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Derived Category Methods in Commutative Algebra

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Preface

Homological algebra originated in late 19^{th} century topology. Homological studies of algebraic objects, such as rings and modules, only got under way in the middle of the 20^{th} century—Cartan and Eilenberg's classic text "Homological Algebra" serves as a historic marker. The utility of homological methods in commutative algebra was firmly established in the mid 1950s through the proofs of Krull's conjectures on regular local rings, which were achieved by Auslander and Buchsbaum and by Serre.

The technically more involved methods of derived categories came into commutative algebra through the work of Grothendieck and his school. This happened only some ten years after the initial successes of classic homological algebra; an early, and still central, text is Hartshorne's exposition "Residues and Duality" of Grothendieck's 1963–64 seminar at Harvard. Applications of derived categories in commutative algebra have grown steadily, albeit slowly, since the mid 1960s. One reason for the modest pace has, possibly, been the absence of a coherent introduction to the topic. There are several excellent textbooks from which one can learn about classic homological algebra and its applications, but to become an efficient practitioner of derived category methods in commutative algebra one must be well-versed in a train of research articles and lecture notes, including unpublished ones. Textbooks on the foundations have only emerged in this century: Neeman's book "Triangulated Categories" is from 2001, "Categories and Sheaves" by Kashiwara and Schapira is from 2006, and Yekutieli's "Derived Categories" is from 2020.

With this book, we aim to provide an accessible and coherent introduction to derived category methods—in the past known as hyperhomological algebra—and their applications in commutative algebra. We want to make the case that these methods, compared to those of classic homological algebra, provide broader and stronger results with cleaner proofs—this to an extent that outweighs the effort it requires to master them. Moreover, there are important results in commutative algebra whose natural habitat is the derived category. The Local Duality Theorem, for example, has an elegant formulation in the derived category, but only for a limited class of rings does it have a satisfactory one within classic homological algebra.

The book is intended to double as a graduate textbook and a work of reference. It is organized into three parts: "Foundations", "Tools", and "Applications". In the first part, we introduce the fundamental homological machinery and construct the derived category over a ring. The second part continues with a systematic study of functors and invariants of utility in ring theory. In the third part we assemble textbook applications of derived category methods in commutative ring theory into a treatise on homological commutative algebra.

The organization of the book serves several purposes. Readers familiar with derived categories may skip “Foundations”. The tools are kept separate from their applications in order to develop them in higher generality; this should make “Foundations” and “Tools” useful, not only to commutative algebraists, but also to readers from neighboring fields. The third part “Applications” is intended to be a high-level primer to homological aspects of commutative algebra.

We have learned the material in this book from our teachers, our collaborators, and other colleagues—we hope they will find that we have done it justice.

The forerunners of this book are two collections of lecture notes by H.-B.F., that were used at the University of Copenhagen from 1982. Indeed, L.W.C. and H.H. were first introduced to the subject through these notes, and through the notes “Differential Graded Homological Algebra” by Avramov, H.-B.F., and Halperin. Following the untimely passing of H.-B.F. in 2014, L.W.C. and H.H. have striven to complete it in the style and spirit already established. Several versions of the lecture notes mentioned above have been circulated widely over the years, and it is hoped that their many readers will recognize H.-B.F.’s voice in the book.

Preliminary versions of this book have since 2006 been used in graduate classes and lectures at Beijing Normal University, Texas Tech University, and at the University of Nebraska-Lincoln. We thank the many students who gave their comments on these early versions of the manuscript. Thanks are also due to the anonymous refeerees and to colleagues who have provided their comments and answered our questions along the way; they include H. Faridian, J. Faucett, L. Ferraro, A. Hardesty, C.U. Jensen, S. Jøndrup, F. Köksal, L. Liang, Q. Pan, G. Piepmeyer, P. Thompson, Y. Wang, D. Wu, and A. Yekutieli. Finally, we thank our successive editors at Springer-Verlag: Karen Borthwick, Joerg Sixt, and last but not least Remi Lodh, who saw the project through to completion.

Great strides were made on the book during sojourns at Banff International Research Station, Centre Internationale de Rencontres Mathématiques (Luminy), Centro Internazionale per la Ricerca Matematica (Trento), and Mathematisches Forschungsinstitut Oberwolfach, as well as during L.W.C.’s visits to the University of Copenhagen and H.H.’s visits to Texas Tech University; the support and hospitality of these institutions is greatly appreciated.

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Lubbock and Copenhagen, *L.W.C.*
March 2024 *H.-B.F.*
H.H.

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Introduction

The year 1956 saw the first major applications of homological algebra in commutative ring theory. In seminal papers by Auslander and Buchsbaum [10] and Serre [226], homological algebra was used to characterize regular local rings in a way that opened to proofs of conjectures by Krull [163, 164]. This breakthrough right away established homological algebra as a powerful research tool in commutative ring theory and, in the words of Kaplansky [155], "marked a turning point of the subject." It coincided with the appearance of the magnum opus "Homological Algebra" [48], by Cartan and Eilenberg, which made the tools of classic homological algebra broadly available. The present book deals with the more advanced methods of derived categories and their applications in commutative algebra; applications to other areas of mathematics than ring theory are beyond the scope of the book.

Theme and Goal

Under the heading "Hyperhomology", a predecessor to the framework of derived categories was briefly treated by Cartan and Eilenberg in the final chapter of [48]. Yet, it was the work of Grothendieck—in particular, Hartshorne's notes "Residues and Duality" [114] published in 1966—that truly brought derived category methods into algebraic geometry and commutative algebra. In the 1970s, works by Iversen [141] and Roberts [213] emphasized the utility of these methods in commutative algebra, and since then their importance has grown steadily.

THE THEME OF THIS BOOK, as indicated in the Preface, is that derived category methods have significant advantages over those of classic homological algebra, which they extend. To facilitate a discussion of their similarities and differences, we sketch elements of the two pieces of machinery in the algebraic context.

Classic homological algebra is used to study modules through the behavior, notably vanishing, derived functors on modules. Let $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ be a well-behaved functor on the category of modules over a ring R. The value of the n^{th} left derived functor $\mathrm{L}_n\,\mathrm{F}$ on a module M is computed follows:

(1) Choose any free resolution $L_\bullet$ of M.
(2) Apply the functor F to this resolution to get a complex $\mathrm{F}(L_\bullet)$ of R-modules.
(3) By definition, $\mathrm{L}_n\,\mathrm{F}(M)$ is the homology module $\mathrm{H}_n(\mathrm{F}(L_\bullet))$.

This procedure determines $\mathrm{L}_n\,\mathrm{F}(M)$ up to isomorphism in $\mathcal{M}(R)$, and a similar one for homomorphisms establishes $\mathrm{L}_n\,\mathrm{F}$ as a functor. The purpose of step (3) is to return an object in $\mathcal{M}(R)$; this is not only aesthetically pleasing but also useful, as it allows the procedure to be iterated. However, the honest output is the complex $\mathrm{F}(L_\bullet)$.

Derived category methods are used to study R-modules and complexes of such through the behavior, notably boundedness, derived functors on complexes. For a well-behaved functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ on the category of complexes of R-modules, the value of the left derived functor LF on a complex $M_\bullet$ is computed as follows:

(1) Choose any semi-free resolution $L_\bullet$ of the complex $M_\bullet$.
(2) By definition, $\mathsf{LF}(M_\bullet)$ is the complex $\mathrm{F}(L_\bullet)$ of R-modules.

This procedure only determines $\mathsf{LF}(M_\bullet)$ up to homology isomorphism in $\mathcal{C}(R)$. Thus, the homology $\mathrm{H}(\mathsf{LF}(M_\bullet))$ is a well-defined object in $\mathcal{C}(R)$, but $\mathsf{LF}(M_\bullet)$ itself is not. This is overcome by passage to the derived category $\mathcal{D}(R)$, which has the same objects as $\mathcal{C}(R)$ but more morphisms—enough to make all homology isomorphisms invertible. In this category, $\mathsf{LF}(M_\bullet)$ is unique up to isomorphism, and LF is a functor.

A module M can be viewed as a complex, and in that perspective $\mathrm{H}(\mathsf{LF}(M))$ is nothing but the assembly of all the modules $\mathrm{L}_n\,\mathrm{F}(M)$. The theme of the book, in a nutshell, is that working with the functors LF on $\mathcal{D}(R)$, as opposed to $\mathrm{L}_n\,\mathrm{F}$ on $\mathcal{M}(R)$, yields broader and stronger results, even for modules. An early example of the case in point is Roberts' "no holes" theorem [213] in local algebra, which gave new insight into the structure of injective resolutions of modules. Another example comes from algebraic geometry. Grothendieck's localization problem for flat local homomorphisms was stated in EGA [112, §7], but a solution was only obtained 30 years later, when Avramov and Foxby [24] applied derived category methods to study a wider class of ring homomorphisms.

THE GOAL OF THIS BOOK is to expound the utility of derived category methods in ring theory. For commutative rings, the goal is further to collect textbook applications of these methods into a compact, yet comprehensive, introduction to homological aspects of commutative algebra. The book is intended to serve as a graduate textbook as well as a work of reference for professional mathematicians.

Contents and Organization

The facets of the goal, as described above, are reflected in the organization of the book. First we build the framework, that is, the derived category over a ring. Within this framework, we develop a set of efficacious tools, such as derived functors, fundamental transformations of such functors, categorical equivalences, and homological invariants. Finally, we apply these tools to prove classic and modern results in commutative algebra. The body of the book hence consists of three parts.

Part I "Foundations" presents the primary machinery of homological algebra. Chapter 1 is focused on the category of modules over a ring. For convenience, its first section recounts a few basic concepts and results that we expect the reader to be familiar with, such as bimodule structures and diagram lemmas. Chapters 2–4 treat the category of complexes of modules; the treatment includes categorical constructions, important functors, and special types of morphisms. Resolutions play a key role in homological algebra; they are the topic of Chap. 5. The derived category, which provides the framework for the rest of the book, is constructed in Chap. 6.

Part II "Tools" is devoted to homological invariants of complexes and studies of special complexes, morphisms, and functors in the derived category. Chapter 7 covers the derived Hom and tensor product functors; together with the standard isomorphisms that link them, these functors are key players in the rest of the book. Homological dimensions are developed in Chaps. 8 and 9, and they are tied in to the so-called evaluation morphisms. Together with the standard isomorphisms from Chap. 7, the evaluation morphisms play a central role in our approach to the material in Part III. Another powerful technical tool, dualizing complexes, is developed in Chap. 10, where also Grothendieck Duality, Morita Equivalence, and Foxby–Sharp Equivalence are treated. Torsion, completion, and the associated (co)homology theories are introduced in Chap. 11.

Part III "Applications" focuses on the homological theory of commutative Noetherian rings. It opens with a "Brief for Commutative Ring Theorists" (Chap. 12), which recapitulates central results from Part II in the simpler form they take in the setting of commutative Noetherian rings. In Chap. 13 we further develop the theory of derived torsion and section functors and treat the Greenlees–May Equivalence. Chapter 14 extends classic notions from commutative algebra, such as support, Krull dimension, and depth to objects in the derived category, while Chap. 15 centers on concepts of support that more relevant in the derived category setting than the classic notion. With these concepts in place, the tools from Part II are applied to prove time-honored and recent theorems on commutative rings and their modules. Chapter 16 treats homological invariants of modules over local rings, and in the subseqeuent chapters—most conspicuously in Chap. 17—they drive the study of homological invariants over general rings through the local–global principle. Chap. 18 has several of the classic big results in the field such as the Matlis and Grothendieck Duality Theorems, the Local Duality Theorem, the New Intersection Theorem, and the related homological characterizations of Cohen–Macaulay rings. Chapter 19 further develops the theory of Gorenstein homological dimensions and the homological characterizations of Gorenstein rings. Regular rings are the topic of the final Chap. 20.

The choice of topics is detailed in the synopses that open each section; they are also embedded in the table of contents.

A few topics that one might expect to find in this book, spectral sequences for example, are absent. The simplistic reason is that we manage without them. For example, standard arguments based on collapsing spectral sequences coming from double complexes are naturally replaced by isomorphisms in the derived category. More to the point, our rationale is that, within the scope of this book, we see no way to improve on existing expositions of these topics, and since we do not need them, we have decided to avoid them all together.

At the same time, the book covers at least one topic that could be considered non-standard: Gorenstein homological dimensions. The motivation for including this theory is two-fold. Significant progress has been made within the last three decades and derived category methods have been crucial for the successes of this research. The literature is ponderous to penetrate, and the existing surveys [52, 60] are not up to date or present no proofs.

The derived category over a ring is a particular example of a triangulated category. Other such categories—stable module categories and singularity categories to name a few—play significant roles in contemporary algebra and representation theory, but they are not treated in this book.

Exposition

We have striven to make the book coherent and self-contained. Assuming some background knowledge—specified in the Reader's Guide below—we provide full proofs of all statements in the book, though with the following qualification pertaining to Part III: In addition to the dimension theory for commutative Noetherian rings and the classic theory of support for modules, which we consider prerequisites, we use the Artin–Rees Lemma [208] and the existence of big Cohen–Macaulay modules over commutative Noetherian local rings. A proof of the Artin–Reese Lemma can be found in standard texts on commutative algebra such as Matsumura's [182]. Hochster [123, 126] proved the existence of big Cohen–Macaulay modules in the equicharacteristic case, and André [4] proved their existence in general.

Constructions and results that are required to keep the book self-contained, but might otherwise disrupt the flow of the material, have been relegated to appendices.

To keep the text accessible to a wide audience—and to avoid working in too many different contexts—we have resisted temptations to increase the level of generality beyond what is justified by the goal of the book, even when it would come at low or no cost. To begin with, several statements in Parts I and II are proved for Noetherian rings though they could be extended to coherent rings. Further, in the construction of the homotopy category over a ring, one could easily replace the module category with any other additive category. Similarly, parts of the material on standard isomorphisms and evaluation morphisms could be developed in the more general context of a closed monoidal category. In the same vein, we emphasize explicit constructions over axiomatic approaches. For example, we use resolutions to establish certain properties of the derived category over a ring, though they could be deduced from formal properties of triangulated categories.

THE MATERIAL in this textbook is not new, and the fact that references are sparse in the main text should *not* be interpreted as the authors' subtle claim for credit. In fact, every significant statement in the text is either folklore or can be traced to the papers and books listed in the Bibliography. Most references to the literature are found in the Remarks that accompany the main text. We have not attempted to systematically trace the origins of all major results; the selection of literature to cite only reveals our personal preferences and does not aspire to establish priority. The Bibliography is meant as a gateway to the vast literature, and the reader is encouraged to interpret "see $[n]$" to mean start from $[n]$ and follow the relevant citations therein. Among the texts that have inspired the contents, we emphasize two that have also influenced the exposition: "Homological Algebra" [48] by Cartan and Eilenberg and "Differential Graded Homological Algebra" [25] by Avramov, Foxby, and Halperin.

Reader's Guide

This book is written for researchers and advanced graduate students, so the reader should possess the mathematical maturity of a doctoral student of algebra.

THE PREREQUISITES include familiarity with basic notions from set theory, category theory, ring theory and, for Part III, commutative algebra. Notice, though, that we assume no prerequisites in homological algebra.

Commutative algebraists who are familiar with derived categories and want to dive into Part III will find an extract of essential results from Parts I and II in Chap. 12 and the final section of Chap. 15.

TEXT ELEMENTS that are typeset in small font are auxiliary. They provide commentary and perspective to the main text, but they can be read or skipped at will, as the main text does not depend on them.

THE CHAPTERS of the book can, of course, be read sequentially, with the occasional foray into an appendix. That being said, substantial references to Chap. 9 only occur in Chap. 19, and references to Chap. 10 only appear in Chaps. 18 and 19.

THE CONVENTIONS we employ are explained right after this introduction.

A GLOSSARY is provided towards the end of the book. It lists terms that are used but not defined in the text along with their definitions and references to textbooks. We refer to Lam's books "A First Course in Noncommutative Rings" and "Lectures on Modules and Rings" [167, 168] for notions in ring theory, to "Categories for the Working Mathematician" [175] by MacLane for concepts in category theory, and to Matsumura's "Commutative Ring Theory" [182] for notions in commutative algebra.

A LIST OF SYMBOLS follows the glossary; it includes most symbols used—and certainly all those defined—in the book.

EXERCISES are included at the end of each section. In addition to the purpose that exercises ordinarily serve in a textbook, they supplement the text in three ways.

- Examples in the text are kept as elementary as reasonably possible; exercises are used to develop more interesting ones.
- To some results in the main text there are parallel or dual versions that are not needed in the exposition; these are relegated to exercises.
- Exercises are used to explore directions beyond the scope of the book; some of these come with references to the literature to facilitate further study.

We emphasize that the main text does not depend on the exercises.

Teacher's Guide

The nature of the subject leaves a teacher of homological algebra with hard choices to make about the amount of detail to present in lectures. An over- or underemphasis on technical details can easily create an irksome feeling that the topic belongs to accounting or religion rather than mathematics. The favored compromise is to treat some constructions and arguments in full detail and leave it to the students to reassure themselves that the balance of the material is also solid. We facilitate this style by supplying a fair and consistent amount of detail; certainly more than one would attempt to include in lectures. With some restraint, we follow the tradition for recycling arguments by saying that a proof is "parallel" or "dual" to one that has already been presented. We apply this technique when consecutive statements have similar proofs, but we avoid the wholesale application that would dispense with the theory of injective modules in one sentence, "dual to the projective case."

For pedagogical reasons, we have included several exercises that ask the students to perform elementary verifications and computations that are omitted from the text. These exercises are easily recognized as they open with a reference like (Cf. 1.1.1). The details examined in these exercises are ones that a professional mathematician can fill in on the fly, so we stand by the claim made right above: The main text does not depend on the exercises. Another class of exercises explore elementary versions of constructions and arguments that are treated later in the text; they are not marked in any particular way, but they should be easily recognizable to the teacher.

Conventions and Notation

Throughout the book, the symbols Q, R, S, and T denote non-zero associative unital rings. They are assumed to be algebras over a common commutative unital ring $\Bbbk$, and homomorphisms between them are tacitly assumed to be $\Bbbk$-algebra homomorphisms. The generic choice of $\Bbbk$ is the ring of integers, but in concrete settings other choices may be useful. For example, in studies of algebras over a field $\boldsymbol{k}$, the natural choice is $\Bbbk = \boldsymbol{k}$, and in studies of Artin A-algebras, the commutative Artinian ring A is the natural candidate for $\Bbbk$. In a different direction, the center of R is a possible choice for $\Bbbk$, when one studies ring homomorphisms $R \longrightarrow S$.

Ideals in a ring are subsets that are both left ideals and right ideals. Similarly, a ring is called Artinian/Noetherian/perfect etc. if it is both left and right Artinian/Noetherian/perfect etc. From Chap. 11 and onwards, the rings Q, R, S, and T are assumed to be commutative and such distinctions conveniently disappear.

Modules are assumed to be unitary, and by convention the ring acts on the left. That is, an R-module is a left R-module. Right R-modules are, consequently, considered to be modules (i.e. left modules) over R°, the opposite ring of R. Thus, a left ideal in R is a submodule of the R-module R, while a right ideal in R is a submodule of the R°-module R.

Functors are by convention covariant. Replacing one category by its opposite, a contravariant functor $\mathcal{U} \longrightarrow \mathcal{V}$ is hence considered to be a (covariant) functor $\mathcal{U}^{\mathrm{op}} \longrightarrow \mathcal{V}$ or $\mathcal{U} \longrightarrow \mathcal{V}^{\mathrm{op}}$. For a natural transformation $\tau\colon \mathrm{F} \longrightarrow \mathrm{G}$ of functors $\mathcal{U} \longrightarrow \mathcal{V}$ we write $\tau^{\mathrm{op}}\colon \mathrm{G}^{\mathrm{op}} \longrightarrow \mathrm{F}^{\mathrm{op}}$ for the natural transformation of opposite functors $\mathcal{U}^{\mathrm{op}} \longrightarrow \mathcal{V}^{\mathrm{op}}$. On endofunctors $\mathcal{U} \longrightarrow \mathcal{U}$ and their transformations the '${}^{\mathrm{op}}$' is usually suppressed.

The notation is either standard or explained in the text, starting here:

$(\ll 0)\ \gg 0$	sufficiently (small) large	$\mathbb{C}$	complex numbers
$\rightarrowtail$	injective map	$\mathbb{N}$ ($\mathbb{N}_0$)	natural numbers (and 0)
$\twoheadrightarrow$	surjective map	$\mathbb{Q}$	rational numbers
$(\subset)\ \subseteq$	(proper) subset	$\mathbb{R}$	real numbers
$\setminus$	difference of sets	$\mathbb{Z}$	integers
$\uplus$	disjoint union of sets	Id	identity functor
$\times$	cartesian product	card	cardinality
$\cong$	isomorphism		

Part I
Foundations

The objective of the six chapters that lay immediately ahead is to construct the derived category of the module category over a ring. In order to not loose track of the overall aim—applications of homological algebra in (commutative) ring theory—we suspend work on the construction every now and then to tie up connections to ring theory as they come within reach. Such connections include homological characterizations of principal ideal domains, semi-simple rings, von Neumann regular rings, perfect rings, and semi-perfect rings—and in exercises, hereditary rings. Most of these excursions into ring theory are short, but the homological characterizations of (semi-)perfect rings, including Bass' Theorem P, are intertwined with the constructions of minimal resolutions, which is the topic of the rather lengthy Appn. B.

Homological algebra has a reputation for being (tediously) technical, per se. It is somewhat befitting, and we shall make no attempt to hide or trivialize the theory's nature. Rather, we expound it as we adopt the point of view that the fabric of homological algebra is technical constructions and statements about them. With this unapologetic style we aspire to put a transparent hood on the homological machine.

Chapter 1
Modules

The collection of all R-modules and all homomorphisms of R-modules forms an Abelian category with set indexed products and coproducts; it is denoted $\mathcal{M}(R)$. We take this for granted, and the first section of this chapter sums up the key properties of this category that we build on throughout the book.

Although the applications of homological algebra that we ultimately aim for are to modules, it is advantageous to work with the wider notion of complexes. Modules are elementary complexes, or complexes are modules with extra structure, depending on the point of view. In any event, there is a decision to make about when to leave modules and pass to complexes. We make this transition as soon as we have established a baseline of homological module theory that allows us to interpret results about complexes in the realm of modules. That baseline is established in this first chapter, and the transition to complexes takes place in Chap. 2.

1.1 Prerequisites

Synopsis. Five Lemma; Snake Lemma; Hom; tensor product; restriction of scalars; linear category; linear functor; biproduct; product; coproduct; direct sum; bimodule; split exact sequence; (half/left/right) exact functor; faithful functor.

The primary purpose of this section is to remind the reader of some basic material and, in that process, to introduce the accompanying symbols and nomenclature. The material in this section is used throughout the book and usually without reference.

1.1.1 Definition. A sequence of R-modules is a, possibly infinite, diagram in $\mathcal{M}(R)$,

$$(1.1.1.1) \qquad \cdots \longrightarrow M^0 \xrightarrow{\alpha^0} M^1 \xrightarrow{\alpha^1} M^2 \xrightarrow{\alpha^2} \cdots ;$$

it is called *exact* if $\operatorname{Im} \alpha^{n-1} = \operatorname{Ker} \alpha^n$ holds for all n. Notice that (1.1.1.1) is exact if and only if every sequence $0 \to \operatorname{Im} \alpha^{n-1} \to M^n \to \operatorname{Im} \alpha^n \to 0$ is exact. An exact sequence of the form $0 \to M' \to M \to M'' \to 0$ is called a *short exact sequence*.

Two sequences $\{\alpha^n : M^n \to M^{n+1}\}_{n\in\mathbb{Z}}$ and $\{\beta^n : N^n \to N^{n+1}\}_{n\in\mathbb{Z}}$ of R-modules are called *isomorphic* if there exists a family of isomorphisms $\{\varphi^n\}_{n\in\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} M^n \arrow[r, "\alpha^n"] \arrow[d, "\varphi^n"'] & M^{n+1} \arrow[d, "\varphi^{n+1}"] \\ N^n \arrow[r, "\beta^n"] & N^{n+1} \end{tikzcd}$$

is commutative for every $n \in \mathbb{Z}$.

Diagram lemmas are workhorses of homological algebra. Two of the most frequently used are known as the Five Lemma and the Snake Lemma.

1.1.2 Five Lemma. *Consider a commutative diagram in $\mathcal{M}(R)$ with exact rows,*

$$\begin{tikzcd} M^1 \arrow[r] \arrow[d, "\varphi^1"] & M^2 \arrow[r] \arrow[d, "\varphi^2"] & M^3 \arrow[r] \arrow[d, "\varphi^3"] & M^4 \arrow[r] \arrow[d, "\varphi^4"] & M^5 \arrow[d, "\varphi^5"] \\ N^1 \arrow[r] & N^2 \arrow[r] & N^3 \arrow[r] & N^4 \arrow[r] & N^5 . \end{tikzcd}$$

(a) *If φ^1 is surjective, and φ^2 and φ^4 are injective, then φ^3 is injective.*
(b) *If φ^5 is injective, and φ^2 and φ^4 are surjective, then φ^3 is surjective.*
(c) *If φ^1, φ^2, φ^4, and φ^5 are isomorphisms, then φ^3 is an isomorphism.*

1.1.3 Kernel Lemma. *Consider a commutative diagram in $\mathcal{M}(R)$,*

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[d, "\varphi''"] \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' , \end{tikzcd}$$

with exact rows. There is an exact sequence,

$$\operatorname{Ker}\varphi' \xrightarrow{\alpha'} \operatorname{Ker}\varphi \xrightarrow{\alpha} \operatorname{Ker}\varphi'' .$$

If α' is injective, then so is the restriction $\alpha' : \operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi$.

1.1.4 Cokernel Lemma. *Consider a commutative diagram in $\mathcal{M}(R)$,*

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' , \end{tikzcd}$$

with exact rows. There is an exact sequence,

$$\operatorname{Coker}\varphi' \xrightarrow{\bar{\beta}'} \operatorname{Coker}\varphi \xrightarrow{\bar{\beta}} \operatorname{Coker}\varphi'' .$$

If β is surjective, then so is the induced homomorphism $\bar{\beta} : \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$.

The two results above are fused in the Snake Lemma 1.1.6 below.

1.1.5 Construction. Consider a commutative diagram in $\mathcal{M}(R)$,

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' , \end{tikzcd}$$

with exact rows. Given an element $x'' \in \operatorname{Ker}\varphi''$ choose, by surjectivity of α, a preimage x in M. Set $y = \varphi(x)$ and note that y belongs to $\operatorname{Ker}\beta$, by commutativity of the right-hand square. By exactness at N choose $y' \in N'$ with $\beta'(y') = y$. It is straightforward to verify that the element $[y']_{\operatorname{Im}\varphi'}$ in $\operatorname{Coker}\varphi'$ does not depend on the choices of x and y', so this procedure defines a map $\delta\colon \operatorname{Ker}\varphi'' \to \operatorname{Coker}\varphi'$.

1.1.6 Snake Lemma. *The map $\delta\colon \operatorname{Ker}\varphi'' \to \operatorname{Coker}\varphi'$ defined in* 1.1.5 *is a homomorphism of R-modules, called the connecting homomorphism, and there is an exact sequence in $\mathcal{M}(R)$,*

$$\operatorname{Ker}\varphi' \xrightarrow{\alpha'} \operatorname{Ker}\varphi \xrightarrow{\alpha} \operatorname{Ker}\varphi'' \xrightarrow{\delta} \operatorname{Coker}\varphi' \xrightarrow{\bar\beta'} \operatorname{Coker}\varphi \xrightarrow{\bar\beta} \operatorname{Coker}\varphi'' .$$

Moreover, if α' is injective, then so is the restriction $\alpha'\colon \operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi$, and if β is surjective, then so is the induced homomorphism $\bar\beta\colon \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$.

For historical reasons, the morphisms in $\mathcal{M}(R)$ are called homomorphisms and the hom-sets are written Hom_R.

1.1.7 Homomorphisms. From the hom-sets in $\mathcal{M}(R)$ one can construct a functor

$$\operatorname{Hom}_R(-,-) : \mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\Bbbk) .$$

For homomorphisms $\alpha\colon M' \to M$ and $\beta\colon N \to N'$ of R-modules, the functor acts as follows,

$$\operatorname{Hom}_R(\alpha,\beta) :\ \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(M',N') \quad \text{is given by} \quad \vartheta \longmapsto \beta\vartheta\alpha .$$

1.1.8 Example. Let $\mathfrak{a}$ be a left ideal in R and M an R-module. There is an isomorphism of $\Bbbk$-modules $\operatorname{Hom}_R(R/\mathfrak{a}, M) \to (0 :_M \mathfrak{a})$ given by $\alpha \mapsto \alpha([1]_{\mathfrak{a}})$. The inverse function maps an element m in $(0 :_M \mathfrak{a})$ to the homomorphism $R/\mathfrak{a} \to M$ given by $[r]_{\mathfrak{a}} \mapsto rm$. In particular, for a cyclic R-module $M = R/\mathfrak{b}$ one gets

$$\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong (0 :_{R/\mathfrak{b}} \mathfrak{a}) \cong (\mathfrak{b} :_R \mathfrak{a})/\mathfrak{b} .$$

1.1.9 Tensor Product. The tensor product of modules yields a functor

$$- \otimes_R - : \mathcal{M}(R^{\mathrm{o}}) \times \mathcal{M}(R) \longrightarrow \mathcal{M}(\Bbbk) .$$

For a homomorphism $\alpha\colon M \to M'$ of R^{o}-modules and a homomorphism $\beta\colon N \to N'$ of R-modules, the functor acts as follows,

$$\alpha \otimes_R \beta : M \otimes_R N \longrightarrow M' \otimes_R N' \quad \text{is given by} \quad m \otimes n \longmapsto \alpha(m) \otimes \beta(n) .$$

1.1.10 Example. Let $\mathfrak{b}$ be a right ideal in R and M an R-module. There is an isomorphism of $\Bbbk$-modules $R/\mathfrak{b} \otimes_R M \to M/\mathfrak{b}M$ given by $[r]_\mathfrak{b} \otimes m \mapsto [rm]_{\mathfrak{b}M}$. The inverse function maps an element $[m]_{\mathfrak{b}M}$ in $M/\mathfrak{b}M$ to the element $[1]_\mathfrak{b} \otimes m$ in $R/\mathfrak{b} \otimes_R M$. In particular, for a cyclic R-module $M = R/\mathfrak{a}$ there are isomorphisms

$$R/\mathfrak{b} \otimes_R R/\mathfrak{a} \cong \frac{R/\mathfrak{a}}{\mathfrak{b}(R/\mathfrak{a})} \cong \frac{R/\mathfrak{a}}{(\mathfrak{b}+\mathfrak{a})/\mathfrak{a}} \cong R/(\mathfrak{a}+\mathfrak{b}) \,.$$

The localization of a module over a commutative ring at a multiplicative subset can be realized as a tensor product.

1.1.11 Example. Assume that R is commutative. Let U be a multiplicative subset of R and M an R-module. There is an isomorphism $U^{-1}R \otimes_R M \to U^{-1}M$ of $U^{-1}R$-modules, given by $\frac{r}{u} \otimes m \mapsto \frac{rm}{u}$, and it is natural in M.

1.1.12 Restriction of scalars I. Let $\varphi\colon R \to S$ be a ring homomorphism; it induces an R-module structure on every S-module and an R^{o}-module structure on every S^{o}-module. The forgetful functors

$$\mathrm{res}_R^S : \mathcal{M}(S) \longrightarrow \mathcal{M}(R) \qquad \text{and} \qquad \mathrm{res}_{R^{\mathrm{o}}}^{S^{\mathrm{o}}} : \mathcal{M}(S^{\mathrm{o}}) \longrightarrow \mathcal{M}(R^{\mathrm{o}})$$

that assign to an S-module (S^{o}-module) the R-module (R^{o}-module) with the action induced by φ are called *restriction of scalars* along φ. At the level of symbols these functors are mostly suppressed, but when we write "over R" or "as an R-module" about an S-module, it means that the restriction of scalars functor is being applied.

Linearity

Because R is assumed to be a $\Bbbk$-algebra, the module category $\mathcal{M}(R)$ is $\Bbbk$-linear in the following sense.

1.1.13 Definition. A category $\mathcal{U}$ is called $\Bbbk$-*linear* if it satisfies the next conditions.

(1) For every pair of objects M and N in $\mathcal{U}$, the hom-set $\mathcal{U}(M,N)$ is a $\Bbbk$-module, and composition of morphisms $\mathcal{U}(M,N) \times \mathcal{U}(L,M) \to \mathcal{U}(L,N)$ is $\Bbbk$-bilinear.

(2) There is a *zero object,* 0, in $\mathcal{U}$. That is, for each object M in $\mathcal{U}$ there is a unique morphism $M \to 0$ and a unique morphism $0 \to M$.

(3) For every pair of objects M and N in $\mathcal{U}$ there is a *biproduct,* $M \oplus N$, in $\mathcal{U}$. That is, given M and N there is a diagram in $\mathcal{U}$,

$$M \underset{\varepsilon^M}{\overset{\varpi^M}{\rightleftarrows}} M \oplus N \underset{\varepsilon^N}{\overset{\varpi^N}{\rightleftarrows}} N \,,$$

such that $\varpi^M \varepsilon^M = 1^M$, $\varpi^N \varepsilon^N = 1^N$, and $\varepsilon^M \varpi^M + \varepsilon^N \varpi^N = 1^{M \oplus N}$ hold. Here 1^X denotes the *identity morphism* of the object X.

A category that satisfies (1) is called $\Bbbk$-*prelinear*. It is evident that the opposite category of a $\Bbbk$-(pre)linear category is $\Bbbk$-(pre)linear.

The module category $\mathcal{M}(R)$ has additional structure; indeed, it is Abelian, and so is the category $\mathcal{C}(R)$ of R-complexes, which is the topic of the next chapter. However, the homotopy category $\mathcal{K}(R)$ and the derived category $\mathcal{D}(R)$, which are constructed in Chap. 6, are not Abelian but triangulated. All four categories are $\Bbbk$-linear, hence the focus on that notion.

1.1.14 Direct Sum. Let $\mathcal{U}$ be a $\Bbbk$-linear category. The biproduct $\oplus$ is both a product and a coproduct, and it is elementary to verify that it is associative. For a finite set U and a family $\{M^u\}_{u\in U}$ of objects in $\mathcal{U}$, the notation $\bigoplus_{u\in U} M^u$ for the iterated biproduct is, therefore, unambiguous. The object $M = \bigoplus_{u\in U} M^u$ is called the *direct sum* of the family $\{M^u\}_{u\in U}$, and each object M^u is called a *direct summand* of M.

It is handy to identify a morphism $\alpha\colon \bigoplus_{v=1}^n M^v \to \bigoplus_{u=1}^m N^u$ by an $m\times n$ matrix,

$$\begin{matrix} M_1 \\ \oplus \\ \vdots \\ \oplus \\ M_n \end{matrix} \xrightarrow{\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}} \begin{matrix} N_1 \\ \oplus \\ \vdots \\ \oplus \\ N_m \end{matrix} ,$$

with $\alpha_{uv} = \varpi^{N^u}\alpha\varepsilon^{M^v}$, where ε^{M^v} and ϖ^{N^u} are the morphisms from 1.1.13.

The homomorphism functor 1.1.7 and the tensor product functor 1.1.9 are both $\Bbbk$-multilinear in the following sense.

1.1.15 Definition. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between $\Bbbk$-prelinear categories is called $\Bbbk$-*linear* if it satisfies the following conditions.

(1) $\mathrm{F}(\alpha+\beta) = \mathrm{F}(\alpha) + \mathrm{F}(\beta)$ for all parallel morphisms α and β in $\mathcal{U}$.
(2) $\mathrm{F}(x\alpha) = x\,\mathrm{F}(\alpha)$ for all morphisms α in $\mathcal{U}$ and all $x \in \Bbbk$.

Let $\mathcal{U}^1, \dots, \mathcal{U}^n$ and $\mathcal{V}$ be $\Bbbk$-prelinear categories. A functor

$$\mathrm{F} : \mathcal{U}^1 \times \cdots \times \mathcal{U}^n \longrightarrow \mathcal{V}$$

is called $\Bbbk$-*multilinear* if it is $\Bbbk$-linear in each variable.

There is a unique ring homomorphism $\mathbb{Z} \to \Bbbk$; therefore, every $\Bbbk$-linear category/functor is $\mathbb{Z}$-linear in a canonical way.

1.1.16 Definition. A $\mathbb{Z}$-prelinear category is also called *preadditive*. A $\mathbb{Z}$-linear category/functor is also called *additive*.

1.1.17 Definition. Let M and N be objects in an additive category $\mathcal{U}$. The *zero morphism* from M to N is the composite of the unique morphisms $M \to 0$ and $0 \to N$; this morphism is denoted by 0.

1.1.18. Let $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ be an additive functor between additive categories. It takes zero morphisms in $\mathcal{U}$ to zero morphisms in $\mathcal{V}$. In particular, F takes zero objects in $\mathcal{U}$ to zero objects in $\mathcal{V}$.

Products, Coproducts, and Direct Sums of Modules

1.1.19 Product. For a family $\{N^u\}_{u\in U}$ in $\mathcal{M}(R)$, the product $\prod_{u\in U} N^u$ in $\mathcal{M}(R)$ is the Cartesian product of the underlying sets, with the R-module structure given by coordinatewise operations, together with the projections $\varpi^u\colon \prod_{u\in U} N^u \twoheadrightarrow N^u$. Indeed, given a family of homomorphisms $\{\alpha^u\colon M \to N^u\}_{u\in U}$, the map α defined by $m \mapsto (\alpha^u(m))_{u\in U}$ is the unique homomorphism that makes the diagram

$$\begin{tikzcd} & \prod_{u\in U} N^u \arrow[d, two heads, "\varpi^u"] \\ M \arrow[ur, dotted, "\alpha"] \arrow[r, "\beta^u"'] & N^u \end{tikzcd}$$

commutative for every $u \in U$. For a family $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ of homomorphisms, the product $\prod_{u\in U} \alpha^u\colon \prod_{u\in U} M^u \to \prod_{u\in U} N^u$ is given by $(m^u)_{u\in U} \mapsto (\alpha^u(m^u))_{u\in U}$.

If one has $M^u = M$ for every $u \in U$, then the product $\prod_{u\in U} M^u$ is denoted M^U and called the *U-fold product* of M.

Remark. Other names for the product are 'categorical product' and 'direct product'.

1.1.20 Coproduct. For a family $\{M^u\}_{u\in U}$ in $\mathcal{M}(R)$, the coproduct $\coprod_{u\in U} M^u$ is the submodule

$$\left\{ (m^u)_{u\in U} \in \prod_{u\in U} M^u \;\middle|\; m^u = 0 \text{ for all but finitely many } u \in U \right\}$$

of the product, together with the injections $\varepsilon^u\colon M^u \rightarrowtail \coprod_{u\in U} M^u$. Every element of $\coprod_{u\in U} M^u$ has the form $\sum_{u\in U} \varepsilon^u(m^u)$ for a unique family $(m^u)_{u\in U}$ with $m^u = 0$ for all but finitely many $u \in U$. Given a family of homomorphisms $\{\alpha^u\colon M^u \to N\}_{u\in U}$, the map α defined by $\sum_{u\in U} \varepsilon^u(m^u) \mapsto \sum_{u\in U} \alpha^u(m^u)$ is the unique homomorphism that makes the following diagram commutative

$$\begin{tikzcd} M^u \arrow[r, tail, "\varepsilon^u"] \arrow[d, "\alpha^u"'] & \coprod_{u\in U} M^u \arrow[dl, dotted, "\alpha"] \\ N & \end{tikzcd}$$

for every $u \in U$. For a family $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ of homomorphisms the coproduct $\coprod_{u\in U} \alpha^u\colon \coprod_{u\in U} M^u \to \coprod_{u\in U} N^u$ is given by $\sum_{u\in U} \varepsilon^u(m^u) \mapsto \sum_{u\in U} \alpha^u(m^u)$.

If one has $M^u = M$ for every $u \in U$, then the coproduct $\coprod_{u\in U} M^u$ is denoted $M^{(U)}$ and called the *U-fold coproduct* of M.

Remark. Other names for the coproduct are 'categorical sum' and 'direct sum'; we reserve the latter for the iterated biproduct; see 1.1.14.

1.1.21 Direct Sum. If U is a finite set, then the product and the coproduct of a family $\{M^u\}_{u\in U}$ in $\mathcal{M}(R)$ coincide, and this module $\prod_{u\in U} M^u = \coprod_{u\in U} M^u$ is the iterated biproduct $M = \bigoplus_{u\in U} M^u$. Per 1.1.14 one refers to M as the *direct sum* of the family $\{M^u\}_{u\in U}$ and to each module M^u as a *direct summand* of M.

1.1.22. Let $\{M^u\}_{u\in U}$ be a family of objects in a $\Bbbk$-linear category $\mathcal{U}$. Notice that for every finite subset U' of U one has $\prod_{u\in U} M^u \cong (\bigoplus_{u\in U'} M^u) \oplus \prod_{u\in U\setminus U'} M^u$ and $\coprod_{u\in U} M^u \cong (\bigoplus_{u\in U'} M^u) \oplus \coprod_{u\in U\setminus U'} M^u$.

1.1.23 Definition. Let M be an R-module and $\{M^u\}_{u\in U}$ a family of submodules of M. The *sum* of the family, written $\sum_{u\in U} M^u$, is the image of the homomorphism,

$$\coprod_{u\in U} M^u \longrightarrow M \qquad \text{given by} \qquad \sum_{u\in U} \varepsilon^u(m^u) \longmapsto \sum_{u\in U} m^u \ ;$$

if it is injective, then the family $\{M^u\}_{u\in U}$ is called *independent*.

Remark. In some texts, an independent family of submodules is said to form an 'internal direct sum'. We have, though, reserved 'direct sum' to mean a finite (co)product; see 1.1.14.

1.1.24 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of submodules of an R-module M.*

(a) *The following conditions are equivalent.*

  (i) *The family $\{M^u\}_{u\in U}$ is independent.*
  (ii) $(\sum_{u\in U\setminus\{w\}} M^u) \cap M^w = 0$ *for every $w \in U$.*
  (iii) $(\sum_{u\in V} M^u) \cap M^w = 0$ *for every finite subset $V \subseteq U$ and $w \in U \setminus V$.*

(b) *If U is infinite, $M^u \neq 0$ holds for all $u \in U$, and the family $\{M^u\}_{u\in U}$ is independent, then M is neither Artinian nor Noetherian.*

Proof. (a): The implication (ii) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (ii): If $\{M^u\}_{u\in U}$ is independent, then there is by 1.1.23 an isomorphism $\coprod_{u\in U} M^u \cong \sum_{u\in U} M^u$ that identifies the submodule $\varepsilon^u(M^u)$ of $\coprod_{u\in U} M^u$ with the submodule M^u of $\sum_{u\in U} M^u$. For every element $w \in U$ one evidently has $(\sum_{u\in U\setminus\{w\}} \varepsilon^u(M^u)) \cap \varepsilon^w(M^w) = 0$ and, therefore, $(\sum_{u\in U\setminus\{w\}} M^u) \cap M^w = 0$.

(iii) $\Rightarrow$ (i): Every element $x \in \coprod_{u\in U} M^u$ has the form $x = \sum_{u\in X} \varepsilon^u(m^u)$ for some finite subset $X \subseteq U$ and $m^u \in M^u$. If x belongs to the kernel of the homomorphism in 1.1.23, then one has $\sum_{u\in X} m^u = 0$ in $\sum_{u\in U} M^u$. Thus, for every $w \in X$ the element $m^w = -\sum_{u\in X\setminus\{w\}} m^u$ belongs to the module $(\sum_{u\in X\setminus\{w\}} M^u) \cap M^w$, which is zero by assumption. Hence $m^w = 0$ holds for every $w \in X$ and thus $x = 0$, so the homomorphism in 1.1.23 is injective.

(b): Let $\{u_1, u_2, \dots\}$ be a countable subset of U. For every $s \geqslant 1$ set $X^s = \sum_{i\leqslant s} M^{u_i}$ and $Y^s = \sum_{i\geqslant s} M^{u_i}$. If $X^s = X^{s+1}$ holds, then one has $M^{u_{s+1}} \subseteq \sum_{i\leqslant s} M^{u_i}$ and thus $(\sum_{i\leqslant s} M^{u_i}) \cap M^{u_{s+1}} = M^{u_{s+1}} \neq 0$, which contradicts the assumption that the family $\{M^u\}_{u\in U}$ is independent. Now $X^1 \subset X^2 \subset X^3 \subset \cdots$ is a strictly increasing family of submodules of M, so M is not Noetherian. If $Y^s = Y^{s+1}$ holds, then one has $M^{u_s} \subseteq \sum_{i\geqslant s+1} M^{u_i}$ and thus $(\sum_{i\geqslant s+1} M^{u_i}) \cap M^{u_s} = M^{u_s} \neq 0$, which contradicts the assumption that the family is independent. Now $Y^1 \supset Y^2 \supset Y^3 \supset \cdots$ is a strictly decreasing family of submodules of M, whence M is not Artinian. □

An Abelian group M that is both an R-module and an S-module is called an *R–S-bimodule* if the two module structures are compatible, i.e. $s(rm) = r(sm)$ holds for all $r \in R$, $s \in S$, and $m \in M$. A homomorphism of R–S-bimodules is a homomorphism of Abelian groups that is both R- and S-linear.

The convention to identify right modules with left modules over the opposite ring is convenient for abstract considerations. However, in concrete computations with elements in, say, an R–S^{o}-bimodule it often adds clarity to write the S-action on the right; the bimodule condition then reads $(rm)s = r(ms)$.

An R–R^{o}-bimodule is called *symmetric* if $rm = mr$ holds for all $r \in R$ and $m \in M$. If R is commutative, then every R-module has a canonical structure of a symmetric R–R-bimodule. When no other convention is specified, modules over a commutative ring R are tacitly considered to be symmetric bimodules—this is discussed further with the passage to commutative Noetherian rings in Chap. 12.

As R and S are $\Bbbk$-algebras, an R–S^{o}-bimodule is a $\Bbbk$–$\Bbbk$-bimodule. We assume that this structure is symmetric; i.e. we only consider $\Bbbk$-symmetric R–S^{o}-bimodules. In particular, an R-module is an R–$\Bbbk$-bimodule, and an S^{o}-module is a $\Bbbk$–S^{o}-bimodule.

1.1.25 Example. For numbers $m, n \in \mathbb{N}$ let $\mathrm{M}_{m\times n}(R)$ denote the R–R^{o}-bimodule of $m \times n$-matrices with entries from R. Set $M = \mathrm{M}_{1\times n}(R)$ and $N = \mathrm{M}_{n\times 1}(R)$, and let Q denote the ring $\mathrm{M}_{n\times n}(R)$. Now M is an R–Q^{o}-bimodule and N is a Q–R^{o}-bimodule.

1.1.26. There is an equivalence of $\Bbbk$-linear Abelian categories,

$$\{R\text{–}S^{\mathrm{o}}\text{-bimodules and their homomorphisms}\} \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{M}(R \otimes_{\Bbbk} S^{\mathrm{o}}) \,.$$

The functor F assigns to an R–S^{o}-bimodule M the $R \otimes_{\Bbbk} S^{\mathrm{o}}$-module with action given by $(r \otimes s)m = rms$. Conversely, G assigns to an $R \otimes_{\Bbbk} S^{\mathrm{o}}$-module M the R–S^{o}-bimodule with R-action given by $rm = (r \otimes 1)m$ and right S-action given by $ms = (1 \otimes s)m$. Homomorphisms are treated analogously. Notice that G assigns to an $R \otimes_{\Bbbk} S^{\mathrm{o}}$-module the R- and S^{o}-module structures obtained by restriction of scalars along the canonical ring homomorphisms $R \to R \otimes_{\Bbbk} S^{\mathrm{o}}$ and $S^{\mathrm{o}} \to R \otimes_{\Bbbk} S^{\mathrm{o}}$.

1.1.27 Definition. Let $\mathcal{M}(R\text{–}S^{\mathrm{o}})$ denote the category $\mathcal{M}(R \otimes_{\Bbbk} S^{\mathrm{o}})$.

Per 1.1.26 the category $\mathcal{M}(R\text{–}S^{\mathrm{o}})$ is naturally identified with the category of R–S^{o}-bimodules and their homomorphisms.

1.1.28 Enveloping Algebra. The ring $R \otimes_{\Bbbk} R^{\mathrm{o}}$ is called the *enveloping algebra* of R and denoted R^{e}. Per 1.1.26 the category of R–R^{o}-bimodules is equivalent to the category of R^{e}-modules.

1.1.29 Restriction of scalars II. Let $\varphi\colon R \to S$ be a ring homomorphism. Restriction of scalars along the composites of φ and the canonical ring homomorphisms $S \to S \otimes_{\Bbbk} T^{\mathrm{o}}$ and $S^{\mathrm{o}} \to (S \otimes_{\Bbbk} T^{\mathrm{o}})^{\mathrm{o}} = T \otimes_{\Bbbk} S^{\mathrm{o}}$ yield functors $\mathcal{M}(S\text{–}T^{\mathrm{o}}) \to \mathcal{M}(R)$ and $\mathcal{M}(T\text{–}S^{\mathrm{o}}) \to \mathcal{M}(R^{\mathrm{o}})$. When we refer to, say, the ring $S \otimes_{\Bbbk} T^{\mathrm{o}}$ or a module over it as an R-module, it means that the former functor is applied.

CAVEAT. An endomorphism $\varphi\colon R \to R$ induces two R–R^{o}-bimodule structures on R; if φ is not the identity, then neither is symmetric—not even if R is commutative.

The set $\mathrm{Hom}_R(M,N)$ of R-linear maps between R-modules is a $\Bbbk$-module but it has, in general, no built-in R-module structure. The reason is, so to say, that the R-actions on M and N are consumed by the R-linearity of the maps. Similarly, the R-balancedness consumes the R^{o}- and R-actions on the factors in a tensor product $M \otimes_R N$ and leaves only a $\Bbbk$-module. If R is commutative, then one can take $\Bbbk = R$. At work here is the tacit assumption that a module over a commutative ring R is a symmetric R–R-bimodule. Also in a non-commutative setting, access to richer module structures on hom-sets and tensor products is via bimodules.

1.1.30 Addendum (to 1.1.7). If M is an R–Q^{o}-bimodule and N an R–S^{o}-bimodule, then the $\Bbbk$-module $\mathrm{Hom}_R(M,N)$ has a Q–S^{o}-bimodule structure given by

$$(q\varphi)(m) \;=\; \varphi(mq) \qquad\text{and}\qquad (\varphi s)(m) \;=\; (\varphi(m))s\,.$$

Moreover, if $\alpha\colon M \longrightarrow M'$ is a homomorphism of R–Q^{o}-bimodules, and $\beta\colon N \longrightarrow N'$ is a homomorphism of R–S^{o}-bimodules, then $\mathrm{Hom}_R(\alpha,\beta)$, as defined in 1.1.7, is a homomorphism of Q–S^{o}-bimodules. Thus, there is an induced $\Bbbk$-bilinear functor,

$$\mathrm{Hom}_R(-,-) : \mathcal{M}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{M}(R\text{–}S^{\mathrm{o}}) \;\longrightarrow\; \mathcal{M}(Q\text{–}S^{\mathrm{o}})\,.$$

1.1.31 Example. Let $R \longrightarrow S$ be a ring homomorphism and consider S as an R–S^{o}-bimodule; see 1.1.29. Now $\mathrm{Hom}_R(S,-)$ is a functor from $\mathcal{M}(R)$ to $\mathcal{M}(S)$.

1.1.32 Example. Set $Q = \mathrm{M}_{n\times n}(R)$ and $M = \mathrm{M}_{1\times n}(R)$ as in 1.1.25. An application of 1.1.30 with $S = \Bbbk$ yields a functor $\mathrm{Hom}_R(M,-)\colon \mathcal{M}(R) \to \mathcal{M}(Q)$. Another application of 1.1.30, this time with $S = \mathrm{M}_n(R) = Q$, shows that $\mathrm{Hom}_R(M,-)$ is a functor from $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ to $\mathcal{M}(Q\text{–}Q^{\mathrm{o}})$. In particular, $\mathrm{Hom}_R(M,M)$ has the structure of a Q–Q^{o}-bimodule.

1.1.33 Addendum (to 1.1.9). If M is a Q–R^{o}-bimodule and N is an R–S^{o}-bimodule, then the $\Bbbk$-module $M \otimes_R N$ has a Q–S^{o}-bimodule structure given by

$$q(m \otimes n) \;=\; (qm) \otimes n \qquad\text{and}\qquad (m \otimes n)s \;=\; m \otimes (ns)\,.$$

Moreover, if $\alpha\colon M \longrightarrow M'$ is a homomorphism of Q–R^{o}-bimodules, and $\beta\colon N \longrightarrow N'$ is a homomorphism of R–S^{o}-bimodules, then $\alpha \otimes \beta$, as defined in 1.1.9, is a homomorphism of Q–S^{o}-bimodules. Thus, there is an induced $\Bbbk$-bilinear functor,

$$-\otimes_R - : \mathcal{M}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{M}(R\text{–}S^{\mathrm{o}}) \;\longrightarrow\; \mathcal{M}(Q\text{–}S^{\mathrm{o}})\,.$$

1.1.34 Example. Let $R \longrightarrow S$ be a ring homomorphism and consider S as an S–R^{o}-bimodule; see 1.1.29. Now $S \otimes_R -$ is a functor from $\mathcal{M}(R)$ to $\mathcal{M}(S)$.

1.1.35 Example. Set $Q = \mathrm{M}_{n\times n}(R)$, $M = \mathrm{M}_{1\times n}(R)$, and $N = \mathrm{M}_{n\times 1}(R)$ as in 1.1.25. An application of 1.1.33 with $S = \Bbbk$ yields a functor $N \otimes_R -\colon \mathcal{M}(R) \to \mathcal{M}(Q)$. Another application, this time with $S = \mathrm{M}_{n\times n}(R) = Q$, shows that $N \otimes_R -$ is a functor from $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ to $\mathcal{M}(Q\text{–}Q^{\mathrm{o}})$. In particular, $N \otimes_R M$ is a Q–Q^{o}-bimodule.

EXACTNESS

Though we are not concerned with abstract Abelian categories, the language is useful for the following reason. The convention that every functor is covariant forces one to consider, for example, Hom_R as a functor on the product category $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R)$ which is Abelian but not a module category.

REMARK. The class of all Abelian categories is closed under products and opposites of categories, that is, if $\mathcal{U}$ and $\mathcal{V}$ are Abelian categories, then so are $\mathcal{U} \times \mathcal{V}$ and $\mathcal{U}^{\mathrm{op}}$. The class of module categories—i.e. categories that are equivalent to $\mathcal{M}(R)$ for some ring R—is also closed under products; indeed $\mathcal{M}(R) \times \mathcal{M}(S)$ is equivalent to $\mathcal{M}(R \times S)$; see E 1.1.21. However, $\mathcal{M}(R)^{\mathrm{op}}$ is not a module category; see E 1.1.23. Yet, by the Freyd–Mitchell Embedding Theorem, see [102, 4.4 and 7.1] and [184], every Abelian category is, in fact, a full subcategory of a module category.

1.1.36 Split Exact Sequences. An exact sequence $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ in $\mathcal{M}(R)$ is called *split* if it satisfies the following equivalent conditions.

(i) There exist homomorphisms $\varrho \colon M \to M'$ and $\sigma \colon M'' \to M$ such that one has
$$\varrho\alpha' = 1^{M'}, \quad \alpha'\varrho + \sigma\alpha = 1^{M}, \quad \text{and} \quad \alpha\sigma = 1^{M''}.$$

(ii) There exists a homomorphism $\varrho \colon M \to M'$ such that $\varrho\alpha' = 1^{M'}$.

(iii) There exists a homomorphism $\sigma \colon M'' \to M$ such that $\alpha\sigma = 1^{M''}$.

(iv) The sequence is isomorphic to $0 \longrightarrow M' \xrightarrow{\varepsilon} M' \oplus M'' \xrightarrow{\varpi} M'' \longrightarrow 0$, where ε and ϖ are the injection and the projection, respectively.

Moreover, if $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ is split exact, then also the sequence $0 \longrightarrow M'' \xrightarrow{\sigma} M \xrightarrow{\varrho} M' \longrightarrow 0$, where ϱ and σ are as in part (i), is split exact.

REMARK. There exist non-split short exact sequences of $\mathbb{Z}$-modules $0 \to M' \to M \to M'' \to 0$ with $M \cong M' \oplus M''$, via some isomorphism. By a result of Miyata [185] this phenomenon can not occur for finitely generated modules over a commutative Noetherian ring.

The definitions 1.1.1 and 1.1.36 of (split) exact sequences in $\mathcal{M}(R)$ make sense in any Abelian category. A (split) exact sequence in $\mathcal{M}(R)^{\mathrm{op}}$ is just a (split) exact sequence in $\mathcal{M}(R)$ with the arrows reversed.

1.1.37 Definition. Let M and N be R-modules. An *extension of M by N* is an exact sequence of R-modules of the form $0 \to N \to X \to M \to 0$; it is called a *trivial extension* if the sequence is split.

REMARK. The terminology in 1.1.37 may appear awkward as X contains N, not M; a justification is provided in the Remark after 7.3.30. While this terminology is standard, there are authors, for example Rotman [219], who call a sequence $0 \to N \to X \to M \to 0$ an 'extension of N by M'.

1.1.38. Let $\mathrm{F} \colon \mathcal{U} \to \mathcal{V}$ be an additive functor between Abelian categories. For every split exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$ the induced sequence in $\mathcal{V}$, $0 \longrightarrow \mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'') \longrightarrow 0$, is split exact.

1.1.39 Example. The assignments $M \mapsto R \oplus M$ and $\alpha \mapsto 1^R \oplus \alpha$ define a functor $\mathrm{F} \colon \mathcal{M}(R) \to \mathcal{M}(R)$, which is not additive, as $\mathrm{F}(0)$ is non-zero; cf. 1.1.18.

A frequently used consequence of 1.1.38 is that additive functors preserve biproducts and hence direct sums; cf. 1.1.13 and 1.1.14. In particular, there is a canonical isomorphism $\mathrm{F}(\oplus_{u\in U} M^u) \cong \oplus_{u\in U} \mathrm{F}(M^u)$ for an additive functor F and a finite family of objects $\{M^u\}_{u\in U}$.

1.1.40 Half Exactness. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is called *half exact* if for every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$, the sequence $\mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'')$ in $\mathcal{V}$ is exact. A half exact functor is additive.

It is not too hard to construct an additive functor that is not half exact. An example of such a functor comes up naturally in 11.1.32. See also E 2.2.4.

The Hom functor 1.1.7 is left exact in the following sense.

1.1.41 Left Exactness. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is called *left exact* if it satisfies the following equivalent conditions.

(*i*) For every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$, the sequence $0 \longrightarrow \mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'')$ in $\mathcal{V}$ is exact.

(*ii*) For every (left) exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M''$ in $\mathcal{U}$, the sequence $0 \longrightarrow \mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'')$ in $\mathcal{V}$ is exact.

Remark. If $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ is an additive left exact functor that preserves products, then there is an R–S^{o}-bimodule N and a natural isomorphism $\mathrm{G} \cong \mathrm{Hom}_R(-, N)$. If $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(\mathbb{Z})$ is an additive left exact functor that preserves limits, then there exists an R-module M and a natural isomorphism $\mathrm{F} \cong \mathrm{Hom}_R(M, -)$; see Watts [252].

The tensor product functor 1.1.9 is right exact in the following sense.

1.1.42 Right Exactness. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is called *right exact* if it satisfies the following equivalent conditions.

(*i*) For every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$, the sequence $\mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'') \longrightarrow 0$ in $\mathcal{V}$ is exact.

(*ii*) For every (right) exact sequence $M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$, the sequence $\mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'') \longrightarrow 0$ in $\mathcal{V}$ is exact.

Remark. If $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ is an additive right exact functor that preserves coproducts, then there exists an S–R^{o}-bimodule M and a natural isomorphism $\mathrm{F} \cong M \otimes_R -$; see Watts [252].

1.1.43 Exactness. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is called *exact* if it satisfies the following equivalent conditions.

(*i*) For every short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ in $\mathcal{U}$, the sequence $0 \longrightarrow \mathrm{F}(M') \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M'') \longrightarrow 0$ in $\mathcal{V}$ is exact.

(*ii*) F preserves exactness of sequences.

(*iii*) F is left exact and right exact.

It is not too hard to construct a half exact functor that is neither left nor right exact. See 2.2.22 and E 8.2.2 for examples of such a functors.

FAITHFULNESS

An additive functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between additive categories is faithful if for every morphism α in $\mathcal{U}$ one has $\mathrm{F}(\alpha) = 0$ in $\mathcal{V}$ only if $\alpha = 0$ in $\mathcal{U}$. In that case it follows that for objects M in $\mathcal{U}$ one has $\mathrm{F}(M) \cong 0$ in $\mathcal{V}$ only if $M \cong 0$ in $\mathcal{U}$.

1.1.44 Faithful Exactness. An additive functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is called *faithfully exact* if it satisfies the following equivalent conditions.

(i) F is exact and faithful.

(ii) F is exact and for every M in $\mathcal{U}$ one has $\mathrm{F}(M) \cong 0$ in $\mathcal{V}$ only if $M \cong 0$ in $\mathcal{U}$.

Faithful functors have convenient cancellation properties.

1.1.45. Let $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ be an additive functor between Abelian categories. If F is faithfully exact and $M' \to M \to M''$ is a sequence in $\mathcal{U}$, then exactness of the induced sequence $\mathrm{F}(M') \to \mathrm{F}(M) \to \mathrm{F}(M'')$ in $\mathcal{V}$ implies exactness of $M' \to M \to M''$.

Let $\mathrm{G}\colon \mathcal{V} \to \mathcal{W}$ be a faithfully exact functor between Abelian categories. The composite $\mathrm{GF}\colon \mathcal{U} \to \mathcal{W}$ is then (faithfully) exact if and only if F is (faithfully) exact.

Recall that a functor is conservative if it reflects isomorphisms.

1.1.46 Conservative Functor. A faithful functor reflects monomorphisms and epimorphisms. Consequently, if a category $\mathcal{U}$ is *balanced*, meaning that a morphism in $\mathcal{U}$ is an isomorphism if (and only if) it is both a monomorphism and an epimorphism, then a faithful functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is conservative. Every Abelian category is balanced, and by E.24 so is every triangulated category. Thus, if $\mathcal{U}$ is Abelian or triangulated, then every faithful functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is conservative. As demonstrated in 6.4.37 and 7.3.37, a conservative functor need not be faithful.

1.1.47 Example. Given a ring homomorphism $\varphi\colon R \to S$, the restriction of scalars functors $\mathrm{res}^S_R\colon \mathcal{M}(S) \to \mathcal{M}(R)$ and $\mathrm{res}^{S^{\mathrm{o}}}_{R^{\mathrm{o}}}\colon \mathcal{M}(S^{\mathrm{o}}) \to \mathcal{M}(R^{\mathrm{o}})$ from 1.1.12 are easily seen to be faithfully exact and conservative; the latter property also follows from the former in view of 1.1.46. In particular, $\mathrm{res}^R_{\Bbbk}\colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$ has these properties.

EXERCISES

E 1.1.1 (Cf. 1.1.47) Let $\varphi\colon R \to S$ be a ring homomorphism. Show that the restriction functor $\mathrm{res}^S_R\colon \mathcal{M}(S) \to \mathcal{M}(R)$ is faithfully exact and conservative.

E 1.1.2 Show that a morphism in $\mathcal{M}(R)$ is a monomorphism if and only if it is injective, and that it is an epimorphism if and only if it is surjective.

E 1.1.3 Show that the category $\mathcal{M}(R)$ is Abelian.

E 1.1.4 Show that in the category of unital rings the embedding $\mathbb{Z} \rightarrowtail \mathbb{Q}$ is both a monomorphism and an epimorphism though not an isomorphism.

E 1.1.5 Consider a commutative diagram in $\mathcal{M}(R)$,

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[d, "\varphi''"] \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' . \end{tikzcd}$$

Assuming that the lower row is exact and $\alpha\alpha' = 0$ holds, show that there is a unique homomorphism $\varphi' \colon M' \to N'$ such that one has $\beta'\varphi' = \varphi\alpha'$.

E 1.1.6 Consider a commutative diagram in $\mathcal{M}(R)$,

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\psi'"] & M \arrow[r, "\alpha"] \arrow[d, "\psi"] & M'' \arrow[r] & 0 \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' . \end{tikzcd}$$

Assuming that the upper row is exact and $\beta\beta' = 0$ holds, show that there is a unique homomorphism $\psi'' \colon M'' \to N''$ such that one has $\psi''\alpha = \beta\psi$.

E 1.1.7 Let

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[d, "\varphi''"] \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \end{tikzcd}$$

be a commutative diagram in $\mathcal{M}(R)$ with exact rows. (a) Show that if α is surjective, then the sequence $\operatorname{Coker}\varphi' \to \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$ is exact. (b) Show that if β' is injective, then the sequence $\operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi \to \operatorname{Ker}\varphi''$ is exact.

E 1.1.8 Determine the inverse of the map $\operatorname{Hom}_R(R/\mathfrak{a}, M) \to (0 :_M \mathfrak{a})$ given in 1.1.8.

E 1.1.9 Determine the inverse of the map $R/\mathfrak{b} \otimes_R M \to M/\mathfrak{b}M$ given in 1.1.10.

E 1.1.10 Let $\mathfrak{b}$ be a right ideal in R and M an R-module. Show that for every submodule $K \subseteq M$ there is an isomorphism of $\Bbbk$-modules $R/\mathfrak{b} \otimes_R M/K \cong M/(\mathfrak{b}M + K)$.

E 1.1.11 Let $\{M^u\}_{u\in U}$ be a family of R-modules and set $M = \coprod_{u\in U} M^u$. Show that the family $\{\varepsilon^u(M^u)\}_{u\in U}$ of submodules of M is independent.

E 1.1.12 Let $\{\alpha^u \colon M^u \to N^u\}_{u\in U}$ be a family of homomorphisms in $\mathcal{M}(R)$.

(a) Show that $\prod_{u\in U} \alpha^u$, as defined in 1.1.19, is the unique homomorphism α that makes the following diagram commutative for every $u \in U$.

$$\begin{tikzcd} \prod_{u\in U} M^u \arrow[r, dotted, "\alpha"] \arrow[d, two heads] & \prod_{u\in U} N^u \arrow[d, two heads] \\ M^u \arrow[r, "\alpha^u"] & N^u \end{tikzcd}$$

(b) Show that $\coprod_{u\in U} \alpha^u$, as defined in 1.1.20, is the unique homomorphism α that makes the following diagram commutative for every $u \in U$.

$$\begin{tikzcd} M^u \arrow[r, tail] \arrow[d, "\alpha^u"'] & \coprod_{u\in U} M^u \arrow[d, dotted, "\alpha"] \\ N^u \arrow[r, tail] & \coprod_{u\in U} N^u \end{tikzcd}$$

E 1.1.13 Show that R is an R–R^{o}-bimodule, but only a symmetric one if R is commutative.

E 1.1.14 Show that the enveloping algebra $(R^{\mathrm{o}})^{\mathrm{e}}$ is the opposite ring of R^{e} and isomorphic to R^{e}.

E 1.1.15 Show that if an R-module is a sum of simple submodules, then it is semi-simple.

E 1.1.16 For Q and M as in 1.1.32 decide if $\operatorname{Hom}_R(M,M)$ and Q are isomorphic Q–Q^{o}-bimodules.

E 1.1.17 For Q, M, and N as in 1.1.35 decide if $N \otimes_R M$ and Q are isomorphic Q–Q^{o}-bimodules.

E 1.1.18 Let M be an R-module. (a) Show that $S = \operatorname{Hom}_R(M,M)$ is a $\Bbbk$-algebra with multiplication given by composition of homomorphisms. (b) Show that M is an S-module.

E 1.1.19 Show that the endomorphism algebra $\operatorname{Hom}_R(R,R)$, see E 1.1.18, is isomorphic to R^{o}.

E 1.1.20 In the algebra $M = \mathrm{M}_{2\times 2}(\mathbb{R})$, consider the subset

$$C = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} .$$

(a) Show that C is a subring isomorphic to $\mathbb{C}$. (b) Show that the $\mathbb{C}$-actions afforded by the canonical ring homomorphism $\mathbb{C} \to M$ make M a non-symmetric $\mathbb{C}$–$\mathbb{C}$-bimodule.

E 1.1.21 Consider the functors $\mathrm{F}\colon \mathcal{M}(R) \times \mathcal{M}(S) \to \mathcal{M}(R \times S)$ given by $\mathrm{F}(M, N) = M \times N$ and $\mathrm{G}\colon \mathcal{M}(R \times S) \to \mathcal{M}(R) \times \mathcal{M}(S)$ given by $\mathrm{G}(X) = ((1,0)X, (0,1)X)$. Show that F and G yield an equivalence of categories.

E 1.1.22 Let U be a set with $\operatorname{card} U \geqslant \max\{\operatorname{card} R, \aleph_0\}$. Show that $R^{(U)}$ has cardinality strictly less than R^U and conclude that there exists no surjective maps $R^{(U)} \to R^U$.

E 1.1.23 Consider the category $\mathcal{V} = \mathcal{M}(R)^{\mathrm{op}}$. Coproducts in $\mathcal{M}(R)$ yields products in $\mathcal{V}$, denoted $\prod^{\mathcal{V}}$, and products in $\mathcal{M}(R)$ yield coproducts in $\mathcal{V}$, denoted $\coprod^{\mathcal{V}}$. (a) Show that for every family $\{X_u\}_{u\in U}$ of objects in $\mathcal{V}$ there exists an epimorphism $\coprod^{\mathcal{V}}_{u\in U} X_u \to \prod^{\mathcal{V}}_{u\in U} X_u$. (b) Use E 1.1.22 to show that unless R is the zero ring, there exists no ring S such that $\mathcal{V}$ is equivalent to $\mathcal{M}(S)$.

E 1.1.24 Show that the sequence of $\mathbb{Z}$-modules $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \xrightarrow{\beta} (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \longrightarrow 0$, with $\alpha(m) = (2m, 0, 0, \dots)$ and $\beta(n, x_1, x_2, \dots) = ([n]_{2\mathbb{Z}}, x_1, x_2, \dots)$, is exact but not split.

E 1.1.25 Show that the $\mathbb{Z}$-submodule $\mathbb{Z}^{(\mathbb{N})}$ of $\mathbb{Z}^{\mathbb{N}}$ is not a direct summand.

E 1.1.26 (a) Show that a half exact functor between Abelian categories is additive. (b) For every $\mathbb{Z}$-module M set $\mathrm{F}(M) = 2M$, and for every homomorphism $\alpha\colon M \to N$ of $\mathbb{Z}$-modules let $\mathrm{F}(\alpha)\colon 2M \to 2N$ be the restriction of α. Show that $\mathrm{F}\colon \mathcal{M}(\mathbb{Z}) \to \mathcal{M}(\mathbb{Z})$ is an additive functor but not half exact.

E 1.1.27 (Cf. 1.1.43) Show that the three conditions on $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ in 1.1.43 in are equivalent.

E 1.1.28 Show that a functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ between Abelian categories is right exact if and only if the opposite functor $\mathrm{F}^{\mathrm{op}}\colon \mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is left exact.

E 1.1.29 Let $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ and $\mathrm{G}\colon \mathcal{V} \to \mathcal{W}$ be functors between Abelian categories. (a) Show that if both functors are right exact, then GF is right exact. (b) Show that if both functors are left exact, then GF is left exact. (c) Show that if one functor is left exact and the other is right exact, then GF need not be half exact.

E 1.1.30 Let M be an R–R^{o}-bimodule. (a) Show that $M \otimes_{\Bbbk} M$ has two compatible structures as a module over R^{e}, an "inner" and an "outer". (b) Show that the assignments $M \mapsto \operatorname{Hom}_{R^e}(R, M \otimes_{\Bbbk} M)$ and $\alpha \mapsto \operatorname{Hom}_{R^e}(R, \alpha \otimes_{\Bbbk} \alpha)$ define an endofunctor on $\mathcal{M}(R^{\mathrm{e}})$ that is not additive.

E 1.1.31 Let M and N be R-modules. (a) Show that if $\operatorname{Hom}_R(M, -)$ and $\operatorname{Hom}_R(N, -)$ are naturally isomorphic, then M and N are isomorphic. (b) Show that if $\operatorname{Hom}_R(-, M)$ and $\operatorname{Hom}_R(-, N)$ are naturally isomorphic, then M and N are isomorphic.

1.2 Standard Isomorphisms

Synopsis. Unitor; counitor; commutativity; associativity; swap; adjunction; base change; cobase change.

Under suitable assumptions about bimodule structures, it makes sense to consider composites like $\operatorname{Hom}(X \otimes M, N)$ and $\operatorname{Hom}(M, \operatorname{Hom}(X, N))$, and they turn out to be isomorphic. At work here is adjunction, one of several natural isomorphisms of composites of Hom and tensor product functors, which are the focus of this section.

We start by recalling that both functors $R \otimes_R -$ and $\operatorname{Hom}_R(R, -)$ are naturally isomorphic to the identity functor on $\mathcal{M}(R)$.

UNITOR AND COUNITOR

1.2.1. Let M be an R-module. There is an isomorphism of R-modules,

$$\mu_R^M : R \otimes_R M \longrightarrow M \qquad \text{given by} \qquad \mu_R^M(r \otimes m) = rm\,,$$

called the *unitor*. It is natural in M, and if M is an R–S^{o}-bimodule, then it is an isomorphism of R–S^{o}-bimodules.

1.2.2. Let M be an R-module. There are isomorphisms of R-modules,

$$\epsilon_R^M : M \longrightarrow \operatorname{Hom}_R(R, M) \qquad \text{given by} \qquad \epsilon_R^M(m)(r) = rm\,,$$

called the *counitor*. It is natural in M, and if M is an R–S^{o}-bimodule, then it is an isomorphism of R–S^{o}-bimodules.

The tensor product behaves as one would expect of a multiplicative product. Being additive, the tensor product distributes over direct sums, and the first two standard isomorphisms below show that it is commutative and associative.

COMMUTATIVITY

1.2.3 Proposition. *Let M be an R^{o}-module and N an R-module. The commutativity map,*

$$\upsilon^{MN} : M \otimes_R N \longrightarrow N \otimes_{R^{\mathrm{o}}} M \qquad \textit{given by} \qquad \upsilon^{MN}(m \otimes n) = n \otimes m\,,$$

is an isomorphism of $\Bbbk$-modules, and it is natural in M and N. Moreover, if M is in $\mathcal{M}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{M}(R\text{–}S^{\mathrm{o}})$, then υ^{MN} is an isomorphism in $\mathcal{M}(Q\text{–}S^{\mathrm{o}})$.

PROOF. For every R^{o}-module M and every R-module N, the map π from $M \times N$ to $N \otimes_{R^{\mathrm{o}}} M$ given by $(m, n) \mapsto n \otimes m$ is $\Bbbk$-bilinear and middle R-linear. Indeed,

$$\begin{aligned}
\pi(m, n + n') &= (n + n') \otimes m = n \otimes m + n' \otimes m = \pi(m, n) + \pi(m, n') \\
\pi(m + m', n) &= n \otimes (m + m') = n \otimes m + n \otimes m' = \pi(m, n) + \pi(m', n) \\
\pi(mr, n) &= n \otimes mr = rn \otimes m = \pi(m, rn) \\
\pi(mx, n) &= \pi(m, xn) = xn \otimes m = x(n \otimes m) = x\pi(m, n)
\end{aligned}$$

holds for all $m \in M$, $n \in N$, $r \in R$, and $x \in \Bbbk$. Thus, υ^{MN} is a homomorphism of $\Bbbk$-modules.

Let $\alpha\colon M \to M'$ be a homomorphism of R^{o}-modules and $\beta\colon N \to N'$ a homomorphism of R-modules. The diagram

$$\begin{tikzcd}
M \otimes_R N \arrow[r, "\upsilon^{MN}"] \arrow[d, "\alpha\otimes\beta"'] & N \otimes_{R^{\mathrm{o}}} M \arrow[d, "\beta\otimes\alpha"] \\
M' \otimes_R N' \arrow[r, "\upsilon^{M'N'}"] & N' \otimes_{R^{\mathrm{o}}} M'
\end{tikzcd}$$

is commutative, as one has

$$\begin{aligned}(\upsilon^{M'N'} \circ (\alpha \otimes_R \beta))(m \otimes n) &= \upsilon^{M'N'}(\alpha(m) \otimes \beta(n)) \\ &= \beta(n) \otimes \alpha(m) \\ &= (\beta \otimes_{R^{\mathrm{o}}} \alpha)(n \otimes m) \\ &= ((\beta \otimes_{R^{\mathrm{o}}} \alpha) \circ \upsilon^{MN})(m \otimes n)\end{aligned}$$

for all $m \in M$ and all $n \in N$. Thus, υ is a natural transformation of functors from $\mathcal{M}(R^{\mathrm{o}}) \times \mathcal{M}(R)$ to $\mathcal{M}(\Bbbk)$.

For M in $\mathcal{M}(R^{\mathrm{o}})$ and N in $\mathcal{M}(R)$ the map from $N \otimes_{R^{\mathrm{o}}} M$ to $M \otimes_R N$ given by $n \otimes m \mapsto m \otimes n$ is an inverse of υ^{MN}, so υ^{MN} is an isomorphism of $\Bbbk$-modules.

If M is a Q–R^{o}-bimodule and N is an R–S^{o}-bimodule, then $M \otimes_R N$ and $N \otimes_{R^{\mathrm{o}}} M$ are Q–S^{o}-bimodules. The computation

$$\upsilon^{MN}(q(m \otimes n)s) = \upsilon^{MN}(qm \otimes ns) = ns \otimes qm = q(n \otimes m)s = q(\upsilon^{MN}(m \otimes n))s\,,$$

which holds for all $q \in Q$, $s \in S$, $m \in M$, and $n \in N$, shows that the isomorphism υ^{MN} is Q- and S^{o}-linear. □

Associativity

1.2.4 Proposition. *Let M be an R^{o}-module, X an R–S^{o}-bimodule, and N an S-module. The associativity map,*

$$\omega^{MXN} : (M \otimes_R X) \otimes_S N \longrightarrow M \otimes_R (X \otimes_S N)\,,$$

given by

$$\omega^{MXN}((m \otimes x) \otimes n) = m \otimes (x \otimes n)$$

is an isomorphism of $\Bbbk$-modules, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{M}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}T^{\mathrm{o}})$, then ω^{MXN} is an isomorphism in $\mathcal{M}(Q\text{–}T^{\mathrm{o}})$.

Proof. Proceeding as in the proof of 1.2.3, it is straightforward to verify that ω is a natural transformation of functors from $\mathcal{M}(R^{\mathrm{o}}) \times \mathcal{M}(R\text{–}S^{\mathrm{o}}) \times \mathcal{M}(S)$ to $\mathcal{M}(\Bbbk)$. Further, for modules M, X, and N as in the statement, the assignment $m \otimes (x \otimes n) \mapsto (m \otimes x) \otimes n$ defines a map $M \otimes_R (X \otimes_S N) \to (M \otimes_R X) \otimes_S N$; it is an inverse of ω^{MXN} which, therefore, is an isomorphism of $\Bbbk$-modules.

If M is in $\mathcal{M}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}T^{\mathrm{o}})$, then the modules $(M \otimes_R X) \otimes_S N$ and $M \otimes_R (X \otimes_S N)$ are in $\mathcal{M}(Q\text{–}T^{\mathrm{o}})$, and a computation similar to the one performed in the proof of 1.2.3 shows that ω^{MXN} is Q- and T^{o}-linear. □

Caveat. Let X and Y be R^{o}-modules and Z an R-module. The innocent looking computation

$$X \otimes_\Bbbk (Y \otimes_R Z) \cong (X \otimes_\Bbbk Y) \otimes_R Z \cong (Y \otimes_\Bbbk X) \otimes_R Z \cong Y \otimes_\Bbbk (X \otimes_R Z)$$

based on associativity 1.2.4 and commutativity 1.2.3 yields $0 \cong \Bbbk$ when applied to $R = \Bbbk[x] = X$, $Y = R/(x)$, and $Z = R/(x-1)$. Indeed, the ideals (x) and $(x-1)$ are comaximal in R, so $Y \otimes_R Z$ is 0, see 1.1.10; at the same time Y and Z are both isomorphic to $\Bbbk$ as $\Bbbk$-modules, whence one has $Y \otimes_\Bbbk (X \otimes_R Z) \cong \Bbbk \otimes_\Bbbk (R \otimes_R \Bbbk) \cong \Bbbk$. The issue is that the R–R-bimodule $X \otimes_\Bbbk Y$ is not symmetric; the first isomorphism is valid under the R-module structure coming from Y, and the last isomorphism is valid under the R-module structure coming from X.

Swap

For modules M, N, and X the bilinear maps $M \times N \longrightarrow X$ are in one-to-one correspondence with elements in $\operatorname{Hom}(M, \operatorname{Hom}(N, X))$ and $\operatorname{Hom}(N, \operatorname{Hom}(M, X))$. Swap compares these two hom-sets directly.

1.2.5 Proposition. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S^{o}-module. The swap map,*

$$\zeta^{MXN} : \operatorname{Hom}_R(M, \operatorname{Hom}_{S^{\mathrm{o}}}(N, X)) \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(N, \operatorname{Hom}_R(M, X)) ,$$

given by

$$\zeta^{MXN}(\psi)(n)(m) = \psi(m)(n)$$

is an isomorphism of $\Bbbk$-modules, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(T\text{–}S^{\mathrm{o}})$, then ζ^{MXN} is an isomorphism in $\mathcal{M}(Q\text{–}T^{\mathrm{o}})$.

Proof. It is straightforward to verify that ζ is a natural transformation of functors from $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R\text{–}S^{\mathrm{o}}) \times \mathcal{M}(S^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{M}(\Bbbk)$. Further, for modules M, X, and N as in the statement, it is immediate that the swap map ζ^{NXM} is an inverse of ζ^{MXN}.

If M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(T\text{–}S^{\mathrm{o}})$, then $\operatorname{Hom}_R(M, \operatorname{Hom}_{S^{\mathrm{o}}}(N, X))$ and $\operatorname{Hom}_{S^{\mathrm{o}}}(N, \operatorname{Hom}_R(M, X))$ are Q–T^{o}-bimodules. The computation

$$\begin{aligned}
\zeta^{MXN}(q\psi t)(n)(m) &= (q\psi t)(m)(n) \\
&= \psi(mq)(tn) \\
&= \zeta^{MXN}(\psi)(tn)(mq) \\
&= (q(\zeta^{MXN}(\psi))t)(n)(m) ,
\end{aligned}$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \operatorname{Hom}_R(M, \operatorname{Hom}_{S^{\mathrm{o}}}(N, X))$, $m \in M$, and $n \in N$, shows that the isomorphism ζ^{MXN} is Q- and T^{o}-linear. □

Adjunction

The next isomorphism expresses that Hom and tensor product are adjoint functors.

