathsf{U}(\mathsf{A})$ for $n \gg 0$. By Lemma 2.3 the category $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$ is closed under direct summands, and so is $\mathsf{U} \cap \mathsf{V}$. It follows that X and X' are isomorphic to soft truncations below, so without loss of generality we assume that $X_n = 0 = X'_n$ holds for $n \ll 0$. As above there exists a V-quasi-isomorphism $\alpha\colon X' \to X$. Thus $\mathrm{Cone}(\alpha)$ is a bounded below V-acyclic complex of right U-Gorenstein objects, so $\mathrm{Z}_n(\mathrm{Cone}(\alpha))$ is right U-Gorenstein for every $n \in \mathbb{Z}$ by Lemma 2.4(a,1). For $n \gg 0$ the exact sequence $0 \to X \to \mathrm{Cone}(\alpha) \to \Sigma X' \to 0$ now yields an exact sequence $0 \to \mathrm{Z}_n(X) \to \mathrm{Z}_n(\mathrm{Cone}(\alpha)) \to \mathrm{Z}_{n-1}(X') \to 0$, and another application of Lemma 2.4(a,1) yields $\mathrm{Z}_n(X) \in \mathsf{RGor}_\mathsf{U}(\mathsf{A})$.

$(iv) \Longrightarrow (i)$: In view of Proposition 1.2 there exists a complex X in $\mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U}\cap\mathsf{V})$ with $V \cong X$ in $\mathsf{D}_\mathrm{b}(\mathsf{V})$, and as above we can without loss of generality assume that $X_n = 0$ holds for $n \ll 0$. By (iv) the cycle object $\mathrm{Z}_n(X)$ is right U-Gorenstein for some $n \gg 0$, so the complex $0 \to \mathrm{Z}_n(X) \to X_n \to \cdots$ is a bounded complex that is isomorphic to V in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ and whose objects belong to $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$. $\square$

2.7 Definition. Let (U, V) be a complete hereditary cotorsion pair in A. We consider the following subcategories of $\mathsf{D}_\mathrm{b}(\mathsf{V})$ and $\mathsf{D}_\mathrm{b}(\mathsf{U})$:

$$\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V}) := \{V \in \mathsf{D}_\mathrm{b}(\mathsf{V}) \mid V \text{ is } \mathsf{RGor}_\mathsf{U}(\mathsf{A})\text{-perfect}\} \quad \text{and}$$
$$\mathsf{Perf}_{\mathsf{LGor}_\mathsf{V}(\mathsf{A})}(\mathsf{U}) := \{U \in \mathsf{D}_\mathrm{b}(\mathsf{U}) \mid U \text{ is } \mathsf{LGor}_\mathsf{V}(\mathsf{A})\text{-perfect}\}\,.$$

2.8 Proposition. *Let (U, V) be a complete hereditary cotorsion pair in A.*

(a) *The category $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ is thick in $\mathsf{D}_\mathrm{b}(\mathsf{V})$, and it is the smallest triangulated subcategory of $\mathsf{D}_\mathrm{b}(\mathsf{V})$ that contains the objects from $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$.*

(b) *The category $\mathsf{Perf}_{\mathsf{LGor}_\mathsf{V}(\mathsf{A})}(\mathsf{U})$ is thick in $\mathsf{D}_\mathrm{b}(\mathsf{U})$, and it is the smallest triangulated subcategory of $\mathsf{D}_\mathrm{b}(\mathsf{U})$ that contains the objects from $\mathsf{LGor}_\mathsf{V}(\mathsf{A})$.*

Proof. We prove part (a); the proof of (b) is similar. Evidently $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ is additive and closed under shifts and isomorphisms.

Let $V' \to V \to V'' \to \Sigma V'$ be a triangle in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ with $V', V \in \mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$. By Proposition 1.2, there is a triangle $X' \xrightarrow{\alpha} X \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma X'$ in $\mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U} \cap \mathsf{V})$ and an isomorphism in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ of triangles:

$$\begin{array}{ccccccc} V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & \Sigma V' \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ X' & \longrightarrow & X & \longrightarrow & \mathrm{Cone}(\alpha) & \longrightarrow & \Sigma X'\,. \end{array}$$

Since the category $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ is closed under isomorphisms and V' and V belong to $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$, so do X and X'. Consequently, there is an exact sequence

$$0 \longrightarrow X \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma X' \longrightarrow 0$$

of complexes in $\mathsf{C}(\mathsf{V})$. The sequence $0 \to \mathrm{Z}_n(X) \to \mathrm{Z}_n(\mathrm{Cone}(\alpha)) \to \mathrm{Z}_n(\Sigma X') \to 0$ is exact for all $n \gg 0$, and $\mathrm{Z}_n(X)$ and $\mathrm{Z}_n(\Sigma X')$ belong to $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$ by Theorem 2.6. It now follows from [14, Lem. 2.10] that $\mathrm{Z}_n(\mathrm{Cone}(\alpha))$ belongs to $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$ for $n \gg 0$. Thus $\mathrm{Cone}(\alpha)$ belongs to $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$, again by Theorem 2.6. It follows that $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ is a triangulated subcategory of $\mathsf{D}_\mathrm{b}(\mathsf{V})$.

The triangulated subcategory $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ holds the objects from $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$. Let $\langle\mathsf{RGor}_\mathsf{U}(\mathsf{A})\rangle$ be the smallest triangulated subcategory of $\mathsf{D}_\mathrm{b}(\mathsf{V})$ that contains the objects from $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$, one then has $\langle\mathsf{RGor}_\mathsf{U}(\mathsf{A})\rangle \subseteq \mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$. Let V be an object in $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$. There exists a bounded complex X with $V \cong X$ in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ such that each X_n is in $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$. Without loss of generality we may assume that one has $X_0 \neq 0$ and $X_n = 0$ for $n < 0$. Set $s = \sup\{n \mid X_n \neq 0\}$; we proceed by induction on s. If $s = 0$, clearly $X \in \langle \mathsf{RGor}_\mathsf{U}(\mathsf{A})\rangle$. For $s > 0$, there is a triangle in $\mathsf{D}_\mathrm{b}(\mathsf{V})$

$$X_{\leqslant s-1} \longrightarrow X \longrightarrow \Sigma^s X_s \longrightarrow \Sigma X_{\leqslant s-1} \,.$$

By the induction hypothesis, both $X_{\leqslant s-1}$ and $\Sigma^s X_s$ belong to the triangulated category $\langle \mathsf{RGor}_\mathsf{U}(\mathsf{A})\rangle$, and hence so does X.

It remains to show that $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ is closed under direct summands. Let $V, V' \in \mathsf{D}_\mathrm{b}(\mathsf{V})$ and assume that $V \oplus V'$ is $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$-perfect. There are V-quasi-isomorphisms $X \to V$ and $X' \to V'$ in $\mathsf{K}(\mathsf{V})$ with $X, X' \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U} \cap \mathsf{V})$, this follows from the dual to [23, Lem. 4.1]. It follows that $X \oplus X' \to V \oplus V'$ is a V-quasi-isomorphism in $\mathsf{K}(\mathsf{V})$. As $V \oplus V'$ is $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$-perfect and $X \oplus X' \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U} \cap \mathsf{V})$, Theorem 2.6 implies that $\mathrm{Z}_n(X) \oplus \mathrm{Z}_n(X') \cong \mathrm{Z}_n(X \oplus X')$ belongs to $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$ for $n \gg 0$. Thus $\mathrm{Z}_n(X)$ and $\mathrm{Z}_n(X')$ belong to $\mathsf{RGor}_\mathsf{U}(\mathsf{A})$ for $n \gg 0$ by Lemma 2.3. Again from Theorem 2.6 it follows that V and V' belong to $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$. □

Given [14, Thm. 2.11], Theorem 2.6, and Proposition 2.8 one could obtain the first equivalence in part (a) below from a result of Iyama and Yang [19, Cor. 2.2], which elaborates on an example by Keller and Vossieck [25].