1.2.6 Proposition. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S-module. The adjunction map,*

$$\rho^{MXN} : \operatorname{Hom}_R(X \otimes_S N, M) \longrightarrow \operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M)) ,$$

given by

$$\rho^{MXN}(\psi)(n)(x) = \psi(x \otimes n)$$

is an isomorphism of $\Bbbk$-modules, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}T^{\mathrm{o}})$, then ρ^{MXN} is an isomorphism in $\mathcal{M}(T\text{–}Q^{\mathrm{o}})$.

Proof. It is straightforward to verify that ρ is a natural transformation of functors from $\mathcal{M}(R) \times \mathcal{M}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{M}(S)^{\mathrm{op}}$ to $\mathcal{M}(\Bbbk)$. Further, for modules M, X, and N as in the statement, it is elementary to verify that the map

$$\varkappa : \operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M)) \longrightarrow \operatorname{Hom}_R(X \otimes_S N, M)$$

given by $\varkappa(\varphi)(x \otimes n) = \varphi(n)(x)$ is an inverse of ρ^{MXN}.

If M is in $\mathcal{M}(R\text{–}T^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}Q^{\mathrm{o}})$, then $\operatorname{Hom}_R(X \otimes_S N, M)$ is a Q–T^{o}-bimodule and so is $\operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M))$. The computation

$$\begin{aligned}
\rho^{MXN}(q\psi t)(n)(x) &= (q\psi t)(x \otimes n) \\
&= (\psi(x \otimes nq))t \\
&= (\rho^{MXN}(\psi)(nq)(x))t \\
&= ((q(\rho^{MXN}(\psi)))(n)(x))t \\
&= (q(\rho^{MXN}(\psi))t)(n)(x)\,,
\end{aligned}$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \operatorname{Hom}_R(X \otimes_S N, M)$, $n \in N$, and $x \in X$, shows that the isomorphism ρ^{MXN} is Q- and T^{o}-linear. □

Caveat. Let Y and Z be R-modules and X an R^{o}-module. The innocent looking computation

$$\operatorname{Hom}_{\Bbbk}(X \otimes_R Y, Z) \;\cong\; \operatorname{Hom}_R(Y, \operatorname{Hom}_{\Bbbk}(X, Z)) \;\cong\; \operatorname{Hom}_{\Bbbk}(X, \operatorname{Hom}_R(Y, Z))$$

based on adjunction 1.2.6 and swap 1.2.5 yields $\Bbbk[x] \cong 0$ when applied to $R = \Bbbk[x] = X = Z$ and $Y = R/(x)$. Indeed, one has $Y \cong \Bbbk$, so the left-hand side is isomorphic to $\operatorname{Hom}_{\Bbbk}(\Bbbk, R) \cong R$ and the right-hand side vanishes as one has $\operatorname{Hom}_R(Y, Z) \cong \operatorname{Hom}_R(\Bbbk, R) = 0$. The issue is that the R–R-bimodule $\operatorname{Hom}_{\Bbbk}(X, Z)$ is not symmetric; the first isomorphism is valid under the R-module structure coming from X, and the second isomorphism is valid under the R-module structure coming from Z.

Base Change and Cobase Change

1.2.7 Proposition. *Let $\varphi\colon R \to S$ be a ring homomorphism.*

(a) *With S considered as an S–R^{o}-bimodule, the functor*

$$S \otimes_R - : \mathcal{M}(R) \longrightarrow \mathcal{M}(S)$$

is left adjoint to the restriction of scalars functor $\operatorname{res}^S_R\colon \mathcal{M}(S) \to \mathcal{M}(R)$.

(b) *With S considered as an R–S^{o}-bimodule, the functor*

$$\operatorname{Hom}_R(S, -) : \mathcal{M}(R) \longrightarrow \mathcal{M}(S)$$

is right adjoint to the restriction of scalars functor $\operatorname{res}^S_R\colon \mathcal{M}(S) \to \mathcal{M}(R)$.

Proof. (a): Consider S with the S–R^{o}-bimodule structure induced by φ; see 1.1.12. The functor res^S_R is isomorphic to $\operatorname{Hom}_S(S, -)\colon \mathcal{M}(S) \to \mathcal{M}(R)$, see 1.1.30, and it follows from adjunction 1.2.6 that this functor has the asserted left adjoint.

(b): Consider S with the R–S^{o}-bimodule structure induced by φ; see 1.1.12. The functor res^S_R is isomorphic to $S \otimes_S -\colon \mathcal{M}(S) \to \mathcal{M}(R)$, see 1.1.33, and it follows from adjunction 1.2.6 that this functor has the asserted right adjoint. □

1.2.8 Definition. Let $\varphi\colon R \to S$ be a ring homomorphism. The functor $S \otimes_R -$ from 1.2.7(a) is called *base change* along φ, and the functor $\operatorname{Hom}_R(S, -)$ from part (b) is called *cobase change* along φ.

Remark. Other terms for base change and cobase change that better capture the connections to restriction of scalars are ‘extension of scalars’ and ‘coextension of scalars’; nevertheless we opt for the shorter terms.

Exercises

E 1.2.1 Determine the inverse maps of the unitor 1.2.1 and the counitor 1.2.2.

E 1.2.2 Let α be an $m \times n$ matrix with entries in R. (a) Show that $R^m \xrightarrow{\cdot\alpha} R^n$, i.e. the map given by right multiplication by α, is a homomorphism of R-modules. (b) Let M be an R-module and show that $\operatorname{Hom}_R(\cdot\alpha, M)$ is a homomorphism of $\Bbbk$-modules naturally identified with $M^n \xrightarrow{\alpha\cdot} M^m$.

E 1.2.3 Let $\Bbbk$ be a field and set $(-)^* = \operatorname{Hom}_\Bbbk(-,\Bbbk)$. Let L be a $\Bbbk$-vector space with basis $\{e_u\}_{u\in U}$. For each $u \in U$ let $e_u^*\colon L \to \Bbbk$ be the functional given by $e_u^*(e_v) = \delta_{uv}$. (a) Show that the assignment $e_u \mapsto e_u^*$ defines a homomorphism $\epsilon\colon L \to L^*$ of $\Bbbk$-vector spaces. (b) Show that ϵ is an isomorphism if L has finite rank. (c) Assume that L has rank at least 2; show that there is an automorphism $\alpha\colon L \to L$ such that the following diagram is not commutative,

$$\begin{tikzcd} L \arrow[r, "\epsilon"] \arrow[d, "\alpha"'] & L^* \\ L \arrow[r, "\epsilon"] & L^* \arrow[u, "\alpha^*"'] \end{tikzcd} .$$

E 1.2.4 Let $\Bbbk$ be a field and set $(-)^* = \operatorname{Hom}_\Bbbk(-,\Bbbk)$. Let M be a $\Bbbk$-vector space. (a) For $m \in M$, show that the map $\varepsilon^m\colon M^* \to \Bbbk$ given by $\varphi \mapsto \varphi(m)$ is an element in $M^{**} = (M^*)^*$. (b) Show that the map $\delta^M\colon M \to M^{**}$ given by $m \mapsto \varepsilon^m$ is $\Bbbk$-linear. (c) Show that $\delta\colon \operatorname{Id}_{\mathcal{M}(\Bbbk)} \to (-)^{**}$ is a natural transformation of functors from $\mathcal{M}(\Bbbk)$ to $\mathcal{M}(\Bbbk)$.

E 1.2.5 Let M be an R^{o}-module, X an R–S^{o}-bimodule, and N an S-module. Show that $(M \otimes_R X) \otimes_S N$ is an R^{c}–S^{c}-bimodule, where R^{c} is the center of R.

E 1.2.6 Assume that R is commutative. Let M be an R-module and show that the functor $M \otimes_R -$ is left adjoint for $\operatorname{Hom}_R(M,-)$

E 1.2.7 Let X be an R–S^{o}-bimodule. Show that the functor $\operatorname{Hom}_R(-,X)\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S^{\mathrm{o}})$ is a right adjoint for $\operatorname{Hom}_{S^{\mathrm{o}}}(-,X)^{\mathrm{op}}$.

E 1.2.8 Let X be an R–S^{o}-bimodule and consider the map $R \to \operatorname{Hom}_{S^{\mathrm{o}}}(X,X)$ that maps r to multiplication by r on X. Show that it is a homomorphism of R–R^{o}-bimodules.

E 1.2.9 Let M be an R-module, X an R–S^{o}-bimodule, and N an S-module. Without using 1.2.3–1.2.5, show that there is a natural isomorphism of $\Bbbk$-modules

$$\operatorname{Hom}_R(N \otimes_{S^{\mathrm{o}}} X, M) \longrightarrow \operatorname{Hom}_S(N, \operatorname{Hom}_R(X,M)) .$$

E 1.2.10 Show that under suitable assumptions one can derive swap 1.2.5 from adjunction 1.2.6.

1.3 Exact Functors and Classes of Modules

Synopsis. Basis; free module; unique extension property; finite generation of Hom and tensor product; projective module; injective module; lifting property; semi-simple ring; Baer’s criterion; finitely presented module; flat module; von Neumann regular ring; vanishing of functor.

We start by recalling the language of generators of modules and ideals.

1.3.1 Definition. Let M be an R-module and X a subset of M. The submodule,

$$R\langle X\rangle = \left\{ \sum_{x\in X} r_x x \;\middle|\; r_x \in R \text{ and } r_x = 0 \text{ for all but finitely many } x \in X \right\},$$

of M is called the *submodule generated by* X. For an indexed set $X = \{x_u\}_{u\in U}$ one often writes $R\langle x_u \mid u \in U\rangle$ instead of $R\langle X\rangle$; in particular, for a finite set $X = \{x_1, \ldots, x_n\}$ one writes $R\langle x_1, \ldots, x_n\rangle$. By convention, $R\langle\varnothing\rangle$ is the zero module 0.

If $R\langle X\rangle = M$ holds, then X is called a *set of generators* for M; if no proper subset of X generates M, then X is called a *minimal* set of generators for M. If M has a finite set of generators, then M is called *finitely generated*.

If M is generated by one element, then M is called *cyclic*. A cyclic R-module generated by an element x is isomorphic to $R/\mathfrak{a}$ for the left ideal $\mathfrak{a} = (0 :_R x)$ in R.

1.3.2. Let M be an R-module. For a subset X of M one has $R\langle X\rangle = \sum_{x\in X} R\langle x\rangle$; for a family of submodules $\{M^u\}_{u\in U}$ of M one has $\sum_{u\in U} M^u = R\langle\bigcup_{u\in U} M^u\rangle$, cf. 1.1.23.

1.3.3 Definition. Left ideals and right ideals in R generated by elements $x_1, \ldots, x_n$ are denoted $R(x_1, \ldots, x_n)$ and $(x_1, \ldots, x_n)R$, respectively. The abridged notations Rx and xR are used for principal left and right ideals. If R is commutative, then the ideal $R(x_1, \ldots, x_n) = (x_1, \ldots, x_n)R$ is written $(x_1, \ldots, x_n)$, and a principal ideal may be written using any of the notations Rx, xR, and (x). The zero ideal is usually written 0 but occasionally (0) when the simpler notation could lead to an ambiguity.

REMARK. Though left and right ideals in R are submodules of the R-module R and the R^{o}-module R, respectively, it would be awkward to insist on applying the notation from 1.3.1 to ideals. Indeed, a principal right ideal would be written $R^{\mathrm{o}}\langle x\rangle$; even worse, writing $\Bbbk\langle x_1, x_2\rangle$ for the ideal in $\Bbbk$ generated by elements x_1 and x_2 would conflict with the standard notation for a free $\Bbbk$-algebra.

FREE MODULES

The gateway to projective objects, and also to injective objects, in module categories is free modules.

1.3.4 Definition. Let L be an R-module and $E = \{e_u\}_{u\in U}$ a set of generators for L. Every element in L can then be expressed on the form $\sum_{u\in U} r_u e_u$; if this expression is unique, then E is called a *basis* for L, and L is called *free*. By convention, the zero module is free with the empty set as basis.

For a set E, not a priori assumed to be a subset of a module, the free R-module with basis E is denoted $R\langle E\rangle$.

1.3.5 Example. If R is a division ring, then every R-module is free; indeed, every module over a division ring has a basis.

The $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z}$ is not free; indeed, a set of generators for M must include $[1]_{2\mathbb{Z}}$, and one has $0[1]_{2\mathbb{Z}} = 2[1]_{2\mathbb{Z}}$.

Free modules have the following unique extension property.

1.3.6 Proposition. *Let L be a free R-module with basis $E = \{e_u\}_{u\in U}$ and M an R-module. For every map $\alpha\colon E \to M$ of sets there is a unique R-module homomorphism, $\widetilde{\alpha}\colon L \to M$, such that the diagram*

$$\begin{tikzcd} E \arrow[r, tail] \arrow[d, "\alpha"'] & L \arrow[dl, dotted, "\widetilde{\alpha}"] \\ M & \end{tikzcd}$$

is commutative; the homomorphism is given by $\widetilde{\alpha}(\sum_{u\in U} r_u e_u) = \sum_{u\in U} r_u \alpha(e_u)$.

PROOF. The assertion is immediate from the definition, 1.3.4, of a free module. □

REMARK. The unique extension property characterizes free modules; see E 1.3.3.

1.3.7. Let L be a free R-module with basis $\{e_u\}_{u\in U}$ and $\{f_u\}_{u\in U}$ be the standard basis for $R^{(U)}$. There is an isomorphism, $L \to R^{(U)}$, given by $e_u \mapsto f_u$.

Important families of rings—commutative rings, left Noetherian rings, and local rings included—have the Invariant Basis Number property (IBN).

1.3.8 Definition. A free R-module is said to have *finite rank* if it is finitely generated; that is, it has a finite basis. A free R-module that is not finitely generated is said to have *infinite rank*. For a finitely generated free module L over a ring R that has IBN, the *rank* of L, written $\operatorname{rank}_R L$, is the number of elements in a basis for L.

REMARK. If U is infinite, then one has $R^{(U)} \cong R^{(V)}$ only if the sets U and V have the same cardinality, even if R does not have IBN; see [167, §1A].

1.3.9. Let $\{L^u\}_{u\in U}$ be a family of free R-modules with bases $\{E^u\}_{u\in U}$. The coproduct $\coprod_{u\in U} L^u$ is then a free R-module with basis $\bigcup_{u\in U} \varepsilon^u(E^u)$, where ε^u is the injection $L^u \rightarrowtail \coprod_{u\in U} L^u$. Notice that if U is a finite set, each module L^u is finitely generated, and R has IBN, then one has $\operatorname{rank}_R(\bigoplus_{u\in U} L^u) = \sum_{u\in U} \operatorname{rank}_R L^u$.

1.3.10. Assume that R is commutative. Let L and L' be free R-modules with bases $\{e_u\}_{u\in U}$ and $\{f_v\}_{v\in V}$, it is elementary to verify that the R-module $L \otimes_R L'$ is free with basis $\{e_u \otimes f_v\}_{u\in U,\, v\in V}$. Thus, if L and L' are finitely generated, then one has $\operatorname{rank}_R(L \otimes_R L') = (\operatorname{rank}_R L)(\operatorname{rank}_R L')$.

1.3.11 Theorem. *If R is a principal left ideal domain, then every submodule of a free R-module is free.*

PROOF. Let L be a free R-module with basis $\{e_u\}_{u\in U}$ and M a submodule of L. Choose a well-ordering $\leqslant$ on U. For $u \in U$ define submodules of L as follows:

$$L^{<u} = R\langle e_v \mid v < u\rangle \qquad \text{and} \qquad L^{\leqslant u} = R\langle e_v \mid v \leqslant u\rangle\,.$$

Let $u \in U$ be given. Every element l in $L^{\leqslant u}$ has a unique decomposition $l = l' + re_u$ with $l' \in L^{<u}$ and $r \in R$, so there is a split exact sequence of R-modules,

$$0 \longrightarrow L^{<u} \longrightarrow L^{\leqslant u} \xrightarrow{\varpi_u} R\langle e_u\rangle \longrightarrow 0\,,$$

where ϖ_u is given by $l' + re_u \mapsto re_u$. Let φ_u be the restriction of ϖ_u to $M \cap L^{\leqslant u}$; one has $\operatorname{Ker}\varphi_u = M \cap \operatorname{Ker}\varpi_u = M \cap L^{<u}$. The image of φ_u is a submodule of $R\langle e_u \rangle$ and, hence, isomorphic to a left ideal in R. It follows from the assumption on R that $\operatorname{Im}\varphi_u$ is cyclic and free, so choose $x_u \in R$ with $\operatorname{Im}\varphi_u = R\langle x_u e_u \rangle$ and note that if x_u is non-zero, then $\{x_u e_u\}$ is a basis for $\operatorname{Im}\varphi_u$. There is an exact sequence,

$$0 \longrightarrow M \cap L^{<u} \longrightarrow M \cap L^{\leqslant u} \xrightarrow{\varphi_u} R\langle x_u e_u \rangle \longrightarrow 0\,,$$

and it is split. Indeed, if $x_u \neq 0$ choose an element f_u in $M \cap L^{\leqslant u}$ with $\varphi_u(f_u) = x_u e_u$, and if $x_u = 0$ set $f_u = 0$. The assignment $x_u e_u \mapsto f_u$ then defines a right inverse homomorphism to φ_u; cf. 1.3.6. Thus, one has $M \cap L^{\leqslant u} = (M \cap L^{<u}) \oplus R\langle f_u \rangle$; that is, every element m in $M \cap L^{\leqslant u}$ has a unique decomposition $m = m' + rf_u$ with $m' \in M \cap L^{<u}$ and $r \in R$.

Set $U' = \{u \in U \mid x_u \neq 0\}$; we argue that $\{f_u\}_{u \in U'}$ is a basis for M. To see that every linear combination of the elements f_u is unique, suppose that there is a relation $r_1 f_{u_1} + \cdots + r_n f_{u_n} = 0$ with $r_i \in R$ and $u_i \in U'$. One can assume that the elements u_i are ordered $u_1 < \cdots < u_n$, and thus consider the relation in $M \cap L^{\leqslant u_n}$. Applying φ_{u_n} to the relation one gets $r_n x_{u_n} e_{u_n} = 0$. As u_n is in U', the singleton $\{x_{u_n} e_{u_n}\}$ is a basis for $\operatorname{Im}\varphi_u$, whence one has $r_n = 0$. Continuing in this manner, one gets $r_n = \cdots = r_1 = 0$, as desired. If $F = R\langle f_u \mid u \in U' \rangle$ were a proper submodule of M, then there would be a least u in U' such that $M \cap L^{\leqslant u}$ contains an element m not in F. This element has a unique decomposition $m = m' + rf_u$ with $m' \in M \cap L^{<u}$ and $r \in R$. The element m' is in $M \cap L^{\leqslant v}$ for some $v < u$ and hence in F by minimality of u. However, rf_u is also in F, so one has $m = m' + rf_u \in F$; a contradiction. □

Remark. A commutative ring is a principal ideal domain if (and only if) every submodule of a free module is free; see E 11.2.11.

The content of the next result is often phrased as: the category of R-modules has enough free modules.

1.3.12 Lemma. *Let M be an R-module; there is a surjective homomorphism $L \to M$ of R-modules where L is free. Moreover, if M is generated by n elements, then one can choose L such that it has a basis with n elements.*

Proof. Choose a set G of generators for M and let $E = \{e_g \mid g \in G\}$ be an abstract set. Consider the free R-module $L = R\langle E \rangle$ and define by 1.3.6 a homomorphism $\pi\colon L \to M$ by $\pi(\sum_{g \in G} r_g e_g) = \sum_{g \in G} r_g g$; it is surjective by the assumption on G. □

Finite Generation of Hom and Tensor Product

Under suitable assumptions on the ring, the Hom and tensor product functors restrict to the subcategories of finitely generated modules.

1.3.13 Proposition. *Assume that S is right Noetherian. Let M be an R-module and X an R–S^{o}-bimodule. If M is finitely generated and X is finitely generated over S^{o}, then the S^{o}-module $\operatorname{Hom}_R(M, X)$ is finitely generated.*

Proof. Choose by 1.3.12 a surjective homomorphism of R-modules $L \to M$, where L is free and finitely generated; say, $L \cong R^n$ as R-modules. Apply the left exact functor $\operatorname{Hom}_R(-, X)$ to get an injective homomorphism $\operatorname{Hom}_R(M, X) \to \operatorname{Hom}_R(L, X)$. The counitor 1.2.2 and additivity of the Hom functor yield an isomorphism of S^{o}-modules, $\operatorname{Hom}_R(L, X) \cong X^n$. Thus, $\operatorname{Hom}_R(M, X)$ is a submodule of a finitely generated S^{o}-module and hence finitely generated, as S is right Noetherian. □

Caveat. For M and X as in 1.3.13 the S-module $\operatorname{Hom}_R(X, M)$ need not be finitely generated, not even if R and S are Noetherian; see E 1.3.32.

1.3.14 Proposition. *Let N be an S-module and X an R–S^{o}-bimodule. If N is finitely generated and X is finitely generated over R, then the R-module $X \otimes_S N$ is finitely generated.*

Proof. Choose by 1.3.12 a surjective homomorphism of S-modules $L \to N$, where L is free and finitely generated; say, $L \cong S^n$ as S-modules. Apply the right exact functor $X \otimes_S -$ to get a surjective homomorphism $X \otimes_S L \to X \otimes_S N$. The unitor 1.2.1 and additivity of the tensor product yield an isomorphism of R-modules, $X \otimes_S L \cong X^n$. Thus, $X \otimes_S N$ is a homomorphic image of a finitely generated R-module and hence finitely generated. □

1.3.15 Corollary. *Let $\varphi\colon R \to S$ be a ring homomorphism and N an S-module. If S is finitely generated as an R-module, then N is finitely generated as an R-module if and only if it is finitely generated as an S-module.*

Proof. If N is finitely generated as an S-module, then it follows from 1.3.14 applied with $X = S$ that N is finitely generated as an R-module. On the other hand, the R-action on N factors through S, so every set of elements that generates N as an R-module also generates N as an S-module. □

Projective Modules

For an R-module M, the functors $\operatorname{Hom}_R(M, -)$, $\operatorname{Hom}_R(-, M)$, and $- \otimes_R M$ are, in general, not exact. Modules that make one or more of these functors exact are of particular interest and play a pivotal role in homological algebra.

1.3.16 Definition. An R-module P is called *projective* if the functor $\operatorname{Hom}_R(P, -)$ from $\mathcal{M}(R)$ to $\mathcal{M}(\Bbbk)$ is exact. If the functor $\operatorname{Hom}_R(P, -)$ is faithfully exact, then P is called *faithfully projective.*

Part (*ii*) below captures the lifting property of projective modules, which amounts to the definition of projective objects in a general category.

1.3.17 Proposition. *For an R-module P, the following conditions are equivalent.*

(*i*) *P is projective.*

(*ii*) *For every homomorphism* $\alpha\colon P \to N$ *and every surjective homomorphism* $\beta\colon M \to N$*, there exists a homomorphism* $\gamma\colon P \to M$ *such that the diagram*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\gamma"'] \arrow[d, "\alpha"] \\ M \arrow[r, two heads, "\beta"'] & N \end{tikzcd}$$

in $\mathcal{M}(R)$ *is commutative*; *that is, there is an equality* $\alpha = \beta\gamma$*.*

(*iii*) *Every exact sequence* $0 \to M' \to M \to P \to 0$ *of* R*-modules is split.*

(*iv*) P *is a direct summand of a free* R*-module.*

Proof. Conditions (*i*) and (*ii*) are equivalent, as the Hom functor is left exact.

(*ii*) $\Rightarrow$ (*iii*): Let β denote the homomorphism $M \twoheadrightarrow P$. by (*ii*) there exists a homomorphism $\gamma\colon P \to M$ such that $1^P = \beta\gamma$ holds, whence the sequence is split.

(*iii*) $\Rightarrow$ (*iv*): Choose by 1.3.12 a surjective homomorphism $\beta\colon L \to P$, where L is free. The associated exact sequence $0 \to \operatorname{Ker}\beta \to L \to P \to 0$ is split, so P is a direct summand of L.

(*iv*) $\Rightarrow$ (*ii*) As the Hom functor is additive, it is sufficient to prove that every free R-module has the lifting property. Let $\beta\colon M \twoheadrightarrow N$ and $\alpha\colon L \to N$ be homomorphisms of R-modules. Assume that L is free with basis $E = \{e_u \mid u \in U\}$. For every u in U, choose by surjectivity of β a preimage $m_u \in M$ of $\alpha(e_u)$. The map $E \to M$ given by $e_u \mapsto m_u$ extends by the unique extension property 1.3.6 to a homomorphism $\gamma\colon L \to M$ with $\alpha = \beta\gamma$. □

1.3.18 Corollary. *Every free* R*-module is projective.*

Proof. The assertion is immediate from the equivalence of (*i*) and (*iv*) in 1.3.17. □

1.3.19 Example. With R considered as an R–$\Bbbk$-bimodule there is a natural isomorphism of functors $\operatorname{Hom}_R(R, -) \cong \operatorname{res}^R_{\Bbbk}$, so it follows from 1.1.47 that R is a faithfully projective R-module. As R is a direct summand of every free R-module $L \neq 0$, see 1.3.7, it follows from 1.3.18 and additivity of the Hom functor that every such module is faithfully projective.

The next result is commonly referred to as Eilenberg's swindle, though Lam [169] perfer's Eilenberg's trick.

1.3.20 Corollary. *Let* P *be an* R*-module. If* P *is projective, then there exists a free* R*-module* L *such that there is an isomorphism* $P \oplus L \cong L$*.*

Proof. Assuming that P is projective, there exists by 1.3.17 a projective R-module P' such that $P \oplus P'$ is free. The R-module $L = (P \oplus P')^{(\mathbb{N})}$ is free by 1.3.9 and, evidently, one has $L \cong P \oplus L$. □

A homomorphism $P \twoheadrightarrow M$ with P projective is called a projective precover of M; cf. C.8. By 1.3.12 and 1.3.18 every R-module has a projective precover.

1.3.21 Corollary. *If R is a principal left ideal domain, then an R-module is projective if and only if it is free.*

Proof. Combine 1.3.11 and 1.3.17. □

Remark. That every projective module is free is also true over a local ring, see Kaplansky [154], and over a polynomial algebra $\Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is a field, see Quillen [205], and Suslin [242]. The result for the polynomial algebra resolved "Serre's problem on projective modules"; Lam gives a thorough account of its history in [169]. Bass [31] shows that not finitely generated projective modules in many cases are free.

1.3.22 Example. In a product ring $R \times S$, the ideal $\mathfrak{a} = R \times 0$ is a projective module; if S is non-zero then $\mathfrak{a}$ is not free.

1.3.23 Example. In the commutative ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$, the subset $\mathfrak{a} = \{a + b\sqrt{-5} \mid a = b \mod 2\}$ is an ideal. The elements $1 \pm \sqrt{-5}$ in $\mathfrak{a}$ have no common factor, so $\mathfrak{a}$ is not a principal ideal, whence it is not free. Yet, the map $\mathbb{Z}[\sqrt{-5}] \oplus \mathbb{Z}[\sqrt{-5}] \to \mathfrak{a} \oplus \mathfrak{a}$ given by $(r, s) \mapsto (2r + (1 + \sqrt{-5})s, 2s + (1 - \sqrt{-5})r)$ is an isomorphism of $\mathbb{Z}[\sqrt{-5}]$-modules, so $\mathfrak{a}$ is a direct summand of a free module and hence projective.

Remark. A classic example of a non-free projective module can be found, for example, in Eisenbud's book [78, 19.4]. Let R be the coordinate ring of the 2-sphere, $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, and consider the homomorphism $\varphi \colon R \to R^3$ given by $r \mapsto (rx, ry, rz)$ where, by a standard abuse of notation, x, y, and z now denote the cosets in R of the indeterminates. The sequence $0 \longrightarrow R \xrightarrow{\varphi} R^3 \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0$ is split exact, so the module $\operatorname{Coker}\varphi$ is projective; one can, however, show that it is not free.

1.3.24 Proposition. *Let $\{P^u\}_{u \in U}$ be a family of R-modules. The coproduct $\coprod_{u \in U} P^u$ is projective if and only if each module P^u is projective.*

Proof. If the coproduct $\coprod_{u \in U} P^u$ is projective, then by 1.3.17 it is a direct summand of a free module, and hence so is each module P^u. Conversely, if each module P^u is projective and hence a direct summand of a free module L^u, then the coproduct $\coprod_{u \in U} P^u$ is a direct summand of the free module $\coprod_{u \in U} L^u$; cf. 1.3.9. □

Caveat. A product of projective modules need not be projective; see E 1.3.15.

Injective Modules

Injective modules are categorically dual to projective modules.

1.3.25 Definition. An R-module I is called *injective* if the functor $\operatorname{Hom}_R(-, I)$ from $\mathcal{M}(R)^{\mathrm{op}}$ to $\mathcal{M}(\Bbbk)$ is exact. If the functor $\operatorname{Hom}_R(-, I)$ is faithfully exact, then I is called *faithfully injective.*

1.3.26. An R-module I is injective if and only if it has the following lifting property, which amounts to the definition of an injective object in a general category. Given a homomorphism $\alpha \colon K \to I$ and an injective homomorphism $\beta \colon K \to M$, there exists a homomorphism $\gamma \colon M \to I$ such that the diagram

$$\begin{tikzcd} K \arrow[r, tail, "\beta"] \arrow[d, "\alpha"'] & M \arrow[dl, dotted, "\gamma"] \\ I & \end{tikzcd}$$

in $\mathcal{M}(R)$ is commutative; that is, there is an equality $\gamma\beta = \alpha$.

REMARK. The epimorphisms in the category $\mathcal{M}(R)$ are exactly the surjective homomorphisms; see E 1.1.2. In view of this and 1.3.17, an R-module P is projective if and only if the functor $\operatorname{Hom}_R(P, -) : \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$ takes epimorphisms to epimorphisms.

The epimorphisms in the category $\mathcal{M}(R)^{\mathrm{op}}$ correspond to monomorphisms—which by E 1.1.2 are the injective homomorphisms—in $\mathcal{M}(R)$. In view of this and 1.3.26, an R-module I is injective if and only if the functor $\operatorname{Hom}_R(-, I) : \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(\Bbbk)$ takes epimorphisms to epimorphisms.

1.3.27 Proposition. *Let $\{I^u\}_{u\in U}$ be a family of R-modules. The product $\prod_{u\in U} I^u$ is injective if and only if each module I^u is injective.*

PROOF. Let $\beta : K \to M$ be an injective homomorphism of R-modules and $\{I^u\}_{u\in U}$ a family of R-modules; set $I = \prod_{u\in U} I^u$. Assume first that each module I^u is injective. Let a homomorphism $\alpha : K \to I$ be given. For each $u \in U$ set $\alpha^u = \varpi^u\alpha$, where ϖ^u is the projection $I \twoheadrightarrow I^u$. By assumption there exist homomorphisms $\gamma^u : M \to I^u$, such that $\gamma^u\beta = \alpha^u$ holds for every $u \in U$. The unique homomorphism $\gamma : M \to I$ with $\gamma^u = \varpi^u\gamma$ now satisfies $\gamma\beta = \alpha$, so I is injective.

Assume now that I is injective, fix an element $u \in U$, and let ε^u denote the injection $I^u \rightarrowtail I$. Given a homomorphism $\alpha : K \to I^u$, one has a homomorphism $\varepsilon^u\alpha$ from K to I. By injectivity of I there is a homomorphism $\gamma : M \to I$ such that $\gamma\beta = \varepsilon^u\alpha$ holds and, therefore, one has $(\varpi^u\gamma)\beta = \alpha$. □

CAVEAT. A coproduct of injective modules need not be injective; see E 1.3.21.

1.3.28 Theorem. *The following conditions are equivalent.*

(*i*) *R is semi-simple.*
(*ii*) *Every short exact sequence of R-modules is split.*
(*iii*) *Every R-module is projective.*
(*iv*) *Every R-module is injective.*

PROOF. Recall that R being semi-simple means that every submodule M' of an R-module M is a direct summand of that module; conditions (*i*) and (*ii*) are, therefore, equivalent. It is evident that (*ii*) implies (*iii*) and (*iv*). The converse implications follow from 1.3.17 and 1.3.26. □

REMARK. The Artin–Wedderburn Theorem asserts that a ring is semi-simple if and only if it is isomorphic to a product $\mathrm{M}_{n_1\times n_1}(D_1) \times \cdots \times \mathrm{M}_{n_k\times n_k}(D_k)$ where $D_1, \ldots, D_k$ are division rings; see [168, §3]. In particular, a commutative ring is semi-simple if and only if it is a finite product of fields. An arbitrary product of fields is a von Neumann regular ring; see 1.3.45.

1.3.29 Example. Every module over a division ring is injective, as division rings are semi-simple.

The next result is known as Baer's criterion; it goes back to [28].

1.3.30 Theorem. *Let I be an R-module, $\mathfrak{a}$ a left ideal in R, and $\iota\colon \mathfrak{a} \rightarrowtail R$ the embedding. The module I is injective if and only if for every R-module homomorphism $\varphi\colon \mathfrak{a} \to I$ there exists a homomorphism $\varphi'\colon R \to I$ with $\varphi'\iota = \varphi$.*

Proof. The "only if" part of the statement follows from 1.3.26. To prove "if", let $\alpha\colon K \to I$ and $\beta\colon K \to M$ be homomorphisms of R-modules and assume that β is injective. Denote by G the set of all homomorphisms $\gamma'\colon M' \to I$ with $\operatorname{Im}\beta \subseteq M'$ and $\gamma'\beta = \alpha$. Since $\alpha\beta^{-1}\colon \operatorname{Im}\beta \to I$ belongs to G, this set is non-empty. By declaring $(\gamma'\colon M' \to I) \leqslant (\gamma''\colon M'' \to I)$ if $M' \subseteq M''$ and $\gamma''|_{M'} = \gamma'$, the set G becomes inductively ordered. Hence Zorn's lemma guarantees the existence of a maximal element $\gamma''\colon M'' \to I$. To finish the proof, we show the equality $M'' = M$. Assume, towards a contradiction, that M'' is a proper submodule of M and choose an element $m \in M \setminus M''$. The set $\mathfrak{a} = (M'' :_R m)$ is a left ideal in R. The map $\varphi\colon \mathfrak{a} \to I$ given by $\varphi(r) = \gamma''(rm)$ is an R-module homomorphism, so by assumption it has a lift $\varphi'\colon R \to I$. It follows from the definition of $\mathfrak{a}$ that the map $\gamma'\colon M'' + R\langle m\rangle \to I$ given by $\gamma'(m'' + rm) = \gamma''(m'') + \varphi'(r)$ is well-defined. It is evidently a homomorphism whose restriction to M'' is γ'', so one has $\gamma'\beta = \gamma''\beta = \alpha$. Hence γ' belongs to G and satisfies $\gamma' > \gamma''$, which contradicts the maximality of γ''. □

Remark. It appears that Baer's criterion has no real counterpart for projective modules; perhaps E 1.4.7 comes as close as one can get.

1.3.31. Let R be a domain. Recall that an R-module M is called *divisible* if $rM = M$ holds for all $r \neq 0$ in R. Every injective R-module I is divisible. Indeed, right-multiplication by $r \neq 0$ yields an injective homomorphism $R \xrightarrow{r} R$ of R-modules, so the induced homomorphism $\operatorname{Hom}_R(R, I) \to \operatorname{Hom}_R(R, I)$ of $\Bbbk$-modules is surjective, whence one has $rI = I$; cf. 1.2.2. The converse statement in 1.3.32 below, however, hinges crucially on the principal ideal hypothesis, as illustrated in 1.3.34.

1.3.32 Proposition. *Assume that R is a principal left ideal domain. An R-module is injective if and only if it is divisible. Moreover, every quotient of an injective R-module is injective.*

Proof. Once the first claim is proved, the second one follows, as the divisibility property is inherited by quotient modules. Every injective R-module is divisible by 1.3.31. Let I be a divisible R-module and $\mathfrak{a}$ a left ideal in R. By assumption there exists an element $x \in R$ with $Rx = \mathfrak{a}$. A homomorphism of R-modules $\alpha\colon \mathfrak{a} \to I$ is determined by the value $\alpha(x) = i$, and it can be lifted to a homomorphism $R \to I$ as there exists an element $i' \in I$ with $xi' = i$. Thus, it follows from Baer's criterion 1.3.30 that I is injective. □

The field of fractions of an integral domain is clearly divisible and, in fact, injective. For principal ideal domains this follows from 1.3.32 and here is the general case:

1.3.33 Proposition. *Assume that R is an integral domain. A divisible and torsion-free R-module is injective*; *in particular, the field of fractions of R is injective.*

Proof. Let M be a divisible and torsion-free R-module. Let $\mathfrak{a}$ be an ideal in R and $\varphi\colon \mathfrak{a} \to M$ a homomorphism. For elements $r \neq 0 \neq r'$ in $\mathfrak{a}$ there exist by divisibility of M elements m and m' with $\varphi(r) = rm$ and $\varphi(r') = r'm'$. As R is commutative, one now has $r'rm = \varphi(r'r) = \varphi(rr') = rr'm' = r'rm'$, and since M is torsion-free, this implies $m = m'$. Thus the morphism $\widetilde{\varphi}\colon R \to M$ given by $\widetilde{\varphi}(r) = rm$ extends φ, so M is injective by 1.3.30. □

1.3.34 Example. Let $\Bbbk$ be a field and consider the polynomial algebra $R = \Bbbk[x,y]$; its field of fractions is the field $Q = \Bbbk(x,y)$ of rational functions. Evidently, Q and, therefore, Q/R are divisible R-modules. However, Q/R is not injective. Indeed, consider map $R \to Q/R$ given by

$$f \longmapsto \left[\frac{f(x,0)}{xy}\right]_R .$$

While it is additive, it is not R-linear. However, its restriction to the ideal $\mathfrak{M}$ of polynomials with zero constant term (called the 'irrelevant maximal ideal') is R-linear. This homomorphism $\mathfrak{M} \to Q/R$ does not extend to a homomorphism $\varphi\colon R \to Q/R$. If it did, then $\varphi(1)$, which has the form $[gh^{-1}]_R$ for some g and h in R, would satisfy $y\varphi(1) = \varphi(y) = [0]_R$ and $x\varphi(1) = \varphi(x) = [y^{-1}]_R$. The first equation shows that there would be a $k \in R$ with $ygh^{-1} = k$ and, therefore, $\varphi(1) = [gh^{-1}]_R = [ky^{-1}]_R$. To satisfy the second equation there would be an l in R with $xky^{-1} = y^{-1} + l$, that is, $xk = 1 + ly$, which is absurd.

Faithful Injectivity

It follows from 1.3.32 that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules; one can say even more about the latter.

1.3.35 Proposition. *The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is faithfully injective.*

Proof. Let G be a non-zero $\mathbb{Z}$-module and choose an element $g \neq 0$ in G. Define a homomorphism ξ from the cyclic submodule $\mathbb{Z}\langle g\rangle$ to $\mathbb{Q}/\mathbb{Z}$ as follows. If g is torsion set $\xi(g) = [\frac{1}{n}]_{\mathbb{Z}}$, where n is the least positive integer with $ng = 0$. If g is not torsion, set $\xi(g) = [\frac{1}{2}]_{\mathbb{Z}}$. By the lifting property 1.3.26 there is a homomorphism $G \to \mathbb{Q}/\mathbb{Z}$ that restricts to ξ on $\mathbb{Z}\langle g\rangle$, whence $\operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ is non-zero. □

1.3.36 Lemma. *Let E be an injective $\Bbbk$-module. The R–R°-bimodule $\operatorname{Hom}_{\Bbbk}(R,E)$ is injective over R and over R°, and if E is faithfully injective, then so is $\operatorname{Hom}_{\Bbbk}(R,E)$.*

Proof. By adjunction 1.2.6 and the unitor 1.2.1 there are natural isomorphisms of functors from $\mathcal{M}(R)$ to $\mathcal{M}(\Bbbk)$,

$$\operatorname{Hom}_R(-, \operatorname{Hom}_{\Bbbk}(R,E)) \;\cong\; \operatorname{Hom}_{\Bbbk}(R \otimes_R -, E) \;\cong\; \operatorname{Hom}_{\Bbbk}(-, E) .$$

It follows that if E is (faithfully) injective, then so is the R-module $\operatorname{Hom}_{\Bbbk}(R,E)$, cf. 1.3.25. Similarly, the R°-module $\operatorname{Hom}_{\Bbbk}(R,E)$ is (faithfully) injective. □

For $\Bbbk = \mathbb{Z}$ a natural choice of a faithfully injective $\Bbbk$-module is $\mathbb{Q}/\mathbb{Z}$; see 1.3.35. For any choice of $\Bbbk$, the $\Bbbk$-module $\operatorname{Hom}_{\mathbb{Z}}(\Bbbk, \mathbb{Q}/\mathbb{Z})$ is faithfully injective by 1.3.36. For specific choices of $\Bbbk$ other candidates present themselves: If, say, $\Bbbk$ is a field, then $\Bbbk$ is itself faithfully injective $\Bbbk$-module.

1.3.37 Definition. Let $\mathbb{E}$ denote a faithfully injective $\Bbbk$-module.

1.3.38. It follows from 1.3.36 that the R–R^{o}-bimodule $\operatorname{Hom}_{\Bbbk}(R, \mathbb{E})$ is faithfully injective over R and over R^{o}.

See C.3 for a different approach to the existence of faithfully injective modules.

Finitely Presented Modules

1.3.39 Definition. Let M be an R-module. By 1.3.12 there exist free R-modules L and L' such that there is an exact sequence

$$L' \longrightarrow L \longrightarrow M \longrightarrow 0\,;$$

it is called a *free presentation* of M. If M has a free presentation with L and L' finitely generated, then M is called *finitely presented*.

Every finitely presented module is finitely generated; the converse holds over left Noetherian rings; in fact, it characterizes left Noetherian rings.

1.3.40 Lemma. *Let M be a finitely presented R-module.*

(a) *For every surjective homomorphism $\pi\colon L \to M$ with L finitely generated and free, the submodule* $\operatorname{Ker}\pi$ *is finitely generated.*
(b) *For every finitely generated submodule N of M, the quotient module M/N is finitely presented.*

Proof. As M is finitely presented, there exists a surjective homomorphism of R-modules $\pi'\colon L' \to M$ with L' finitely generated free and $\operatorname{Ker}\pi'$ finitely generated.

(a): Denote by X the kernel of the homomorphism $L \oplus L' \to M$ that maps (l, l') to $\pi(l) - \pi'(l')$. As π and π' are surjective, the canonical homomorphisms $X \to L$ and $X \to L'$ that map (l, l') to l and l', respectively, are surjective; notice that they have kernels $0 \oplus \operatorname{Ker}\pi'$ and $\operatorname{Ker}\pi \oplus 0$. It follows from 1.3.17 that there are isomorphisms $\operatorname{Ker}\pi' \oplus L \cong X \cong \operatorname{Ker}\pi \oplus L'$. Thus, $\operatorname{Ker}\pi$ is a direct summand of the finitely generated module $\operatorname{Ker}\pi' \oplus L$ and hence finitely generated.

(b): There is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & L' \arrow[r, "1^{L'}"] \arrow[d, two heads, "\pi'"] & L' \arrow[r] \arrow[d, two heads, "\varkappa\pi'"] & 0 \\
0 \arrow[r] & N \arrow[r] & M \arrow[r, "\varkappa"] & M/N \arrow[r] & 0\,,
\end{tikzcd}$$

where $\varkappa$ is the canonical surjection. By the Snake Lemma 1.1.6 there is an exact sequence $0 \to \operatorname{Ker}\pi' \to \operatorname{Ker}(\varkappa\pi') \to N \to 0$. As $\operatorname{Ker}\pi'$ and N are finitely generated, it follows that also $\operatorname{Ker}(\varkappa\pi')$ is finitely generated, whence M/N is finitely presented. □

Flat Modules

1.3.41 Definition. An R-module F is called *flat* if the functor $- \otimes_R F$ from $\mathcal{M}(R^{\mathrm{o}})$ to $\mathcal{M}(\Bbbk)$ is exact. If $- \otimes_R F$ is faithfully exact, then F is called *faithfully flat*.

1.3.42 Example. Assume that R is commutative and let U be a multiplicative subset of R. As localization at U is an exact functor, the isomorphism in 1.1.11 shows that the R-module $U^{-1}R$ is flat. In particular, if R is an integral domain, then its field of fractions is flat as an R-module.

1.3.43 Example. It is elementary to verify that every non-zero free R-module is faithfully flat. Hence, by additivity of the tensor product, every projective R-module is flat; see 1.3.17. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat by 1.3.42, but it is not faithfully flat, and in view of 1.3.21 it is elementary to show that it is not projective either.

1.3.44 Lemma. *Let M be an R-module and K a submodule of M. If the quotient module M/K is flat, then $\mathfrak{b}K = \mathfrak{b}M \cap K$ holds for every right ideal $\mathfrak{b}$ in R.*

Proof. Let ι be the embedding $\mathfrak{b} \rightarrowtail R$. For every module X set $\mu_{\mathfrak{b}}^X = \mu_R^X \circ (\iota \otimes_R X)$, where μ_R^X is the unitor 1.2.1. By assumption, the homomorphism $\iota \otimes_R M/K$ is injective, hence so is $\mu_{\mathfrak{b}}^{M/K}$. There is a commutative diagram with exact rows,

$$\begin{tikzcd}
& \mathfrak{b} \otimes_R K \arrow[r] \arrow[d, "\mu_{\mathfrak{b}}^K"] & \mathfrak{b} \otimes_R M \arrow[r] \arrow[d, "\mu_{\mathfrak{b}}^M"] & \mathfrak{b} \otimes_R M/K \arrow[r] \arrow[d, "\mu_{\mathfrak{b}}^{M/K}"] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r] & M/K \arrow[r] & 0\,,
\end{tikzcd}$$

where the upper row is obtained by application of $\mathfrak{b} \otimes_R -$ to the lower row. One has $\operatorname{Im} \mu_{\mathfrak{b}}^K = \mathfrak{b}K$ and $\operatorname{Im} \mu_{\mathfrak{b}}^M = \mathfrak{b}M$, so it follows from the Snake Lemma 1.1.6 that the canonical homomorphism $K/\mathfrak{b}K \to M/\mathfrak{b}M$ is injective. This yields the inclusion $K \cap \mathfrak{b}M \subseteq \mathfrak{b}K$; the opposite inclusion is trivial, so equality holds. □

Our first application of 1.3.44 is to von Neumann regular rings.

1.3.45 Example. Let $\{R^u\}_{u\in U}$ be a family of von Neumann regular rings, for example fields. Let $x = (x_u)_{u\in U}$ be an element in the product ring $R = \bigtimes_{u\in U} R^u$ and set $r = (r_u)_{u\in U}$, where r_u satisfies $x_u = x_u r_u x_u$ in R^u. One thus has $x = xrx$ in R, so R is a von Neumann regular ring.

1.3.46 Proposition. *If every R-module is flat, then R is von Neumann regular.*

Proof. Let x be an element in R and set $M = R$, $K = Rx$, and $\mathfrak{b} = xR$; one now has $x \in \mathfrak{b}M \cap K$. If every R-module is flat, then 1.3.44 yields $x \in \mathfrak{b}K = (xR)(Rx)$, so R is von Neumann regular. □

The von Neumann regular rings are, in fact, precisely the rings over which every module is flat; this is proved in 8.5.8.

Remark. Von Neumann regular rings are also called 'absolutely flat' rings.

The next theorem is an easy consequence of a theorem due to Govorov and Lazard, see 5.5.10. Here we present a direct proof.

1.3.47 Theorem. *For an R-module F, the following conditions are equivalent.*

(*i*) *F is a finitely presented flat R-module.*

(*ii*) *F is a finitely generated projective R-module.*

(*iii*) *F is a direct summand of a finitely generated free R-module.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let $L' \to L \to F \to 0$ be a presentation with L and L' finitely generated free R-modules. It follows that F and the kernel K of the surjection $L \to F$ are finitely generated, so there is a short exact sequence $0 \to K \to L \to F \to 0$ of finitely generated R-modules. Showing that F is projective is by 1.3.17 tantamount to showing that this sequence is split. To this end it suffices to construct a homomorphism $\varrho\colon L \to K$ with $\varrho|_K = 1^K$.

Let $\{e_1, \dots, e_m\}$ be a basis for L and $\{k_1, \dots, k_n\}$ a set of generators for the submodule K. Let $k \in K$ be given; we start by constructing a homomorphism $\varrho_k\colon L \to K$ with $\varrho_k(k) = k$. Write k in terms of the basis: $k = r_1 e_1 + \cdots + r_m e_m$. Let $\mathfrak{b}$ be the right ideal in R generated by $r_1, \dots, r_m$; one has $k \in K \cap \mathfrak{b}L$, whence k is in $\mathfrak{b}K$ by 1.3.44. It follows that there are elements $b_i \in \mathfrak{b}$ and $x_{ji} \in R$ such that the following equalities hold,

$$(\dagger) \qquad k \;=\; \sum_{i=1}^{n} b_i k_i \;=\; \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} r_j x_{ji}\Big) k_i \;=\; \sum_{j=1}^{m} r_j \Big(\sum_{i=1}^{n} x_{ji} k_i\Big)\,.$$

Define ϱ_k by $e_j \mapsto \sum_{i=1}^{n} x_{ji} k_i$, then $\varrho_k(k) = \varrho_k(\sum_{j=1}^{m} r_j e_j) = k$ holds by ($\dagger$).

To construct a homomorphism $\varrho\colon L \to K$ whose restriction to K is the identity, it suffices to construct ϱ with $\varrho(k_i) = k_i$ for all the generators $k_1, \dots, k_n$. Proceed by induction on n; the construction above settles the base case $n = 1$. For $n > 1$ there exists by the same construction a homomorphism ϱ_{k_n} that fixes k_n. For $i < n$ set $k_i' = k_i - \varrho_{k_n}(k_i)$. By the induction hypothesis, there is a homomorphism $\varrho'\colon L \to K$ with $\varrho'(k_i') = k_i'$ for $i < n$. Now, set $\varrho = \varrho' - \varrho'\varrho_{k_n} + \varrho_{k_n}$; one has

$$\varrho(k_n) \;=\; \varrho'(k_n) - \varrho'(k_n) + k_n \;=\; k_n,$$

and for $i < n$ one has

$$\varrho(k_i) \;=\; \varrho'(k_i) - \varrho'\varrho_{k_n}(k_i) + \varrho_{k_n}(k_i) \;=\; \varrho'(k_i') + \varrho_{k_n}(k_i) \;=\; k_i' + \varrho_{k_n}(k_i) \;=\; k_i\,.$$

(*ii*) $\Rightarrow$ (*iii*): By 1.3.12 there is a short exact sequence $0 \to K \to L \to F \to 0$, where L is a finitely generated free R-module, and by 1.3.17 the sequence is split.

(*iii*) $\Rightarrow$ (*i*): There is a split exact sequence $0 \to K \to L \to F \to 0$ of R-modules, where L is finitely generated and free. In particular, F is flat by 1.3.43. The module K is also a direct summand of L, in particular it is finitely generated, so by 1.3.12 there is a finitely generated free R-module L' and a surjective homomorphism $L' \to K$. Thus, one has a free presentation $L' \to L \to F \to 0$. □

A characterization in non-homological terms of flat modules over principal ideal domains—akin to 1.3.21 and 1.3.32—is given in 11.2.31.

Flat–Injective Duality

Projective and injective objects are categorically dual. In module categories there is another important duality between flat and injective modules, which is rooted in the adjointness of Hom and tensor product.

1.3.48 Proposition. *For an R-module F, the following conditions are equivalent.*

(*i*) *F is flat.*

(*ii*) *For every right ideal $\mathfrak{b}$ in R, the homomorphism $\iota \otimes_R F$, induced by the embedding $\iota\colon \mathfrak{b} \rightarrowtail R$, is injective.*

(*iii*) *The R°-module $\operatorname{Hom}_{\Bbbk}(F, E)$ is injective for every injective $\Bbbk$-module E.*

(*iv*) *The R°-module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is injective.*

Moreover, the R°-module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is faithfully injective if and only if F is a faithfully flat R-module.

Proof. Condition (*iv*) clearly follows from (*iii*), and it follows from the definition 1.3.41 that (*i*) implies (*ii*).

(*ii*) $\Rightarrow$ (*iii*): Apply the exact functors $- \otimes_R F$ followed by $(-)^{\vee} = \operatorname{Hom}_{\Bbbk}(-, E)$ to the short exact sequence $0 \to \mathfrak{b} \to R \to R/\mathfrak{b} \to 0$. This yields the upper row in the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (R/\mathfrak{b} \otimes_R F)^{\vee} & \longrightarrow & (R \otimes_R F)^{\vee} & \longrightarrow & (\mathfrak{b} \otimes_R F)^{\vee} & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_{R^{\circ}}(R/\mathfrak{b}, F^{\vee}) & \longrightarrow & \operatorname{Hom}_{R^{\circ}}(R, F^{\vee}) & \longrightarrow & \operatorname{Hom}_{R^{\circ}}(\mathfrak{b}, F^{\vee}) & \longrightarrow & 0\,. \end{array}$$

The vertical isomorphisms follow from commutativity 1.2.3 and adjunction 1.2.6. By commutativity of the diagram, the lower row is an exact sequence. Thus, the R°-module $F^{\vee} = \operatorname{Hom}_{\Bbbk}(F, E)$ is injective by Baer's criterion 1.3.30.

(*iv*) $\Rightarrow$ (*i*): By adjunction 1.2.6 and commutativity 1.2.3 there is a natural isomorphism of functors from $\mathcal{M}(R^{\circ})$ to $\mathcal{M}(\Bbbk)$,

$$(\flat) \qquad \operatorname{Hom}_{R^{\circ}}(-, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \;\cong\; \operatorname{Hom}_{\Bbbk}(- \otimes_R F, \mathbb{E})\,.$$

The left-hand functor is by assumption exact, and $\operatorname{Hom}_{\Bbbk}(-, \mathbb{E})$ is faithfully exact; it follows that $- \otimes_R F$ is exact, whence F is flat.

It also follows from ($\flat$) that the functor $\operatorname{Hom}_{R^{\circ}}(-, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}))$ is faithfully exact if and only if $- \otimes_R F$ is so. This proves the last assertion. ◻

1.3.49 Corollary. *Let $\mathfrak{b}$ be a right ideal in R and F a flat R-module. The canonical map $\mathfrak{b} \otimes_R F \to \mathfrak{b}F$ induced by the unitor is an isomorphism of $\Bbbk$-modules.*

Proof. By 1.3.48 one has an injective homomorphism $\mathfrak{b} \otimes_R F \rightarrowtail R \otimes_R F \xrightarrow{\cong} F$ with image $\mathfrak{b}F$, so one has $\mathfrak{b} \otimes_R F \cong \mathfrak{b}F$. ◻

In Sect. 5.3 we make extensive use of the following special case.

1.3.50 Corollary. *Let L be a free R^{o}-module. If L is non-zero, then the R-module* $\mathrm{Hom}_{\Bbbk}(L,\mathbb{E})$ *is faithfully injective.*

Proof. The assertion is immediate from 1.3.43 and 1.3.48. □

Vanishing of Functors

1.3.51 Lemma. *Let $\mathcal{U}$ be an Abelian category and* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{U}$ *a half exact functor. One has* $\mathrm{F}(M) = 0$ *for every finitely generated R-module M if and only if* $\mathrm{F}(R/\mathfrak{a}) = 0$ *holds for every left ideal* $\mathfrak{a}$ *in R.*

Proof. The "only if" part is trivial. Assume that $\mathrm{F}(R/\mathfrak{a}) = 0$ holds for every left ideal $\mathfrak{a}$ in R. Let M be an R-module generated by elements $x_1,\dots,x_n$ and proceed by induction on the number, n, of generators. For $n = 1$ one has $M \cong R/(0 :_R x_1)$, whence $\mathrm{F}(M) = 0$ holds by assumption. For $n > 1$ set $N = R\langle x_1,\dots,x_{n-1}\rangle$; the quotient module M/N is then generated by $[x_n]_N$. Apply F to the exact sequence $0 \to N \to M \to M/N \to 0$ to get the exact sequence $\mathrm{F}(N) \longrightarrow \mathrm{F}(M) \longrightarrow \mathrm{F}(M/N)$ where $\mathrm{F}(M/N) = 0$ holds by assumption and $\mathrm{F}(N) = 0$ holds by the induction hypothesis. It follows that also $\mathrm{F}(M)$ is zero. □

1.3.52 Lemma. *Let $\mathcal{U}$ be an Abelian category and* $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{U}$ *a half exact functor. One has* $\mathrm{G}(M) = 0$ *for every finitely generated R-module M if and only if* $\mathrm{G}(R/\mathfrak{a}) = 0$ *holds for every left ideal* $\mathfrak{a}$ *in R.*

Proof. Apply 1.3.51 to the opposite functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{M}(R) \to \mathcal{U}^{\mathrm{op}}$. □

Exercises

E 1.3.1 Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module and that $\mathbb{R}$ is not a finitely generated $\mathbb{Q}$-module.

E 1.3.2 Let L be an R-module and $E = \{e_u\}_{u\in U}$ a set of generators for L. Show that every element in L can be expressed uniquely on the form $\sum_{u\in U} r_u e_u$ if and only if some element in L can be expressed uniquely on that form.

E 1.3.3 Let L be an R-module and E a subset of L. Show that if every map from E to an R-module M extends uniquely to a homomorphism $L \to M$ of R-modules, then E is a basis for L; in particular, L is free.

E 1.3.4 Assume that R is commutative. Show that if every cyclic R-module is free, then R is a field.

E 1.3.5 Let $\Bbbk$ be a field and M a $\Bbbk$-vector space of infinite rank. Show that the endomorphism ring $\mathrm{Hom}_{\Bbbk}(M,M)$ does not have IBN.

E 1.3.6 Show that every division ring has IBN.

E 1.3.7 Assume that R is commutative. Let L and L' be finitely generated free R-modules, show that the R-module $\mathrm{Hom}_R(L,L')$ is free, and find its rank as a function of the ranks of L and L'.

E 1.3.8 Denote by $\mathcal{S}$ the category of sets. Show that the functor $\mathcal{S} \to \mathcal{M}(R)$ given by $U \mapsto R^{(U)}$ is a left adjoint for the forgetful functor $\mathcal{M}(R) \to \mathcal{S}$.

E 1.3.9 Let L be a free R-module with basis $\{e_u\}_{u\in U}$ and K a submodule of L generated by elements $\{k_v\}_{v\in V}$. (a) Show that if V is a finite set, then there is a finite subset U' of U such that K is contained in $L' = R\langle e_u \mid u \in U'\rangle$. (b) Show that if V is infinite, then there is a subset U' of U with $\operatorname{card} U' \leqslant \operatorname{card} V$ such that K is contained in $L' = R\langle e_u \mid u \in U'\rangle$.

E 1.3.10 Let M be an R^{o}-module generated by elements $x_1, \dots, x_m$ and N an R-module generated by $y_1, \dots, y_n$. Assume that R is generated as a $\Bbbk$-module by $r_1, \dots, r_l$. Show that the elements in the set $\{x_i \otimes r_h y_j \mid 1 \leqslant h \leqslant l,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ generate the $\Bbbk$-module $M \otimes_R N$.

E 1.3.11 Assume that R is commutative. Let L be a free R-module with basis $\{e_1, \dots, e_n\}$, let $l_1, \dots, l_n$ be elements in L, and write $l_j = \sum_{j=1}^{n} a_{ij} e_j$. Show that $\{l_1, \dots, l_n\}$ is a basis for L if and only if the matrix $(a_{ij})_{1\leqslant i,j\leqslant n}$ is invertible.

E 1.3.12 Show that a direct summand of a projective/injective/flat R-module is projective/injective/flat.

E 1.3.13 Dualize the proof of 1.3.27 to show that a coproduct of projective R-modules is projective. This provides an alternative proof of 1.3.24.

E 1.3.14 (Cf. 1.3.43) Show that every non-zero free R-module is faithfully flat, and show that $\mathbb{Q}$ as $\mathbb{Z}$-module is flat but not projective.

E 1.3.15 Show that $\mathbb{Z}^{\mathbb{N}}$ is not a projective $\mathbb{Z}$-module.

E 1.3.16 Show that $\mathbb{Z}$ is not a homomorphic image of an injective $\mathbb{Z}$-module.

E 1.3.17 Let R be left hereditary. Show that a submodule of a projective R-module is projective.

E 1.3.18 Let P be an R-module and P' a $\Bbbk$-module. Show: (a) If P and P' are free/projective, then the R-module $P \otimes_{\Bbbk} P'$ is free/projective. (b) If P' is faithfully projective, then the R-module $P \otimes_{\Bbbk} P'$ is (faithfully) projective if and only if P is (faithfully) projective.

E 1.3.19 Let F be an R-module and F' a flat $\Bbbk$-module. Show: (a) If F is flat, then the R-module $F \otimes_{\Bbbk} F'$ is flat. (b) If F' is faithfully flat, then the R-module $F \otimes_{\Bbbk} F'$ is (faithfully) flat if and only if F is (faithfully) flat.

E 1.3.20 Let $0 \to K \to L \to M \to 0$ be an exact sequence of R-modules with L free. If K is finitely generated, then M is said to be *finitely related*. Show that a finitely related flat R-module is projective. (A countably related flat module is almost projective; see D.9.)

E 1.3.21 In the ring $R = \bigtimes_{n\in\mathbb{N}} \mathbb{Q}$ consider the ideals $\mathfrak{a}^i = \{(q_n)_{n\in\mathbb{N}} \in R \mid q_n = 0 \text{ for all } n \neq i\}$ for every $i \in \mathbb{N}$. Show that each $\mathfrak{a}^i$ is an injective R-module but that $\coprod_{i\in\mathbb{N}} \mathfrak{a}^i$ is not. Show also that the R-module R is injective.

E 1.3.22 Let I be an R-module and P a projective $\Bbbk$-module. Show: (a) If I is injective, then the R-module $\operatorname{Hom}_{\Bbbk}(P, I)$ is injective. (b) If P is faithfully projective, then the R-module $\operatorname{Hom}_{\Bbbk}(P, I)$ is (faithfully) injective if and only if I is (faithfully) injective.

E 1.3.23 (a) Show that $\operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Z}) = 0$ holds for every injective $\mathbb{Z}$-module I. (b) Consider the full subcategory of $\mathcal{M}(\mathbb{Z})$ whose objects are the injective $\mathbb{Z}$-modules. Show that in this category the canonical homomorphism $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism and an epimorphism but not an isomorphism. *Hint:* 1.3.32.

E 1.3.24 Let $R \to S$ be a ring homomorphism. Show that if P is a free/(faithfully) projective R-module, then the S-module $S \otimes_R P$ is free/(faithfully) projective.

E 1.3.25 Let $R \to S$ be a ring homomorphism. Show that S is faithfully flat as an R-module if and only if the map $R \to S$ is injective and S/R is a flat R-module.

E 1.3.26 Let $R \to S$ be a ring homomorphism. Show that R has IBN if S has IBN.

E 1.3.27 Let $R \to S$ be a ring homomorphism. Show that if F is a (faithfully) flat R-module, then the S-module $S \otimes_R F$ is (faithfully) flat.

E 1.3.28 Let $R \to S$ be a ring homomorphism. Show that if I is a (faithfully) injective R-module, then the S-module $\operatorname{Hom}_R(S, I)$ is (faithfully) injective.

E 1.3.29 Let M be an R-module and E a faithfully injective R-module. Show that for every $m \neq 0$ in M there exists a homomorphism $\varphi \in \operatorname{Hom}_R(M, E)$ with $\varphi(m) \neq 0$.

E 1.3.30 Give a proof of 1.3.36 that does not use adjunction 1.2.6. *Hint:* Let $\widetilde{\gamma}\colon M \to E$ be a non-zero homomorphism of $\Bbbk$-modules. For $m \in M$ consider the map $\gamma_m\colon R \to E$ defined by $\gamma_m(r) = \widetilde{\gamma}(rm)$. Show that $m \mapsto \gamma_m$ is a non-zero homomorphism of R-modules. To prove injectivity, turn a map to $\operatorname{Hom}_{\Bbbk}(R, E)$ into a map to E by evaluation at 1.

E 1.3.31 There exists a family $\{A_u\}_{u\in U}$ of infinite subsets of $\mathbb{N}$ such that $\operatorname{card} U = 2^{\aleph_0}$ and $A_u \cap A_v$ is finite for all $u \neq v$ in U; see for example Sierpiński [232, IV.14]. Let $\Bbbk$ be a field and use this fact to show that a basis for $\Bbbk^{\mathbb{N}}$ has cardinality at least $2^{\aleph_0}$.

E 1.3.32 Let $\Bbbk$ be a field and set $R = \Bbbk[x]$. Show that $\operatorname{Hom}_{\Bbbk}(R, \Bbbk)$ is not a finitely generated R-module. *Hint:* E 1.3.31.

1.4 Evaluation Homomorphisms

Synopsis. Biduality; tensor evaluation; homomorphism evaluation.

With projective, injective, and flat modules now available, we continue the comparisons, started in Sect. 1.2, of composites of Hom and tensor product functors.

1.4.1 Definition. Let L be a free R-module with basis $\{e_u\}_{u\in U}$. For each $u \in U$ let $e_u^* \in \operatorname{Hom}_R(L, R)$ be given by $e_v \mapsto \delta_{uv}$, where δ denotes the Kronecker delta.

If $\{e_u\}_{u\in U}$ is a basis for a vector space L of finite rank, then the family of functionals $\{e_u^*\}_{u\in U}$ is a basis for the dual space L^*, and it is called the *dual basis*.

Biduality

For a vector space M of finite rank there is a canonical isomorphism from M to its double dual space M^{**}; it is given by evaluation: A basis element e_v is mapped to the functional given by $e_u^* \mapsto e_u^*(e_v) = \delta_{uv}$. This isomorphism is an instance of the *biduality* map that we consider next.

1.4.2 Lemma. *Let X be an R–S^{o}-bimodule. For an R-module M, the biduality map,*

$$\delta_X^M : M \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M, X), X)$$

given by

$$\delta_X^M(m)(\psi) = \psi(m)\,,$$

is a homomorphism of R-modules, and it is natural in M. Moreover, if M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$, then δ_X^M is a homomorphism in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$.

Proof. It is straightforward to verify that δ is a natural transformation of endofunctors on $\mathcal{M}(R)$; see the proof of 1.2.3.

If M is an R–Q^{o}-bimodule, then $\operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M, X), X)$ is an R–Q^{o}-bimodule as well. For $q \in Q$, $m \in M$, and $\psi \in \operatorname{Hom}_R(M, X)$ one has

$$\delta_X^M(mq)(\psi) = \psi(mq) = (q\psi)(m) = \delta_X^M(m)(q\psi) = (\delta_X^M(m)q)(\psi)\,;$$

that is, the homomorphism δ_X^M is Q^{o}-linear. $\square$

1.4.3 Proposition. *Let P be a finitely generated projective R-module. The R^{o}-module* $\mathrm{Hom}_R(P,R)$ *is finitely generated and projective, and biduality*

$$\delta_R^P : P \longrightarrow \mathrm{Hom}_{R^{\mathrm{o}}}(\mathrm{Hom}_R(P,R),R)$$

is an isomorphism.

Proof. By 1.3.47 the module P is a direct summand of a finitely generated free R-module L. Let $\{e_u\}_{u\in U}$ be a basis for L. The R^{o}-module $L^* = \mathrm{Hom}_R(L,R)$ is free with basis $\{e_u^*\}_{u\in U}$; see 1.4.1. Indeed, for every element φ in L^* one has $\varphi = \sum_{u\in U} e_u^*\varphi(e_u)$. The Hom functor is additive, so the R^{o}-module $\mathrm{Hom}_R(P,R)$ is a direct summand of L^*, whence it is finitely generated and projective.

Again because the Hom functor is additive, it is sufficient to show that δ_R^L is an isomorphism for a cyclic free module $L = R\langle e\rangle$. Let e^* be the functional $L \to R$ given by $e^*(e) = 1$; every element ψ in $\mathrm{Hom}_R(L,R)$ has the form e^*a with $a = \psi(e)$. As $\delta_R^L(e)(e^*) = e^*(e) = 1$ holds, the homomorphism δ_R^L is injective. Let ϑ be a homomorphism in $\mathrm{Hom}_{R^{\mathrm{o}}}(\mathrm{Hom}_R(L,R),R)$; for every $\psi \in \mathrm{Hom}_R(L,R)$ one has $\vartheta(\psi) = \vartheta(e^*\psi(e)) = \vartheta(e^*)\psi(e) = \vartheta(e^*)\delta_R^L(e)(\psi)$, and hence $\vartheta = b\delta_R^L(e) = \delta_R^L(be)$ with $b = \vartheta(e^*)$, so δ_R^L is surjective as well. □

Tensor Evaluation

For a vector space M of finite rank there is an isomorphism $M^* \otimes M \to \mathrm{Hom}(M,M)$; it assigns to a basis element $e_u^* \otimes e_v$ the linear map given by $e_w \mapsto e_u^*(e_w)e_v = \delta_{uw}e_v$. It is a special case of a map called *tensor evaluation*.

1.4.4 Lemma. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S-module. The tensor evaluation map,*

$$\theta^{MXN} : \mathrm{Hom}_R(M,X)\otimes_S N \longrightarrow \mathrm{Hom}_R(M, X\otimes_S N)$$

given by

$$\theta^{MXN}(\psi\otimes n)(m) = \psi(m)\otimes n\,,$$

is a homomorphism of $\Bbbk$-modules, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}T^{\mathrm{o}})$, then θ^{MXN} is a homomorphism in $\mathcal{M}(Q\text{–}T^{\mathrm{o}})$.

Proof. It is straightforward to verify that θ is a natural transformation of functors from $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R\text{–}S^{\mathrm{o}}) \times \mathcal{M}(S)$ to $\mathcal{M}(\Bbbk)$; see the proof of 1.2.3.

If M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(S\text{–}T^{\mathrm{o}})$, then $\mathrm{Hom}_R(M,X)\otimes_S N$ is a Q–T^{o}-bimodule and so is $\mathrm{Hom}_R(M, X\otimes_S N)$. The computation

$$\begin{aligned}
\theta^{MXN}(q(\psi\otimes n)t)(m) &= \theta^{MXN}(q\psi\otimes nt)(m)\\
&= (q\psi)(m)\otimes nt\\
&= \psi(mq)\otimes nt\\
&= (\psi(mq)\otimes n)t\\
&= (\theta^{MXN}(\psi\otimes n)(mq))t
\end{aligned}$$

$$= \ (q(\theta^{MXN}(\psi \otimes n))t)(m) \ ,$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \mathrm{Hom}_R(M,X)$, $m \in M$, and $n \in N$, shows that the homomorphism θ^{MXN} is Q- and T^{o}-linear. □

1.4.5 Example. Set $R = S = \Bbbk = \mathbb{Z}$. For the $\mathbb{Z}$-modules $M = \mathbb{Z}/2\mathbb{Z} = N$ and $X = \mathbb{Z}$, the homomorphism θ^{MXN} maps from 0 to $\mathbb{Z}/2\mathbb{Z}$, so it is not an isomorphism.

1.4.6 Proposition. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S-module. Tensor evaluation* 1.4.4*,*

$$\theta^{MXN} : \mathrm{Hom}_R(M,X) \otimes_S N \longrightarrow \mathrm{Hom}_R(M, X \otimes_S N) \ ,$$

is an isomorphism under any one of the following conditions.

(a) *M or N is finitely generated and projective.*
(b) *M is projective and N is finitely presented.*
(c) *M is finitely presented and N is flat.*

Proof. (a): For every R–S^{o}-bimodule X and S-module N the counitor 1.2.2 yields a commutative diagram,

$$\begin{tikzcd} \mathrm{Hom}_R(R,X) \otimes_S N \arrow[rr, "\theta^{RXN}"] & & \mathrm{Hom}_R(R, X \otimes_S N) \\ & X \otimes_S N \ , \arrow[ul, "\cong", "\epsilon^{X} \otimes N"'] \arrow[ur, "\cong"', "\epsilon^{X \otimes N}"] & \end{tikzcd}$$

which shows that θ^{RXN} is an isomorphism. A similar diagram involving the unitor 1.2.1 shows that θ^{MXS} is an isomorphism for every R-module M and every R–S^{o}-bimodule X. By additivity of the involved functors, it now follows that θ^{MXN} is an isomorphism if M or N is finitely generated and projective.

(b): Choose a presentation of N by finitely generated free S-modules

$$(\diamond) \qquad L' \longrightarrow L \longrightarrow N \longrightarrow 0 \ .$$

Consider the following diagram, which is commutative as θ is natural by 1.4.4.

$$\begin{tikzcd} \mathrm{Hom}_R(M,X) \otimes_S L' \arrow[r] \arrow[d, "\cong"', "\theta^{MXL'}"] & \mathrm{Hom}_R(M,X) \otimes_S L \arrow[r] \arrow[d, "\cong"', "\theta^{MXL}"] & \mathrm{Hom}_R(M,X) \otimes_S N \arrow[r] \arrow[d, "\theta^{MXN}"] & 0 \\ \mathrm{Hom}_R(M, X \otimes_S L') \arrow[r] & \mathrm{Hom}_R(M, X \otimes_S L) \arrow[r] & \mathrm{Hom}_R(M, X \otimes_S N) \arrow[r] & 0 \ . \end{tikzcd}$$

Either row in this diagram is exact. Indeed, they are obtained by applying the right exact functors $\mathrm{Hom}_R(M,X) \otimes_S -$ and $\mathrm{Hom}_R(M, X \otimes_S -)$ to ($\diamond$). Right exactness of the latter functor hinges on the assumption that M is projective. The maps $\theta^{MXL'}$ and θ^{MXL} are isomorphisms by part (a), and it follows from the Five Lemma that θ^{MXN} is an isomorphism.

(c): Choose a presentation $L' \to L \to M \to 0$ of M by finitely generated free R-modules. The functors $\mathrm{Hom}_R(-,X) \otimes_S N$ and $\mathrm{Hom}_R(-, X \otimes_S N)$ are left exact;

left exactness of the former functor hinges on the assumption that N is flat. As in the proof of (b) one thus gets the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(M,X)\otimes_S N \arrow[r] \arrow[d, "\theta^{MXN}"] & \operatorname{Hom}_R(L,X)\otimes_S N \arrow[r] \arrow[d, "\theta^{LXN}", "\cong"'] & \operatorname{Hom}_R(L',X)\otimes_S N \arrow[d, "\theta^{L'XN}", "\cong"'] \\
0 \arrow[r] & \operatorname{Hom}_R(M,X\otimes_S N) \arrow[r] & \operatorname{Hom}_R(L,X\otimes_S N) \arrow[r] & \operatorname{Hom}_R(L',X\otimes_S N)\,.
\end{tikzcd}$$

The maps θ^{LXN} and $\theta^{L'XN}$ are isomorphisms by part (a), and it follows from the Five Lemma that θ^{MXN} is an isomorphism. □

Homomorphism Evaluation

For a vector space M of finite rank there is an isomorphism $M \otimes M \to \operatorname{Hom}(M^*, M)$; it assigns to a basis element $e_v \otimes e_w$ the linear map given by $e_u^* \mapsto e_u^*(e_w)e_v = \delta_{uw}e_v$. It is a special case of a map called *homomorphism evaluation*.

1.4.7 Lemma. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S^{o}-module. The homomorphism evaluation map*

$$\eta^{MXN} : N \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M)$$

given by

$$\eta^{MXN}(n \otimes \psi)(\vartheta) = \psi\vartheta(n)$$

is a homomorphism of $\Bbbk$-modules, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(T\text{–}S^{\mathrm{o}})$, then η^{MXN} is a homomorphism in $\mathcal{M}(T\text{–}Q^{\mathrm{o}})$.

Proof. It is straightforward to verify that η is a natural transformation of functors from $\mathcal{M}(R) \times \mathcal{M}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{M}(S^{\mathrm{o}})$ to $\mathcal{M}(\Bbbk)$. See the proof of 1.2.3.

If M is in $\mathcal{M}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{M}(T\text{–}S^{\mathrm{o}})$, then $N \otimes_S \operatorname{Hom}_R(X, M)$ is a T–Q^{o}-bimodule, and so is $\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M)$. The computation

$$\begin{aligned}
\eta^{MXN}(t(n \otimes \psi)q)(\vartheta) &= \eta^{MXN}(tn \otimes \psi q)(\vartheta) \\
&= (\psi q)\vartheta(tn) \\
&= (\psi\vartheta(tn))q \\
&= (\psi(\vartheta t)(n))q \\
&= (\eta^{MXN}(n \otimes \psi)(\vartheta t))q \\
&= (t(\eta^{MXN}(n \otimes \psi))q)(\vartheta)\,,
\end{aligned}$$

which holds for all $q \in Q$, $t \in T$, $\psi \in \operatorname{Hom}_R(X, M)$, $n \in N$, and $\vartheta \in \operatorname{Hom}_{S^{\mathrm{o}}}(N, X)$, shows that the homomorphism η^{MXN} is T- and Q^{o}-linear. □

1.4.8 Example. Set $R = S = \Bbbk = \mathbb{Z}$. For the $\mathbb{Z}$-modules $M = \mathbb{Z}/2\mathbb{Z} = N$ and $X = \mathbb{Z}$, the homomorphism η^{MXN} maps from $\mathbb{Z}/2\mathbb{Z}$ to 0, so it is not an isomorphism.

1.4.9 Proposition. *Let M be an R-module, X an R–S^{o}-bimodule, and N an S^{o}-module. Homomorphism evaluation* 1.4.7*,*

$$\eta^{MXN} : N \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M) ,$$

is an isomorphism under either one of the following conditions.

(a) *N is finitely generated and projective.*
(b) *N is finitely presented and M is injective.*

Proof. (a): Using the unitor 1.2.1 and counitor 1.2.2 it is elementary to verify that η^{RXN} is an isomorphism for all R–S^{o}-bimodules X and all S^{o}-modules N. The claim then follows by additivity of the involved functors.

(b): Choose a presentation of N by finitely generated free S^{o}-modules

$$(\star) \qquad L' \longrightarrow L \longrightarrow N \longrightarrow 0 .$$

Consider the following diagram, which is commutative as η is natural by 1.4.7.

$$\begin{tikzcd}
L' \otimes_S \operatorname{Hom}_R(X, M) \arrow[r] \arrow[d, "\eta^{MXL'}", "\cong"'] & L \otimes_S \operatorname{Hom}_R(X, M) \arrow[r] \arrow[d, "\eta^{MXL}", "\cong"'] & N \otimes_S \operatorname{Hom}_R(X, M) \arrow[r] \arrow[d, "\eta^{MXN}"] & 0 \\
\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(L', X), M) \arrow[r] & \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(L, X), M) \arrow[r] & \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M) \arrow[r] & 0 .
\end{tikzcd}$$

Either row in this diagram is exact. Indeed, they are obtained by applying the right exact functors $- \otimes_S \operatorname{Hom}_R(X, M)$ and $\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(-, X), M)$ to $(\star)$. Right exactness of the latter functor hinges on the assumption that M is injective. The maps $\eta^{MXL'}$ and η^{MXL} are isomorphisms by part (a), and it follows from the Five Lemma that η^{MXN} is an isomorphism. $\square$

Exercises

E 1.4.1 Let L be a finitely generated free R-module with basis $\{e_u\}_{u \in U}$; show that the functionals e_u^* form a basis for the R^{o}-module $\operatorname{Hom}_R(L, R)$.

E 1.4.2 Let P be a finitely generated projective R-module; show that $\operatorname{Hom}_R(P, R)$ is a projective R^{o}-module.

E 1.4.3 Assume that R is right Noetherian and let M be a finitely generated R-module. Show that the R^{o}-module $\operatorname{Hom}_R(M, R)$ is finitely generated.

E 1.4.4 Set $E = \operatorname{Hom}_{\Bbbk}(R, \mathbb{E})$; show that biduality δ_E^M is injective for every R-module M.

E 1.4.5 Show that biduality 1.4.2 need not be injective nor surjective.

E 1.4.6 Let M be an R-module. Show that there is an injective homomorphism of R-modules $M \to I$ where I is injective.

E 1.4.7 Show that an R-module P is projective if and only if every homomorphism $I \twoheadrightarrow N$ with I injective induces a surjective homomorphism $\operatorname{Hom}_R(P, I) \to \operatorname{Hom}_R(P, N)$.

E 1.4.8 Show that R is left hereditary if and only if every quotient of an injective R-module is injective.

E 1.4.9 Show that an R-module is injective if and only if it is a direct summand of a module $\operatorname{Hom}_{\Bbbk}(L, \mathbb{E})$ where L is a free R^{o}-module.

E 1.4.10 For an R-module I, show that the next conditions are equivalent. (*i*) I is injective. (*ii*) For every left ideal $\mathfrak{a}$ in R, the homomorphism $\operatorname{Hom}_R(\iota, I)$, induced by the embedding $\iota\colon \mathfrak{a} \rightarrowtail R$, is surjective. (*iii*) Every exact sequence $0 \to I \to M \to M'' \to 0$ is split.

E 1.4.11 Assume that R is left Noetherian; let I be an R-module and F a flat $\Bbbk$-module. Show: (a) If I is injective, then so is the R-module $I \otimes_\Bbbk F$. (b) If F is faithfully flat, then the R-module $I \otimes_\Bbbk F$ is (faithfully) injective if and only if I is (faithfully) injective.

E 1.4.12 Let $\Bbbk$ be a field and $\{V_i\}_{i\in\mathbb{Z}}$ and $\{W_j\}_{j\in\mathbb{Z}}$ be families of finite rank $\Bbbk$-vector spaces. Show that the vector spaces $\coprod_{i\in\mathbb{Z}} V_i$ and $\prod_{j\in\mathbb{Z}} W_j$ are isomorphic if and only if the sets $I = \{i \in \mathbb{Z} \mid V_i \neq 0\}$ and $J = \{j \in \mathbb{Z} \mid W_j \neq 0\}$ are finite and one has $\sum_{i\in I} \operatorname{rank}_\Bbbk V_i = \sum_{j\in J} \operatorname{rank}_\Bbbk W_j$. *Hint:* E 1.3.31.

E 1.4.13 Let $\Bbbk$ be a field and M, N, and $X \neq 0$ be $\Bbbk$-vector spaces. Show that tensor evaluation $\theta^{MXN}\colon \operatorname{Hom}_\Bbbk(M,X) \otimes_\Bbbk N \to \operatorname{Hom}_\Bbbk(M, X \otimes_\Bbbk N)$ is an isomorphism if and only if M or N has finite rank. *Hint:* E 1.4.12.

E 1.4.14 Let $\Bbbk$ be a field and M, $X \neq 0$, and $N \neq 0$ be $\Bbbk$-vector spaces. Show that homomorphism evaluation $\eta^{MXN}\colon M \otimes_\Bbbk \operatorname{Hom}_\Bbbk(X,N) \to \operatorname{Hom}_\Bbbk(\operatorname{Hom}_\Bbbk(M,X),N)$ is an isomorphism if and only if M has finite rank. *Hint:* E 1.4.12.

E 1.4.15 Show that biduality 1.4.2 yields the unit of the adjunction from E 1.2.7.

E 1.4.16 Let X be an R–S^{o}-bimodule. Show that the unit of and counit of the adjunction 1.2.6 are the unique homomorphisms that make the following diagrams commutative

$$\begin{tikzcd}
S \otimes_S N \arrow[r, "\chi\otimes N"] \arrow[d, "\mu^N"', "\cong"] & \operatorname{Hom}_R(X,X) \otimes_S N \arrow[d, "\theta^{XXN}"] \\
N \arrow[r] & \operatorname{Hom}_R(X, X \otimes_S N)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes_S \operatorname{Hom}_R(X,M) \arrow[rr] \arrow[d, "\eta^{MXX}"'] & & M \\
\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(X,X),M) \arrow[rr, "\operatorname{Hom}(\chi,M)"] & & \operatorname{Hom}_R(R,M) \arrow[u, "\cong"', "\epsilon^M"]
\end{tikzcd} .$$

Here χ is the morphism from E 1.2.8.

E 1.4.17 Use homomorphism evaluation 1.4.7 to show that every finitely presented flat R-module is projective.

E 1.4.18 Let $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ be a right exact functor. Show that if $\mathrm{F}(R)$ is a finitely generated S-module, then $\mathrm{F}(M)$ is finitely generated for every finitely generated R-module M.

E 1.4.19 Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be a left exact functor. Show that S is left Noetherian and $\mathrm{G}(R)$ is a finitely generated S-module, then $\mathrm{G}(M)$ is finitely generated for every finitely generated R-module M.

E 1.4.20 Let $\tau\colon \mathrm{E} \to \mathrm{F}$ be a natural transformation of right exact functors $\mathcal{M}(R) \to \mathcal{M}(S)$. Show that if τ^R is an isomorphism, then τ^M is an isomorphism for every finitely presented R-module M.

E 1.4.21 Let $\tau\colon \mathrm{G} \to \mathrm{J}$ be a natural transformation of left exact functors $\mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$. Show that if τ^R is an isomorphism, then τ^M is an isomorphism for every finitely presented R-module M.

Chapter 2
Complexes

We now shift focus to complexes of modules. They are the objects in the derived category, and our first step towards that category is to study the Abelian category of complexes and their morphisms.

2.1 Definitions and Examples

SYNOPSIS. Graded module; graded homomorphism; complex; chain map; category of complexes of bimodules; Five Lemma; Snake Lemma; (degreewise) split exact sequence; $\natural$-functor.

Many modules carry an intrinsic structure: a grading. By imposing an additional structure, a square zero endomap that respects the grading, one arrives at the notion of a complex. The zero map respects any grading, so one can always consider a graded module as a complex. After a short opening discussion of graded modules, we move on to complexes, which abound in mathematics. We illustrate the concept with examples from algebra, geometry, and topology.

GRADED MODULES

2.1.1 Definition. Let U be a set. An R-module M is called *U-graded* if there exists a family $\{M_u\}_{u \in U}$ of submodules of M such that $M = \coprod_{u \in U} M_u$. A $\mathbb{Z}$-graded module is simply called a *graded module*.

2.1.2 Example. Considered as an $\Bbbk$-module, the polynomial algebra $M = \Bbbk[x]$ is graded with $M_v = 0$ for $v < 0$ and, in the notation from 1.3.1, $M_v = \Bbbk\langle x^v \rangle$ for $v \geqslant 0$.

2.1.3 Definition. Let $M = \coprod_{v \in \mathbb{Z}} M_v$ be a graded R-module. The submodule M_v is called the *module in degree v*. An element m in M that belongs to a submodule M_v is said to be *homogeneous* of degree v. Thus, the zero element is homogeneous of every degree. A homogeneous element $m \neq 0$ is homogeneous of exactly one degree, which is called the *degree* of m and denoted $|m|$; in symbols:

$$|m| = v \iff m \in M_v \,.$$

All subsequent formulas that involve the degree $|m|$ of an arbitrary homogeneous element m are valid no matter what value one assigns to $|0|$.

2.1.4 Definition. Let M and N be graded R-modules. Denote by $\operatorname{Hom}_R(M, N)$ the graded $\Bbbk$-module with

$$\operatorname{Hom}_R(M, N)_p = \prod_{v \in \mathbb{Z}} \operatorname{Hom}_R(M_v, N_{v+p}) \,;$$

it is called the *graded* Hom of M and N. An element of $\operatorname{Hom}_R(M, N)_p$ is called a *(graded) homomorphism of degree p*.

2.1.5. Let $M = \coprod_{v \in \mathbb{Z}} M_v$ be a graded R-module and N an R-module. A homomorphism α from M to N is identified with the family $(\alpha_v)_{v \in \mathbb{Z}}$ in $\prod_{v \in \mathbb{Z}} \operatorname{Hom}_R(M_v, N)$, given by $\alpha_v = \alpha\varepsilon_v$ where ε_v is the injection $M_v \rightarrowtail M$. Under this identification homomorphisms $\alpha\colon M \to N$ with $\alpha(M_v) \subseteq N_{v+p}$ for all $v \in \mathbb{Z}$ are in one-to-one correspondence with elements in $\operatorname{Hom}_R(M, N)_p$. This explains the terminology in 2.1.4.

2.1.6 Example. Set $M = \Bbbk[x]$ as in 2.1.2. The derivative $\frac{d}{dx}$ yields a homomorphism $M \to M$ of degree -1.

For graded R-modules M and N the symbol $\operatorname{Hom}_R(M, N)$ has two possible interpretations; namely the $\Bbbk$-module of all R-linear maps from M to N, and the graded $\Bbbk$-module defined in 2.1.4. The two interpretations do, in general, not agree, but whenever we consider graded modules, we have the second in mind.

2.1.7. Let L, M, and N be graded R-modules and p and q be integers. There is a $\Bbbk$-bilinear composition rule for graded homomorphisms:

$$\operatorname{Hom}_R(M, N)_p \times \operatorname{Hom}_R(L, M)_q \longrightarrow \operatorname{Hom}_R(L, N)_{p+q} \,.$$

For $\alpha = (\alpha_v)_{v \in \mathbb{Z}}$ and $\beta = (\beta_v)_{v \in \mathbb{Z}}$ it is given by $(\alpha, \beta) \mapsto \alpha\beta = (\alpha_{v+q}\beta_v)_{v \in \mathbb{Z}}$. Notice that the composite of homomorphisms of degree 0 is a homomorphism of degree 0.

2.1.8 Definition. A homomorphism of degree 0 is called a *morphism* of graded R-modules. The category of graded R-modules and their morphisms is denoted $\mathcal{M}_{\mathrm{gr}}(R)$.

2.1.9 Example. Let M be a graded R-module and r an element of R. The map $r^M\colon M \to M$ given by $m \mapsto rm$ for $m \in M$ is called a *homothety*. It is a morphism in $\mathcal{M}_{\mathrm{gr}}(\Bbbk)$, and if r is central, e.g. if r comes from $\Bbbk$, then r^M is a morphism in $\mathcal{M}_{\mathrm{gr}}(R)$.

Notice that the notation r^M for a homothety is in line with the notation 1^M for the identity morphism.

2.1.10 Definition. Let M be a graded R-module. A *graded submodule* of M is a graded R-module K that is a submodule of M as an R-module such that the embedding $K \to M$ is a morphism of graded R-modules.

Given a graded submodule K of a graded R-module M, the *graded quotient* is the graded module $M/K = \coprod_{v\in\mathbb{Z}} M_v/K_v$, and the canonical surjection $M \to M/K$ is a morphism of graded R-modules.

2.1.11 Example. Let $\alpha\colon M \to N$ be a graded homomorphism of graded R-modules. The kernel $\operatorname{Ker}\alpha = \{m \in M \mid \alpha(m) = 0\}$ is a graded submodule of M, and the image $\operatorname{Im}\alpha = \{\alpha(m) \mid m \in M\}$ is a graded submodule of N. The cokernel is the graded quotient module $\operatorname{Coker}\alpha = N/\operatorname{Im}\alpha$.

2.1.12. With the notions of kernels and cokernels from 2.1.11, the category $\mathcal{M}_{\mathrm{gr}}(R)$ is $\Bbbk$-linear and Abelian. The biproduct $M \oplus N$ of $M = \coprod_{v\in\mathbb{Z}} M_v$ and $N = \coprod_{v\in\mathbb{Z}} N_v$ in $\mathcal{M}_{\mathrm{gr}}(R)$ is the graded module $\coprod_{v\in\mathbb{Z}}(M_v \oplus N_v)$; one refers to $M \oplus N$ as a *graded direct sum* and to M and N as *graded direct summands*. See also 1.1.14.

2.1.13. There are exact $\Bbbk$-linear functors between Abelian categories,

$$\mathcal{M}(R) \rightleftarrows \mathcal{M}_{\mathrm{gr}}(R)\,.$$

The functor from $\mathcal{M}(R)$ to $\mathcal{M}_{\mathrm{gr}}(R)$ is full and faithful; it equips an R-module M with the trivial grading $M_0 = M$ and $M_v = 0$ for $v \neq 0$, and it acts analogously on homomorphisms. The functor from $\mathcal{M}_{\mathrm{gr}}(R)$ to $\mathcal{M}(R)$ forgets the grading.

At the level of symbols, applications of these functors is suppressed. However, when we write e.g. “as an R-module” about a graded R-module, it means that the forgetful functor is applied.

The category $\mathcal{M}_{\mathrm{gr}}(R)$ also has products and coproducts (and limits and colimits); they are all treated within the context of complexes; see Chap. 3.

2.1.14 Definition. Let M be a graded R^{o}-module and N a graded R-module. Denote by $M \otimes_R N$ the graded $\Bbbk$-module with

$$(M \otimes_R N)_p \;=\; \coprod_{v\in\mathbb{Z}} M_v \otimes_R N_{p-v}\,;$$

it is called the *graded tensor product* of M and N. Notice that if m and n are homogeneous elements in M and N, then $m \otimes n$ is homogeneous in $M \otimes_R N$ of degree $|m \otimes n| = |m| + |n|$.

The graded tensor product has the expected universal property.

2.1.15 Proposition. *Let M be a graded R^{o}-module, N a graded R-module, X a graded $\Bbbk$-module, and p an integer. For every family of $\Bbbk$-bilinear and middle R-linear maps*

$$\Big\{\varPhi_v :\ \biguplus_{i\in\mathbb{Z}} M_i \times N_{v-i} \;\longrightarrow\; X_{v+p}\Big\}_{v\in\mathbb{Z}}$$

there is a unique degree p homomorphism of graded $\Bbbk$-modules

$$\varphi : M \otimes_R N \;\longrightarrow\; X$$

with $\varphi(m \otimes n) = \varPhi_v(m, n)$ for all elements $m \in M$ and $n \in N$ with $|m \otimes n| = v$.

Proof. It is evident that such a morphism is unique if it exists. For existence, it is sufficient to verify that for each integer v there is a homomorphism of $\Bbbk$-modules $\varphi_v\colon (M \otimes_R N)_v = \coprod_{i\in\mathbb{Z}} M_i \otimes_R N_{v-i} \to X_{v+p}$ with $\varphi_v(m \otimes n) = \Phi_v(m,n)$. Fix v; for each $i \in \mathbb{Z}$ it follows from the universal property of tensor products that there is a homomorphism $\varphi_v^i\colon M_i \otimes_R N_{v-i} \to X_{v+p}$ with $\varphi_v^i(m \otimes n) = \Phi_v(m,n)$. Now the universal property of coproducts yields the desired homomorphism φ_v. □

2.1.16 Definition. Let $\mathcal{M}_{\mathrm{gr}}(R\text{–}S^{\mathrm{o}})$ denote the category $\mathcal{M}_{\mathrm{gr}}(R \otimes_\Bbbk S^{\mathrm{o}})$.

2.1.17. Graded R–S^{o}-bimodules and graded homomorphisms of such are defined as in 2.1.3 and 2.1.4. The category $\mathcal{M}_{\mathrm{gr}}(R\text{–}S^{\mathrm{o}})$ is naturally identified with the category whose objects are graded R–S^{o}-bimodules, and whose morphisms are R- and S^{o}-linear homomorphisms of degree 0; cf. 1.1.26.

2.1.18 Addendum (to 2.1.4 and 2.1.14). If M is a graded R–Q^{o}-bimodule and N is a graded R–S^{o}-bimodule, then it follows from 1.1.30 that the graded $\Bbbk$-module $\operatorname{Hom}_R(M,N)$ has a graded Q–S^{o}-bimodule structure.

If M is a graded Q–R^{o}-bimodule and N is a graded R–S^{o}-bimodule, then the graded $\Bbbk$-module $M \otimes_R N$ has a graded Q–S^{o}-bimodule structure; cf. 1.1.33.

2.1.19 Addendum (to 2.1.15). If M is in $\mathcal{M}_{\mathrm{gr}}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{M}_{\mathrm{gr}}(R\text{–}S^{\mathrm{o}})$, then the tensor product $M \otimes_R N$ is a graded Q–S^{o}-bimodule; see 2.1.18. If X is also in $\mathcal{M}_{\mathrm{gr}}(Q\text{–}S^{\mathrm{o}})$ and $\{\Phi_v\colon \biguplus_{i\in\mathbb{Z}} M_i \times N_{v-i} \to X_{v+p}\}_{v\in\mathbb{Z}}$ is a family of Q- and S^{o}-linear maps that are middle R-linear, then $\varphi\colon M \otimes_R N \to X$ is a morphism in $\mathcal{M}_{\mathrm{gr}}(Q\text{–}S^{\mathrm{o}})$.

It follows from 2.1.7 that if M is a graded R-module, then $\operatorname{Hom}_R(M,M)$ has a graded $\Bbbk$-algebra structure with multiplication given by composition of homomorphisms. The tensor powers of a module can also be assembled into a graded algebra.

2.1.20 Example. Assume that R is commutative and let M be an R-module. Consider the graded R-module $\mathrm{T}^R(M)$ with

$$\mathrm{T}^R_p(M) = 0 \text{ for } p < 0\,, \quad \mathrm{T}^R_0(M) = R\,, \quad\text{and}\quad \mathrm{T}^R_p(M) = M^{\otimes p} \text{ for } p > 0\,;$$

where $M^{\otimes p} = M \otimes_R \cdots \otimes_R M$ is the p-fold tensor product of M. The module $\mathrm{T}^R_p(M)$ is called the p^{th} *tensor power* of M. With multiplication given by concatenation of elementary tensors, $\mathrm{T}^R(M)$ is a graded R-algebra called the *tensor algebra* of M. To be precise, the product of elements $x_1 \otimes \cdots \otimes x_p$ in $\mathrm{T}^R_p(M)$ and $y_1 \otimes \cdots \otimes y_q$ in $\mathrm{T}^R_q(M)$ is the element $x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q$ in $\mathrm{T}^R_{p+q}(M)$.

Denote by $\mathfrak{H}$ the ideal in $\mathrm{T}^R(M)$ generated by the set $\{x \otimes y - y \otimes x \mid x, y \in M\}$ of homogeneous elements of degree 2; the graded quotient algebra $\mathrm{S}^R(M) = \mathrm{T}^R(M)/\mathfrak{H}$ is called the *symmetric algebra* of M, and the module $\mathrm{S}^R_p(M)$ in degree p is called the p^{th} *symmetric power* of M. One writes $x_1 \cdots x_p$ for the coset $[x_1 \otimes \cdots \otimes x_p]_{\mathfrak{H}}$ in $\mathrm{S}^R(M)$; notice that $\mathrm{S}^R(M)$ is commutative.

Denote by $\mathfrak{I}$ the ideal in $\mathrm{T}^R(M)$ generated by the set $\{x \otimes x \mid x \in M\}$ of homogeneous elements of degree 2; the graded quotient algebra $\bigwedge^R(M) = \mathrm{T}^R(M)/\mathfrak{I}$ is called the *exterior algebra* of M, and the module $\bigwedge^R_p(M)$ in degree p is called the

p^{th} *exterior power* of M. One writes $x_1 \wedge \cdots \wedge x_p$ for the coset $[x_1 \otimes \cdots \otimes x_p]_{\mathfrak{I}}$ in $\bigwedge^R(M)$ and it is called the *wedge product* of the elements $x_1, \ldots, x_p$. As elements of the form $x \otimes y + y \otimes x = (x+y) \otimes (x+y) - x \otimes x - y \otimes y$ belong to $\mathfrak{I}$, one has $x_1 \wedge \cdots \wedge x_i \wedge x_{i+1} \wedge \cdots \wedge x_p = -x_1 \wedge \cdots \wedge x_{i+1} \wedge x_i \wedge \cdots \wedge x_p$.

REMARK. The algebras in 2.1.20 have universal properties; see E 2.1.7–E 2.1.9.

2.1.21 Example. Assume that R is commutative and let L be a free R-module with basis $x_1, \ldots, x_n$. The p^{th} tensor power $\mathrm{T}^R_p(L)$ is by 1.3.10 free of rank n^p.

The symmetric algebra $\mathrm{S}^R(L)$ is evidently isomorphic to the polynomial algebra $R[x_1, \ldots, x_n]$, so the monomials in $x_1, \ldots, x_n$ of degree p form a basis for the p^{th} symmetric power of L, whence one has $\operatorname{rank}_R \mathrm{S}^R_p(L) = \binom{n-1+p}{n-1}$.

The p^{th} exterior power $\bigwedge^R_p(L)$ is evidently generated by the $\binom{n}{p}$ elements

$$\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p} \mid 1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n\} \,.$$

To see that they form a basis notice the isomorphism

$$\textstyle\bigwedge^R_p(L) \;\cong\; \bigwedge^R_p(R\langle x_1, \ldots, x_{n-1}\rangle) \oplus \bigwedge^R_{p-1}(R\langle x_1, \ldots, x_{n-1}\rangle)$$

and conclude by induction on n that one has $\operatorname{rank}_R \bigwedge^R_p(L) = \binom{n-1}{p} + \binom{n-1}{p-1} = \binom{n}{p}$.

We return to the exterior algebra of a free module in 2.1.25.

COMPLEXES

2.1.22 Definition. An *R-complex*, also called a *complex of R-modules,* is a graded R-module M equipped with a homomorphism $\partial^M \colon M \to M$ of degree -1, called the *differential*, that satisfies $\partial^M \partial^M = 0$. It can be visualized as follows:

$$\cdots \longrightarrow M_{v+1} \xrightarrow{\partial^M_{v+1}} M_v \xrightarrow{\partial^M_v} M_{v-1} \longrightarrow \cdots \,.$$

The module M_v is the module in degree v; the homomorphism $\partial^M_v \colon M_v \to M_{v-1}$ is the v^{th} differential, and $\partial^M_v \partial^M_{v+1} = 0$ holds for all $v \in \mathbb{Z}$.

Given an R-complex M, the underlying graded R-module is denoted $M^\natural$.

As highlighted by Dold [73] an elementary complex serves to illustrate many aspects of the homological theory of complexes.

2.1.23 Example. Over the ring $\mathbb{Z}/4\mathbb{Z}$ consider the graded module with $\mathbb{Z}/4\mathbb{Z}$ in each degree. Endowed with the degree -1 homomorphism that in each degree is multiplication by 2, it is a $\mathbb{Z}/4\mathbb{Z}$-complex,

$$\cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots \,,$$

called the *Dold complex*.

REMARK. Another word for complex is 'differential graded module'. In the notation for complexes introduced in 2.1.22, degrees are written as subscripts and descend in the direction of the arrows; this is known as "homological notation". In "cohomological notation", degrees are written as superscripts and ascend in the direction of the arrows; in this notation, a complex M is visualized as:

$$\cdots \longrightarrow M^{v-1} \xrightarrow{\partial_M^{v-1}} M^v \xrightarrow{\partial_M^v} M^{v+1} \longrightarrow \cdots .$$

In the literature, there is a strong tradition for employing homological (cohomological) notation for complexes that are bounded below (above) in the sense of 2.5.2; such complexes are often referred to as 'chain (cochain) complexes'. Switching between homological and cohomological notation, it is standard to set $M_v = M^{-v}$. We have no proclivity for complexes bounded on either side, but we settle on homological notation and only deviate from it in a few examples, such as 2.1.27 below.

2.1.24 Definition. Let M be an R-complex and $u \geqslant w$ be integers. The complex M is said to be *concentrated in degrees* $u, \ldots, w$ if $M_v = 0$ holds for all $v \notin \{w, \ldots, u\}$; it is then visualized like this:

$$0 \longrightarrow M_u \longrightarrow M_{u-1} \longrightarrow \cdots \longrightarrow M_{w+1} \longrightarrow M_w \longrightarrow 0 .$$

2.1.25 Example. Assume that R is commutative. Let M be an R-module and $\varepsilon\colon M \to R$ a homomorphism. Recall from 2.1.20 that $\mathrm{T}^R(M)$ and $\bigwedge^R(M)$ are the tensor algebra and the exterior algebra of M. Consider the degree -1 homomorphism $\delta\colon \mathrm{T}^R(M) \to \bigwedge^R(M)$ given by

$$\delta(x_1 \otimes \cdots \otimes x_p) \;=\; \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i) x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p .$$

Every element in the ideal $\mathfrak{I} \subseteq \mathrm{T}^R(M)$ from 2.1.20 is an R-linear combination of elements of the form $y_1 \otimes \cdots \otimes y_p \otimes x \otimes x \otimes z_1 \otimes \cdots \otimes z_q$; since δ is zero on such elements, it factors through the exterior algebra, yielding a degree -1 homomorphism $\partial\colon \bigwedge^R(M) \to \bigwedge^R(M)$ given by

$$\partial(x_1 \wedge \cdots \wedge x_p) \;=\; \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i) x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p .$$

This homomorphism is square zero; indeed, one has

$$\begin{aligned}
&\partial\partial(x_1 \wedge \cdots \wedge x_p) \\
&= \sum_{i=1}^{p} (-1)^{i+1} \varepsilon(x_i) \partial(x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p) \\
&= \sum_{j<i} (-1)^{(i+1)(j+1)} \varepsilon(x_i)\varepsilon(x_j) x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_p \\
&\quad + \sum_{i<k} (-1)^{(i+1)k} \varepsilon(x_i)\varepsilon(x_k) x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{k-1} \wedge x_{k+1} \wedge \cdots \wedge x_p \\
&= 0 .
\end{aligned}$$

The R-complex with underlying graded module $\bigwedge^R(M)$ and differential ∂ is called the *Koszul complex* over ε and denoted $\mathrm{K}^R(\varepsilon)$.

For a sequence $x_1, \ldots, x_n$ in R, one writes $\mathrm{K}^R(x_1, \ldots, x_n)$ for the Koszul complex over the canonical homomorphism from the free module $R\langle e_1, \ldots, e_n \rangle$ to the ideal $(x_1, \ldots, x_n)$. Notice that in this important special case, the differential is given by

$$\partial^{\mathrm{K}^R(x_1,\dots,x_n)}(e_{h_1} \wedge \cdots \wedge e_{h_p}) = \sum_{i=1}^{p} (-1)^{i+1} x_i e_{h_1} \wedge \cdots \wedge e_{h_{i-1}} \wedge e_{h_{i+1}} \wedge \cdots \wedge e_{h_p} .$$

It is standard to set $\mathrm{K}^R_v(x_1,\dots,x_n) = \mathrm{K}^R(x_1,\dots,x_n)_v$, and the superscript R may be suppressed if the ring is understood. Recall from 2.1.21 that $\mathrm{K}^R_v(x_1,\dots,x_n)$ is a free R-module of rank $\binom{n}{v}$ and notice that the differential on $\mathrm{K}^R(x_1,\dots,x_n)$ is represented by matrices with entries in the ideal $(x_1,\dots,x_n)$.

2.1.26 Example. Consider for each $n \in \mathbb{N}_0$ the standard n-simplex,

$$\Delta^n = \Big\{ (t_0,\dots,t_n) \in \mathbb{R}^{n+1} \;\Big|\; \sum_{j=0}^{n} t_j = 1 \text{ and } t_0,\dots,t_n \geqslant 0 \Big\} .$$

For $n \geqslant 1$ and $i \in \{0,\dots,n\}$ the i^{th} face map $\epsilon^n_i \colon \Delta^{n-1} \to \Delta^n$ is given by

$$(t_0,\dots,t_{n-1}) \longmapsto (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1}) .$$

Let X be a topological space. Denote by $\mathrm{C}(\Delta^n, X)$ the set of all continuous maps from Δ^n to X, and consider the free Abelian group $\mathrm{S}_n(X) = \mathbb{Z}\langle \mathrm{C}(\Delta^n, X)\rangle$ on this set. The elements of $\mathrm{S}_n(X)$ are known as singular n-chains. For $n \geqslant 1$, the map

$$\mathrm{C}(\Delta^n, X) \longrightarrow \mathrm{S}_{n-1}(X) \qquad \text{given by} \qquad \sigma \longmapsto \sum_{i=0}^{n} (-1)^i \sigma \epsilon^n_i$$

extends uniquely by 1.3.6 to a group homomorphism $\partial^X_n \colon \mathrm{S}_n(X) \to \mathrm{S}_{n-1}(X)$. Set $\partial^X_0 = 0$, then $\partial^X_0 \partial^X_1 = 0$ holds; for $n \geqslant 1$ and a basis element σ in $\mathrm{S}_{n+1}(X)$ one has

$$\begin{aligned}
\partial^X_n \partial^X_{n+1}(\sigma) &= \partial^X_n\Big(\sum_{i=0}^{n+1} (-1)^i \sigma \epsilon^{n+1}_i\Big) \\
&= \sum_{j=0}^{n} (-1)^j \Big(\sum_{i=0}^{n+1} (-1)^i \sigma \epsilon^{n+1}_i\Big) \epsilon^n_j \\
&= \sum_{j<i} (-1)^{i+j} \sigma \epsilon^{n+1}_i \epsilon^n_j + \sum_{i \leqslant j} (-1)^{i+j} \sigma \epsilon^{n+1}_i \epsilon^n_j .
\end{aligned}$$

For $0 \leqslant j < i \leqslant n+1$ one has $\epsilon^{n+1}_i \epsilon^n_j = \epsilon^{n+1}_j \epsilon^n_{i-1}$ and, therefore,

$$\sum_{j<i} (-1)^{i+j} \sigma \epsilon^{n+1}_i \epsilon^n_j = \sum_{j<i} (-1)^{i+j} \sigma \epsilon^{n+1}_j \epsilon^n_{i-1} = \sum_{j \leqslant i} (-1)^{i+j+1} \sigma \epsilon^{n+1}_j \epsilon^n_i .$$

In combination, these displays yield $\partial^X_n \partial^X_{n+1}(\sigma) = 0$, and thus one has a $\mathbb{Z}$-complex,

$$\mathrm{S}(X) = \cdots \longrightarrow \mathrm{S}_2(X) \xrightarrow{\partial^X_2} \mathrm{S}_1(X) \xrightarrow{\partial^X_1} \mathrm{S}_0(X) \longrightarrow 0 .$$

This complex is called the *singular chain complex* of the space X.

Let A be an Abelian group. Application of the functors $- \otimes_{\mathbb{Z}} A$ and $\operatorname{Hom}_{\mathbb{Z}}(-, A)$ to the complex $\mathrm{S}(X)$ in each degree yields complexes called the *singular chain* and *singular cochain complex* of X with coefficients in A.

2.1.27 Example. Let M be a smooth d-dimensional real manifold. For a point x on M, denote by $\mathrm{C}^\infty_x(M)$ the germ of smooth functions at x; it is an $\mathbb{R}$-algebra. A linear map $v \colon \mathrm{C}^\infty_x(M) \to \mathbb{R}$ with $v(fg) = v(f)g(x) + f(x)v(g)$ for all $f, g \in \mathrm{C}^\infty_x(M)$ is called a derivation at x. The set M_x of all derivations is a d-dimensional vector space, called the tangent space at x. Given a chart $\varphi = (\varphi^1, \dots, \varphi^d)\colon U \to \mathbb{R}^d$, where U is a neighborhood of x, one can form a basis $\frac{\partial}{\partial\varphi^1}\big|_x, \dots, \frac{\partial}{\partial\varphi^d}\big|_x$ for M_x; for f in $\mathrm{C}^\infty_x(M)$ the derivation $\frac{\partial}{\partial\varphi^i}\big|_x$ is defined by

$$\frac{\partial}{\partial\varphi^i}\Big|_x(f) \;=\; \mathrm{D}_i(f \circ \varphi^{-1})\Big|_{\varphi(x)} \;,$$

where $\mathrm{D}_i(\,\cdot\,)\big|_{\varphi(x)}$ is the i^{th} derivative at the point $\varphi(x) \in \mathbb{R}^d$. The dual space M_x^* is called the cotangent space at x, and the dual basis is denoted by $(\mathrm{d}\varphi^1)_x, \dots, (\mathrm{d}\varphi^d)_x$. More generally, for a smooth function $f\colon U \to \mathbb{R}$ its differential at x is the functional $(\mathrm{d}f)_x \in M_x^*$ given by $(\mathrm{d}f)_x(v) = v(f)$ for $v \in M_x$.

Fix $k \in \{0, \dots, d\}$ and denote by $\bigwedge_k^{\mathbb{R}}(M_x^*)$ the k^{th} exterior power of M_x^*, which is a vector space of dimension $q = \binom{d}{k}$. One can show that the family $\{\bigwedge_k^{\mathbb{R}}(M_x^*)\}_{x\in M}$ can be assembled to a smooth vector bundle E^k on M of rank q; the total space is $E^k = \biguplus_{x\in M} \bigwedge_k^{\mathbb{R}}(M_x^*)$ and the bundle projection $\pi\colon E^k \twoheadrightarrow M$ maps $\bigwedge_k^{\mathbb{R}}(M_x^*)$ to x.

A differential form of degree k is a smooth section of $\pi\colon E^k \twoheadrightarrow M$, i.e. a smooth map $\omega\colon M \to E^k$ with $\pi\omega = 1^M$. The vector space of all such maps is denoted $\Omega^k(M)$.

As one has $\bigwedge_0^{\mathbb{R}}(M_x^*) = \mathbb{R}$, the bundle $E^0 = M \times \mathbb{R}$ is trivial. Thus, an element in $\Omega^0(M)$ is nothing but a smooth map of the form $(1^M, f)\colon M \to M \times \mathbb{R}$. Hence one naturally identifies $\Omega^0(M)$ with the set $\mathrm{C}^\infty(M)$ of smooth functions $M \to \mathbb{R}$.

As one has $\bigwedge_1^{\mathbb{R}}(M_x^*) = M_x^*$, an element in $\Omega^1(M)$ is a smooth map $\omega\colon M \to E^1$ such that $\omega_x = \omega(x)$ belongs to M_x^* for every $x \in M$. In particular, $\omega = \mathrm{d}f$ is a differential form of degree 1 for every f in $\mathrm{C}^\infty(M)$. Thus, the differential yields a map $\Omega^0(M) \to \Omega^1(M)$. This map $\mathrm{d}^0 = \mathrm{d}$ is part of the *de Rham complex*, which traditionally is written in cohomological notation,

$$\Omega(M) \;=\; 0 \longrightarrow \Omega^0(M) \xrightarrow{\;\mathrm{d}^0\;} \Omega^1(M) \xrightarrow{\;\mathrm{d}^1\;} \Omega^2(M) \xrightarrow{\;\mathrm{d}^2\;} \cdots \;.$$

The differential on $\Omega(M)$ is called the exterior derivative. To define it, notice that the wedge product $\wedge\colon \bigwedge_i^{\mathbb{R}}(M_x^*) \times \bigwedge_j^{\mathbb{R}}(M_x^*) \to \bigwedge_{i+j}^{\mathbb{R}}(M_x^*)$ on the exterior algebra induces a pairing $\wedge\colon \Omega^i(M) \times \Omega^j(M) \to \Omega^{i+j}(M)$, defined by $(\omega \wedge \psi)_x = \omega_x \wedge \psi_x$. The exterior derivative is defined recursively by the formula

$$\mathrm{d}^{i+j}(\omega \wedge \psi) \;=\; \mathrm{d}^i(\omega) \wedge \psi + (-1)^i \omega \wedge \mathrm{d}^j(\psi)$$

for $\omega \in \Omega^i(M)$ and $\psi \in \Omega^j(M)$.

2.1.28 Definition. A *homomorphism of complexes* is a graded homomorphism of the underlying graded modules; see 2.1.4. Let M and N be R-complexes, a *chain map* $M \to N$ is homomorphism $\alpha\colon M \to N$ that satisfies

$$\partial^N \alpha \;=\; (-1)^{|\alpha|} \alpha \partial^M \;.$$

This means that every square in the next diagram commutes up to the sign $(-1)^{|\alpha|}$.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{v+1} \arrow[r, "\partial^M_{v+1}"] \arrow[d, "\alpha_{v+1}"'] \arrow[dr, phantom, "(-1)^{|\alpha|}"] & M_v \arrow[r, "\partial^M_v"] \arrow[d, "\alpha_v"] \arrow[dr, phantom, "(-1)^{|\alpha|}"] & M_{v-1} \arrow[r] \arrow[d, "\alpha_{v-1}"] & \cdots \\
\cdots \arrow[r] & N_{v+1+|\alpha|} \arrow[r, "\partial^N_{v+1+|\alpha|}"'] & N_{v+|\alpha|} \arrow[r, "\partial^N_{v+|\alpha|}"'] & N_{v-1+|\alpha|} \arrow[r] & \cdots ,
\end{tikzcd}$$

A chain map of degree 0 is called a *morphism* of R-complexes.

REMARK. The sign above follows the Koszul Sign Convention: A sign $(-1)^{mn}$ is introduced when elements of degree m and n are interchanged; in this case, $(-1)^{|\alpha||\partial|} = (-1)^{-|\alpha|} = (-1)^{|\alpha|}$.

2.1.29 Example. Let M be an R-complex. The differential ∂^M is a chain map of R-complexes of degree -1. For a central element $x \in R$ the homothety x^M is a morphism of R-complexes; see 2.1.9.

2.1.30 Example. Assume that R is commutative. Let

$$(x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n, y_1, \ldots, y_m)$$

be ideals in R, with the notation from 2.1.25 it is elementary to verify that the map $\mathrm{K}^R(x_1, \ldots, x_n) \to \mathrm{K}^R(x_1, \ldots, x_n, y_1, \ldots, y_m)$ given by $x_1 \wedge \cdots \wedge x_p \mapsto x_1 \wedge \cdots \wedge x_p$ is a morphism of R-complexes.

2.1.31. It is straightforward to verify that a composite of chain maps is a chain map. In particular, a composite of morphisms is a morphism; cf. 2.1.7.

2.1.32 Definition. The category R-complexes and their morphisms is denoted $\mathcal{C}(R)$.

2.1.33 Definition. Let M be an R-complex. A graded submodule K of $M^\natural$, with the property that $\partial^M(K)$ is contained in K, is an R-complex when endowed with the differential $\partial^M|_K$; it is called a *subcomplex* of M. In this case, the embedding $K \rightarrowtail M$ is a morphism of R-complexes.

Given a subcomplex K of M, the differential ∂^M induces a differential on the graded module $M^\natural/K^\natural$. The resulting complex M/K is called a *quotient complex*. Note that the canonical map $M \twoheadrightarrow M/K$ is a morphism of R-complexes.

2.1.34 Example. Let M be a smooth real manifold. Recall from 2.1.26 that the modules in the singular chain complex $\mathrm{S}(M)$ are $\mathrm{S}_n(M) = \mathbb{Z}\langle \mathrm{C}(\Delta^n, M)\rangle$ for $n \geqslant 0$ and $\mathrm{S}_n(M) = 0$ for $n < 0$. It is immediate from the definitions that the graded submodule $\mathrm{S}^\infty(M)$ with $\mathrm{S}^\infty_n(M) = \mathbb{Z}\langle \mathrm{C}^\infty(\Delta^n, M)\rangle$ for $n \geqslant 0$ is a subcomplex of $\mathrm{S}(M)$.

With the notation from 2.1.26 and 2.1.27 there exists a morphism of $\mathbb{R}$-complexes $\Omega(M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{S}^\infty(M), \mathbb{R})$ that maps a differential form ω of degree n to the group homomorphism $\mathrm{S}^\infty_n(M) \to \mathbb{R}$ given by $\sigma \mapsto \int_\sigma \omega = \int_{\Delta^n} \sigma^*\omega$.

2.1.35. Let $\alpha\colon M \to N$ be a chain map of R-complexes. It is straightforward to verify that $\operatorname{Ker}\alpha$, $\operatorname{Im}\alpha$, and $\operatorname{Coker}\alpha$ are R-complexes, see 2.1.11, and that the category $\mathcal{C}(R)$ is $\Bbbk$-linear and Abelian. The biproduct of complexes M and N is the graded direct sum $M^\natural \oplus N^\natural$ from 2.1.12 endowed with the differential $\partial^M \oplus \partial^N$.

2.1.36. There are exact $\Bbbk$-linear functors between Abelian categories,

$$\mathcal{M}_{\mathrm{gr}}(R) \underset{(-)^\natural}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{C}(R)\;.$$

The functor from $\mathcal{M}_{\mathrm{gr}}(R)$ to $\mathcal{C}(R)$ is full and faithful; it equips a graded module M with the trivial differential $\partial^M = 0$, and it is the identity on morphisms. The functor $(-)^\natural$ from $\mathcal{C}(R)$ to $\mathcal{M}_{\mathrm{gr}}(R)$ forgets the differential on R-complexes, and it is the identity on morphisms. At the level of symbols, applications of these functors is often suppressed. When we write e.g. "as an R-complex" about a graded R-module, it means that the functor $\mathcal{M}_{\mathrm{gr}}(R) \to \mathcal{C}(R)$ is applied.

In particular, graded R-modules are isomorphic if and only if they are isomorphic as R-complexes. Further, a morphism $M \to N$ of R-complexes is an isomorphism if and only if it is an isomorphism of the underlying graded modules.

The module category $\mathcal{M}(R)$ is a full subcategory of $\mathcal{C}(R)$ via the full and faithful functors $\mathcal{M}(R) \to \mathcal{M}_{\mathrm{gr}}(R) \to \mathcal{C}(R)$; see 2.1.13.

2.1.37 Definition. Let $\mathcal{C}(R\text{–}S^{\mathrm{o}})$ denote the category $\mathcal{C}(R \otimes_{\Bbbk} S^{\mathrm{o}})$.

2.1.38. Complexes of R–S^{o}-bimodules as well as homomorphisms and chain maps of such are defined as in 2.1.22 and 2.1.28. The category $\mathcal{C}(R\text{–}S^{\mathrm{o}})$ is naturally identified with the category whose objects are complexes of R–S^{o}-bimodules (with R- and S^{o}-linear differentials), and whose morphisms are R- and S^{o}-linear chain maps of degree 0; cf. 2.1.17.

Diagram Lemmas

In the next several paragraphs, in 2.1.39–2.1.47 to be precise, the category of complexes could be replaced by any Abelian category, and in that sense they repeat material from Sect. 1.1. We include them, nevertheless, for ease of reference, and we provide proofs, because the material is central.

2.1.39 Definition. A sequence of R-complexes is a, possibly infinite, diagram,

$$\cdots \longrightarrow M^0 \xrightarrow{\alpha^0} M^1 \xrightarrow{\alpha^1} M^2 \longrightarrow \cdots\;, \tag{2.1.39.1}$$

in $\mathcal{C}(R)$; it is called *exact* if $\operatorname{Im}\alpha^{n-1} = \operatorname{Ker}\alpha^n$ holds for all n. Notice that (2.1.39.1) is exact if and only if every sequence $0 \to \operatorname{Im}\alpha^{n-1} \to M^n \to \operatorname{Im}\alpha^n \to 0$ is exact. An exact sequence $0 \to M' \to M \to M'' \to 0$ is called a *short exact sequence*.

Two sequences $\{\alpha^n : M^n \to M^{n+1}\}_{n\in\mathbb{Z}}$ and $\{\beta^n : N^n \to N^{n+1}\}_{n\in\mathbb{Z}}$ of R-complexes are called *isomorphic* if there exists a family of isomorphisms $\{\varphi^n\}_{n\in\mathbb{Z}}$ such that the diagram

$$\begin{array}{ccc} M^n & \xrightarrow{\alpha^n} & M^{n+1} \\ {\scriptstyle\varphi^n}\downarrow & & \downarrow{\scriptstyle\varphi^{n+1}} \\ N^n & \xrightarrow{\beta^n} & N^{n+1} \end{array}$$

is commutative for every $n \in \mathbb{Z}$.

2.1.40. Exactness of a sequence in $\mathcal{C}(R)$ does not depend on the differentials, so it can be detected degreewise. That is, (2.1.39.1) is exact if and only the sequence

$$\cdots \longrightarrow M_v^0 \xrightarrow{\alpha_v^0} M_v^1 \xrightarrow{\alpha_v^1} M_v^2 \xrightarrow{\alpha_v^2} \cdots$$

in $\mathcal{M}(R)$ is exact for every $v \in \mathbb{Z}$.

The next result is known as the Five Lemma.

2.1.41 Lemma. *Consider a commutative diagram in $\mathcal{C}(R)$ with exact rows,*

$$\begin{tikzcd} M^1 \arrow[r] \arrow[d, "\varphi^1"] & M^2 \arrow[r] \arrow[d, "\varphi^2"] & M^3 \arrow[r] \arrow[d, "\varphi^3"] & M^4 \arrow[r] \arrow[d, "\varphi^4"] & M^5 \arrow[d, "\varphi^5"] \\ N^1 \arrow[r] & N^2 \arrow[r] & N^3 \arrow[r] & N^4 \arrow[r] & N^5 \,. \end{tikzcd}$$

(a) *If φ^1 is surjective, and φ^2 and φ^4 are injective, then φ^3 is injective.*
(b) *If φ^5 is injective, and φ^2 and φ^4 are surjective, then φ^3 is surjective.*
(c) *If φ^1, φ^2, φ^4, and φ^5 are isomorphisms, then φ^3 is an isomorphism.*

Proof. (a): The assumptions imply for each $v \in \mathbb{Z}$ that the homomorphism φ_v^1 is surjective, and the homomorphisms φ_v^2 and φ_v^4 are injective. Thus, the Five Lemma for modules 1.1.2 applied to the degree v part of the given diagram yields that φ_v^3 is injective. Since this holds for every $v \in \mathbb{Z}$, it follows that φ^3 is injective.

The proofs of parts (b) and (c) are similar. □

2.1.42 Lemma. *Consider a commutative diagram in $\mathcal{C}(R)$,*

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[d, "\varphi''"] \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \,, \end{tikzcd}$$

with exact rows. There is an exact sequence,

$$\operatorname{Ker}\varphi' \xrightarrow{\alpha'} \operatorname{Ker}\varphi \xrightarrow{\alpha} \operatorname{Ker}\varphi'' \,.$$

If α' is injective, then so is the restriction $\alpha' \colon \operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi$.

Proof. The assertion follows by degreewise application of 1.1.3. □

2.1.43 Lemma. *Consider a commutative diagram in $\mathcal{C}(R)$,*

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \,, \end{tikzcd}$$

with exact rows. There is an exact sequence,

$$\operatorname{Coker}\varphi' \xrightarrow{\bar{\beta}'} \operatorname{Coker}\varphi \xrightarrow{\bar{\beta}} \operatorname{Coker}\varphi'' .$$

If β is surjective, then so is the induced homomorphism $\bar{\beta}\colon \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$.

Proof. The assertion follows by degreewise application of 1.1.4. □

2.1.44 Construction. Consider a commutative diagram in $\mathcal{C}(R)$,

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' , & \end{tikzcd}$$

with exact rows. In each degree v it yields a commutative diagram in $\mathcal{M}(R)$, to which 1.1.5 applies to give a homomorphism $\delta_v\colon \operatorname{Ker}\varphi''_v \to \operatorname{Coker}\varphi'_v$. Denote by $\delta\colon (\operatorname{Ker}\varphi'')^\natural \to (\operatorname{Coker}\varphi')^\natural$ the morphism $(\delta_v)_{v\in\mathbb{Z}}$ of graded R-modules.

The next result is known as the Snake Lemma; the morphism δ is called the *connecting morphism*.

2.1.45 Lemma. *The morphism δ of graded R-modules, defined in* 2.1.44*, is a morphism of R-complexes, and there is an exact sequence in $\mathcal{C}(R)$,*

$$\operatorname{Ker}\varphi' \xrightarrow{\alpha'} \operatorname{Ker}\varphi \xrightarrow{\alpha} \operatorname{Ker}\varphi'' \xrightarrow{\delta} \operatorname{Coker}\varphi' \xrightarrow{\bar{\beta}'} \operatorname{Coker}\varphi \xrightarrow{\bar{\beta}} \operatorname{Coker}\varphi'' .$$

Moreover, if α' is injective then so is the restricted morphism $\alpha'\colon \operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi$, and if β is surjective, then so is the induced morphism $\bar{\beta}\colon \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$.

Proof. By the Snake Lemma for modules, 1.1.6, there is an exact sequence

$$\begin{aligned} &(\operatorname{Ker}\varphi')_v \xrightarrow{\alpha'_v} (\operatorname{Ker}\varphi)_v \xrightarrow{\alpha_v} (\operatorname{Ker}\varphi'')_v \xrightarrow{\delta_v} \\ &\qquad\qquad (\operatorname{Coker}\varphi')_v \xrightarrow{\bar{\beta}'_v} (\operatorname{Coker}\varphi)_v \xrightarrow{\bar{\beta}_v} (\operatorname{Coker}\varphi'')_v \end{aligned}$$

for every $v \in \mathbb{Z}$. The restricted maps $\alpha'\colon \operatorname{Ker}\varphi' \to \operatorname{Ker}\varphi$ and $\alpha\colon \operatorname{Ker}\varphi \to \operatorname{Ker}\varphi''$ are morphisms because $\operatorname{Ker}\varphi'$, $\operatorname{Ker}\varphi$, and $\operatorname{Ker}\varphi''$ are subcomplexes. Similarly, $\bar{\beta}' = (\bar{\beta}'_v)_{v\in\mathbb{Z}}\colon \operatorname{Coker}\varphi' \to \operatorname{Coker}\varphi$ and $\bar{\beta} = (\bar{\beta}_v)_{v\in\mathbb{Z}}\colon \operatorname{Coker}\varphi \to \operatorname{Coker}\varphi''$ are morphisms. It remains to show that δ is a morphism. For a homogeneous element $m'' \in \operatorname{Ker}\varphi''$ one has $\delta(m'') = [n']_{\operatorname{Im}\varphi'}$ for a homogeneous element $n' \in N'$ that satisfies $\beta'(n') = \varphi(m)$ for some $m \in M$ with $\alpha(m) = m''$; see 1.1.5. Now one has

$$\partial^{\operatorname{Coker}\varphi'}(\delta(m'')) \;=\; \partial^{\operatorname{Coker}\varphi'}([n']_{\operatorname{Im}\varphi'}) \;=\; [\partial^{N'}(n')]_{\operatorname{Im}\varphi'} .$$

The element $\partial^{N'}(n')$ satisfies

$$\beta'(\partial^{N'}(n')) \;=\; \partial^{N}(\beta'(n')) \;=\; \partial^{N}(\varphi(m)) \;=\; \varphi(\partial^{M}(m))$$

and $\alpha(\partial^M(m)) = \partial^{M''}(\alpha(m)) = \partial^{M''}(m'')$ holds. Thus, by the definition of δ one has $\delta(\partial^{M''}(m'')) = [\partial^{N'}(n')]_{\operatorname{Im}\varphi'} = \partial^{\operatorname{Coker}\varphi'}(\delta(m''))$. □

Split Exact Sequences

2.1.46 Definition. An exact sequence $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ in $\mathcal{C}(R)$ is called *split* if there exist morphisms $\varrho\colon M \to M'$ and $\sigma\colon M'' \to M$ such that one has

$$\varrho\alpha' = 1^{M'}, \quad \alpha'\varrho + \sigma\alpha = 1^{M}, \quad \text{and} \quad \alpha\sigma = 1^{M''}.$$

2.1.47 Proposition. *Let* $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ *be an exact sequence in* $\mathcal{C}(R)$. *The following conditions are equivalent.*

(*i*) *The sequence is split.*

(*ii*) *There exists a morphism* $\varrho\colon M \to M'$ *such that* $\varrho\alpha' = 1^{M'}$.

(*iii*) *There exists a morphism* $\sigma\colon M'' \to M$ *such that* $\alpha\sigma = 1^{M''}$.

(*iv*) *The sequence is isomorphic to* $0 \longrightarrow M' \xrightarrow{\varepsilon} M' \oplus M'' \xrightarrow{\varpi} M'' \longrightarrow 0$, *where* ε *and* ϖ *are the injection and the projection, respectively.*

Moreover, if $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ *is split exact, then also the sequence* $0 \longrightarrow M'' \xrightarrow{\sigma} M \xrightarrow{\varrho} M' \longrightarrow 0$, *where* ϱ *and* σ *are as in* 2.1.46, *is split exact.*

Proof. Conditions (*ii*) and (*iii*) follow from (*i*). To see that (*ii*) implies (*iv*), let $\varrho\colon M \to M'$ be a morphism with $\varrho\alpha' = 1^{M'}$. There is then a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d, equal] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varepsilon^{M'}"] & M' \oplus M'' \arrow[r, "\varpi^{M''}"] & M'' \arrow[r] & 0\,,
\end{tikzcd}$$

in $\mathcal{C}(R)$, where φ is given by $m \mapsto (\varrho(m), \alpha(m))$; it follows from the Five Lemma 2.1.41 that φ is an isomorphism. A parallel argument shows that (*iii*) implies (*iv*).

Given a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'", "\cong"'] & M \arrow[r, "\alpha"] \arrow[d, "\varphi", "\cong"'] & M'' \arrow[r] \arrow[d, "\varphi''", "\cong"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varepsilon^{M'}"] & M' \oplus M'' \arrow[r, "\varpi^{M''}"] & M'' \arrow[r] & 0\,,
\end{tikzcd}$$

set $\varrho = (\varphi')^{-1}\varpi^{M'}\varphi$ and $\sigma = \varphi^{-1}\varepsilon^{M''}\varphi''$. Now one has $\varrho\alpha' = 1^{M'}$, and $\alpha\sigma = 1^{M''}$, and $\alpha'\varrho + \sigma\alpha = \alpha'(\varphi')^{-1}\varpi^{M'}\varphi + \varphi^{-1}\varepsilon^{M''}\varphi''\alpha = \varphi^{-1}(\varepsilon^{M'}\varpi^{M'} + \varepsilon^{M''}\varpi^{M''})\varphi = 1^{M}$, so (*iv*) implies (*i*).

Finally, assume that $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ is split exact and let ϱ and σ be as in 2.1.46. The sequence $0 \longrightarrow M'' \xrightarrow{\sigma} M \xrightarrow{\varrho} M' \longrightarrow 0$ is exact. Indeed ϱ is injective and σ is surjective; moreover, $\varrho\sigma = \varrho\sigma\alpha\sigma = \varrho(1^M - \alpha'\varrho)\sigma = 0$ holds, and for $m \in M$ with $\varrho(m) = 0$ one has $m = (\alpha'\varrho + \sigma\alpha)(m) = \sigma\alpha(m)$. Thus the sequence is split exact by 2.1.46. □

2.1.48. A functor $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ extends to a functor $\mathcal{C}(R) \to \mathcal{C}(S)$ that is also denoted F and acts as follows: For a morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ and $v \in \mathbb{Z}$ one has

$$\mathrm{F}(M)_v = \mathrm{F}(M_v)\,, \quad \partial_v^{\mathrm{F}(M)} = \mathrm{F}(\partial_v^M)\,, \quad \text{and} \quad \mathrm{F}(\alpha)_v = \mathrm{F}(\alpha_v)\,.$$

A natural transformation $\tau\colon \mathrm{E} \to \mathrm{F}$ of functors $\mathcal{M}(R) \to \mathcal{M}(S)$ extends to a natural transformation of the extended functors $\mathrm{E}, \mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ given by $(\tau^M)_v = \tau^{M_v}$.

A functor $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ extends to a functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$, also denoted G, which acts as follows: For a morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)^{\mathrm{op}}$ and $v \in \mathbb{Z}$ one has

$$\mathrm{G}(N)_v = \mathrm{G}(N_{-v})\,, \quad \partial_v^{\mathrm{G}(N)} = \mathrm{G}(\partial_{-v+1}^N)\,, \quad \text{and} \quad \mathrm{G}(\alpha)_v = \mathrm{G}(\alpha_{-v})\,.$$

A natural transformation $\tau\colon \mathrm{G} \to \mathrm{J}$ of functors $\mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ extends to a natural transformation of the extended functors $\mathrm{G}, \mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ given by $(\tau^M)_v = \tau^{M_v}$.

If the original functor is $\Bbbk$-linear then so is the extended functor, and if the original functor is (left/right) exact, then so is the extended functor.

2.1.49 Example. Let $\varphi\colon R \to S$ be a ring homomorphism. Base change and cobase change along φ, see 1.2.7, extend to functors

$$S \otimes_R - : \mathcal{C}(R) \longrightarrow \mathcal{C}(S) \qquad \text{and} \qquad \mathrm{Hom}_R(S, -) : \mathcal{C}(R) \longrightarrow \mathcal{C}(S)\,.$$

Also their right/left adjoint functor, restriction of scalars res_R^S, and its companion extend to functors $\mathrm{res}_R^S\colon \mathcal{C}(S) \to \mathcal{C}(R)$ and $\mathrm{res}_{R^{\mathrm{o}}}^{S^{\mathrm{o}}}\colon \mathcal{C}(S^{\mathrm{o}}) \to \mathcal{C}(R^{\mathrm{o}})$; at the level of symbols, the restriction of scalars functors are usually suppressed. These restriction functors are easily seen to be faithfully exact and conservative; the latter property also follows from the former in view of 1.1.46.

2.1.50 Example. Assume that R is commutative and let U be a multiplicative subset of R. The localization functor,

$$U^{-1} : \mathcal{M}(R) \longrightarrow \mathcal{M}(U^{-1}R)\,,$$

which is exact, extends per 2.1.48 to an exact functor $U^{-1}\colon \mathcal{C}(R) \to \mathcal{C}(U^{-1}R)$.

For every R-complex M there is by 1.1.11 and 2.1.49 an isomorphism,

$$U^{-1}M \cong U^{-1}R \otimes_R M\,,$$

of $U^{-1}R$-complexes which is natural in M.

2.1.51 Definition. A short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}(R)$ is called *degreewise split* if the exact sequence $0 \to M'^{\natural} \to M^{\natural} \to M''^{\natural} \to 0$ is split.

Only exact functors $\mathcal{C}(R) \to \mathcal{C}(S)$ preserve short exact sequences of complexes. It follows, however, from the next lemma that every functor F that satisfies a natural condition—the differential on a complex M has no influence on the graded module underlying $\mathrm{F}(M)$—preserves degreewise split exact sequences.

2.1.52 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ is called a $\natural$-*functor* if it is additive and $(-)^{\natural} \circ \mathrm{F}$ and $(-)^{\natural} \circ \mathrm{F} \circ (-)^{\natural}$ are naturally isomorphic functors from $\mathcal{C}(R)$ to $\mathcal{M}_{\mathrm{gr}}(S)$.

A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is called a $\natural$-*functor* if it is additive and $(-)^{\natural} \circ \mathrm{G}$ and $(-)^{\natural} \circ \mathrm{G} \circ ((-)^{\natural})^{\mathrm{op}}$ are naturally isomorphic functors from $\mathcal{U}^{\mathrm{op}}$ to $\mathcal{M}_{\mathrm{gr}}(S)$.

2.1.53 Example. Every functor on R-complexes that is extended from an additive functor on R-modules, as described in 2.1.48, is per construction a $\natural$-functor.

Further examples of $\natural$-functors are given in 2.3.12/2.3.13 and 2.4.11/2.4.12. A few functors that are not $\natural$-functors can be found in 2.2.18.

2.1.54 Lemma. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a* $\natural$*-functor. For every degreewise split exact sequence of* R*-complexes,* $0 \to M' \to M \to M'' \to 0$*, the induced sequence of* S*-complexes,* $0 \to \mathrm{F}(M') \to \mathrm{F}(M) \to \mathrm{F}(M'') \to 0$*, is degreewise split exact.*

Proof. It follows from the assumptions on F that the sequence

$$(\star) \qquad 0 \longrightarrow (\mathrm{F}(M'))^{\natural} \longrightarrow (\mathrm{F}(M))^{\natural} \longrightarrow (\mathrm{F}(M''))^{\natural} \longrightarrow 0$$

is isomorphic to

$$(\diamond) \qquad 0 \longrightarrow (\mathrm{F}(M'^{\natural}))^{\natural} \longrightarrow (\mathrm{F}(M^{\natural}))^{\natural} \longrightarrow (\mathrm{F}(M''^{\natural}))^{\natural} \longrightarrow 0$$

in $\mathcal{M}_{\mathrm{gr}}(S)$. The sequence $(\diamond)$ arises from application of the additive functor $(-)^{\natural} \circ \mathrm{F}$ to the split exact sequence $0 \to M'^{\natural} \to M^{\natural} \to M''^{\natural} \to 0$. Therefore, $(\diamond)$ is split exact, and hence so is the isomorphic sequence $(\star)$. $\square$

2.1.55 Lemma. *Let* $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be a* $\natural$*-functor. For every degreewise split exact sequence of* R*-complexes,* $0 \to M' \to M \to M'' \to 0$*, the induced sequence of* S*-complexes,* $0 \to \mathrm{G}(M'') \to \mathrm{G}(M) \to \mathrm{G}(M') \to 0$*, is degreewise split exact.*

Proof. The assertion follows from an argument similar to the proof of 2.1.54. $\square$

Exercises

E 2.1.1 Let $M = \coprod_{v\in\mathbb{Z}} M_v$ be a graded R-module and K a submodule of M. Show that the following conditions are equivalent. (*i*) K is a graded submodule of M. (*ii*) K is generated by homogeneous elements. (*iii*) The equality $K = \coprod_{v\in\mathbb{Z}} K \cap M_v$ holds.

E 2.1.2 Let n be an integer. (a) Show that there is a full and faithful functor $\mathcal{M}(R) \to \mathcal{M}_{\mathrm{gr}}(R)$ that equips an R-module M with the grading $M_n = M$ and $M_v = 0$ for $n \neq v$. (b) Show that there is a functor $\mathcal{M}_{\mathrm{gr}}(R) \to \mathcal{M}(R)$ that forgets the modules M_v for $v \neq n$.

E 2.1.3 Let $\varphi\colon M \to N$ be a graded homomorphism. Show that if φ is a bijective map, then its inverse is a graded homomorphism of degree $-|\varphi|$.

E 2.1.4 Assume that R is commutative. Show that for a free R-module L of rank n the tensor algebra $\mathrm{T}^R(L)$ is the free R-algebra on n indeterminates.

E 2.1.5 Assume that R is commutative. Show that the assignment $M \mapsto \mathrm{T}^R(M)$ yields a functor from $\mathcal{M}(R)$ to the category of R-algebras and that it is left adjoint to the forgetful functor.

E 2.1.6 Let R be an integral domain with field of fractions Q. Show that one has $\bigwedge_2^R(Q) = 0$.

E 2.1.7 Assume that R is commutative. Let M be an R-module and $\iota\colon M \to \mathrm{T}^R(M)$ the canonical map. Show that for every R-algebra A and every R-linear map $\alpha\colon M \to A$, the assignment $x_1 \otimes \cdots \otimes x_p \mapsto \alpha(x_1)\cdots\alpha(x_p)$ defines a homomorphism $\widetilde{\alpha}\colon \mathrm{T}^R(M) \to A$ of R-algebras with $\widetilde{\alpha}\iota = \alpha$ and that $\widetilde{\alpha}$ is unique with this property.

E 2.1.8 Assume that R is commutative. Let M be an R-module and $\iota\colon M \to \mathrm{S}^R(M)$ the canonical map. Show that for every commutative R-algebra A and every R-linear map $\alpha\colon M \to A$, the assignment $x_1\cdots x_p \mapsto \alpha(x_1)\cdots\alpha(x_p)$ defines a homomorphism $\widetilde{\alpha}\colon \mathrm{S}^R(M) \to A$ of R-algebras with $\widetilde{\alpha}\iota = \alpha$ and that $\widetilde{\alpha}$ is unique with this property.

E 2.1.9 Assume that R is commutative. Let M be an R-module and $\iota\colon M \to \bigwedge^R(M)$ the canonical map; note that $\iota(x)\iota(x) = x \wedge x = 0$ holds for all x in M. Show that for every R-algebra A and every R-linear map $\alpha\colon M \to A$ with $\alpha(x)\alpha(x) = 0$ for all x in M, the assignment $x_1 \wedge \cdots \wedge x_p \mapsto \alpha(x_1)\cdots\alpha(x_p)$ defines a homomorphism $\widetilde{\alpha}$ of R-algebras with $\widetilde{\alpha}\iota = \alpha$ and that $\widetilde{\alpha}$ is unique with that property.

E 2.1.10 Let $n \geqslant 2$ be an integer. Compute the algebras $\mathrm{T}^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$ and $\bigwedge^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z})$.

E 2.1.11 Show that the Koszul complexes $\mathrm{K}^{\mathbb{Z}}(2,3)$ and $\mathrm{K}^{\mathbb{Z}}(4,5)$ are not isomorphic.

E 2.1.12 Assume that R is commutative. For elements x and y in R, show that the complexes $\mathrm{K}^R(x,y)$ and $\mathrm{K}^R(y,x)$ are isomorphic.

E 2.1.13 Assume that R is commutative, let M be an R-module and $\varepsilon\colon M \to R$ a homomorphism. Show that the differential ∂ on the Koszul complex $\mathrm{K}^R(\varepsilon)$ from 2.1.25 satisfies the Leibniz Rule: $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$ for homogeneous elements a and b.

E 2.1.14 Let $f\colon X \to Y$ be a continuous map between topological spaces; that is, a morphism in the category $\mathcal{T}$ of topological spaces. (a) Show that the assignment $\sigma \mapsto f\sigma$ defines a morphism of singular chain complexes $\mathrm{S}(X) \to \mathrm{S}(Y)$. (b) Show that there is a functor $\mathcal{T} \to \mathcal{C}(\mathbb{Z})$ that assigns to a topological space X its singular chain complex $\mathrm{S}(X)$.

E 2.1.15 Fix $v \in \mathbb{Z}$. Show that the functions $\mathcal{C}(R) \to \mathcal{M}(R)$ given by $M \mapsto M_v$ and $\alpha \mapsto \alpha_v$ constitute a $\Bbbk$-linear exact functor.

E 2.1.16 Show that the category $\mathcal{C}(R)$ is isomorphic to $\mathcal{M}_{\mathrm{gr}}(R[x]/(x^2))$.

E 2.1.17 Show that a morphism in $\mathcal{C}(R)$ is a monomorphism if and only if it is injective, and show that it is an epimorphism if and only if it is surjective.

E 2.1.18 (Cf. 2.1.9) Let x be a central element in R and M a graded R-module. Show that the homothety $x^M\colon M \to M$ is a morphism of graded R-modules.

E 2.1.19 (Cf. 2.1.31) Let $\beta\colon L \to M$ and $\alpha\colon M \to N$ be chain maps of R-complexes; show that the composite $\alpha\beta\colon L \to N$ is a chain map of R-complexes of degree $|\alpha| + |\beta|$.

E 2.1.20 (Cf. 2.1.35) Let $\alpha\colon M \to N$ be a chain map of R-complexes. Show that $\operatorname{Ker}\alpha$ is a subcomplex of M and that $\operatorname{Im}\alpha$ is a subcomplex of N.

E 2.1.21 (Cf. 2.1.35) Verify that $\mathcal{C}(R)$ is an Abelian category.

E 2.1.22 (Cf. 2.1.36) Let α be a morphism of R-complexes. Show that α is an isomorphism of R-complexes if and only if it is an isomorphism of the underlying graded R-modules.

E 2.1.23 Show that every split exact sequence in $\mathcal{C}(R)$ is degreewise split. Is the converse true?

E 2.1.24 Give a proof of 2.1.55.

2.2 Homology

Synopsis. Shift; homology; connecting morphism; homotopy.

The requirement that the differential ∂ on a complex has to be square zero yields an inclusion $\operatorname{Im}\partial \subseteq \operatorname{Ker}\partial$; the homology of the complex is the quotient $\operatorname{Ker}\partial/\operatorname{Im}\partial$. The implications of vanishing of homology depend on the underlying complex, but it is often illustrative to interpret homology as a measure of deviation from a paragon or obstructions to an aim. For instance, the homology of the de Rham complex $\Omega(M)$ from 2.1.27 detects closed differential forms on the manifold M that are not exact. The homology of the singular chain complex $\mathrm{S}(X)$ from 2.1.26 tells whether it is possible to contract the space X to a single point, and much more. The homology of the Koszul complex $\mathrm{K}^R(x_1,\ldots,x_n)$ from 2.1.25 shows, among other things, if one or more of the elements $x_1,\ldots,x_n$ are non-zerodivisors.

SHIFT

Before we proceed with the precise definition of homology, we formalize the intuitively natural process of renumbering the modules in a complex.

2.2.1 Definition. Let M be an R-complex and s an integer. The *s-fold shift* of M is the complex $\Sigma^s M$ given by

$$(\Sigma^s M)_v = M_{v-s} \qquad \text{and} \qquad \partial_v^{\Sigma^s M} = (-1)^s \partial_{v-s}^M$$

for all $v \in \mathbb{Z}$. For a homomorphism $\alpha = (\alpha_v)_{v\in\mathbb{Z}} \colon M \to N$ of R-complexes, the s-fold shift $\Sigma^s\alpha \colon \Sigma^s M \to \Sigma^s N$ is the homomorphism with $(\Sigma^s\alpha)_v = \alpha_{v-s}$ for all $v \in \mathbb{Z}$.

REMARK. Other words for shift are ‘suspension’ and ‘translation’.

2.2.2 Example. Every functor F on R-complexes that is extended from a functor on R-modules, as described in 2.1.48, commutes with shift; that is $\mathrm{F}\Sigma = \Sigma\mathrm{F}$. Further, for an extended natural transformation τ of extended functors one has $\tau\Sigma = \Sigma\tau$.

2.2.3. For an R-complex M one has $\partial^{\Sigma^s M} = (-1)^s\Sigma^s\partial^M$. For a homomorphism α of R-complexes one has $|\Sigma^s\alpha| = |\alpha|$, and for composable homomorphisms α and β one has $(\Sigma^s\beta)(\Sigma^s\alpha) = \Sigma^s(\beta\alpha)$. It follows that if $\alpha\colon M \to N$ is a chain map of R-complexes then so is $\Sigma^s\alpha$; indeed, there are equalities

$$\begin{aligned}\partial^{\Sigma^s N}(\Sigma^s\alpha) &= (-1)^s(\Sigma^s\partial^N)(\Sigma^s\alpha)\\ &= (-1)^s\Sigma^s(\partial^N\alpha)\\ &= (-1)^{|\alpha|}(-1)^s\Sigma^s(\alpha\partial^M)\\ &= (-1)^{|\Sigma^s\alpha|}(\Sigma^s\alpha)\partial^{\Sigma^s M}\,.\end{aligned}$$

In particular, Σ^s takes morphisms to morphisms, and it follows that Σ^s is an exact $\Bbbk$-linear automorphism on $\mathcal{C}(R)$ with inverse Σ^{-s}. Evidently, Σ^s is the s-fold composite of the functor $\Sigma = \Sigma^1$.

2.2.4 Definition. Let M be an R-complex and s an integer. There is a canonical chain map $\varsigma_s^M \colon M \to \Sigma^s M$ of degree s that maps a homogeneous element m in M to the corresponding element of degree $|m| + s$ in $\Sigma^s M$. The map is invertible, and $(\varsigma_s^M)^{-1}$ is the chain map $\varsigma_{-s}^{\Sigma^s M} \colon \Sigma^s M \to M$.

By means of shift, every chain map can be identified with a morphism.

2.2.5. Notice that for every homomorphism $\alpha\colon M \to N$ of R-complexes the diagram

(2.2.5.1)
$$\begin{tikzcd} M \arrow[r, "\varsigma_s^M"] \arrow[d, "\alpha"] & \Sigma^s M \arrow[r, "\varsigma_{-s}^{\Sigma^s M}"] \arrow[d, "\Sigma^s\alpha"] & M \arrow[d, "\alpha"] \\ N \arrow[r, "\varsigma_s^N"] & \Sigma^s N \arrow[r, "\varsigma_{-s}^{\Sigma^s N}"] & N \end{tikzcd}$$

is commutative. The degree shifting maps are occasionally suppressed. In particular, a homomorphism (chain map) $\beta\colon M \to N$ can be identified with the homomorphism (chain map) $\beta\varsigma^{\Sigma^s M}_{-s}\colon \Sigma^s M \to N$ of degree $|\beta|-s$ or $\varsigma^N_s\beta\colon M \to \Sigma^s N$ of degree $|\beta|+s$.

Homology

2.2.6. Let M be an R-complex. The differential ∂^M is a chain map, so it follows from 2.1.35 that $\operatorname{Ker}\partial^M$ and $\operatorname{Im}\partial^M$ are subcomplexes of M. As $\partial^M\partial^M = 0$ holds, there is an inclusion $\operatorname{Im}\partial^M \subseteq \operatorname{Ker}\partial^M$, and the induced differential on either subcomplex is 0. Likewise, the differential on the quotient complex $\operatorname{Coker}\partial^M$ is 0.

2.2.7 Definition. Let M be an R-complex; set

$$\mathrm{Z}(M) = \operatorname{Ker}\partial^M\,, \quad \mathrm{B}(M) = \operatorname{Im}\partial^M\,, \quad \text{and} \quad \mathrm{C}(M) = \operatorname{Coker}\partial^M\,.$$

Notice that $\mathrm{C}(M)$ is the quotient complex $M/\mathrm{B}(M)$. Elements in the subcomplex $\mathrm{Z}(M)$ are called *cycles*, and elements in $\mathrm{B}(M)$ are called *boundaries*. The module of homogeneous cycles in M of degree v, that is, the module in degree v in the complex $\mathrm{Z}(M)$, is written $\mathrm{Z}_v(M)$. Similarly the modules in the complexes $\mathrm{B}(M)$ and $\mathrm{C}(M)$ are written $\mathrm{B}_v(M)$ and $\mathrm{C}_v(M)$. As these complexes have trivial differentials, they could alternately be defined by specifying their modules as follows,

$$\mathrm{Z}_v(M) = \operatorname{Ker}\partial^M_v\,, \quad \mathrm{B}_v(M) = \operatorname{Im}\partial^M_{v+1}\,, \quad \text{and} \quad \mathrm{C}_v(M) = \operatorname{Coker}\partial^M_{v+1}\,.$$

The quotient

$$\mathrm{H}(M) = \mathrm{Z}(M)/\mathrm{B}(M)$$

is called the *homology* complex of M. The module in degree v is written $\mathrm{H}_v(M)$ and called the v^{th} *homology module* of M. If $\mathrm{H}(M) = 0$ holds, then M is called *acyclic*.

Notice that an R-complex is acyclic (see 2.2.7) as an object in $\mathcal{C}(R)$ if and only if it is exact (see 1.1.1) when regarded as a sequence in $\mathcal{M}(R)$.

Remark. Other words for acyclic are 'exact' and 'homologically trivial'. For a complex written in cohomological notation $M = \cdots \to M^{v-1} \to M^v \to M^{v+1} \to \cdots$ it is standard to write the homology complex $\mathrm{H}(M)$ in cohomological notation as well. The module in cohomological degree v of $\mathrm{H}(M)$ is denoted $\mathrm{H}^v(M)$ and called the 'v^{th} cohomology module' of M.

The terms 'boundary' and 'cycle' are borrowed from specific homology theories, where this terminology comes natural. Briefly: paths π and π' in $\mathbb{R}^2$ are deemed equivalent if they differ by a boundary; that is, the concatenation of π and $-\pi'$ is the boundary of a region. The reason is that by Stokes' theorem the path integral of an exact differential form is then the same along π and π'. In a topological space that has a structure of a simplicial complex—a higher triangulation; cf. 2.1.26—it is compelling to use the name cycle for a potential boundary. For a rigid, but still brief, explanation see the first section of Rotman's book [220]. Weibel's historical survey [254] has plenty of illuminating details.

2.2.8 Example. The Dold complex in 2.1.23 is acyclic.

2.2.9 Example. Assume that R is commutative. The Koszul homology of a sequence $x_1, \ldots, x_n$ in R is the homology of the Koszul complex $\mathrm{K}^R(x_1, \ldots, x_n)$ from 2.1.25. As the image in $\mathrm{K}_0^R(x_1, \ldots, x_n) = R$ of $\partial_1^{\mathrm{K}^R(x_1,\ldots,x_n)}$ is the ideal $(x_1, \ldots, x_n)$, the homology in degree 0 is

$$\mathrm{H}_0(\mathrm{K}^R(x_1, \ldots, x_n)) = R/(x_1, \ldots, x_n) .$$

For a single element x the Koszul complex $\mathrm{K}^R(x)$ has the form

$$(2.2.9.1) \qquad 0 \longrightarrow R\langle e \rangle \xrightarrow{\partial} R \longrightarrow 0 ,$$

where $\partial(e) = x$. Hence one has $\mathrm{H}_1(\mathrm{K}^R(x)) \cong (0 :_R x)$. In particular, this homology module is zero if and only if x is not a zerodivisor in R.

The Koszul complex $K = \mathrm{K}^R(x_1, x_2)$ has the form

$$(2.2.9.2) \qquad 0 \longrightarrow R\langle e_1 \wedge e_2 \rangle \xrightarrow{\partial_2^K} R\langle e_1, e_2 \rangle \xrightarrow{\partial_1^K} R \longrightarrow 0 ,$$

where $\partial_1^K(k_1e_1 + k_2e_2) = k_1x_1 + k_2x_2$ and $\partial_2^K(e_1 \wedge e_2) = x_1e_2 - x_2e_1$. The cycles in degree 2 are elements $ke_1 \wedge e_2$ with $kx_1 = 0 = kx_2$; hence there is an isomorphism $\mathrm{H}_2(K) \cong (0 :_R x_1) \cap (0 :_R x_2) = (0 :_R (x_1, x_2))$. The cycles in degree 1 embody the relations between x_1 and x_2 while the boundaries embody the simple so-called Koszul relation $x_1x_2 - x_2x_1 = 0$. Notice that $\mathrm{H}_2(K)$ is zero if x_1 or x_2 is not a zerodivisor in R. The homology module $\mathrm{H}_1(K)$ is zero if and only if $[x_2]_{(x_1)}$ is not a zerodivisor in the quotient ring $R/(x_1)$ and $[x_1]_{(x_2)}$ is not a zerodivisor in $R/(x_2)$.

The complex $K = \mathrm{K}^R(x_1, \ldots, x_n)$ is concentrated in degrees $n, \ldots, 1, 0$. The homology module $\mathrm{H}_n(K)$ is isomorphic to the ideal of common annihilators of the elements $x_1, \ldots, x_n$. The module $\mathrm{H}_1(K)$ is the module generated by all relations between the generators modulo the one generated by the simple ones $x_ix_j - x_jx_i = 0$. In intermediate degrees $n > i > 1$, the module $\mathrm{H}_i(K)$ is generated by "higher" relations between the generators, and the boundaries that are factored out are simple "higher" relations usually also referred to as 'Koszul relations'.

2.2.10 Example. The *singular homology* $\mathrm{H}_*(X; A)$ of a topological space X with coefficients in an Abelian group A is the homology of the singular chain complex $\mathrm{S}(X) \otimes_{\mathbb{Z}} A$ from 2.1.26. Singular homology with coefficients in $\mathbb{Z}$ is written $\mathrm{H}_*(X)$. The singular homology group $\mathrm{H}_n(X)$ detects "n-dimensional holes" in X.

For every space X, the Abelian group $\mathrm{H}_0(X)$ is free, and its rank (possibly an infinite cardinal) is the number of path components of X. The singular chain complex of a one-point space pt is

$$\mathrm{S}(pt) = \cdots \longrightarrow \mathbb{Z}\langle \sigma_2 \rangle \xrightarrow{\partial_2^{pt}} \mathbb{Z}\langle \sigma_1 \rangle \xrightarrow{\partial_1^{pt}} \mathbb{Z}\langle \sigma_0 \rangle \longrightarrow 0 .$$

where σ_n for each $n \geqslant 0$ denotes the unique (continuous) map $\Delta^n \to pt$ from the standard n-simplex to a point. The definition 2.1.26 yields

$$\partial_n^{pt} = 0 \quad \text{for } n \text{ odd} \qquad \text{and} \qquad \partial_n^{pt}(\sigma_n) = \sigma_{n-1} \quad \text{for } n \text{ even} ,$$

and thus one has $\mathrm{H}_0(pt) \cong \mathbb{Z}$ and $\mathrm{H}_n(pt) = 0$ for all $n \neq 0$. Every contractible space has the same singular homology as pt; see 4.3.5. For $m > 0$ the singular homology of the m-sphere S^m is $\mathrm{H}_0(S^m) \cong \mathbb{Z} \cong \mathrm{H}_m(S^m)$ and $\mathrm{H}_n(S^m) = 0$ for all $n \notin \{0, m\}$.

2.2.11 Example. The *de Rham cohomology* $\mathrm{H}_{\mathrm{dR}}(M)$ of a smooth real manifold M is the cohomology of the de Rham complex $\Omega(M)$ from 2.1.27; it detects closed differential forms on M that are not exact. A closed 0-form is constant on every connected component, so the rank of the real vector space $\mathrm{H}^0_{\mathrm{dR}}(M)$ is the number of connected components of M. The cohomology group $\mathrm{H}^1_{\mathrm{dR}}(M)$ is zero if and only if every closed differential 1-form is the differential of a smooth function $f\colon M \to \mathbb{R}$.

The de Rham complex $\Omega(S^1)$ of the 1-sphere is concentrated in degrees 1 and 0; in particular, the cohomology modules $\mathrm{H}^n_{\mathrm{dR}}(S^1)$ are zero for $n \notin \{0, 1\}$. As S^1 is connected, one has $\mathrm{H}^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. Every 1-form is closed ($\mathrm{d}^1 = 0$). Despite the appearance, $d\theta$, the differential of the polar "coordinate function" θ is not exact, so $\mathrm{H}^1_{\mathrm{dR}}(M)$ is non-zero and, in fact, isomorphic to $\mathbb{R}$.

The de Rham cohomology $\mathrm{H}^n_{\mathrm{dR}}(B^n)$ of the open unit ball in $\mathbb{R}^n$ is zero for $n \neq 0$.

Functoriality

The subcomplexes and (sub-)quotient complexes introduced in 2.2.7 commingle in canonical exact sequences.

2.2.12 Proposition. *Let M be an R-complex. The following sequences of subcomplexes and (sub-)quotient complexes of M are exact.*

(a) $0 \longrightarrow \mathrm{Z}(M) \longrightarrow M \longrightarrow \Sigma\mathrm{B}(M) \longrightarrow 0\,.$

(b) $0 \longrightarrow \mathrm{B}(M) \longrightarrow M \longrightarrow \mathrm{C}(M) \longrightarrow 0\,.$

(c) $0 \longrightarrow \mathrm{B}(M) \longrightarrow \mathrm{Z}(M) \longrightarrow \mathrm{H}(M) \longrightarrow 0\,.$

(d) $0 \longrightarrow \mathrm{H}(M) \longrightarrow \mathrm{C}(M) \longrightarrow \Sigma\mathrm{B}(M) \longrightarrow 0\,.$

Proof. The differential ∂^M is per 2.1.29 and 2.2.7 a morphism $M \to \Sigma M$ with kernel $\mathrm{Z}(M)$ and image $\mathrm{B}(M)$. This explains the sequence (a), and also (b) and (c) follow straight from the definitions. As $\mathrm{B}(M)$ is contained in $\mathrm{Z}(M)$ the differential induces a morphism $\bar{\partial}^M\colon M/\mathrm{B}(M) = \mathrm{C}(M) \to \Sigma M$ with the same image as ∂^M and kernel $\mathrm{Z}(M)/\mathrm{B}(M) = \mathrm{H}(M)$; this explains the sequence (d). □

2.2.13 Proposition. *Every morphism $M \to N$ of R-complexes restricts to morphisms $\mathrm{Z}(M) \to \mathrm{Z}(N)$ and $\mathrm{B}(M) \to \mathrm{B}(N)$. Whence, Z, B, and C are $\Bbbk$-linear endofunctors on $\mathcal{C}(R)$. Moreover, there are equalities $\mathrm{Z}\Sigma = \Sigma\mathrm{Z}$, $\mathrm{B}\Sigma = \Sigma\mathrm{B}$, and $\mathrm{C}\Sigma = \Sigma\mathrm{C}$ of endofunctors on $\mathcal{C}(R)$.*

Proof. Let $\alpha\colon M \to N$ be a morphism of R-complexes. As $\alpha\partial^M = \partial^N\alpha$ holds, it follows that α restricts to a morphism of cycle complexes $\mathrm{Z}(M) \to \mathrm{Z}(N)$ and to a morphism of boundary complexes $\mathrm{B}(M) \to \mathrm{B}(N)$. In particular, α induces a morphism, $\bar{\alpha}$, of cokernel complexes $\mathrm{C}(M) \to \mathrm{C}(N)$. This explains how Z, B, and

C act on morphisms; in symbols: $\mathrm{Z}(\alpha) = \alpha|_{\mathrm{Z}(M)}$, $\mathrm{B}(\alpha) = \alpha|_{\mathrm{B}(M)}$, and $\mathrm{C}(\alpha) = \bar{\alpha}$. It is straightforward to verify that they are $\Bbbk$-linear functors.