2.9 Theorem. *Let (U, V) be a complete hereditary cotorsion pair in A.*

(a) *There are triangulated equivalences*

$$\mathsf{StRGor}_\mathsf{U}(\mathsf{A}) \simeq \frac{\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})}{\mathsf{K}_\mathrm{b}(\mathsf{U} \cap \mathsf{V})} \qquad \text{and} \qquad \mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{V}) \simeq \frac{\mathsf{D}_\mathrm{b}(\mathsf{V})}{\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})} \,.$$

(b) *There are triangulated equivalences*

$$\mathsf{StLGor}_\mathsf{V}(\mathsf{A}) \simeq \frac{\mathsf{Perf}_{\mathsf{LGor}_\mathsf{V}(\mathsf{A})}(\mathsf{U})}{\mathsf{K}_\mathrm{b}(\mathsf{U} \cap \mathsf{V})} \qquad \text{and} \qquad \mathsf{D}^{\mathrm{def}}_{\mathrm{Inj}}(\mathsf{U}) \simeq \frac{\mathsf{D}_\mathrm{b}(\mathsf{U})}{\mathsf{Perf}_{\mathsf{LGor}_\mathsf{V}(\mathsf{A})}(\mathsf{U})} \,.$$

Proof. We prove part (a); the equivalences in part (b) have similar proofs. By Proposition 2.8 it suffices to show that the quotient $\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})/\mathsf{K}_\mathrm{b}(\mathsf{U} \cap \mathsf{V})$ is the essential image of the functor $\mathrm{F}_{\mathrm{Prj}} : \mathsf{StRGor}_\mathsf{U}(\mathsf{A}) \to \mathsf{D}_\mathrm{b}(\mathsf{V})/\mathsf{K}_\mathrm{b}(\mathsf{U} \cap \mathsf{V})$. Let $V \in \mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$, by Proposition 1.2 there is an isomorphism $V \cong X$ in $\mathsf{D}_\mathrm{b}(\mathsf{V})$ for some $X \in \mathsf{K}_{(\mathsf{V}\text{-ac}),+}(\mathsf{U} \cap \mathsf{V})$. By Theorem 2.6 there exists an integer n such that $\mathrm{Z}_n(X) \in \mathsf{RGor}_\mathsf{U}(\mathsf{A})$ and $X_{\supseteq n}$ is V-acyclic; indeed, the cycle objects are right U-Gorenstein by Lemma 2.4(a,1). Let $\widetilde{T}$ be a right U-totally acyclic complex with $\mathrm{Z}_0(\widetilde{T}) = \mathrm{Z}_n(X)$. Splicing together $\Sigma^n \widetilde{T}_{\leqslant 0}$ and $X_{\geqslant n+1}$ we obtain a right U-totally acyclic complex T with $T_{\geqslant n+1} = X_{\geqslant n+1}$. By definition one has $\mathrm{F}_{\mathrm{Prj}}(\mathrm{Z}_0(T)) = T_{\geqslant 0}$, and there are isomorphisms $T_{\geqslant 0} \cong T_{\geqslant n+1} \cong X$ in $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V})$.

Since the projective objects in V are those in $\mathsf{U} \cap \mathsf{V}$, see Proposition 1.7, the equivalence $\mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{V}) \simeq \mathsf{D}_\mathrm{b}(\mathsf{V})/\mathsf{Perf}_{\mathsf{RGor}_\mathsf{U}(\mathsf{A})}(\mathsf{V})$ follows from Definition 1.3(a) and Definition 2.2. □

2.10 Corollary. *Let (U, V) be a complete hereditary cotorsion pair in A.*

(a) *The following conditions are equivalent:*

  (*i*) $\mathrm{F}_{\mathrm{Prj}} \colon \mathsf{StRGor}_\mathsf{U}(\mathsf{A}) \to \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V})$ *is a triangulated equivalence.*

  (*ii*) *One has* $\mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{V}) = 0$.

(*iii*) *Every object in* $\mathsf{D_b}(\mathsf{V})$ *is* $\mathsf{RGor_U}(\mathsf{A})$*-perfect.*
(*iv*) *Every object in* V *is* $\mathsf{RGor_U}(\mathsf{A})$*-perfect.*

(b) *The following conditions are equivalent:*
(*i*) $\mathrm{F_{Inj}}\colon \mathsf{StLGor_V}(\mathsf{A}) \to \mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})$ *is a triangulated equivalence.*
(*ii*) *One has* $\mathsf{D}^{\mathrm{def}}_{\mathrm{Inj}}(\mathsf{U}) = 0$.
(*iii*) *Every object in* $\mathsf{D_b}(\mathsf{U})$ *is* $\mathsf{LGor_V}(\mathsf{A})$*-perfect.*
(*iv*) *Every object in* U *is* $\mathsf{LGor_V}(\mathsf{A})$*-perfect.*

Proof. We prove part (a); the proof of part (b) is similar. The equivalence of the first three conditions follows immediately from Definition 2.2 and Theorem 2.9. The equivalence of (*iii*) and (*iv*) follows from Proposition 2.8, as the smallest triangulated subcategory of $\mathsf{D_b}(\mathsf{V})$ that contains V is $\mathsf{D_b}(\mathsf{V})$. □

2.11 Remark. Let (U,V) be a cotorsion pair in A. When one combines the previous corollary with the equivalence from [14, Thm. 3.8], it transpires that if every object in V is $\mathsf{RGor_U}(\mathsf{A})$-perfect, then there are triangulated equivalences

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V}) \;\simeq\; \mathsf{StRGor_U}(\mathsf{A}) \;\simeq\; \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{V})\,.$$

Similarly, if every object in U is $\mathsf{LGor_V}(\mathsf{A})$-perfect, then there are triangulated equivalences

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U}) \;\simeq\; \mathsf{StLGor_V}(\mathsf{A}) \;\simeq\; \mathsf{K}^{\mathrm{L}}_{\mathsf{V}\text{-tac}}(\mathsf{U}\cap\mathsf{V})\,.$$

In [14] right and left totally acyclic complexes and right and left Gorenstein objects are defined for any subcategory of A. Given a cotorsion pair (U,V) in A, the subcategory $\mathsf{U}\cap\mathsf{V}$ is self-orthogonal, and for such a category the right/left distinctions disappear, see [14, Def. 1.6 and 2.1], and one speaks of $\mathsf{U}\cap\mathsf{V}$-totally acyclic complexes and $\mathsf{U}\cap\mathsf{V}$-Gorenstein objects. The homotopy category of $\mathsf{U}\cap\mathsf{V}$-totally acyclic complexes is denoted $\mathsf{K_{tac}}(\mathsf{U}\cap\mathsf{V})$ and the category of $\mathsf{U}\cap\mathsf{V}$-Gorenstein objects is denoted $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A})$.