For every R-complex M and $v \in \mathbb{Z}$ one has

$$\mathrm{Z}_v(\Sigma M) = \operatorname{Ker}\partial_v^{\Sigma M} = \operatorname{Ker}(-\partial_{v-1}^{M}) = \operatorname{Ker}\partial_{v-1}^{M} = \mathrm{Z}_{v-1}(M) = (\Sigma\mathrm{Z}(M))_v\,,$$

so $\mathrm{Z}(\Sigma M) = \Sigma\mathrm{Z}(M)$ holds. For every morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ one now gets

$$\mathrm{Z}(\Sigma\alpha) = (\Sigma\alpha)|_{\mathrm{Z}(\Sigma M)} = (\Sigma\alpha)|_{\Sigma\mathrm{Z}(M)} = \Sigma(\alpha|_{\mathrm{Z}(M)}) = \Sigma\mathrm{Z}(\alpha)\,;$$

so $\mathrm{Z}\Sigma = \Sigma\mathrm{Z}$ as functors. Similar arguments yield $\mathrm{B}\Sigma = \Sigma\mathrm{B}$ and $\mathrm{C}\Sigma = \Sigma\mathrm{C}$. □

2.2.14. Let $\alpha\colon M \to N$ be a morphism of R-complexes. It follows from 2.2.13 that α induces morphism of R-complexes,

$$\mathrm{H}(\alpha)\colon\ \mathrm{H}(M) \longrightarrow \mathrm{H}(N)\,,$$

which is given by the assignment $[z]_{\mathrm{B}(M)} \mapsto [\alpha(z)]_{\mathrm{B}(N)}$ for $z \in \mathrm{Z}(M)$.

The equivalence class $[z]_{\mathrm{B}(M)}$ is called the *homology class* of z. Hereafter we drop the subscript on homology classes and write $[z]$ for the homology class of a cycle z.

By the definition of $\mathrm{H}(\alpha)$ there is a commutative diagram,

(2.2.14.1)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}(M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{Z}(M) \arrow[r] \arrow[d, "\alpha"] & \mathrm{H}(M) \arrow[r] \arrow[d, "\mathrm{H}(\alpha)"] & 0 \\
0 \arrow[r] & \mathrm{B}(N) \arrow[r] & \mathrm{Z}(N) \arrow[r] & \mathrm{H}(N) \arrow[r] & 0\,.
\end{tikzcd}$$

2.2.15 Theorem. *Homology* H *is a* $\Bbbk$*-linear endofunctor on* $\mathcal{C}(R)$*. Moreover, there is an equality* $\mathrm{H}\Sigma = \Sigma\mathrm{H}$ *of endofunctors on* $\mathcal{C}(R)$*.*

Proof. It is straightforward to verify that $\mathrm{H}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ is a $\Bbbk$-linear functor. For every R-complex M, exactness of the shift functor and 2.2.13 yield:

$$\begin{aligned}
\mathrm{H}(\Sigma M) &= \mathrm{Z}(\Sigma M)/\mathrm{B}(\Sigma M)\\
&= (\Sigma\mathrm{Z}(M))/(\Sigma\mathrm{B}(M))\\
&= \Sigma(\mathrm{Z}(M)/\mathrm{B}(M))\\
&= \Sigma\mathrm{H}(M)\,.
\end{aligned}$$

Similarly, one has $\mathrm{H}(\Sigma\alpha) = \Sigma\mathrm{H}(\alpha)$ for every morphism α in $\mathcal{C}(R)$. □

2.2.16 Lemma. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* R*-complexes.*

(a) *For every* $v \in \mathbb{Z}$ *there is an exact sequence*

$$0 \to \mathrm{Z}_v(M') \to \mathrm{Z}_v(M) \to \mathrm{Z}_v(M'') \to \mathrm{H}_{v-1}(M') \to \mathrm{H}_{v-1}(M) \to \mathrm{H}_{v-1}(M'').$$

In particular, the functors Z_v *and* Z *are left exact.*

(b) *For every $v \in \mathbb{Z}$ there is an exact sequence*

$$\mathrm{H}_{v+1}(M') \to \mathrm{H}_{v+1}(M) \to \mathrm{H}_{v+1}(M'') \to \mathrm{C}_v(M') \to \mathrm{C}_v(M) \to \mathrm{C}_v(M'') \to 0\,.$$

In particular, the functors C_v *and* C *are right exact.*

PROOF. For every $v \in \mathbb{Z}$ the Snake Lemma 1.1.6 applies to the diagram,

$$\begin{tikzcd}
0 \arrow[r] & M'_v \arrow[r] \arrow[d, "\partial_v^{M'}"] & M_v \arrow[r] \arrow[d, "\partial_v^{M}"] & M''_v \arrow[r] \arrow[d, "\partial_v^{M''}"] & 0 \\
0 \arrow[r] & M'_{v-1} \arrow[r] & M_{v-1} \arrow[r] & M''_{v-1} \arrow[r] & 0\,,
\end{tikzcd}$$

and yields an exact sequence

$$0 \to \mathrm{Z}_v(M') \to \mathrm{Z}_v(M) \to \mathrm{Z}_v(M'') \to \mathrm{C}_{v-1}(M') \to \mathrm{C}_{v-1}(M) \to \mathrm{C}_{v-1}(M'') \to 0\,.$$

Thus, every functor Z_v is left exact, and it follows that Z, which is computed degreewise, is left exact. Similarly, each functor C_v and hence C is right exact.

To finish part (a) apply the Snake Lemma 1.1.6 to the diagram,

$$\begin{tikzcd}
0 \arrow[r] & M'_v \arrow[r] \arrow[d, "\partial_v^{M'}"] & M_v \arrow[r] \arrow[d, "\partial_v^{M}"] & M''_v \arrow[r] \arrow[d, "\partial_v^{M''}"] & 0 \\
0 \arrow[r] & \mathrm{Z}_{v-1}(M') \arrow[r] & \mathrm{Z}_{v-1}(M) \arrow[r] & \mathrm{Z}_{v-1}(M'')\,. &
\end{tikzcd}$$

To finish part (b) apply the Snake Lemma 1.1.6 to the diagram,

$$\begin{tikzcd}
 & \mathrm{C}_{v+1}(M') \arrow[r] \arrow[d, "\bar{\partial}_{v+1}^{M'}"] & \mathrm{C}_{v+1}(M) \arrow[r] \arrow[d, "\bar{\partial}_{v+1}^{M}"] & \mathrm{C}_{v+1}(M'') \arrow[r] \arrow[d, "\bar{\partial}_{v+1}^{M''}"] & 0 \\
0 \arrow[r] & M'_v \arrow[r] & M_v \arrow[r] & M''_v \arrow[r] & 0\,.
\end{tikzcd}$$

□

REMARK. The functor B is not even half exact, but it preserves surjective morphisms; see E 2.2.2.

2.2.17 Example. For an R-complex Z with zero differential one has $\mathrm{H}(Z) = Z$. In particular, $\mathrm{H}(\mathrm{H}(M)) = \mathrm{H}(M)$ holds for every complex.

2.2.18 Example. The functors $\mathrm{B}, \mathrm{C}, \mathrm{H}, \mathrm{Z}\colon \mathcal{C}(R) \to \mathcal{C}(R)$, see 2.2.13 and 2.2.15, are $\Bbbk$-linear but not $\natural$-functors. Indeed, with $D = 0 \longrightarrow R \xrightarrow{=} R \longrightarrow 0$, concentrated in degrees 1 and 0, the exact sequence $\eta = 0 \to R \to D \to \Sigma R \to 0$ in $\mathcal{C}(R)$ is degreewise split, but none of the sequences $\mathrm{B}(\eta)$, $\mathrm{C}(\eta)$, $\mathrm{H}(\eta)$, and $\mathrm{Z}(\eta)$ are even exact. The assertion now follows from 2.1.54.

An exact functor $\mathcal{C}(R) \to \mathcal{C}(S)$ need not commute with homology. Far from, actually, and most of Chap. 5 is devoted to a study of exact functors that, at least, preserve acyclicity of complexes. Functors that are extended from exact functors on modules do, however, commute with homology.

2.2.19. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ that per 2.1.48 is extended from an exact functor $\mathcal{M}(R) \to \mathcal{M}(S)$ commutes with homology. Indeed, it is straightforward to verify that for every $v \in \mathbb{Z}$ one has

$$\begin{aligned} \mathrm{Z}_v(\mathrm{F}(M)) &\cong \mathrm{F}(\mathrm{Z}_v(M)), & \mathrm{B}_v(\mathrm{F}(M)) &\cong \mathrm{F}(\mathrm{B}_v(M)), \\ \mathrm{C}_v(\mathrm{F}(M)) &\cong \mathrm{F}(\mathrm{C}_v(M)), \quad \text{and} & \mathrm{H}_v(\mathrm{F}(M)) &\cong \mathrm{F}(\mathrm{H}_v(M)); \end{aligned}$$

similarly $\mathrm{H}(\mathrm{F}(\alpha)) = \mathrm{F}(\mathrm{H}(\alpha))$ for every morphism α in $\mathcal{C}(R)$.

Analogously, for a functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ that is extended from an exact functor $\mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ one has

$$\begin{aligned} \mathrm{Z}_v(\mathrm{G}(M)) &\cong \mathrm{G}(\mathrm{C}_{-v}(M)), & \mathrm{B}_v(\mathrm{G}(M)) &\cong \mathrm{G}(\mathrm{B}_{-v-1}(M)), \\ \mathrm{C}_v(\mathrm{G}(M)) &\cong \mathrm{G}(\mathrm{Z}_{-v}(M)), \quad \text{and} & \mathrm{H}_v(\mathrm{G}(M)) &\cong \mathrm{G}(\mathrm{H}_{-v}(M)); \end{aligned}$$

similarly $\mathrm{H}(\mathrm{G}(\alpha)) = \mathrm{G}(\mathrm{H}(\alpha))$ for every morphism α in $\mathcal{C}(R)$.

Connecting Morphism in Homology

The homology functor is only half exact, but the Snake Lemma facilitates a closer comparison of the homology of complexes in a short exact sequence.

2.2.20 Construction. Consider a short exact sequence of R-complexes,

$$0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0\,.$$

It induces a commutative diagram in $\mathcal{C}(R)$ with exact rows,

$$\begin{tikzcd} & \mathrm{C}(M') \arrow[r, "\bar{\alpha}'"] \arrow[d, "\varsigma_1^{M'}\bar{\partial}^{M'}"] & \mathrm{C}(M) \arrow[r, "\bar{\alpha}"] \arrow[d, "\varsigma_1^{M}\bar{\partial}^{M}"] & \mathrm{C}(M'') \arrow[r] \arrow[d, "\varsigma_1^{M''}\bar{\partial}^{M''}"] & 0 \\ 0 \arrow[r] & \Sigma\mathrm{Z}(M') \arrow[r, "\Sigma\alpha'"] & \Sigma\mathrm{Z}(M) \arrow[r, "\Sigma\alpha"] & \Sigma\mathrm{Z}(M'')\,, \end{tikzcd}$$

and the Snake Lemma 2.1.45 yields an exact sequence

$$\mathrm{H}(M') \xrightarrow{\mathrm{H}(\alpha')} \mathrm{H}(M) \xrightarrow{\mathrm{H}(\alpha)} \mathrm{H}(M'') \xrightarrow{\eth} \Sigma\mathrm{H}(M') \xrightarrow{\Sigma\mathrm{H}(\alpha')} \Sigma\mathrm{H}(M)\,, \tag{2.2.20.1}$$

where the connecting morphism in homology, $\eth$, maps a homology class $[z'']$ with $z'' = \alpha(m)$ to $[z']$ with $(\Sigma\alpha')(z') = \varsigma_1^M\partial^M(m)$; cf. 2.1.44 and 1.1.5.

The complexes in the exact sequence (2.2.20.1) have zero differentials, and it is often written as an exact sequence in $\mathcal{M}(R)$:

$$\cdots \xrightarrow{\eth_{v+1}} \mathrm{H}_v(M') \xrightarrow{\mathrm{H}_v(\alpha')} \mathrm{H}_v(M) \xrightarrow{\mathrm{H}_v(\alpha)} \mathrm{H}_v(M'') \xrightarrow{\eth_v} \mathrm{H}_{v-1}(M') \xrightarrow{\mathrm{H}_{v-1}(\alpha')} \cdots\,. \tag{2.2.20.2}$$

The connecting morphism is natural in the following sense.

2.2.21 Proposition. *For every commutative diagram of R-complexes*

$$
\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \arrow[r] & 0
\end{tikzcd}
$$

with exact rows, there is a commutative diagram in $\mathcal{C}(R)$ *with exact rows,*

$$
\begin{tikzcd}
\mathrm{H}(M') \arrow[r, "\mathrm{H}(\alpha')"] \arrow[d, "\mathrm{H}(\varphi')"] & \mathrm{H}(M) \arrow[r, "\mathrm{H}(\alpha)"] \arrow[d, "\mathrm{H}(\varphi)"] & \mathrm{H}(M'') \arrow[r, "\eth"] \arrow[d, "\mathrm{H}(\varphi'')"] & \Sigma\mathrm{H}(M') \arrow[r, "\Sigma\mathrm{H}(\alpha')"] \arrow[d, "\Sigma\mathrm{H}(\varphi')"] & \Sigma\mathrm{H}(M) \arrow[d, "\Sigma\mathrm{H}(\varphi)"] \\
\mathrm{H}(N') \arrow[r, "\mathrm{H}(\beta')"] & \mathrm{H}(N) \arrow[r, "\mathrm{H}(\beta)"] & \mathrm{H}(N'') \arrow[r, "\delta"] & \Sigma\mathrm{H}(N') \arrow[r, "\Sigma\mathrm{H}(\beta')"] & \Sigma\mathrm{H}(N)\,;
\end{tikzcd}
$$

here $\eth$ *and* δ *are the connecting morphisms from* 2.2.20. *Equivalently, there is a commuative diagram in* $\mathcal{M}(R)$ *with exact rows*

$$
\begin{tikzcd}
\cdots \arrow[r, "\eth_{v+1}"] & \mathrm{H}_v(M') \arrow[r, "\mathrm{H}_v(\alpha')"] \arrow[d, "\mathrm{H}_v(\varphi')"] & \mathrm{H}_v(M) \arrow[r, "\mathrm{H}_v(\alpha)"] \arrow[d, "\mathrm{H}_v(\varphi)"] & \mathrm{H}_v(M'') \arrow[r, "\eth_v"] \arrow[d, "\mathrm{H}_v(\varphi'')"] & \mathrm{H}_{v-1}(M') \arrow[r, "\mathrm{H}_{v-1}(\alpha')"] \arrow[d, "\mathrm{H}_{v-1}(\varphi')"] & \cdots \\
\cdots \arrow[r, "\delta_{v+1}"] & \mathrm{H}_v(N') \arrow[r, "\mathrm{H}_v(\beta')"] & \mathrm{H}_v(N) \arrow[r, "\mathrm{H}_v(\beta)"] & \mathrm{H}_v(N'') \arrow[r, "\delta_v"] & \mathrm{H}_{v-1}(N') \arrow[r, "\mathrm{H}_{v-1}(\beta')"] & \cdots
\end{tikzcd}
$$

PROOF. In view of 2.2.15 and 2.2.20 it remains to prove that the square

$$
(\flat) \qquad
\begin{tikzcd}
\mathrm{H}(M'') \arrow[r, "\eth"] \arrow[d, "\mathrm{H}(\varphi'')"'] & \Sigma\mathrm{H}(M') \arrow[d, "\Sigma\mathrm{H}(\varphi')"] \\
\mathrm{H}(N'') \arrow[r, "\delta"] & \Sigma\mathrm{H}(N')
\end{tikzcd}
$$

is commutative. To this end, let $[z'']$ be an element in $\mathrm{H}(M'') = \operatorname{Ker}\bar{\partial}^{M''}$. By the definition of the connecting morphism $\eth$, see 2.2.20, one has $\eth([z'']) = [z']$, where $z' \in \Sigma\mathrm{Z}(M')$ satisfies $(\Sigma\alpha')(z') = \varsigma_1^M\bar{\partial}^M([x]_{\mathrm{B}(M)})$ for some element $x \in M$ with $\bar{\alpha}([x]_{\mathrm{B}(M)}) = [\alpha(x)]_{\mathrm{B}(M'')} = [z'']$. Thus, one has $(\Sigma\mathrm{H}(\varphi')\eth)([z'']) = [(\Sigma\varphi')(z')]$. The cycle $(\Sigma\varphi')(z')$ satisfies

$$
\begin{aligned}
(\Sigma\beta')(\Sigma\varphi')(z') &= (\Sigma\varphi)(\Sigma\alpha')(z') \\
&= (\Sigma\varphi)\varsigma_1^M\bar{\partial}^M([x]_{\mathrm{B}(M)}) \\
&= \varsigma_1^N\bar{\partial}^N([\varphi(x)]_{\mathrm{B}(N)})\,,
\end{aligned}
$$

where the last equality follows from (2.2.5.1). The element $[\varphi(x)]_{\mathrm{B}(N)}$ satisfies

$$
\begin{aligned}
\bar{\beta}([\varphi(x)]_{\mathrm{B}(N)}) &= [\beta\varphi(x)]_{\mathrm{B}(N'')} \\
&= [\varphi''\alpha(x)]_{\mathrm{B}(N'')} \\
&= \mathrm{H}(\varphi'')([\alpha(x)]_{\mathrm{B}(M'')}) \\
&= \mathrm{H}(\varphi'')([z''])\,.
\end{aligned}
$$

By the definition of δ one now has $(\delta\,\mathrm{H}(\varphi''))([z'']) = [(\Sigma\varphi')(z')]$. That is, the square $(\flat)$ is commutative. □

It is evident from 2.2.21 that homology is a half exact functor, as the next example shows it is neither left nor right exact.

2.2.22 Example. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of R-complexes. If the complex N is acyclic but N', N'' are not, then the induced sequence $0 \to \mathrm{H}(N') \to 0 \to \mathrm{H}(N'') \to 0$ is neither exact at $\mathrm{H}(N')$ nor at $\mathrm{H}(N'')$.

Homotopy

Like many other notions in homological algebra, homotopy originates in topology. Homotopy works behind the scenes: it is preserved by most functors of interest, but it has zero homological footprint.

2.2.23 Definition. A chain map of R-complexes $\alpha\colon M \to N$ is called *null-homotopic* if there exists a homomorphism $\sigma\colon M^\natural \to N^\natural$ of degree $|\alpha|+1$,

$$\begin{tikzcd}
\cdots \arrow[r] & M_{v+1} \arrow[r, "\partial^M_{v+1}"] \arrow[d, "\alpha_{v+1}"] & M_v \arrow[r, "\partial^M_v"] \arrow[d, "\alpha_v"] \arrow[dl, dotted, "\sigma_v"] & M_{v-1} \arrow[r] \arrow[d, "\alpha_{v-1}"] \arrow[dl, dotted, "\sigma_{v-1}"] & \cdots \\
\cdots \arrow[r] & N_{v+1+|\alpha|} \arrow[r, "\partial^N_{v+1+|\alpha|}"'] & N_{v+|\alpha|} \arrow[r, "\partial^N_{v+|\alpha|}"'] & N_{v-1+|\alpha|} \arrow[r] & \cdots ,
\end{tikzcd}$$

such that the equality $\alpha = \partial^N\sigma + (-1)^{|\alpha|}\sigma\partial^M$ holds.

Two chain maps of R-complexes $\alpha, \alpha'\colon M \to N$ of the same degree are called *homotopic*, in symbols $\alpha \sim \alpha'$, if $\alpha - \alpha'$ is null-homotopic. A homomorphism σ with $\alpha - \alpha' = \partial^N\sigma + (-1)^{|\alpha|}\sigma\partial^M$ is called a *homotopy* from α to α'.

2.2.24 Example. Let M be an R-module; the identity morphism for the complex $0 \longrightarrow M \xrightarrow{=} M \longrightarrow 0$ is null-homotopic. If the complex is concentrated in degrees, say, 1 and 0, then the required homotopy is σ with $\sigma_0 = 1^M$ and $\sigma_v = 0$ for $v \neq 0$. Complexes with this property are treated towards the end of Sect. 4.2.

The next proposition shows that homotopy yields a congruence relation in $\mathcal{C}(R)$, i.e. an equivalence relation that is compatible with $\Bbbk$-linearity and composition.

2.2.25 Proposition. *Let L, M, and N be R-complexes. For every integer n, homotopy '$\sim$' is an equivalence relation on the set of chain maps $M \to N$ of degree n. Further,*

(a) *Let $\alpha, \alpha', \beta, \beta'\colon M \to N$ be chain maps of the same degree. If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then one has $\alpha + x\beta \sim \alpha' + x\beta'$ for every $x \in \Bbbk$.*
(b) *Let $\beta, \beta'\colon L \to M$ and $\alpha, \alpha'\colon M \to N$ be chain maps. If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then one has $\alpha\beta \sim \alpha'\beta'$.*

Proof. (a): Let σ be a homotopy from α to α' and τ a homotopy from β to β'. It is immediate that $\sigma + x\tau$ is a homotopy from $\alpha + x\beta$ to $\alpha' + x\beta'$.

In now follows that '$\sim$' is transitive: For chain maps $\gamma \sim \gamma'$ and $\gamma' \sim \gamma''$ one has $\gamma - \gamma'' = (\gamma - \gamma') + (\gamma' - \gamma'') \sim 0 + 0 = 0$ and hence $\gamma \sim \gamma''$. As it is evident from the definition, 2.2.23, that $\sim$ is reflexive and symmetric, it is an equivalence relation.

(b): Let σ be a homotopy from α to α' and τ a homotopy from β to β'. Using the identity $\alpha\beta - \alpha'\beta' = \alpha(\beta - \beta') + (\alpha - \alpha')\beta'$ it is straightforward to verify that degree $|\alpha| + |\beta| + 1$ homomorphism $(-1)^{|\alpha|}\alpha\tau + \sigma\beta'$ is a homotopy from $\alpha\beta$ to $\alpha'\beta'$. ◻

2.2.26 Proposition. *Let $\alpha, \beta \colon M \to N$ be morphisms of R-complexes. If one has $\alpha \sim \beta$, then $\mathrm{H}(\alpha) = \mathrm{H}(\beta)$ holds.*

Proof. If $\alpha \colon M \to N$ is null-homotopic, then it follows directly from the definition, 2.2.23, that there is an inclusion $\alpha(\mathrm{Z}(M)) \subseteq \mathrm{B}(N)$, so the induced morphism $\mathrm{H}(\alpha)$ is 0. As homology is an additive functor, see 2.2.15, the assertion follows. ◻

2.2.27 Example. Consider an R-complex

$$M = 0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0\,.$$

One has $\mathrm{H}(1^M) = 0$ if and only if M is exact as a sequence in $\mathcal{M}(R)$, while 1^M is null-homotopic if and only if M is split exact as a sequence in $\mathcal{M}(R)$.

2.2.28 Definition. A diagram in $\mathcal{C}(R)$,

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & M' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ N & \xrightarrow{\beta} & N'\,, \end{array}$$

is called *commutative up to homotopy* if one has $\varphi'\alpha \sim \beta\varphi$.

Notice that, in view of 2.2.26, application of the homology functor H to a diagram in $\mathcal{C}(R)$ that is commutative up to homotopy yields a commutative diagram.

Exercises

E 2.2.1 Consider the morphism of $\mathbb{Z}$-complexes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/30\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/30\mathbb{Z} & \xrightarrow{10} & \mathbb{Z}/30\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 5} & & \downarrow{\scriptstyle 3} & & \\ 0 & \longrightarrow & \mathbb{Z}/30\mathbb{Z} & \xrightarrow{10} & \mathbb{Z}/30\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/30\mathbb{Z} & \longrightarrow & 0\,. \end{array}$$

Describe the induced morphism in homology.

E 2.2.2 Show that a surjective morphism of R-complexes is surjective on boundaries and show that a morphism that is injective on cycles is injective.

E 2.2.3 Show that a morphism of R-complexes is surjective if it is surjective on boundaries and on cycles.

E 2.2.4 (Cf. 2.2.13) Show that $\mathrm{B} \colon \mathcal{C}(R) \to \mathcal{C}(R)$ is a $\Bbbk$-linear functor that is not half exact.

E 2.2.5 (Cf. 2.2.15) Show that homology $\mathrm{H} \colon \mathcal{C}(R) \to \mathcal{C}(R)$ is a $\Bbbk$-linear half exact functor.

E 2.2.6 (Cf. 2.2.19) Let $\mathrm{F} \colon \mathcal{M}(R) \to \mathcal{M}(S)$ be a functor and consider the extended functor $\mathcal{C}(R) \to \mathcal{C}(S)$; see 2.1.48. Let M be an R-complex. (a) Show that if F is left exact, then $\mathrm{Z}_v(\mathrm{F}(M)) \cong \mathrm{F}(\mathrm{Z}_v(M))$ holds for every $v \in \mathbb{Z}$. (b) Show that if F is right exact, then $\mathrm{C}_v(\mathrm{F}(M)) \cong \mathrm{F}(\mathrm{C}_v(M))$ holds for every $v \in \mathbb{Z}$. (c) Conclude that if F is exact, then one has $\mathrm{B}_v(\mathrm{F}(M)) \cong \mathrm{F}(\mathrm{B}_v(M))$ and $\mathrm{H}_v(\mathrm{F}(M)) \cong \mathrm{F}(\mathrm{H}_v(M))$ for every $v \in \mathbb{Z}$.

E 2.2.7 (Cf. 2.2.19) Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be a functor and consider the extended functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$; see 2.1.48. Let M be an R-complex. (a) Show that if G is left exact, then $\mathrm{Z}_v(\mathrm{G}(M)) \cong \mathrm{G}(\mathrm{C}_{-v}(M))$ holds for every $v \in \mathbb{Z}$. (b) Show that if G is right exact, then $\mathrm{C}_v(\mathrm{G}(M)) \cong \mathrm{G}(\mathrm{Z}_{-v}(M))$ holds for every $v \in \mathbb{Z}$. (c) Conclude that if G is exact, then $\mathrm{B}_v(\mathrm{G}(M)) \cong \mathrm{G}(\mathrm{B}_{-v-1}(M))$ and $\mathrm{H}_v(\mathrm{G}(M)) \cong \mathrm{G}(\mathrm{H}_{-v}(M))$ for every $v \in \mathbb{Z}$.

E 2.2.8 Let $0 \to M^1 \to \cdots \to M^n \to 0$ be an exact sequence of R-complexes. Show that if $n-1$ of the complexes M^u are acyclic, then they are all acyclic.

E 2.2.9 Let M be an R-complex. Show that for every $v \in \mathbb{Z}$ with $\mathrm{H}_v(M) = 0$ there is an isomorphism $\mathrm{C}_v(M) \cong \mathrm{B}_{v-1}(M)$.

E 2.2.10 Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-complexes and $v \in \mathbb{Z}$. Show that if $\mathrm{H}_v(M'') = 0$ holds, then $0 \to \mathrm{Z}_v(M') \to \mathrm{Z}_v(M) \to \mathrm{Z}_v(M'') \to 0$ is exact.

E 2.2.11 Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-complexes and $v \in \mathbb{Z}$. Show that if $\mathrm{H}_v(M') = 0$ holds, then $0 \to \mathrm{C}_v(M') \to \mathrm{C}_v(M) \to \mathrm{C}_v(M'') \to 0$ is exact.

E 2.2.12 Let $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ be an exact sequence of R-complexes; show that the following conditions are equivalent. (*i*) $\mathrm{H}(\alpha')$ is injective. (*ii*) $\mathrm{H}(\alpha)$ is surjective. (*iii*) The sequence $0 \longrightarrow \mathrm{H}(M') \xrightarrow{\mathrm{H}(\alpha')} \mathrm{H}(M) \xrightarrow{\mathrm{H}(\alpha)} \mathrm{H}(M'') \longrightarrow 0$ is exact.

E 2.2.13 Let $\Bbbk$ be a field and $M = 0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0$ a complex of $\Bbbk$-vector spaces of finite rank. Show that the equality $\sum_{v=0}^{n}(-1)^v \operatorname{rank}_{\Bbbk} M_v = \sum_{v=0}^{n}(-1)^v \operatorname{rank}_{\Bbbk} \mathrm{H}_v(M)$ holds.

E 2.2.14 Assume that R is semi-simple. Show that for every R-complex M there are morphisms $\alpha\colon \mathrm{H}(M) \to M$ and $\beta\colon M \to \mathrm{H}(M)$ such that $\mathrm{H}(\alpha)$ and $\mathrm{H}(\beta)$ are isomorphisms.

E 2.2.15 Let f and g be elements in the polynomial ring $\mathbb{Z}[x]$ and denote by K the Koszul complex $\mathrm{K}^{\mathbb{Z}[x]}(f,g)$. (a) Show that if $\mathrm{H}_0(K)$ is zero, then K is acyclic. (b) Show that if $\mathrm{H}_1(K)$ is zero, then $\mathrm{H}_2(K)$ is zero as well.

E 2.2.16 Let $f, g\colon X \to Y$ be continuous maps between topological spaces and assume that f and g are homotopic in the topological sense; that is, there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Show that the induced morphisms $\mathrm{S}(f), \mathrm{S}(g)\colon \mathrm{S}(X) \to \mathrm{S}(Y)$ in $\mathcal{C}(\mathbb{Z})$ are homotopic. Is the converse true?

E 2.2.17 Let M be an R-complex. Show that ∂^M is a null-homotopic chain map.

E 2.2.18 Show that a homotopy between homotopic chain maps need not be unique.

E 2.2.19 Assume that R is commutative. Let x_1 and x_2 be elements in R and set $K = \mathrm{K}^R(x_1, x_2)$. Show that the homothety x^K is null-homotopic for every $x \in (x_1, x_2)$.

E 2.2.20 Assume that R is commutative and let M be an R-complex. Show that $\{x \in R \mid x^M \sim 0\}$ is an ideal in R and notice that it coincides with $(0 :_R M)$ if M is a module.

E 2.2.21 Show that the identity morphism on the Koszul complex $\mathrm{K}^{\mathbb{Z}}(2,3)$ is null-homotopic.

E 2.2.22 Show that the cokernel functor is left adjoint to the inclusion $\mathcal{M}_{\mathrm{gr}}(R) \to \mathcal{C}(R)$.

E 2.2.23 Show that the cycle functor is right adjoint to the inclusion $\mathcal{M}_{\mathrm{gr}}(R) \to \mathcal{C}(R)$.

2.3 Homomorphisms

Synopsis. Hom complex; chain map; homotopy; the functor Hom; exactness of ~.

Complexes M and N are graded modules with differentials, and those differentials induce a differential on the graded module $\mathrm{Hom}_R(M,N)$, which then becomes the *Hom complex.* Recall from 2.1.28 that the homogeneous elements in $\mathrm{Hom}_R(M,N)$ are homomorphisms $M \to N$.

2.3.1 Definition. For R-complexes M and N, the complex $\operatorname{Hom}_R(M,N)$ is the $\Bbbk$-complex with underlying graded module

$$\operatorname{Hom}_R(M,N)^\natural = \operatorname{Hom}_R(M^\natural, N^\natural)$$

and differential given by

$$\partial^{\operatorname{Hom}_R(M,N)}(\alpha) = \partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M$$

for a homogeneous element α.

2.3.2. It is elementary to verify that $\partial^{\operatorname{Hom}_R(M,N)}$ is square zero. Indeed, one has

$$\begin{aligned}\partial^{\operatorname{Hom}_R(M,N)}\partial^{\operatorname{Hom}_R(M,N)}(\alpha) &= \partial^{\operatorname{Hom}_R(M,N)}(\partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M)\\ &= \partial^N(\partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M)\\ &\qquad - (-1)^{|\alpha|-1}(\partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M)\partial^M\\ &= 0\,.\end{aligned}$$

The next result interprets boundaries and cycles in Hom complexes in familiar terms and provides, thus, a retrospective motivation for the definition of the differential on Hom complexes.

2.3.3 Proposition. *Let M and N be R-complexes.*

(a) *A homomorphism $M \to N$ is a cycle in the complex* $\operatorname{Hom}_R(M,N)$ *if and only if it is a chain map.*
(b) *A homomorphism $M \to N$ is a boundary in the complex* $\operatorname{Hom}_R(M,N)$ *if and only if it is a null-homotopic chain map.*

Proof. The assertions are immediate from the definition of the differential on the Hom complex; see 2.3.1. □

2.3.4. Let $L \xrightarrow{\beta} M \xrightarrow{\alpha} N$ be homomorphisms of R-complexes. The differentials on the Hom complexes interact with the composition rule from 2.1.7 as follows,

$$\begin{aligned}\partial^{\operatorname{Hom}_R(L,N)}(\alpha\beta) &= \partial^N\alpha\beta - (-1)^{|\alpha\beta|}\alpha\beta\partial^L\\ &= (\partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M)\beta + (-1)^{|\alpha|}\alpha(\partial^M\beta - (-1)^{|\beta|}\beta\partial^L)\\ &= \partial^{\operatorname{Hom}_R(M,N)}(\alpha)\beta + (-1)^{|\alpha|}\alpha\partial^{\operatorname{Hom}_R(L,M)}(\beta)\,.\end{aligned}$$

Assume that α and β are chain maps. It follows from the identity above and 2.3.3 that $\alpha\beta$ is a chain map; this recovers 2.1.31.

Remark. A *differential graded* (for short, DG) $\Bbbk$-algebra is a $\Bbbk$-complex A endowed with a graded $\Bbbk$-algebra structure, such that the Leibniz Rule, $\partial^A(ab) = \partial^A(a)b + (-1)^{|a|}a\partial^A(b)$, holds for all homogeneous elements $a, b \in A$. In other words: the differential ∂^A is a derivation on A. For an R-complex M, it follows from 2.1.7 and 2.3.4 that the complex $\operatorname{Hom}_R(M,M)$ is a DG $\Bbbk$-algebra. Moreover, the Koszul complex from 2.1.25 is a DG $\Bbbk$-algebra, see E 2.1.13, and so is the de Rham complex under the wedge product as well as the singular cochain complex $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{S}(X),\Bbbk)$ from 2.1.26 endowed with the so-called cup product. An algebra structure on a complex A induces an algebra structure, notably a product, in homology; this yields another tool for investigating $\mathrm{H}(A)$.

FUNCTORIALITY

We now set out to prove that Hom is a functor. Our efforts culminate in Theorem 2.3.10; the intermediate paragraphs establish the necessary definitions and verify the technical requirements. We start by defining how Hom acts on homomorphisms, in particular, on morphisms.

2.3.5 Definition. Let $\alpha\colon M' \to M$ and $\beta\colon N \to N'$ be homomorphisms of R-complexes. Denote by $\operatorname{Hom}_R(\alpha,\beta)$ the degree $|\alpha|+|\beta|$ homomorphism of $\Bbbk$-complexes

$$\operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(M',N') \quad \text{given by} \quad \vartheta \longmapsto (-1)^{|\alpha|(|\beta|+|\vartheta|)}\beta\vartheta\alpha\ .$$

Furthermore, set $\operatorname{Hom}_R(\alpha,N) = \operatorname{Hom}_R(\alpha,1^N)$ and $\operatorname{Hom}_R(M,\beta) = \operatorname{Hom}_R(1^M,\beta)$.

2.3.6. It is straightforward to verify that the assignment $(\alpha,\beta) \mapsto \operatorname{Hom}_R(\alpha,\beta)$, for homomorphisms α and β of R-complexes, is $\Bbbk$-bilinear. It is also immediate from the definition that one has

$$\operatorname{Hom}_R(1^M,1^N) \;=\; 1^{\operatorname{Hom}_R(M,N)}\ .$$

For homomorphisms $M'' \xrightarrow{\alpha} M \xrightarrow{\alpha'} M'$ and $N' \xrightarrow{\beta'} N \xrightarrow{\beta} N''$ of R-complexes there is an equality

$$\operatorname{Hom}_R(\alpha'\alpha,\beta\beta') \;=\; (-1)^{|\alpha'|(|\alpha|+|\beta|)}\operatorname{Hom}_R(\alpha,\beta)\operatorname{Hom}_R(\alpha',\beta')\ .$$

Indeed, for every homomorphism $\vartheta\colon M' \to N'$ one has

$$\begin{aligned}
\operatorname{Hom}_R(\alpha'\alpha,\beta\beta')(\vartheta) &= (-1)^{|\alpha'\alpha|(|\beta\beta'|+|\vartheta|)}(\beta\beta')\vartheta(\alpha'\alpha)\\
&= (-1)^{|\alpha'\alpha|(|\beta\beta'|+|\vartheta|)}(-1)^{|\alpha|(|\beta|+|\beta'\vartheta\alpha'|)}\operatorname{Hom}_R(\alpha,\beta)(\beta'\vartheta\alpha')\\
&= (-1)^{|\alpha'|(|\alpha|+|\beta|)}(-1)^{|\alpha'|(|\beta'|+|\vartheta|)}\operatorname{Hom}_R(\alpha,\beta)(\beta'\vartheta\alpha')\\
&= (-1)^{|\alpha'|(|\alpha|+|\beta|)}\operatorname{Hom}_R(\alpha,\beta)\operatorname{Hom}_R(\alpha',\beta')(\vartheta)\ .
\end{aligned}$$

In particular, there are equalities,

$$\begin{aligned}
\operatorname{Hom}_R(\alpha'\alpha,N) &= (-1)^{|\alpha'||\alpha|}\operatorname{Hom}_R(\alpha,N)\operatorname{Hom}_R(\alpha',N)\ ,\\
\operatorname{Hom}_R(M,\beta\beta') &= \operatorname{Hom}_R(M,\beta)\operatorname{Hom}_R(M,\beta')\ ,
\end{aligned}$$

and, furthermore, a commutative diagram of homomorphisms of $\Bbbk$-complexes,

$$\begin{tikzcd}
\operatorname{Hom}_R(M,N) \arrow[r, "\operatorname{Hom}(M,\beta)"] \arrow[d, "(-1)^{|\alpha||\beta|}\operatorname{Hom}(\alpha,N)"'] \arrow[dr, "\operatorname{Hom}(\alpha,\beta)"] & \operatorname{Hom}_R(M,N'') \arrow[d, "\operatorname{Hom}(\alpha,N'')"] \\
\operatorname{Hom}_R(M'',N) \arrow[r, "\operatorname{Hom}(M'',\beta)"] & \operatorname{Hom}_R(M'',N'')\ .
\end{tikzcd}$$

2.3.7 Lemma. *Let $\alpha\colon M' \to M$ and $\beta\colon N \to N'$ be homomorphisms of R-complexes. With $H = \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_R(M,N),\operatorname{Hom}_R(M',N'))$ there is an equality*

$$\begin{aligned}
&\partial^H(\operatorname{Hom}_R(\alpha,\beta))\\
&\qquad = \operatorname{Hom}_R(\partial^{\operatorname{Hom}_R(M',M)}(\alpha),\beta) + (-1)^{|\alpha|}\operatorname{Hom}_R(\alpha,\partial^{\operatorname{Hom}_R(N,N')}(\beta))\ .
\end{aligned}$$

Proof. Let $\vartheta\colon M \to N$ be a homomorphism. The definitions yield

$$\begin{aligned}
&(\partial^{H}(\operatorname{Hom}_R(\alpha,\beta)))(\vartheta)\\
&\qquad = (\partial^{\operatorname{Hom}_R(M',N')}\operatorname{Hom}_R(\alpha,\beta))(\vartheta)\\
&\qquad\qquad - (-1)^{|\alpha|+|\beta|}(\operatorname{Hom}_R(\alpha,\beta)\partial^{\operatorname{Hom}_R(M,N)})(\vartheta)\\
&\qquad = \partial^{\operatorname{Hom}_R(M',N')}((-1)^{|\alpha|(|\beta|+|\vartheta|)}\beta\vartheta\alpha)\\
&\qquad\qquad - (-1)^{|\alpha|+|\beta|}\operatorname{Hom}_R(\alpha,\beta)(\partial^{N}\vartheta - (-1)^{|\vartheta|}\vartheta\partial^{M})\\
&\qquad = (-1)^{|\alpha|(|\beta|+|\vartheta|)}(\partial^{N'}\beta\vartheta\alpha - (-1)^{|\beta|+|\vartheta|+|\alpha|}\beta\vartheta\alpha\partial^{M'})\\
&\qquad\qquad - (-1)^{|\alpha|+|\beta|}(-1)^{|\alpha|(|\beta|+|\vartheta|-1)}\beta(\partial^{N}\vartheta - (-1)^{|\vartheta|}\vartheta\partial^{M})\alpha\,.
\end{aligned}$$

Further, one has

$$\begin{aligned}
&(\operatorname{Hom}_R(\partial^{\operatorname{Hom}_R(M',M)}(\alpha),\beta) + (-1)^{|\alpha|}\operatorname{Hom}_R(\alpha,\partial^{\operatorname{Hom}_R(N,N')}(\beta)))(\vartheta)\\
&\qquad = (-1)^{(|\alpha|-1)(|\beta|+|\vartheta|)}\beta\vartheta\partial^{\operatorname{Hom}_R(M',M)}(\alpha)\\
&\qquad\qquad + (-1)^{|\alpha|}(-1)^{|\alpha|(|\beta|-1+|\vartheta|)}\partial^{\operatorname{Hom}_R(N,N')}(\beta)\vartheta\alpha\\
&\qquad = (-1)^{(|\alpha|-1)(|\beta|+|\vartheta|)}\beta\vartheta(\partial^{M}\alpha - (-1)^{|\alpha|}\alpha\partial^{M'})\\
&\qquad\qquad + (-1)^{|\alpha|}(-1)^{|\alpha|(|\beta|-1+|\vartheta|)}(\partial^{N'}\beta - (-1)^{|\beta|}\beta\partial^{N})\vartheta\alpha\,.
\end{aligned}$$

Compare the two expressions above to see that they are identical. □

Now we focus on chain maps.

2.3.8 Proposition. *Let α and β be chain maps of R-complexes. The homomorphism $\operatorname{Hom}_R(\alpha,\beta)$ is a chain map of degree $|\alpha|+|\beta|$, and if α or β is null-homotopic, then $\operatorname{Hom}_R(\alpha,\beta)$ is null-homotopic.*

Proof. Let $\alpha\colon M' \to M$ and $\beta\colon N \to N'$ be chain maps; by 2.3.3 they are cycles in the complexes $\operatorname{Hom}_R(M',M)$ and $\operatorname{Hom}_R(N,N')$. That is, one has $\partial^{\operatorname{Hom}_R(M',M)}(\alpha) = 0$ and $\partial^{\operatorname{Hom}_R(N,N')}(\beta) = 0$. With $H = \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_R(M,N),\operatorname{Hom}_R(M',N'))$, 2.3.7 yields $\partial^{H}(\operatorname{Hom}_R(\alpha,\beta)) = 0$, so $\operatorname{Hom}_R(\alpha,\beta)$ is a chain map by 2.3.3.

If α is null-homotopic, then $\alpha = \partial^{\operatorname{Hom}_R(M',M)}(\vartheta)$ holds for a ϑ in $\operatorname{Hom}_R(M',M)$; see 2.3.3. Now 2.3.7 yields $\operatorname{Hom}_R(\alpha,\beta) = \partial^{H}(\operatorname{Hom}_R(\vartheta,\beta))$, whence $\operatorname{Hom}_R(\alpha,\beta)$ is null-homotopic. Similarly, if one has $\beta = \partial^{\operatorname{Hom}_R(N,N')}(\vartheta)$ for some ϑ in $\operatorname{Hom}_R(N,N')$, then 2.3.7 yields $\operatorname{Hom}_R(\alpha,\beta) = \partial^{H}(\operatorname{Hom}_R(\alpha,(-1)^{|\alpha|}\vartheta))$. □

2.3.9 Corollary. *Let $\alpha,\alpha'\colon M' \to M$ and $\beta,\beta'\colon N \to N'$ be chain maps of R-complexes. If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then one has $\operatorname{Hom}_R(\alpha,\beta) \sim \operatorname{Hom}_R(\alpha',\beta')$.*

Proof. By bilinearity one has

$$\operatorname{Hom}_R(\alpha,\beta) - \operatorname{Hom}_R(\alpha',\beta') = \operatorname{Hom}_R(\alpha-\alpha',\beta) + \operatorname{Hom}_R(\alpha',\beta-\beta')\,,$$

and by 2.3.8 that is a sum of null-homotopic chain maps; now invoke 2.2.25. □

The functor described in the next theorem is called the *Hom functor*. The last equality requires a preparatory remark. In the $\Bbbk$-linear category $\mathcal{C}(R)$, each hom-set

$\mathcal{C}(R)(M,N)$ is a $\Bbbk$-module. Homotopy '$\sim$' is by 2.2.25 a congruence relation on $\mathcal{C}(R)(M,N)$, so the set $\mathcal{C}(R)(M,N)/\!\sim$ of equivalence classes is a $\Bbbk$-module under induced addition, $[\alpha]_\sim + [\beta]_\sim = [\alpha+\beta]_\sim$, and $\Bbbk$-multiplication $x[\alpha]_\sim = [x\alpha]_\sim$.

2.3.10 Theorem. *The functions*

$$\operatorname{Hom}_R(-,-) : \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \longrightarrow \mathcal{C}(\Bbbk)\,,$$

defined on objects in 2.3.1 *and on morphisms in* 2.3.5*, constitute a* $\Bbbk$*-bilinear and left exact functor.*

For R-complexes M and N, the $\Bbbk$*-complex* $\operatorname{Hom}_R(M,N)$ *and the* $\Bbbk$*-module* $\mathcal{C}(R)(M,N)$ *are related by the following identities,*

$$\begin{aligned} \mathrm{Z}_0(\operatorname{Hom}_R(M,N)) &= \mathcal{C}(R)(M,N) \quad \textit{and} \\ \mathrm{H}_0(\operatorname{Hom}_R(M,N)) &= \mathcal{C}(R)(M,N)/\!\sim\ . \end{aligned}$$

Proof. It follows from 2.3.1 and 2.3.8 that Hom_R takes objects and morphisms in $\mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R)$ to objects and morphisms in $\mathcal{C}(\Bbbk)$. The functoriality and the $\Bbbk$-bilinearity are established in 2.3.6. To prove left exactness, let

$$0 \longrightarrow (M',N') \xrightarrow{(\alpha',\beta')} (M,N) \xrightarrow{(\alpha,\beta)} (M'',N'') \longrightarrow 0$$

be an exact sequence in $\mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R)$. It is sufficient to verify that the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M',N')_v \xrightarrow{\operatorname{Hom}(\alpha',\beta')_v} \operatorname{Hom}_R(M,N)_v \xrightarrow{\operatorname{Hom}(\alpha,\beta)_v} \operatorname{Hom}_R(M'',N'')_v$$

in $\mathcal{M}(\Bbbk)$ is exact for every $v \in \mathbb{Z}$. By 2.1.4 and 2.3.5 such a sequence is a product of exact sequences; indeed $\operatorname{Hom}_R(\alpha',\beta')_v$ is the product

$$\prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(\alpha'_i,\beta'_{i+v}) : \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(M'_i,N'_{i+v}) \longrightarrow \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(M_i,N_{i+v})\,,$$

and $\operatorname{Hom}_R(\alpha,\beta)_v$ has a similar form; it follows that the sequence above is exact.

Finally, the equalities of $\Bbbk$-modules follow from 2.3.3. □

Remark. The proof above uses the fact, immediate from 1.1.19, that a product of exact sequences in $\mathcal{M}(R)$ is exact. This property does not hold in every Abelian category; a counterexample is the category of sheaves of Abelian groups on a suitable topological space; see Grothendieck [110] or Krause [160].

2.3.11 Addendum (to 2.3.10). If M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$, then it follows from 2.1.38 and 2.1.18 that $\operatorname{Hom}_R(M,N)^\natural$ is a graded Q–S^{o}-bimodule, and it is elementary to verify that the differential $\partial^{\operatorname{Hom}_R(M,N)}$ is Q- and S^{o}-linear. That is, $\operatorname{Hom}_R(M,N)$ is an object in $\mathcal{C}(Q\text{–}S^{\mathrm{o}})$. For morphisms α in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ and β in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$ it is straightforward to verify that $\operatorname{Hom}_R(\alpha,\beta)$ is a morphism in $\mathcal{C}(Q\text{–}S^{\mathrm{o}})$. Thus, there is an induced $\Bbbk$-bilinear functor,

$$\operatorname{Hom}_R(-,-) : \mathcal{C}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{C}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{C}(Q\text{–}S^{\mathrm{o}})\,.$$

2.3.12 Proposition. *If M is an R-complex, then* $\operatorname{Hom}_R(M,-)$ *is a* $\natural$*-functor and for every degreewise split exact sequence* $0 \to N' \to N \to N'' \to 0$ *in* $\mathcal{C}(R)$*,* *the sequence* $0 \to \operatorname{Hom}_R(M,N') \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N'') \to 0$ *is degreewise split exact.*

Proof. It follows from 2.3.1 and 2.3.5 that $\operatorname{Hom}_R(M,-)$ is a $\natural$-functor. The last assertion is a special case of 2.1.54. □

2.3.13 Proposition. *If N is an R-complex, then $\operatorname{Hom}_R(-,N)$ is a $\natural$-functor and for every degreewise split exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}(R)$, the sequence $0 \to \operatorname{Hom}_R(M'',N) \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M',N) \to 0$ is degreewise split exact.*

Proof. It follows from 2.3.1 and 2.3.5 that $\operatorname{Hom}_R(-,N)$ is a $\natural$-functor. The last assertion is a special case of 2.1.55. □

Hom and Shift

The shift functor allows one to consider chain maps as morphisms; for this reason, and for deeper reasons that become apparent in Chap. 6, we record how shift interacts with Hom.

2.3.14 Proposition. *Let M and N be R-complexes and s an integer. The composite $\operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M},N)\varsigma_s^{\Sigma^{-s}\operatorname{Hom}(M,N)}$ is an isomorphism of $\Bbbk$-complexes,*

$$\Sigma^{-s}\operatorname{Hom}_R(M,N) \xrightarrow{\cong} \operatorname{Hom}_R(\Sigma^s M,N)\,,$$

and it is natural in M and N.

Proof. Recall from 2.2.4 that $\varsigma_s^{\Sigma^{-s}\operatorname{Hom}(M,N)}$ is an invertible chain map of degree s and that $\varsigma_{-s}^{\Sigma^s M}$ is a degree $-s$ invertible chain map with inverse ς_s^M. It follows from 2.3.6 and 2.3.8 that $\operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M},N)$ is a degree $-s$ invertible chain map with inverse $(-1)^s\operatorname{Hom}_R(\varsigma_s^M,N)$. Thus, the composite $\operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M},N)\varsigma_s^{\Sigma^{-s}\operatorname{Hom}(M,N)}$ is an invertible chain map of degree 0, i.e. an isomorphism in the category $\mathcal{C}(\Bbbk)$.

To prove that this isomorphism is natural in M and N, let $\alpha\colon M' \to M$ and $\beta\colon N \to N'$ be morphisms of R-complexes. It suffices to show that the following diagram of homomorphisms of $\Bbbk$-complexes is commutative,

$$\begin{tikzcd}
\Sigma^{-s}\operatorname{Hom}_R(M,N) \arrow[r, "\varsigma_s^{\Sigma^{-s}\operatorname{Hom}(M,N)}"] \arrow[d, "\Sigma^{-s}\operatorname{Hom}(\alpha,\beta)"] & \operatorname{Hom}_R(M,N) \arrow[r, "\operatorname{Hom}(\varsigma_{-s}^{\Sigma^s M},N)"] \arrow[d, "\operatorname{Hom}(\alpha,\beta)"] & \operatorname{Hom}_R(\Sigma^s M,N) \arrow[d, "\operatorname{Hom}(\Sigma^s\alpha,\beta)"] \\
\Sigma^{-s}\operatorname{Hom}_R(M',N') \arrow[r, "\varsigma_s^{\Sigma^{-s}\operatorname{Hom}(M',N')}"'] & \operatorname{Hom}_R(M',N') \arrow[r, "\operatorname{Hom}(\varsigma_{-s}^{\Sigma^s M'},N')"'] & \operatorname{Hom}_R(\Sigma^s M',N')\,.
\end{tikzcd}$$

The left-hand square is commutative by (2.2.5.1). The equality $\alpha\varsigma_{-s}^{\Sigma^s M'} = \varsigma_{-s}^{\Sigma^s M}(\Sigma^s\alpha)$ from (2.2.5.1) conspires with 2.3.6 to give

$$\begin{aligned}
\operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M'},N')\operatorname{Hom}_R(\alpha,\beta) &= \operatorname{Hom}_R(\alpha\varsigma_{-s}^{\Sigma^s M'},\beta) \\
&= \operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M}(\Sigma^s\alpha),\beta)
\end{aligned}$$

$$= \operatorname{Hom}_R(\Sigma^s\alpha, \beta)\operatorname{Hom}_R(\varsigma_{-s}^{\Sigma^s M}, N)\ .$$

This shows that the right-hand square is commutative. □

2.3.15. If one suppresses the degree changing maps, then the isomorphism in 2.3.14 is given by the assignment $\vartheta \mapsto (-1)^{s|\vartheta|}\vartheta$, where $|\vartheta|$ is the degree of ϑ as an element of $\operatorname{Hom}_R(M,N)$; see 2.3.5. Similarly, the isomorphism in 2.3.16 below is the identity upon suppression of the degree changing chain maps.

2.3.16 Proposition. *Let M and N be R-complexes and s an integer. The composite $\varsigma_s^{\operatorname{Hom}(M,N)}\operatorname{Hom}_R(M, \varsigma_{-s}^{\Sigma^s N})$ is an isomorphism of $\Bbbk$-complexes,*

$$\operatorname{Hom}_R(M, \Sigma^s N) \xrightarrow{\cong} \Sigma^s \operatorname{Hom}_R(M,N)\ ,$$

and it is natural in M and N.

Proof. The assertions follow from an argument parallel to the proof of 2.3.14. □

Exactness

2.3.17 Proposition. *Let M be an R-complex and $0 \to N' \to N \to N'' \to 0$ a sequence of R-complexes. If for all $v, i \in \mathbb{Z}$ the sequence of modules,*

$$0 \longrightarrow \operatorname{Hom}_R(M_v, N'_i) \longrightarrow \operatorname{Hom}_R(M_v, N_i) \longrightarrow \operatorname{Hom}_R(M_v, N''_i) \longrightarrow 0\ ,$$

is exact, then $0 \to \operatorname{Hom}_R(M,N') \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N'') \to 0$ is an exact sequence of complexes.

Proof. By 2.1.40 exactness of a sequence of complexes can be verified degreewise. Fix $p \in \mathbb{Z}$; it follows from 2.1.4 and 2.3.5 that the sequence of modules, $0 \to \operatorname{Hom}_R(M,N')_p \to \operatorname{Hom}_R(M,N)_p \to \operatorname{Hom}_R(M,N'')_p \to 0$, is the product of $0 \to \operatorname{Hom}_R(M_v, N'_{v+p}) \to \operatorname{Hom}_R(M_v, N_{v+p}) \to \operatorname{Hom}_R(M_v, N''_{v+p}) \to 0$ for all $v \in \mathbb{Z}$. By assumption all these sequences are exact, hence so is the product. □

2.3.18 Example. Let P be a complex of projective R-modules. For every exact sequence $0 \to N' \to N \to N'' \to 0$ of R-complexes the sequence of $\Bbbk$-complexes, $0 \to \operatorname{Hom}_R(P,N') \to \operatorname{Hom}_R(P,N) \to \operatorname{Hom}_R(P,N'') \to 0$, is exact by 2.3.17.

2.3.19 Proposition. *Let N be an R-complex and $0 \to M' \to M \to M'' \to 0$ a sequence in $\mathcal{C}(R)$. If for all $v, i \in \mathbb{Z}$ the sequence of modules,*

$$0 \longrightarrow \operatorname{Hom}_R(M''_v, N_i) \longrightarrow \operatorname{Hom}_R(M_v, N_i) \longrightarrow \operatorname{Hom}_R(M'_v, N_i) \longrightarrow 0\ ,$$

is exact, then $0 \to \operatorname{Hom}_R(M'',N) \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M',N) \to 0$, is an exact sequence of complexes.

Proof. The assertion follows from an argument parallel to the proof of 2.3.17. □

2.3.20 Example. Let I be a complex of injective R-modules. For every exact sequence $0 \to M' \to M \to M'' \to 0$ of R-complexes the sequence of $\Bbbk$-complexes, $0 \to \operatorname{Hom}_R(M'',I) \to \operatorname{Hom}_R(M,I) \to \operatorname{Hom}_R(M',I) \to 0$, is exact by 2.3.19.

EXERCISES

E 2.3.1 Apply 2.3.4 to prove that if $\beta\colon L \to M$ and $\alpha\colon M \to N$ are chain maps and one of them is null-homotopic, then so is the composite $\alpha\beta$. From there recover 2.2.25(b).

E 2.3.2 Generalize the identity in 2.3.4 as follows. If $M^1 \xrightarrow{\alpha^1} M^2 \xrightarrow{\alpha^2} \cdots \longrightarrow M^n \xrightarrow{\alpha^n} M^{n+1}$ are homomorphisms of R-complexes, then one has

$$\begin{aligned}&\partial^{\operatorname{Hom}_R(M^1,M^{n+1})}(\alpha^n\cdots\alpha^2\alpha^1)\\ &\qquad= \textstyle\sum_{i=1}^n(-1)^{|\alpha^{i+1}|+\cdots+|\alpha^n|}\alpha^n\cdots\alpha^{i+1}\partial^{\operatorname{Hom}_R(M^i,M^{i+1})}(\alpha^i)\alpha^{i-1}\cdots\alpha^1\,.\end{aligned}$$

E 2.3.3 Give a proof of part (b) in 2.3.9.

E 2.3.4 For R-complexes M and N consider the homomorphisms $\partial^{\operatorname{Hom}_R(M,N)}$, $\operatorname{Hom}_R(\partial^M,N)$, and $\operatorname{Hom}_R(M,\partial^N)$ of degree -1 from the complex $\operatorname{Hom}_R(M,N)$ to itself. Verify that

$$\partial^{\operatorname{Hom}_R(M,N)} = \operatorname{Hom}_R(M,\partial^N) - \operatorname{Hom}_R(\partial^M,N)\,.$$

E 2.3.5 (Cf. 2.3.11) Let α be a homomorphism of complexes of R–Q^{o}-bimodules and β a homomorphism of complexes of R–S^{o}-bimodules. Show that $\operatorname{Hom}_R(\alpha,\beta)$ is a homomorphism of complexes of Q–S^{o}-bimodules.

E 2.3.6 Give a proof of 2.3.16.

E 2.3.7 Let M be an R-module and consider the functor $\operatorname{Hom}_R(M,-)\colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$. Show that the extended functor $\mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ described in 2.1.48 agrees with the functor $\operatorname{Hom}_R(M,-)$ defined in this section. Is the parallel statement true for $\operatorname{Hom}_R(-,M)$?

E 2.3.8 Let $\eta = 0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-complexes. Show that the next conditions are equivalent. (*i*) η is degreewise split. (*ii*) $\operatorname{Hom}_R(\eta,N)$ is exact for every R-module/complex N. (*iii*) $\operatorname{Hom}_R(N,\eta)$ is exact for every R-module/complex N.

2.4 Tensor Products

SYNOPSIS. Tensor product complex; chain map; homotopy; the functor ⊗; exactness of ∼.

The graded tensor product of two complexes can be endowed with a differential constructed from the differentials of the factors.

2.4.1 Definition. Let M be an R^{o}-complex and N an R-complex. The *tensor product complex* $M \otimes_R N$ is the $\Bbbk$-complex with underlying graded module

$$(M \otimes_R N)^\natural = M^\natural \otimes_R N^\natural$$

and differential given by

$$\partial^{M\otimes_R N}(m \otimes n) = \partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n)$$

for homogeneous elements m and n.

2.4.2. It is elementary to verify that $\partial^{M\otimes_R N}$ is square zero. Indeed, one has

$$\begin{aligned}\partial^{M\otimes_R N}\partial^{M\otimes_R N}(m\otimes n) &= \partial^{M\otimes_R N}(\partial^M(m)\otimes n + (-1)^{|m|}m\otimes\partial^N(n))\\ &= \partial^M\partial^M(m)\otimes n + (-1)^{|m|-1}\partial^M(m)\otimes\partial^N(n)\\ &\qquad + (-1)^{|m|}(\partial^M(m)\otimes\partial^N(n) + (-1)^{|m|}m\otimes\partial^N\partial^N(n))\\ &= 0\,.\end{aligned}$$

2.4.3 Example. Assume that R is commutative. For elements x_1 and x_2 in R, the tensor product of Koszul complexes $\mathrm{K}^R(x_1) \otimes_R \mathrm{K}^R(x_2)$ is isomorphic to $\mathrm{K}^R(x_1, x_2)$. Per (2.2.9.1) one has

$$0 \longrightarrow R\langle e_1\rangle \otimes_R R\langle e_2\rangle \xrightarrow{\partial_2} (R\langle e_1\rangle \otimes_R R) \oplus (R \otimes_R R\langle e_2\rangle) \xrightarrow{\partial_1} R \otimes_R R \longrightarrow 0$$

with

$$\partial_2(e_1 \otimes e_2) = (-e_1 \otimes x_2, x_1 \otimes e_2) \quad \text{and} \quad \partial_1(e_1 \otimes k_2, k_1 \otimes e_2) = x_1 \otimes k_2 + k_1 \otimes x_2\ .$$

Similarly, consider $\mathrm{K}^R(x_1, x_2)$ to be generated by the free module $R\langle f_1, f_2\rangle$. Comparing to (2.2.9.2) it is now elementary to verify that the assignments

$$e_1 \otimes e_2 \longmapsto f_1 \wedge f_2\,, \quad (e_1 \otimes k_2, k_1 \otimes e_2) \longmapsto k_2 f_1 + k_1 f_2\,, \quad \text{and} \quad k_1 \otimes k_2 \longmapsto k_1 k_2$$

define an isomorphism $\mathrm{K}^R(x_1) \otimes_R \mathrm{K}^R(x_2) \longrightarrow \mathrm{K}^R(x_1, x_2)$.

Functoriality

The next several paragraphs lead up to Theorem 2.4.9, which shows that the tensor product yields a functor.

2.4.4 Definition. Let $\alpha\colon M \to M'$ be a homomorphism of R^{o}-complexes and $\beta\colon N \to N'$ a homomorphism of R-complexes. Denote by $\alpha \otimes_R \beta$ the degree $|\alpha|+|\beta|$ homomorphism of $\Bbbk$-complexes

$$M \otimes_R N \longrightarrow M' \otimes_R N' \quad \text{given by} \quad m \otimes n \longmapsto (-1)^{|\beta||m|}\alpha(m) \otimes \beta(n)\ .$$

Furthermore, set $\alpha \otimes_R N = \alpha \otimes_R 1^N$ and $M \otimes_R \beta = 1^M \otimes_R \beta$.

2.4.5. It is straightforward to verify that the assignment $(\alpha, \beta) \mapsto \alpha \otimes_R \beta$, for homomorphisms α and β of complexes, is $\Bbbk$-bilinear. It is also immediate from the definition that one has

$$1^M \otimes_R 1^N = 1^{M \otimes_R N}\ .$$

For homomorphisms $M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M''$ of R^{o}-complexes and homomorphisms $N' \xrightarrow{\beta'} N \xrightarrow{\beta} N''$ of R-complexes there is an equality

$$(\alpha\alpha') \otimes_R (\beta\beta') = (-1)^{|\alpha'||\beta|}(\alpha \otimes_R \beta)(\alpha' \otimes_R \beta')\ .$$

Indeed, for homogeneous elements m' in M' and n' in N' one has

$$\begin{aligned}
&((\alpha\alpha') \otimes_R (\beta\beta'))(m' \otimes n') \\
&\qquad = (-1)^{|\beta\beta'||m'|}(\alpha\alpha')(m') \otimes (\beta\beta')(n') \\
&\qquad = (-1)^{|\beta\beta'||m'|}(-1)^{|\beta||\alpha'(m')|}(\alpha \otimes_R \beta)(\alpha'(m') \otimes \beta'(n')) \\
&\qquad = (-1)^{|\alpha'||\beta|}(-1)^{|\beta'||m'|}(\alpha \otimes_R \beta)(\alpha'(m') \otimes \beta'(n')) \\
&\qquad = (-1)^{|\alpha'||\beta|}(\alpha \otimes_R \beta)((\alpha' \otimes_R \beta')(m' \otimes n'))\ .
\end{aligned}$$

In particular, there are equalities,

$$\begin{aligned}(\alpha\alpha') \otimes_R N &= (\alpha \otimes_R N)(\alpha' \otimes_R N)\,,\\ M \otimes_R (\beta\beta') &= (M \otimes_R \beta)(M \otimes_R \beta')\,,\end{aligned}$$

and, furthermore, a commutative diagram of homomorphisms of $\Bbbk$-complexes,

(2.4.5.1)
$$\begin{tikzcd} M \otimes_R N \arrow[r, "M\otimes\beta"] \arrow[d, "(-1)^{|\alpha||\beta|}\alpha\otimes N"'] \arrow[dr, "\alpha\otimes\beta"] & M \otimes_R N'' \arrow[d, "\alpha\otimes N''"] \\ M'' \otimes_R N \arrow[r, "M''\otimes\beta"] & M'' \otimes_R N''\,. \end{tikzcd}$$

2.4.6 Lemma. *Let $\alpha\colon M \to M'$ be a homomorphism of R^{o}-complexes and $\beta\colon N \to N'$ a homomorphisms of R-complexes. With $H = \operatorname{Hom}_{\Bbbk}(M \otimes_R N, M' \otimes_R N')$ one has*

$$\partial^H(\alpha \otimes_R \beta) = \partial^{\operatorname{Hom}_{R^{\mathrm{o}}}(M,M')}(\alpha) \otimes_R \beta + (-1)^{|\alpha|}\alpha \otimes_R \partial^{\operatorname{Hom}_R(N,N')}(\beta)\,.$$

Proof. Let m and n be homogeneous elements in M and N. There are equalities,

$$\begin{aligned}(\partial^H&(\alpha \otimes_R \beta))(m \otimes n)\\
&= (\partial^{M'\otimes_R N'}(\alpha \otimes_R \beta) - (-1)^{|\alpha|+|\beta|}(\alpha \otimes_R \beta)\partial^{M\otimes_R N})(m \otimes n)\\
&= (-1)^{|\beta||m|}\partial^{M'\otimes_R N'}(\alpha(m) \otimes \beta(n))\\
&\qquad - (-1)^{|\alpha|+|\beta|}(\alpha \otimes_R \beta)(\partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n))\\
&= (-1)^{|\beta||m|}(\partial^{M'}\alpha(m) \otimes \beta(n) + (-1)^{|\alpha|+|m|}\alpha(m) \otimes \partial^{N'}\beta(n))\\
&\qquad - (-1)^{|\alpha|+|\beta|}((-1)^{|\beta|(|m|-1)}\alpha\partial^M(m) \otimes \beta(n)\\
&\qquad\qquad + (-1)^{|m|}(-1)^{|\beta||m|}\alpha(m) \otimes \beta\partial^N(n))\,.\end{aligned}$$

Further, one has

$$\begin{aligned}(\partial^{\operatorname{Hom}_{R^{\mathrm{o}}}(M,M')}&(\alpha) \otimes_R \beta + (-1)^{|\alpha|}\alpha \otimes_R \partial^{\operatorname{Hom}_R(N,N')}(\beta))(m \otimes n)\\
&= (-1)^{|\beta||m|}(\partial^{\operatorname{Hom}_{R^{\mathrm{o}}}(M,M')}(\alpha))(m) \otimes \beta(n)\\
&\qquad + (-1)^{|\alpha|}(-1)^{(|\beta|-1)|m|}\alpha(m) \otimes (\partial^{\operatorname{Hom}_R(N,N')}(\beta))(n)\\
&= (-1)^{|\beta||m|}(\partial^{M'}\alpha - (-1)^{|\alpha|}\alpha\partial^M)(m) \otimes \beta(n)\\
&\qquad + (-1)^{|\alpha|}(-1)^{(|\beta|-1)|m|}\alpha(m) \otimes (\partial^{N'}\beta - (-1)^{|\beta|}\beta\partial^N)(n)\,.\end{aligned}$$

Compare the two expressions above to see that they are identical. □

2.4.7 Proposition. *Let $\alpha\colon M \to M'$ be a chain map of R^{o}-complexes and $\beta\colon N \to N'$ a chain map of R-complexes. The homomorphism $\alpha \otimes_R \beta$ is a chain map of degree $|\alpha| + |\beta|$, and if α or β is null-homotopic, then $\alpha \otimes_R \beta$ is null-homotopic.*

Proof. By 2.3.3 the chain maps α and β are cycles in the complexes $\operatorname{Hom}_{R^{\mathrm{o}}}(M, M')$ and $\operatorname{Hom}_R(N, N')$. That is, one has $\partial^{\operatorname{Hom}_{R^{\mathrm{o}}}(M,M')}(\alpha) = 0$ and $\partial^{\operatorname{Hom}_R(N,N')}(\beta) = 0$. Denote by H the complex $\operatorname{Hom}_{\Bbbk}(M \otimes_R N, M' \otimes_R N')$, 2.4.6 yields $\partial^H(\alpha \otimes_R \beta) = 0$, so $\alpha \otimes_R \beta$ is a chain map by 2.3.3.

If α is null-homotopic, then $\alpha = \partial^{\operatorname{Hom}_{R^{\circ}}(M,M')}(\vartheta)$ for some $\vartheta \in \operatorname{Hom}_{R^{\circ}}(M,M')$; see 2.3.3. Now 2.4.6 yields $\alpha \otimes_R \beta = \partial^H(\vartheta \otimes_R \beta)$, whence $\alpha \otimes_R \beta$ is null-homotopic. Similarly, $\beta = \partial^{\operatorname{Hom}_R(N,N')}(\vartheta)$ implies $\alpha \otimes_R \beta = \partial^H((-1)^{|\alpha|}\alpha \otimes_R \vartheta)$. □

2.4.8 Corollary. *Let $\alpha, \alpha' \colon M \to M'$ and $\beta, \beta' \colon N \to N'$ be chain maps of R°- and R-complexes, respectively. If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then one has $\alpha \otimes_R \beta \sim \alpha' \otimes_R \beta'$.*

Proof. Bilinearity yields

$$\alpha \otimes_R \beta - \alpha' \otimes_R \beta' \;=\; (\alpha - \alpha') \otimes_R \beta + \alpha' \otimes_R (\beta - \beta')\;,$$

and by 2.4.7 this is a sum of null-homotopic chain maps; now invoke 2.2.25. □

The functor described in the next theorem is called the *tensor product functor*.

2.4.9 Theorem. *The functions*

$$-\otimes_R - : \mathcal{C}(R^{\circ}) \times \mathcal{C}(R) \;\longrightarrow\; \mathcal{C}(\Bbbk)\;,$$

defined on objects in 2.4.1 *and on morphisms in* 2.4.4*, constitute a $\Bbbk$-bilinear and right exact functor.*

Proof. It follows from 2.4.1 and 2.4.7 that $-\otimes_R-$ takes objects and morphisms in $\mathcal{C}(R^{\circ}) \times \mathcal{C}(R)$ to objects and morphisms in $\mathcal{C}(\Bbbk)$. The functoriality and the $\Bbbk$-bilinearity are established in 2.4.5. To prove right exactness, let

$$0 \longrightarrow (M', N') \xrightarrow{(\alpha',\beta')} (M, N) \xrightarrow{(\alpha,\beta)} (M'', N'') \longrightarrow 0$$

be an exact sequence in $\mathcal{C}(R^{\circ}) \times \mathcal{C}(R)$. It is sufficient to verify that the sequence

$$(\diamond) \qquad (M' \otimes_R N')_v \xrightarrow{(\alpha'\otimes\beta')_v} (M \otimes_R N)_v \xrightarrow{(\alpha\otimes\beta)_v} (M'' \otimes_R N'')_v \longrightarrow 0$$

in $\mathcal{M}(\Bbbk)$ is exact for every $v \in \mathbb{Z}$. By 2.1.14 and 2.4.4 such a sequence is a coproduct of exact sequences; indeed $(\alpha' \otimes_R \beta')_v$ is the coproduct

$$\coprod_{i\in\mathbb{Z}} \alpha'_i \otimes_R \beta'_{v-i} : \coprod_{i\in\mathbb{Z}} M'_i \otimes_R N'_{v-i} \;\longrightarrow\; \coprod_{i\in\mathbb{Z}} M_i \otimes_R N_{v-i}\;,$$

and $(\alpha \otimes_R \beta)_v$ has a similar form; it follows that $(\diamond)$ is exact. □

2.4.10 Addendum (to 2.4.9). If M is in $\mathcal{C}(Q\text{–}R^{\circ})$ and N is in $\mathcal{C}(R\text{–}S^{\circ})$, then it follows from 2.1.38 and 2.1.18 that $(M \otimes_R N)^{\natural}$ is a graded Q–S°-bimodule, and it is elementary to verify that $\partial^{M\otimes_R N}$ is Q- and S°-linear. That is, $M \otimes_R N$ is an object in $\mathcal{C}(Q\text{–}S^{\circ})$. For morphisms α in $\mathcal{C}(Q\text{–}R^{\circ})$ and β in $\mathcal{C}(R\text{–}S^{\circ})$ it is elementary to verify that $\alpha \otimes_R \beta$ is a morphism in $\mathcal{C}(Q\text{–}S^{\circ})$. Thus, there is an induced $\Bbbk$-bilinear functor

$$-\otimes_R - : \mathcal{C}(Q\text{–}R^{\circ}) \times \mathcal{C}(R\text{–}S^{\circ}) \;\longrightarrow\; \mathcal{C}(Q\text{–}S^{\circ})\;.$$

2.4.11 Proposition. *If M is an R°-complex, then $M \otimes_R -$ is a $\natural$-functor and for every degreewise split exact sequence $0 \to N' \to N \to N'' \to 0$ in $\mathcal{C}(R)$, the sequence $0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$ is degreewise split exact.*

Proof. It follows from 2.4.1 and 2.4.4 that $M \otimes_R -$ is a $\natural$-functor. The last assertion is a special case of 2.1.54. □

2.4.12 Proposition. *If N is an R-complex, then $- \otimes_R N$ is a $\natural$-functor and for every degreewise split exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}(R^{\mathrm{o}})$, the sequence $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ is degreewise split exact.*

Proof. It follows from 2.4.1 and 2.4.4 that $- \otimes_R M$ is a $\natural$-functor. The last assertion is a special case of 2.1.55. □

Tensor Product and Shift

Recall from 2.2.4 that for every complex X and every integer s there is an invertible chain map $\varsigma_s^X \colon X \to \Sigma^s X$. If one suppresses these maps, then the isomorphism in 2.4.13 below is given by the assignment $m \otimes n \mapsto (-1)^{s|m|} m \otimes n$.

2.4.13 Proposition. *Let M be an R^{o}-complex, N an R-complex, and s an integer. The composite $\varsigma_s^{M \otimes N} \circ (M \otimes_R \varsigma_{-s}^{\Sigma^s N})$ is an isomorphism of $\Bbbk$-complexes,*

$$M \otimes_R \Sigma^s N \xrightarrow{\cong} \Sigma^s (M \otimes_R N) \,,$$

and it is natural in M and N.

Proof. As recalled above, $\varsigma_s^{M \otimes N}$ is an invertible chain map of degree s, and $\varsigma_{-s}^{\Sigma^s N}$ is a degree $-s$ invertible chain map with inverse ς_s^N; see 2.2.4. It follows from 2.4.5 and 2.4.7 that $M \otimes_R \varsigma_{-s}^{\Sigma^s N}$ is a degree $-s$ invertible chain map with inverse $M \otimes_R \varsigma_s^N$. Thus, the composite $\varsigma_s^{M \otimes N} \circ (M \otimes_R \varsigma_{-s}^{\Sigma^s N})$ is an invertible chain map of degree 0, i.e. an isomorphism in the category $\mathcal{C}(\Bbbk)$.

To prove that this isomorphism is natural in M and N, let $\alpha \colon M \to M'$ and $\beta \colon N \to N'$ be morphisms of complexes. It suffices to show that the following diagram of homomorphisms of $\Bbbk$-complexes is commutative,

$$\begin{tikzcd}
M \otimes_R \Sigma^s N \arrow[r, "M \otimes \varsigma_{-s}^{\Sigma^s N}"] \arrow[d, "\alpha \otimes \Sigma^s \beta"] & M \otimes_R N \arrow[r, "\varsigma_s^{M \otimes N}"] \arrow[d, "\alpha \otimes \beta"] & \Sigma^s (M \otimes_R N) \arrow[d, "\Sigma^s(\alpha \otimes \beta)"] \\
M' \otimes_R \Sigma^s N' \arrow[r, "M' \otimes \varsigma_{-s}^{\Sigma^s N'}"] & M' \otimes_R N' \arrow[r, "\varsigma_s^{M' \otimes N'}"] & \Sigma^s (M' \otimes_R N') \,.
\end{tikzcd}$$

The equality $\beta \varsigma_{-s}^{\Sigma^s N} = \varsigma_{-s}^{\Sigma^s N'} \Sigma^s \beta$ from (2.2.5.1) conspires with 2.4.5 to give

$$\begin{aligned}
(\alpha \otimes_R \beta)(M \otimes_R \varsigma_{-s}^{\Sigma^s N}) &= \alpha \otimes_R (\beta \varsigma_{-s}^{\Sigma^s N}) \\
&= \alpha \otimes_R (\varsigma_{-s}^{\Sigma^s N'} \Sigma^s \beta) \\
&= (M' \otimes_R \varsigma_{-s}^{\Sigma^s N'})(\alpha \otimes_R \Sigma^s \beta) \,.
\end{aligned}$$

This shows that the left-hand square is commutative. The right-hand square is commutative by (2.2.5.1). □

If one suppresses the degree changing chain maps ς defined in 2.2.4, then the isomorphism in 2.4.14 below is an equality.

2.4.14 Proposition. *Let M be an R^{o}-complex, N an R-complex, and s an integer. The composite $\varsigma_s^{M\otimes N} \circ (\varsigma_{-s}^{\Sigma^s M} \otimes_R N)$ is an isomorphism of $\Bbbk$-complexes,*

$$\Sigma^s M \otimes_R N \xrightarrow{\cong} \Sigma^s (M \otimes_R N) \,,$$

and it is natural in M and N.

Proof. An argument parallel to the proof of 2.4.13 applies. □

Exactness

2.4.15 Proposition. *Let M be an R^{o}-complex and $0 \to N' \to N \to N'' \to 0$ a sequence of R-complexes. If for all $v, i \in \mathbb{Z}$ the sequence of modules,*

$$0 \longrightarrow M_v \otimes_R N'_i \longrightarrow M_v \otimes_R N_i \longrightarrow M_v \otimes_R N''_i \longrightarrow 0 \,,$$

is exact, then the sequence $0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$ is exact.

Proof. By 2.1.40 exactness of a sequence of complexes can be verified degree-wise. Fix $p \in \mathbb{Z}$; it follows from 2.1.14 and 2.4.4 that the sequence of modules, $0 \to (M \otimes_R N')_p \to (M \otimes_R N)_p \to (M \otimes_R N'')_p \to 0$, is the coproduct of the sequences $0 \to M_v \otimes_R N'_{v-p} \to M_v \otimes_R N_{v-p} \to M_v \otimes_R N''_{v-p} \to 0$ for all $v \in \mathbb{Z}$. By assumption each of these sequences is exact, hence so is the coproduct. □

2.4.16 Proposition. *Let N be an R-complex and $0 \to M' \to M \to M'' \to 0$ a sequence of R^{o}-complexes. If for all $v, i \in \mathbb{Z}$ the sequence of modules,*

$$0 \longrightarrow M'_v \otimes_R N_i \longrightarrow M_v \otimes_R N_i \longrightarrow M''_v \otimes_R N_i \longrightarrow 0 \,,$$

is exact, then the sequence $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ is exact.

Proof. An argument parallel to the proof of 2.4.15 applies. □

2.4.17 Example. Let F be a complex of flat R-modules. For every exact sequence $0 \to M' \to M \to M'' \to 0$ of R^{o}-complexes the induced sequence of $\Bbbk$-complexes, $0 \to M' \otimes_R F \to M \otimes_R F \to M'' \otimes_R F \to 0$, is exact.

Exercises

E 2.4.1 Let M be an R^{o}-complex and N an R-complex. Consider the degree -1 homomorphisms $\partial^{M\otimes_R N}$, $\partial^M \otimes_R N$, and $M \otimes_R \partial^N$ from $M \otimes_R N$ to itself. Verify the identity

$$\partial^{M\otimes_R N} = \partial^M \otimes_R N + M \otimes_R \partial^N \,.$$

E 2.4.2 Give a proof of part (b) in 2.4.8.

E 2.4.3 (Cf. 2.4.10) Let α be a homomorphism of complexes of Q–R^{o}-bimodules and β a homomorphism of complexes of R–S^{o}-bimodules. Show that $\alpha \otimes_R \beta$ is a homomorphism of complexes of Q–S^{o}-bimodules.

E 2.4.4 Let M be an R^{o}-module and consider the functor $M \otimes_R - \colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$. Show that the extended functor $\mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ described in 2.1.48 agrees with the functor $M \otimes_R -$ defined in this section. Is the similar statement true for an R-module N and $- \otimes_R N$?

2.5 Boundedness and Finiteness

SYNOPSIS. Degreewise finite generation/presentation; boundedness (above/below); supremum; infimum; amplitude; boundedness of Hom complex; boundedness of tensor product complex; hard/soft truncation; graded basis; graded-free module.

In this section, we examine conditions on complexes that naturally extend finite generation and finite presentation of modules.

2.5.1 Definition. An R-complex M is called *degreewise finitely generated* if the R-module M_v is finitely generated for every $v \in \mathbb{Z}$. Similarly, M is called *degreewise finitely presented* if the R-module M_v is finitely presented for every $v \in \mathbb{Z}$.

2.5.2 Definition. An R-complex M is called *bounded above* if $M_v = 0$ holds for $v \gg 0$, *bounded below* if $M_v = 0$ holds for $v \ll 0$, and *bounded* if it is bounded above and below.

REMARK. Other terms for bounded above/below are 'left/right bounded'.

2.5.3. Suppressing the full and faithful functors $\mathcal{M}(R) \longrightarrow \mathcal{M}_{\mathrm{gr}}(R) \longrightarrow \mathcal{C}(R)$, see 2.1.13 and 2.1.35, an R-module M is considered as an R-complex $0 \to M \to 0$ concentrated in degree 0. Conversely, an R-complex M that is concentrated in degree 0 is identified with the module M_0; this amounts to suppressing the forgetful functors $\mathcal{C}(R) \longrightarrow \mathcal{M}_{\mathrm{gr}}(R) \longrightarrow \mathcal{M}(R)$.

More generally, a diagram in $\mathcal{M}(R)$,

$$M^0 \xrightarrow{\alpha^0} \cdots \longrightarrow M^{p-1} \xrightarrow{\alpha^{p-1}} M^p \;,$$

such that $\alpha^n \alpha^{n-1} = 0$ holds for all $n \in \{1, \ldots, p-1\}$, can be considered, for every $u \in \mathbb{Z}$, as an R-complex concentrated in degrees $p+u, \ldots, u$.