2.12 Proposition. *Let* (U,V) *be a complete hereditary cotorsion pair in* A*.*

(a) *The following conditions are equivalent:*
(*i*) $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{V}$.
(*ii*) $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) = \mathsf{RGor_U}(\mathsf{A})$.

Moreover, when these conditions hold one has $\mathsf{LGor_V}(\mathsf{A}) = \mathsf{U}\cap\mathsf{V}$.

(b) *The following conditions are equivalent:*
(*i*) $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{U}$.
(*ii*) $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) = \mathsf{LGor_V}(\mathsf{A})$.

Moreover, when these conditions hold one has $\mathsf{RGor_U}(\mathsf{A}) = \mathsf{U}\cap\mathsf{V}$.

Proof. We prove (a); the proof of (b) is similar.

(*ii*)$\Longrightarrow$(*i*): The containment $\mathsf{RGor_U}(\mathsf{A}) \subseteq \mathsf{V}$ holds by definition.

(*i*)$\Longrightarrow$(*ii*): The containment $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \supseteq \mathsf{RGor_U}(\mathsf{A})$ holds by [14, Rem. 1.8]. If one has $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{V}$, then every $\mathsf{U}\cap\mathsf{V}$-totally acyclic complex is right U-totally acyclic, and hence (*ii*) follows.

To see that the moreover statement holds under these conditions, recall from [14, Def. 1.1 and Exa. 1.2] that there are containments $\mathsf{U}\cap\mathsf{V} \subseteq \mathsf{LGor_V}(\mathsf{A}) \subseteq \mathsf{U}$.

Further, $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A})$ holds by [14, Rem. 1.8], so in view of (i) one now has $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{V}$ and hence $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A}) = \mathsf{U} \cap \mathsf{V}$. □

2.13 Remark. Let (U,V) be a cotorsion pair in A. For the self-orthogonal class $\mathsf{U}\cap\mathsf{V}$, the equivalences in Remark 2.11 simplify as follows:

If every object in V is $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-perfect and $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{V}$, then there are triangulated equivalences

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{V}) \simeq \mathsf{StGor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \simeq \mathsf{K}_{\mathrm{tac}}(\mathsf{U}\cap\mathsf{V})\,.$$

Similarly, if every object in U is $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-perfect and $\mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \subseteq \mathsf{U}$, then there are triangulated equivalences

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U}) \simeq \mathsf{StGor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \simeq \mathsf{K}_{\mathrm{tac}}(\mathsf{U}\cap\mathsf{V})\,.$$

2.14 Example. Let A be a ring and consider the cotorsion pair $(\mathsf{Prj}(A), \mathsf{Mod}(A))$. A $\mathsf{Prj}(A)$-Gorenstein module is a Gorenstein projective module in the classic sense, see [14, Exa. 2.5]. If every A-module has finite Gorenstein projective dimension, then the categories in the first display in Remark 2.13 are "big" versions of the equivalent categories (⋄) from the introduction.

3. Gorenstein dimensions for everyone

Assume that A is Grothendieck and (U,V) is a complete hereditary cotorsion pair in A. Under this extra, but not too restrictive, assumption on A one can extend the notions of $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$- and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-perfection to the derived category of A: To every complex M of objects from A we assign two numbers, the $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective dimension and the $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-injective dimension, and for objects in $\mathsf{D}_{\mathrm{b}}(\mathsf{V})$ and $\mathsf{D}_{\mathrm{b}}(\mathsf{U})$ these dimensions are finite if and only if the objects are $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$- and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-perfect, respectively.

The category $\mathsf{C}(\mathsf{A})$ of chain complexes is also Grothendieck, and it follows from work of Gillespie [17, Prop. 3.6] and Šťovíček [33, Prop. 7.14] that (U,V) induces two complete hereditary cotorsion pairs in $\mathsf{C}(\mathsf{A})$:

$$(\mathsf{semi\text{-}U}, \mathsf{V\text{-}ac}) \qquad \text{and} \qquad (\mathsf{U\text{-}ac}, \mathsf{semi\text{-}V})\,. \tag{3.0.1}$$

The complexes in $\mathsf{V\text{-}ac}$ are acyclic complexes of objects in V with cycle objects in V, i.e. the V-acyclic complexes in $\mathsf{K}(\mathsf{V})$. The semi-U complexes, i.e. the objects in $\mathsf{semi\text{-}U}$, are complexes U of objects in U with the property that $\mathrm{Hom}_{\mathsf{A}}(U,V)$ is acyclic for every V-acyclic complex V. The classes $\mathsf{U\text{-}ac}$ and $\mathsf{semi\text{-}V}$ are defined similarly. We call a complex in $\mathsf{semi\text{-}U} \cap \mathsf{semi\text{-}V}$ a *semi*-U-V *complex.*

Completeness of the cotorsion pairs in (3.0.1) yields:

3.1 Fact. For every A-complex M there are exact sequences of A-complexes,

$$0 \longrightarrow V' \longrightarrow U \xrightarrow{\pi} M \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow M \xrightarrow{\iota} V \longrightarrow U' \longrightarrow 0\,,$$

where U is $\mathsf{semi\text{-}U}$, V is $\mathsf{semi\text{-}V}$, V' is V-acyclic, U' is U-acyclic, and the homomorphisms π and ι are quasi-isomorphisms. See [17, Prop. 3.6] and [33, Prop. 7.14].

3.2 Example. A complex U of objects in U with $\mathrm{C}_n(U) \in \mathsf{U}$ for $n \ll 0$ is a semi-U complex and a complex V of objects in V with $\mathrm{Z}_n(V) \in \mathsf{V}$ for $n \gg 0$ is a semi-V complex. This follows from [15, Props. A.1 and A.3][1].

[1] The proofs of [15, Props. A.1 and A.3] apply to complexes of objects in any abelian category.

In particular, a bounded below complex of objects in U is a semi-U complex and a bounded above complex of objects in V is a semi-V complex; cf. [17, Lem. 3.4].

3.3 Definition. Let M be an A-complex. A *semi-U-V replacement of M* is a semi-U-V complex that is isomorphic to M in the derived category $\mathsf{D}(\mathsf{A})$.

Some technical results about the cotorsion pairs (3.0.1) and semi-U-V complexes have been relegated to an appendix. The first step towards defining the $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-injective dimensions is to notice that every A-complex has a semi-U-V replacement.

3.4 Remark. Per Lemma A.4(a) every semi-V complex V has a semi-U-V replacement, as there is even a surjective quasi-isomorphism $W \xrightarrow{\simeq} V$ with W semi-U-V. Similarly, by Lemma A.4(b), every semi-U complex U has a semi-U-V replacement as there is even an injective quasi-isomorphism $U \xrightarrow{\simeq} W'$ with W' semi-U-V. These combine to show that every A-complex M has a semi-U-V replacement:

Let M be an A-complex. By Fact 3.1 there is a semi-V complex V and a quasi-isomorphism $M \xrightarrow{\simeq} V$, and Lemma A.4(a) yields a semi-U-V complex W and a quasi-isomorphism $W \xrightarrow{\simeq} V$. Thus W is a semi-U-V replacement of M.

One could also start with a quasi-isomorphism $U \xrightarrow{\simeq} M$ as in Lemma A.4(a) and get a quasi-isomorphism $U \xrightarrow{\simeq} W'$ from Lemma A.4(b).