REMARK. A complex $0 \to M \to 0$ with the module M in degree zero is called a 'stalk complex'.

The supremum and infimum of a complex, which are defined next, capture its homological position; the amplitude captures its homological size.

2.5.4 Definition. Let M be an R-complex. The *supremum* and *infimum* of M are defined as follows,

$$\sup M \;=\; \sup\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\} \quad \text{and} \quad \inf M \;=\; \inf\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\} \;.$$

Adopting the conventions $\sup \varnothing = -\infty$ and $\inf \varnothing = -\infty$ one has $\sup M = -\infty$ and $\inf M = \infty$ if M is acyclic. The *amplitude* of M is the difference

$$\operatorname{amp} M \;=\; \sup M - \inf M \;;$$

by the conventions above one has $\operatorname{amp} M = -\infty$ if M is acyclic.

REMARK. Given a complex M one may, depending on the context, take more interest in the extent of the underlying graded module $M^{\natural}$ or the extent of the homology $\mathrm{H}(M)$. Some authors define the supremum and infimum of a complex M based on $M^{\natural}$ as opposed to the homology. With that definition the invariants defined above are the supremum and infimum of $\mathrm{H}(M)$. With 2.5.4 we claim the simpler notation for the notion used more frequently in this book; as a graded module is equal to its homology it only takes addition of the symbold $\natural$ to make the switch; see 2.5.5 below.

2.5.5. Let M be an R-complex. For every integer s one evidently has

$$\sup \Sigma^s M \;=\; \sup M + s\,, \quad \inf \Sigma^s M \;=\; \inf M + s\,, \quad \text{and} \quad \operatorname{amp} \Sigma^s M \;=\; \operatorname{amp} M\,.$$

Moreover, one has $\mathrm{H}(M^\natural) = M^\natural$ and, therefore,

$$\sup M^\natural \;=\; \sup\{v \in \mathbb{Z} \mid M_v \neq 0\} \quad \text{and} \quad \inf M^\natural \;=\; \inf\{v \in \mathbb{Z} \mid M_v \neq 0\}\,.$$

Thus, a complex $M \neq 0$ is bounded above if and only if $\sup M^\natural$ is not ∞, and M is bounded below if and only if $\inf M^\natural$ is not $-\infty$.

2.5.6 Proposition. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-complexes. The following inequalities hold.*

$$\begin{aligned} \sup M' &\leqslant \max\{\sup M, \sup M'' - 1\}\,, & \inf M' &\geqslant \min\{\inf M, \inf M'' - 1\}\,,\\ \sup M &\leqslant \max\{\sup M', \sup M''\}\,, & \inf M &\geqslant \min\{\inf M', \inf M''\}\,,\\ \sup M'' &\leqslant \max\{\sup M' + 1, \sup M\}\,, \quad \textit{and} & \inf M'' &\geqslant \min\{\inf M' + 1, \inf M\}\,. \end{aligned}$$

In particular, if two of the complexes M', M, and M'' are acyclic, then so is the third.

Proof. The inequalities follow from the exact sequence (2.2.20.2). □

2.5.7 Proposition. *Let M be an R-complex.*

(a) *For every projective R-module P the next inequalities hold,*

$$-\sup \operatorname{Hom}_R(P, M) \;\geqslant\; -\sup M \quad \textit{and} \quad -\inf \operatorname{Hom}_R(P, M) \;\leqslant\; -\inf M\,;$$

if P is faithfully projective, then equalities hold. In particular, if P is faithfully projective, then $\operatorname{Hom}_R(P, M)$ is acyclic if and only if M is acyclic.

(b) *For every injective R-module E the next inequalities hold,*

$$-\sup \operatorname{Hom}_R(M, E) \;\geqslant\; \inf M \quad \textit{and} \quad -\inf \operatorname{Hom}_R(M, E) \;\leqslant\; \sup M\,;$$

if E is faithfully injective, then equalities hold. In particular, if E is faithfully injective, then $\operatorname{Hom}_R(M, E)$ is acyclic if and only if M is acyclic.

(c) *For every flat R°-module F the next inequalities hold,*

$$\inf\,(F \otimes_R M) \;\geqslant\; \inf M \quad \textit{and} \quad \sup\,(F \otimes_R M) \;\leqslant\; \sup M\,;$$

if F is faithfully flat, then equalities hold. In particular, if F is faithfully flat, then $F \otimes_R M$ is acyclic if and only if M is acyclic.

Proof. The assertions are immediate from 2.2.19. □

HOM COMPLEX

The Hom functor does not commute with homology, but a map compares the homology of a Hom complex to Hom of the homology complexes. This map is sometimes referred to as the Künneth map for Hom because of its resemblance to 2.5.14.

2.5.8 Proposition. *Let M and N be R-complexes. There is a morphism,*

$$\mathrm{H}(\mathrm{Hom}_R(M,N)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}(M),\mathrm{H}(N))\;,$$

of $\Bbbk$-complexes with degree v component

$$\mathrm{H}_v(\mathrm{Hom}_R(M,N)) \to \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(\mathrm{H}_i(M),\mathrm{H}_{i+v}(N)) \quad \textit{given by} \quad [\alpha] \mapsto (\mathrm{H}_i(\alpha))_{i\in\mathbb{Z}}\;.$$

PROOF. Let α be a chain map of degree v; it can by 2.2.5 be seen as a morphism $\alpha\colon M \to \Sigma^{-v}N$. For every $i \in \mathbb{Z}$ it induces by 2.2.15 a homomorphism of R-modules,

$$\mathrm{H}_i(\alpha)\colon\; \mathrm{H}_i(M) \;\longrightarrow\; \mathrm{H}_i(\Sigma^{-v}N) \;=\; \mathrm{H}_{i+v}(N)\;.$$

In view of this and 2.3.3(a) there is for every $i \in \mathbb{Z}$ a homomorphism of $\Bbbk$-modules,

$$\mathrm{Z}_v(\mathrm{Hom}_R(M,N)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_i(M),\mathrm{H}_{i+v}(N)) \quad \text{given by} \quad \alpha \mapsto \mathrm{H}_i(\alpha)\;.$$

By 2.3.3(b) and 2.2.26 the submodule $\mathrm{B}_v(\mathrm{Hom}_R(M,N)) = \mathrm{B}_0(\mathrm{Hom}_R(M,\Sigma^{-v}N))$ is contained in the kernel of this homomorphism, and hence there are homomorphisms

$$\mathrm{H}_v(\mathrm{Hom}_R(M,N)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_i(M),\mathrm{H}_{i+v}(N)) \quad \text{given by} \quad [\alpha] \mapsto \mathrm{H}_i(\alpha)\;.$$

The assertion now follows from 1.1.19. □

REMARK. If R is semi-simple, then the morphism from 2.5.8 is an isomorphism; see E 2.5.3. In general, if all the modules M_v and $\mathrm{B}_v(M)$ are projective, then this morphism is surjective and its kernel can be explicitly described; this is the content of the Universal Coefficient Theorem. It is in some places called the Künneth Formula for Cohomology; see also the Remark after 2.5.14.

The map in 2.5.8 need neither be surjective nor injective.

2.5.9 Example. Let M be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ and N the complex $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$, concentrated in degrees 1 and 0. Evidently, $\mathrm{Hom}_{\mathbb{Z}}(M,N) = 0$ holds and, therefore, $\mathrm{H}_0(\mathrm{Hom}_{\mathbb{Z}}(M,N)) = 0$. At the same time, one has

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_0(M),\mathrm{H}_0(N)) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \;\cong\; \mathbb{Z}/2\mathbb{Z}\;,$$

so the map from 2.5.8 is not surjective in degree 0. With $N' = 0 \to \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$, concentrated in degrees 0 and -1, one has $\mathrm{Hom}_{\mathbb{Z}}(M,N') \cong \Sigma^{-1}(\mathbb{Z}/2\mathbb{Z})$ and, therefore, $\mathrm{H}_{-1}(\mathrm{Hom}_{\mathbb{Z}}(M,N')) \cong \mathbb{Z}/2\mathbb{Z}$ compared to

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_0(M),\mathrm{H}_{-1}(N')) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},0) \;=\; 0\;.$$

Thus the map from 2.5.8 is not injective in degree -1.

2.5.10 Lemma. *Let M and N be R-complexes with $M_v = 0$ for all $v < 0$ and $N_v = 0$ for all $v > 0$. The homomorphism of $\Bbbk$-modules,*

$$\mathrm{H}_0(\mathrm{Hom}_R(M,N)) \longrightarrow \mathrm{Hom}_R(\mathrm{H}_0(M),\mathrm{H}_0(N)) \quad \textit{given by} \quad [\alpha] \longmapsto \mathrm{H}_0(\alpha)$$

is an isomorphism.

PROOF. It follows from the assumptions on M and N that the map in question is the degree 0 component of the morphism from 2.5.8. It further follows that the only degree 1 homomorphism from M to N is the zero map. In particular, the only null-homotopic morphism $M \to N$ is the zero map. Hence one has $\mathrm{H}_0(\mathrm{Hom}_R(M,N)) = \mathcal{C}(R)(M,N)$ by 2.3.10. This $\Bbbk$-module may be identified with

$$\{\beta \in \mathrm{Hom}_R(M_0,N_0) \mid \beta\partial_1^M = 0 = \partial_0^N\beta\}\,,$$

again by the assumptions on M and N, which also yield $\mathrm{H}_0(M) = \mathrm{C}_0(M)$ and $\mathrm{H}_0(N) = \mathrm{Z}_0(N)$. Let $\pi_0\colon M_0 \twoheadrightarrow \mathrm{C}_0(M)$ be the quotient map and $\iota_0\colon \mathrm{Z}_0(N) \rightarrowtail N_0$ the embedding; the homomorphism in question is now identified with

$$(\star) \qquad \{\beta \in \mathrm{Hom}_R(M_0,N_0) \mid \beta\partial_1^M = 0 = \partial_0^N\beta\} \longrightarrow \mathrm{Hom}_R(\mathrm{C}_0(M),\mathrm{Z}_0(N))$$

which maps β to the unique homomorphism $\tilde{\beta}$ that makes the diagram

$$\begin{tikzcd} M_0 \arrow[r, "\beta"] \arrow[d, two heads, "\pi_0"'] & N_0 \\ \mathrm{C}_0(M) \arrow[r, "\tilde{\beta}"] & \mathrm{Z}_0(N) \arrow[u, tail, "\iota_0"'] \end{tikzcd}$$

commutative. It remains to note that $(\star)$ is an isomorphism; its inverse maps an element $\gamma \in \mathrm{Hom}_R(\mathrm{C}_0(M),\mathrm{Z}_0(N))$ to the composite $\iota_0\gamma\pi_0$. $\square$

2.5.11. Let M and N be R-complexes. Suppose there exist integers w and u such that one has $M_v = 0$ for all $v < u$ and $N_v = 0$ for all $v > w$. For each $v \in \mathbb{Z}$ the module $\mathrm{Hom}_R(M,N)_v$ is then a direct sum

$$\mathrm{Hom}_R(M,N)_v = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_i,N_{i+v}) = \bigoplus_{i=u}^{w-v} \mathrm{Hom}_R(M_i,N_{i+v})\,.$$

If one has $w - v < u$, then $\mathrm{Hom}_R(M,N)_v = 0$ holds.

2.5.12 Proposition. *Let M and N be R-complexes. If M is bounded below and N is bounded above, then the complex $\mathrm{Hom}_R(M,N)$ is bounded above. More precisely, if one has $M_v = 0$ for all $v < u$ and $N_v = 0$ for all $v > s$, then the next assertions hold.*

(a) $\mathrm{Hom}_R(M,N)_{-v} = 0$ *for* $v < u - s$.
(b) $\mathrm{Hom}_R(M,N)_{-(u-s)} = \mathrm{Hom}_R(M_u,N_s)$.
(c) $\mathrm{H}_{-(u-s)}(\mathrm{Hom}_R(M,N)) \cong \mathrm{Hom}_R(\mathrm{H}_u(M),\mathrm{H}_s(N))$.

PROOF. Parts (a) and (b) are immediate from 2.5.11.

(c): By 2.2.15, 2.3.14, and 2.3.16 there is an isomorphism

$$\mathrm{H}_{-(u-s)}(\mathrm{Hom}_R(M,N)) = \mathrm{H}_0(\Sigma^{u-s}\mathrm{Hom}_R(M,N)) \cong \mathrm{H}_0(\mathrm{Hom}_R(\Sigma^{-u}M,\Sigma^{-s}N))\,.$$

The complexes $\Sigma^{-u}M$ and $\Sigma^{-s}N$ are concentrated in non-negative and non-positive degrees, respectively, so by 2.5.10 there is an isomorphism

$$\begin{aligned}\mathrm{H}_0(\mathrm{Hom}_R(\Sigma^{-u}M, \Sigma^{-s}N)) &\cong \mathrm{Hom}_R(\mathrm{H}_0(\Sigma^{-u}M), \mathrm{H}_0(\Sigma^{-s}N))\\ &= \mathrm{Hom}_R(\mathrm{H}_u(M), \mathrm{H}_s(N))\,. \qquad \square\end{aligned}$$

2.5.13 Proposition. *Assume that S is right Noetherian. Let M be a bounded below R-complex and X a bounded above complex of R–S^{o}-bimodules. If M is degreewise finitely generated and X is degreewise finitely generated over S^{o}, then the S^{o}-complex $\mathrm{Hom}_R(M, X)$ is bounded above and degreewise finitely generated.*

Proof. For every $v \in \mathbb{Z}$ and $i \in \mathbb{Z}$ the S^{o}-module $\mathrm{Hom}_R(M_i, X_{i+v})$ is finitely generated; see 1.3.13. By assumption there exist integers w and u such that $M_v = 0$ for all $v < u$ and $X_v = 0$ for all $v > w$. It follows from 2.5.11 that the module $\mathrm{Hom}_R(M, X)_v$ is finitely generated for every v, and by 2.5.12 it is zero for $v > w - u$. $\square$

Tensor Product Complex

The tensor product does not commute with homology, but the so-called Künneth map compares the homology of a tensor product complex to the tensor product of the homology complexes.

2.5.14 Proposition. *Let M be an R^{o}-complex and N an R-complex. There is a morphism of $\Bbbk$-complexes,*

$$\mathrm{H}(M) \otimes_R \mathrm{H}(N) \longrightarrow \mathrm{H}(M \otimes_R N)\,,$$

with degree v component

$$\coprod_{i\in\mathbb{Z}}(\mathrm{H}_i(M) \otimes_R \mathrm{H}_{v-i}(N)) \longrightarrow \mathrm{H}_v(M \otimes_R N) \quad \textit{given by} \quad \varepsilon^i([m]\otimes[n]) \longmapsto [m\otimes n]\,.$$

Proof. From the definition of the differential on the complex $M \otimes_R N$, see 2.4.1, it follows for every $i \in \mathbb{Z}$ that the middle R-linear map $M_i \times N_{v-i} \to (M \otimes_R N)_v$ given by $(m, n) \mapsto m \otimes n$ restrict to maps

$$(\dagger)\quad \mathrm{Z}_i(M) \times \mathrm{Z}_{v-i}(N) \to \mathrm{Z}_v(M \otimes_R N) \;\text{ and }\; \mathrm{B}_i(M) \times \mathrm{B}_{v-i}(N) \to \mathrm{B}_v(M \otimes_R N)\,.$$

For the second map, note that for $m = \partial^M_{i+1}(m')$ in $\mathrm{B}_i(M)$ and $n = \partial^N_{v-i+1}(n')$ in $\mathrm{B}_{v-i}(N)$ one has

$$\begin{aligned}\partial^{M\otimes_R N}_{v+1}(m' \otimes n) &= \partial^M_{i+1}(m') \otimes n + (-1)^{i+1} m' \otimes \partial^N_{v-i}(n)\\ &= m \otimes n + (-1)^{i+1} m' \otimes \partial^N_{v-i}(\partial^N_{v-i+1}(n'))\\ &= m \otimes n\,,\end{aligned}$$

so $m \otimes n$ belongs to $\mathrm{B}_v(M \otimes_R N)$. For every $i \in \mathbb{Z}$ it follows from (†) that there is a well-defined middle R-linear map $\mathrm{H}_i(M) \times \mathrm{H}_{v-i}(N) \to \mathrm{H}_v(M \otimes_R N)$ given by $([m], [n]) \mapsto [m\otimes n]$, which by the universal property of the tensor product, 2.1.15, induces a homomorphism

$$\mathrm{H}_i(M) \otimes_R \mathrm{H}_{v-i}(N) \longrightarrow \mathrm{H}_v(M \otimes_R N) \quad \text{given by} \quad [m] \otimes [n] \longmapsto [m \otimes n]\ .$$

The assertion now follows from 1.1.20. □

Remark. If R is semi-simple, then the morphism from 2.5.14 is an isomorphism; see E 2.5.4. In general, if all the modules M_v and $\mathrm{B}_v(M)$ are flat, then this morphism is injective and its cokernel can be explicitly described; this is the content of the Künneth Formula, see e.g. Weibel [253, 3.6]. It is in some places called the Universal Coefficient Theorem for Homology; see also the Remark after 2.5.8.

The map in 2.5.14 need neither be surjective nor injective.

2.5.15 Example. Let M be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ and N the complex $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$, concentrated in degrees 1 and 0. One has $M \otimes_{\mathbb{Z}} N \cong \Sigma(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$ and, therefore, $\mathrm{H}_1(M \otimes_{\mathbb{Z}} N) \cong \mathbb{Z}/2\mathbb{Z}$. At the same time, one has

$$\mathrm{H}_0(M) \otimes_{\mathbb{Z}} \mathrm{H}_1(N) \;=\; \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 0 \;=\; 0\ ,$$

so the map from 2.5.14 is not surjective in degree 1. With $N' = 0 \to \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$, concentrated in degrees 0 and -1, one has $M \otimes_{\mathbb{Z}} N \cong 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{=} \mathbb{Z}/2\mathbb{Z} \to 0$ and, therefore, $\mathrm{H}_0(M \otimes_{\mathbb{Z}} N) = 0$ compared to

$$\mathrm{H}_0(M) \otimes_{\mathbb{Z}} \mathrm{H}_0(N) \;=\; \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 2\mathbb{Z} \;\cong\; \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \;\cong\; \mathbb{Z}/2\mathbb{Z}\ .$$

Thus the map from 2.5.14 is not injective in degree 0.

2.5.16 Lemma. *Let M be an R^{o}-complex and N an R-complex with $M_v = 0 = N_v$ for all $v < 0$. The homomorphism of $\Bbbk$-modules,*

$$\mathrm{H}_0(M) \otimes_R \mathrm{H}_0(N) \longrightarrow \mathrm{H}_0(M \otimes_R N) \quad \textit{given by} \quad [m] \otimes [n] \longmapsto [m \otimes n]\ ,$$

is an isomorphism.

Proof. It follows from the assumptions on M and N that the map in question is the degree 0 component of the morphism from 2.5.14; we proceed to establish an inverse. The assumptions further yield $\mathrm{H}_0(M) = \mathrm{C}_0(M)$, $\mathrm{H}_0(N) = \mathrm{C}_0(N)$, and $\mathrm{H}_0(M \otimes_R N) = \mathrm{C}_0(M \otimes_R N) = (M_0 \otimes_R N_0)/\mathrm{B}_0(M \otimes_R N)$; see 2.4.1. The map

$$(\flat) \quad M_0 \otimes_R N_0 \to \mathrm{C}_0(M) \otimes_R \mathrm{C}_0(N) \quad \text{given by} \quad m \otimes n \mapsto [m]_{\mathrm{B}_0(M)} \otimes [n]_{\mathrm{B}_0(N)}$$

is zero on $\mathrm{B}_0(M \otimes_R N)$. Indeed, one has $(M \otimes_R N)_1 = (M_1 \otimes_R N_0) \oplus (M_0 \otimes_R N_1)$ and for $m' \otimes n \in (M_1 \otimes_R N_0)$ and $m \otimes n' \in (M_0 \otimes_R N_1)$ there are equalities,

$$\partial_1^{M \otimes_R N}(m' \otimes n) \;=\; \partial_1^M(m') \otimes n - m' \otimes \partial_0^N(n) \;=\; \partial_1^M(m') \otimes n \quad \text{and}$$
$$\partial_1^{M \otimes_R N}(m \otimes n') \;=\; \partial_0^M(m) \otimes n' + m \otimes \partial_1^N(n') \;=\; m \otimes \partial_1^N(n')\ .$$

Both elements $\partial_1^M(m') \otimes n$ and $m \otimes \partial_1^N(n')$ are mapped to zero by ($\flat$). It follows that ($\flat$) induces a homomorphism $\mathrm{C}_0(M \otimes_R N) \to \mathrm{C}_0(M) \otimes_R \mathrm{C}_0(N)$, which is given by $[m \otimes n]_{\mathrm{B}_0(M \otimes_R N)} \mapsto [m]_{\mathrm{B}_0(M)} \otimes [n]_{\mathrm{B}_0(N)}$; this is the desired inverse. □

2.5.17. Let M be an R°-complex and N an R-complex. Suppose there exist integers u and w such that $M_v = 0$ for $v < u$ and $N_v = 0$ for $v < w$. For each $v \in \mathbb{Z}$ the module $(M \otimes_R N)_v$ is then a direct sum

$$(M \otimes_R N)_v \;=\; \coprod_{i\in\mathbb{Z}} M_i \otimes_R N_{v-i} \;=\; \bigoplus_{i=u}^{v-w} M_i \otimes_R N_{v-i}\;.$$

If one has $v - w < u$, then $(M \otimes_R N)_v = 0$ holds.

2.5.18 Proposition. *Let M be an R°-complex and N an R-complex. If M and N are bounded below, then the complex $M \otimes_R N$ is a bounded below. More precisely, if one has $M_v = 0$ for $v < u$ and $N_v = 0$ for $v < w$, then the next assertions hold.*

(a) $(M \otimes_R N)_v = 0$ *for* $v < u + w$.
(b) $(M \otimes_R N)_{u+w} = M_u \otimes_R N_w$.
(c) $\mathrm{H}_{u+w}(M \otimes_R N) \cong \mathrm{H}_u(M) \otimes_R \mathrm{H}_w(N)$.

Proof. Parts (a) and (b) are immediate from 2.5.17.
(c): By 2.2.15, 2.4.13, and 2.4.14 there is an isomorphism

$$\mathrm{H}_{u+w}(M \otimes_R N) \;=\; \mathrm{H}_0(\Sigma^{-u-w}(M \otimes_R N)) \;\cong\; \mathrm{H}_0((\Sigma^{-u}M) \otimes_R (\Sigma^{-w}N))\;.$$

The complexes $\Sigma^{-u}M$ and $\Sigma^{-w}N$ are concentrated in non-negative degrees, so by 2.5.16 there is an isomorphism

$$\begin{aligned}\mathrm{H}_0((\Sigma^{-u}M) \otimes_R (\Sigma^{-w}N)) \;&\cong\; \mathrm{H}_0(\Sigma^{-u}M) \otimes_R \mathrm{H}_0(\Sigma^{-w}N)\\ &=\; \mathrm{H}_u(M) \otimes_R \mathrm{H}_w(N)\;. \qquad \square\end{aligned}$$

2.5.19 Proposition. *Let N be a bounded below S-complex and X a bounded below complex of R–S°-bimodules. If N is degreewise finitely generated and X is degreewise finitely generated over R, then the R-complex $X \otimes_S N$ is bounded below and degreewise finitely generated.*

Proof. For every $v \in \mathbb{Z}$ and $i \in \mathbb{Z}$ the R-module $X_i \otimes_S N_{v-i}$ is finitely generated; see 1.3.14. By assumption there exist integers u and w such that $X_v = 0$ for $v < u$ and $N_v = 0$ for $v < w$. It follows from 2.5.17 that the module $(X \otimes_S N)_v$ is finitely generated for every v, and by 2.5.18 it is zero for $v < u + w$. □

Truncations

To handle unbounded complexes it is at times convenient to cut them into bounded pieces. The instruments for such procedures are known as truncations.

2.5.20 Definition. Let M be an R-complex and n an integer. The *hard truncation above* of M at n is the complex $M_{\leqslant n}$ defined by

$$(M_{\leqslant n})_v \;=\; \begin{cases} 0 & \text{for } \; v > n \\ M_v & \text{for } \; v \leqslant n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\leqslant n}} \;=\; \begin{cases} 0 & \text{for } \; v > n \\ \partial_v^M & \text{for } \; v \leqslant n\,. \end{cases}$$

It can be visualized as follows,

$$M_{\leqslant n} \;=\; 0 \longrightarrow M_n \xrightarrow{\partial_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} M_{n-2} \longrightarrow \cdots \,.$$

Similarly, the *hard truncation below* of M at n is the complex $M_{\geqslant n}$ defined by

$$(M_{\geqslant n})_v \;=\; \begin{cases} M_v & \text{for } v \geqslant n \\ 0 & \text{for } v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\geqslant n}} \;=\; \begin{cases} \partial_v^M & \text{for } v > n \\ 0 & \text{for } v \leqslant n\,. \end{cases}$$

It can be visualized as follows,

$$M_{\geqslant n} \;=\; \cdots \longrightarrow M_{n+2} \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \longrightarrow 0\,.$$

Let $\alpha\colon M \to N$ be a morphism of R-complexes and n an integer. The symbol $\alpha_{\leqslant n}$ denotes the induced morphism $M_{\leqslant n} \to N_{\leqslant n}$, and $\alpha_{\geqslant n}$ is defined similarly.

2.5.21. Let n be an integer. Hard truncation above $(-)_{\leqslant n}$ and hard truncation below $(-)_{\geqslant n}$ are evidently $\Bbbk$-linear $\natural$-functors from $\mathcal{C}(R)$ to $\mathcal{C}(R)$.

2.5.22. Let M be an R-complex and n an integer. The truncation $M_{\leqslant n}$ is a subcomplex of M, and the quotient complex $M/M_{\leqslant n}$ is the truncation $M_{\geqslant n+1}$. In particular, there is a degreewise split exact sequence of R-complexes,

$$0 \longrightarrow M_{\leqslant n} \longrightarrow M \longrightarrow M_{\geqslant n+1} \longrightarrow 0\,.$$

2.5.23 Definition. Let M be an R-complex and n an integer.

The *soft truncation above* of M at n is the R-complex $M_{\subseteq n}$ defined by

$$(M_{\subseteq n})_v \;=\; \begin{cases} 0 & \text{for } v > n \\ \mathrm{C}_n(M) & \text{for } v = n \\ M_v & \text{for } v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\subseteq n}} \;=\; \begin{cases} 0 & \text{for } v > n \\ \bar{\partial}_n^M & \text{for } v = n \\ \partial_v^M & \text{for } v < n \end{cases}$$

where $\bar{\partial}_n^M\colon \mathrm{C}_n(M) \to M_{n-1}$ is the homomorphism induced by ∂_n^M. The truncated complex can be visualized as follows,

$$M_{\subseteq n} \;=\; 0 \longrightarrow \mathrm{C}_n(M) \xrightarrow{\bar{\partial}_n^M} M_{n-1} \xrightarrow{\partial_{n-1}^M} M_{n-2} \longrightarrow \cdots \,.$$

The canonical map $M \twoheadrightarrow M_{\subseteq n}$ is denoted $\tau_{\subseteq n}^M$.

Similarly, the *soft truncation below* of M at n is the R-complex $M_{\supseteq n}$ defined by:

$$(M_{\supseteq n})_v \;=\; \begin{cases} M_v & \text{for } v > n \\ \mathrm{Z}_n(M) & \text{for } v = n \\ 0 & \text{for } v < n \end{cases} \qquad \text{and} \qquad \partial_v^{M_{\supseteq n}} \;=\; \begin{cases} \partial_v^M & \text{for } v > n \\ 0 & \text{for } v \leqslant n\,. \end{cases}$$

The truncated complex can be visualized as follows,

$$M_{\supseteq n} \;=\; \cdots \longrightarrow M_{n+2} \xrightarrow{\partial_{n+2}^M} M_{n+1} \xrightarrow{\partial_{n+1}^M} \mathrm{Z}_n(M) \longrightarrow 0\,.$$

The canonical map $M_{\supseteq n} \rightarrowtail M$ is denoted $\tau_{\supseteq n}^M$.

Let $\alpha\colon M \to N$ be a morphism of R-complexes and n an integer. The symbol $\alpha_{\subseteq n}$ denotes the induced morphism $M_{\subseteq n} \to N_{\subseteq n}$, and $\alpha_{\supseteq n}$ is defined similarly.

2.5.24 Proposition. *Let M be an R-complex and n an integer. Soft truncation above $(-)_{\subseteq n}$ is a $\Bbbk$-linear endofunctor on $\mathcal{C}(R)$, and the following assertions hold.*

(a) *The canonical map $\tau_{\subseteq n}^{M} \colon M \twoheadrightarrow M_{\subseteq n}$ from* 2.5.23 *is a surjective morphism of R-complexes, i.e. $M_{\subseteq n}$ is a quotient complex of M.*

(b) *For every $v \leqslant n$ the induced map $\mathrm{H}_v(\tau_{\subseteq n}^{M})$ is an isomorphism of R-modules.*

(c) *One has*

$$\operatorname{Ker}(\tau_{\subseteq n}^{M}) \;=\; \cdots \longrightarrow M_{n+2} \xrightarrow{\partial_{n+2}^{M}} M_{n+1} \xrightarrow{\partial_{n+1}^{M}} \mathrm{B}_n(M) \longrightarrow 0\,,$$

and if $\mathrm{H}_n(M) = 0$ *holds, then* $\operatorname{Ker}(\tau_{\subseteq n}^{M})$ *is the complex* $M_{\supseteq n}$.

Proof. It is immediate from the definition that $(-)_{\subseteq n}$ is a $\Bbbk$-linear functor.

(a): To see that the surjective map $\tau_{\subseteq n}^{M} \colon M \twoheadrightarrow M_{\subseteq n}$ is a morphism of R-complexes, notice that the composite $(\tau_{\subseteq n}^{M})_n \partial_{n+1}^{M}$ is zero and recall that the definition of $\bar{\partial}_n^{M}$ is $\bar{\partial}_n^{M}([x]_{\mathrm{B}_n(M)}) = \partial_n^{M}(x)$ for $x \in M_n$.

(b): One has $\mathrm{H}_n(M_{\subseteq n}) = \operatorname{Ker} \bar{\partial}_n^{M} = \mathrm{H}_n(M)$ and $\mathrm{H}_v(\tau_{\subseteq n}^{M}) = \mathrm{H}_v(1^{M})$ for $v < n$.

(c): It is evident from the definition that the kernel of $\tau_{\subseteq n}^{M}$ has the asserted form. Finally, $\mathrm{H}_n(M) = 0$ means $\mathrm{B}_n(M) = \mathrm{Z}_n(M)$, so the kernel complex is $M_{\supseteq n}$. □

2.5.25 Proposition. *Let M be an R-complex and n an integer. Soft truncation below $(-)_{\supseteq n}$ is a $\Bbbk$-linear endofunctor on $\mathcal{C}(R)$, and the following assertions hold.*

(a) *The canonical map $\tau_{\supseteq n}^{M} \colon M_{\supseteq n} \rightarrowtail M$ from* 2.5.23 *is an injective morphism of R-complexes, i.e. $M_{\supseteq n}$ is a subcomplex of M.*

(b) *For every $v \geqslant n$ the induced map $\mathrm{H}_v(\tau_{\supseteq n}^{M})$ is an isomorphism of R-modules.*

(c) *There is an isomorphism,*

$$\operatorname{Coker}(\tau_{\supseteq n}^{M}) \;\cong\; 0 \longrightarrow \mathrm{B}_{n-1}(M) \longrightarrow M_{n-1} \xrightarrow{\partial_{n-1}^{M}} M_{n-2} \longrightarrow \cdots\,,$$

and if $\mathrm{H}_n(M) = 0$ *holds, then* $\operatorname{Coker}(\tau_{\supseteq n}^{M})$ *is isomorphic to the complex* $M_{\subseteq n}$.

Proof. It is immediate from the definition that $(-)_{\supseteq n}$ is a $\Bbbk$-linear functor.

(a): To see that the injective map $\tau_{\supseteq n}^{M} \colon M_{\supseteq n} \rightarrowtail M$ is a morphism of R-complexes, notice that the composite $\partial_n^{M}(\tau_{\supseteq n}^{M})_n$ is zero.

(b): Evidently one has $\mathrm{H}_n(M_{\supseteq n}) = \mathrm{H}_n(M)$ and $\mathrm{H}_v(\tau_{\supseteq n}^{M}) = \mathrm{H}_v(1^{M})$ for $v > n$.

(c): It follows from 2.2.12(a) that the cokernel of $\tau_{\supseteq n}^{M}$ has the asserted form. Finally, if $\mathrm{H}_n(M) = 0$ holds, then one has $\mathrm{B}_{n-1}(M) \cong \mathrm{C}_n(M)$ by 2.2.12(d). □

Graded-Free Modules

Free modules play a pivotal role in the development of homological algebra in module categories. Complexes of free modules play a similar central role in the homological theory of complexes. For now we contend ourselves with realizing every complex M as a homomorphic image of a complex L of free modules. Later, in Chap. 5, we refine this construction to ensure that L is homologically indistinguishable from M.

2.5.26 Definition. Let L be a graded R-module. A set $E = \{e_u\}_{u\in U}$ of generators for L, a basis for L in particular, is called *graded* if each element e_u is homogeneous. The module L is called *graded-free* if has a graded basis.

For a graded set E, not a priori assumed to be a subset of a module, the graded-free R-module with graded basis E is denoted $R\langle E\rangle$.

2.5.27 Proposition. *A graded R-module L is graded-free if and only if the R-module L_v is free for every $v \in \mathbb{Z}$.*

Proof. If each module L_v is free with basis E_v, then $E = \bigcup_{v\in\mathbb{Z}} E_v$ is a graded basis for L. For the converse, let E be a graded basis for L and fix v. Every element in L_v is a unique linear combination of elements in E. Only elements of degree v occur with non-zero coefficients, so the elements of degree v in E form a basis for L_v. $\square$

2.5.28. Let M be a graded R-module. In view of 2.5.27 it follows from 1.3.12 that there is a surjective morphism $L \to M$ of graded R-modules with L graded-free. If M is degreewise finitely generated, then L can be chosen degreewise finitely generated.

To realize a complex as the image of a complex of free modules, one has to take the differentials into account. To this end, the next construction is key.

2.5.29 Construction. For an R-module F and an integer v, let $\mathrm{D}^v(F)$ denote the *disk complex* $0 \longrightarrow F \xrightarrow{=} F \longrightarrow 0$ concentrated in degrees v and $v-1$.

Let M be an R-complex. For every homomorphism $\varphi\colon F \to M_v$ of R-modules, there is a morphism of R-complexes, $\mathrm{D}^v(F) \to M$, given by the diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & F \arrow[r, "1^F"] \arrow[d, "\varphi"] & F \arrow[r] \arrow[d, "\partial_v^M \varphi"] & 0 & \\
\cdots \arrow[r] & M_{v+1} \arrow[r, "\partial_{v+1}^M"] & M_v \arrow[r, "\partial_v^M"] & M_{v-1} \arrow[r, "\partial_{v-1}^M"] & M_{v-2} \arrow[r] & \cdots\,.
\end{tikzcd}$$

Choose for every $v \in \mathbb{Z}$ a surjective homomorphism $\varphi^v\colon F^v \to M_v$ with F^v free, and do it such that F^v has a basis with $n \geqslant 0$ elements if M_v is generated by n elements; cf. 1.3.12. Consider the complex L given by

$$L_v = \begin{matrix} F^v \\ \oplus \\ F^{v+1} \end{matrix} \qquad \text{and} \qquad \partial_v^L = \begin{pmatrix} 0 & 0 \\ 1^{F^v} & 0 \end{pmatrix},$$

and the map $\pi\colon L \to M$ given by $\pi_v = (\varphi^v \;\; \partial_{v+1}^M \varphi^{v+1})$; it is straightforward to verify that π is a morphism of complexes.

Remark. Presumably, disk complexes are called so because they are basic examples of contractible complexes, see 4.3.23, just as topological disks are elementary examples of contractible spaces.

2.5.30 Lemma. *Let M be an R-complex. The complex L and morphism $\pi\colon L \to M$ constructed in* 2.5.29 *have the following properties.*

(a) *The morphism π is surjective.*

(b) *The complex L is acyclic, and the graded module $L^\natural$ is graded-free.*
(c) *If M is degreewise finitely generated, then L is degreewise finitely generated.*
(d) *If M is bounded (above/below), then L is bounded (above/below).*

PROOF. As the morphisms φ^v are surjective, so is π; this proves part (a).

It is evident from the definition of ∂^L that L is acyclic. Each module L_v is a direct sum of free modules, F^v and F^{v+1}, and hence free. Thus $L^\natural$ is graded-free by 2.5.27; this proves (b). Moreover, if M_v is finitely generated (zero), then F^v is finitely generated (zero), and parts (c) and (d) follow. □

Notice that in each degree v the exact sequence of R-complexes in part (*iii*) below affords a free presentation of the module M_v.

2.5.31 Proposition. *For an R-complex M, the following conditions are equivalent.*
(*i*) *M is degreewise finitely presented.*
(*ii*) *There exists an R-complex L of finitely generated free modules and a surjective morphism $L \to M$ whose kernel is degreewise finitely generated.*
(*iii*) *There exist R-complexes L and L' of finitely generated free modules and an exact sequence of R-complexes $L' \to L \to M \to 0$.*

Moreover, if M is bounded (above/below) and degreewise finitely presented, then the complexes L and L' in (iii) can be chosen to be bounded (above/below).

PROOF. The implication (*iii*) $\Rightarrow$ (*i*) is trivial.

(*i*) $\Rightarrow$ (*ii*): By 2.5.30 there exists a surjective morphism $\pi\colon L \to M$ where L is a complex of finitely generated free R-modules and, moreover, if M is bounded (above/below), then so is L. Since each M_v is finitely presented, it follows from 1.3.40 that $\operatorname{Ker}\pi$ is degreewise finitely generated.

(*ii*) $\Rightarrow$ (*iii*): By assumption there exists a short exact sequence of R-complexes $0 \to K \to L \to M \to 0$ where L is a complex of finitely generated free modules, and K is degreewise finitely generated. By 2.5.30 there is a surjective morphism $L' \to K$ where L' is a complex of finitely generated free modules. Moreover, if M and hence L and K are bounded (above/below), then so is L'. The composite $L' \twoheadrightarrow K \rightarrowtail L$ now yields the left-hand morphism in an exact sequence $L' \to L \to M \to 0$. □

EXERCISES

E 2.5.1 Let M be an R-complex. Show that if $M^\natural$ is finitely generated in $\mathcal{M}_{\mathrm{gr}}(R)$, then M is degreewise finitely generated. Show that the converse is not true.

E 2.5.2 Show that the isomorphism classes of degreewise finitely presented complexes form a set.

E 2.5.3 Assume that R is semi-simple. Show that the morphism in 2.5.8 is an isomorphism for all R-complexes.

E 2.5.4 Assume that R is semi-simple. Show that the morphism in 2.5.14 is an isomorphism for all R-complexes.

E 2.5.5 Let M be a bounded above R-complex and N a bounded below R-complex. Establish a result about $\operatorname{Hom}_R(M, N)$ akin to 2.5.12(a,b); how about part (c)?

E 2.5.6 Let M be a bounded above R°-complex and N a bounded above R-complex. Establish a result about $M \otimes_R N$ akin to 2.5.18(a,b); how about part (c)?

E 2.5.7 (Cf. 2.5.22) Let M be an R-complex and n an integer. Show that the canonical sequence $0 \to M_{\leqslant n} \to M \to M_{\geqslant n+1} \to 0$ is degreewise split exact and decide if it is split exact.

E 2.5.8 For $n \in \mathbb{Z}$ define full subcategories of $\mathcal{C}(R)$ by specifying their objects as follows,

$$\mathcal{C}_{\leqslant n}(R) = \{M \in \mathcal{C}(R) \mid M_v = 0 \text{ for all } v > n\} \quad \text{and}$$
$$\mathcal{C}_{\geqslant n}(R) = \{M \in \mathcal{C}(R) \mid M_v = 0 \text{ for all } v < n\}\ .$$

Show that the functors $(-)_{\leqslant n}$ and $(-)_{\geqslant n}$ are right and left adjoints for the inclusion functors $\mathcal{C}_{\leqslant n}(R) \to \mathcal{C}(R)$ and $\mathcal{C}_{\geqslant n}(R) \to \mathcal{C}(R)$, respectively.

E 2.5.9 Show that soft truncation above and below are right and left exact, respectively, endofunctors on $\mathcal{C}(R)$.

E 2.5.10 Show that soft truncations are not $\natural$-functors.

E 2.5.11 Show that soft truncations commute with homology.

E 2.5.12 Let M and N be R-complexes, the former concentrated in non-negative degrees. Show that for integers $m \geqslant n$ one has

$$\operatorname{Hom}_R(M, N)_{\geqslant m} = \operatorname{Hom}_R(M, N_{\geqslant n})_{\geqslant m} \quad \text{and}$$
$$\operatorname{Hom}_R(N, M)_{\leqslant n} = \operatorname{Hom}_R(N_{\geqslant -m}, M)_{\leqslant n}\ .$$

E 2.5.13 Let M and N be R-complexes, the latter concentrated in non-positive degrees. Show that for integers $m \geqslant n$ one has

$$\operatorname{Hom}_R(M, N)_{\geqslant m} = \operatorname{Hom}_R(M_{\leqslant -n}, N)_{\geqslant m} \quad \text{and}$$
$$\operatorname{Hom}_R(N, M)_{\leqslant n} = \operatorname{Hom}_R(N, M_{\leqslant m})_{\leqslant n}\ .$$

E 2.5.14 Let M and N be R-complexes, the former concentrated in non-negative degrees. Show that for integers $m \geqslant n$ one has $(M \otimes_R N)_{\leqslant n} = (M \otimes_R N_{\leqslant m})_{\leqslant n}$.

E 2.5.15 Let M and N be R-complexes, the latter concentrated in non-positive degrees. Show that for integers $m \geqslant n$ one has $(M \otimes_R N)_{\geqslant m} = (M_{\geqslant n} \otimes_R N)_{\geqslant m}$.

E 2.5.16 Let M be an R-complex. Show that for every integer n with $\mathrm{H}_n(M) = 0$ there is an exact sequence $0 \to M_{\supseteq n} \to M \to M_{\subseteq n} \to 0$.

E 2.5.17 Give an example of a graded R-module that is free but not graded-free.

E 2.5.18 Let L be a graded R-module and $E = \{e_u\}_{u \in U}$ a subset of L consisting of homogeneous elements. Show that L is graded-free with graded basis E if and only if it has the following graded unique extension property: Given a graded R-module M and a graded map $\alpha\colon E \to M$ there exists a unique graded homomorphism $\widetilde{\alpha}\colon L \to M$ with $\widetilde{\alpha}|_E = \alpha$.

Chapter 3
Categorical Constructions

Products and coproducts are categorical devices defined by universal properties. In the first section of this chapter, it is established that the category of R-complexes has products and coproducts. As this category is Abelian, it then follows from general principles that it has limits and colimits. In the remaining sections we give a detailed treatment of (co)limits over preordered sets, which is sufficient for our purposes.

3.1 Products and Coproducts

Synopsis. (Co)product; universal property; functor that preserves (co)products; direct sum.

In the category of complexes, like in the category of modules, products and coproducts are palpable objects. After describing their construction, we consider their interactions with the functors from Chap. 2. Homology and shift preserve products and coproducts alike. The Hom functor preserves products, while the tensor product functor preserves coproducts. Under assumptions of boundedness and finiteness, Hom also preserves coproducts and the tensor product preserves products.

Coproducts

3.1.1 Construction. Let $\{M^u\}_{u\in U}$ be a family of R-complexes. One defines an R-complex $\coprod_{u\in U} M^u$ by setting

$$\Bigl(\coprod_{u\in U} M^u\Bigr)_v = \coprod_{u\in U} M^u_v \qquad \text{and} \qquad \partial_v^{\coprod_{u\in U} M^u} = \coprod_{u\in U} \partial_v^{M^u}\,,$$

where the right-hand side of either equality is given by the coproduct in $\mathcal{M}(R)$. For each $u \in U$ the injections $\varepsilon^u_v : M^u_v \rightarrowtail \coprod_{u\in U} M^u_v$ in $\mathcal{M}(R)$ yield an injective morphism

$$\varepsilon^u : M^u \rightarrowtail \coprod_{u\in U} M^u \tag{3.1.1.1}$$

of R-complexes given by $\varepsilon^u(m^u_v) = \varepsilon^u_v(m^u_v)$ on homogeneous elements $m^u_v \in M^u_v$.

It is straightforward to verify that every element in M^u has the form $\sum_{u\in U} \varepsilon^u(m^u)$ for a unique family of elements $m^u \in M^u$ with $m^u = 0$ for all but finitely many $u \in U$.

The next theorem confirms that the complex constructed above indeed has the universal property of a coproduct in the category of complexes. Notice from the statement that for a morphism $\coprod_{u\in U} M^u \to N$ the image of an element $\sum_{u\in U} \varepsilon^u(m^u)$ in the coproduct is fully determined by the images of the components $\varepsilon^u(m^u)$. Hereafter we mostly describe morphisms out of coproducts in that way, see e.g. 3.1.8.

3.1.2 Theorem. *For a family $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$, the complex $\coprod_{u\in U} M^u$ together with the morphisms $\{\varepsilon^u\}_{u\in U}$, constructed in* 3.1.1*, is the coproduct of $\{M^u\}_{u\in U}$.*

For every family $\{\alpha^u : M^u \to N\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$, the unique morphism α that makes the diagram

$$\begin{tikzcd} M^u \arrow[r, tail, "\varepsilon^u"] \arrow[d, "\alpha^u"'] & \coprod_{u\in U} M^u \arrow[dl, dotted, "\alpha"] \\ N & \end{tikzcd}$$

commutative for every $u \in U$ is given by $\sum_{u\in U} \varepsilon^u(m^u) \mapsto \sum_{u\in U} \alpha^u(m^u)$.

Proof. The assignment $\sum_{u\in U} \varepsilon^u(m^u) \mapsto \sum_{u\in U} \alpha^u(m^u)$ yields by 3.1.1 a well-defined map $\alpha\colon \coprod_{u\in U} M^u \to N$, and it is straightforward to verify that it is a morphism of R-complexes. The equality $\alpha^u = \alpha\varepsilon^u$ holds for all $u \in U$ by definition of α. Any morphism $\alpha'\colon \coprod_{u\in U} M^u \to N$ with $\alpha^u = \alpha'\varepsilon^u$ for all $u \in U$ satisfies $\alpha'(\sum_{u\in U} \varepsilon^u(m^u)) = \sum_{u\in U} \alpha'\varepsilon^u(m^u) = \sum_{u\in U} \alpha^u(m^u)$, and hence $\alpha' = \alpha$. □

3.1.3. It follows from 3.1.1 and 3.1.2 that the full subcategory $\mathcal{M}_{\mathrm{gr}}(R)$ of $\mathcal{C}(R)$ is closed under coproducts in $\mathcal{C}(R)$. In particular, the category $\mathcal{M}_{\mathrm{gr}}(R)$ has coproducts.

3.1.4 Definition. Let M be an R-complex and $\{M^u\}_{u\in U}$ a family of subcomplexes of M. The *sum* of the family, written $\sum_{u\in U} M^u$, is the image of the morphism,

$$\coprod_{u\in U} M^u \longrightarrow M \qquad \text{given by} \qquad \sum_{u\in U} \varepsilon^u(m^u) \longmapsto \sum_{u\in U} m^u \,;$$

if it is injective, then the family $\{M^u\}_{u\in U}$ is called *independent*.

Below, and throughout this chapter, we use the same symbol, ε^u, for the injections in all coproducts.

3.1.5. As is the case for the coproduct in any category, the coproduct in $\mathcal{C}(R)$ acts on morphisms: If $\{\alpha^u : M^u \to N^u\}_{u\in U}$ is a family of morphisms of complexes, then $\coprod_{u\in U} \alpha^u : \coprod_{u\in U} M^u \to \coprod_{u\in U} N^u$ is the unique morphism given by $\varepsilon^u(m^u) \mapsto \varepsilon^u\alpha^u(m^u)$ for every $u \in U$ and $m^u \in M^u$; see 3.1.2.

3.1.6 Proposition. *Let $\{\alpha^u : M^u \to N^u\}_{u\in U}$ and $\{\beta^u : N^u \to X^u\}_{u\in U}$ be families of morphisms of R-complexes. The sequence*

$$\coprod_{u\in U} M^u \xrightarrow{\coprod_{u\in U}\alpha^u} \coprod_{u\in U} N^u \xrightarrow{\coprod_{u\in U}\beta^u} \coprod_{u\in U} X^u$$

is exact if and only if the sequence $M^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} X^u$ *is exact for every* $u \in U$.

Proof. The assertion is immediate from the definitions of the involved complexes and morphisms; see 3.1.1 and 3.1.5. □

3.1.7 Proposition. *Let* $\{\alpha^u : M^u \to N\}_{u\in U}$ *be a family of morphisms in* $\mathcal{C}(R)$. *If each* α^u *is null-homotopic, then so is the canonical morphism* $\alpha : \coprod_{u\in U} M^u \to N$.

Furthermore, a coproduct of null-homotopic morphisms is null-homotopic.

Proof. By assumption there are degree 1 homomorphisms $\tau^u : M^u \to N$ such that $\alpha^u = \partial^N\tau^u + \tau^u\partial^{M^u}$ holds for every $u \in U$. Consider each homomorphism τ^u as a morphism $M^{u\natural} \to \Sigma^{-1}N^\natural$ of graded R-modules; cf. 2.2.5. Set $M = \coprod_{u\in U} M^u$; as $M^\natural$ together with the injections $\{\varepsilon^u : M^{u\natural} \rightarrowtail M^\natural\}_{u\in U}$ is the coproduct of $\{M^{u\natural}\}_{u\in U}$ in $\mathcal{M}_{\mathrm{gr}}(R)$, see 3.1.3, there is a morphism $\tau : M^\natural \to \Sigma^{-1}N^\natural$ with $\tau\varepsilon^u = \tau^u$ for all $u \in U$. Viewing τ as a degree 1 homomorphism $M \to N$, one has

$$\alpha\varepsilon^u = \alpha^u = \partial^N\tau^u + \tau^u\partial^{M^u} = \partial^N\tau\varepsilon^u + \tau\varepsilon^u\partial^{M^u} = (\partial^N\tau + \tau\partial^M)\varepsilon^u$$

for every $u \in U$; here the third equality holds by definition of τ, and the last equality holds as ε^u is a morphism. Since $\partial^N\tau + \tau\partial^M$ is a morphism of R-complexes, the universal property of coproducts yields $\alpha = \partial^N\tau + \tau\partial^M$. That is, α is null-homotopic.

If $\{\alpha^u : M^u \to N^u\}_{u\in U}$ is a family of null-homotopic morphisms, then the composites $M^u \to N^u \rightarrowtail \coprod_{u\in U} N^u$ are null-homotopic by 2.2.25. Thus, the canonical morphism $\coprod_{u\in U}\alpha^u : \coprod_{u\in U} M^u \to \coprod_{u\in U} N^u$ is null-homotopic; cf. 3.1.5. □

Functors that Preserve Coproducts

3.1.8. Let $\{M^u\}_{u\in U}$ be a family of R-complexes and $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(S)$ a functor. The universal property of the coproduct yields is a canonical morphism in $\mathcal{C}(S)$,

(3.1.8.1) $$\coprod_{u\in U} \mathrm{F}(M^u) \longrightarrow \mathrm{F}\Big(\coprod_{u\in U} M^u\Big) \quad \text{given by} \quad \varepsilon^u(x^u) \longmapsto \mathrm{F}(\varepsilon^u)(x^u)$$

for $u \in U$ and $x^u \in \mathrm{F}(M^u)$; see 3.1.2. Recall that F is said to preserve coproducts if this map is an isomorphism for every family $\{M^u\}_{u\in U}$. In this case, F also preserves coproducts of morphisms; cf. 3.1.5. Further, given a natural transformation $\tau : \mathrm{E} \to \mathrm{F}$ of functors $\mathcal{C}(R) \to \mathcal{C}(S)$, it is straightforward to verify that there is a commutative diagram in $\mathcal{C}(S)$,

(3.1.8.2)
$$\begin{tikzcd} \coprod_{u\in U} \mathrm{E}(M^u) \arrow[r, "\coprod_{u\in U}\tau^{M^u}"] \arrow[d] & \coprod_{u\in U} \mathrm{F}(M^u) \arrow[d] \\ \mathrm{E}\Big(\coprod_{u\in U} M^u\Big) \arrow[r, "\tau^{\coprod_{u\in U} M^u}"] & \mathrm{F}\Big(\coprod_{u\in U} M^u\Big), \end{tikzcd}$$

where the vertical maps are the canonical morphisms.

The next result shows that the shift functor from 2.2.1 preserves coproducts.

3.1.9 Proposition. *Let s be an integer and $\{M^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(R)$,*

$$\coprod_{u\in U} \Sigma^s M^u \longrightarrow \Sigma^s \coprod_{u\in U} M^u \,, \tag{3.1.9.1}$$

given by $\varepsilon^u(x^u) \mapsto (\Sigma^s\varepsilon^u)(x^u)$ for $u \in U$ and $x^u \in \Sigma^s M^u$, is an isomorphism.

Proof. The morphism (3.1.9.1) is the identity map. □

The next result shows that the cycle, boundary, cokernel, and homology functors from 2.2.7 preserve coproducts.

3.1.10 Proposition. *For every family $\{M^u\}_{u\in U}$ of R-complexes, the following four canonical morphisms in $\mathcal{C}(R)$ are isomorphisms*:

(a) $\displaystyle\coprod_{u\in U} \mathrm{Z}(M^u) \longrightarrow \mathrm{Z}\Big(\coprod_{u\in U} M^u\Big)$ *given by* $\varepsilon^u(z^u) \longmapsto \mathrm{Z}(\varepsilon^u)(z^u)$.

(b) $\displaystyle\coprod_{u\in U} \mathrm{B}(M^u) \longrightarrow \mathrm{B}\Big(\coprod_{u\in U} M^u\Big)$ *given by* $\varepsilon^u(b^u) \longmapsto \mathrm{B}(\varepsilon^u)(b^u)$.

(c) $\displaystyle\coprod_{u\in U} \mathrm{C}(M^u) \longrightarrow \mathrm{C}\Big(\coprod_{u\in U} M^u\Big)$ *given by* $\varepsilon^u(c^u) \longmapsto \mathrm{C}(\varepsilon^u)(c^u)$.

(d) $\displaystyle\coprod_{u\in U} \mathrm{H}(M^u) \longrightarrow \mathrm{H}\Big(\coprod_{u\in U} M^u\Big)$ *given by* $\varepsilon^u(h^u) \longmapsto \mathrm{H}(\varepsilon^u)(h^u)$.

Proof. The morphisms (a) and (b) are the identity maps. The short exact sequences $0 \to \mathrm{B}(M^u) \to M^u \to \mathrm{C}(M^u) \to 0$, see 2.2.12(b), yield a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \coprod_{u\in U} \mathrm{B}(M^u) \arrow[r] \arrow[d] & \coprod_{u\in U} M^u \arrow[r] \arrow[d, equal] & \coprod_{u\in U} \mathrm{C}(M^u) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{B}\big(\coprod_{u\in U} M^u\big) \arrow[r] & \coprod_{u\in U} M^u \arrow[r] & \mathrm{C}\big(\coprod_{u\in U} M^u\big) \arrow[r] & 0 \,.
\end{tikzcd}$$

In this diagram, the rows are exact by 3.1.6 and 2.2.12(b). The leftmost vertical map is (b), which is an isomorphism, and the rightmost vertical map (c). It now follows from the Five Lemma 2.1.41 that (c) is an isomorphism. A similar argument, using the exact sequences $0 \to \mathrm{B}(M^u) \to \mathrm{Z}(M^u) \to \mathrm{H}(M^u) \to 0$ from 2.2.12(c), shows that (d) is an isomorphism. □

3.1.11 Corollary. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes. There are equalities,*

$$\sup\Big(\coprod_{u\in U} M^u\Big) = \sup_{u\in U}\{\sup M^u\} \quad \textit{and} \quad \inf\Big(\coprod_{u\in U} M^u\Big) = \inf_{u\in U}\{\inf M^u\}\,;$$

in particular, $\coprod_{u\in U} M^u$ is acyclic if and only if each complex M^u is acyclic.

Proof. The isomorphism $\mathrm{H}(\coprod_{u\in U} M^u) \cong \coprod_{u\in U} \mathrm{H}(M^u)$ from 3.1.10(d) yields both equalities. $\square$

The next results show that the tensor product functor 2.4.9 preserves coproducts.

3.1.12 Proposition. *Let N be an R-complex and $\{M^u\}_{u\in U}$ a family of R^{o}-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$\coprod_{u\in U}(M^u \otimes_R N) \longrightarrow \Big(\coprod_{u\in U} M^u\Big) \otimes_R N\,, \tag{3.1.12.1}$$

given by $\varepsilon^u(t^u) \mapsto (\varepsilon^u \otimes_R N)(t^u)$ for $u \in U$ and $t^u \in M^u \otimes_R N$, is an isomorphism.

Proof. To show that (3.1.12.1) is an isomorphism, it suffices to construct an inverse at the level of graded modules. The map $\biguplus_{i\in\mathbb{Z}}(\coprod_{u\in U} M^u)_i \times N_{v-i} \to \coprod_{u\in U}(M^u \otimes_R N)_v$ defined by $(\varepsilon^u(m^u), n) \mapsto \varepsilon^u(m^u \otimes n)$ is $\Bbbk$-bilinear and middle R-linear. By 2.1.15 there is a unique morphism $((\coprod_{u\in U} M^u) \otimes_R N)^\natural \to \coprod_{u\in U}(M^u \otimes_R N)^\natural$ of graded $\Bbbk$-modules that maps the element $\varepsilon^u(m^u) \otimes n$ to $\varepsilon^u(m^u \otimes n)$. This gives the desired inverse of the canonical morphism. $\square$

3.1.13 Proposition. *Let M be an R^{o}-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$\coprod_{u\in U}(M \otimes_R N^u) \longrightarrow M \otimes_R \coprod_{u\in U} N^u\,, \tag{3.1.13.1}$$

given by $\varepsilon^u(t^u) \mapsto (M \otimes_R \varepsilon^u)(t^u)$ for $u \in U$ and $t^u \in M \otimes_R N^u$, is an isomorphism.

Proof. The assertion follows from an argument similar to the proof of 3.1.12. $\square$

Remark. For an R^{o}-complex M, the functor $M \otimes_R -\colon \mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ has a right adjoint, namely the functor $\mathrm{Hom}_\Bbbk(M,-)$; see 4.5.14. From this fact alone, it follows that $M \otimes_R -$ preserves coproducts; see E 3.1.6.

Products

3.1.14 Construction. Let $\{N^u\}_{u\in U}$ be a family of R-complexes. One defines an R-complex $\prod_{u\in U} N^u$ by setting

$$\Big(\prod_{u\in U} N^u\Big)_v = \prod_{u\in U} N^u_v \qquad \text{and} \qquad \partial_v^{\prod_{u\in U} N^u} = \prod_{u\in U} \partial_v^{N^u}\,,$$

where the right-hand side of either equality is given by the product in $\mathcal{M}(R)$. For every $u \in U$ the projections $\varpi^u_v\colon \prod_{u\in U} N^u_v \twoheadrightarrow N^u_v$ in $\mathcal{M}(R)$ yield a surjective morphism

$$\varpi^u\colon \prod_{u\in U} N^u \twoheadrightarrow N^u \tag{3.1.14.1}$$

of R-complexes given by $\varpi^u((n^u_v)_{u\in U}) = \varpi^u_v((n^u_v)_{u\in U}) = n^u_v$ on homogeneous elements $(n^u_v)_{u\in U} \in (\prod_{u\in U} N^u)_v = \prod_{u\in U} N^u_v$.

It is straightforward to verify that every element $n \in \prod_{u\in U} N^u$ has the form $n = (n^u)_{u\in U} = (\varpi^u(n))_{u\in U}$, where $n^u = \varpi^u(n)$ belongs to N^u.

The next theorem confirms that the complex constructed above indeed has the universal property of a product in the category of complexes.

3.1.15 Theorem. *For a family $\{N^u\}_{u\in U}$ in $\mathcal{C}(R)$, the complex $\prod_{u\in U} N^u$ together with the morphisms $\{\varpi^u\}_{u\in U}$, constructed in 3.1.14, is the product of $\{N^u\}_{u\in U}$.*

For every family $\{\alpha^u : M \to N^u\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$, the unique morphism α that makes the diagram

$$\begin{tikzcd} & \prod_{u\in U} N^u \arrow[d, two heads, "\varpi^u"] \\ M \arrow[ur, dotted, "\alpha"] \arrow[r, "\alpha^u"'] & N^u \end{tikzcd}$$

commutative for every $u \in U$ is given by $m \mapsto (\alpha^u(m))_{u\in U}$.

Proof. The assignment $m \mapsto (\alpha^u(m))_{u\in U}$ yields a map $\alpha : M \to \prod_{u\in U} N^u$, see 3.1.14, and it is straightforward to verify that it is a morphism of R-complexes. The equality $\alpha^u = \varpi^u\alpha$ holds for all $u \in U$ by definition of α. Any morphism $\alpha' : M \to \prod_{u\in U} N^u$ with $\alpha^u = \varpi^u\alpha'$ for all $u \in U$ satisfies $\alpha'(m) = (\varpi^u\alpha'(m))_{u\in U} = (\alpha^u(m))_{u\in U}$, and hence $\alpha' = \alpha$. □

3.1.16. It follows from 3.1.14 and 3.1.15 that the full subcategory $\mathcal{M}_{\mathrm{gr}}(R)$ of $\mathcal{C}(R)$ is closed under products in $\mathcal{C}(R)$. In particular, the category $\mathcal{M}_{\mathrm{gr}}(R)$ has products.

3.1.17. As is the case for the product in any category, the product in $\mathcal{C}(R)$ also acts on morphisms: If $\{\alpha^u : M^u \to N^u\}_{u\in U}$ is a family of morphisms of R-complexes, then $\prod_{u\in U} \alpha^u : \prod_{u\in U} M^u \to \prod_{u\in U} N^u$ is the unique morphism given by $(m^u)_{u\in U} \mapsto (\alpha^u(m^u))_{u\in U}$ for every $(m^u)_{u\in U} \in \prod_{u\in U} M^u$; see 3.1.15.

3.1.18 Proposition. *Let $\{\alpha^u : M^u \to N^u\}_{u\in U}$ and $\{\beta^u : N^u \to X^u\}_{u\in U}$ be families of morphisms of R-complexes. The sequence*

$$\prod_{u\in U} M^u \xrightarrow{\prod_{u\in U}\alpha^u} \prod_{u\in U} N^u \xrightarrow{\prod_{u\in U}\beta^u} \prod_{u\in U} X^u$$

is exact if and only if the sequence $M^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} X^u$ is exact for every $u \in U$.

Proof. The assertion is immediate from the definitions of the involved complexes and morphisms; see 3.1.14 and 3.1.17. □

3.1.19 Proposition. *Let $\{\alpha^u : M \to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{C}(R)$. If each α^u is null-homotopic, then so is the canonical morphism $\alpha : M \to \prod_{u\in U} N^u$.*

Moreover, a product of null-homotopic morphisms is null-homotopic.

Proof. By assumption there are degree 1 homomorphisms $\tau^u : M \to N^u$ such that $\alpha^u = \partial^{N^u}\tau^u + \tau^u\partial^M$ holds for every $u \in U$. Consider each homomorphism τ^u as a morphism $\Sigma M^\natural \to N^{u\natural}$ of graded R-modules; cf. 2.2.5. Set $N = \prod_{u\in U} N^u$; as $N^\natural$ together with the projections $\{\varpi^u : N^\natural \twoheadrightarrow N^{u\natural}\}_{u\in U}$ is the product of $\{N^{u\natural}\}_{u\in U}$ in

$\mathcal{M}_{\mathrm{gr}}(R)$, there is a morphism $\tau\colon \Sigma M^{\natural} \to N^{\natural}$ with $\varpi^u\tau = \tau^u$ for all $u \in U$. Viewing τ as a degree 1 homomorphism $M \to N$, one has

$$\varpi^u\alpha \;=\; \alpha^u \;=\; \partial^{N^u}\tau^u + \tau^u\partial^M \;=\; \partial^{N^u}\varpi^u\tau + \varpi^u\tau\partial^M \;=\; \varpi^u(\partial^N\tau + \tau\partial^M)$$

for every $u \in U$; here the third equality holds by definition of τ, and the last equality holds as ϖ^u is a morphism. Since $\partial^N\tau + \tau\partial^M$ is a morphism of R-complexes, the universal property of products yields $\alpha = \partial^N\tau + \tau\partial^M$. That is, α is null-homotopic.

If $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ is a family of null-homotopic morphisms, then the composites $\prod_{u\in U} M^u \twoheadrightarrow M^u \to N^u$ are null-homotopic by 2.2.25. Thus, the canonical morphism $\prod_{u\in U}\alpha^u\colon \prod_{u\in U} M^u \to \prod_{u\in U} N^u$ is null-homotopic; cf. 3.1.17. □

Functors that Preserve Products

Below, and throughout this chapter, we use the same symbol ϖ^u for the projections in all products.

3.1.20. Let $\{N^u\}_{u\in U}$ be a family of R-complexes and $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ a functor. The universal property of the product yields a canonical morphism in $\mathcal{C}(S)$,

$$\mathrm{F}\Big(\prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U}\mathrm{F}(N^u) \quad \text{given by} \quad x \longmapsto (\mathrm{F}(\varpi^u)(x))_{u\in U}\;; \tag{3.1.20.1}$$

see 3.1.15. Recall that F is said to preserve products if this map is an isomorphism for every family $\{N^u\}_{u\in U}$. In this case, F also preserves products of morphisms; cf. 3.1.17. Further, given a natural transformation $\tau\colon \mathrm{E} \to \mathrm{F}$ of functors $\mathcal{C}(R) \to \mathcal{C}(S)$, it is straightforward to verify that there is a commutative diagram in $\mathcal{C}(S)$,

$$\begin{tikzcd} \mathrm{E}\Big(\prod_{u\in U} N^u\Big) \arrow[r, "\tau^{\prod_{u\in U} N^u}"] \arrow[d] & \mathrm{F}\Big(\prod_{u\in U} N^u\Big) \arrow[d] \\ \prod_{u\in U}\mathrm{E}(N^u) \arrow[r, "\prod_{u\in U}\tau^{N^u}"] & \prod_{u\in U}\mathrm{F}(N^u)\,, \end{tikzcd} \tag{3.1.20.2}$$

where the vertical maps are the canonical morphisms.

The next result shows that the shift functor from 2.2.1 preserves products.

3.1.21 Proposition. *Let s be an integer and $\{N^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(R)$,*

$$\Sigma^s\prod_{u\in U} N^u \longrightarrow \prod_{u\in U}\Sigma^s N^u\,, \tag{3.1.21.1}$$

given by $x \mapsto ((\Sigma^s\varpi^u)(x))_{u\in U}$ for $x \in \Sigma^s\prod_{u\in U} N^u$, is an isomorphism.

Proof. The morphism (3.1.21.1) is the identity map. □

The next result shows that the cycle, boundary, cokernel, and homology functors from 2.2.7 preserve products.

3.1.22 Proposition. *For every family $\{N^u\}_{u\in U}$ of R-complexes, the following four canonical morphisms in $\mathcal{C}(R)$ are isomorphisms*:

(a) $$\mathrm{Z}\Big(\prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U} \mathrm{Z}(N^u) \quad \textit{given by} \quad z \longmapsto (\mathrm{Z}(\varpi^u)(z))_{u\in U}\,.$$

(b) $$\mathrm{B}\Big(\prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U} \mathrm{B}(N^u) \quad \textit{given by} \quad b \longmapsto (\mathrm{B}(\varpi^u)(b))_{u\in U}\,.$$

(c) $$\mathrm{C}\Big(\prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U} \mathrm{C}(N^u) \quad \textit{given by} \quad c \longmapsto (\mathrm{C}(\varpi^u)(c))_{u\in U}\,.$$

(d) $$\mathrm{H}\Big(\prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U} \mathrm{H}(N^u) \quad \textit{given by} \quad h \longmapsto (\mathrm{H}(\varpi^u)(h))_{u\in U}\,.$$

PROOF. The morphisms (a) and (b) are the identity maps. The short exact sequences $0 \to \mathrm{B}(N^u) \to N^u \to \mathrm{C}(N^u) \to 0$, see 2.2.12(b), yield a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}\big(\prod_{u\in U} N^u\big) \arrow[r] \arrow[d] & \prod_{u\in U} N^u \arrow[r] \arrow[d, equal] & \mathrm{C}\big(\prod_{u\in U} N^u\big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{u\in U} \mathrm{B}(N^u) \arrow[r] & \prod_{u\in U} N^u \arrow[r] & \prod_{u\in U} \mathrm{C}(N^u) \arrow[r] & 0\,.
\end{tikzcd}$$

In this diagram, the rows are exact by 2.2.12(b) and 3.1.18. The leftmost vertical map is (b), which is an isomorphism, and the rightmost vertical map is (c). It now follows from the Five Lemma 2.1.41 that (c) is an isomorphism. A similar argument, using the exact sequences $0 \to \mathrm{B}(N^u) \to \mathrm{Z}(N^u) \to \mathrm{H}(N^u) \to 0$ from 2.2.12(c), shows that (d) is an isomorphism. □

3.1.23 Corollary. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes. There are equalities,*

$$\sup\Big(\prod_{u\in U} M^u\Big) = \sup_{u\in U}\{\sup M^u\} \quad \textit{and} \quad \inf\Big(\prod_{u\in U} M^u\Big) = \inf_{u\in U}\{\inf M^u\}\,;$$

in particular, $\prod_{u\in U} M^u$ is acyclic if and only if each complex M^u is acyclic.

PROOF. The isomorphism $\mathrm{H}(\prod_{u\in U} M^u) \cong \prod_{u\in U} \mathrm{H}(M^u)$ from 3.1.22(d) yields both equalities. □

If M is an object in a $\Bbbk$-linear category $\mathcal{U}$, then the functor $\mathcal{U}(M,-)\colon \mathcal{U} \to \mathcal{M}(\Bbbk)$ preserves products. In particular, for every M in $\mathcal{C}(R)$, the functor $\mathcal{C}(R)(M,-) = \mathrm{Z}_0(\mathrm{Hom}_R(M,-))$ from $\mathcal{C}(R)$ to $\mathcal{M}(\Bbbk)$ preserves products; cf. 2.3.10. The next result is stronger, it shows that the functor $\mathrm{Hom}_R(M,-)$ preserves products.

3.1.24 Proposition. *Let M be an R-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$\operatorname{Hom}_R\Big(M, \prod_{u\in U} N^u\Big) \longrightarrow \prod_{u\in U} \operatorname{Hom}_R(M, N^u)\,, \tag{3.1.24.1}$$

given by $\vartheta \mapsto (\operatorname{Hom}_R(M, \varpi^u)(\vartheta))_{u\in U} = (\varpi^u \vartheta)_{u\in U}$*, is an isomorphism.*

Proof. Assign to an element $(\vartheta^u)_{u\in U}$ in $\prod_{u\in U} \operatorname{Hom}_R(M, N^u)$ the homomorphism ϑ in $\operatorname{Hom}_R(M, \prod_{u\in U} N^u)$ that maps an element $m \in M$ to $(\vartheta^u(m))_{u\in U}$ in $\prod_{u\in U} N^u$. This assignment defines an inverse to (3.1.24.1). $\square$

Remark. For an R-complex M, the functor $\operatorname{Hom}_R(M, -)\colon \mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ has a left adjoint, namely the functor $M \otimes_\Bbbk -$; see 4.5.14. From this fact alone, it follows that $\operatorname{Hom}_R(M, -)$ preserves products; see E 3.1.12.

3.1.25. Coproducts in $\mathcal{C}(R)$ correspond to products in $\mathcal{C}(R)^{\mathrm{op}}$. I.e. let $\{M^u\}_{u\in U}$ be a family of R-complexes whose coproduct in $\mathcal{C}(R)$ is M with injections $\varepsilon^u\colon M^u \to M$. The R-complex M is also an object in $\mathcal{C}(R)^{\mathrm{op}}$ and the morphisms ε^u in $\mathcal{C}(R)$ correspond to morphisms $\varepsilon^u\colon M \to M^u$ in $\mathcal{C}(R)^{\mathrm{op}}$. Since the family $\{\varepsilon^u\colon M^u \to M\}_{u\in U}$ in $\mathcal{C}(R)$ has the universal property of a coproduct, the family $\{\varepsilon^u\colon M \to M^u\}_{u\in U}$ in $\mathcal{C}(R)^{\mathrm{op}}$ has the universal property of a product in $\mathcal{C}(R)^{\mathrm{op}}$ and vice versa.

3.1.26. Let $\{M^u\}_{u\in U}$ be a family of R-complexes and $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ a functor. The canonical morphism from G of the product of $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)^{\mathrm{op}}$ to the product of $\{\mathrm{G}(M^u)\}_{u\in U}$ in $\mathcal{C}(R)$ takes per 3.1.25 the form

$$\mathrm{G}\Big(\coprod_{u\in U} M^u\Big) \longrightarrow \prod_{u\in U} \mathrm{G}(M^u) \quad \text{given by} \quad x \longmapsto (\mathrm{G}(\varepsilon^u)(x))_{u\in U}\,, \tag{3.1.26.1}$$

where $\coprod_{u\in U} M^u$ is the coproduct in $\mathcal{C}(R)$ with injections $\varepsilon^u\colon M^u \to \coprod_{u\in U} M^u$. Recall that $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is said to preserve products if this morphism is an isomorphism for every family $\{M^u\}_{u\in U}$ of R-complexes. Further, given a natural transformation $\tau\colon \mathrm{G} \to \mathrm{J}$ of functors $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$, the diagram in 3.1.20 takes the form

$$\begin{tikzcd}
\mathrm{G}\Big(\coprod_{u\in U} M^u\Big) \arrow[r, "\tau^{\coprod_{u\in U} M^u}"] \arrow[d] & \mathrm{J}\Big(\coprod_{u\in U} M^u\Big) \arrow[d] \\
\prod_{u\in U} \mathrm{G}(M^u) \arrow[r, "\prod_{u\in U} \tau^{M^u}"] & \prod_{u\in U} \mathrm{J}(M^u)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms.

The next result and 3.1.24 show that the functor $\operatorname{Hom}_R(-,-)$ preserves products in each variable.

3.1.27 Proposition. *Let N be an R-complex and $\{M^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in* $\mathcal{C}(\Bbbk)$*,*

$$\operatorname{Hom}_R\Big(\coprod_{u\in U} M^u, N\Big) \longrightarrow \prod_{u\in U} \operatorname{Hom}_R(M^u, N)\,, \tag{3.1.27.1}$$

given by $\vartheta \mapsto (\operatorname{Hom}_R(\varepsilon^u, N)(\vartheta))_{u\in U} = (\vartheta\varepsilon^u)_{u\in U}$*, is an isomorphism.*

Proof. Assign to every family $(\vartheta^u)_{u\in U}$ in $\prod_{u\in U}\operatorname{Hom}_R(M^u,N)$ the homomorphism in $\operatorname{Hom}_R(\coprod_{u\in U} M^u, N)$ given by $\varepsilon^u(m^u)\mapsto\vartheta^u(m^u)$. This assignment defines an inverse to (3.1.27.1). $\square$

Boundedness and Finiteness

3.1.28. For every family $\{M^u\}_{u\in U}$ of R-complexes, the coproduct $\coprod_{u\in U} M^u$ is a subcomplex of the product $\prod_{u\in U} M^u$, cf. 1.1.20. If U is a finite set, then the product and the coproduct of $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$ coincide, and this complex $\prod_{u\in U} M^u = \coprod_{u\in U} M^u$ is the iterated biproduct $M=\bigoplus_{u\in U} M^u$. Per 1.1.14 one calls M the *direct sum* of the family $\{M^u\}_{u\in U}$ and each complex M^u is called a *direct summand* of M.

3.1.29 Example. There is no isomorphism of $\mathbb{Z}$-modules,

$$\mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/2^n\mathbb{Z}\longrightarrow\prod_{n\in\mathbb{N}}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2^n\mathbb{Z})\,.$$

Indeed, one has $\prod_{n\in\mathbb{Z}}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2^n\mathbb{Z})=0$, see 1.1.10. The family $\{\mathbb{Z}\twoheadrightarrow\mathbb{Z}/2^n\mathbb{Z}\}_{n\in\mathbb{N}}$ of canonical homomorphisms induces by the universal property of the product an injective homomorphism $\mathbb{Z}\to\prod_{n\in\mathbb{N}}\mathbb{Z}/2^n\mathbb{Z}$. As the $\mathbb{Z}$-module $\mathbb{Q}$ is flat, see 1.3.42, there is an injective homomorphism $\mathbb{Q}\to\mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/2^n\mathbb{Z}$, in particular, $\mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/2^n\mathbb{Z}$ is non-zero.

Under suitable finiteness conditions on M the functor $M\otimes_R -$ preserves products.

3.1.30 Proposition. *Let M be a bounded and degreewise finitely presented R°-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$M\otimes_R\prod_{u\in U}N^u\longrightarrow\prod_{u\in U}(M\otimes_R N^u)\,, \tag{3.1.30.1}$$

given by $t\mapsto((M\otimes_R\varpi^u)(t))_{u\in U}$, is an isomorphism.

Proof. First we prove the result in the special case where $M=L$ is a bounded complex of finitely generated free R°-modules. Write $\varkappa^L$ for the morphism (3.1.30.1); we argue that $\varkappa^L_v$ is an isomorphism for every $v\in\mathbb{Z}$. One can assume that L is non-zero, and since L is bounded, the quantities $\imath=\inf L^\natural$ and $\jmath=\sup L^\natural$ from 2.5.5 are integers. For every $v\in\mathbb{Z}$ one has

$$\Big(L\otimes_R\prod_{u\in U}N^u\Big)_v=\bigoplus_{i=\imath}^{\jmath}\Big(L_i\otimes_R\Big(\prod_{u\in U}N^u\Big)_{v-i}\Big)=\bigoplus_{i=\imath}^{\jmath}\Big(L_i\otimes_R\prod_{u\in U}N^u_{v-i}\Big)$$

and

$$\Big(\prod_{u\in U}(L\otimes_R N^u)\Big)_v=\prod_{u\in U}\bigoplus_{i=\imath}^{\jmath}(L_i\otimes_R N^u_{v-i})\cong\bigoplus_{i=\imath}^{\jmath}\prod_{u\in U}(L_i\otimes_R N^u_{v-i})\,,$$

where the isomorphism follows as it is elementary to check that a product and a direct sum may be interchanged, cf. 3.4.14. Via these isomorphisms, $\varkappa^L_v$ is identified with $\bigoplus_{i=\imath}^{\jmath}\varkappa_{v,i}$ where $\varkappa_{v,i}$ is the canonical morphism $L_i\otimes_R(\prod_{u\in U}N^u)_{v-i}\to\prod_{u\in U}(L_i\otimes_R N^u_{v-i})$. To see that each $\varkappa_{v,i}$ is an isomorphism, assume that the finitely generated free R°-module L_i has a basis with n_i elements. Write $L_i = \bigoplus_{\ell=1}^{n_i} L_{i,\ell}$ where each $L_{i,\ell}$ is a free R°-module having a basis with one element. Now one has

$$L_i \otimes_R \big(\prod_{u\in U} N^u\big)_{v-i} \cong \bigoplus_{\ell=1}^{n_i} \big(L_{i,\ell} \otimes_R \big(\prod_{u\in U} N^u\big)_{v-i}\big) \quad \text{and}$$
$$\prod_{u\in U} (L_i \otimes_R N^u_{v-i}) \cong \bigoplus_{\ell=1}^{n_i} \prod_{u\in U} (L_{i,\ell} \otimes_R N^u_{v-i})\,,$$

and via these isomorphisms $\varkappa_{v,i}$ is identified with $\bigoplus_{\ell=1}^{n_i} \varkappa_{v,i,\ell}$ where $\varkappa_{v,i,\ell}$ is the canonical morphism $L_{i,\ell} \otimes_R (\prod_{u\in U} N^u)_{v-i} \to \prod_{u\in U}(L_{i,\ell} \otimes_R N^u_{v-i})$. Using the unitor 1.2.1 and the fact that each $L_{i,\ell}$ is isomorphic to R (as an R°-module), it follows that each $\varkappa_{v,i,\ell}$ is an isomorphism, and hence so is $\varkappa_{v,i} = \bigoplus_{\ell=1}^{n_i} \varkappa_{v,i,\ell}$.

In the general case where M is bounded and degreewise finitely presented, apply 2.5.31 to get an exact sequence $L' \to L \to M \to 0$ where L and L' are bounded complexes of finitely generated free R°-modules. In the commutative diagram below, the vertical maps are the canonical morphisms (3.1.30.1) and the rows are exact by right exactness of the tensor products 2.4.9 and exactness of products 3.1.18,

$$\begin{tikzcd}
L' \otimes_R \prod_{u\in U} N^u \arrow[r] \arrow[d, "\cong"'] & L \otimes_R \prod_{u\in U} N^u \arrow[r] \arrow[d, "\cong"'] & M \otimes_R \prod_{u\in U} N^u \arrow[r] \arrow[d] & 0 \\
\prod_{u\in U} (L' \otimes_R N^u) \arrow[r] & \prod_{u\in U} (L \otimes_R N^u) \arrow[r] & \prod_{u\in U} (M \otimes_R N^u) \arrow[r] & 0\,.
\end{tikzcd}$$

As argued above, the left-hand and middle vertical maps are isomorphisms, and hence, by the Five Lemma 2.1.41, so is the right-hand vertical map. □

Remark. An R°-module M is finitely presented if and only if (3.1.30.1) is an isomorphism for every family $\{N^u\}_{u\in U}$ of R-complexes; see E 3.1.15. There exists a $\mathbb{Q}$-module M such that $M \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathbb{N}}$ and $(M \otimes_{\mathbb{Q}} \mathbb{Q})^{\mathbb{N}}$ are isomorphic whilst the canonical map $M \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathbb{N}} \to (M \otimes_{\mathbb{Q}} \mathbb{Q})^{\mathbb{N}}$ is not an isomorphism; see E 3.1.13.

3.1.31 Proposition. *Let N be a bounded and degreewise finitely presented R-complex and $\{N^u\}_{u\in U}$ a family of R°-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

(3.1.31.1) $$\Big(\prod_{u\in U} M^u\Big) \otimes_R N \longrightarrow \prod_{u\in U} (M^u \otimes_R N)\,,$$

given by $t \mapsto ((\varpi^u \otimes_R N)(t))_{u\in U}$, is an isomorphism.

Proof. The assertion follows from an argument similar to the proof of 3.1.30. □

3.1.32 Example. The canonical homomorphism of R-modules,

(3.1.32.1) $$\operatorname{Hom}_R(R^{(\mathbb{N})}, R)^{(\mathbb{N})} \longrightarrow \operatorname{Hom}_R(R^{(\mathbb{N})}, R^{(\mathbb{N})})\,,$$

given by $\varepsilon^n(\vartheta^n) \mapsto \operatorname{Hom}_R(R^{(\mathbb{N})}, \varepsilon^n)(\vartheta^n) = \varepsilon^n\vartheta^n$ for elements $n \in \mathbb{N}$ and $\vartheta^n \in \operatorname{Hom}_R(R^{(\mathbb{N})}, R)$, is not surjective. Indeed, the identity map on $R^{(\mathbb{N})}$ is not in the image of (3.1.32.1).

Under suitable conditions on M, the functor $\operatorname{Hom}_R(M,-)$ preserves coproducts.

3.1.33 Proposition. *Let M be a bounded and degreewise finitely generated R-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$\coprod_{u\in U}\operatorname{Hom}_R(M,N^u)\longrightarrow \operatorname{Hom}_R\Big(M,\coprod_{u\in U}N^u\Big)\,, \tag{3.1.33.1}$$

given by $\varepsilon^u(\vartheta^u)\mapsto \operatorname{Hom}_R(M,\varepsilon^u)(\vartheta^u)=\varepsilon^u\vartheta^u$ for $u\in U$ and $\vartheta^u\in\operatorname{Hom}_R(M,N^u)$, is an isomorphism.

Proof. Assume that M is bounded and degreewise finitely generated. The module $M^\natural$ is finitely generated, so a homomorphism $\vartheta\colon M\to\coprod_{u\in U}N^u$ factors through a subcomplex $\bigoplus_{u\in U'}N^u$, where U' is a finite subset of U. Let $\varpi^u\colon \bigoplus_{u\in U'}N^u\twoheadrightarrow N^u$ be the projection for $u\in U'$. Assign to a homomorphism ϑ in $\operatorname{Hom}_R(M,\coprod_{u\in U}N^u)$ the element $\sum_{u\in U}\varepsilon^u(\vartheta^u)$ in $\coprod_{u\in U}\operatorname{Hom}_R(M,N^u)$ where $\vartheta^u=\varpi^u\vartheta$ for $u\in U'$ and $\vartheta^u=0$ for $u\notin U'$. This defines an inverse to (3.1.33.1). □

Remark. Rentschler [211] says that a module M is *of type Σ* if the functor $\operatorname{Hom}(M,-)$ preserves coproducts. More recent names used in the literature for such modules are 'small' and 'dually slender', see for example Eklof, Goodearl, and Trlifaj [80]. A dually slender module need not be finitely generated, however, if R is left Noetherian, then every dually slender R-module is finitely generated. See E 3.1.19–E 3.1.22.

Exercises

E 3.1.1 Let $\{\widetilde{\varepsilon}^u\colon M^u\to C\}_{u\in U}$ be a family of morphisms in $\mathcal{C}(R)$ with the property that for every family $\{\alpha^u\colon M^u\to N\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$ there is a unique morphism $\alpha\colon C\to N$ with $\alpha\widetilde{\varepsilon}^u=\alpha^u$ for all $u\in U$. Show that there is an isomorphism $\varphi\colon\coprod_{u\in U}M^u\to C$ with $\varphi\varepsilon^u=\widetilde{\varepsilon}^u$ for all $u\in U$. Conclude that the universal property determines the coproduct uniquely up to isomorphism.

E 3.1.2 Let $\alpha\colon\coprod_{u\in U}M^u\to N$ be the morphism induced by a family $\{\alpha^u\colon M^u\to N\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$. Show that α is surjective if $\bigcup_{u\in U}\operatorname{Im}\alpha^u=N$ holds.

E 3.1.3 (Cf. 3.1.3) Show that the coproduct in $\mathcal{C}(R)$ of a family of graded R-modules is a graded R-module. Conclude, in particular, that the category $\mathcal{M}_{\mathrm{gr}}(R)$ has coproducts.

E 3.1.4 Let U be a set. Show that U-indexed families of R-complexes form an Abelian category and that the product and coproduct are exact functors from this category to $\mathcal{C}(R)$.

E 3.1.5 (Cf. 3.1.8) Verify that the diagram (3.1.8.2) is commutative.

E 3.1.6 Show that every functor $\mathrm{F}\colon\mathcal{C}(R)\to\mathcal{C}(S)$ that has a right adjoint preserves coproducts.

E 3.1.7 Show that for every complex N one has isomorphisms $\coprod_{v\in\mathbb{Z}}\Sigma^v N_v\cong N^\natural\cong\prod_{v\in\mathbb{Z}}\Sigma^v N_v$.

E 3.1.8 Let $\{\widetilde{\varpi}^u\colon P\to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{C}(R)$ with the property that for every family $\{\alpha^u\colon M\to N^u\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$ there is a unique morphism $\alpha\colon M\to P$ with $\widetilde{\varpi}^u\alpha=\alpha^u$ for all $u\in U$. Show that there is an isomorphism $\varphi\colon P\to\prod_{u\in U}N^u$ with $\varpi^u\varphi=\widetilde{\varpi}^u$ for all $u\in U$. Conclude that the universal property determines the product uniquely up to isomorphism.

E 3.1.9 Let $\alpha\colon M\to\prod_{u\in U}N^u$ be the morphism induced by a family $\{\alpha^u\colon M\to N^u\}_{u\in U}$ of morphisms in $\mathcal{C}(R)$. Show that α is injective if $\bigcap_{u\in U}\operatorname{Ker}\alpha^u=0$ holds.

E 3.1.10 (Cf. 3.1.16) Show that the product in $\mathcal{C}(R)$ of a family of graded R-modules is a graded R-module. Conclude, in particular, that the category $\mathcal{M}_{\mathrm{gr}}(R)$ has products.

E 3.1.11 (Cf. 3.1.20) Verify that the diagram (3.1.20.2) is commutative.

E 3.1.12 Show that every functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ that has a left adjoint preserves products.

E 3.1.13 Let $\Bbbk$ be a field. Show that the canonical homomorphism $\Bbbk^{\mathbb{N}} \otimes_{\Bbbk} \Bbbk^{\mathbb{N}} \to (\Bbbk^{\mathbb{N}} \otimes_{\Bbbk} \Bbbk)^{\mathbb{N}}$ is not an isomorphism. Show that $\Bbbk^{\mathbb{N}} \otimes_{\Bbbk} \Bbbk^{\mathbb{N}}$ and $(\Bbbk^{\mathbb{N}} \otimes_{\Bbbk} \Bbbk)^{\mathbb{N}}$ are isomorphic.

E 3.1.14 Let M be an R^{o}-module. Show that the next conditions are equivalent. (*i*) M is finitely generated. (*ii*) The morphism (3.1.30.1) is surjective for every family $\{N^u\}_{u\in U}$. (*iii*) The morphism (3.1.30.1) is surjective for every family $\{N^u\}_{u\in U}$ with $N^u = R$ for all $u \in U$.

E 3.1.15 Let M be an R^{o}-module. Show that the next conditions are equivalent. (*i*) M is finitely presented. (*ii*) The morphism (3.1.30.1) is bijective for every family $\{N^u\}_{u\in U}$. (*iii*) The morphism (3.1.30.1) is bijective for every family $\{N^u\}_{u\in U}$ with $N^u = R$ for all $u \in U$.

E 3.1.16 Show that the canonical morphism (3.1.33.1) is always injective.

E 3.1.17 Let M be a degreewise finitely presented R^{o}-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. Show that if there exists $n \in \mathbb{N}$ such that $n \geqslant \sup\,(N^u)^\natural$ and $\inf\,(N^u)^\natural \geqslant -n$ hold for all $u \in U$, then the canonical morphism (3.1.30.1) is an isomorphism.

E 3.1.18 Let M be a degreewise finitely generated R-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. Show that if there exist $n \in \mathbb{N}$ such that $n \geqslant \sup\,(N^u)^\natural$ and $\inf\,(N^u)^\natural \geqslant -n$ hold for all $u \in U$, then the canonical morphism (3.1.33.1) is an isomorphism.

E 3.1.19 Let M be an R-module. Show that $\operatorname{Hom}_R(M,-)$ preserves coproducts if and only if for every ascending chain $M^1 \subseteq M^2 \subseteq \cdots$ of submodules of M with $M = \bigcup_{n\in\mathbb{N}} M^n$ one has $M = M^n$ for some $n \in \mathbb{N}$.

E 3.1.20 Let M be an R-module. Show that if $\operatorname{Hom}_R(M,-)$ preserves coproducts and M is countably generated, then M is finitely generated.

E 3.1.21 Give an example of an R-module M such that M is not finitely generated and $\operatorname{Hom}_R(M,-)$ preserves coproducts. *Hint:* See e.g. Head [116] or Rentschler [211].

E 3.1.22 Assume that R is left Noetherian and let M be an R-module. Show that $\operatorname{Hom}_R(M,-)$ preserves coproducts if and only if M is finitely generated.

E 3.1.23 Let $\mathcal{V}$ be a category. Show that products in $\mathcal{V}$ correspond to coproducts in the opposite category $\mathcal{V}^{\mathrm{op}}$ and that coproducts in $\mathcal{V}$ correspond to products in $\mathcal{V}^{\mathrm{op}}$.

E 3.1.24 Show that for a family $\{R^u\}_{u\in U}$ of $\Bbbk$-algebras the ring $\bigtimes_{u\in U} R^u$ is the product in the category of $\Bbbk$-algebras.

E 3.1.25 Show that for commutative $\Bbbk$-algebras R and S the ring $R \otimes_{\Bbbk} S$ is the coproduct in the category of commutative $\Bbbk$-algebras.

3.2 Colimits

Synopsis. Direct system; colimit; universal property; functor that preserves colimits; pushout.

A colimit is a quotient of a coproduct. Structurally, our treatment of colimits follows the pattern established in Sect. 3.1. Recall that a set endowed with a reflexive and transitive binary relation '$\leqslant$' is called preordered. Partially ordered sets are, in particular, preordered.

3.2.1 Definition. Let $(U,\leqslant)$ be a preordered set. A *U-direct system* in $\mathcal{C}(R)$ is a family $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ of morphisms in $\mathcal{C}(R)$ with the following properties.

(1) $\mu^{uu} = 1^{M^u}$ for all $u \in U$.

(2) $\mu^{wv}\mu^{vu} = \mu^{wu}$ for all $u \leqslant v \leqslant w$ in U.

Any mention of a U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ includes the tacit assumption that $(U,\leqslant)$ is a preordered set.

3.2.2. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system of R-complexes. Notice that even if $u \leqslant v$ and $v \leqslant u$ hold, one may not have $u = v$ as the relation on U is not assumed to be antisymmetric; however, it follows from 3.2.1 that $\mu^{vu}\colon M^u \to M^v$ is an isomorphism with inverse $\mu^{uv}\colon M^v \to M^u$, so one has $M^u \cong M^v$.

3.2.3 Construction. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. We describe the quotient of the coproduct $\coprod_{u\in U} M^u$ by the subcomplex generated by the set of elements $\{\varepsilon^u(m^u) - \varepsilon^v\mu^{vu}(m^u) \mid m^u \in M^u,\ u \leqslant v\}$ as the cokernel of a morphism between coproducts in $\mathcal{C}(R)$.

Set $\nabla(U) = \{(u,v) \in U \times U \mid u \leqslant v\}$ and set $M^{(u,v)} = M^u$ for all $(u,v) \in \nabla(U)$. The assignment

$$\varepsilon^{(u,v)}(m^{(u,v)}) \longmapsto \varepsilon^u(m^{(u,v)}) - \varepsilon^v\mu^{vu}(m^{(u,v)})\,,$$

where $(u,v) \in \nabla(U)$, $m^{(u,v)} \in M^{(u,v)} = M^u$, and ε is the injection (3.1.1.1), defines by 3.1.2 a morphism of R-complexes

$$\varDelta_\mu\colon \coprod_{(u,v)\in\nabla(U)} M^{(u,v)} \longrightarrow \coprod_{u\in U} M^u\,.$$

Set

$$\operatorname*{colim}_{u\in U} M^u = \operatorname{Coker}\varDelta_\mu\,.$$

Notice that for every $u \in U$ the composite of the injection ε^u with the canonical map onto $\operatorname{colim}_{u\in U} M^u$ is a morphism of R-complexes,

$$\mu^u\colon M^u \longrightarrow \operatorname*{colim}_{u\in U} M^u\,, \tag{3.2.3.1}$$

and $\mu^u = \mu^v\mu^{vu}$ holds for all $u \leqslant v$ in U. Every element in $\operatorname{colim}_{u\in U} M^u$ has the form $\sum_{u\in U}\mu^u(m^u)$ for some element $\sum_{u\in U}\varepsilon^u(m^u)$ in $\coprod_{u\in U} M^u$, and one has

$$\partial^{\operatorname{colim}_{u\in U} M^u}(\mu^u(m^u)) = \mu^u(\partial^{M^u}(m^u))\,.$$

If U is filtered, then every element in $\operatorname{colim}_{u\in U} M^u$ has the form $\mu^v(m^v)$ for some $v \in U$ and $m^v \in M^v$; see 3.3.2.

Remark. Though the complex $\operatorname{colim}_{u\in U} M^u$ depends on the morphisms $\mu^{vu}\colon M^u \to M^v$, it is standard to use this symbol that suppresses the morphisms.

The next definition is justified by 3.2.5; it shows that the complex $\operatorname{colim}_{u\in U} M^u$ and the canonical morphisms μ^u have the universal property of a colimit. In any category, this property determines the colimit uniquely up to isomorphism.

3.2.4 Definition. For a U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$ the complex $\operatorname{colim}_{u\in U} M^u$ together with the canonical morphisms $\{\mu^u\}_{u\in U}$, constructed in 3.2.3, is called the *colimit* of $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$.

Remark. Other names for the colimit defined above are 'inductive limit' and 'injective limit'; other symbols used for this gadget are $\varinjlim$ and inj lim.

3.2.5 Theorem. *Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. The colimit from* 3.2.4 *has the following universal property*: *For every family of morphisms $\{\alpha^u\colon M^u \to N\}_{u\in U}$ in $\mathcal{C}(R)$ with $\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$, there is a unique morphism α that makes the next diagram commutative for all $u \leqslant v$,*

$$\begin{tikzcd}
M^v \arrow[dr, "\mu^v"] \arrow[drr, bend left, "\alpha^v"] & & \\
 & \operatorname*{colim}_{u\in U} M^u \arrow[r, dotted, "\alpha"] & N \\
M^u \arrow[uu, "\mu^{vu}"] \arrow[ur, "\mu^u"'] \arrow[urr, bend right, "\alpha^u"'] & &
\end{tikzcd}$$

The morphism α is given by $\sum_{u\in U}\mu^u(m^u) \mapsto \sum_{u\in U}\alpha^u(m^u)$.

Proof. Let $\{\alpha^u\colon M^u \to N\}_{u\in U}$ be a family of morphisms with $\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$. The equalities $\alpha^u = \alpha^v\mu^{vu}$ ensure that the morphism $\coprod_{u\in U} M^u \to N$ from 3.1.2 factors through $\operatorname{colim}_{u\in U} M^u$ to yield a morphism α with the stipulated definition.

It is evident from the definition that α satisfies $\alpha^u = \alpha\mu^u$ for all $u \in U$. Moreover, for any morphism $\alpha'\colon \operatorname{colim}_{u\in U} M^u \to N$ that satisfies $\alpha^u = \alpha'\mu^u$ for all $u \in U$, one has $\alpha'(\sum_{u\in U}\mu^u(m^u)) = \sum_{u\in U}\alpha'\mu^u(m^u) = \sum_{u\in U}\alpha^u(m^u)$, hence $\alpha' = \alpha$. □

3.2.6. With the notation from 3.1.4 and 3.2.5, notice that one has $\operatorname{Im}\alpha = \sum_{u\in U}\operatorname{Im}\alpha^u$; in particular, α is surjective if $\bigcup_{u\in U}\operatorname{Im}\alpha^u = N$ holds.

3.2.7. It follows readily from 3.2.3 and 3.2.5 that the full subcategories $\mathcal{M}(R)$ and $\mathcal{M}_{\mathrm{gr}}(R)$ of $\mathcal{C}(R)$ are closed under colimits, that is, if $\{\mu_n^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ is a U-direct system in $\mathcal{M}(R)$ or in $\mathcal{M}_{\mathrm{gr}}(R)$, then $\operatorname{colim}_{u\in U} M^u$ belongs to $\mathcal{M}(R)$ or $\mathcal{M}_{\mathrm{gr}}(R)$. It follows that for every U-direct system $\{\mu_n^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$ one has

$$\big(\operatorname*{colim}_{u\in U} M^u\big)_i = \operatorname*{colim}_{u\in U}(M_i^u) \qquad\text{and}\qquad \big(\operatorname*{colim}_{u\in U} M^u\big)^\natural = \operatorname*{colim}_{u\in U}(M^u)^\natural\,.$$

See 3.2.2 to reconcile the next example with the fact that a greatest element of a preordered set need not be unique.

3.2.8 Example. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system of R-complexes. If U has a greatest element, w, then there is an isomorphism,

$$\operatorname*{colim}_{u\in U} M^u \;\cong\; M^w\,.$$

Indeed, the family $\{\mu^{wu}\colon M^u \to M^w\}_{u\in U}$ satisfies $\mu^{wu} = \mu^{wv}\mu^{vu}$ for $u \leqslant v$ in U, so 3.2.5 yields a unique morphism $\varphi\colon \operatorname{colim}_{u\in U} M^u \to M^w$ with $\varphi\mu^u = \mu^{wu}$ for $u \in U$. Evidently $\varphi\mu^w = \mu^{ww} = 1^{M^w}$ holds. To see that φ is an isomorphism with inverse μ^w, notice that for every $u \in U$ one has $\mu^w\varphi\mu^u = \mu^w\mu^{wu} = \mu^u$, so the uniqueness statement in 3.2.5 implies that $\mu^w\varphi$ is the identity on $\operatorname{colim}_{u\in U} M^u$.

3.2.9 Example. Let $\{M^u\}_{u\in U}$ be a family of R-complexes. Endowed with the discrete order, U is a preordered set, and $\{\mu^{uu} = 1^{M^u}\}_{u\in U}$ is a U-direct system with $\operatorname{colim}_{u\in U} M^u = \coprod_{u\in U} M^u$ and $\mu^u = \varepsilon^u$ for all $u \in U$. Thus, a coproduct is a colimit.

As is the case for the colimit in any category, the colimit in $\mathcal{C}(R)$ also acts on morphisms. This is explained in the following definition.

3.2.10 Definition. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ and $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$ be U-direct systems in $\mathcal{C}(R)$. A family of morphisms $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ in $\mathcal{C}(R)$ that satisfy $\nu^{vu}\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$ in U is called a *morphism of U-direct systems*. Such a morphism is called injective (surjective) if each map α^u is injective (surjective).

Given a morphism $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ of U-direct systems, it follows from the universal property of colimits 3.2.5 that the map given by $\mu^u(m^u) \mapsto \nu^u\alpha^u(m^u)$ is the unique morphism that makes the next diagram commutative for all $u \leqslant v$,

$$
\begin{tikzcd}
M^v \arrow[rrr, "\alpha^v"] \arrow[dr, "\mu^v"] & & & N^v \arrow[dl, "\nu^v"'] \\
& \operatorname*{colim}_{u\in U} M^u \arrow[r, dotted] & \operatorname*{colim}_{u\in U} N^u & \\
M^u \arrow[uu, "\mu^{vu}"] \arrow[ur, "\mu^u"'] \arrow[rrr, "\alpha^u"] & & & N^u \arrow[uu, "\nu^{vu}"'] \arrow[ul, "\nu^v"]
\end{tikzcd}
.
$$

This map is called the *colimit* of $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ and denoted $\operatorname{colim}_{u\in U}\alpha^u$.

3.2.11 Example. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. The maps μ^{vu} are morphisms in $\mathcal{C}(R)$, so the family $\{\partial^{M^u}\colon M^u \to \Sigma M^u\}_{u\in U}$ is a morphism of U-direct systems. From the definitions one gets $\operatorname{colim}_{u\in U}\partial^{M^u} = \partial^{\operatorname{colim}_{u\in U} M^u}$.

The next result shows that colimits are right exact. While general colimits are not exact—see 3.2.29 for an example—we show in 3.3.10 that colimits over filtered sets are exact.

3.2.12 Lemma. *Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ and $\{\beta^u\colon N^u \to X^u\}_{u\in U}$ be morphisms of U-direct systems in $\mathcal{C}(R)$. If the sequence*

$$M^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} X^u \longrightarrow 0$$

is exact for every $u \in U$, then the next sequence is exact,

$$\operatorname*{colim}_{u\in U} M^u \xrightarrow{\operatorname{colim}_{u\in U}\alpha^u} \operatorname*{colim}_{u\in U} N^u \xrightarrow{\operatorname{colim}_{u\in U}\beta^u} \operatorname*{colim}_{u\in U} X^u \longrightarrow 0\,.$$

Proof. Let $\nabla(U)$ be as in 3.2.3 and set $\alpha^{(u,v)} = \alpha^u$ and $\beta^{(u,v)} = \beta^u$ for $(u,v) \in \nabla(U)$. By 3.1.6 there is a commutative diagram with exact rows,

$$
\begin{tikzcd}
\coprod_{(u,v)\in\nabla(U)} M^{(u,v)} \arrow[r, "\coprod_{(u,v)\in\nabla(U)}\alpha^{(u,v)}"] \arrow[d, "\Delta_\mu"] & \coprod_{(u,v)\in\nabla(U)} N^{(u,v)} \arrow[r, "\coprod_{(u,v)\in\nabla(U)}\beta^{(u,v)}"] \arrow[d, "\Delta_\nu"] & \coprod_{(u,v)\in\nabla(U)} X^{(u,v)} \arrow[r] \arrow[d, "\Delta_\chi"] & 0 \\
\coprod_{u\in U} M^u \arrow[r, "\coprod_{u\in U}\alpha^u"] & \coprod_{u\in U} N^u \arrow[r, "\coprod_{u\in U}\beta^u"] & \coprod_{u\in U} X^u \arrow[r] & 0
\end{tikzcd}
,
$$

where the vertical morphisms $\varDelta$ are defined in 3.2.3. It follows from 2.1.43 that there is an exact sequence $\operatorname{Coker}\varDelta_\mu \to \operatorname{Coker}\varDelta_\nu \to \operatorname{Coker}\varDelta_\chi \to 0$ in $\mathcal{C}(R)$, which in view of 3.2.3 is the desired one. $\square$

The product of preordered sets U and U' is the cartesian product $U \times U'$ equipped with the product order, i.e. $(u, u') \leqslant (v, v')$ holds in $U \times U'$ if and only if one has $u \leqslant v$ in U and $u' \leqslant v'$ in U'. It is straightforward to verify that $U \times U'$ is a preordered set.

3.2.13 Proposition. *Let U and U' be preordered sets and*

$$\{\mu^{(v,v')(u,u')} \colon M^{(u,u')} \to M^{(v,v')}\}_{(u,u') \leqslant (v,v')}$$

a $(U \times U')$-direct system in $\mathcal{C}(R)$. There are isomorphisms,

$$\operatorname*{colim}_{u' \in U'} \big(\operatorname*{colim}_{u \in U} M^{(u,u')}\big) \;\cong\; \operatorname*{colim}_{(u,u') \in U \times U'} M^{(u,u')} \;\cong\; \operatorname*{colim}_{u \in U} \big(\operatorname*{colim}_{u' \in U'} M^{(u,u')}\big)\,.$$

Proof. Due to the symmetry, it suffices to prove the first isomorphism.

Let $u' \in U'$ and note that $\{\mu^{(v,u')(u,u')} \colon M^{(u,u')} \to M^{(v,u')}\}_{u \leqslant v}$ is a U-direct system; set $X^{u'} = \operatorname{colim}_{u \in U} M^{(u,u')}$ and write $\{\varphi^{u,u'} \colon M^{(u,u')} \to X^{u'}\}_{u \in U}$ for the canonical morphisms; so $\varphi^{v,u'}\mu^{(v,u')(u,u')} = \varphi^{u,u'}$ holds for $u \leqslant v$ in U. Let $u' \leqslant v'$ in U' be given. For all $u \leqslant v$ in U the next diagram is, by assumption, commutative,

$$\begin{tikzcd}
M^{(u,u')} \arrow[r, "\mu^{(u,v')(u,u')}"] \arrow[d, "\mu^{(v,u')(u,u')}"'] & M^{(u,v')} \arrow[d, "\mu^{(v,v')(u,v')}"] \\
M^{(v,u')} \arrow[r, "\mu^{(v,v')(v,u')}"] & M^{(v,v')}\,.
\end{tikzcd}$$

Thus $\{\mu^{(u,v')(u,u')} \colon M^{(u,u')} \to M^{(u,v')}\}_{u \in U}$ is a morphism from the U-direct system $\{\mu^{(v,u')(u,u')} \colon M^{(u,u')} \to M^{(v,u')}\}_{u \leqslant v}$ to $\{\mu^{(v,v')(u,v')} \colon M^{(u,v')} \to M^{(v,v')}\}_{u \leqslant v}$, so 3.2.10 yields an induced morphism $\chi^{v'u'} = \operatorname{colim}_{u \in U} \mu^{(u,v')(u,u')} \colon X^{u'} \to X^{v'}$, which is the unique morphism that makes the diagram

(b)
$$\begin{tikzcd}
M^{(u,u')} \arrow[r, "\mu^{(u,v')(u,u')}"] \arrow[d, "\varphi^{u,u'}"'] & M^{(u,v')} \arrow[d, "\varphi^{u,v'}"] \\
X^{u'} \arrow[r, "\chi^{v'u'}"] & X^{v'}
\end{tikzcd}$$

commutative for every $u \in U$. From the uniqueness of this morphism, it follows that $\{\chi^{v'u'} \colon X^{u'} \to X^{v'}\}_{u' \leqslant v'}$ is a U'-direct system. Set $X = \operatorname{colim}_{u' \in U'} X^{u'}$ and write $\{\chi^{u'} \colon X^{u'} \to X\}_{u' \in U'}$ for the canonical morphisms.

Notice that X is the iterated colimit on the left-hand side of the asserted isomorphism. Set $M = \operatorname{colim}_{(u,u') \in U \times U'} M^{(u,u')}$ and write

$$\{\mu^{(u,u')} \colon M^{(u,u')} \to M\}_{(u,u') \in U \times U'}$$

for the canonical morphisms. Next we show that M is isomorphic to X by constructing a pair of mutually inverse morphisms $\alpha \colon M \to X$ and $\beta \colon X \to M$.

For every $(u,u') \in U \times U'$ set $\alpha^{(u,u')} = \chi^{u'}\varphi^{u,u'} : M^{(u,u')} \to X$. For $(u,u') \leqslant (v,v')$ in $U \times U'$ the definitions and commutativity of ($\flat$) yield:

$$\begin{aligned}\alpha^{(v,v')}\mu^{(v,v')(u,u')} &= \chi^{v'}\varphi^{v,v'}\mu^{(v,v')(u,v')}\mu^{(u,v')(u,u')}\\ &= \chi^{v'}\varphi^{u,v'}\mu^{(u,v')(u,u')}\\ &= \chi^{v'}\chi^{v'u'}\varphi^{u,u'}\\ &= \chi^{u'}\varphi^{u,u'}\\ &= \alpha^{(u,u')}\,.\end{aligned}$$

Thus 3.2.5 yields a unique morphism $\alpha\colon M \to X$ with $\alpha\mu^{(u,u')} = \alpha^{(u,u')} = \chi^{u'}\varphi^{u,u'}$ for every $(u,u') \in U \times U'$.

Let $u' \in U'$; for all $u \leqslant v$ in U one has $\mu^{(v,u')}\mu^{(v,u')(u,u')} = \mu^{(u,u')}$, so by 3.2.5 there exists a unique morphism $\beta^{u'}\colon X^{u'} \to M$ with $\beta^{u'}\varphi^{u,u'} = \mu^{(u,u')}$ for every $u \in U$. Let $u' \leqslant v'$ in U' be given; from the definitions and ($\flat$) it follows that

$$\beta^{v'}\chi^{v'u'}\varphi^{u,u'} = \beta^{v'}\varphi^{u,v'}\mu^{(u,v')(u,u')} = \mu^{(u,v')}\mu^{(u,v')(u,u')} = \mu^{(u,u')} = \beta^{u'}\varphi^{u,u'}$$

holds for every $u \in U$. Thus there is an identity $\beta^{v'}\chi^{v'u'} = \beta^{u'}$ of maps from $X^{u'}$ to M, as $X^{u'}$ is a colimit with canonical maps $\varphi^{u,u'}\colon M^{(u,u')} \to X^{u'}$. Another application of 3.2.5 yields a unique morphism $\beta\colon X \to M$ with $\beta\chi^{u'} = \beta^{u'}$ for every $u' \in U'$.

To verify $\alpha\beta = 1^X$, it is enough to prove $\alpha\beta\chi^{u'} = \chi^{u'}$ for every $u' \in U'$, and to that end it suffices to argue that $\alpha\beta\chi^{u'}\varphi^{u,u'} = \chi^{u'}\varphi^{u,u'}$ for every $u \in U$. And indeed,

$$\alpha\beta\chi^{u'}\varphi^{u,u'} = \alpha\beta^{u'}\varphi^{u,u'} = \alpha\mu^{(u,u')} = \alpha^{(u,u')} = \chi^{u'}\varphi^{u,u'}\,.$$

To verify $\beta\alpha = 1^M$, it suffices to prove $\beta\alpha\mu^{(u,u')} = \mu^{(u,u')}$ for every $(u,u') \in U \times U'$. This hold as $\beta\alpha\mu^{(u,u')} = \beta\chi^{u'}\varphi^{u,u'} = \beta^{u'}\varphi^{u,u'} = \mu^{(u,u')}$. $\square$

3.2.14 Corollary. *Let U and U' be sets and $\{M^{(u,u')}\}_{(u,u') \in U \times U'}$ a family of R-complexes. There are isomorphisms,*

$$\coprod_{u' \in U'}\Big(\coprod_{u \in U} M^{(u,u')}\Big) \cong \coprod_{(u,u') \in U \times U'} M^{(u,u')} \cong \coprod_{u \in U}\Big(\coprod_{u' \in U'} M^{(u,u')}\Big)\,.$$

Proof. The isomorphisms follow immediately from 3.2.9 and 3.2.13. $\square$

Functors that Preserve Colimits

3.2.15 Construction. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and $\{\mu^{vu}\colon M^u \to M^v\}_{u \leqslant v}$ a U-direct system in $\mathcal{C}(R)$. The maps $\{\mathrm{F}(\mu^{vu})\colon \mathrm{F}(M^u) \to \mathrm{F}(M^v)\}_{u \leqslant v}$ form a U-direct system in $\mathcal{C}(S)$; write $\lambda^u\colon \mathrm{F}(M^u) \to \operatorname{colim}_{u \in U} \mathrm{F}(M^u)$ for the canonical morphisms, see (3.2.3.1). As $\mathrm{F}(\mu^u) = \mathrm{F}(\mu^v)\,\mathrm{F}(\mu^{vu})$ holds for all $u \leqslant v$ in U, the universal property of colimits 3.2.5 yields a unique morphism

(3.2.15.1) $$\operatorname*{colim}_{u \in U} \mathrm{F}(M^u) \longrightarrow \mathrm{F}\big(\operatorname*{colim}_{u \in U} M^u\big) \quad \text{given by} \quad \lambda^u(x^u) \longmapsto \mathrm{F}(\mu^u)(x^u)$$

for $u \in U$ and $x^u \in \mathrm{F}(M^u)$, that makes the next diagram commutative for all $u \leqslant v$,

$$\begin{tikzcd}
\mathrm{F}(M^v) \arrow[dr, "\lambda^v"] \arrow[drr, bend left, "\mathrm{F}(\mu^v)"] & & \\
& \operatorname*{colim}_{u\in U} \mathrm{F}(M^u) \arrow[r, dotted] & \mathrm{F}\big(\operatorname*{colim}_{u\in U} M^u\big) \\
\mathrm{F}(M^u) \arrow[uu, "\mathrm{F}(\mu^{vu})"] \arrow[ur, "\lambda^u"'] \arrow[urr, bend right, "\mathrm{F}(\mu^u)"'] & &
\end{tikzcd}$$

3.2.16 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *preserves colimits* if the morphism (3.2.15.1) is an isomorphism for every U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$.

Remark. A functor that preserves colimits is also called 'cocontinuous'.

Even if a functor does not preserve (all) colimits in the sense of 3.2.16, it may still preserve certain types of colimits, meaning that the morphism (3.2.15.1) is an isomorphism for a U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ provided that U is of a certain type. For example, a functor that preserves coproducts, see 3.1.8, preserves colimits over discrete sets. Every right exact functor preserves pushouts, that is, colimits formed over the preordered set in 3.2.24. By 3.3.15 the homology functor preserves colimits over filtered sets, but it does not preserve all colimits.

While 3.2.16 is a condition on objects, it carries over to morphisms.

3.2.17. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ a morphism of U-direct systems in $\mathcal{C}(R)$. It is elementary to see that there is a commutative diagram,

$$\begin{tikzcd}
\operatorname*{colim}_{u\in U} \mathrm{F}(M^u) \arrow[r, "\operatorname{colim}_{u\in U}\mathrm{F}(\alpha^u)"] \arrow[d] & \operatorname*{colim}_{u\in U} \mathrm{F}(N^u) \arrow[d] \\
\mathrm{F}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, "\mathrm{F}(\operatorname{colim}_{u\in U}\alpha^u)"] & \mathrm{F}\big(\operatorname*{colim}_{u\in U} N^u\big)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms (3.2.15.1). Thus, if F preserves colimits, then the morphisms $\operatorname{colim}_{u\in U}\mathrm{F}(\alpha^u)$ and $\mathrm{F}(\operatorname{colim}_{u\in U}\alpha^u)$ are isomorphic.

3.2.18. Let $\tau\colon \mathrm{E} \to \mathrm{F}$ be a natural transformation of functors $\mathcal{C}(R) \to \mathcal{C}(S)$ and $\{M^u\}_{u\in U}$ a family of R-complexes. It is straightforward to verify that there is a commutative diagram in $\mathcal{C}(S)$,

$$\begin{tikzcd}
\operatorname*{colim}_{u\in U} \mathrm{E}(M^u) \arrow[r, "\operatorname{colim}_{u\in U}\tau^{M^u}"] \arrow[d] & \operatorname*{colim}_{u\in U} \mathrm{F}(M^u) \arrow[d] \\
\mathrm{E}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, "\tau^{(\operatorname{colim}_{u\in U} M^u)}"] & \mathrm{F}\big(\operatorname*{colim}_{u\in U} M^u\big)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms see (3.2.15.1).

3.2.19 Lemma. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a functor. If* F *is right exact and preserves coproducts, then it preserves colimits.*

Proof. Let $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. Consider the following commutative diagram in $\mathcal{C}(S)$ where the left-hand and middle vertical maps are given by (3.1.8.1) and the right-hand vertical map is given by (3.2.15.1),

$$\begin{tikzcd}
\coprod\limits_{(u,v)\in\nabla(U)} \mathrm{F}(M^{(u,v)}) \arrow[r, "\varDelta_{\mathrm{F}(\mu)}"] \arrow[d] & \coprod\limits_{u\in U} \mathrm{F}(M^u) \arrow[r] \arrow[d] & \operatorname*{colim}\limits_{u\in U} \mathrm{F}(M^u) \arrow[r] \arrow[d] & 0 \\
\mathrm{F}\big(\coprod\limits_{(u,v)\in\nabla(U)} M^{(u,v)}\big) \arrow[r, "\mathrm{F}(\varDelta_\mu)"] & \mathrm{F}\big(\coprod\limits_{u\in U} M^u\big) \arrow[r] & \mathrm{F}\big(\operatorname*{colim}\limits_{u\in U} M^u\big) \arrow[r] & 0\,.
\end{tikzcd}$$

The rows in this diagram are exact by 3.2.3 and right exactness of F. The left-hand and middle vertical maps are isomorphisms by assumption, so it follows from the Five Lemma 2.1.41 that the right-hand vertical map is an isomorphism. □

The next two results show that the shift and cokernel functors preserve colimits.

3.2.20 Proposition. *Let s be an integer and $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ a U-direct system in $\mathcal{C}(R)$. The canonical morphism in $\mathcal{C}(R)$,*

$$\operatorname*{colim}_{u\in U} \Sigma^s M^u \longrightarrow \Sigma^s \operatorname*{colim}_{u\in U} M^u\,,$$

given by $\lambda^u(x^u) \mapsto (\Sigma^s \mu^u)(x^u)$ for $u \in U$ and $x^u \in \Sigma^s M^u$, is an isomorphism.

Proof. The assertion follows immediately from 3.2.19 and 3.1.9. □

3.2.21 Proposition. *For every U-direct system $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ in $\mathcal{C}(R)$, the canonical morphism in $\mathcal{C}(R)$,*

$$\operatorname*{colim}_{u\in U} \mathrm{C}(M^u) \longrightarrow \mathrm{C}\big(\operatorname*{colim}_{u\in U} M^u\big)\,,$$

given by $\lambda^u(x^u) \mapsto \mathrm{C}(\mu^u)(x^u)$ for $u \in U$ and $x^u \in \mathrm{C}(M^u)$, is an isomorphism.

Proof. The assertion follows immediately from 3.2.19, 2.2.16, and 3.1.10(c). □

The next results show that the tensor product functor 2.4.9 preserves colimits.

3.2.22 Proposition. *Let N be an R-complex and $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ a U-direct system in $\mathcal{C}(R^{\mathrm{o}})$. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$(3.2.22.1) \qquad \operatorname*{colim}_{u\in U} (M^u \otimes_R N) \longrightarrow \big(\operatorname*{colim}_{u\in U} M^u\big) \otimes_R N\,,$$

given by $\lambda^u(t^u) \mapsto (\mu^u \otimes_R N)(t^u)$ for $u \in U$ and $t^u \in M^u \otimes_R N$, is an isomorphism.

Proof. The assertion follows immediately from 3.2.19, 2.4.9, and 3.1.12. □

3.2.23 Proposition. *Let M be an R^{o}-complex and $\{\nu^{vu} \colon N^u \to N^v\}_{u\in U}$ a U-direct system in $\mathcal{C}(R)$. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$(3.2.23.1) \qquad \operatorname*{colim}_{u\in U} (M \otimes_R N^u) \longrightarrow M \otimes_R \big(\operatorname*{colim}_{u\in U} N^u\big)\,,$$

given by $\lambda^u(t^u) \mapsto (M \otimes_R \nu^u)(t^u)$ for $u \in U$ and $t^u \in M \otimes_R N^u$, is an isomorphism.

PROOF. The assertion follows immediately from 3.2.19, 2.4.9, and 3.1.13. $\square$

PUSHOUTS

Simple non-trivial colimits arise from three term direct systems $M \leftarrow X \rightarrow N$.

3.2.24 Construction. Let $U = \{u, v, w\}$ be a set, preordered as follows $v \geqslant u \leqslant w$. Given a diagram $M \xleftarrow{\alpha} X \xrightarrow{\beta} N$ in $\mathcal{C}(R)$, set

$$M^v = M\,, \quad M^u = X\,, \quad M^w = N\,,$$
$$\mu^{vv} = 1^M\,, \quad \mu^{vu} = \alpha\,, \quad \mu^{uu} = 1^X\,, \quad \mu^{wu} = \beta\,, \quad \text{and} \quad \mu^{ww} = 1^N\,.$$

This defines a U-direct system in $\mathcal{C}(R)$. It is straightforward to verify that the colimit of this system is the cokernel of the morphism $(-\alpha\ \ \beta)\colon X \rightarrow M \oplus N$.

3.2.25 Definition. For a diagram $M \xleftarrow{\alpha} X \xrightarrow{\beta} N$ in $\mathcal{C}(R)$, the colimit of the U-direct system in 3.2.24 is called the *pushout* of (α, β) and denoted $M \sqcup_X N$. Let

$$\alpha' : N \longrightarrow M \sqcup_X N \qquad \text{and} \qquad \beta' : M \longrightarrow M \sqcup_X N$$

be the canonical morphisms from (3.2.3.1); they are given by $n \mapsto [(0,n)]_{\operatorname{Im}(-\alpha\ \beta)}$ and $m \mapsto [(m,0)]_{\operatorname{Im}(-\alpha\ \beta)}$.

REMARK. As for the colimit, the notation for the pushout suppresses the morphisms. Other names for the pushout are 'fibered coproduct', 'fibered sum', 'Cocartesian square', and 'amalgamated product'.

3.2.26. Given morphisms $\alpha\colon X \rightarrow M$ and $\beta\colon X \rightarrow N$ in $\mathcal{C}(R)$, the pushouts of (α, β) and (β, α) are isomorphic via the map that comes from the canonical isomorphism $M \oplus N \cong N \oplus M$.

3.2.27. Adopt the notation from 3.2.25. Given a diagram $M \xrightarrow{\beta''} Y \xleftarrow{\alpha''} N$ in $\mathcal{C}(R)$ with $\beta''\alpha = \alpha''\beta$, it follows from 3.2.5 that the assignment

$$[(m,n)]_{\operatorname{Im}(-\alpha\ \beta)} = \alpha'(n) + \beta'(m) \longmapsto \alpha''(n) + \beta''(m)$$

defines the unique morphism $M \sqcup_X N \rightarrow Y$ that makes the next diagram commute,

$$\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & N \arrow[d, "\alpha'"] \arrow[ddr, bend left, "\alpha''"] & \\
M \arrow[r, "\beta'"] \arrow[drr, bend right, "\beta''"'] & M \sqcup_X N \arrow[dr, dotted] & \\
& & Y\,.
\end{tikzcd}$$

3.2.28 Theorem. *Adopt the notation from* 3.2.25. *There is a commutative diagram in* $\mathcal{C}(R)$ *with exact rows and columns,*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & \operatorname{Ker}\alpha \arrow[r, "\bar{\beta}"] \arrow[d] & \operatorname{Ker}\alpha' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d, "\bar{\alpha}"] & X \arrow[r, "\beta"] \arrow[d, "\alpha"] & N \arrow[r] \arrow[d, "\alpha'"] & \operatorname{Coker}\beta \arrow[r] \arrow[d, "\bar{\bar{\alpha}}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta' \arrow[r] \arrow[d] & M \arrow[r, "\beta'"] \arrow[d] & M \sqcup_X N \arrow[r] \arrow[d] & \operatorname{Coker}\beta' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Coker}\alpha \arrow[r, "\bar{\bar{\beta}}"] \arrow[d] & \operatorname{Coker}\alpha' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0\,, &
\end{tikzcd}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are the induced morphisms on kernels, and $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are the induced morphisms on cokernels. In particular, $\bar{\alpha}$ and $\bar{\beta}$ are surjective, $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are isomorphisms, and the following assertions hold.

(a) *If α is injective, then α' is injective.*
(b) *α is surjective if and only if α' is surjective.*
(c) *If β is injective, then β' is injective.*
(d) *β is surjective if and only if β' is surjective.*

Proof. With $Z = \operatorname{Ker}\bar{\alpha}$ the sequence $0 \to Z \to \operatorname{Ker}\alpha \to \operatorname{Ker}\alpha'$ is exact by 2.1.42.

For $m \in M$ one has $\beta'(m) = [(m,0)]_{\operatorname{Im}(-\alpha\ \beta)}$; so if m is in $\operatorname{Ker}\beta'$, then there exists $x \in X$ with $(m,0) = (-\alpha(x), \beta(x))$. Thus $-x$ belongs to $\operatorname{Ker}\beta$ with $\alpha(-x) = m$, which shows that $\bar{\alpha}$ is surjective. By symmetry, see 3.2.26, $\bar{\beta}$ is surjective.

For $n \in N$ one has $\bar{\bar{\alpha}}([n]_{\operatorname{Im}\beta}) = [\alpha'(n)]_{\operatorname{Im}\beta'}$. Thus, if $[n]_{\operatorname{Im}\beta}$ belongs to $\operatorname{Ker}\bar{\bar{\alpha}}$, then $\alpha'(n) = \beta'(m)$ holds for some $m \in M$. Consequently, $[(-m,n)]_{\operatorname{Im}(-\alpha\ \beta)} = \alpha'(n) - \beta'(m) = 0$ and hence also $n \in \operatorname{Im}\beta$. This proves that $\bar{\bar{\alpha}}$ is injective. Notice that every element $z \in M \sqcup_X N$ has the form $z = \alpha'(n) + \beta'(m)$ for some $m \in M$ and $n \in N$. It follows that $[z]_{\operatorname{Im}\beta'} = [\alpha'(n)]_{\operatorname{Im}\beta'} = \bar{\bar{\alpha}}([n]_{\operatorname{Im}\beta})$, so $\bar{\bar{\alpha}}$ is surjective and hence an isomorphism. By symmetry, $\bar{\bar{\beta}}$ is an isomorphim as well.

The assertions (a)–(d) are direct consequences of the established diagram. □

The following example shows that colimits are not left exact.

3.2.29 Example. The embeddings below form an injective morphism of pushout diagrams of $\mathbb{Z}$-modules; that is, an injective morphism of U-direct systems as in 3.2.10, where U is the preordered set described in 3.2.24.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[rr] \arrow[dd] & & 0 \\
2\mathbb{Z} \arrow[ur, tail] \arrow[rr, crossing over] \arrow[dd] & & 0 \arrow[ur, tail] & \\
 & \mathbb{Z} & & \\
\mathbb{Z} \arrow[ur, tail] & & &
\end{tikzcd}$$

The colimit $\mathbb{Z} \sqcup_{2\mathbb{Z}} 0 \to \mathbb{Z} \sqcup_{\mathbb{Z}} 0$ of this morphism is $\mathbb{Z}/2\mathbb{Z} \to 0$, which is not injective.

EXERCISES

E 3.2.1 Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. Let $\{\widetilde{\mu}^u\colon M^u \to C\}_{u\in U}$ be a family of morphisms that satisfy the following conditions. (1) One has $\widetilde{\mu}^u = \widetilde{\mu}^v\mu^{vu}$ for all $u \leqslant v$. (2) For every family $\{\alpha^u\colon M^u \to N\}_{u\in U}$ of morphisms with $\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$ there exists a unique morphism $\alpha\colon C \to N$ with $\alpha\widetilde{\mu}^u = \alpha^u$ for all $u \in U$. Show that there is an isomorphism $\varphi\colon \operatorname{colim}_{u\in U} M^u \to C$ with $\varphi\mu^u = \widetilde{\mu}^u$ for every $u \in U$. Conclude that the universal property determines the colimit uniquely up to isomorphism.

E 3.2.2 (Cf. 3.2.7) Show that the colimit in $\mathcal{C}(R)$ of a direct system of morphisms of graded R-modules is a graded R-module. Conclude, in particular, that $\mathcal{M}_{\mathrm{gr}}(R)$ has colimits.

E 3.2.3 (Cf. 3.2.7) Show that the colimit in $\mathcal{C}(R)$ of a direct system of homomorphisms of R-modules is an R-module. Conclude, in particular, that the category $\mathcal{M}(R)$ has colimits.

E 3.2.4 Fix a preordered set U. Show that U-direct systems in $\mathcal{C}(R)$ and their morphisms form an Abelian category and that the colimit is a right exact functor from this category to $\mathcal{C}(R)$.

E 3.2.5 Generalize the result in E 3.1.6 by showing that every functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ that has a right adjoint preserves colimits.

E 3.2.6 (Cf. 3.2.24) Verify the isomorphism $\operatorname{colim}_{u\in U} M^u \cong \operatorname{Coker}(-\alpha\ \ \beta)$ in 3.2.24.

E 3.2.7 (a) Consider the diagram in 3.2.28 in the case where $\mathfrak{a}$ and $\mathfrak{b}$ are left ideals in R and $\alpha\colon R \twoheadrightarrow R/\mathfrak{a}$ and $\beta\colon R \twoheadrightarrow R/\mathfrak{b}$ are the canonical homomorphisms. Show that $Z \neq 0$.
(b) In the following two diagrams, the solid parts are given. Show that they can be completed to commutative diagrams with exact rows and columns, as depicted.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dotted] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, dotted] & D \arrow[r] \arrow[d, equal, dotted] & 0 \\
0 \arrow[r, dotted] & M \arrow[r, dotted] \arrow[d] & M \sqcup_X N \arrow[r, dotted] \arrow[d, dotted] & D \arrow[r, dotted] & 0 \\
 & C \arrow[r, equal, dotted] \arrow[d] & C \arrow[d, dotted] & & \\
 & 0 & 0 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d, dotted] & \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal, dotted] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r, dotted] & L \arrow[r, dotted] & M \arrow[r, dotted] \arrow[d] & M \sqcup_X N \arrow[r, dotted] \arrow[d, dotted] & 0 \\
 & & C \arrow[r, equal, dotted] \arrow[d] & C \arrow[d, dotted] & \\
 & & 0 & 0\,. &
\end{tikzcd}
$$

E 3.2.8 (Cf. 3.2.17) Verify that the diagram in 3.2.17 is commutative.

E 3.2.9 (Cf. 3.2.18) Verify that the diagram in 3.2.18 is commutative.

3.3 Filtered Colimits

SYNOPSIS. Filtered colimit; functor that preserves filtered colimits; well-ordered colimit; telescope.

Additional properties of the index set U, beyond being preordered, can translate into additional properties of the colimit of a U-direct system. We consider the cases where the index set is filtered, well-ordered, or simply $\mathbb{Z}$.

3.3.1 Definition. A colimit of a U-direct system is called *filtered* if the preordered set $(U, \leqslant)$ is filtered.

REMARK. Another name used for a filtered colimit is ‘direct limit’.

3.3.2 Lemma. *Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. If U is filtered, then the following assertions hold.*

(a) *For every $m \in \operatorname{colim}_{u\in U} M^u$ there exist $v \in U$ and $m^v \in M^v$ with $\mu^v(m^v) = m$.*
(b) *If $m^v \in M^v$ satisfies $\mu^v(m^v) = 0$, then one has $\mu^{wv}(m^v) = 0$ for some $w \in U$ with $v \leqslant w$.*
(c) *If $m^u \in M^u$ and $m^v \in M^v$ satisfy $\mu^u(m^u) = \mu^v(m^v)$, then there exists $w \in U$ with $u \leqslant w$ and $v \leqslant w$ such that $\mu^{wu}(m^u) = \mu^{wv}(m^v)$ holds.*

Proof. (a): By 3.2.3 every $m \in \operatorname{colim}_{u\in U} M^u$ has the form $m = \sum_{u\in U} \mu^u(m^u)$ where $m^u \neq 0$ for only finitely many $u \in U$. As U is filtered there exists $v \in U$ such that $v \geqslant u$ holds for all $u \in U$ with $m^u \neq 0$. Now $m = \sum_{u\leqslant v} \mu^v\mu^{vu}(m^u) = \mu^v(\sum_{u\leqslant v} \mu^{vu}(m^u))$.

(b): If $\mu^v(m^v) = 0$ holds in $\operatorname{colim}_{u\in U} M^u$, then by 3.2.3 there is an equality

$$(\diamond) \qquad \varepsilon^v(m^v) \;=\; \sum_{(t,u)\in\nabla(U)} \varepsilon^t(m^{(t,u)}) - \varepsilon^u\mu^{ut}(m^{(t,u)})$$

in $\coprod_{s\in U} M^s$ where $m^{(t,u)} \neq 0$ holds for only finitely many $(t,u) \in \nabla(U)$. As U is filtered there exists $w \in U$ such that $w \geqslant v$ and $w \geqslant u$ for all $u \in U$ satisfying $m^{(t,u)} \neq 0$ (for some t). Now apply the morphism $\coprod_{s\in U} M^s \to M^w$, given by $\varepsilon^s(m^s) \mapsto \mu^{ws}(m^s)$ for $s \leqslant w$ and $\varepsilon^s(m^s) \mapsto 0$ otherwise, to both sides in $(\diamond)$ to get

$$\mu^{wv}(m^v) \;=\; \sum_{(t,u)\in\nabla(U)} \mu^{wt}(m^{(t,u)}) - \mu^{wu}\mu^{ut}(m^{(t,u)}) \;=\; 0\,.$$

(c): Choose $t \in U$ with $t \geqslant u, v$; now one has

$$\mu^t(\mu^{tu}(m^u) - \mu^{tv}(m^v)) \;=\; \mu^u(m^u) - \mu^v(m^v) \;=\; 0\,.$$

By part (b) there exists $w \in U$ with $w \geqslant t$ and

$$0 \;=\; \mu^{wt}(\mu^{tu}(m^u) - \mu^{tv}(m^v)) \;=\; \mu^{wu}(m^u) - \mu^{wv}(m^v)\,. \qquad \square$$

3.3.3 Example. Let $(U, \leqslant)$ be a preordered filtered set and $\{M^u\}_{u\in U}$ a family of subcomplexes of an R-complex M with $M^u \subseteq M^v$ for $u \leqslant v$ in U. There is a U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ where μ^{vu} is the embedding, and one has

$$\operatorname*{colim}_{u\in U} M^u \;\cong\; \bigcup_{u\in U} M^u\,.$$

Indeed, for u in U let $\iota^u\colon M^u \to \bigcup_{u\in U} M^u$ be the embedding. One has $\iota^v\mu^{vu} = \iota^u$ for $u \leqslant v$ in U, so 3.2.5 yields a unique morphism $\iota\colon \operatorname{colim}_{u\in U} M^u \to \bigcup_{u\in U} M^u$ with $\iota\mu^u = \iota^u$ for $u \in U$; it is surjective by 3.2.6. To see that ι is injective note that by 3.3.2(a) every element $x \in \operatorname{colim}_{u\in U} M^u$ has the form $x = \mu^u(m)$ for some $u \in U$ and $m \in M^u$. Now one has $\iota(x) = \iota\mu^u(m) = \iota^u(m)$, so $\iota(x) = 0$ implies $m = 0$ and hence $x = \mu^u(m) = 0$.

Every complex is a filtered colimit of bounded above subcomplexes.

3.3.4 Example. Let M be an R-complex and consider the $\mathbb{Z}$-direct system of subcomplexes $\{\mu^{vu}\colon M_{\leqslant u} \rightarrowtail M_{\leqslant v}\}_{u\leqslant v}$ where $M_{\leqslant u}$ is the hard truncation, see 2.5.20, and μ^{vu} is the embedding. By 3.3.3 one has

$$\operatorname*{colim}_{u\in\mathbb{Z}} M_{\leqslant u} \;\cong\; \bigcup_{u\in\mathbb{Z}} M_{\leqslant u} \;=\; M\,.$$

A classic application of colimits is to write an arbitrary module as a filtered colimit of finitely generated (sub)modules. A version of this for complexes is given in the next example. Stronger results can be found in 3.3.21 and 3.3.22.

3.3.5 Example. Let M be an R-complex and U the set of all bounded and degreewise finitely generated subcomplexes of M. The set $(U, \subseteq)$ is partially ordered, and for subcomplexes M' and M'' in U also the subcomplex $M' + M''$ is in U, so U is filtered. For the U-direct system $\{M' \rightarrowtail M''\}_{M'\subseteq M''}$ of embeddings, 3.3.3 yields an isomorphism,

$$\operatorname*{colim}_{M'\in U} M' \;\cong\; \bigcup_{M'\in U} M' \;=\; M\,.$$

The equality holds as each homogeneous element m' in M belongs to some $M' \in U$; for example, with $d = |m'|$ one can take M' to be the subcomplex

$$M' \;=\; 0 \longrightarrow R\langle m'\rangle \xrightarrow{\partial_d^M} R\langle \partial_d^M(m')\rangle \longrightarrow 0\,.$$

Every subsequence of a convergent sequence of numbers is convergent with the same limit; colimits of complexes behave similarly.

3.3.6 Definition. Let $(U, \leqslant)$ be a preordered filtered set. A subset V of U is called *cofinal* if for every $u \in U$ there is a $v \in V$ with $u \leqslant v$.

Note that if U and V are as in the definition, then V is filtered.

3.3.7 Proposition. *Let $(U, \leqslant)$ be a preordered filtered set and V a cofinal subset of U. For every U-direct system $\{\mu^{vu} \colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$ there is an isomorphism* $\operatorname{colim}_{v\in V} M^v \cong \operatorname{colim}_{u\in U} M^u$.

Proof. For $u \in U$ denote by μ^u is the canonical morphism $M^u \to \operatorname{colim}_{u\in U} M^u$, and for $v \in V$ denote by $\widetilde{\mu}^v$ the canonical morphism $M^v \to \operatorname{colim}_{v\in V} M^v$. For $v \leqslant w$ in V one has $\mu^v = \mu^w\mu^{wv}$, so by the universal property of colimits there is a morphism $\mu\colon \operatorname{colim}_{v\in V} M^v \to \operatorname{colim}_{u\in U} M^u$. For every m in $\operatorname{colim}_{u\in U} M^u$ there is by 3.3.2 a u in U and an element m^u in M^u with $\mu^u(m^u) = m$. As V is cofinal in U, there is a $v \in V$ with $v \geqslant u$, such that $m = \mu^v\mu^{vu}(m^u)$ holds. Thus μ is surjective; see 3.2.6.

Let m be an element in $\operatorname{Ker}\mu$. As V is filtered, there exists a $v \in V$ and an element m^v in M^v with $\widetilde{\mu}^v(m^v) = m$. Now one has $0 = \mu(m) = \mu\widetilde{\mu}^v(m^v) = \mu^v(m^v)$. It follows from 3.3.2 that there is a w in U with $\mu^{wv}(m^v) = 0$. As V is cofinal in U one may assume that w is in V, and then $m = \widetilde{\mu}^v(m^v) = \widetilde{\mu}^w\mu^{wv}(m^v) = 0$ holds. $\square$

3.3.8 Proposition. *Let $\{\mu^{vu} \colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$ and $\{\alpha^u \colon M^u \to N\}_{u\in U}$ a family of morphisms with $\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$. If U is filtered and the subset* $\{u \in U \mid \alpha^u \text{ is injective}\}$ *is cofinal, then the morphism* $\alpha\colon \operatorname{colim}_{u\in U} M^u \to N$ *from* 3.2.5 *is injective.*

Proof. Let m be an element in $\operatorname{Ker}\alpha$. By 3.3.2 there is a $u \in U$ and an $m^u \in M^u$ with $\mu^u(m^u) = m$; by assumption there is a $v \geqslant u$ such that α^v is injective. Now one has $m = \mu^v\mu^{vu}(m^u)$ and $0 = \alpha(m) = \alpha\mu^v\mu^{vu}(m^u) = \alpha^v\mu^{vu}(m^u)$, so $m = 0$. $\square$

A coproduct is a special case of a filtered colimit.

3.3.9 Example. Let $\{M^u\}_{u\in U}$ be a family of R-complexes and Υ the set of all finite subsets of U; it is partially ordered under inclusion and filtered. For the Υ-direct system $\{\mu^{WV}\colon \bigoplus_{u\in V} M^u \to \bigoplus_{u\in W} M^u\}_{V\subseteq W}$ of subcomplexes of $\coprod_{u\in U} M^u$, where μ^{WV} is the embedding, 3.3.3 yields an isomorphism,

$$\operatorname*{colim}_{V\in\Upsilon}\Big(\bigoplus_{u\in V} M^u\Big) \cong \bigcup_{V\in\Upsilon}\Big(\bigoplus_{u\in V} M^u\Big) = \coprod_{u\in U} M^u\,.$$

Thus, a coproduct is a filtered colimit.

The next result shows that filtered colimits are exact.

3.3.10 Proposition. *Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ and $\{\beta^u\colon N^u \to X^u\}_{u\in U}$ be morphisms of U-direct systems in $\mathcal{C}(R)$. If U is filtered and the sequence*

$$0 \longrightarrow M^u \xrightarrow{\;\alpha^u\;} N^u \xrightarrow{\;\beta^u\;} X^u \longrightarrow 0$$

is exact for every $u \in U$, then the following sequence is exact,

$$0 \longrightarrow \operatorname*{colim}_{u\in U} M^u \xrightarrow{\operatorname{colim}_{u\in U}\alpha^u} \operatorname*{colim}_{u\in U} N^u \xrightarrow{\operatorname{colim}_{u\in U}\beta^u} \operatorname*{colim}_{u\in U} X^u \longrightarrow 0\,.$$

Proof. By 3.2.12 it is sufficient to prove that $\alpha = \operatorname{colim}_{u\in U}\alpha^u$ is injective. Write $\mu^{uv}\colon M^u \to M^v$ and $\nu^{uv}\colon N^u \to N^v$ for the morphisms in the direct systems. Let $m \in \operatorname{Ker}\alpha$ and choose by 3.3.2 a v in U and an element m^v in M^v with $\mu^v(m^v) = m$; now one has $0 = \alpha\mu^v(m^v) = \nu^v\alpha^v(m^v)$. By 3.3.2 there is a $w \in U$ with $w \geqslant v$ and $0 = \nu^{wv}\alpha^v(m^v) = \alpha^w\mu^{wv}(m^v)$. Since α^w is injective, one has $\mu^{wv}(m^v) = 0$ and, therefore, $m = \mu^v(m^v) = \mu^w\mu^{wv}(m^v) = 0$. □

3.3.11 Proposition. *Let $(U,\leqslant)$ be a preordered filtered set and $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ and $\{\beta^u\colon X^u \to Y^u\}_{u\in U}$ be morphisms of U-direct systems in $\mathcal{C}(R)$. The morphism*

$$\operatorname*{colim}_{u\in U}(\alpha^u \otimes_R \beta^u)\colon\; \operatorname*{colim}_{u\in U}(M^u \otimes_R X^u) \longrightarrow \operatorname*{colim}_{u\in U}(N^u \otimes_R Y^u)$$

is isomorphic to

$$\big(\operatorname{colim}_{u\in U} M^u\big) \otimes_R \big(\operatorname{colim}_{v\in U} X^v\big) \xrightarrow{(\operatorname{colim}_{u\in U}\alpha^u)\otimes_R(\operatorname{colim}_{v\in U}\beta^v)} \big(\operatorname{colim}_{u\in U} N^u\big) \otimes_R \big(\operatorname{colim}_{v\in U} Y^v\big)\,.$$

Proof. Let $\Delta(U) \subseteq U\times U$ be the diagonal, that is, $\Delta(U) = \{(u,u) \mid u \in U\}$. Clearly, the first morphism $\operatorname{colim}_{u\in U}(\alpha^u \otimes_R \beta^u)$ is the same as $\operatorname{colim}_{(u,v)\in\Delta(U)}(\alpha^u \otimes_R \beta^v)$. For the second morphism there are by 3.2.22, 3.2.23, and 3.2.13 identifications,

$$\begin{aligned}
\big(\operatorname*{colim}_{u\in U}\alpha^u\big) \otimes_R \big(\operatorname*{colim}_{v\in U}\beta^v\big) &\cong \operatorname*{colim}_{u\in U}\big(\alpha^u \otimes_R \big(\operatorname*{colim}_{v\in U}\beta^v\big)\big)\\
&\cong \operatorname*{colim}_{u\in U}\big(\operatorname*{colim}_{v\in U}(\alpha^u \otimes_R \beta^v)\big)\\
&\cong \operatorname*{colim}_{(u,v)\in U\times U}(\alpha^u \otimes_R \beta^v)\,.
\end{aligned}$$

Thus, to finish the proof it must be argued that the maps $\operatorname{colim}_{(u,v)\in\Delta(U)}(\alpha^u \otimes_R \beta^v)$ and $\operatorname{colim}_{(u,v)\in U\times U}(\alpha^u \otimes_R \beta^v)$ are isomorphic. To this end it suffices by 3.3.7 to show that the diagonal $\Delta(U)$ is cofinal in the product $U \times U$ equipped with the product order. This is straightforward to verify. Indeed, given $(u,v) \in U \times U$ there exists, as U is filtered, $w \in U$ with $u \leqslant w$ and $v \leqslant w$. Now (w,w) is in $\Delta(U)$ and one has $(u,v) \leqslant (w,w)$ in $U \times U$. □

Functors that Preserve Filtered Colimits

The following definition shoul be compared to 3.2.16.

3.3.12 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ said to *preserve filtered colimits* if the map (3.2.15.1) is an isomorphism for every U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$ with U filtered.

3.3.13 Definition. Let $\mathrm{E}, \mathrm{F}, \mathrm{G}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be functors and let $\sigma\colon \mathrm{E} \to \mathrm{F}$ and $\tau\colon \mathrm{F} \to \mathrm{G}$ be natural transformations. The sequence of functors $\mathrm{E} \xrightarrow{\sigma} \mathrm{F} \xrightarrow{\tau} \mathrm{G}$ is said to be *exact* if the sequence

$$\mathrm{E}(M) \xrightarrow{\sigma^M} \mathrm{F}(M) \xrightarrow{\tau^M} \mathrm{G}(M)$$

in $\mathcal{C}(S)$ is exact for every R-complex M.

Remark. Functors $\mathcal{C}(R) \to \mathcal{C}(S)$ and natural transformations among them constitute an Abelian category, and thus one can talk about exact sequences in this category in the sense of 1.1.43. One can verify that exactness in this sense agrees with the definition above.

3.3.14 Lemma. *Let* E, F, *and* G *be functors from* $\mathcal{C}(R)$ *to* $\mathcal{C}(S)$.

(a) *If the sequence* $0 \to \mathrm{E} \to \mathrm{F} \to \mathrm{G}$ *is exact and* F *and* G *preserve filtered colimits, then* E *preserves filtered colimits.*

(b) *If the sequence* $0 \to \mathrm{E} \to \mathrm{F} \to \mathrm{G} \to 0$ *is exact and* E *and* G *preserve filtered colimits, then* F *preserves filtered colimits.*

(c) *If the sequence* $\mathrm{E} \to \mathrm{F} \to \mathrm{G} \to 0$ *is exact and* E *and* F *preserve colimits, then* G *preserves filtered colimits.*

Proof. (a): Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$ with U filtered. Write $\sigma\colon \mathrm{E} \to \mathrm{F}$ and $\tau\colon \mathrm{F} \to \mathrm{G}$ for the given natural transformations and notice that $\{\sigma^{M^u}\colon \mathrm{E}(M^u) \to \mathrm{F}(M^u)\}_{u\in U}$ and $\{\tau^{M^u}\colon \mathrm{F}(M^u) \to \mathrm{G}(M^u)\}_{u\in U}$ are morphisms of U-direct systems in $\mathcal{C}(S)$. In the commutative diagram below, the vertical morphisms are given by (3.2.15.1) and the rows are exact by assumption and 3.3.10.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname*{colim}_{u\in U} \mathrm{E}(M^u) \arrow[r, "\operatorname{colim}_{u\in U}\sigma^{M^u}"] \arrow[d] & \operatorname*{colim}_{u\in U} \mathrm{F}(M^u) \arrow[r, "\operatorname{colim}_{u\in U}\tau^{M^u}"] \arrow[d] & \operatorname*{colim}_{u\in U} \mathrm{G}(M^u) \arrow[d] \\
0 \arrow[r] & \mathrm{E}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, "\sigma^{(\operatorname{colim}_{u\in U} M^u)}"] & \mathrm{F}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, "\tau^{(\operatorname{colim}_{u\in U} M^u)}"] & \mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) .
\end{tikzcd}$$

The assertion now follows from the Five Lemma 2.1.41.

(b): An argument similar to the proof of part (a) applies.

(c): An argument similar to the proof of part (a) applies, but in this case only right exactness of colimits is needed, so one appeals to 3.2.12 in place of 3.3.10. □

The next result shows that the cycle, the boundary, the cokernel, and the homology functors from 2.2.7 preserve filtered colimits.

3.3.15 Proposition. *Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. If U is filtered, then the following four canonical morphisms in $\mathcal{C}(R)$ are isomorphisms*:

(a) $\operatorname*{colim}_{u\in U} \mathrm{Z}(M^u) \longrightarrow \mathrm{Z}\big(\operatorname*{colim}_{u\in U} M^u\big)$ *given by* $\lambda^u(z^u) \longmapsto \mathrm{Z}(\mu^u)(z^u)$.

(b) $\operatorname*{colim}_{u\in U} \mathrm{B}(M^u) \longrightarrow \mathrm{B}\big(\operatorname*{colim}_{u\in U} M^u\big)$ *given by* $\lambda^u(b^u) \longmapsto \mathrm{B}(\mu^u)(b^u)$.

(c) $\operatorname*{colim}_{u\in U} \mathrm{C}(M^u) \longrightarrow \mathrm{C}\big(\operatorname*{colim}_{u\in U} M^u\big)$ *given by* $\lambda^u(c^u) \longmapsto \mathrm{C}(\mu^u)(c^u)$.

(d) $\operatorname*{colim}_{u\in U} \mathrm{H}(M^u) \longrightarrow \mathrm{H}\big(\operatorname*{colim}_{u\in U} M^u\big)$ *given by* $\lambda^u(h^u) \longmapsto \mathrm{H}(\mu^u)(h^u)$.

Proof. It follows from 3.2.21 that the cokernel functor, C, preserves (filtered) colimits, and hence the leftmost map in (c) is an isomorphism. By 2.2.12(b) there is an exact sequence $0 \to \mathrm{B} \to \mathrm{Id}_{\mathcal{C}(R)} \to \mathrm{C} \to 0$ of endofunctors on $\mathcal{C}(R)$, so by 3.3.14(a) the boundary functor, B, preserves filtered colimits. By 2.2.12(a) and 2.2.12(d) there are also exact sequences of functors $0 \to \mathrm{Z} \to \mathrm{Id}_{\mathcal{C}(R)} \to \Sigma\mathrm{B} \to 0$, and $0 \to \mathrm{H} \to \mathrm{C} \to \Sigma\mathrm{B} \to 0$; thus 3.2.20 and 3.3.14(a) imply that the cycle functor, Z, and the homology functor, H, preserve filtered colimits. □

3.3.16 Corollary. *Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. If U is filtered, then there are inequalities,*

$$\sup\big(\operatorname*{colim}_{u\in U} M^u\big) \leqslant \sup_{u\in U}\{\sup M^u\} \quad \textit{and} \quad \inf\big(\operatorname*{colim}_{u\in U} M^u\big) \geqslant \inf_{u\in U}\{\inf M^u\}\ ;$$

in particular, if each complex M^u is acyclic, then $\operatorname{colim}_{u\in U} M^u$ *is acyclic.*

Proof. By 3.3.15(d) there is for every $v \in \mathbb{Z}$ an isomorphism $\mathrm{H}_v(\operatorname{colim}_{u\in U} M^u) \cong \operatorname{colim}_{u\in U} \mathrm{H}_v(M^u)$. Thus, if $\mathrm{H}_v(\operatorname{colim}_{u\in U} M^u) \neq 0$ holds, then one has $\mathrm{H}_v(M^u) \neq 0$ for some $u \in U$ and, therefore,

$$v \leqslant \sup\{\sup M^u \mid u \in U\} \quad \text{and} \quad v \geqslant \inf\{\inf M^u \mid u \in U\}\ . \qquad \square$$

Under conditions on M the functor $\mathrm{Hom}_R(M,-)$ preserves filtered colimits.

3.3.17 Proposition. *Let M be an R-complex and $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R)$. If U is filtered and M is bounded and degreewise finitely presented, then the canonical morphism in $\mathcal{C}(\Bbbk)$,*

(3.3.17.1)
$$\operatorname*{colim}_{u\in U} \mathrm{Hom}_R(M,N^u) \longrightarrow \mathrm{Hom}_R\big(M,\operatorname*{colim}_{u\in U} N^u\big)\ ,$$

given by $\lambda^u(\vartheta^u) \mapsto \mathrm{Hom}_R(M,\nu^u)(\vartheta^u) = \nu^u\vartheta^u$ for $u \in U$ and $\vartheta^u \in \mathrm{Hom}_R(M,N^u)$, is an isomorphism.

Proof. Let L be a bounded complex of finitely generated projective R-modules. The functor $\operatorname{Hom}_R(L,-)$ is exact by 2.3.18 and it preserves coproducts by 3.1.33; hence it also preserves (filtered) colimits by 3.2.19. If M is bounded and degreewise finitely presented, then by 2.5.31 there exists an exact sequence $L' \to L \to M \to 0$ where L and L' are bounded complexes of finitely generated free R-modules. Hence there is an exact sequence $0 \to \operatorname{Hom}_R(M,-) \to \operatorname{Hom}_R(L,-) \to \operatorname{Hom}_R(L',-)$ of functors from $\mathcal{C}(R)$ to $\mathcal{C}(\Bbbk)$, and the assertion now follows from 3.3.14(a). $\square$

The requirement in 3.3.17 that M be bounded and degreewise finitely presented cannot be relaxed; see 3.3.23.

Bounded and Degreewise Finitely Presented Complexes

The next construction is the first step towards a theorem that says that every complex is a filtered colimit of bounded and degreewise finitely presented complexes.

3.3.18 Construction. Let M be an R-complex and Δ a set of R-complexes. Denote by Ξ the set of all morphisms $\xi\colon D_\xi \to M$ whose domain D_ξ belongs to Δ. Set $X = \coprod_{\xi\in\Xi} D_\xi$ and let $\pi\colon X \to M$ be the unique morphism whose composite with the injection $D_\xi \rightarrowtail X$ equals ξ for every $\xi \in \Xi$; see 3.1.2. Set $\widehat{X} = X^{(\mathbb{N})}$ and let $\hat{\pi}\colon \widehat{X} \to M$ be the morphism whose composite with every injection $X \rightarrowtail \widehat{X}$ is π.

A family $K = \{K_n\}_{n\in\mathbb{N}}$ of finite subsets $K_n \subseteq \Xi$, such that K_n is non-empty for only finitely many $n \in \mathbb{N}$, is called a *string*. To a string K one associates the following subcomplex of $\widehat{X}$,

$$\widehat{X}_K = \coprod_{n\in\mathbb{N}} \Big(\bigoplus_{\xi\in K^n} D_\xi \Big) .$$

Let U be the set of pairs (K,Y) where K is a string and Y is a degreewise finitely generated subcomplex of $\widehat{X}_K \cap \operatorname{Ker}\hat{\pi}$. For elements (K,Y) and (K',Y') in U declare $(K,Y) \leqslant (K',Y')$ if one has $K_n \subseteq K'_n$ for all $n \in \mathbb{N}$ and $Y \subseteq Y'$. Evidently, the set $(U,\leqslant)$ is partially ordered. It is also filtered as $(K\cup K', Y+Y')$, where $K \cup K'$ is the string $\{K_n \cup K'_n\}_{n\in\mathbb{N}}$, dominates both (K,Y) and (K',Y'). For every element (K,Y) in U set $M^{(K,Y)} = \widehat{X}_K/Y$. For $(K,Y) \leqslant (K',Y')$ there is a morphism,

$$\mu^{(K',Y')(K,Y)} : M^{(K,Y)} \longrightarrow M^{(K',Y')} \qquad \text{given by} \qquad [x]_Y \longmapsto [x]_{Y'} .$$

It is straightforward to verify that these morphisms form a U-direct system in $\mathcal{C}(R)$. For every (K,Y) in U the restriction of $\hat{\pi}\colon \widehat{X} \to M$ to $\widehat{X}_K$ is zero on Y, and thus there is a morphism,

$$\alpha^{(K,Y)} : M^{(K,Y)} \longrightarrow M \qquad \text{given by} \qquad [x]_Y \longmapsto \hat{\pi}(x) .$$

For elements $(K,Y) \leqslant (K',Y')$ in U one clearly has $\alpha^{(K,Y)} = \alpha^{(K',Y')}\mu^{(K',Y')(K,Y)}$, so by the universal property of colimits 3.2.5 there is a morphism,

$$\alpha : \operatorname*{colim}_{(K,Y)\in U} M^{(K,Y)} \longrightarrow M ,$$

that satisfies $\alpha\mu^{(K,Y)} = \alpha^{(K,Y)}$ for all $(K,Y) \in U$.

3.3.19 Proposition. *Let M be an R-complex and Δ a set of R-complexes. The complexes and morphisms constructed in* 3.3.18 *have the following properties.*

(a) *The morphism α is injective.*
(b) *If every homogeneous element $m \in M$ belongs to the image of a morphism $D \to M$ with $D \in \Delta$, then the morphism α is surjective.*
(c) *If every complex D in Δ is bounded* (*above/below*), *then the complex $M^{(K,Y)}$ is bounded* (*above/below*) *for every $(K,Y) \in U$.*
(d) *If every complex D in Δ is degreewise finitely presented, then the complex $M^{(K,Y)}$ is degreewise finitely presented for every $(K,Y) \in U$.*

PROOF. (a): Since α is, in particular, a morphism of graded modules, it suffices to show that $z = 0$ is the only homogeneous element in $\operatorname{colim}_{(K,Y)\in U} M^{(K,Y)}$ with $\alpha(z) = 0$. Set $w = |z|$ and assume that $\alpha(z) = 0$ holds. By 3.3.2 there is an element (K,Y) in U and a z' in $M^{(K,Y)}$ with $|z'| = w$ and $\mu^{(K,Y)}(z') = z$. It follows that one has

$$(\dagger) \qquad \alpha^{(K,Y)}(z') = \alpha\mu^{(K,Y)}(z') = \alpha(z) = 0\,.$$

Write $z' = [x]_Y$ for some homogeneous element $x \in \widehat{X}_K$ with $|x| = w$. Now

$$Y' = 0 \longrightarrow R\langle x\rangle \xrightarrow{\partial^{\widehat{X}}_w} R\langle \partial^{\widehat{X}}_w(x)\rangle \longrightarrow 0$$

is a subcomplex of $\widehat{X}_K$, concentrated in degrees w and $w-1$; evidently it is degreewise finitely generated. By definition, $\alpha^{(K,Y)}(z') = \hat{\pi}(x)$ holds, so it follows from ($\dagger$) that x belongs to $\operatorname{Ker}\hat{\pi}$. As $\hat{\pi}\colon \widehat{X} \to M$ is a morphism of complexes, it follows that Y' is a subcomplex of $\operatorname{Ker}\hat{\pi}$. Consequently, $(K, Y + Y')$ is an element in U. By construction one has

$$\mu^{(K,Y+Y')(K,Y)}(z') = [x]_{Y+Y'} = 0\,,$$

and therefore also

$$z = \mu^{(K,Y)}(z') = \mu^{(K,Y+Y')}\mu^{(K,Y+Y')(K,Y)}(z') = \mu^{(K,Y+Y')}(0) = 0\,.$$

(b): Let m be in M. One can assume that m is homogeneous, and it is enough to argue that m is in the image of one of the morphisms $\alpha^{(K,Y)}$; see 3.2.6. By definition, $\alpha^{(K,0)}$ is the restricted morphism $\hat{\pi}\colon \widehat{X}_K \to M$, so it suffices to show that m is in the image of this map for some string K. By assumption there exists a morphism $\xi\colon D_\xi \to M$ with $D_\xi \in \Delta$ such that m is in $\operatorname{Im}\xi$. Define a string K by setting $K_1 = \{\xi\}$ and $K_n = \varnothing$ for $n > 1$. For this string one has $\widehat{X}_K = D_\xi$, and the restriction of $\hat{\pi}$ to this subcomplex is the morphism ξ, which has m in its image.

(c): For every (K,Y) in U the complex $M^{(K,Y)}$ is a quotient of a direct sum of complexes from Δ, so if they are all bounded (above/below) then so is $M^{(K,Y)}$.

(d): Assume that every complex D in Δ is degreewise finitely presented. For a string K, the complex $\widehat{X}_K$ is a direct sum of complexes from Δ and hence degreewise finitely presented. For every degreewise finitely generated subcomplex of Y of $\widehat{X}$ it follows from 1.3.40 that $M^{(K,Y)} = \widehat{X}_K/Y$ is degreewise finitely presented. $\square$

3.3.20. Let M be an R-complex and m a homogeneous element in M. Denote by M' the subcomplex of M whose underlying graded module is $R\langle m, \partial^M(m)\rangle$. It follows from 2.5.30 that m is in the image of a morphism $\mathrm{D}^{|m|}(R) \to M' \rightarrowtail M$.

3.3.21 Theorem. *Every R-complex is isomorphic to a filtered colimit of bounded and degreewise finitely presented R-complexes.*

Proof. Let Δ be the set $\{\mathrm{D}^u(R) \mid u \in \mathbb{Z}\}$ of bounded degreewise finitely presented R-complexes. Let M be an R-complex; by 3.3.20 every homogeneous element m in M is in the image of a morphism $\mathrm{D}^{|m|}(R) \to M$. The assertion now follows from 3.3.19. ◻

The next corollary also follows directly from 3.3.19 applied with $\Delta = \{R\}$.

3.3.22 Corollary. *Every R-module is isomorphic to a filtered colimit of finitely presented R-modules.*

Proof. Let M be an R-module. By 3.3.21 there is an isomorphism of R-complexes $M \cong \operatorname{colim}_{u\in U} M^u$ where $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ is a filtered U-direct system of degreewise finitely presented complexes. The degree 0 component yields an isomorphism between M and the colimit of the U-direct system $\{\mu_0^{vu}\colon M_0^u \to M_0^v\}_{u\leqslant v}$. ◻

Before proving a more detailed version of 3.3.21 we supplement 3.3.17.

3.3.23 Theorem. *For an R-complex M, the following conditions are equivalent.*

(*i*) *M is bounded and degreewise finitely presented.*
(*ii*) *The functor* $\operatorname{Hom}_R(M,-)$ *preserves filtered colimits.*
(*iii*) *The functor* $\mathcal{C}(R)(M,-)$ *preserves filtered colimits.*

Proof. The implication (*i*) ⟹ (*ii*) is proved in 3.3.17.

(*ii*) ⟹ (*iii*): By 3.3.15(a) the functor Z_0 preserves filtered colimits. From this fact and (*ii*), it follows that the composite functor $\mathrm{Z}_0(\operatorname{Hom}_R(M,-))$ preserves filtered colimits, and by 2.3.10 this functor is nothing but $\mathcal{C}(R)(M,-)$.