The classic notion of Gorenstein projective dimension of modules and complexes over a ring A is a special case of the $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective dimension defined below: Apply the definition with $\mathsf{A} = \mathsf{Mod}(A)$ and $(\mathsf{U},\mathsf{V}) = (\mathsf{Prj}(A), \mathsf{Mod}(A))$; see also [14, Exa. 1.7] and [12, Rmk. 4.8].

3.5 Definition. Let (U,V) be a complete hereditary cotorsion pair in A and M an A-complex.

(a) The *$\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective dimension of M* is given by

$$\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, M = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{There is a semi-}\mathsf{U}\text{-}\mathsf{V}\text{ replacement } W \text{ of } M \text{ with} \\ \mathrm{H}_i(W) = 0 \text{ for all } i > n \text{ and } \mathrm{C}_n(W) \text{ in } \mathsf{RGor}_{\mathsf{U}}(\mathsf{A}) \end{array} \right\}$$

with $\inf \varnothing = \infty$ by convention.

(b) The *$\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-injective dimension of M* is given by

$$\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})\text{-id}\, M = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{There is a semi-}\mathsf{U}\text{-}\mathsf{V}\text{ replacement } W \text{ of } M \text{ with} \\ \mathrm{H}_i(W) = 0 \text{ for all } i < -n \text{ and } \mathrm{Z}_{-n}(W) \text{ in } \mathsf{LGor}_{\mathsf{V}}(\mathsf{A}) \end{array} \right\}$$

with $\inf \varnothing = \infty$ by convention.

Note that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, M = -\infty = \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})\text{-id}\, M$ holds if M is acyclic. We continue with a caveat that compares to [12, Rmk. 5.9]: An object of $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective dimension 0 need not belong to $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$; similarly for $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$.

3.6 Remark. For every non-zero object $U \in \mathsf{U}$ one has $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, U = 0$. Indeed, the inequality "$\geqslant$" holds as $\mathrm{H}_0(U) \neq 0$ and the opposite inequality holds as one can construct a semi-U-V replacement of U concentrated in non-positive degrees: Completeness of (U,V) yields a coresolution of U by objects in $\mathsf{U} \cap \mathsf{V}$ that is concentrated in non-positive degrees and whose cycle objects are in U; it is a semi-U-V complex by Example 3.2.

Similarly, $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})\text{-id}\, V = 0$ holds for every non-zero object $V \in \mathsf{V}$.

3.7 Lemma. *Let (U,V) be a complete hereditary cotorsion pair in A and M an A-complex with a semi-U-V replacement W.*

(a) *For every integer n with $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, M \leqslant n$ one has $\mathrm{C}_n(W) \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$.*
(b) *For every integer n with $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})\text{-id}\, M \leqslant n$ one has $\mathrm{Z}_{-n}(W) \in \mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$.*

Proof. We prove part (a); the proof of part (b) is similar. One can assume that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, M = g$ holds for some integer g. By assumption there exists a semi-U-V replacement W' of M with $\mathrm{C}_g(W')$ in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ and $\mathrm{H}_n(W') = 0$ for $n > g$. By induction it follows from Lemma 2.4(a) that $\mathrm{C}_n(W')$ belongs to $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for every $n \geqslant g$. Let W be any semi-U-V replacement of M. It follows from Proposition A.7 and Lemma 2.3 that $\mathrm{C}_n(W)$ belongs to $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ for every $n \geqslant g$. □

3.8 Theorem. *Let (U,V) be a complete hereditary cotorsion pair in A.*

(a) *An object $V \in \mathsf{D}_{\mathrm{b}}(\mathsf{V})$ is $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-perfect if and only if it has finite $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective dimension.*
(b) *An object $U \in \mathsf{D}_{\mathrm{b}}(\mathsf{U})$ is $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-perfect if and only if it has finite $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-injective dimension.*

Proof. We prove part (a); the proof of part (b) is similar. Without loss of generality, let V be a bounded complex of objects in V. There exists by Theorem A.8(a) a bounded below semi-U-V replacement W of V. The assertion now follows from Theorem 2.6 and Lemma 3.7 as one has $\mathrm{Z}_n(W) \cong \mathrm{C}_{n+1}(W)$ for $n > \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(V) \neq 0\}$. □

3.9 Remark. Let (U,V) be a complete hereditary cotorsion pair in A. Standard arguments, see [12, Thm. 4.5], based on Lemma 2.4(a,2) and 2.4(b,2) show that the $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$-projective and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$-injective dimensions of an A-complex M when finite can be detected in terms of vanishing of cohomology:

$$\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\text{-pd}\, M = \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_{\mathsf{A}}(M,W) \neq 0 \text{ for some } W \in \mathsf{U}\cap\mathsf{V}\} \quad \text{and}$$
$$\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})\text{-id}\, M = \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_{\mathsf{A}}(W,M) \neq 0 \text{ for some } W \in \mathsf{U}\cap\mathsf{V}\}\,.$$

4. Gorenstein rings and schemes

Our main interest here is schemes, but we warm up with rings.

Let A be a ring. The cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ in the Grothendieck category $\mathsf{Mod}(A)$ is complete and hereditary and hence gives rise to the $\mathsf{RGor}_{\mathsf{Flat}}(A)$-projective dimension. By [14, Prop. 4.2] this cotorsion pair satisfies the conditions in Proposition 2.12(a), and the $\mathsf{RGor}_{\mathsf{Flat}}(A)$-projective dimension was studied in [12] under the name "Gorenstein flat-cotorsion dimension;" see also [12, Rmk. 4.8]. In this section we keep with that terminology as it more descriptive, and we write $\mathsf{FlatCot}(A)$ for the intersection $\mathsf{Flat}(A) \cap \mathsf{Cot}(A)$.

If every A-module has finite Gorenstein flat-cotorsion dimension, then, as noted in Remark 2.13, there are triangulated equivalences

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(A)) \;\simeq\; \mathsf{StGor}_{\mathsf{FlatCot}}(A) \;\simeq\; \mathsf{K}_{\mathrm{tac}}(\mathsf{FlatCot}(A))\,.$$

We proceed to identify a class of rings over which these equivalences are realized.

Recall that a commutative noetherian ring is called *Iwanaga–Gorenstein* if it has finite self-injective dimension. The next theorem collects characterizations of such rings in terms of (Gorenstein) flat–cotorsion theory that closely mirror classic characterizations in terms of (Gorenstein) projectivity. In particular, (ii) is the analogue of Buchweitz's equivalence between the stable category of finitely generated Gorenstein projective modules and the singularity category [7, Thm. 4.4.1]. Condition (iii) is analogous to [6, Thm. 3.6] and compares to [20, Rmk. 5.6]. Conditions (iv) and (v) compare to Auslander and Bridger's [1, Thm. 4.20].