(*iii*) ⟹ (*i*): By 3.3.21 there is a filtered U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ of bounded degreewise finitely presented R-complexes with $M \cong \operatorname{colim}_{u\in U} M^u$. By assumption, the canonical morphism

$$\varphi\colon \operatorname*{colim}_{u\in U} \mathcal{C}(R)(M,M^u) \longrightarrow \mathcal{C}(R)\bigl(M, \operatorname*{colim}_{u\in U} M^u\bigr) \cong \mathcal{C}(R)(M,M)$$

is an isomorphism. Write

$$\mu^u\colon M^u \longrightarrow \operatorname*{colim}_{u\in U} M^u \cong M \quad\text{and}\quad \lambda^u\colon \mathcal{C}(R)(M,M^u) \longrightarrow \operatorname*{colim}_{u\in U} \mathcal{C}(R)(M,M^u)$$

for the canonical morphisms. By the diagram in 3.2.15 one has $\varphi\lambda^u = \mathcal{C}(R)(M,\mu^u)$ for all $u \in U$. Surjectivity of φ yields an element $\varkappa \in \operatorname{colim}_{u\in U} \mathcal{C}(R)(M,M^u)$ with $\varphi(\varkappa) = 1^M$. By 3.3.2 one has $\varkappa = \lambda^u(\psi^u)$ for some $u \in U$ and $\psi^u \in \mathcal{C}(R)(M,M^u)$. Consequently, there are equalities $\mu^u\psi^u = \mathcal{C}(R)(M,\mu^u)(\psi^u) = \varphi\lambda^u(\psi^u) = \varphi(\varkappa) = 1^M$. It follows that M is a direct summand of M^u, and since M^u is bounded and degreewise finitely presented, so is M. ◻

Remark. Following the standard definition—see for example Crawley-Boevey [71] or Krause [159]—the finitely presented objects in $\mathcal{C}(R)$ are exactly the complexes M that satisfy condition (*iii*) in 3.3.23. Thus by 3.3.21 the category $\mathcal{C}(R)$ is locally finitely presented.

The next result, which subsumes 3.3.21, lays the foundation for the proof of Govorov and Lazard's theorem in Sect. 5.5.

3.3.24 Theorem. *Let M be an R-complex and Λ a class of degreewise finitely presented R-complexes. If every morphism of R-complexes $\varphi\colon N \to M$ where N is degreewise finitely presented admits a factorization in $\mathcal{C}(R)$,*

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "\kappa"'] & & M \\ & L \arrow[ur, "\lambda"'] & \end{tikzcd}$$

with L in Λ, then M is isomorphic to a filtered colimit of complexes from Λ.

Moreover, if the complexes in Λ are bounded (*above/below*) *then it is sufficient that the factorization exists for morphisms $\varphi\colon N \to M$ where the degreewise finitely resented complex N is bounded* (*above/below*).

Proof. Every homogeneous element $m \in M$ is, as noted in 3.3.20, in the image of a morphism $\mathrm{D}^{|m|}(R) \to M$. It follows from the assumptions that m is in the image of a morphism $L \to M$ with $L \in \Lambda$. Let Δ be a set of representatives for the isomorphism classes in Λ. It follows from 3.3.19 that M is isomorphic to the colimit of the filtered U-direct system of degreewise finitely presented R-complexes

$$\left\{\mu^{(K',Y')(K,Y)} : M^{(K,Y)} \longrightarrow M^{(K',Y')}\right\}_{(K,Y)\leqslant(K',Y')}$$

constructed in 3.3.18. Notice, also from 3.3.19, that if the complexes in Λ are bounded (above/below) then also the complexes in this U-direct system are bounded (above/below). To prove the assertion, it suffices by 3.3.7 to show that the subset of U of elements (K, Y) such that $M^{(K,Y)}$ belongs to Δ is cofinal in U. To this end, let (K, Y) be in U; by assumption there exists a factorization,

$$\begin{tikzcd} M^{(K,Y)} \arrow[rr, "\alpha^{(K,Y)}"] \arrow[dr, "\kappa"'] & & M \\ & L \arrow[ur, "\lambda"'] & \end{tikzcd}$$

with L in Λ. One can assume that L is in Δ. Pick any $p \in \mathbb{N}$ such that $K_p = \emptyset$ holds. Define a string K' by setting $K'_p = \{\lambda\}$ and $K'_n = K_n$ for $n \neq p$. With the notation from 3.3.18 one has $\widehat{X}_{K'} = \widehat{X}_K \oplus L$, where L is considered as a subcomplex of the p^{th} copy of X in $\widehat{X}$. Let $\varrho\colon \widehat{X}_K \twoheadrightarrow \widehat{X}_K/Y = M^{(K,Y)}$ be the canonical morphism. The morphism

$$(\flat) \qquad \widehat{X}_{K'} = \widehat{X}_K \oplus L \longrightarrow L \qquad \text{given by} \qquad (x, l) \longmapsto \kappa\varrho(x) + l$$

has as kernel the subcomplex

$$Y' = \{(x, -\kappa\varrho(x)) \in \widehat{X}_K \oplus L \mid x \in \widehat{X}_K\}\,.$$

Since $\widehat{X}_K$ is degreewise finitely generated, so is Y'. By definition, $\hat{\pi} = \alpha^{(K,Y)}\varrho$ on $\widehat{X}_K$, and the restriction of $\hat{\pi}$ to L is λ. For every $(x, -\kappa\varrho(x))$ in Y' there are now equalities,

$$\hat{\pi}(x, -\kappa\varrho(x)) = \hat{\pi}(x) - \hat{\pi}\kappa\varrho(x) = \hat{\pi}(x) - \lambda\kappa\varrho(x) = \hat{\pi}(x) - \alpha^{(K,Y)}\varrho(x) = 0\,.$$

Thus Y' is contained in $\operatorname{Ker}\hat{\pi}$ and, therefore, (K', Y') is an element in U. For x in Y one has $\kappa\varrho(x) = \kappa(0) = 0$, so Y is contained in Y' and $(K, Y) \leqslant (K', Y')$ holds. It remains to see that $M^{(K',Y')}$ belongs to Δ. As the morphism (b) is surjective with kernel Y', it follows that there is an isomorphism $M^{(K',Y')} = \widehat{X}_{K'}/Y' \cong L$. □

Well-Ordered Colimits

3.3.25 Definition. Let λ be an ordinal. A λ-*sequence* of R-complexes is a λ-direct system $\{\mu^{\beta\alpha} \colon M^\alpha \to M^\beta\}_{\alpha\leqslant\beta<\lambda}$ in $\mathcal{C}(R)$ as in 3.2.1, indexed by the well-ordered set λ, with the additional property that for every limit ordinal $\beta < \lambda$ the canonical morphism $\varphi_\beta \colon \operatorname{colim}_{\alpha<\beta} M^\alpha \to M^\beta$ is an isomorphism.

3.3.26. Let λ be an ordinal and $\{\mu^{\beta\alpha} \colon M^\alpha \to M^\beta\}_{\alpha\leqslant\beta<\lambda}$ a λ-sequence. If $\beta < \lambda$ is a limit ordinal and $\mu^\alpha_\beta \colon M^\alpha \to \operatorname{colim}_{\alpha<\beta} M^\alpha$ for $\alpha < \beta$ is the canonical morphism, then φ_β from 3.3.25 is the unique morphism with $\varphi_\beta\mu^\alpha_\beta = \mu^{\beta\alpha}$ for $\alpha < \beta$, see 3.2.4.

3.3.27 Example. A continuous chain $\{M^\alpha\}_{\alpha<\lambda}$ of subcomplexes of an R-complex M, see D.1, yields in view of 3.3.3 a λ-sequence $\{\mu^{\beta\alpha} \colon M^\alpha \rightarrowtail M^\beta\}_{\alpha\leqslant\beta<\lambda}$ where $\mu^{\beta\alpha}$ is the embedding. On the other hand, if $\{\mu^{\beta\alpha} \colon M^\alpha \to M^\beta\}_{\alpha\leqslant\beta<\lambda}$ a λ-sequence with colimit M and canonical maps $\mu^\alpha \colon M^\alpha \to M$, then $\{\operatorname{Im}\mu^\alpha\}_{\alpha<\lambda}$ is a continuous chain of subcomplexes of M with $\bigcup_{\alpha<\lambda}\operatorname{Im}\mu^\alpha = M$, see 3.3.2 and 3.3.26.

A subset V of a preordered set $(U, \leqslant)$ is preordered with the induced order, but notice that $(V, \leqslant)$ need not be filtered even if $(U, \leqslant)$ is filtered. However, a modest enlargement of V will, in fact, be filtered. The next lemmas go back to Maeda [177, Appn. II]; we learned it from Jensen [148, §1] and Enochs and López-Ramos [88]. The proofs below follow Adámek and Rosický [2, 1.A].

3.3.28 Lemma. *Let $(U, \leqslant)$ be a preordered filtered set and $V \subseteq U$ a subset.*

(a) *If V is finite, then there exists a finite subset $V \subseteq V' \subseteq U$ with $(V', \leqslant)$ filtered.*
(b) *If V is infinite, then there exists a subset $V \subseteq V' \subseteq U$ with* $\operatorname{card} V' = \operatorname{card} V$ *and $(V', \leqslant)$ filtered.*

Proof. (a): As V is finite and U is filtered there exists an element $u \in U$ with $v \leqslant u$ for every $v \in V$. Now set $V' = V \cup \{u\}$.

(b): Set $V_0 = V$. Let $n \in \mathbb{N}_0$ and assume that a subset $V_n \subseteq U$ has been constructed. As U is filtered there exists for each pair of elements $x, y \in V_n$ an element $u_{x,y} \in U$ with $x \leqslant u_{x,y}$ and $y \leqslant u_{x,y}$. Set $V_{n+1} = V_n \cup \{u_{x,y} \mid x, y \in V_n\}$ and note that one has $\operatorname{card} V_{n+1} = \operatorname{card} V_n$. The set $V' = \bigcup_{n\in\mathbb{N}_0} V_n$ has the asserted properties. □

Note that in the next result we slightly abuse notation: The symbol "$\leqslant$" is used both for the preorder on the set U and the well-order on the limit ordinal λ.

3.3.29 Lemma. *Let $(U, \leqslant)$ an infinite preordered filtered set and λ the initial ordinal of cardinality* $\operatorname{card} U$. *There is a family $\{U_\alpha\}_{\alpha<\lambda}$ of subsets of U with these properties*:

(a) *$U_0 = \varnothing$ and $\bigcup_{\alpha<\lambda} U_\alpha = U$.*
(b) *For all ordinals $\alpha \leqslant \beta < \lambda$ one has $U_\alpha \subseteq U_\beta$.*
(c) *For each limit ordinal $\beta < \lambda$ one has $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$.*
(d) *For each ordinal $\alpha < \lambda$ the set $(U_\alpha, \leqslant)$ is filtered with* $\operatorname{card} U_\alpha < \operatorname{card} U$.

Proof. To each subset $V \subseteq U$ associate a subset $V' \subseteq U$ as described in 3.3.28. Further, well-order U and write $U = \{u_\alpha \mid \alpha < \lambda\}$. By transfinite induction one defines the family $\{U_\alpha\}_{\alpha<\lambda}$ as follows:

- $U_0 = \varnothing$.
- $U_{\alpha+1} = (U_\alpha \cup \{u_\alpha\})'$ for every ordinal α with $\alpha + 1 < \lambda$.
- $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$ for every limit ordinal $\beta < \lambda$.

Now every element u_α in U is an element of the set $U_{\alpha+1}$. The sets $U_{\alpha+1}$ are filtered by construction, and a union of an increasing chain of filtered sets is filtered. Finally, for infinite sets U_α one has $\operatorname{card} U_{\alpha+1} = \operatorname{card} U_\alpha$, and for $\beta < \lambda$ the the union $\bigcup_{\alpha<\beta} U_\alpha$, where each set U_α has $\operatorname{card} U_\alpha < \operatorname{card} U$, again has cardinality less than $\operatorname{card} U$. □

The gist of the next result is that filtered colimits can be reduced to well-ordered colimits, and filtered colimits of monomorphisms can be reduced to unions of continuous chains; we learned this from Enochs and López-Ramos [88]. The proof we give follows Adámek and Rosický [2, 1.A]. To parse conditions (*ii*) and (*ii′*), recall from 3.3.25 and D.1 the definitions of λ-sequences and continuous chains.

3.3.30 Theorem. *Let $\mathcal{X}$ be a class of R-complexes.*

(a) *The following conditions are equivalent.*
   (*i*) *For every U-direct system $\{\mu^{vu} : M^u \longrightarrow M^v\}_{u\leqslant v}$ of complexes in $\mathcal{X}$ with U filtered, one has* $\operatorname{colim}_{u\in U} M^u \in \mathcal{X}$.
   (*ii*) *For every ordinal λ and λ-sequence $\{\mu^{\beta\alpha} : M^\alpha \longrightarrow M^\beta\}_{\alpha\leqslant\beta<\lambda}$ of complexes in $\mathcal{X}$, one has* $\operatorname{colim}_{\alpha<\lambda} M^\alpha \in \mathcal{X}$.

(b) *The following conditions are equivalent.*
   (*i′*) *For every U-direct system $\{\mu^{vu} : M^u \rightarrowtail M^v\}_{u\leqslant v}$ of complexes in $\mathcal{X}$ with U filtered and each μ^{vu} is injective, one has* $\operatorname{colim}_{u\in U} M^u \in \mathcal{X}$.
   (*ii′*) *For every R-complex M that is the union of a continuous chain $\{M^\alpha\}_{\alpha<\lambda}$ of subcomplexes with $M^\alpha \in \mathcal{X}$ for every $\alpha < \lambda$, one has $M \in \mathcal{X}$.*

Proof. The implication (*i*) $\Rightarrow$ (*ii*) is trivial. To show that condition (*ii*) implies (*i*) we argue by transfinite induction on the cardinality of the preordered filtered set $(U, \leqslant)$. If U is finite, then U has a greatest element, w, so one has $\operatorname{colim}_{u\in U} M^u \cong M^w \in \mathcal{X}$ by 3.2.8. Now let U be infinite and assume that for every preordered filtered set V with $\operatorname{card} V < \operatorname{card} U$ the colimit of every V-direct system in $\mathcal{X}$ belongs to $\mathcal{X}$. Let λ be the initial ordinal of cardinality $\operatorname{card} U$ and let $\{\mu^{vu} : M^u \longrightarrow M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{X}$. The goal is to show that $\operatorname{colim}_{u \in U} M^u$ is in $\mathcal{X}$. Let $\{U_\alpha\}_{\alpha<\lambda}$ be a family as in 3.3.29 and set $U_\lambda = U$. For $\alpha \leqslant \lambda$ set

$$M_\alpha = \operatorname*{colim}_{u \in U_\alpha} M^u \quad \text{with canonical maps} \quad \mu^u_\alpha \colon M^u \longrightarrow M_\alpha \quad \text{for } \alpha \leqslant \lambda \text{ and } u \in U_\alpha\,.$$

Note that one has $M_\lambda = \operatorname{colim}_{u \in U} M^u$, so the goal is to prove that M_λ belongs to $\mathcal{X}$. Recall from 3.3.29(d) that for $\alpha < \lambda$ the set $(U_\alpha, \leqslant)$ is filtered with $\operatorname{card} U_\alpha < \operatorname{card} U$, so the induction hypothesis yields $M_\alpha \in \mathcal{X}$ for every $\alpha < \lambda$. By 3.2.3 the canonical morphisms above satisfy

$$(\flat) \qquad \mu^v_\alpha \mu^{vu} = \mu^u_\alpha \quad \text{for } \alpha \leqslant \lambda \text{ and } u \leqslant v \text{ in } U_\alpha\,.$$

For $\alpha \leqslant \beta \leqslant \lambda$ one has $U_\alpha \subseteq U_\beta$; indeed, for $\beta < \lambda$ this inclusion holds by 3.3.29(b), and for $\beta = \lambda$ it holds by 3.3.29(a) as one has $U_\lambda = U$. Thus $(\flat)$ and the universal property 3.2.5 yield a unique morphism,

$$(\diamond) \qquad \sigma_{\beta\alpha} \colon M_\alpha \longrightarrow M_\beta \quad \text{satisfying} \quad \sigma_{\beta\alpha}\mu^u_\alpha = \mu^u_\beta \quad \text{for } \alpha \leqslant \beta \leqslant \lambda \text{ and } u \in U_\alpha\,.$$

By assumption, $M_\alpha \in \mathcal{X}$ holds for $\alpha < \lambda$. We now show that $\{\sigma_{\beta\alpha} \colon M_\alpha \to M_\beta\}_{\alpha\leqslant\beta<\lambda}$ is a λ-sequence, and hence a λ-sequence in $\mathcal{X}$. For ordinals $\alpha \leqslant \beta \leqslant \gamma \leqslant \lambda$ and $u \in U_\alpha$ there are by $(\diamond)$ equalities,

$$\sigma_{\alpha\alpha}\mu^u_\alpha = \mu^u_\alpha = 1^{M_\alpha}\mu^u_\alpha \quad \text{and} \quad \sigma_{\gamma\beta}\sigma_{\beta\alpha}\mu^u_\alpha = \sigma_{\gamma\beta}\mu^u_\beta = \mu^u_\gamma = \sigma_{\gamma\alpha}\mu^u_\alpha\,,$$

so the uniqueness of the morphisms in $(\diamond)$ implies that one has

$$(\star) \qquad \sigma_{\alpha\alpha} = 1^{M_\alpha} \quad \text{and} \quad \sigma_{\gamma\beta}\sigma_{\beta\alpha} = \sigma_{\gamma\alpha} \quad \text{for } \alpha \leqslant \beta \leqslant \gamma \leqslant \lambda\,.$$

This shows that $\{\sigma_{\beta\alpha} \colon M_\alpha \to M_\beta\}_{\alpha\leqslant\beta<\lambda}$ is a λ-direct system, see 3.2.1. Now, let $\kappa \leqslant \lambda$ be a limit ordinal and notice that λ itself is a limit ordinal. With a slight variation of the notation from 3.3.25 and 3.3.26 there are canonical morphisms,

$$(\dagger\dagger) \qquad \sigma^\kappa_\alpha \colon M_\alpha \longrightarrow \operatorname*{colim}_{\alpha<\kappa} M_\alpha \quad \text{satisfying} \quad \sigma^\kappa_\beta\sigma_{\beta\alpha} = \sigma^\kappa_\alpha \quad \text{for } \alpha \leqslant \beta < \kappa \leqslant \lambda\,,$$

$$(\ddagger\ddagger) \qquad \varphi^\kappa \colon \operatorname*{colim}_{\alpha<\kappa} M_\alpha \longrightarrow M_\kappa \quad \text{satisfying} \quad \varphi^\kappa\sigma^\kappa_\alpha = \sigma_{\kappa\alpha} \quad \text{for } \alpha < \kappa \leqslant \lambda\,.$$

Next we argue that φ^κ from $(\ddagger\ddagger)$ is an isomorphism for every limit ordinal $\kappa \leqslant \lambda$. Once this has been shown, the proof of the implication (*ii*) $\Longrightarrow$ (*i*) is complete. Indeed, the fact that φ^κ is an isomorphism for all limit ordinals $\kappa < \lambda$ shows that the already constructed λ-direct system $\{\sigma_{\beta\alpha} \colon M_\alpha \to M_\beta\}_{\alpha\leqslant\beta<\lambda}$ in $\mathcal{X}$ is a λ-sequence, see 3.3.25 and 3.3.26, and condition (*ii*) therefore implies that $\operatorname{colim}_{\alpha<\lambda} M_\alpha$ is in $\mathcal{X}$. As also φ^λ is an isomorphism, it follows that M_λ is in $\mathcal{X}$, as desired.

Let $\kappa \leqslant \lambda$ be a limit ordinal; to show that φ^κ from $(\ddagger\ddagger)$ is an isomorphism we construct its inverse. To this end, note that one has $U_\kappa = \bigcup_{\alpha<\kappa} U_\alpha$; indeed, for $\kappa < \lambda$ this equality holds by 3.3.29(c), and for $\kappa = \lambda$ it holds by 3.3.29(a) as $U_\lambda = U$. Thus, every element $u \in U_\kappa$ is in U_α for some $\alpha < \kappa$. If u is in both U_α and U_β for $\alpha, \beta < \kappa$ there is an equality $\sigma^\kappa_\alpha\mu^u_\alpha = \sigma^\kappa_\beta\mu^u_\beta$ of morphisms $M^u \to \operatorname{colim}_{\alpha<\kappa} M_\alpha$; indeed, as λ well-ordered one can assume that $\alpha \leqslant \beta$ holds, so $(\dagger\dagger)$ and $(\diamond)$ yield $\sigma^\kappa_\alpha\mu^u_\alpha = \sigma^\kappa_\beta\sigma_{\beta\alpha}\mu^u_\alpha = \sigma^\kappa_\beta\mu^u_\beta$. Thus for every $u \in U_\kappa$ we can define a morphism

$$(\flat\flat) \qquad \psi^u \colon M^u \to \operatorname*{colim}_{\alpha<\kappa} M_\alpha \quad \text{by setting} \quad \psi^u = \sigma^\kappa_\alpha \mu^u_\alpha \ \text{ for } \alpha < \kappa \text{ with } u \in U_\alpha\,.$$

For $u \leqslant v$ in U_κ one has $\psi^v \mu^{vu} = \psi^u$; indeed, for any choice of $\alpha < \kappa$ with $u, v \in U_\alpha$ one gets from $(\flat\flat)$ and $(\flat)$ the equalities $\psi^v \mu^{vu} = \sigma^\kappa_\alpha \mu^v_\alpha \mu^{vu} = \sigma^\kappa_\alpha \mu^u_\alpha = \psi^u$. By definition, one has $M_\kappa = \operatorname{colim}_{u \in U_\kappa} M^u$ with canonical morphisms $\mu^u_\kappa \colon M^u \to M_\kappa$ for $u \in U_\kappa$, so the universal property 3.2.5 yields a unique morphism,

$$(\diamond\diamond) \qquad \psi^\kappa \colon M_\kappa \to \operatorname*{colim}_{\alpha<\kappa} M_\alpha \quad \text{satisfying} \quad \psi^\kappa \mu^u_\kappa = \psi^u \ \text{ for } \ u \in U_\kappa\,.$$

We now show that ψ^κ is the inverse of φ^κ from $(\ddagger\ddagger)$. To see that $\psi^\kappa \varphi^\kappa$ is the identity on $\operatorname{colim}_{\alpha<\kappa} M_\alpha$ it suffices by $(\dagger\dagger)$ and the uniqueness part of 3.2.5 to show that for every $\alpha < \kappa$ there is an equality $\psi^\kappa \varphi^\kappa \sigma^\kappa_\alpha = \sigma^\kappa_\alpha$ of morphisms $M_\alpha \to \operatorname{colim}_{\alpha<\kappa} M_\alpha$. Recall that one has $M_\alpha = \operatorname{colim}_{u \in U_\alpha} M^u$ with canonical morphisms $\mu^u_\alpha \colon M^u \to M_\alpha$ for $u \in U_\alpha$. Thus, to prove $\psi^\kappa \varphi^\kappa \sigma^\kappa_\alpha = \sigma^\kappa_\alpha$ it is enough to argue that $\psi^\kappa \varphi^\kappa \sigma^\kappa_\alpha \mu^u_\alpha = \sigma^\kappa_\alpha \mu^u_\alpha$ holds for every $u \in U_\alpha$. This equality follows from $(\ddagger\ddagger)$, $(\diamond)$, $(\diamond\diamond)$, and $(\flat\flat)$:

$$\psi^\kappa \varphi^\kappa \sigma^\kappa_\alpha \mu^u_\alpha = \psi^\kappa \sigma_{\kappa\alpha} \mu^u_\alpha = \psi^\kappa \mu^u_\kappa = \psi^u = \sigma^\kappa_\alpha \mu^u_\alpha\,.$$

Similarly, to see that $\varphi^\kappa \psi^\kappa$ is the identity on $M_\kappa = \operatorname{colim}_{u \in U_\kappa} M^u$ it suffices to show that for each $u \in U_\kappa$ one has $\varphi^\kappa \psi^\kappa \mu^u_\kappa = \mu^u_\kappa$. Given $u \in U_\kappa$ choose $\alpha < \kappa$ with $u \in U_\alpha$; now $(\diamond\diamond)$, $(\flat\flat)$, $(\ddagger\ddagger)$, and $(\diamond)$ yield equalities,

$$\varphi^\kappa \psi^\kappa \mu^u_\kappa = \varphi^\kappa \psi^u = \varphi^\kappa \sigma^\kappa_\alpha \mu^u_\alpha = \sigma_{\kappa\alpha} \mu^u_\alpha = \mu^u_\kappa\,.$$

This completes the proof of part (a).

For part (b) we first show that (ii') is equivalent to the following condition:

(ii'') For every ordinal λ and λ-sequence $\{\mu^{\beta\alpha} \colon M^\alpha \rightarrowtail M^\beta\}_{\alpha \leqslant \beta < \lambda}$ of complexes in $\mathcal{X}$ where each morphism $\mu^{\beta\alpha}$ is injective, one has $\operatorname{colim}_{\alpha<\lambda} M^\alpha \in \mathcal{X}$.

The implication $(ii'') \Rightarrow (ii')$ follows from 3.3.27 and 3.3.3. To show the converse implication, let $\{\mu^{\beta\alpha} \colon M^\alpha \rightarrowtail M^\beta\}_{\alpha \leqslant \beta < \lambda}$ be a λ-sequence in $\mathcal{X}$ where each $\mu^{\beta\alpha}$ is injective. Set $M = \operatorname{colim}_{\alpha<\lambda} M^\alpha$ and for $\alpha < \lambda$ let $\mu^\alpha \colon M^\alpha \to M$ be the canonical morphism. Each μ^α is injective by 3.3.2(b), so one has $\operatorname{Im} \mu^\alpha \cong M^\alpha \in \mathcal{X}$. Further, $\{\operatorname{Im} \mu^\alpha\}_{\alpha<\lambda}$ is a continuous chain of subcomplexes of M with and $\bigcup_{\alpha<\lambda} \operatorname{Im} \mu^\alpha = M$, see 3.3.27. Thus, if (ii') holds it follows that the complex $M = \operatorname{colim}_{\alpha<\lambda} M^\alpha$ is in $\mathcal{X}$.

We finish the proof by showing that conditions (i') and (ii'') are equivalent. The implication $(i') \Rightarrow (ii'')$ is trivial. To see that (ii'') implies (i'), recall that in the proof of part (a) we constructed from a U-direct system $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ in $\mathcal{X}$ a λ-sequence $\{\sigma_{\beta\alpha} \colon M_\alpha \to M_\beta\}_{\alpha \leqslant \beta < \lambda}$ in $\mathcal{X}$ with $\operatorname{colim}_{u \in U} M^u \cong \operatorname{colim}_{\alpha<\lambda} M_\alpha$. Thus, to prove the implication $(ii'') \Rightarrow (i')$ it suffices to show that if each morphism μ^{vu} is injective, then so is every morphism $\sigma_{\beta\alpha}$. If each μ^{vu} is injective it follows from 3.3.2(b) that the canonical morphism $\mu^u_\beta \colon M^u \to M_\beta = \operatorname{colim}_{u \in U_\beta} M^u$ is injective for every $\beta < \lambda$ and $u \in U_\beta$. Let $\alpha \leqslant \beta < \lambda$ be given. By construction, see $(\diamond)$, the morphism $\sigma_{\beta\alpha} \colon M_\alpha = \operatorname{colim}_{u \in U_\alpha} M^u \to M_\beta$ is the unique one induced per 3.2.5 by the family $\{\mu^u_\beta \colon M^u \to M_\beta\}_{u \in U_\alpha}$, so it follows from 3.3.8 that $\sigma_{\beta\alpha}$ is injective. $\square$

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A frequently occurring form of colimits stems from sequences of morphisms.

3.3.31 Construction. Let $\{\kappa^u : M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a sequence of morphisms in $\mathcal{C}(R)$. It determines a $\mathbb{Z}$-direct system $\{\mu^{vu} : M^u \to M^v\}_{u\leqslant v}$ as follows: set

$$\mu^{uu} = 1^{M^u} \quad \text{for all } u \text{ in } \mathbb{Z} \qquad \text{and} \qquad \mu^{vu} = \kappa^{v-1}\cdots\kappa^u \quad \text{for all } u < v \text{ in } \mathbb{Z}\,.$$

Given additional sequences $\{\lambda^u : N^u \to N^{u+1}\}_{u\in\mathbb{Z}}$ and $\{\alpha^u : M^u \to N^u\}_{u\in\mathbb{Z}}$ of morphisms, such that $\alpha^{u+1}\kappa^u = \lambda^u\alpha^u$ holds for all $u \in \mathbb{Z}$, it is elementary to verify that $\{\alpha^u\}_{u\in\mathbb{Z}}$ is a morphism of the direct systems determined by $\{\kappa^u\}_{u\in\mathbb{Z}}$ and $\{\lambda^u\}_{u\in\mathbb{Z}}$.

For example, the $\mathbb{Z}$-direct system in 3.3.4 arises, as described above, from the sequence $\cdots \rightarrowtail M_{\leqslant -1} \rightarrowtail M_{\leqslant 0} \rightarrowtail M_{\leqslant 1} \rightarrowtail \cdots$.

3.3.32 Definition. A sequence $\{\kappa^u : M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ of morphisms in $\mathcal{C}(R)$ with $M^u = 0$ for $u \ll 0$ is called a *telescope* in $\mathcal{C}(R)$. The colimit, $\operatorname{colim}_{u\in\mathbb{Z}} M^u$, of the associated $\mathbb{Z}$-direct system, see 3.3.31, is called the *colimit* of the telescope.

Given telescopes $\{\kappa^u : M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ and $\{\lambda^u : N^u \to N^{u+1}\}_{u\in\mathbb{Z}}$, a sequence of morphisms $\{\alpha^u : M^u \to N^u\}_{u\in\mathbb{Z}}$ that satisfy $\alpha^{u+1}\kappa^u = \lambda^u\alpha^u$ for all $u \in \mathbb{Z}$ is called a *morphism of telescopes*. The morphism,

$$\operatorname*{colim}_{u\in\mathbb{Z}} \alpha^u : \operatorname*{colim}_{u\in\mathbb{Z}} M^u \longrightarrow \operatorname*{colim}_{u\in\mathbb{Z}} N^u\,,$$

see 3.3.31 and 3.2.10, is called the *colimit* of $\{\alpha^u : M^u \to N^u\}_{u\in\mathbb{Z}}$.

3.3.33. Let $\{\kappa^u : M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a telescope and $\{\mu^{vu} : M^u \to M^v\}_{u\leqslant v}$ the associated $\mathbb{Z}$-direct system in $\mathcal{C}(R)$. Given an R-complex N and a sequence of morphisms $\{\alpha^u : M^u \to N\}_{u\in\mathbb{Z}}$ that satisfy $\alpha^u = \alpha^{u+1}\kappa^u$ for all $u \in \mathbb{Z}$, one has $\alpha^u = \alpha^v\mu^{vu}$ for all $u \leqslant v$. By the universal property of colimits, there is a morphism $\alpha\colon \operatorname{colim}_{u\in\mathbb{Z}} M^u \to N$ in $\mathcal{C}(R)$ with properties as described in 3.2.5.

3.3.34 Example. Let $M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots$ be an ascending chain of subcomplexes of an R-complex M. The embeddings $M^u \rightarrowtail M^{u+1}$ define a telescope whose colimit is isomorphic to the subcomplex $\bigcup_{u\in\mathbb{Z}} M^u$. This is a special case of 3.3.3.

3.3.35 Example. Assume that R is commutative. Let M be an R-module and x an element in R; set $X = \{x^n \mid n \geqslant 0\}$. Consider the following commutative diagram of R-modules,

$$\begin{tikzcd}
 & & & & C \arrow[dd, dotted, "\alpha"] \\
M \arrow[r, "\kappa^0 = x"] \arrow[urrrr, bend left, "\mu^0"] \arrow[drrrr, bend right, "\alpha^0 = 1"'] & M \arrow[r, "\kappa^1 = x"] \arrow[urrr, bend left, "\mu^1"] \arrow[drrr, bend right, "\alpha^1 = \frac{1}{x}"'] & M \arrow[r, "\kappa^2 = x"] \arrow[urr, "\mu^2"] \arrow[drr, "\alpha^2 = \frac{1}{x^2}"'] & \cdots & \\
 & & & & X^{-1}M
\end{tikzcd}$$

The row is the telescope where each homomorphism κ^u is the homothety x^M, the module C is the colimit of the telescope with canonical homomorphisms μ^u and α^u multiplies with $\frac{1}{x^u}$. As one has $\alpha^u = \alpha^{u+1}\kappa^u$ there is a unique homomorphism $\alpha\colon C \to X^{-1}M$ with $\alpha\mu^u = \alpha^u$ for all u; see 3.3.33. We argue that α is an isomorphism. Note that $\alpha\mu^u(m) = \alpha^u(m) = \frac{m}{x^u}$, so α is surjective. By 3.3.2 every element $z \in C$ has the form $z = \mu^u(m)$ for some u and some $m \in M$, so if z is in the kernel of α, then $0 = \alpha(z) = \alpha\mu^u(m) = \alpha^u(m) = \frac{m}{x^u}$ holds in $X^{-1}M$. This means that $x^v m = 0$ in M for some v, and it follows that

$$z = \mu^u(m) = \mu^{u+v}\kappa^{u+v-1}\cdots\kappa^{u+1}\kappa^u(m) = \mu^{u+v}(x^v m) = \mu^{u+v}(0) = 0$$

holds, so α is injective.

3.3.36 Proposition. *Let $\{\kappa^u\colon M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a telescope in $\mathcal{C}(R)$.*

(a) *If $\kappa^u = 0$ holds for infinitely many $u > 0$, then one has $\operatorname{colim}_{u\in\mathbb{Z}} M^u = 0$.*
(b) *If there exists an integer w such that κ^u is bijective for all $u \geqslant w$, then the canonical map $M^w \to \operatorname{colim}_{u\in\mathbb{Z}} M^u$ is an isomorphism.*

Proof. Let $\{\mu^{vu}\colon M^u \to M^v\}_{u\in\mathbb{Z}}$ be the direct system associated to the telescope.

(a): By 3.3.2 every element m in $\operatorname{colim}_{u\in\mathbb{Z}} M^u$ has the form $m = \mu^u(m^u)$ for some $u \in \mathbb{Z}$ and $m^u \in M^u$. It follows from the assumption that the map μ^{vu} is zero for some $v \geqslant u$, and consequently one has $m = \mu^u(m^u) = \mu^v(\mu^{vu}(m^u)) = 0$.

(b): Define a sequence $\{\alpha^u\colon M^u \to M^w\}_{u\in\mathbb{Z}}$ of morphisms in $\mathcal{C}(R)$ as follows: Set $\alpha^w = 1^{M^w}$, set $\alpha^u = \kappa^{w-1}\cdots\kappa^u$ for $u < w$ and $\alpha^u = (\kappa^{u-1}\cdots\kappa^w)^{-1}$ for $u > w$. By construction $\alpha^u = \alpha^{u+1}\kappa^u$ holds for all $u \in \mathbb{Z}$, so by 3.3.33 there is a morphism $\alpha\colon \operatorname{colim}_{u\in\mathbb{Z}} M^u \to M^w$, given by $\mu^u(m^u) \mapsto \alpha^u(m^u)$. Evidently one has $\alpha\mu^w = \alpha^w = 1^{M^w}$. It follows from 3.3.8 that α is injective, and hence it is the inverse of μ^w. □

3.3.37 Proposition. *Let $\{\kappa^u\colon M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a telescope in $\mathcal{C}(R)$. With*

$$\Delta^0_\kappa\colon \coprod_{u\in\mathbb{Z}} M^u \longrightarrow \coprod_{u\in\mathbb{Z}} M^u \quad \text{given by} \quad \varepsilon^u(m^u) \longmapsto \varepsilon^u(m^u) - \varepsilon^{u+1}\kappa^u(m^u)$$

one has $\operatorname{colim}_{u\in\mathbb{Z}} M^u = \operatorname{Coker}\Delta^0_\kappa$. Moreover, the morphism Δ^0_κ is injective.

Proof. As in 3.3.31 let $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ be the direct system associated with the telescope and consider the morphism Δ_μ; see 3.2.3. Evidently, the image of Δ_μ in $\coprod_{u\in\mathbb{N}} M^u$ contains the image of Δ^0_κ. On the other hand, for integers $u \leqslant v$ an element $\varepsilon^u(m^{(u,v)}) - \varepsilon^v\mu^{vu}(m^{(u,v)})$ in the image of Δ_μ can with $m = m^{(u,v)}$ be rewritten as

$$\begin{aligned}
\varepsilon^u(m) - \varepsilon^v\mu^{vu}(m) &= \varepsilon^u(m) - \varepsilon^v\kappa^{v-1}\cdots\kappa^u(m)\\
&= \varepsilon^u(m) - \varepsilon^{u+1}\kappa^u(m) + \varepsilon^{u+1}\kappa^u(m) -\\
&\qquad \cdots + \varepsilon^{v-1}\kappa^{v-2}\cdots\kappa^u(m) - \varepsilon^v\kappa^{v-1}\cdots\kappa^u(m)\\
&= \Delta^0_\kappa(\varepsilon^u(m) + \varepsilon^{u+1}\kappa^u(m) + \cdots + \varepsilon^{v-1}\kappa^{v-2}\cdots\kappa^u(m))\,.
\end{aligned}$$

Thus, the morphisms Δ_μ and Δ^0_κ have the same image and hence the same cokernel. To see that Δ^0_κ is injective, let $s = \sum_{u\in\mathbb{Z}} \varepsilon^u(m^u)$ be a non-zero element in $\coprod_{u\in\mathbb{Z}} M^u$

and w the least index with $m^w \neq 0$. It is now immediate from the definition of the map that w^{th} term of $\varDelta^0_\kappa(s)$ is $m^w - \kappa^{w-1}(m^{w-1}) = m^w \neq 0$. □

Exercises

E 3.3.1 Let M be a finitely presented R-module and $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ a U-direct system of R-modules. Show that if U is filtered and there is an isomorphism $\operatorname{colim}_{u\in U} M^u \cong M$, then M is a direct summand of one of the modules M^u. *Hint:* 3.3.17.

E 3.3.2 Show that an R-module F is flat if and only if the morphism $\iota \otimes_R F$ induced by the embedding $\iota\colon \mathfrak{b} \to R$ is injective for every finitely generated right ideal $\mathfrak{b}$ in R.

E 3.3.3 Assume that R is right coherent; let I be an R^{o}-module and E an injective $\Bbbk$-module. Show: (a) If I is injective, then the R-module $\operatorname{Hom}_\Bbbk(I, E)$ is flat. (b) If E is faithfully injective, then the R-module $\operatorname{Hom}_\Bbbk(I, E)$ is (faithfully) flat, if and only if I is (faithfully) injective.

E 3.3.4 Assume that R is right coherent; let F be an R-module and P a projective $\Bbbk$-module. Show: (a) If F is flat, then the R-module $\operatorname{Hom}_\Bbbk(P, F)$ is flat. (b) If P is faithfully projective, then the R-module $\operatorname{Hom}_\Bbbk(P, F)$ is (faithfully) flat, if and only if F is (faithfully) flat.

E 3.3.5 Show that every R-module is a filtered colimit of finitely presented modules. *Hint:* Let $\{m_u\}_{u\in U}$ generate M, let L be free with basis $\{e_u\}_{u\in U}$, and consider the canonical exact sequence $0 \to K \to L \xrightarrow{\pi} M \to 0$. For subsets $V \subseteq U$ set $L_V = R\langle e_u \mid u \in V\rangle$. Put a partial order on the set of all pairs (V, H) where $V \subseteq U$ is finite and $H \subseteq K \cap L_V$ is a finitely generated submodule, such that it becomes filtered. Set $M^{(V,H)} = L_V/H$ and show that M is the colimit of a direct system of these modules.

E 3.3.6 Assume that R is left Noetherian. Show that every filtered colimit of injective R-modules is an injective R-module.

E 3.3.7 Show that a filtered colimit of flat R-modules is flat.

E 3.3.8 Show that the following conditions on R are equivalent. (*i*) R is von Neumann regular. (*ii*) $R/\mathfrak{b}$ is a flat R^{o}-module for every (finitely generated) right ideal $\mathfrak{b}$ in R. (*iii*) Every R-module is flat.

E 3.3.9 Show that a left Noetherian and von Neumann regular ring is semi-simple.

E 3.3.10 As in 3.3.31 let $\{\kappa^u\colon M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a sequence (not necessarily a telescope) of morphisms in $\mathcal{C}(R)$. Show that the colimit of the associated direct system does not depend on κ^u for $u \ll 0$.

E 3.3.11 Show that every complex is the colimit of a telescope of bounded below complexes.

E 3.3.12 Show that for a U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$, the morphism $\varDelta_\mu$ is not injective.

E 3.3.13 Let $\{\kappa^u\colon M^u \to M^{u+1}\}_{u\in\mathbb{Z}}$ be a telescope in $\mathcal{C}(R)$ with colimit M. Set $N^u = M^{u+1}$ and $\lambda^u = \kappa^{u+1}$; show that $\{\lambda^u\colon N^u \to N^{u+1}\}_{u\in\mathbb{Z}}$ is a telescope with colimit M and that the morphisms $\alpha^u = \kappa^u$ from M^u to N^u form a morphism of telescopes with colimit 1^M.

E 3.3.14 Let M be an R-module. Show that M is finitely presented if and only if (3.3.17.1) is bijective for every U-direct system $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$. *Hint:* E 3.3.5 and E 3.3.1.

E 3.3.15 Let M be an R-module. Show that M is finitely generated if and only if (3.3.17.1) is injective for every U-direct system $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$.

3.4 Limits

Synopsis. Inverse system; limit; universal property; functor that preserves limits; pullback.

Limits are categorically dual to colimits; they are subcomplexes of products. Their theory starts out parallel to the theory for colimits, however, the two deviate at one important point: While colimits often are exact, limits rarely are.

3.4.1 Definition. Let $(U, \leqslant)$ be a preordered set. A U-*inverse system* in $\mathcal{C}(R)$ is a family $\{\nu^{uv} \colon N^v \to N^u\}_{u \leqslant v}$ of morphisms in $\mathcal{C}(R)$ with the following properties.

(1) $\nu^{uu} = 1^{N^u}$ for all $u \in U$.

(2) $\nu^{uv}\nu^{vw} = \nu^{uw}$ for all $u \leqslant v \leqslant w$ in U.

Any mention of a U-inverse system $\{\nu^{uv} \colon N^v \to N^u\}_{u \leqslant v}$ includes the tacit assumption that $(U, \leqslant)$ is a preordered set.

3.4.2. Let $\{\nu^{uv} \colon N^v \to N^u\}_{u \leqslant v}$ be a U-inverse system of R-complexes. Notice that even if $u \leqslant v$ and $v \leqslant u$ hold, one may not have $u = v$ as the relation on U is not assumed to be antisymmetric; however, it follows from 3.4.1 that $\nu^{uv} \colon N^v \to N^u$ is an isomorphism with inverse $\nu^{vu} \colon N^u \to N^v$, so one has $N^u \cong N^v$.

3.4.3 Construction. Let $\{\nu^{uv} \colon N^v \to N^u\}_{u \leqslant v}$ be a U-inverse system in $\mathcal{C}(R)$. We describe the subcomplex of the product $\prod_{u \in U} N^u$ generated by the elements $(n^u)_{u \in U}$ with $n^u = \nu^{uv}(n^v)$ for $u \leqslant v$ as the kernel of a morphism between products in $\mathcal{C}(R)$.

Indeed, let $\nabla(U)$ be as in 3.2.3 and set $N^{(u,v)} = N^u$ for all $(u, v) \in \nabla(U)$. The assignment

$$(n^u)_{u \in U} \longmapsto (n^u - \nu^{uv}(n^v))_{(u,v) \in \nabla(U)}$$

defines by 3.1.15 a morphism of R-complexes

$$\varDelta^{\nu} \colon \prod_{u \in U} N^u \longrightarrow \prod_{(u,v) \in \nabla(U)} N^{(u,v)} .$$

Set

$$\lim_{u \in U} N^u = \operatorname{Ker} \varDelta^{\nu} .$$

Note that for every $u \in U$, restriction of the projection (3.1.14.1) yields a morphism of R-complexes,

$$\nu^u \colon \lim_{u \in U} N^u \longrightarrow N^u , \tag{3.4.3.1}$$

and $\nu^u = \nu^{uv}\nu^v$ holds for all $u \leqslant v$.

Remark. As for colimits, the standard notation suppresses the morphisms $\nu^{uv} \colon N^v \to N^u$, though the complex $\lim_{u \in U} N^u$ depends on them.

The next definition is justified by 3.4.5, which shows that the complex $\lim_{u \in U} M^u$ and the projections ν^u have the universal property that defines a limit. In any category, this property determines the limit uniquely up to isomorphism.

3.4.4 Definition. For a U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ in $\mathcal{C}(R)$ the complex $\lim_{u\in U} N^u$ together with the canonical morphisms $\{\nu^u\}_{u\in U}$, constructed in 3.4.3, is called the *limit* of $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ in $\mathcal{C}(R)$.

Remark. Other names for the limit defined above are 'inverse limit' and 'projective limit'; other symbols used for this gadget are $\varprojlim$ and proj lim.

3.4.5 Theorem. *Let $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ be a U-inverse system in $\mathcal{C}(R)$. The limit from* 3.4.4 *has the following universal property: For every family of morphisms $\{\alpha^u\colon M \to N^u\}_{u\in U}$ in $\mathcal{C}(R)$ with $\alpha^u = \nu^{uv}\alpha^v$ for all $u \leqslant v$, there is a unique morphism α that makes the next diagram commutative for all $u \leqslant v$,*

$$\begin{tikzcd}
 & & N^v \arrow[dd, "\nu^{uv}"] \\
M \arrow[r, dotted, "\alpha"] \arrow[urr, bend left, "\alpha^v"] \arrow[drr, bend right, "\alpha^u"'] & \lim\limits_{u\in U} N^u \arrow[ur, "\nu^v"] \arrow[dr, "\nu^u"'] & \\
 & & N^u\,.
\end{tikzcd}$$

The morphism α is given by $m \mapsto (\alpha^u(m))_{u\in U}$.

Proof. Let $\{\alpha^u\colon M \to N^u\}_{u\in U}$ be a family of morphisms with $\alpha^u = \nu^{uv}\alpha^v$ for all $u \leqslant v$. The equalities $\alpha^u = \nu^{uv}\alpha^v$ ensure that the morphism $\alpha\colon M \to \prod_{u\in U} N^u$ from 3.1.15, given by $m \mapsto (\alpha^u(m))_{u\in U}$, maps to the subcomplex $\lim_{u\in U} N^u$.

It is evident from the definition that α satisfies $\alpha^u = \nu^u\alpha$ for all $u \in U$. Moreover, for any morphism $\alpha'\colon M \to \lim_{u\in U} N^u$ that satisfies $\alpha^u = \nu^u\alpha'$ for all $u \in U$, one has $\alpha'(m) = (\nu^u(\alpha'(m))_{u\in U} = (\alpha^u(m))_{u\in U} = \alpha(m)$, so α is unique. $\square$

3.4.6. With the notation from 3.4.5, notice that one has $\operatorname{Ker}\alpha = \bigcap_{u\in U}\operatorname{Ker}\alpha^u$; in particular, α is injective if and only if $\bigcap_{u\in U}\operatorname{Ker}\alpha^u = 0$ holds.

3.4.7. It follows readily from 3.4.3 and 3.4.5 that the full subcategories $\mathcal{M}(R)$ and $\mathcal{M}_{\mathrm{gr}}(R)$ of $\mathcal{C}(R)$ are closed under limits, that is, if $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ is a U-inverse system in $\mathcal{M}(R)$ or in $\mathcal{M}_{\mathrm{gr}}(R)$, then $\lim_{u\in U} N^u$ belongs to $\mathcal{M}(R)$ or $\mathcal{M}_{\mathrm{gr}}(R)$. It follows that for every U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ in $\mathcal{C}(R)$ one has

$$\big(\lim_{u\in U} N^u\big)_i = \lim_{u\in U}(N^u_i) \qquad\text{and}\qquad \big(\lim_{u\in U} N^u\big)^\natural = \lim_{u\in U}(N^u)^\natural\,.$$

3.4.8 Example. Let $\{N^u\}_{u\in U}$ be a family of R-complexes. Endowed with the discrete order, U is a preordered set, and $\{\nu^{uu} = 1^{N^u}\}_{u\in U}$ is a U-inverse system with $\lim_{u\in U} N^u = \prod_{u\in U} N^u$ and $\nu^u = \varpi^u$ for all $u \in U$. Thus, every product is a limit.

Every complex is a limit of bounded below quotient complexes.

3.4.9 Example. Let N be an R-complex; there is a $\mathbb{Z}$-inverse system in $\mathcal{C}(R)$,

$$\{\nu^{uv}\colon N_{\geqslant -v} \twoheadrightarrow N_{\geqslant -u}\}_{u\leqslant v}\,.$$

The canonical surjections $\beta^u\colon N \twoheadrightarrow N_{\geqslant -u}$ satisfy $\beta^u = \nu^{uv}\beta^v$ for all $u \leqslant v$, so by the universal property of limits 3.4.5 there is a unique morphism $\beta\colon N \to \lim_{u\in\mathbb{Z}} N_{\geqslant -u}$

given by $\beta(n) = (\beta^u(n))_{u\in\mathbb{Z}}$ for $n \in N$. It is injective by 3.4.6. To see that β is surjective, let $(n^u)_{u\in\mathbb{Z}}$ in $\lim_{u\in\mathbb{Z}} N_{\geqslant -u}$ be homogeneous of degree $-w$. In particular, every n^u is a homogeneous element in $N_{\geqslant -u}$ of degree $-w$, that is, $n^u \in (N_{\geqslant -u})_{-w}$. It follows that $n^u = 0$ holds if $w > u$. For $w \leqslant u$ one has $\nu^{wu}(n^u) = n^w$ by 3.4.3, which just means that $n^u = n^w$ as $\nu^{wu}_{-w}\colon (N_{\geqslant -u})_{-w} \to (N_{\geqslant -w})_{-w}$ is the identity map. The element $n = n^w$ is in $(N_{\geqslant -w})_{-w} = N_{-w}$, that is, n is a homogeneous element in N of degree $-w$, and one has $\beta(n) = (n^u)_{u\in\mathbb{Z}}$. Indeed, $\beta^u(n) = 0 = n^u$ holds for $w > u$, and $\beta^u(n) = n = n^w = n^u$ holds for $w \leqslant u$. Thus, β is an isomorphism.

As is the case for the limit in any category, the limit in $\mathcal{C}(R)$ also acts on morphisms. This is explained in the following definition.

3.4.10 Definition. Let $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ and $\{\mu^{uv}\colon M^v \to M^u\}_{u\leqslant v}$ be U-inverse systems in $\mathcal{C}(R)$. A family of morphisms $\{\beta^u\colon N^u \to M^u\}_{u\in U}$ in $\mathcal{C}(R)$ that satisfies $\beta^u\nu^{uv} = \mu^{uv}\beta^v$ for all $u \leqslant v$ is called a *morphism of U-inverse systems*. Such a morphism is called injective (surjective) if each map β^u is injective (surjective).

Given a morphism $\{\beta^u\colon N^u \to M^u\}_{u\in U}$ of U-inverse systems, it follows from the universal property of limits 3.4.5 that the map given by $(n^u)_{u\in U} \mapsto (\beta^u(n^u))_{u\in U}$ is the unique morphism that makes the next diagram commutative for all $u \leqslant v$ in U,

$$\begin{tikzcd}
N^v \arrow[rrr, "\beta^v"] \arrow[ddd, "\nu^{uv}"'] & & & M^v \arrow[ddd, "\mu^{uv}"] \\
 & \lim\limits_{u\in U} N^u \arrow[ul, "\nu^v"'] \arrow[r, dotted] \arrow[dl, "\nu^u"] & \lim\limits_{u\in U} M^u \arrow[ur, "\mu^v"] \arrow[dr, "\mu^u"'] & \\
 & & & \\
N^u \arrow[rrr, "\beta^u"'] & & & M^u\,.
\end{tikzcd}$$

This morphism is called the *limit* of $\{\beta^u\colon N^u \to M^u\}_{u\in U}$ and denoted $\lim_{u\in U} \beta^u$.

3.4.11 Example. Let $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ be a U-inverse system. Because the maps ν^{uv} are morphisms in $\mathcal{C}(R)$, the family $\{\partial^{N^u}\colon N^u \to \Sigma N^u\}_{u\in U}$ is a morphism of U-inverse systems. From the definitions one has $\lim_{u\in U} \partial^{N^u} = \partial^{\lim_{u\in U} N^u}$.

The next result shows that limits are left exact. Exactness of limits is a delicate issue, not to say a rare occurrence. A sufficient condition for exactness of certain limits is given in 3.5.17, and an example of a non-exact limit is given in 3.5.18.

3.4.12 Lemma. *Let $\{\alpha^u\colon X^u \to N^u\}_{u\in U}$ and $\{\beta^u\colon N^u \to M^u\}_{u\in U}$ be morphisms of U-inverse systems in $\mathcal{C}(R)$. If the sequence*

$$0 \longrightarrow X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u$$

is exact for every $u \in U$, then the next sequence is exact,

$$0 \longrightarrow \lim_{u\in U} X^u \xrightarrow{\lim_{u\in U}\alpha^u} \lim_{u\in U} N^u \xrightarrow{\lim_{u\in U}\beta^u} \lim_{u\in U} M^u\,.$$

Proof. Let $\nabla(U)$ be as in 3.2.3 and set $\alpha^{(u,v)} = \alpha^u$ and $\beta^{(u,v)} = \beta^u$ for $(u,v) \in \nabla(U)$. In view of 3.1.18 and 3.4.10 there is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \prod_{u\in U} X^u \arrow[r, "\prod_{u\in U}\alpha^u"] \arrow[d, "\varDelta^\chi"] & \prod_{u\in U} N^u \arrow[r, "\prod_{u\in U}\beta^u"] \arrow[d, "\varDelta^\nu"] & \prod_{u\in U} M^u \arrow[d, "\varDelta^\mu"] \\
0 \arrow[r] & \prod_{(u,v)\in\nabla(U)} X^{(u,v)} \arrow[r, "\prod_{(u,v)\in\nabla(U)}\alpha^{(u,v)}"] & \prod_{(u,v)\in\nabla(U)} N^{(u,v)} \arrow[r, "\prod_{(u,v)\in\nabla(U)}\beta^{(u,v)}"] & \prod_{(u,v)\in\nabla(U)} M^{(u,v)}\,,
\end{tikzcd}$$

where the vertical morphisms $\varDelta$ are defined in 3.4.3. It follows from 2.1.42 that there is an exact sequence $0 \longrightarrow \operatorname{Ker}\varDelta^\chi \longrightarrow \operatorname{Ker}\varDelta^\nu \longrightarrow \operatorname{Ker}\varDelta^\mu$ in $\mathcal{C}(R)$, which in view of 3.4.3 is the desired one. $\square$

The product of preordered sets U and U' is the cartesian product $U\times U'$ equipped with the product order, i.e. $(u,u') \leqslant (v,v')$ holds in $U\times U'$ if and only if one has $u\leqslant v$ in U and $u'\leqslant v'$ in U'. It is elementary to see that $U\times U'$ is a preordered set.

3.4.13 Proposition. *Let U and U' be preordered sets and*

$$\bigl\{\nu^{(u,u')(v,v')}\colon N^{(v,v')} \longrightarrow N^{(u,u')}\bigr\}_{(u,u')\leqslant(v,v')}$$

a $(U\times U')$-inverse system in $\mathcal{C}(R)$. There are isomorphisms,

$$\lim_{u'\in U'}\bigl(\lim_{u\in U} N^{(u,u')}\bigr) \;\cong\; \lim_{(u,u')\in U\times U'} N^{(u,u')} \;\cong\; \lim_{u\in U}\bigl(\lim_{u'\in U'} N^{(u,u')}\bigr)\,.$$

Proof. Due to the symmetry, it suffices to prove the first isomorphism.

Let $u'\in U'$ and note that $\{\nu^{(u,u')(v,u')}\colon N^{(v,u')} \longrightarrow N^{(u,u')}\}_{u\leqslant v}$ is a U-inverse system. Set $X^{u'} = \lim_{u\in U} N^{(u,u')}$ and write $\{\varphi^{u,u'}\colon X^{u'} \longrightarrow N^{(u,u')}\}_{u\in U}$ for the canonical morphisms; now $\nu^{(u,u')(v,u')}\varphi^{v,u'} = \varphi^{u,u'}$ holds for $u\leqslant v$ in U. Let $u'\leqslant v'$ in U' be given. For all $u\leqslant v$ in U the next diagram is, by assumption, commutative,

$$\begin{tikzcd}
N^{(v,v')} \arrow[r, "\nu^{(v,u')(v,v')}"] \arrow[d, "\nu^{(u,v')(v,v')}"'] & N^{(v,u')} \arrow[d, "\nu^{(u,u')(v,u')}"] \\
N^{(u,v')} \arrow[r, "\nu^{(u,u')(u,v')}"] & N^{(u,u')}\,.
\end{tikzcd}$$

Thus $\{\nu^{(u,u')(u,v')}\colon N^{(u,u')} \longrightarrow N^{(u,v')}\}_{u\in U}$ is a morphism from the U-inverse system $\{\nu^{(u,v')(v,v')}\colon N^{(v,v')} \longrightarrow N^{(u,v')}\}_{u\leqslant v}$ to $\{\nu^{(u,u')(v,u')}\colon N^{(v,u')} \longrightarrow N^{(u,u')}\}_{u\leqslant v}$, so 3.4.10 yields an induced morphism $\chi^{u'v'} = \lim_{u\in U}\nu^{(u,u')(u,v')}\colon N^{v'} \longrightarrow N^{u'}$, which is the unique morphism that makes the diagram

(★)
$$\begin{tikzcd}
X^{v'} \arrow[r, "\chi^{u'v'}"] \arrow[d, "\varphi^{u,v'}"'] & X^{u'} \arrow[d, "\varphi^{u,u'}"] \\
N^{(u,v')} \arrow[r, "\nu^{(u,u')(u,v')}"] & N^{(u,u')}
\end{tikzcd}$$

commutative for every $u\in U$. From the uniqueness of this morphism, it follows that $\{\chi^{u'v'}\colon X^{v'} \longrightarrow X^{u'}\}_{u'\leqslant v'}$ is a U'-inverse system. Set $X = \lim_{u'\in U'} X^{u'}$ and write $\{\chi^{u'}\colon X \longrightarrow X^{u'}\}_{u'\in U'}$ for the canonical morphisms.

Notice that X is the iterated limit on the left-hand side of the desired isomorphism. Set $N = \lim_{(u,u')\in U\times U'} N^{(u,u')}$ and write $\{\nu^{(u,u')}\colon N \to N^{(u,u')}\}_{(u,u')\in U\times U'}$ for the canonical morphisms. Next we show that N is isomorphic to X by constructing a pair of mutually inverse morphisms $\alpha\colon X \to N$ and $\beta\colon N \to X$.

For every $(u,u') \in U \times U'$ set $\alpha^{(u,u')} = \varphi^{u,u'}\chi^{u'}\colon X \to N^{(u,u')}$. For $(u,u') \leqslant (v,v')$ in $U \times U'$ the definitions and commutativity of $(\star)$ yield:

$$\begin{aligned}
\nu^{(u,u')(v,v')}\alpha^{(v,v')} &= \nu^{(u,u')(u,v')}\nu^{(u,v')(v,v')}\varphi^{v,v'}\chi^{v'}\\
&= \nu^{(u,u')(u,v')}\varphi^{u,v'}\chi^{v'}\\
&= \varphi^{u,u'}\chi^{u'v'}\chi^{v'}\\
&= \varphi^{u,u'}\chi^{u'}\\
&= \alpha^{(u,u')}\,.
\end{aligned}$$

Thus 3.4.5 yields a unique morphism $\alpha\colon X \to N$ with $\nu^{(u,u')}\alpha = \alpha^{(u,u')} = \varphi^{u,u'}\chi^{u'}$ for every $(u,u') \in U \times U'$.

Let $u' \in U'$. For all $u \leqslant v$ in U one has $\nu^{(u,u')(v,u')}\nu^{(v,u')} = \nu^{(u,u')}$, so by 3.4.5 there exists a unique morphism $\beta^{u'}\colon N \to X^{u'}$ with $\varphi^{u,u'}\beta^{u'} = \nu^{(u,u')}$ for every $u \in U$. Let $u' \leqslant v'$ in U' be given; it follows from $(\star)$ and the definitions that

$$\varphi^{u,u'}\chi^{u'v'}\beta^{v'} = \nu^{(u,u')(u,v')}\varphi^{u,v'}\beta^{v'} = \nu^{(u,u')(u,v')}\nu^{(u,v')} = \nu^{(u,u')} = \varphi^{u,u'}\beta^{u'}$$

holds for every $u \in U$. Thus there is an identity $\chi^{u'v'}\beta^{v'} = \beta^{u'}$ of maps from N to $X^{u'}$, as $X^{u'}$ is a limit with canonical morphisms $\varphi^{u,u'}$. Another application of 3.4.5 yields a unique morphism $\beta\colon N \to X$ with $\chi^{u'}\beta = \beta^{u'}$ for every $u' \in U'$.

To verify $\beta\alpha = 1^X$, it is enough to prove $\chi^{u'}\beta\alpha = \chi^{u'}$ for every $u' \in U'$, and to that end it suffices to argue that $\varphi^{u,u'}\chi^{u'}\beta\alpha = \varphi^{u,u'}\chi^{u'}$ for every $u \in U$. And indeed,

$$\varphi^{u,u'}\chi^{u'}\beta\alpha = \varphi^{u,u'}\beta^{u'}\alpha = \nu^{(u,u')}\alpha = \alpha^{(u,u')} = \varphi^{u,u'}\chi^{u'}\,.$$

To verify $\alpha\beta = 1^N$, it suffices to prove $\nu^{(u,u')}\alpha\beta = \nu^{(u,u')}$ for every $(u,u') \in U\times U'$. This hold as $\nu^{(u,u')}\alpha\beta = \varphi^{u,u'}\chi^{u'}\beta = \varphi^{u,u'}\beta^{u'} = \nu^{(u,u')}$. □

3.4.14 Corollary. *Let U and U' be sets and $\{M^{(u,u')}\}_{(u,u')\in U\times U'}$ a family of R-complexes. There are isomorphisms,*

$$\prod_{u'\in U'}\Big(\prod_{u\in U} M^{(u,u')}\Big) \cong \prod_{(u,u')\in U\times U'} M^{(u,u')} \cong \prod_{u\in U}\Big(\prod_{u'\in U'} M^{(u,u')}\Big)\,.$$

Proof. The isomorphisms follow immediately from 3.4.8 and 3.4.13. □

3.4.15 Proposition. *Let $(U,\leqslant)$ be a preordered filtered set and V a cofinal subset of U. For every U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ in $\mathcal{C}(R)$ there is an isomorphism $\lim_{v\in V} N^v \cong \lim_{u\in U} N^u$.*

Proof. For $u \in U$ denote by ν^u the canonical morphism $\lim_{u\in U} N^u \to N^u$, and for $v \in V$ denote by $\widetilde{\nu}^v$ the canonical morphism $\lim_{v\in V} N^v \to N^v$. For $v \leqslant w$ in V one has $\nu^v = \nu^{vw}\nu^w$, so by the universal property of limits there is a morphism $\nu\colon \lim_{u\in U} N^u \to \lim_{v\in V} N^v$ with $\widetilde{\nu}^v\nu = \nu^v$ for all $v \in V$. To see that ν is injective, let $n \in \operatorname{Ker}\nu$ and $u \in U$; we verify that $\nu^u(n) = 0$ holds. As V is cofinal in U there exits a w in V with $u \leqslant w$, and now one has $\nu^u(n) = \nu^{uw}\nu^w(n) = \nu^{uw}\widetilde{\nu}^w\nu(n) = 0$. To see that ν is surjective, let $x = (x^v)_{v\in V}$ be an element in $\lim_{v\in V} N^v$. For every $u \in U$ and elements v and w in V with $u \leqslant v \leqslant w$ one has

$$\nu^{uv}(x^v) - \nu^{uw}(x^w) \;=\; \nu^{uv}(x^v) - \nu^{uv}\nu^{vw}(x^w) \;=\; \nu^{uv}(x^v - \nu^{vw}(x^w)) \;=\; 0\,.$$

As V is filtered it follows that for every $u \in U$ one has $\nu^{uv}(x^v) = \nu^{uv'}(x^{v'})$ for all elements v and v' in V with $u \leqslant v$ and $u \leqslant v'$; indeed, $\nu^{uv}(x^v) = \nu^{uw}(x^w) = \nu^{uv'}(x^{v'})$ holds for every $w \in V$ with $v \leqslant w$ and $v' \leqslant w$. Now consider the element $n = (n^u)_{u\in U}$ in $\prod_{u\in U} N^u$ given by $n^u = \nu^{uv}(x^v)$ for any choice of $v \in V$ with $u \leqslant v$. For $(u,v) \in \nabla(U)$ choose w in V with $v \leqslant w$. One now has

$$n^u - \nu^{uv}(n^v) \;=\; \nu^{uw}(x^w) - \nu^{uv}\nu^{vw}(x^w) \;=\; \nu^{uw}(x^w) - \nu^{uw}(x^w) \;=\; 0\,,$$

so n belongs to $\lim_{u\in U} N^u$. For $v \in V$ one has $\widetilde{\nu}^v\nu(n) = \nu^v(n) = n^v = \nu^{vv}(x^v) = x^v$, so $\nu(n) = x$ holds as desired. □

Functors that Preserve Limits

3.4.16. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ a U-inverse system in $\mathcal{C}(R)$ with limit as in 3.4.4. The maps $\{\mathrm{F}(\nu^{uv})\colon \mathrm{F}(N^v) \to \mathrm{F}(N^u)\}_{u\leqslant v}$ form a U-inverse system in $\mathcal{C}(S)$; write $\lambda^u\colon \lim_{u\in U}\mathrm{F}(N^u) \to \mathrm{F}(N^u)$ for the canonical morphism, see (3.4.3.1). As $\mathrm{F}(\nu^u) = \mathrm{F}(\nu^{uv})\,\mathrm{F}(\nu^v)$ holds for all $u \leqslant v$ in U, the universal property of limits 3.4.5 yields a unique morphism

$$\mathrm{F}\big(\lim_{u\in U} N^u\big) \longrightarrow \lim_{u\in U}\mathrm{F}(N^u) \quad\text{ given by }\quad x \longmapsto (\mathrm{F}(\nu^u)(x))_{u\in U} \tag{3.4.16.1}$$

for $x \in \mathrm{F}(\lim_{u\in U} N^u)$, that makes the next diagram commutative for all $u \leqslant v$,

$$\begin{tikzcd}
 & & \mathrm{F}(N^v) \arrow[d, "\mathrm{F}(\nu^{uv})"] \\
\mathrm{F}\big(\lim\limits_{u\in U} N^u\big) \arrow[r, dotted] \arrow[urr, bend left, "\mathrm{F}(\nu^v)"] \arrow[drr, bend right, "\mathrm{F}(\nu^u)"'] & \lim\limits_{u\in U}\mathrm{F}(N^u) \arrow[ur, "\lambda^v"] \arrow[dr, "\lambda^u"'] & \\
 & & \mathrm{F}(N^u)\,.
\end{tikzcd}$$

3.4.17 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *preserves limits* if the morphism in (3.4.16.1) is an isomorphism for every U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ in $\mathcal{C}(R)$.

Remark. A functor that preserves limits is also called 'continuous'.

Even if a functor does not preserve (all) limits in the sense of 3.4.17, it may still preserve certain types of limits, meaning that the morphism in (3.4.16.1) is an isomorphism for every U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ provided that U is of a certain type. For example, every left exact functor preserves pullbacks, that is, limits formed over the preordered set in 3.4.30.

While 3.4.17 is a condition on objects, it carries over to morphisms.

3.4.18. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor and $\{\beta^u\colon N^u \to M^u\}_{u\in U}$ a morphism of U-inverse systems in $\mathcal{C}(R)$. It is easy to see that there is a commutative diagram,

$$\begin{tikzcd}
\mathrm{F}\bigl(\lim\limits_{u\in U} N^u\bigr) \arrow[r, "\mathrm{F}(\lim_{u\in U}\beta^u)"] \arrow[d] & \mathrm{F}\bigl(\lim\limits_{u\in U} M^u\bigr) \arrow[d] \\
\lim\limits_{u\in U} \mathrm{F}(N^u) \arrow[r, "\lim_{u\in U}\mathrm{F}(\beta^u)"] & \lim\limits_{u\in U} \mathrm{F}(M^u)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms (3.4.16.1). Thus, if F preserves limits, then the morphisms $\mathrm{F}(\lim_{u\in U}\beta^u)$ and $\lim_{u\in U}\mathrm{F}(\beta^u)$ are isomorphic.

3.4.19. Let $\tau\colon \mathrm{E} \to \mathrm{F}$ be a natural transformation of functors $\mathcal{C}(R) \to \mathcal{C}(S)$ and $\{N^u\}_{u\in U}$ a family of R-complexes. It is straightforward to verify that there is a commutative diagram in $\mathcal{C}(S)$,

$$\begin{tikzcd}
\mathrm{E}\bigl(\lim\limits_{u\in U} N^u\bigr) \arrow[r, "\tau^{(\lim_{u\in U} N^u)}"] \arrow[d] & \mathrm{F}\bigl(\lim\limits_{u\in U} N^u\bigr) \arrow[d] \\
\lim\limits_{u\in U} \mathrm{E}(N^u) \arrow[r, "\lim_{u\in U}\tau^{N^u}"] & \lim\limits_{u\in U} \mathrm{F}(N^u)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms; see (3.4.16.1).

3.4.20 Lemma. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a functor. If* F *is left exact and preserves products, then it preserves limits.*

Proof. Let $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ be a U-inverse system in $\mathcal{C}(R)$. Consider the following commutative diagram in $\mathcal{C}(S)$ where the middle and right-hand vertical maps are given by (3.1.20.1) and the left-hand vertical map is given by (3.4.16.1),

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{F}\bigl(\lim\limits_{u\in U} N^u\bigr) \arrow[r] \arrow[d] & \mathrm{F}\bigl(\prod\limits_{u\in U} N^u\bigr) \arrow[r, "\mathrm{F}(\varDelta^\nu)"] \arrow[d] & \mathrm{F}\bigl(\prod\limits_{(u,v)\in\nabla(U)} N^{(u,v)}\bigr) \arrow[d] \\
0 \arrow[r] & \lim\limits_{u\in U} \mathrm{F}(N^u) \arrow[r] & \prod\limits_{u\in U} \mathrm{F}(N^u) \arrow[r, "\varDelta^{\mathrm{F}(\nu)}"] & \prod\limits_{(u,v)\in\nabla(U)} \mathrm{F}(N^{(u,v)})\,.
\end{tikzcd}$$

The rows in this diagram are exact by 3.4.3 and left exactness of F. The middle and right-hand vertical maps are isomorphisms by assumption, so it follows from the Five Lemma 2.1.41 that the left-hand vertical map is an isomorphism. □

The next two results show that the shift and kernel functors preserves limits.

3.4.21 Proposition. *Let s be an integer and* $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ *a U-inverse system in* $\mathcal{C}(R)$*. The canonical morphism in* $\mathcal{C}(R)$*,*

$$\Sigma^s \lim_{u\in U} N^u \longrightarrow \lim_{u\in U} \Sigma^s N^u\,,$$

given by $x \mapsto ((\Sigma^s \nu^u)(x))_{u\in U}$*, is an isomorphism.*

Proof. The assertion is immediate from 3.4.20 and 3.1.21. □

3.4.22 Proposition. *For every* U*-inverse system* $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ *in* $\mathcal{C}(R)$*, the canonical morphism in* $\mathcal{C}(R)$*,*

$$\mathrm{Z}\big(\lim_{u\in U} N^u\big) \longrightarrow \lim_{u\in U} \mathrm{Z}(N^u)\,,$$

given by $x \mapsto (\mathrm{Z}(\nu^u)(x))_{u\in U}$*, is an isomorphism.*

Proof. The assertion is immediate from 3.4.20, 2.2.16, and 3.1.22(a). □

3.4.23 Proposition. *Let* M *be an* R*-complex and* $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ *a* U*-inverse system in* $\mathcal{C}(R)$*. The canonical morphism in* $\mathcal{C}(\Bbbk)$*,*

$$\mathrm{Hom}_R\big(M, \lim_{u\in U} N^u\big) \longrightarrow \lim_{u\in U} \mathrm{Hom}_R(M, N^u)\,,$$

given by $\vartheta \mapsto (\mathrm{Hom}_R(M,\nu^u)(\vartheta))_{u\in U} = (\nu^u\vartheta)_{u\in U}$*, is an isomorphism.*

Proof. The assertion is immediate from 3.4.20, 2.3.10, and 3.1.24. □

3.4.24. Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor and $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R)$ with colimit as in 3.2.4. The maps $\{\mathrm{G}(\mu^{vu})\colon \mathrm{G}(M^v) \to \mathrm{G}(M^u)\}_{u\leqslant v}$ form a U-inverse system in $\mathcal{C}(S)$; write $\lambda^u\colon \lim_{u\in U}\mathrm{G}(M^u) \to \mathrm{G}(M^u)$ for the canonical morphisms, see (3.4.3.1). As $\mathrm{G}(\mu^u) = \mathrm{G}(\mu^{vu})\,\mathrm{G}(\mu^v)$ holds for all $u \leqslant v$ in U, the universal property of limits 3.4.5 yields a unique morphism

$$\mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) \longrightarrow \lim_{u\in U}\mathrm{G}(M^u) \quad \text{given by} \quad x \longmapsto (\mathrm{G}(\mu^u)(x))_{u\in U} \tag{3.4.24.1}$$

for $x \in \mathrm{G}(\operatorname{colim}_{u\in U} M^u)$, that makes the next diagram commutative for all $u \leqslant v$,

$$\begin{tikzcd}
 & & \mathrm{G}(M^v) \arrow[dd, "\mathrm{G}(\mu^{vu})"] \\
\mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, dotted] \arrow[urr, bend left, "\mathrm{G}(\mu^v)"] \arrow[drr, bend right, "\mathrm{G}(\mu^u)"'] & \lim_{u\in U}\mathrm{G}(M^u) \arrow[ur, "\lambda^v"] \arrow[dr, "\lambda^u"'] & \\
 & & \mathrm{G}(M^u)\,.
\end{tikzcd}$$

3.4.25 Definition. A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *preserves limits* if the morphism in (3.4.24.1) is an isomorphism for every U-direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ in $\mathcal{C}(R)$.

While 3.4.25 is a condition on objects, it carries over to morphisms.

3.4.26. Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor and $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ a morphism of U-direct systems in $\mathcal{C}(R)$. It is easy to verify that there is a commutative diagram,

$$\begin{tikzcd}
\mathrm{G}\big(\operatorname*{colim}_{u\in U} N^u\big) \arrow[r, "\mathrm{G}(\operatorname{colim}_{u\in U}\alpha^u)"] \arrow[d] & \mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[d] \\
\lim_{u\in U} \mathrm{G}(N^u) \arrow[r, "\lim_{u\in U}\mathrm{G}(\alpha^u)"] & \lim_{u\in U} \mathrm{G}(M^u)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms (3.4.24.1). Thus, if G preserves limits, then the morphisms $\mathrm{G}(\operatorname{colim}_{u\in U}\alpha^u)$ and $\lim_{u\in U}\mathrm{G}(\alpha^u)$ are isomorphic.

3.4.27. Let $\tau\colon \mathrm{G}\to\mathrm{J}$ be a natural transformation of functors $\mathcal{C}(R)^{\mathrm{op}}\to\mathcal{C}(S)$ and $\{M^u\}_{u\in U}$ a family of R-complexes. It is straightforward to verify that there is a commutative diagram in $\mathcal{C}(S)$,

$$\begin{tikzcd}
\mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r, "\tau^{(\operatorname{colim}_{u\in U} M^u)}"] \arrow[d] & \mathrm{J}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[d] \\
\lim_{u\in U} \mathrm{G}(M^u) \arrow[r, "\lim_{u\in U}\tau^{M^u}"] & \lim_{u\in U} \mathrm{J}(M^u)\,,
\end{tikzcd}$$

where the vertical maps are the canonical morphisms (3.4.24.1).

3.4.28 Lemma. *Let* $\mathrm{G}\colon\mathcal{C}(R)^{\mathrm{op}}\to\mathcal{C}(S)$ *be a functor. If* G *is left exact and preserves products, then it preserves limits.*

Proof. Let $\{\mu^{vu}\colon M^u\to M^v\}_{u\leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. Consider the following commutative diagram in $\mathcal{C}(S)$ where the middle and right-hand vertical maps are given by (3.1.26.1) and the left-hand vertical map is given by (3.4.24.1),

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{G}\big(\operatorname*{colim}_{u\in U} M^u\big) \arrow[r] \arrow[d] & \mathrm{G}\big(\coprod_{u\in U} M^u\big) \arrow[r, "\mathrm{G}(\varDelta_\mu)"] \arrow[d] & \mathrm{G}\big(\coprod_{(u,v)\in\nabla(U)} M^{(u,v)}\big) \arrow[d] \\
0 \arrow[r] & \lim_{u\in U}\mathrm{G}(M^u) \arrow[r] & \prod_{u\in U}\mathrm{G}(M^u) \arrow[r, "\varDelta^{\mathrm{G}(\mu)}"] & \prod_{(u,v)\in\nabla(U)}\mathrm{G}(M^{(u,v)})\,.
\end{tikzcd}$$

The rows in this diagram are exact by 3.2.3, 3.4.3, and left exactness of G. The middle and right-hand vertical maps are isomorphisms by assumption, so it follows from the Five Lemma 2.1.41 that the left-hand vertical map is an isomorphism. □

Together with 3.4.23 the next result shows that the Hom functor preserves limits.

3.4.29 Proposition. *Let N be an R-complex and $\{\mu^{vu}\colon M^u\to M^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R)$. The canonical morphism in $\mathcal{C}(\Bbbk)$,*

$$\operatorname{Hom}_R\big(\operatorname*{colim}_{u\in U} M^u, N\big)\longrightarrow \lim_{u\in U}\operatorname{Hom}_R(M^u,N)\,,$$

given by $\vartheta\mapsto(\operatorname{Hom}_R(\mu^u,N)(\vartheta))_{u\in U}=(\vartheta\mu^u)_{u\in U}$, is an isomorphism.

Proof. The assertion is immediate from 3.4.28, 2.3.10, and 3.1.27. □

PULLBACKS

Simple non-trivial limits arise from three term inverse systems $N \to Y \leftarrow M$.

3.4.30 Construction. Let $U = \{u, v, w\}$ be a set, preordered as follows $v \geqslant u \leqslant w$. Given a diagram $N \xrightarrow{\beta} Y \xleftarrow{\alpha} M$ in $\mathcal{C}(R)$, set

$$N^v = N\,, \quad N^u = Y\,, \quad N^w = M\,,$$
$$\nu^{vv} = 1^N\,, \quad \nu^{uv} = \beta\,, \quad \nu^{uu} = 1^Y\,, \quad \nu^{uw} = \alpha\,, \quad \text{and} \quad \nu^{ww} = 1^M\,.$$

This defines a U-inverse system in $\mathcal{C}(R)$. It is straightforward to verify that the limit of this system is the kernel of the morphism $(\beta\ {-\alpha})^{\mathrm{T}} : N \oplus M \to Y$.

3.4.31 Definition. For a diagram $N \xrightarrow{\beta} Y \xleftarrow{\alpha} M$ in $\mathcal{C}(R)$, the limit of the U-inverse system in 3.4.30 is called the *pullback* of (β, α) and denoted $N \sqcap_Y M$. Let

$$\alpha' : N \sqcap_Y M \longrightarrow N \qquad \text{and} \qquad \beta' : N \sqcap_Y M \longrightarrow M$$

be the canonical morphisms (3.4.3.1); they are given by $(n, m) \mapsto n$ and $(n, m) \mapsto m$.

REMARK. As for the limit, the notation for the pullback suppresses the morphisms. Other names for the pullback are 'fibered product' and 'Cartesian square'.

3.4.32. Given morphisms $\alpha : M \to Y$ and $\beta : N \to Y$ in $\mathcal{C}(R)$, the pullbacks of (β, α) and (α, β) are isomorphic via the map that comes from the canonical isomorphism $N \oplus M \cong M \oplus N$.

3.4.33. Adopt the notation from 3.4.31. Given a diagram $N \xleftarrow{\alpha''} X \xrightarrow{\beta''} M$ in $\mathcal{C}(R)$ with $\beta\alpha'' = \alpha\beta''$, it follows from 3.4.5 that the assignment

$$x \longmapsto (\alpha''(n), \beta''(m))$$

defines the unique morphism that makes the next diagram commutative,

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left, "\beta''"] \arrow[ddr, bend right, "\alpha''"'] & & \\
& N \sqcap_Y M \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & M \arrow[d, "\alpha"] \\
& N \arrow[r, "\beta"] & Y\,.
\end{tikzcd}$$

3.4.34 Theorem. *Adopt the notation from* 3.4.31. *There is a commutative diagram in* $\mathcal{C}(R)$ *with exact rows and columns,*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Ker}\alpha' \arrow[r, "\bar{\beta}"] \arrow[d] & \operatorname{Ker}\alpha \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta' \arrow[r] \arrow[d, "\bar{\alpha}"] & N \sqcap_Y M \arrow[r, "\beta'"] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, "\alpha"] & \operatorname{Coker}\beta' \arrow[r] \arrow[d, "\bar{\bar{\alpha}}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & N \arrow[r, "\beta"] \arrow[d] & Y \arrow[r] \arrow[d] & \operatorname{Coker}\beta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Coker}\alpha' \arrow[r, "\bar{\bar{\beta}}"] \arrow[d] & \operatorname{Coker}\alpha \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0\,, &
\end{tikzcd}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are the induced morphisms on kernels and $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are the induced morphisms on cokernels. In particular, $\bar{\alpha}$ and $\bar{\beta}$ are isomorphisms, $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are injective, and the following assertions hold.

(a) *If α is surjective, then α' is surjective.*
(b) *α is injective if and only if α' is injective.*
(c) *If β is surjective, then β' is surjective.*
(d) *β is injective if and only if β' is injective.*

Proof. With $W = \operatorname{Coker}\bar{\bar{\alpha}}$ the sequence $\operatorname{Coker}\alpha' \to \operatorname{Coker}\alpha \to W \to 0$ is exact by 2.1.43.

For $m \in M$ one has $\bar{\bar{\alpha}}([m]_{\operatorname{Im}\beta'}) = [\alpha(m)]_{\operatorname{Im}\beta}$, so if $[m]_{\operatorname{Im}\beta'}$ is in $\operatorname{Ker}\bar{\bar{\alpha}}$, then there exists $n \in N$ with $\beta(n) = \alpha(m)$. Thus (n, m) is an element in $N \sqcap_Y M$ with $\beta'(n, m) = m$, so $m \in \operatorname{Im}\beta'$. Therefore $\bar{\bar{\alpha}}$ is injective. By symmetry, $\bar{\bar{\beta}}$ is injective, see 3.4.32.