4.1 Theorem. *Let A be commutative noetherian; the next conditions are equivalent.*

- (i) *A is Iwanaga–Gorenstein.*
- (ii) *The functor $\mathrm{F}_{\mathrm{Prj}}\colon \mathsf{StGor}_{\mathsf{FlatCot}}(A) \longrightarrow \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(A))$ yields a triangulated equivalence.*
- (iii) *One has $\mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{Cot}(A)) = 0$.*
- (iv) *Every cotorsion A-module has finite Gorenstein flat-cotorsion dimension.*
- (v) *Every A-module has finite Gorenstein flat-cotorsion dimension.*
- (vi) *Every A-complex with bounded above homology has finite Gorenstein flat-cotorsion dimension.*

Proof. Conditions (ii), (iii), and (iv) are equivalent by Corollary 2.10 and Theorem 3.8. Conditions (i), (v), and (vi) are equivalent by [12, Cor. 5.10]. Evidently (v) implies (iv). For the converse let M be an A-module and consider an exact sequence $0 \to M \to C \to F \to 0$ where C is cotorsion and F is flat. By [12, Cor. 5.8] an A-module has finite Gorenstein flat-cotorsion dimension if and only if it has finite Gorenstein flat dimension. It now follows from Holm [18, Thm. 3.15] that also M has finite Gorenstein flat-cotorsion dimension. □

Part of the allure of the category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(A))$ is that it does not rely on projective modules and thus persists in the non-affine setting. Let X denote a scheme with structure sheaf $\mathcal{O}_X$. By a sheaf on X we mean a quasi-coherent sheaf, and $\mathsf{Qcoh}(X)$ denotes the category of such sheaves. We say that X is *semi-separated* if it has an open affine covering with the property that the intersection of any two open affine sets is affine.

Let X be semi-separated noetherian. The flat sheaves and cotorsion sheaves on X form a complete hereditary cotorsion pair $(\mathsf{Flat}(X), \mathsf{Cot}(X))$ in the Grothendieck category $\mathsf{Qcoh}(X)$; see [13, Rmk. 2.4]. It follows from [13, Thm. 3.3] that this pair satisfies the equivalent conditions in Proposition 2.12(a), so as above we refer to the $\mathsf{RGor}_{\mathsf{Flat}}(X)$-projective dimension as the *Gorenstein flat-cotorsion dimension.*

If every cotorsion sheaf on X has finite Gorenstein flat-cotorsion dimension then, as noted in Remark 2.13, there are triangulated equivalences:

$$\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X)) \simeq \mathsf{StGor}_{\mathsf{FlatCot}}(X) \simeq \mathsf{K}_{\mathrm{tac}}(\mathsf{FlatCot}(X))\,.$$

We proceed to identify a class of schemes over which these equivalences are realized. Recall that a semi-separated noetherian scheme X is *Gorenstein* provided that $\mathcal{O}_{X,x}$ is a Gorenstein ring for every $x \in X$. If, in addition, the scheme has finite Krull dimension, then this is equivalent to saying that $\mathcal{O}_X(U)$ is a Gorenstein ring for each open affine set U in some, equivalently every, open affine covering of X.

4.2 Theorem. *Let X be a semi-separated noetherian scheme of finite Krull dimension. The following conditions are equivalent.*

(i) *X is Gorenstein.*

(ii) *The functor*

$$\mathrm{F}_{\mathrm{Prj}}\colon \mathsf{StGor}_{\mathsf{FlatCot}}(X) \longrightarrow \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))$$

yields a triangulated equivalence.

(iii) *One has $\mathsf{D}^{\mathrm{def}}_{\mathrm{Prj}}(\mathsf{Cot}(X)) = 0$.*

(iv) *Every cotorsion sheaf on X has finite Gorenstein flat-cotorsion dimension.*

(v) *Every sheaf on X has finite Gorenstein flat-cotorsion dimension.*

(vi) *Every complex of sheaves on X with bounded above homology has finite Gorenstein flat-cotorsion dimension.*

Proof. Conditions (ii), (iii), and (iv) are equivalent by Corollary 2.10 and Theorem 3.8. Evidently, (vi) implies (v) implies (iv), so it suffices to show that (i) implies (vi) and (iv) implies (i). Denote by d the Krull dimension of X.

(i) $\Longrightarrow$ (vi): Let $\mathscr{M}$ be a complex of sheaves on X with bounded above homology and assume without loss of generality that $\mathrm{H}_n(\mathscr{M}) = 0$ holds for $n > 0$. Let $\mathscr{F}$ be a semi-flat-cotorsion replacement of $\mathscr{M}$. Notice that the truncated complex $\mathscr{F}_{\geqslant 0}$ is a semi-flat-cotorsion replacement of the cokernel $\mathscr{C} = \mathrm{C}_0(\mathscr{F})$; set $\mathscr{G} = \mathrm{Z}_{d-1}(\mathscr{F})$ and consider the exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F}_{d-1} \longrightarrow \cdots \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{C} \longrightarrow 0\,.$$

The truncated complex $\mathscr{F}_{\geqslant d}$ yields a left resolution of $\mathscr{G}$ by cotorsion sheaves; splicing this with a coresolution by injective sheaves, one obtains $\mathscr{G}$ as a cycle in an acyclic complex of cotorsion sheaves, whence $\mathscr{G}$ is cotorsion by [13, Thm. 3.3]. Per [13, Thm. 4.3] it now suffices to show that $\mathscr{G}$ is Gorenstein flat in the sense of [13, Def. 1.2]. Let $\mathcal{U}$ be an open affine covering of X. For every open set $U \in \mathcal{U}$ there is an exact sequence of $\mathcal{O}_X(U)$-modules,

$$0 \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{F}_{d-1}(U) \longrightarrow \cdots \longrightarrow \mathscr{F}_0(U) \longrightarrow \mathscr{C}(U) \longrightarrow 0\,,$$

with $\mathscr{F}_n(U)$ a flat $\mathcal{O}_X(U)$-module for $0 \leqslant n \leqslant d-1$. As $\mathcal{O}_X(U)$ is Gorenstein of Krull dimension at most d, the module $\mathscr{G}(U)$ is Gorenstein flat; see e.g. [16, Thm. 12.3.1]. From [13, Thm. 1.6] it follows that the sheaf $\mathscr{G}$ is Gorenstein flat.

(iv) $\Longrightarrow$ (i): Let $\mathcal{U}$ be an open affine covering of X and fix an open set $U \in \mathcal{U}$. The goal is to prove that the ring $\mathcal{O}_X(U)$ is Gorenstein. As it has finite Krull dimension, it suffices to show that every $\mathcal{O}_X(U)$-module has finite Gorenstein flat dimension; see e.g. [16, Thm. 12.3.1]. Let M be an $\mathcal{O}_X(U)$-module; denote by $\widetilde{M}$ the corresponding sheaf on U and by $(i_U)_*(\widetilde{M})$ the sheaf on X induced by the embedding $i_U\colon U \to X$. As $(\mathsf{Flat}(X), \mathsf{Cot}(X))$ is a complete hereditary cotorsion pair in $\mathsf{Qcoh}(X)$, see [13, Rmk. 2.4], there is a flat resolution $\mathscr{F}$ of $(i_U)_*(\widetilde{M})$ over X constructed by taking special flat precovers. In particular, for $n \geqslant 1$ the sheaf $\mathrm{C}_n(\mathscr{F})$ is cotorsion and $\mathscr{F}_n$ is flat-cotorsion. By assumption $\mathrm{C}_1(\mathscr{F})$ has finite Gorenstein flat-cotorsion dimension, which means that $\mathrm{C}_n(\mathscr{F})$ is Gorenstein flat-cotorsion for some $n \geqslant 1$. In particular, $\mathrm{C}_n(\mathscr{F})$ is by [13, Thm. 4.3] Gorenstein flat. Thus one gets an exact sequence of $\mathcal{O}_X(U)$-modules,

$$0 \longrightarrow \mathrm{C}_n(\mathscr{F})(U) \longrightarrow \mathscr{F}_{n-1}(U) \longrightarrow \cdots \longrightarrow \mathscr{F}_0(U) \longrightarrow M \longrightarrow 0\,,$$

which shows that M has finite Gorenstein flat dimension, cf. [13, Thm. 1.6]. □

4.3 Remark. Let X be a Gorenstein scheme of finite Krull dimension. It follows from [13, Cor. 4.6], [30, Thm. 4.27], and Murfet's [29, Thm. 7.9] that the stable category $\mathsf{StGor}_{\mathsf{FlatCot}}(X)$ is compactly generated. Further, the opposite category of Orlov's singularity category [32, Def. 1.8] is equivalent, up to direct factors, to the subcategory $\mathsf{StGor}_{\mathsf{FlatCot}}(X)^{\mathrm{c}}$ of compact objects. Thus, it follows from Theorem 4.2 that the singularity category $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))$ is compactly generated and that the opposite of Orlov's singularity category, up to direct factors, is equivalent to $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))^{\mathrm{c}}$.

Finally we justify that $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X))$ is a non-affine avatar of $\mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(A)$:

4.4 Theorem. *Let A be a commutative Gorenstein ring of finite Krull dimension and set $X = \mathrm{Spec}(A)$. There are equivalences of triangulated categories*

$$\begin{array}{ccccc} \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(A) & \simeq & \mathsf{StGor}_{\mathsf{Prj}}(A) & \simeq & \mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(A)) \\ & & \wr\!\wr & & \\ \mathsf{D}^{\mathrm{sg}}_{\mathrm{Prj}}(\mathsf{Cot}(X)) & \simeq & \mathsf{StGor}_{\mathsf{FlatCot}}(X) & \simeq & \mathsf{K}_{\mathrm{tac}}(\mathsf{FlatCot}(X))\,. \end{array}$$

Proof. A commutative noetherian ring of finite Krull dimension is Gorenstein if and only if it is Iwanaga–Gorenstein. The horizontal equivalences thus come from the two theorems above combined with Remark 2.13. The vertical equivalence follows from [14, Cor. 5.9]. □

5. Finitistic dimension

As discussed in Section 1, vanishing of singularity categories can capture finiteness of global dimensions. In this section we show how vanishing of a defect category captures finiteness of a finitistic dimension.

In this section A is a left noetherian ring. The *little finitistic dimension* of A, $\mathrm{findim}(A)$, is the supremum of projective dimensions of finitely generated A-modules of finite projective dimension. It is conjectured that $\mathrm{findim}(A) < \infty$ holds for Artin algebras A; see Bass [3] and [2, Conjectures]. Originally raised as a question by Rosenberg and Zelinsky, this is now the Finitistic Dimension Conjecture.

5.1 Definition. Let (U,V) be a complete hereditary cotorsion pair in $\mathsf{Mod}(A)$. Let M be an A-module. The $\mathsf{U}\cap\mathsf{V}$*-injective dimension of M* is defined as

$$\mathsf{U}\cap\mathsf{V}\text{-id}\,M \;=\; \inf\left\{ n\in\mathbb{Z} \;\middle|\; \begin{array}{c}\text{There is a semi-}\mathsf{U}\text{-}\mathsf{V}\text{ replacement } W\\ \text{of } M \text{ with } W_{-i}=0 \text{ for } i>n.\end{array} \right\}.$$

5.2 Lemma. *Let (U,V) be a complete hereditary cotorsion pair in $\mathsf{Mod}(A)$ and M an A-module. There is an inequality*

$$\mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\,M \;\leqslant\; \mathsf{U}\cap\mathsf{V}\text{-id}\,M$$

and equality holds if $\mathsf{U}\cap\mathsf{V}\text{-id}\,M < \infty$.

Proof. Assume $\mathsf{U}\cap\mathsf{V}\text{-id}\,M = n$ holds for some integer n, otherwise the statement is trivial. Let W be a semi-U-V replacement of M with $W_{-i} = 0$ for $i > n$. The inequality holds as every module in $\mathsf{U}\cap\mathsf{V}$ is left V-Gorenstein; see [14, Exa. 2.2]. Let $m \leqslant n$ be such that $\mathrm{H}_{-i}(M) = 0$ for $i > m$. If $\mathrm{Z}_{-m}(W)$ is left V-Gorenstein, then $\mathrm{Z}_{-m}(W) \in \mathsf{U}\cap\mathsf{V}$. Indeed, it has a finite coresolution by modules in $\mathsf{U}\cap\mathsf{V}$ which splits by Lemma 2.4(b). □

5.3 Lemma. *Let* (U,V) *be a complete hereditary cotorsion pair in* $\mathsf{Mod}(A)$*,* M *an* A*-module in* U*, and* $n > 0$ *an integer. One has* $\mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\, M \leqslant n$ *if and only if there exists an exact sequence of* A*-modules*

$$0 \longrightarrow M \longrightarrow G \longrightarrow L \longrightarrow 0$$

with $G \in \mathsf{LGor}_{\mathsf{V}}(A)$*,* $L \in \mathsf{U}$*, and* $\mathsf{U}\cap\mathsf{V}\text{-id}\, L \leqslant n-1$*.*

Proof. There is, by the completeness of (U,V), a coresolution of M consisting of modules in $\mathsf{U}\cap\mathsf{V}$, concentrated in non-positive degrees and with cycle modules in U; it is a semi-U-V complex by Example 3.2. Using this, the “only if” statement is proved the same way as [18, Thm. 2.10]. The “if” statement holds by a standard application of the Horseshoe Lemma, see [16, Lem. 8.2.1 and Rmk. 8.2.2], and Lemma 3.7; it also follows from the second equality in Remark 3.9. □

5.4 Proposition. *Let* (U,V) *be a complete hereditary cotorsion pair in* $\mathsf{Mod}(A)$*. For every module* M *in* $\mathsf{U}\cap{}^{\perp}\mathsf{LGor}_{\mathsf{V}}(A)$ *one has* $\mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\, M = \mathsf{U}\cap\mathsf{V}\text{-id}\, M$*.*

Proof. By Lemma 5.2, it suffices to show that $\mathsf{U}\cap\mathsf{V}\text{-id}\, M \leqslant \mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\, M$ holds. Set $\mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\, M = n$. If $n = 0$, then there is an exact sequence of A-modules

$$0 \longrightarrow H \longrightarrow W \longrightarrow M \longrightarrow 0$$

with $H \in \mathsf{LGor}_{\mathsf{V}}(A)$ and $W \in \mathsf{U}\cap\mathsf{V}$. Since M is in ${}^{\perp}\mathsf{LGor}_{\mathsf{V}}(A)$ the sequence splits, whence $M \in \mathsf{U}\cap\mathsf{V}$. If $n \geqslant 1$, then Lemma 5.3 yields an exact sequence of A-modules, $0 \to M \to G \to L \to 0$, with $G \in \mathsf{LGor}_{\mathsf{V}}(A)$, $L \in \mathsf{U}$, and $\mathsf{U}\cap\mathsf{V}\text{-id}\, L \leqslant n-1$. There is also an exact sequence $0 \to G' \to W' \to G \to 0$ of A-modules with $G' \in \mathsf{LGor}_{\mathsf{V}}(A)$ and $W' \in \mathsf{U}\cap\mathsf{V}$. There is a pullback diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T & \longrightarrow & W' & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0\,. & & & &
\end{array}$$

As W' is in $\mathsf{U}\cap\mathsf{V}$ and $\mathsf{U}\cap\mathsf{V}\text{-id}\, L \leqslant n-1$ holds, exactness of the middle row yields $\mathsf{U}\cap\mathsf{V}\text{-id}\, T \leqslant n$. As the left-hand column splits, one has $\mathsf{U}\cap\mathsf{V}\text{-id}\, M \leqslant n$. □

Let $\mathsf{P}^{<\infty}(A)$ denote the class of finitely generated A-modules of finite projective dimension and (U,V) be the complete cotorsion pair in $\mathsf{Mod}(A)$ cogenerated by $\mathsf{P}^{<\infty}$; that is, $\mathsf{V} = (\mathsf{P}^{<\infty}(A))^{\perp}$ and $\mathsf{U} = {}^{\perp}\mathsf{V}$. This cotorsion pair is also hereditary by a result of Cortés Izurdiaga, Estrada, and Guil Asensio [21, Cor. 2.7].

5.5 Theorem. *Set* $\mathsf{V} = (\mathsf{P}^{<\infty}(A))^{\perp}$ *and* $\mathsf{U} = {}^{\perp}\mathsf{V}$*. The following conditions are equivalent.*

(i) *One has* $\operatorname{findim}(A) < \infty$*.*

(ii) *Every module in* U *is* $\mathsf{LGor}_{\mathsf{V}}(A)$*-perfect.*

Proof. $(i) \Longrightarrow (ii)$: If one has $\operatorname{findim}(A) = n$ for an integer n, then [21, Thm. 3.4] yields $\mathsf{U}\cap\mathsf{V}\text{-id}\, M \leqslant n$ for every module $M \in \mathsf{U}$; in particular M is $\mathsf{LGor}_{\mathsf{V}}(A)$-perfect by Lemma 5.2 and Theorem 3.8.

$(ii) \Longrightarrow (i)$: The coproduct $A^{(A)}$ is a projective A-module and hence belongs to U, so by Theorem 3.8 there is an integer n such that $\mathsf{LGor}_{\mathsf{V}}(A)\text{-id}\, A^{(A)} = n$ holds. As $A^{(A)}$ also belongs to ${}^{\perp}\mathsf{LGor}_{\mathsf{V}}(A)$, one has $\mathsf{U} \cap \mathsf{V}\text{-id}\, A^{(A)} \leqslant n$ by Proposition 5.4. Now apply [21, Thm. 3.4] to conclude that $\operatorname{findim}(A) \leqslant n$ holds. □

5.6 Corollary. *Set* $\mathsf{V} = (\mathsf{P}^{<\infty}(A))^{\perp}$ *and* $\mathsf{U} = {}^{\perp}\mathsf{V}$. *The following conditions are equivalent:*

- (i) *One has* $\operatorname{findim}(A) < \infty$.
- (ii) *The functor* $\mathrm{F}_{\mathrm{Inj}}\colon \mathsf{StLGor}_{\mathsf{V}}(A) \to \mathsf{D}^{\mathrm{sg}}_{\mathrm{Inj}}(\mathsf{U})$ *yields a triangulated equivalence.*
- (iii) *One has* $\mathsf{D}^{\mathrm{def}}_{\mathrm{Inj}}(\mathsf{U}) = 0$.

Proof. Combine Theorem 5.5 and Corollary 2.10. □

5.7 Remark. Another classic conjecture for Artin algebras is the *Wakamatsu tilting conjecture*, which says that any Wakamatsu tilting A-module of finite projective dimension is a tilting A-module; see Beligiannis and Reiten [5, Chap. IV]. Mantese and Reiten [27, Prop. 4.4] showed that the Wakamatsu tilting conjecture is a special case of the Finitistic Dimension Conjecture. A Wakamatsu tilting module gives rise to a cotorsion pair, see Wang, Li, and Hu [34, Thm. 1.3]. If the module has finite projective dimension, then vanishing of an associated defect category betrays if it is tilting. We omit the details which are similar to the arguments above.

Appendix: A Schanuel's lemma for semi-U-V replacements

Assume throughout this appendix that A, as in Section 3, is Grothendieck and let (U,V) be a complete hereditary cotorsion pair in A.

A.1 Lemma. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence in* $\mathsf{C}(\mathsf{A})$.

- (a) *If* M' *is semi-*V *then* M *is semi-*V *if and only if* M'' *is semi-*V.
- (b) *If* M'' *is semi-*U *then* M *is semi-*U *if and only if* M' *is semi-*U.

Proof. We prove part (a); the proof of part (b) is similar. Assume that M' is semi-V. It is in particular a complex of objects from V, so M is a complex of objects from V if and only if M'' is so. Assuming that this is the case, let U be a U-acyclic complex. As M' is a complex of objects from V, there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathsf{A}}(U, M') \longrightarrow \operatorname{Hom}_{\mathsf{A}}(U, M) \longrightarrow \operatorname{Hom}_{\mathsf{A}}(U, M'') \longrightarrow 0\,.$$

By assumption the left-hand complex is acyclic, so the middle complex is acyclic if and only if the right-hand complex is acyclic. □

A.2 Proposition. *The following assertions hold:*

- (a) *An* A*-complex is* U*-acyclic if and only if it is semi-*U *and acyclic.*
- (b) *An* A*-complex is* V*-acyclic if and only if it is semi-*V *and acyclic.*

Moreover, an acyclic semi-U*-*V *complex is contractible.*

Proof. By [17, Lem. 3.10] a U-acyclic complex is semi-U, and a V-acyclic complex is semi-V. The category A has enough injectives, see e.g. Kashiwara and Schapira [22, Thm. 9.6.2], so it follows from [17, Thm. 3.12] that an acyclic semi-U complex is U-acyclic. As $(\mathsf{semi\text{-}U}, \mathsf{V\text{-}ac})$ is complete, see [17, Prop. 3.6] and [33, Prop. 7.14], it follows from [17, Lem. 3.14(1)] that an acyclic semi-V complex is V-acyclic. Finally, an acyclic semi-U-V complex is contractible as it has cycle objects in $\mathsf{U} \cap \mathsf{V}$. □

A.3 Corollary. *The following assertions hold.*

(a) *A quasi-isomorphism of semi-U complexes is a U-quasi-isomorphism.*
(b) *A quasi-isomorphism of semi-V complexes is a V-quasi-isomorphism.*
(c) *A quasi-isomorphism of semi-U-V complexes is a homotopy equivalence.*

Proof. The mapping cone of a quasi-isomorphism of semi-U complexes is acyclic and by Lemma A.1 semi-U, so it is U-acyclic by Proposition A.2. This proves (a) and the proof of (b) is similar. Finally, the mapping cone of a quasi-isomorphism of semi-U-V complexes is an acyclic semi-U-V complex and hence per A.2 contractible. □

A.4 Lemma. *Let M be an A-complex and consider the exact sequences from 3.1,*

$$0 \longrightarrow V' \longrightarrow U \xrightarrow{\pi} M \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow M \xrightarrow{\iota} V \longrightarrow U' \longrightarrow 0\,.$$

(a) *If M is a complex of objects in V, then π is a V-quasi-isomorphism, and if M is semi-V, then U is semi-U-V.*
(b) *If M is a complex of objects in U, then ι is a U-quasi-isomorphism, and if M is semi-U, then V is semi-U-V.*

Proof. We prove part (a); the proof of part (b) is similar. For every object U' in U the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(U', V') \longrightarrow \mathrm{Hom}_{\mathsf{A}}(U', U) \xrightarrow{\mathrm{Hom}_{\mathsf{A}}(U',\pi)} \mathrm{Hom}_{\mathsf{A}}(U', M) \longrightarrow 0$$

is exact and $\mathrm{Hom}_{\mathsf{A}}(U', V')$ is acyclic. It follows that $\mathrm{Hom}_{\mathsf{A}}(U', \pi)$ is a quasi-isomorphism. If M is a complex of objects from V, then so is U, whence $\mathrm{Cone}(\pi)$ is an acyclic complex of objects in V. As $\mathrm{Hom}_{\mathsf{A}}(U', \mathrm{Cone}(\pi))$ is acyclic for every $U' \in \mathsf{U}$ it follows that the cycles of the complex $\mathrm{Cone}(\pi)$ belong to V. Thus π is a V-quasi-isomorphism. Finally, if M is semi-V, then by Proposition A.2 and Lemma A.1 so is U. That is, U is semi-U-V. □

The lemma above provides a semi-U-V replacement of any A-complex; see Remark 3.4. Our next goal is a Schanuel's lemma for semi-U-V replacements. We move towards it with the next result and prove it in Proposition A.7.

A.5 Proposition. *Let U and U' be semi-U complexes and V and V' be a semi-V complexes.*

(a) *Let $\beta\colon U \xrightarrow{\simeq} U'$ be a U-quasi-isomorphism. For every morphism $\alpha\colon U \to V$ there is a morphism $\gamma\colon U' \to V$ with $\gamma\beta \sim \alpha$, and γ is unique up to homotopy.*
(b) *Let $\beta\colon V' \xrightarrow{\simeq} V$ be a V-quasi-isomorphism. For every morphism $\alpha\colon U \to V$ there is a morphism $\gamma\colon U \to V'$ with $\beta\gamma \sim \alpha$, and γ is unique up to homotopy.*

Proof. We prove part (a); the proof of part (b) is similar. The complex $\mathrm{Cone}(\beta)$ is U-acyclic, so the morphism $\mathrm{Hom}_{\mathsf{A}}(\beta, V)$ is a quasi-isomorphism. From this point, the proof of [12, Prop. 1.7] applies verbatim. □

A.6 Lemma. *Let W and W' be semi-U-V complexes. If there is an isomorphism $W \simeq W'$ in $\mathsf{D}(\mathsf{A})$, then there is a homotopy equivalence $W \to W'$ in $\mathsf{C}(\mathsf{A})$.*

Proof. By assumption there exists an A-complex X and a diagram in $\mathsf{C}(\mathsf{A})$:

$$W \xleftarrow{\simeq} X \xrightarrow{\simeq} W' .$$

By Fact 3.1 there is a semi-U complex U and a quasi-isomorphism $U \xrightarrow{\simeq} X$. The composite $U \longrightarrow W$ is by Corollary A.3(a) a U-quasi-isomorphism, so up to homotopy the composite $U \to W'$ lifts to a quasi-isomorphism $W \longrightarrow W'$; see Proposition A.5. By Corollary A.3(c) this quasi-isomorphism is a homotopy equivalence. □

A.7 Proposition. *Let W and W' be semi-U-V complexes isomorphic in $\mathsf{D}(\mathsf{A})$. For every $n \in \mathbb{Z}$ the following assertions hold.*

(a) *There exist objects U and U' in $\mathsf{U} \cap \mathsf{V}$ such that $\mathrm{C}_n(W) \oplus U \cong \mathrm{C}_n(W') \oplus U'$.*
(b) *There exist objects V and V' in $\mathsf{U} \cap \mathsf{V}$ such that $\mathrm{Z}_n(W) \oplus V \cong \mathrm{Z}_n(W') \oplus V'$.*

Proof. We prove part (a); the proof of part (b) is similar. By Lemma A.6 there is a homotopy equivalence $\alpha\colon W \to W'$. It follows from Proposition A.2 that every cycle object $\mathrm{Z}_n(\mathrm{Cone}(\alpha))$ belongs to $\mathsf{U} \cap \mathsf{V}$. The soft truncated morphism $\alpha_{\subseteq n}\colon W_{\subseteq n} \to W'_{\subseteq n}$ is also a homotopy equivalence, so $\mathrm{Cone}(\alpha_{\subseteq n})$, i.e. the complex

$$0 \to \mathrm{C}_n(W) \to \mathrm{C}_n(W') \oplus W_{n-1} \to W'_{n-1} \oplus W_{n-2} \xrightarrow{\partial_{n-1}^{\mathrm{Cone}(\alpha)}} W'_{n-2} \oplus W_{n-3} \to \cdots$$

is contractible. Hence one has $\mathrm{C}_n(W) \oplus \mathrm{Z}_{n-1}(\mathrm{Cone}(\alpha)) \cong \mathrm{C}_n(W') \oplus W_{n-1}$. □

The point of the next result, which is crucial to our proof of Theorem 3.8, is that one can exert some control over the boundedness of semi-U-V-replacements of bounded complexes.

A.8 Theorem. *The following assertions hold.*

(a) *For every bounded below semi-V complex V there exists an exact sequence*

$$0 \longrightarrow V' \longrightarrow W \xrightarrow{\pi} V \longrightarrow 0$$

such that V' is V-acyclic, W is semi-U-V and bounded below, and π is a V-quasi-isomorphism.

(b) *For every bounded above semi-U complex U there exists an exact sequence*

$$0 \longrightarrow U \xrightarrow{\iota} W \longrightarrow U' \longrightarrow 0$$

such that U' is U-acyclic, W is semi-U-V and bounded above, and ι is a U-quasi-isomorphism.

Proof. We prove part (a); the proof of part (b) is similar. There exists by Lemma A.4(a) an exact sequence

$$0 \longrightarrow V' \longrightarrow W \xrightarrow{\pi} V \longrightarrow 0$$

such that V' is V-acyclic, W is semi-U-V and π is a V-quasi-isomorphism. Since V is bounded below, the complexes V' and W agree in low degrees. It suffices to show that $\mathrm{Z}_n(W)$ belongs to $\mathsf{U} \cap \mathsf{V}$ for $n \ll 0$, as one then can replace V' and W with soft truncations $V'_{\supseteq n}$ and $W_{\supseteq n}$, cf. Lemma A.1 and Example 3.2.

For $n \ll 0$ one has $\mathrm{Z}_n(W) = \mathrm{Z}_n(V')$; this is an object in V so it remains to show that it is in U. Completeness of (U, V) provides a resolution U of $\mathrm{Z}_n(W)$ that is concentrated in non-negative degrees, consists of objects in $\mathsf{U} \cap \mathsf{V}$, and has cycle objects in V; it is thus a semi-U-V replacement per Example 3.2. The truncated complex $\Sigma^n(W_{\leqslant n})$ is another semi-U-V replacement of $\mathrm{Z}_n(W)$, cf. Lemma A.1 and Example 3.2. Now Proposition A.7(b) yields objects X and X' in $\mathsf{U} \cap \mathsf{V}$ such that $\mathrm{Z}_n(W) \oplus X \cong U_0 \oplus X'$ holds. As $U_0 \oplus X'$ is in U, so is $\mathrm{Z}_n(W)$. $\square$

Acknowledgment

We thank the anonymous referee for a very thorough reading and numerous suggestions that helped improve the paper.

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