The kernel of β' consists of all pairs $(n, 0)$ in $N \oplus M$ with $\beta(n) = 0$, so evidently the homomorphism $\bar{\alpha}\colon \operatorname{Ker}\beta' \to \operatorname{Ker}\beta$, which is given by $(n, 0) \mapsto n$, is an isomorphism. By symmetry, $\bar{\beta}$ is an isomorphism as well.

The assertions (a)–(d) are direct consequences of the established diagram. □

Exercises

E 3.4.1 Let $\{\nu^{uv}\colon N^v \to N^u\}_{u \leqslant v}$ be a U-inverse system in $\mathcal{C}(R)$. Let $\{\widetilde{\nu}^u\colon L \to N^u\}_{u \in U}$ be a family of morphisms that satisfy the next conditions. (1) One has $\widetilde{\nu}^v = \nu^{uv}\widetilde{\nu}^v$ for all $u \leqslant v$. (2) For every family $\{\alpha^u\colon M \to N^u\}_{u \in U}$ of morphisms with $\alpha^u = \nu^{uv}\alpha^v$ for all $u \leqslant v$ there exists a unique morphism $\alpha\colon M \to L$ with $\widetilde{\nu}^u\alpha = \alpha^u$ for all $u \in U$. Show that there is an isomorphism $\varphi\colon L \to \lim_{u \in U} N^u$ with $\nu^u\varphi = \widetilde{\nu}^u$ for every $u \in U$. Conclude that the universal property determines the limit uniquely up to isomorphism.

E 3.4.2 (Cf. 3.4.7) Show that the limit in $\mathcal{C}(R)$ of an inverse system of morphisms of graded R-modules is a graded R-module. Conclude, in particular, that $\mathcal{M}_{\mathrm{gr}}(R)$ has limits.

E 3.4.3 (Cf. 3.4.7) Show that the limit in $\mathcal{C}(R)$ of a inverse system of homomorphisms of R-modules is an R-module. Conclude, in particular, that the category $\mathcal{M}(R)$ has limits.

E 3.4.4 Fix a preordered set U. Show that U-inverse systems in $\mathcal{C}(R)$ and their morphisms form an Abelian category and that the limit is a left exact functor from this category to $\mathcal{C}(R)$.

E 3.4.5 Let $\{\nu^{uv}\colon N^v \to N^u\}_{u \leqslant v}$ be a U-inverse system in $\mathcal{C}(R)$. Show that if U has a greatest element, w, then there is an isomorphism $\lim_{u\in U} N^u \cong N^w$.

E 3.4.6 Let $(U, \leqslant)$ be a preordered filtered set and $\{N^u\}_{u\in U}$ a family of subcomplexes of an R-complex N with $N^v \subseteq N^u$ for $u \leqslant v$ in U. Show that there is a U-inverse system $\{\nu^{uv}\colon N^v \to N^u\}_{u\leqslant v}$ with $\lim_{u\in U} N^u \cong \bigcap_{u\in U} N^u$.

E 3.4.7 Generalize the result in E 3.1.12 by showing that every functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ that has a left adjoint preserves limits.

E 3.4.8 (a) Show that U-inverse systems in $\mathcal{C}(R)^{\mathrm{op}}$ correspond to U-direct systems in $\mathcal{C}(R)$ and that U-direct systems in $\mathcal{C}(R)^{\mathrm{op}}$ correspond to U-inverse systems in $\mathcal{C}(R)$. (b) Show limits in $\mathcal{C}(R)^{\mathrm{op}}$ correspond to colimits in $\mathcal{C}(R)$ and that colimits in $\mathcal{C}(R)^{\mathrm{op}}$ correspond to limits in $\mathcal{C}(R)$.

E 3.4.9 (Cf. 3.4.30) Verify the isomorphism $\lim_{u\in U} N^u \cong \operatorname{Ker}(\beta - \alpha)$ in 3.4.30.

E 3.4.10 (a) Consider the diagram in 3.4.34 in the case where $\mathfrak{a}$ and $\mathfrak{b}$ are left ideals in R and $\alpha\colon \mathfrak{a} \rightarrowtail R$ and $\beta\colon \mathfrak{b} \rightarrowtail R$ are the canonical homomorphisms. Show that $W \neq 0$.
(b) In the following two diagrams, the solid parts are given. Show that they can be completed to commutative diagrams with exact rows and columns, as depicted.

$$
\begin{tikzcd}
& & 0 \arrow[d, dotted] & 0 \arrow[d] & \\
& & K \arrow[r, dotted, no head, equal] \arrow[d, dotted] & K \arrow[d] & \\
0 \arrow[r, dotted] & L \arrow[r, dotted] \arrow[d, dotted, equal] & N \sqcap_Y M \arrow[r, dotted] \arrow[d, dotted] & M \arrow[r, dotted] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] & N \arrow[r] \arrow[d, dotted] & Y \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 0 \arrow[d, dotted] & 0 \arrow[d] & & \\
& K \arrow[r, dotted, no head, equal] \arrow[d, dotted] & K \arrow[d] & & \\
0 \arrow[r, dotted] & N \sqcap_Y M \arrow[r, dotted] \arrow[d, dotted] & M \arrow[r, dotted] \arrow[d] & D \arrow[r, dotted] \arrow[d, dotted, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d, dotted] & Y \arrow[r] \arrow[d] & D \arrow[r] & 0 \\
& 0 & 0\,. & &
\end{tikzcd}
$$

E 3.4.11 Consider a diagram of R-complexes, not a priori assumed to be commutative,

$$
\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & M \arrow[d, "\gamma"] \\
N \arrow[r, "\delta"] & Y\,.
\end{tikzcd}
$$

(a) Show that it is a pushout diagram if and only if the next sequence is exact,

$$X \xrightarrow{\left(\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right)} \begin{matrix} M \\ \oplus \\ N \end{matrix} \xrightarrow{(\gamma\ -\delta)} Y \longrightarrow 0\,.$$

(b) Show that it is a pullback diagram if and only if the next sequence is exact,

$$0 \longrightarrow X \xrightarrow{\left(\begin{smallmatrix}\beta\\ \alpha\end{smallmatrix}\right)} \begin{matrix} M \\ \oplus \\ N \end{matrix} \xrightarrow{(\gamma\ -\delta)} Y\,.$$

(c) Show that if it is a pushout diagram and α is injective, then it is a pullback diagram.
(d) Show that if it is a pullback diagram and δ is surjective, then it is a pushout diagram.

E 3.4.12 (Cf. 3.4.18) Verify that the diagram in 3.4.18 is commutative.

E 3.4.13 (Cf. 3.4.19) Verify that the diagram in 3.4.19 is commutative.

E 3.4.14 Let $\Bbbk$ be a field and $\mathcal{L}$ the category of commutative local $\Bbbk$-algebras. (a) Show that $\Bbbk$ is a zero object in $\mathcal{L}$. (b) For R and S in $\mathcal{L}$, show that the pullback $R \sqcap_{\Bbbk} S$ in the category of $\Bbbk$-modules yields a product in $\mathcal{L}$. (c) Show that the full subcategory of $\mathcal{L}$ whose objects are integral domains does not have a product.

3.5 Towers and the Mittag-Leffler Condition

SYNOPSIS. Tower; (trivial) Mittag-Leffler Condition.

Inverse systems that are, essentially, sequences of morphisms play a special role for at least two reasons: they occur frequently and they come with a simple sufficient condition for exactness.

3.5.1 Construction. Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a sequence of morphisms in $\mathcal{C}(R)$. It determines a $\mathbb{Z}$-inverse system $\{\nu^{uv} : N^v \to N^u\}_{u\leqslant v}$ as follows: set

$$\nu^{uu} = 1^{N^u} \quad \text{for all } u \text{ in } \mathbb{Z} \qquad \text{and} \qquad \nu^{uv} = \lambda^{u+1}\cdots\lambda^{v} \quad \text{for all } u < v \text{ in } \mathbb{Z}\,.$$

Given additional sequences $\{\kappa^u : M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$ and $\{\beta^u : N^u \to M^u\}_{u\in\mathbb{Z}}$ of morphisms with $\beta^{u-1}\lambda^u = \kappa^u\beta^u$ for all $u \in \mathbb{Z}$, it is elementary to see that $\{\beta^u\}_{u\in\mathbb{Z}}$ is a morphism of the inverse systems determined by $\{\lambda^u\}_{u\in\mathbb{Z}}$ and $\{\kappa^u\}_{u\in\mathbb{Z}}$.

For example, the $\mathbb{Z}$-inverse system in 3.4.9 arises, as described above, from the sequence $\cdots \twoheadrightarrow N_{\geqslant -1} \twoheadrightarrow N_{\geqslant 0} \twoheadrightarrow N_{\geqslant 1} \twoheadrightarrow \cdots$ (where $N_{\geqslant -u}$ is the u^{th} complex).

3.5.2 Definition. A sequence $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ of morphisms in $\mathcal{C}(R)$ with $N^u = 0$ for $u \ll 0$ is called a *tower* in $\mathcal{C}(R)$. The limit, $\lim_{u\in\mathbb{Z}} N^u$, of the associated $\mathbb{Z}$-inverse system, see 3.5.1, is called the *limit* of the tower in $\mathcal{C}(R)$.

Given towers $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ and $\{\kappa^u : M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$ in $\mathcal{C}(R)$, a sequence of morphisms $\{\beta^u : N^u \to M^u\}_{u\in\mathbb{Z}}$ that satisfy $\beta^{u-1}\lambda^u = \kappa^u\beta^u$ for all $u \in \mathbb{Z}$ is called a *morphism of towers*. The morphism $\lim_{u\in\mathbb{Z}}\beta^u : \lim_{u\in\mathbb{Z}} N^u \to \lim_{u\in\mathbb{Z}} M^u$, see 3.5.1 and 3.4.10, is called the *limit* of $\{\beta^u : N^u \to M^u\}_{u\in\mathbb{Z}}$.

3.5.3 Example. Let $p \in \mathbb{N}$. The sequence $\cdots \twoheadrightarrow \mathbb{Z}/p^3\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ is a tower whose limit is the $\mathbb{Z}$-module $\widehat{\mathbb{Z}}_p$ of p-adic integers. It is, in fact, a ring, see 11.1.18; it is mostly studied in the case where p is a prime.

3.5.4. Let $\{\lambda^u : N^u \to N^{u-1}\}_{n\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ and $\{\nu^{uv} : N^v \to N^u\}_{u\leqslant v}$ the associated $\mathbb{Z}$-inverse system. Given an R-complex M and a sequence of morphisms $\{\alpha^u : M \to N^u\}_{u\in\mathbb{Z}}$ that satisfy $\alpha^{u-1} = \lambda^u\alpha^u$ for all $u \in \mathbb{Z}$, one has $\alpha^u = \nu^{uv}\alpha^v$ for all $u \leqslant v$. By the universal property of limits, there is a morphism $\alpha : M \to \lim_{u\in\mathbb{Z}} N^u$ in $\mathcal{C}(R)$ with properties as described in 3.4.5.

3.5.5 Proposition. *Let* $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ *be a tower in* $\mathcal{C}(R)$. *With*

$$\varDelta_0^\lambda : \prod_{u\in\mathbb{Z}} N^u \longrightarrow \prod_{u\in\mathbb{Z}} N^u \quad \textit{given by} \quad (n^u)_{u\in\mathbb{Z}} \longmapsto (n^u - \lambda^{u+1}(n^{u+1}))_{u\in\mathbb{Z}}$$

one has $\lim_{u\in\mathbb{Z}} N^u = \operatorname{Ker}\varDelta_0^\lambda$.

PROOF. As in 3.5.1 let $\{\nu^{uv} : N^v \to N^u\}_{u\leqslant v}$ be the inverse system associated with the tower and consider the morphism $\varDelta^\nu$; see 3.4.3. Since $\nu^{uv} = \lambda^{u+1}\cdots\lambda^v$ holds, an element $(n^u)_{u\in\mathbb{Z}} \in \prod_{u\in\mathbb{Z}} N^u$ satisfies $n^u = \lambda^{u+1}(n^{u+1})$ for all u in $\mathbb{Z}$ if and only if $n^u = \nu^{uv}(n^v)$ holds for all $u \leqslant v$ in $\mathbb{Z}$, and hence $\operatorname{Ker}\varDelta_0^\lambda = \operatorname{Ker}\varDelta^\nu$. $\square$

3.5.6 Example. Let $N^0 \supseteq N^1 \supseteq N^2 \supseteq \cdots$ be a descending chain of R-complexes. The embeddings $\lambda^u \colon N^u \rightarrowtail N^{u-1}$ define a tower. It is evident that the morphism from $\bigcap_{u\in\mathbb{Z}} N^u$ to $\operatorname{Ker} \Delta_0^\lambda$ that maps an element n to the sequence with $n^u = n$ for all $u \geqslant 0$ is an isomorphism. Thus $\lim_{u\in\mathbb{Z}} N^u$ is the intersection $\bigcap_{u\in\mathbb{Z}} N^u$.

3.5.7 Example. Let $\{M^i\}_{i\in\mathbb{N}}$ be a sequence of R-complexes. Set $N^u = \bigoplus_{i=1}^u M^i$ for $u > 0$ and $N^u = 0$ for $u \leqslant 0$; let $\lambda^u \colon N^u \to N^{u-1}$ be the canonical projections. The limit of the tower $\{\lambda^u \colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ is $M = \prod_{i\in\mathbb{N}} M^i$. Indeed, the morphism $\alpha \colon M \to \lim_{u\in\mathbb{Z}} N^u$ determined by the canonical projections $M \twoheadrightarrow N^u$ maps a sequence $(x_i)_{i\geqslant 1}$ to the family of truncated sequences $((x_1,\dots,x_u))_{u\geqslant 1}$; see 3.4.5. It is injective, and from 3.5.5 evidently also surjective.

3.5.8 Proposition. *Let $\{\lambda^u \colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$.*

(a) *If $\lambda^u = 0$ holds for infinitely many $u > 0$, then one has $\lim_{u\in\mathbb{Z}} N^u = 0$.*

(b) *If there exists an integer w such that λ^u is bijective for all $u > w$, then the canonical map $\lim_{u\in\mathbb{Z}} N^u \to N^w$ is an isomorphism.*

Proof. (a): Let $n = (n^u)_{u\in\mathbb{Z}}$ be an element in $\lim_{u\in\mathbb{Z}} N^u$; for $w > u$ one then has $n^u = \lambda^{u+1}\cdots\lambda^w(n^w)$. For every $u \in \mathbb{Z}$ there is by assumption an integer $w > u$ with $\lambda^w = 0$, whence one has $n^u = 0$ and, consequently, $n = 0$.

(b): Define a sequence of morphisms $\{\alpha^u \colon N^w \to N^u\}_{u\in\mathbb{Z}}$ in $\mathcal{C}(R)$ as follows: set $\alpha^w = 1^{N^w}$, set $\alpha^u = \lambda^{u+1}\cdots\lambda^w$ for $u < w$, and set $\alpha^u = (\lambda^{w+1}\cdots\lambda^u)^{-1}$ for $u > w$. By construction $\alpha^{u-1} = \lambda^u\alpha^u$ hoolds for all $u \in \mathbb{Z}$, so by 3.5.4 there is a morphism $\alpha \colon N^w \to \lim_{u\in\mathbb{Z}} N^u$ given by $\alpha(n) = (\alpha^u(n))_{u\in\mathbb{Z}}$. Evidently there are equalities $\nu^w\alpha = \alpha^w = 1^{N^w}$. For $n = (n^u)_{u\in\mathbb{Z}}$ in $\lim_{u\in\mathbb{Z}} N^u$ one has $\alpha\nu^w(n) = \alpha(n^w) = (\alpha^u(n^w))_{u\in\mathbb{Z}} = (n^u)_{u\in\mathbb{Z}} = n$, where the penultimate equality holds by definition of the maps α^u and the complex $\lim_{u\in\mathbb{Z}} N^u$. □

The Mittag-Leffler Condition

The next definition allows for formulations of criteria on towers that ensure that limits are exact and commute with homology.

3.5.9 Definition. A tower $\{\lambda^u \colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ of R-complexes is said to satisfy the Mittag-Leffler Condition if for every $u \in \mathbb{Z}$ the descending chain

$$\operatorname{Im}\lambda^{u+1} \supseteq \operatorname{Im}(\lambda^{u+1}\lambda^{u+2}) \supseteq \operatorname{Im}(\lambda^{u+1}\lambda^{u+2}\lambda^{u+3}) \supseteq \cdots$$

of subcomplexes of N^u becomes stationary, that is, there exists $v > u$ such that for every $w \geqslant v$ one has $\operatorname{Im}(\lambda^{u+1}\cdots\lambda^v\cdots\lambda^w) = \operatorname{Im}(\lambda^{u+1}\cdots\lambda^v)$. This subcomplex of N^u is called the *stable image* of the tower at stage u.

If for every $u \in \mathbb{Z}$ there exists $v > u$ with $\lambda^{u+1}\cdots\lambda^v = 0$, then the condition clearly holds; in this case the tower is said to satisfy the *trivial* Mittag-Leffler Condition.

Remark. A trivial tower is one in which all the morphisms are trivial, that is, zero. A trivial tower trivially satisfies the trivial Mittag-Leffler Condition. Another word for a trivial tower is ‘trump’.

3.5.10 Example. A tower $\{\lambda^u\colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ in which every morphism λ^u is surjective evidently satisfies the Mittag-Leffler Condition.

The next two results give some basic properties of the Mittag-Leffler Condition.

3.5.11 Proposition. *Consider towers in $\mathcal{C}(R)$,*

$$\{X^u \to X^{u-1}\}_{u\in\mathbb{Z}}\,, \quad \{N^u \to N^{u-1}\}_{u\in\mathbb{Z}}\,, \quad \textit{and} \quad \{M^u \to M^{u-1}\}_{u\in\mathbb{Z}}\,.$$

Let $\{\alpha^u\colon X^u \to N^u\}_{u\in\mathbb{Z}}$ and $\{\beta^u\colon N^u \to M^u\}_{u\in\mathbb{Z}}$ be morphisms of towers such that

$$0 \longrightarrow X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u \longrightarrow 0$$

is an exact sequence for every $u \in U$.

(a) *If the tower $\{N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, then so does $\{M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$.*
(b) *If the towers $\{X^u \to X^{u-1}\}_{u\in\mathbb{Z}}$ and $\{M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$ satisfy the Mittag-Leffler Condition, then so does $\{N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$.*

PROOF. Write $\xi^u\colon X^u \to X^{u-1}$, $\lambda^u\colon N^u \to N^{u-1}$, and $\kappa^u\colon M^u \to M^{u-1}$ for the morphisms in the given towers.

(a): For every $u \in \mathbb{Z}$ and every $w > u$, surjectivity of β^w yields

$$\begin{aligned}\operatorname{Im}(\kappa^{u+1}\cdots\kappa^w) &= \operatorname{Im}(\kappa^{u+1}\cdots\kappa^w\beta^w)\\ &= \operatorname{Im}(\beta^u\lambda^{u+1}\cdots\lambda^w)\\ &= \beta^u(\operatorname{Im}(\lambda^{u+1}\cdots\lambda^w))\,.\end{aligned}$$

The assertion immediately follows from this identity.

(b): Given $u \in \mathbb{Z}$ choose $v > u$ such that $\operatorname{Im}(\xi^{u+1}\cdots\xi^v\cdots\xi^w) = \operatorname{Im}(\xi^{u+1}\cdots\xi^v)$ holds for $w \geqslant v$ and choose $q > v$ such that $\operatorname{Im}(\kappa^{v+1}\cdots\kappa^q\cdots\kappa^w) = \operatorname{Im}(\kappa^{v+1}\cdots\kappa^q)$ holds for $w \geqslant q$. For every $w \geqslant q$ one now has

$$\beta^v(\operatorname{Im}(\lambda^{v+1}\cdots\lambda^q\cdots\lambda^w)) = \operatorname{Im}(\kappa^{v+1}\cdots\kappa^q\cdots\kappa^w\beta_w) = \operatorname{Im}(\kappa^{v+1}\cdots\kappa^q)\,,$$

where the last equality holds as β_w is surjective and by the choice of q. In particular, one has $\beta^v(\operatorname{Im}(\lambda^{v+1}\cdots\lambda^q)) = \beta^v(\operatorname{Im}(\lambda^{v+1}\cdots\lambda^q\cdots\lambda^w))$ for $w \geqslant q$, and hence also $\operatorname{Im}(\lambda^{v+1}\cdots\lambda^q) \subseteq \operatorname{Im}(\lambda^{v+1}\cdots\lambda^q\cdots\lambda^w) + \operatorname{Im}\alpha^v$. Applying $\lambda^{u+1}\cdots\lambda^v$ to this yields

$$(\flat)\qquad \operatorname{Im}(\lambda^{u+1}\cdots\lambda^q) \subseteq \operatorname{Im}(\lambda^{u+1}\cdots\lambda^q\cdots\lambda^w) + \operatorname{Im}(\lambda^{u+1}\cdots\lambda^v\alpha^v)\,.$$

The choice of v explains the second equality below,

$$\begin{aligned}\operatorname{Im}(\lambda^{u+1}\cdots\lambda^v\alpha^v) &= \alpha^u(\operatorname{Im}(\xi^{u+1}\cdots\xi^v))\\ &= \alpha^u(\operatorname{Im}(\xi^{u+1}\cdots\xi^v\cdots\xi^w))\\ &= \operatorname{Im}(\lambda^{u+1}\cdots\lambda^v\cdots\lambda^w\alpha^w)\\ &\subseteq \operatorname{Im}(\lambda^{u+1}\cdots\lambda^v\cdots\lambda^w)\,.\end{aligned}$$

Thus $(\flat)$ shows that $\operatorname{Im}(\lambda^{u+1}\cdots\lambda^q) \subseteq \operatorname{Im}(\lambda^{u+1}\cdots\lambda^q\cdots\lambda^w)$ holds for all $w \geqslant q$. $\square$

3.5.12 Proposition. *Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower that satisfies the Mittag-Leffler Condition. There is a tower $\{\tilde{\lambda}^u : \widetilde{N}^u \to \widetilde{N}^{u-1}\}_{u\in\mathbb{Z}}$ with every $\tilde{\lambda}^u$ surjective, a tower $\{\bar{\lambda}^u : \overline{N}^u \to \overline{N}^{u-1}\}_{u\in\mathbb{Z}}$ that satisfies the trivial Mittag-Leffler Condition, and morphisms of towers $\{\iota^u : \widetilde{N}^u \rightarrowtail N^u\}_{u\in\mathbb{Z}}$ and $\{\pi^u : N^u \twoheadrightarrow \overline{N}^u\}_{u\in\mathbb{Z}}$ such that*

$$0 \longrightarrow \widetilde{N}^u \xrightarrow{\iota^u} N^u \xrightarrow{\pi^u} \overline{N}^u \longrightarrow 0$$

is an exact sequence for every $u \in \mathbb{Z}$.

Proof. Let $\widetilde{N}^u$ be the stable image of the given tower at stage u; see 3.5.9. Note that $\lambda^u(\widetilde{N}^u) = \widetilde{N}^{u-1}$ whence λ^u restricts to a surjective morphism $\widetilde{N}^u \to \widetilde{N}^{u-1}$; denote it by $\tilde{\lambda}^u$. Let $\iota^u : \widetilde{N}^u \rightarrowtail N^u$ be the embedding. Set $\overline{N}^u = N^u/\widetilde{N}^u$, let $\pi^u : N^u \twoheadrightarrow \overline{N}^u$ be the quotient map and $\bar{\lambda}^u : \overline{N}^u \to \overline{N}^{u-1}$ the map induced by λ^u. For every $u \in \mathbb{Z}$ one has $\operatorname{Im}(\lambda^{u+1} \cdots \lambda^v) = \widetilde{N}^u$ for some $v > u$, and hence $\bar{\lambda}^{u+1} \cdots \bar{\lambda}^v = 0$. □

3.5.13 Lemma. *Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ that satisfies the trivial Mittag-Leffler Condition. The map Δ_0^λ from* 3.5.5 *is bijective, in particular, one has* $\lim_{u\in\mathbb{Z}} N^u = 0$.

Proof. Let $n = (n^u)_{u\in\mathbb{Z}}$ be an element in $\prod_{u\in\mathbb{Z}} N^u$. For every $u \in \mathbb{Z}$ the sum

$$\alpha^u(n) \;=\; n^u + \lambda^{u+1}(n^{u+1}) + \lambda^{u+1}\lambda^{u+2}(n^{u+2}) + \cdots$$

is finite as the tower satisfies the trivial Mittag-Leffler Condition, and thus $\alpha^u(n)$ is a well-defined element in N^u. The map $\alpha\colon \prod_{u\in\mathbb{Z}} N^u \to \prod_{u\in\mathbb{Z}} N^u$ that sends n to $\alpha(n) = (\alpha^u(n))_{u\in\mathbb{Z}}$ is an inverse of Δ_0^λ. Indeed, the u^{th} component of $\Delta_0^\lambda\alpha(n)$ is

$$\begin{aligned}\alpha^u(n) - \lambda^{u+1}(\alpha^{u+1}(n)) \;=\; & n^u + \lambda^{u+1}(n^{u+1}) + \lambda^{u+1}\lambda^{u+2}(n^{u+2}) + \cdots \\ & - \lambda^{u+1}(n^{u+1} + \lambda^{u+2}(n^{u+2}) + \lambda^{u+2}\lambda^{u+3}(n^{u+3}) + \cdots)\,,\end{aligned}$$

which is n^u, and the u^{th} component of $\alpha\,\Delta_0^\lambda(n)$ is

$$(n^u - \lambda^{u+1}(n^{u+1})) + \lambda^{u+1}(n^{u+1} - \lambda^{u+2}(n^{u+2})) + \lambda^{u+1}\lambda^{u+2}(n^{u+2} - \lambda^{u+3}(n^{u+3})) + \cdots\,,$$

which is also n^u. In particular, $\lim_{u\in\mathbb{Z}} N^u = 0$ holds by 3.5.5. □

3.5.14 Lemma. *Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$. If every morphism λ^u is surjective, then the map Δ_0^λ from* 3.5.5 *is surjective.*

Proof. Given $n = (n^u)_{u\in\mathbb{Z}}$ in $\prod_{u\in\mathbb{Z}} N^u$ we seek an element $x = (x^u)_{u\in\mathbb{Z}}$ with $\Delta_0^\lambda(x) = n$, that is, a solution to the equations $x^u - \lambda^{u+1}(x^{u+1}) = n^u$ for $u \in \mathbb{Z}$. By the definition of a tower, $n^u = 0$ holds for $u \ll 0$, so set $x^u = 0$ for $u \ll 0$. Now let $v \in \mathbb{Z}$ and assume that x^u for $u \leqslant v$ have been constructed such that $x^u - \lambda^{u+1}(x^{u+1}) = n^u$ holds for all $u < v$. Surjectivity of λ^{v+1} yields an element x^{v+1} with $x^v - \lambda^{v+1}(x^{v+1}) = n^v$. This procedure yields the desired element x. □

3.5.15 Proposition. *Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ that satisfies the Mittag-Leffler Condition. The map Δ_0^λ from* 3.5.5 *is surjective.*

Proof. By 3.5.12 there exists a tower $\{\tilde{\lambda}^u\colon \widetilde{N}^u \to \widetilde{N}^{u-1}\}_{u\in\mathbb{Z}}$ with every $\tilde{\lambda}^u$ surjective, a tower $\{\bar{\lambda}^u\colon \overline{N}^u \to \overline{N}^{u-1}\}_{u\in\mathbb{Z}}$ that satisfies the trivial Mittag-Leffler Condition, and morphisms of towers $\{\iota^u\colon \widetilde{N}^u \rightarrowtail N^u\}_{u\in\mathbb{Z}}$ and $\{\pi^u\colon N^u \twoheadrightarrow \overline{N}^u\}_{u\in\mathbb{Z}}$ such that

$$0 \longrightarrow \widetilde{N}^u \xrightarrow{\iota^u} N^u \xrightarrow{\pi^u} \overline{N}^u \longrightarrow 0$$

is an exact sequence for every $u \in \mathbb{Z}$. In the commutative diagram below, the vertical morphisms come from 3.5.5 and the rows are exact by 3.1.18,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod\limits_{u\in\mathbb{Z}} \widetilde{N}^u & \xrightarrow{\prod_{u\in\mathbb{Z}}\iota^u} & \prod\limits_{u\in\mathbb{Z}} N^u & \xrightarrow{\prod_{u\in\mathbb{Z}}\pi^u} & \prod\limits_{u\in\mathbb{Z}} \overline{N}^u & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta_0^{\tilde{\lambda}}} & & \downarrow{\scriptstyle \Delta_0^{\lambda}} & & \downarrow{\scriptstyle \Delta_0^{\bar{\lambda}}} & & \\
0 & \longrightarrow & \prod\limits_{u\in\mathbb{Z}} \widetilde{N}^u & \xrightarrow{\prod_{u\in\mathbb{Z}}\iota^u} & \prod\limits_{u\in\mathbb{Z}} N^u & \xrightarrow{\prod_{u\in\mathbb{Z}}\pi^u} & \prod\limits_{u\in\mathbb{Z}} \overline{N}^u & \longrightarrow & 0\,.
\end{array}$$

The morphism $\Delta_0^{\tilde{\lambda}}$ is surjective by 3.5.14, and $\Delta_0^{\bar{\lambda}}$ is surjective (even bijective) by 3.5.13. It now follows from the Snake Lemma 2.1.45 that Δ_0^{λ} is surjective. □

3.5.16 Corollary. *Let $\{\lambda^u\colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ that satisfies the Mittag-Leffler Condition. There are inequalities,*

$$\sup\big(\lim_{u\in\mathbb{Z}} N^u\big) \leqslant \sup_{u\in\mathbb{Z}}\{\sup N^u\} \quad \textit{and} \quad \inf\big(\lim_{u\in\mathbb{Z}} N^u\big) \geqslant \inf_{u\in\mathbb{Z}}\{\inf N^u\} - 1\,.$$

In particular, if N^u is acyclic for every $u \in \mathbb{Z}$, then $\lim_{u\in\mathbb{Z}} N^u$ is acyclic.

Proof. By 3.5.5 and 3.5.15 there is an exact sequence

$$(\dagger) \qquad 0 \longrightarrow \lim_{u\in\mathbb{Z}} N^u \longrightarrow \prod_{u\in\mathbb{Z}} N^u \xrightarrow{\Delta_0^{\lambda}} \prod_{u\in\mathbb{Z}} N^u \longrightarrow 0\,.$$

As one has

$$\sup\big(\prod_{u\in U} N^u\big) = \sup_{u\in U}\{\sup N^u\} \quad \text{and} \quad \inf\big(\prod_{u\in U} N^u\big) = \inf_{u\in U}\{\inf N^u\}$$

by 3.1.23, the asserted inequalities hold by 2.5.6. □

3.5.17 Theorem. *Consider towers in $\mathcal{C}(R)$,*

$$\{X^u \to X^{u-1}\}_{u\in\mathbb{Z}}\,, \quad \{N^u \to N^{u-1}\}_{u\in\mathbb{Z}}\,, \quad \textit{and} \quad \{M^u \to M^{u-1}\}_{u\in\mathbb{Z}}\,.$$

Let $\{\alpha^u\colon X^u \to N^u\}_{u\in\mathbb{Z}}$ and $\{\beta^u\colon N^u \to M^u\}_{u\in\mathbb{Z}}$ be morphisms of towers such that

$$0 \longrightarrow X^u \xrightarrow{\alpha^u} N^u \xrightarrow{\beta^u} M^u \longrightarrow 0$$

is an exact sequence for every $u \in U$. If the tower $\{X^u \to X^{u-1}\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, then the following sequence is exact,

$$0 \longrightarrow \lim_{u\in\mathbb{Z}} X^u \xrightarrow{\lim_{u\in\mathbb{Z}}\alpha^u} \lim_{u\in\mathbb{Z}} N^u \xrightarrow{\lim_{u\in\mathbb{Z}}\beta^u} \lim_{u\in\mathbb{Z}} M^u \longrightarrow 0\,.$$

PROOF. Write $\xi^u : X^u \to X^{u-1}$, $\lambda^u : N^u \to N^{u-1}$, and $\kappa^u : M^u \to M^{u-1}$ for the maps in the given towers. In the commutative diagram below, the vertical morphisms come from 3.5.5 and the rows are exact by 3.1.18,

$$\begin{tikzcd} 0 \arrow[r] & \prod\limits_{u\in U} X^u \arrow[r, "\prod_{u\in U}\alpha^u"] \arrow[d, "\Delta_0^\xi"] & \prod\limits_{u\in U} N^u \arrow[r, "\prod_{u\in U}\beta^u"] \arrow[d, "\Delta_0^\lambda"] & \prod\limits_{u\in U} M^u \arrow[r] \arrow[d, "\Delta_0^\kappa"] & 0 \\ 0 \arrow[r] & \prod\limits_{u\in U} X^u \arrow[r, "\prod_{u\in U}\alpha^u"] & \prod\limits_{u\in U} N^u \arrow[r, "\prod_{u\in U}\beta^u"] & \prod\limits_{u\in U} M^u \arrow[r] & 0\,. \end{tikzcd}$$

By 3.5.5 and the Snake Lemma 2.1.45 there is an exact sequence,

$$0 \longrightarrow \lim_{u\in\mathbb{Z}} X^u \xrightarrow{\lim \alpha^u} \lim_{u\in\mathbb{Z}} N^u \xrightarrow{\lim \beta^u} \lim_{u\in\mathbb{Z}} M^u \longrightarrow \operatorname{Coker} \Delta_0^\xi\,.$$

As $\{\xi^u : X^u \to X^{u-1}\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, $\operatorname{Coker} \Delta_0^\xi = 0$ holds by 3.5.15, and the desired conclusion follows. $\square$

3.5.18 Example. Let $m, n > 1$ be integers and relatively prime. Consider the following commutative diagram of $\mathbb{Z}$-modules,

$$\begin{tikzcd} & \vdots \arrow[d, "m"] & \vdots \arrow[d, "m"] & \vdots \arrow[d, "m", "\cong"'] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "m"] & \mathbb{Z} \arrow[r] \arrow[d, "m"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "m", "\cong"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "m"] & \mathbb{Z} \arrow[r] \arrow[d, "m"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "m", "\cong"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0\,. \end{tikzcd}$$

The rows are exact and the vertical maps define towers with limits 0, 0, and $\mathbb{Z}/n\mathbb{Z}$, respectively; cf. 3.5.5 and 3.5.8(b). Thus the sequence $0 \to 0 \to 0 \to \mathbb{Z}/n\mathbb{Z} \to 0$ of limits is not exact.

The Mittag-Leffler Condition facilitates computation of homology of limits.

3.5.19 Theorem. *Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ and consider the canonical morphism in $\mathcal{C}(R)$,*

$$\mathrm{H}\big(\lim_{u\in\mathbb{Z}} N^u\big) \longrightarrow \lim_{u\in\mathbb{Z}} \mathrm{H}(N^u)\,, \tag{3.5.19.1}$$

given by $h \mapsto (\mathrm{H}(\nu^u)(h))_{u\in\mathbb{Z}}$. If the tower $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, then the following assertions hold.

(a) *The morphism* (3.5.19.1) *is surjective.*

(b) *Let* $v \in \mathbb{Z}$. *If the tower* $\{\mathrm{H}_{v+1}(\lambda^u)\colon \mathrm{H}_{v+1}(N^u) \to \mathrm{H}_{v+1}(N^{u-1})\}_{u\in\mathbb{Z}}$ *satisfies the Mittag-Leffler Condition, then the degree* v *component of* (3.5.19.1) *is an isomorphism*; *that is, one has* $\mathrm{H}_v(\lim_{u\in\mathbb{Z}} N^u) \cong \lim_{u\in\mathbb{Z}} \mathrm{H}_v(N^u)$.

Proof. (a): As the tower $\{\lambda^u\colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, so does $\{\mathrm{B}(\lambda^u)\colon \mathrm{B}(N^u) \to \mathrm{B}(N^{u-1})\}_{u\in\mathbb{Z}}$ by 3.5.11 and 2.2.12(a). Thus, in the commutative diagram below, where the vertical maps are the canonical morphisms, see (3.4.16.1), the rows are exact by 2.2.12(c) and 3.5.17.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}\big(\lim\limits_{u\in\mathbb{Z}} N^u\big) \arrow[r] \arrow[d, "\varkappa^{\mathrm{B}}"] & \mathrm{C}\big(\lim\limits_{u\in\mathbb{Z}} N^u\big) \arrow[r] \arrow[d, "\varkappa^{\mathrm{Z}}", "\cong"'] & \mathrm{H}\big(\lim\limits_{u\in\mathbb{Z}} N^u\big) \arrow[r] \arrow[d, "\varkappa^{\mathrm{H}}"] & 0 \\
0 \arrow[r] & \lim\limits_{u\in\mathbb{Z}} \mathrm{B}(N^u) \arrow[r] & \lim\limits_{u\in\mathbb{Z}} \mathrm{Z}(N^u) \arrow[r] & \lim\limits_{u\in\mathbb{Z}} \mathrm{H}(N^u) \arrow[r] & 0
\end{tikzcd}$$

As $\varkappa^{\mathrm{Z}}$ is an isomorphism by 3.4.22, the morphism $\varkappa^{\mathrm{H}}$ is surjective.

(b): By the Snake Lemma 2.1.45 applied to the diagram above, proving that the degree v component $\varkappa^{\mathrm{H}}_v$ is injective, and hence by part (a) an isomorphism, is equivalent to showing that $\varkappa^{\mathrm{B}}_v$ is surjective. Consider the towers determined by the families of morphisms $\{\mathrm{B}_{v+1}(\lambda^u)\}_{u\in\mathbb{Z}}$, $\{\mathrm{Z}_{v+1}(\lambda^u)\}_{u\in\mathbb{Z}}$, and $\{\mathrm{H}_{v+1}(\lambda^u)\}_{u\in\mathbb{Z}}$. The canonical exact sequences,

$$0 \longrightarrow \mathrm{B}_{v+1}(N^u) \longrightarrow \mathrm{Z}_{v+1}(N^u) \longrightarrow \mathrm{H}_{v+1}(N^u) \longrightarrow 0\,,$$

yield morphisms of towers. As in the proof of part (a), the tower $\{\mathrm{B}_{v+1}(\lambda^u)\}_{u\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition, and so does $\{\mathrm{H}_{v+1}(\lambda^u)\}_{u\in\mathbb{Z}}$ by assumption. Hence 3.5.11 implies that $\{\mathrm{Z}_{v+1}(\lambda^u)\}_{i\in\mathbb{Z}}$ satisfies the Mittag-Leffler Condition. The short exact sequences from 2.2.12(a),

$$0 \longrightarrow \mathrm{Z}_{v+1}(N^u) \longrightarrow N^u_{v+1} \longrightarrow \mathrm{B}_v(N^u) \longrightarrow 0\,,$$

induce by 3.5.17 the lower exact row in the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}_{v+1}\big(\lim\limits_{u\in\mathbb{Z}} N^u\big) \arrow[r] \arrow[d, "\varkappa^{\mathrm{Z}}_{v+1}", "\cong"'] & \big(\lim\limits_{u\in\mathbb{Z}} N^u\big)_{v+1} \arrow[r] \arrow[d, equal] & \mathrm{B}_v\big(\lim\limits_{u\in\mathbb{Z}} N^u\big) \arrow[r] \arrow[d, "\varkappa^{\mathrm{B}}_v"] & 0 \\
0 \arrow[r] & \lim\limits_{u\in\mathbb{Z}} \mathrm{Z}_{v+1}(N^u) \arrow[r] & \lim\limits_{u\in\mathbb{Z}} (N^u_{v+1}) \arrow[r] & \lim\limits_{u\in\mathbb{Z}} \mathrm{B}_v(N^u) \arrow[r] & 0\,.
\end{tikzcd}$$

The upper row is exact by 2.2.12(a). As the middle map in this diagram is an equality, see 3.4.7, it follows that $\varkappa^{\mathrm{B}}_v$ is surjective, as desired. □

For ease of reference, we record a frequently used corollary to 3.5.19.

3.5.20 Corollary. *Let* $\{\lambda^u\colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ *be a tower in* $\mathcal{C}(R)$. *If this tower and* $\{\mathrm{H}(\lambda^u)\colon \mathrm{H}(N^u) \to \mathrm{H}(N^{u-1})\}_{u\in\mathbb{Z}}$ *both satisfy the Mittag-Leffler Condition, then the canonical map* $\mathrm{H}(\lim_{u\in\mathbb{Z}} N^u) \to \lim_{u\in\mathbb{Z}} \mathrm{H}(N^u)$ *from* (3.5.19.1) *is an isomorphism.*

Proof. The assertion is immediate from 3.5.19. □

3.5.21 Example. The diagram in 3.5.18 can be interpreted as a tower of identical acyclic complexes. The morphisms in the tower are multiplication by m, so it does not satisfy the Mittag-Leffler Condition and, indeed, the limit is not acyclic.

Exercises

E 3.5.1 As in 3.5.1 let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a sequence (not necessarily a tower) of morphisms in $\mathcal{C}(R)$. Show that the limit of the associated inverse system does not depend on λ^u for $u \ll 0$.

E 3.5.2 Show that every complex is the limit of a tower of bounded above complexes.

E 3.5.3 Show that for a U-inverse system $\{\nu^{uv} : N^v \to N^u\}_{u\leqslant v}$ the map Δ^ν is not surjective.

E 3.5.4 Show that the p-adic integers $\widehat{\mathbb{Z}}_p$ from 3.5.3 form a ring isomorphic to $\mathbb{Z}[\![x]\!]/(x-p)$.

E 3.5.5 Let R be a countable integral domain. Consider the set $U = \{(u) \mid u \in R \setminus \{0\}\}$ of principal ideals ordered under reverse inclusion. Show that $\lim_{u\in U} R/(u)$ is not countable. *Hint:* Enumerate the non-units in U and construct a totally ordered cofinal subset of U.

E 3.5.6 Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ and $\{\alpha^u : M \to N^u\}_{u\in\mathbb{Z}}$ a sequence of morphisms with $\alpha^{u-1} = \lambda^u\alpha^u$ for all $u \in \mathbb{Z}$; denote by $\alpha : M \to \lim_{u\in\mathbb{Z}} N^u$ the canonical morphism. Assume that α^u is surjective and $\operatorname{Ker}\alpha^u = \operatorname{Ker}\alpha^{u-1}$ holds for $u \gg 0$. Show that α is surjective.

E 3.5.7 Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ with limit N. Set $M^u = N^{u-1}$ and $\kappa^u = \lambda^{u-1}$; show that $\{\kappa^u : M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$ is a tower with colimit N and that the morphisms $\beta^u = \lambda^u$ from N^u to M^u form a morphism of towers with limit 1^N.

E 3.5.8 Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ with λ^u surjective for all $u \in \mathbb{Z}$. Assume that N^u is acyclic for infinitely many $u > 0$. Show that $\lim_{u\in\mathbb{Z}} N^u$ is acyclic.

E 3.5.9 Let $\{\lambda^u : N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be a tower in $\mathcal{C}(R)$ with λ^u surjective for all $u \in \mathbb{Z}$. Assume that for every $n \in \mathbb{Z}$ there is a $u_n > 0$ such that $\mathrm{H}_n(N^u) = 0$ for all $u > u_n$. Show that $\lim_{u\in\mathbb{Z}} N^u$ is acyclic.

E 3.5.10 Show that the conclusion in 3.5.16 may fail if the homomorphisms λ^u are surjective for infinitely many but not all $u \in \mathbb{Z}$. *Hint:* Set $N = 0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ and consider the tower given by $\cdots \xrightarrow{m} N \xrightarrow{=} N \xrightarrow{m} N \xrightarrow{=} N$; cf. 3.5.18.

Chapter 4
Equivalences and Isomorphisms

This chapter opens with the mapping cone construction. It attaches to every morphism a complex that reflects essential properties of the morphism. The utility of this construction stretches far beyond the category of complexes, indeed, it is the key to defining a triangulated structure on the derived category. Another notion that points towards the derived category is quasi-isomorphisms; these are morphisms that induce isomorphisms in homology. They are introduced in the second section, and homotopy equivalences—an especially robust subclass of quasi-isomorphisms—are treated in Sect. 4.3.

In Sects. 4.4 and 4.5 we extend the standard isomorphisms and evaluation homomorphisms for modules treated in Chap. 1 to the realm of complexes. To some extent this is an exercise in bootstrapping; but it is a critical one as these maps are among our tools of choice when it comes to applications of homological algebra in ring theory.

4.1 Mapping Cone

SYNOPSIS. Mapping cone; ~ sequence; homotopy; Σ-functor.

The mapping cone of a continuous map $f : X \longrightarrow Y$ between topological spaces is a space glued together from X and Y via the map f. Here we explore the algebraic version of this construction. In Chap. 6 the mapping cone is key to the construction of triangulated structures on the homotopy and derived categories.

4.1.1 Definition. Let $\alpha : M \longrightarrow N$ be a morphism of R-complexes. The *mapping cone* of α is the complex with underlying graded module

$$(\operatorname{Cone}\alpha)^\natural = \begin{matrix} N^\natural \\ \oplus \\ \Sigma M^\natural \end{matrix} \qquad \text{and differential} \qquad \partial^{\operatorname{Cone}\alpha} = \begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix}.$$

4.1.2. A straightforward computation shows that $\partial^{\operatorname{Cone}\alpha}$ is square zero,

$$\partial^{\operatorname{Cone}\alpha}\partial^{\operatorname{Cone}\alpha} \;=\; \begin{pmatrix} \partial^N\partial^N & \partial^N\alpha\varsigma^{\Sigma M}_{-1} + \alpha\varsigma^{\Sigma M}_{-1}\partial^{\Sigma M} \\ 0 & \partial^{\Sigma M}\partial^{\Sigma M} \end{pmatrix} \;=\; 0\,;$$

it uses that α is a morphism and that $\varsigma^{\Sigma M}_{-1}$ is a degree -1 chain map.

4.1.3 Lemma. *Let M be an R-complex. For every $x \in \Bbbk$ one has $x\operatorname{H}(\operatorname{Cone} x^M) = 0$. In particular,* $\operatorname{Cone} 1^M$ *is acyclic.*

Proof. The goal is to show that xz is a boundary for every homogeneous cycle z in the complex $\operatorname{Cone} x^M$. For such a cycle $z = (m\;\; \varsigma^M_1(m'))^{\mathsf{T}}$ one has $\partial^M(m) + xm' = 0$ in M, and hence $-\varsigma^M_1\partial^M(m) = x\varsigma^M_1(m')$ in ΣM. In the computation,

$$\begin{pmatrix} \partial^M & x\varsigma^{\Sigma M}_{-1} \\ 0 & \partial^{\Sigma M} \end{pmatrix}\begin{pmatrix} 0 \\ \varsigma^M_1(m) \end{pmatrix} = \begin{pmatrix} xm \\ \partial^{\Sigma M}\varsigma^M_1(m) \end{pmatrix} = \begin{pmatrix} xm \\ -\varsigma^M_1\partial^M(m) \end{pmatrix} = \begin{pmatrix} xm \\ x\varsigma^M_1(m') \end{pmatrix} \;=\; xz\,,$$

the second equality holds as ς^M_1 is a degree 1 chain map. □

In 4.3.31 we show that $\operatorname{Cone} 1^M$ more than acyclic, namely so-called contractible.

4.1.4. For an R-module M the disk complex $\operatorname{D}^v(M)$ from 2.5.29 is $\Sigma^{v-1}\operatorname{Cone} 1^M$.

Given a morphism $\alpha\colon M \to N$ of R-complexes, the embedding of N into $\operatorname{Cone}\alpha$ and the projection of $\operatorname{Cone}\alpha$ onto ΣM are evidently morphisms of R-complexes.

4.1.5 Definition. Let $\alpha\colon M \to N$ be a morphism of R-complexes. The degreewise split exact sequence of R-complexes,

$$0 \longrightarrow N \xrightarrow{\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\;\,1^{\Sigma M})} \Sigma M \longrightarrow 0\,,$$

is called the *mapping cone sequence* of α.

4.1.6 Theorem. *Let $\alpha, \alpha'\colon M \to N$ be morphisms of R-complexes and consider the following diagram whose rows are the mapping cone sequences of α and α',*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & \operatorname{Cone}\alpha \arrow[r] \arrow[d, dotted, "\gamma"] & \Sigma M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & \operatorname{Cone}\alpha' \arrow[r] & \Sigma M \arrow[r] & 0\,. \end{tikzcd}$$

The morphisms α and α' are homotopic if and only if there exists a morphism γ that makes the diagram commutative; *moreover, such a morphism is an isomorphism.*

Proof. It follows from the Five Lemma 2.1.41 that a morphism γ that makes the diagram commutative is an isomorphism. The assignment $\sigma \mapsto \gamma_\sigma$, where

$$\gamma_\sigma \;=\; \begin{pmatrix} 1^N & \sigma\varsigma^{\Sigma M}_{-1} \\ 0 & 1^{\Sigma M} \end{pmatrix}\,,$$

yields a one-to-one correspondence between degree 1 homomorphisms $\sigma\colon M \to N$ and degree 0 homomorphisms that make the diagram commutative. Now one has

$$\begin{aligned}\gamma_\sigma \partial^{\operatorname{Cone}\alpha} - \partial^{\operatorname{Cone}\alpha'}\gamma_\sigma &= \begin{pmatrix}1^N & \sigma\varsigma_{-1}^{\Sigma M}\\ 0 & 1^{\Sigma M}\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma_{-1}^{\Sigma M}\\ 0 & \partial^{\Sigma M}\end{pmatrix} - \begin{pmatrix}\partial^N & \alpha'\varsigma_{-1}^{\Sigma M}\\ 0 & \partial^{\Sigma M}\end{pmatrix}\begin{pmatrix}1^N & \sigma\varsigma_{-1}^{\Sigma M}\\ 0 & 1^{\Sigma M}\end{pmatrix}\\ &= \begin{pmatrix}0 & (\alpha - \sigma\partial^M - \partial^N\sigma - \alpha')\varsigma_{-1}^{\Sigma M}\\ 0 & 0\end{pmatrix},\end{aligned}$$

so γ_σ is a morphism if and only if $\alpha - \alpha' = \partial^N\sigma + \sigma\partial^M$ holds; that is, if and only if σ is a homotopy from α to α'. ◻

An immediate consequence of the theorem is a characterization of null-homotopic maps; related characterizations of other distinguished types of morphisms are given in 4.2.16 and 4.3.30.

4.1.7 Corollary. *A morphism $\alpha\colon M \to N$ of R-complexes is null-homotopic if and only if the mapping cone sequence from* 4.1.5 *is split exact.*

Proof. The mapping cone of the zero morphism is the direct sum $N \oplus \Sigma M$, whence the claim is immediate from 4.1.6 and 2.1.47. ◻

Σ-Functors

This section closes with four technical results whose utility only becomes fully apparent in Chap. 6. The gist is that the mapping cone construction commutes with Hom and tensor product in such a way that mapping cone sequences are preserved.

4.1.8 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ is called a Σ-*functor* if it is additive and there exists a natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$ of functors, such that for every morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ there exists an isomorphism $\breve{\alpha}$ that makes the diagram

$$\begin{tikzcd}
\mathrm{F}(N) \arrow[r, "\mathrm{F}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)"] \arrow[d, equal] & \mathrm{F}(\operatorname{Cone}\alpha) \arrow[r, "\mathrm{F}(0\ 1^{\Sigma M})"] \arrow[d, dotted, "\breve{\alpha}", "\cong"'] & \mathrm{F}(\Sigma M) \arrow[d, "\phi^M", "\cong"'] \\
\mathrm{F}(N) \arrow[r, "\left(\begin{smallmatrix}1^{\mathrm{F}(N)}\\0\end{smallmatrix}\right)"] & \operatorname{Cone}\mathrm{F}(\alpha) \arrow[r, "(0\ 1^{\Sigma\mathrm{F}(M)})"] & \Sigma\mathrm{F}(M)
\end{tikzcd}$$

in $\mathcal{C}(S)$ commutative.

4.1.9 Definition. Let $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms $\phi\colon \mathrm{E}\Sigma \to \Sigma\mathrm{E}$ and $\psi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$. A natural transformation $\tau\colon \mathrm{E} \to \mathrm{F}$ is called a Σ-*transformation* if the next diagram is commutative for every M in $\mathcal{C}(R)$,

$$\begin{tikzcd}
\mathrm{E}(\Sigma M) \arrow[r, "\phi^M", "\cong"'] \arrow[d, "\tau^{\Sigma M}"'] & \Sigma\mathrm{E}(M) \arrow[d, "\Sigma\tau^M"] \\
\mathrm{F}(\Sigma M) \arrow[r, "\psi^M", "\cong"'] & \Sigma\mathrm{F}(M)\,.
\end{tikzcd}$$

That is, $\tau^{\Sigma M}$ and $\Sigma\tau^{M}$ are isomorphic, and the isomorphism is natural in M.

4.1.10 Definition. A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is called a Σ-*functor* if it is additive and there is a natural isomorphism $\psi\colon \Sigma^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ of functors, such that for every morphism $\alpha\colon M \to N$ in $\mathcal{C}(R)$ there exists an isomorphism $\breve{\alpha}$ that makes the diagram

$$\begin{tikzcd}
\Sigma^{-1}\mathrm{G}(M) \arrow[r, "\Sigma^{-1}\left(\begin{smallmatrix}1^{\mathrm{G}(M)}\\ 0\end{smallmatrix}\right)"] \arrow[d, "\psi^{M}", "\cong"'] & \Sigma^{-1}\mathrm{Cone}\,\mathrm{G}(\alpha) \arrow[r, "(0\ 1^{\mathrm{G}(N)})"] \arrow[d, dotted, "\breve{\alpha}", "\cong"'] & \mathrm{G}(N) \arrow[d, equal] \\
\mathrm{G}(\Sigma M) \arrow[r, "\mathrm{G}(0\ 1^{\Sigma M})"] & \mathrm{G}(\mathrm{Cone}\,\alpha) \arrow[r, "\mathrm{G}\left(\begin{smallmatrix}1^{N}\\ 0\end{smallmatrix}\right)"] & \mathrm{G}(N)
\end{tikzcd}$$

in $\mathcal{C}(S)$ commutative.

4.1.11 Definition. Let $\mathrm{G},\mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms $\psi\colon \Sigma^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ and $\phi\colon \Sigma^{-1}\mathrm{J} \to \mathrm{J}\Sigma$. A natural transformation $\tau\colon \mathrm{G} \to \mathrm{J}$ is called a Σ-*transformation* if the next diagram is commutative for every M in $\mathcal{C}(R)$,

$$\begin{tikzcd}
\Sigma^{-1}\mathrm{G}(M) \arrow[r, "\psi^{M}", "\cong"'] \arrow[d, "\Sigma^{-1}\tau^{M}"'] & \mathrm{G}(\Sigma M) \arrow[d, "\tau^{\Sigma M}"] \\
\Sigma^{-1}\mathrm{J}(M) \arrow[r, "\phi^{M}", "\cong"'] & \mathrm{J}(\Sigma M)\,.
\end{tikzcd}$$

That is, $\tau^{\Sigma M}$ and $\Sigma\tau^{M}$ are isomorphic, and the isomorphism is natural in M.

4.1.12. The following assertions are immediate from the definitions of Σ-functors.

(a) Let $\mathcal{C}(Q) \xrightarrow{\mathrm{F}_1} \mathcal{C}(R) \xrightarrow{\mathrm{F}_2} \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms ϕ_1 and ϕ_2. The composite $\mathrm{F}_2\mathrm{F}_1\colon \mathcal{C}(Q) \to \mathcal{C}(S)$ is a Σ-functor with associated natural isomorphism ϕ where ϕ^{M} is the composite

$$\mathrm{F}_2\mathrm{F}_1(\Sigma M) \xrightarrow{\mathrm{F}_2(\phi_1^{M})} \mathrm{F}_2\Sigma\mathrm{F}_1(M) \xrightarrow{\phi_2^{\mathrm{F}_1(M)}} \Sigma\mathrm{F}_2\mathrm{F}_1(M)\,.$$

(b) Let $\mathcal{C}(Q)^{\mathrm{op}} \xrightarrow{\mathrm{G}} \mathcal{C}(R) \xrightarrow{\mathrm{F}} \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms ϕ and ψ. The composite $\mathrm{F}\mathrm{G}\colon \mathcal{C}(Q)^{\mathrm{op}} \to \mathcal{C}(S)$ is a Σ-functor with associated natural isomorphism $\widetilde{\psi}$ where $\widetilde{\psi}^{M}$ is the composite

$$\Sigma^{-1}\mathrm{F}\mathrm{G}(M) \xrightarrow{\Sigma^{-1}\phi^{\Sigma^{-1}\mathrm{G}(M)}} \mathrm{F}\Sigma^{-1}\mathrm{G}(M) \xrightarrow{\mathrm{F}(\psi^{M})} \mathrm{F}\mathrm{G}(\Sigma M)\,.$$

(c) Let $\mathcal{C}(Q) \xrightarrow{\mathrm{F}} \mathcal{C}(R)$ and $\mathcal{C}(R)^{\mathrm{op}} \xrightarrow{\mathrm{G}} \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms ϕ and ψ. The composite $\mathrm{G}\mathrm{F}^{\mathrm{op}}\colon \mathcal{C}(Q)^{\mathrm{op}} \to \mathcal{C}(S)$ is a Σ-functor with associated natural isomorphism $\widetilde{\psi}$ where $\widetilde{\psi}^{M}$ is the composite

$$\Sigma^{-1}\mathrm{G}\mathrm{F}^{\mathrm{op}}(M) \xrightarrow{\psi^{\mathrm{F}(M)}} \mathrm{G}\Sigma\mathrm{F}^{\mathrm{op}}(M) \xrightarrow{\mathrm{G}(\phi^{M})} \mathrm{G}\mathrm{F}^{\mathrm{op}}(\Sigma M)\,.$$

(d) Let $\mathcal{C}(Q)^{\mathrm{op}} \xrightarrow{\mathrm{G}_1} \mathcal{C}(R)$ and $\mathcal{C}(R)^{\mathrm{op}} \xrightarrow{\mathrm{G}_2} \mathcal{C}(S)$ be Σ-functors with associated natural isomorphisms ψ_1 and ψ_2. The composite $\mathrm{G}_2\,\mathrm{G}_1^{\mathrm{op}} \colon \mathcal{C}(Q) \to \mathcal{C}(S)$ is a Σ-functor with associated natural isomorphism ϕ where ϕ^M is the composite

$$\mathrm{G}_2\,\mathrm{G}_1^{\mathrm{op}}(\Sigma M) \xrightarrow{\mathrm{G}_2(\psi_1^M)} \mathrm{G}_2\,\Sigma^{-1}\mathrm{G}_1^{\mathrm{op}}(M) \xrightarrow{\Sigma\psi_2^{\Sigma^{-1}\mathrm{G}_1(M)}} \Sigma\,\mathrm{G}_2\,\mathrm{G}_1^{\mathrm{op}}(M)\,.$$

4.1.13 Proposition. *Let* $\mathrm{E},\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ *be additive functors and* $\tau\colon \mathrm{E} \to \mathrm{F}$ *a natural transformation. The extended functors* $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *from* 2.1.48 *are* Σ*-functors, and the extended natural transformation* τ *is a* Σ*-transformation.*

Proof. The extended functor E is additive, see 2.1.48, and there is an equality $\mathrm{E}(\Sigma M) = \Sigma\mathrm{E}(M)$ for every R-complex M; see 2.2.2. For a morphism $\alpha\colon M \to N$, the definition of the extended functor yields the following identifications,

$$\mathrm{E}(\operatorname{Cone}\alpha) \;=\; \operatorname{Cone}\mathrm{E}(\alpha)\,,\quad \mathrm{E}\begin{pmatrix}1^N\\0\end{pmatrix} \;=\; \begin{pmatrix}1^{\mathrm{E}(N)}\\0\end{pmatrix}\,,\ \text{and}\quad \mathrm{E}\big(0\ 1^{\Sigma M}\big) \;=\; \big(0\ 1^{\Sigma\mathrm{E}(M)}\big)\,.$$

Thus, the identity maps $\varphi^M\colon \mathrm{E}(\Sigma M) \to \Sigma\mathrm{E}(M)$ and $\breve{\alpha}\colon \mathrm{E}(\operatorname{Cone}\alpha) \to \operatorname{Cone}\mathrm{E}(\alpha)$ make the diagram in 4.1.8 commutative. This shows that E is a Σ-functor and similarly so is F. That the extended natural transformation τ is a Σ-transformation now follows immediately from 2.2.2 and 4.1.9. □

4.1.14 Proposition. *Let* $\mathrm{G},\mathrm{J}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ *be additive functors and* $\tau\colon \mathrm{G} \to \mathrm{J}$ *a natural transformation. The extended functors* $\mathrm{G},\mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *from* 2.1.48 *are* Σ*-functors, and the extended natural transformation* τ *is a* Σ*-transformation.*

Proof. The assertions follow from an argument parallel to the proof of 4.1.14. □

We prove below that the Hom and tensor product functors are Σ-functors. The homology functor is $\Bbbk$-linear and commutes with Σ by 2.2.15; but it is not a Σ-functor:

4.1.15 Example. Let M be an R-complex. One has $\mathrm{H}(\operatorname{Cone}1^M) = 0$ by 4.1.3; however, if $\mathrm{H}(M)$ is non-zero, then one has $\operatorname{Cone}\mathrm{H}(1^M) = \operatorname{Cone}1^{\mathrm{H}(M)} \neq 0$; in particular $\mathrm{H}(\operatorname{Cone}1^M)$ and $\operatorname{Cone}\mathrm{H}(1^M)$ are not isomorphic.

4.1.16 Proposition. *Let M be an R-complex. The functor* $\operatorname{Hom}_R(M,-)$ *is a* Σ*-functor with associated natural isomorphism* $\phi\colon\ \operatorname{Hom}_R(M,\Sigma-) \to \Sigma\operatorname{Hom}_R(M,-)$ *given by*

$$\phi^N \;=\; \varsigma_1^{\operatorname{Hom}_R(M,N)}\operatorname{Hom}_R(M,\varsigma_{-1}^{\Sigma N})$$

for every R-complex N. In particular, there is an isomorphism of $\Bbbk$*-complexes,*

$$\operatorname{Cone}\operatorname{Hom}_R(M,\beta) \;\cong\; \operatorname{Hom}_R(M,\operatorname{Cone}\beta)\,,$$

for every morphism β of R-complexes.

Proof. The functor $\operatorname{Hom}_R(M,-)$ is additive by 2.3.10, and ϕ^N is by 2.3.16 an isomorphism and natural in N. To prove that $\operatorname{Hom}_R(M,-)$ is a Σ-functor, it must be shown that for every morphism $\beta\colon N \to N'$ of R-complexes there exists an isomorphism $\breve{\beta}$ that makes the following diagram in $\mathcal{C}(\Bbbk)$ commutative.

$$
\begin{tikzcd}[column sep=huge]
\mathrm{Hom}_R(M,N') \arrow[r, "\mathrm{Hom}\left(M,\left(\begin{smallmatrix}1^{N'}\\0\end{smallmatrix}\right)\right)"] \arrow[d, equal] & \mathrm{Hom}_R(M,\mathrm{Cone}\,\beta) \arrow[r, "\mathrm{Hom}(M,(0\ 1^{\Sigma N}))"] \arrow[d, "\cong"', "\breve{\beta}"] & \mathrm{Hom}_R(M,\Sigma N) \arrow[d, "\cong"', "\phi^N"] \\
\mathrm{Hom}_R(M,N') \arrow[r, "\left(\begin{smallmatrix}1^{\mathrm{Hom}(M,N')}\\0\end{smallmatrix}\right)"] & \mathrm{Cone}\,\mathrm{Hom}_R(M,\beta) \arrow[r, "(0\ 1^{\Sigma\mathrm{Hom}(M,N)})"] & \Sigma\mathrm{Hom}_R(M,N)\,.
\end{tikzcd} \tag{†}
$$

For reasons of simplicity, we construct an isomorphism

$$\gamma : \mathrm{Cone}\,\mathrm{Hom}_R(M,\beta) \longrightarrow \mathrm{Hom}_R(M,\mathrm{Cone}\,\beta)$$

with $\gamma^{-1} = \breve{\beta}$. To this end, observe that on the level of graded modules one has

$$\mathrm{Hom}_R(M,\mathrm{Cone}\,\beta)^\natural = \mathrm{Hom}_R(M^\natural,(\mathrm{Cone}\,\beta)^\natural) = \mathrm{Hom}_R(M^\natural, N'^\natural \oplus \Sigma N^\natural)\,,$$

and similarly,

$$(\mathrm{Cone}\,\mathrm{Hom}_R(M,\beta))^\natural = \mathrm{Hom}_R(M^\natural,N'^\natural) \oplus \Sigma\mathrm{Hom}_R(M^\natural,N^\natural)\,.$$

These equalities, combined with the fact that the functor $\mathrm{Hom}_R(M^\natural,-)$ is additive and ϕ^N is an isomorphism, show that one gets an isomorphism of graded modules, $\gamma\colon (\mathrm{Cone}\,\mathrm{Hom}_R(M,\beta))^\natural \to \mathrm{Hom}_R(M,\mathrm{Cone}\,\beta)^\natural$, by setting

$$\gamma\begin{pmatrix}\vartheta\\ \xi\end{pmatrix} = \begin{pmatrix}\vartheta\\ (\phi^N)^{-1}(\xi)\end{pmatrix} \tag{‡}$$

for homogeneous elements $\vartheta \in \mathrm{Hom}_R(M,N')$ and $\xi \in \Sigma\mathrm{Hom}_R(M,N)$ of the same degree, say, d. Notice that the left-hand column $(\vartheta\ \xi)^{\mathsf{T}}$ in (‡) is a pair of homomorphisms, whereas the right-hand column is a homomorphism $M \to N'\oplus\Sigma N$ of degree d. To show that γ is a morphism, and hence an isomorphism of complexes, note first that the definition of ϕ^N and (2.2.5.1) yield

$$\mathrm{Hom}_R(M,\beta)\varsigma_{-1}^{\Sigma\mathrm{Hom}(M,N)} = \mathrm{Hom}_R(M,\beta\varsigma_{-1}^{\Sigma N}) \circ (\phi^N)^{-1}\,.$$

Using this identity and the fact that $(\phi^N)^{-1}$ is a morphism of complexes, one gets

$$
\begin{aligned}
\partial^{\mathrm{Hom}(M,\mathrm{Cone}\,\beta)}\gamma\begin{pmatrix}\vartheta\\ \xi\end{pmatrix}
&= \begin{pmatrix}\partial^{N'} & \beta\varsigma_{-1}^{\Sigma N}\\ 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}\vartheta\\ (\phi^N)^{-1}(\xi)\end{pmatrix} - (-1)^d\begin{pmatrix}\vartheta\\ (\phi^N)^{-1}(\xi)\end{pmatrix}\partial^M\\
&= \begin{pmatrix}\partial^{N'}\vartheta + \beta\varsigma_{-1}^{\Sigma N}(\phi^N)^{-1}(\xi) - (-1)^d\vartheta\partial^M\\ \partial^{\Sigma N}(\phi^N)^{-1}(\xi) - (-1)^d(\phi^N)^{-1}(\xi)\partial^M\end{pmatrix}\\
&= \begin{pmatrix}(\partial^{N'}\vartheta - (-1)^d\vartheta\partial^M) + \beta\varsigma_{-1}^{\Sigma N}(\phi^N)^{-1}(\xi)\\ \partial^{\mathrm{Hom}(M,\Sigma N)}((\phi^N)^{-1}(\xi))\end{pmatrix}\\
&= \begin{pmatrix}\partial^{\mathrm{Hom}(M,N')}(\vartheta) + \mathrm{Hom}(M,\beta\varsigma_{-1}^{\Sigma N})((\phi^N)^{-1}(\xi))\\ (\phi^N)^{-1}(\partial^{\Sigma\mathrm{Hom}(M,N)}(\xi))\end{pmatrix}\\
&= \gamma\begin{pmatrix}\partial^{\mathrm{Hom}(M,N')}(\vartheta) + \mathrm{Hom}(M,\beta)\varsigma_{-1}^{\Sigma\mathrm{Hom}_R(M,N)}(\xi)\\ \partial^{\Sigma\mathrm{Hom}(M,N)}(\xi)\end{pmatrix}
\end{aligned}
$$

$$= \gamma\partial^{\operatorname{Cone}\operatorname{Hom}(M,\beta)} \begin{pmatrix} \vartheta \\ \xi \end{pmatrix} .$$

Thus γ is a morphism of complexes.

It remains to verify that the diagram (†) is commutative. The computation

$$\begin{aligned} \operatorname{Hom}_R(M, (0\ 1^{\Sigma N}))\gamma \begin{pmatrix} \vartheta \\ \xi \end{pmatrix} &= \begin{pmatrix} 0 & 1^{\Sigma N} \end{pmatrix} \begin{pmatrix} \vartheta \\ (\phi^N)^{-1}(\xi) \end{pmatrix} \\ &= (\phi^N)^{-1}(\xi) \\ &= (\phi^N)^{-1} \begin{pmatrix} 0 & 1^{\Sigma \operatorname{Hom}(M,N)} \end{pmatrix} \begin{pmatrix} \vartheta \\ \xi \end{pmatrix} \end{aligned}$$

shows that the right-hand square in (†) is commutative. A similar simple computation shows that the left-hand square is commutative. □

4.1.17 Proposition. *Let N be an R-complex. The functor* $\operatorname{Hom}_R(-,N)$ *is a* Σ*-functor with associated natural isomorphism* $\psi\colon \Sigma^{-1}\operatorname{Hom}_R(-,N) \to \operatorname{Hom}_R(\Sigma-,N)$ *given by*

$$\psi^M = \operatorname{Hom}_R(\varsigma_{-1}^{\Sigma M}, N)\varsigma_1^{\Sigma^{-1}\operatorname{Hom}_R(M,N)}$$

for every R-complex M. In particular there is an isomorphism of $\Bbbk$-complexes,

$$\Sigma^{-1}\operatorname{Cone}\operatorname{Hom}_R(\alpha,N) \cong \operatorname{Hom}_R(\operatorname{Cone}\alpha, N) ,$$

for every morphism α of R-complexes.

Proof. The functor $\operatorname{Hom}_R(-,N)$ is additive by 2.3.10, and ψ^M is by 2.3.14 an isomorphism and natural in M. To prove that $\operatorname{Hom}_R(-,N)$ is a Σ-functor, one argues as in the proof of 4.1.16. □

Remark. The natural transformation ϕ in 4.1.16 depends on M. The notation does not reflect this because there is no formal requirement of naturalness in M. Nevertheless, ϕ *is* natural in M; see E 4.1.5. A similar comment applies to the transformation ψ in 4.1.17. Moreover, the transformations ϕ and ψ are compatible in a sense that is also clarified in E 4.1.5.

4.1.18 Proposition. *Let M be an R^{o}-complex. The functor $M \otimes_R -$ is a* Σ*-functor with associated natural transformation $\phi\colon M \otimes_R \Sigma- \longrightarrow \Sigma(M \otimes_R -)$ given by*

$$\phi^N = \varsigma_1^{M\otimes_R N}(M \otimes_R \varsigma_{-1}^{\Sigma N})$$

for every R-complex N. In particular, there is an isomorphism of $\Bbbk$-complexes,

$$\operatorname{Cone}(M \otimes_R \beta) \cong M \otimes_R \operatorname{Cone}\beta ,$$

for every morphism β of R-complexes.

Proof. The functor $M \otimes_R -$ is additive by 2.4.9, and ϕ^N is by 2.4.13 an isomorphism and natural in N. To prove that $M \otimes_R -$ is a Σ-functor, it must be shown that for every morphism $\beta\colon N \to N'$ of R-complexes there exists an isomorphism $\breve{\beta}$ that makes the following diagram in $\mathcal{C}(\Bbbk)$ commutative.

$$
(\dagger)\qquad
\begin{tikzcd}
M \otimes_R N' \arrow[r, "M\otimes\left(\begin{smallmatrix}1^{N'}\\0\end{smallmatrix}\right)"] \arrow[d, equal] & M \otimes_R \operatorname{Cone}\beta \arrow[r, "M\otimes(0\ 1^{\Sigma N})"] \arrow[d, "\breve{\beta}", "\cong"'] & M \otimes_R \Sigma N \arrow[d, "\phi^N", "\cong"'] \\
M \otimes_R N' \arrow[r, "\left(\begin{smallmatrix}1^{M\otimes N'}\\0\end{smallmatrix}\right)"] & \operatorname{Cone}(M \otimes_R \beta) \arrow[r, "(0\ 1^{\Sigma(M\otimes N)})"] & \Sigma(M \otimes_R N)\,.
\end{tikzcd}
$$

To define $\breve{\beta}$, note that on the level of graded modules one has

$$(M \otimes_R \operatorname{Cone}\beta)^\natural \;=\; M^\natural \otimes_R (\operatorname{Cone}\beta)^\natural \;=\; M^\natural \otimes_R (N'^\natural \oplus \Sigma N^\natural)\,,$$

and

$$(\operatorname{Cone}(M \otimes_R \beta))^\natural \;=\; (M^\natural \otimes_R N'^\natural) \oplus \Sigma(M^\natural \otimes_R N^\natural)\,.$$

These equalities, combined with the fact that the functor $M^\natural \otimes_R -$ is additive and ϕ^N is an isomorphism, show that one defines an isomorphism of graded modules, $\breve{\beta}\colon (M \otimes_R \operatorname{Cone}\beta)^\natural \to (\operatorname{Cone}(M \otimes_R \beta))^\natural$, by setting

$$\breve{\beta}\left(m \otimes \begin{pmatrix} n' \\ n \end{pmatrix}\right) \;=\; \begin{pmatrix} m \otimes n' \\ \phi^N(m \otimes n) \end{pmatrix}$$

for an elementary tensor in $M \otimes_R \operatorname{Cone}\beta$ with $n \in \Sigma N$ and $n' \in N'$. To show that $\breve{\beta}$ is a morphism, and hence an isomorphism of complexes, note first that the definition of ϕ^N and (2.2.5.1) yield

$$(M \otimes_R \beta)\varsigma_{-1}^{\Sigma(M\otimes N)}\phi^N \;=\; M \otimes_R (\beta\varsigma_{-1}^{\Sigma N})\,.$$

Using this identity and the fact that ϕ^N is a morphism of complexes, one gets

$$
\begin{aligned}
&\partial^{\operatorname{Cone}(M\otimes\beta)}\breve{\beta}\left(m \otimes \begin{pmatrix} n' \\ n \end{pmatrix}\right) \\
&\quad= \begin{pmatrix} \partial^{M\otimes N'} & (M \otimes \beta)\varsigma_{-1}^{\Sigma(M\otimes N)} \\ 0 & \partial^{\Sigma(M\otimes N)} \end{pmatrix} \begin{pmatrix} m \otimes n' \\ \phi^N(m \otimes n) \end{pmatrix} \\
&\quad= \begin{pmatrix} \partial^{M\otimes N'}(m \otimes n') + (M \otimes (\beta\varsigma_{-1}^{\Sigma N}))(m \otimes n) \\ \partial^{\Sigma(M\otimes N)}\phi^N(m \otimes n) \end{pmatrix} \\
&\quad= \begin{pmatrix} \partial^M(m) \otimes n' + (-1)^{|m|} m \otimes \partial^{N'}(n') + (-1)^{|m|} m \otimes \beta\varsigma_{-1}^{\Sigma N}(n) \\ \phi^N\partial^{M\otimes\Sigma N}(m \otimes n) \end{pmatrix} \\
&\quad= \begin{pmatrix} \partial^M(m) \otimes n' + (-1)^{|m|} m \otimes (\partial^{N'}(n') + \beta\varsigma_{-1}^{\Sigma N}(n)) \\ \phi^N(\partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^{\Sigma N}(n)) \end{pmatrix} \\
&\quad= \breve{\beta}\left(\partial^M(m) \otimes \begin{pmatrix} n' \\ n \end{pmatrix} + (-1)^{|m|} m \otimes \begin{pmatrix} \partial^{N'}(n') + \beta\varsigma_{-1}^{\Sigma N}(n) \\ \partial^{\Sigma N}(n) \end{pmatrix}\right) \\
&\quad= \breve{\beta}\partial^{M\otimes\operatorname{Cone}\beta}\left(m \otimes \begin{pmatrix} n' \\ n \end{pmatrix}\right)\,.
\end{aligned}
$$

Thus $\breve{\beta}$ is a morphism of complexes.

It remains to verify that the diagram (†) is commutative. The computation

$$\begin{aligned}
\begin{pmatrix}0 & 1^{\Sigma(M\otimes N)}\end{pmatrix}\breve{\beta}\left(m\otimes\begin{pmatrix}n'\\ n\end{pmatrix}\right) &= \begin{pmatrix}0 & 1^{\Sigma(M\otimes N)}\end{pmatrix}\begin{pmatrix}m\otimes n'\\ \phi^N(m\otimes n)\end{pmatrix}\\
&= \phi^N(m\otimes n)\\
&= \phi^N\big(M\otimes(0\ 1^{\Sigma N})\big)\left(m\otimes\begin{pmatrix}n'\\ n\end{pmatrix}\right)
\end{aligned}$$

shows that the right-hand square in (†) is commutative. A similar simple computation shows that the left-hand square is commutative. □

Remark. The natural transformation ϕ in 4.1.18 depends on M. The notation does not reflect this because there is no formal requirement of naturalness in M. Nevertheless, ϕ *is* natural in M; see E 4.1.6. A similar comment applies to the transformation ψ in 4.1.19. Moreover, the transformations ϕ and ψ are compatible in a sense that is also clarified in E 4.1.6.

4.1.19 Proposition. *Let N be an R-complex. The functor $-\otimes_R N$ is a Σ-functor with associated natural transformation $\psi\colon \Sigma-\otimes_R N \to \Sigma(-\otimes_R N)$ given by*

$$\psi^M = \varsigma_1^{M\otimes_R N}(\varsigma_{-1}^{\Sigma M}\otimes_R N)$$

for every R^{o}-complex M. In particular, there is an isomorphism of $\Bbbk$-complexes,

$$\operatorname{Cone}(\alpha\otimes_R N) \cong (\operatorname{Cone}\alpha)\otimes_R N\,,$$

for every morphism α of R^{o}-complexes.

Proof. The functor $-\otimes_R N$ is additive by 2.4.9, and ψ^M is by 2.4.14 an isomorphism and natural in M. To see that $-\otimes_R N$ is a Σ-functor, argue as in the proof of 4.1.18. □

Exercises

E 4.1.1 Let $\alpha\colon M\to N$ be a homomorphism of R-modules; examine the homology of Cone α.

E 4.1.2 Let α be a morphism of R-complexes; establish an isomorphism $\operatorname{Cone}(\Sigma\alpha)\cong\Sigma\operatorname{Cone}\alpha$.

E 4.1.3 Let $\alpha\colon M\to N$ be a morphism in $\mathcal{C}(R)$. Show that for isomorphisms $\varphi\colon M'\to M$ and $\psi\colon N\to N'$ the complexes Cone α and $\operatorname{Cone}(\psi\alpha\varphi)$ are isomorphic.

E 4.1.4 Assume that R is commutative. For elements x_1 and x_2 in R, examine the mapping cones $K = \operatorname{Cone} x_1^R$ and $K' = \operatorname{Cone} x_2^K$ of the homotheties x_1^R and x_2^K. Compare with the Koszul complexes $\mathrm{K}^R(x_1)$ and $\mathrm{K}^R(x_1,x_2)$; see 2.1.25.

E 4.1.5 Let M and N be R-complexes. Denote the natural transformations from 4.1.16 and 4.1.17 by ϕ_M and ψ_N, respectively.

(a) Show that ϕ and ψ are natural in M and N; that is, show that the diagrams

$$\begin{tikzcd}
\operatorname{Hom}_R(M,\Sigma N) \arrow[r, "\phi_M^N"] \arrow[d, "\operatorname{Hom}(\alpha,\Sigma N)"'] & \Sigma\operatorname{Hom}_R(M,N) \arrow[d, "\Sigma\operatorname{Hom}(\alpha,N)"] \\
\operatorname{Hom}_R(M',\Sigma N) \arrow[r, "\phi_{M'}^N"] & \Sigma\operatorname{Hom}_R(M',N)
\end{tikzcd}$$

and

$$\begin{CD} \Sigma^{-1}\operatorname{Hom}_R(M,N) @>{\psi_N^M}>> \operatorname{Hom}_R(\Sigma M,N) \\ @V{\Sigma^{-1}\operatorname{Hom}(M,\beta)}VV @VV{\operatorname{Hom}_R(\Sigma M,\beta)}V \\ \Sigma^{-1}\operatorname{Hom}_R(M,N') @>{\psi_{N'}^M}>> \operatorname{Hom}_R(\Sigma M,N') \end{CD}$$

are commutative for all morphisms $\alpha\colon M'\to M$ and $\beta\colon N\to N'$ in $\mathcal{C}(R)$.

(b) Show that the transformations ϕ and ψ are compatible in the sense that the next diagram is commutative,

$$\begin{CD} \Sigma^{-1}\operatorname{Hom}_R(M,\Sigma N) @>{\Sigma^{-1}\phi_M^N}>> \operatorname{Hom}_R(M,N) \\ @V{\psi_{\Sigma N}^M}VV @VV{\Sigma\psi_N^M}V \\ \operatorname{Hom}_R(\Sigma M,\Sigma N) @>{\phi_{\Sigma M}^N}>> \Sigma\operatorname{Hom}_R(\Sigma M,N) \end{CD} \;.$$

E 4.1.6 Let M be an R°-complex and N an R-complex. Denote the natural transformations from 4.1.18 and 4.1.19 by ϕ_M and ψ_N, respectively.

(a) Show that ϕ and ψ are natural in M and N; that is, show that the diagrams

$$\begin{CD} M\otimes_R\Sigma N @>{\phi_M^N}>> \Sigma(M\otimes_R N) \\ @V{\alpha\otimes\Sigma N}VV @VV{\Sigma(\alpha\otimes N)}V \\ M'\otimes_R\Sigma N @>{\phi_{M'}^N}>> \Sigma(M'\otimes_R N) \end{CD} \quad\text{and}\quad \begin{CD} \Sigma M\otimes_R N @>{\psi_N^M}>> \Sigma(M\otimes_R N) \\ @V{\Sigma M\otimes\beta}VV @VV{\Sigma(M\otimes\beta)}V \\ \Sigma M\otimes_R N' @>{\psi_{N'}^M}>> \Sigma(M\otimes_R N') \end{CD}$$

are commutative for all morphisms $\alpha\colon M\to M'$ in $\mathcal{C}(R^{\circ})$ and $\beta\colon N\to N'$ in $\mathcal{C}(R)$.

(b) Show that the transformations ϕ and ψ are compatible in the sense that the next diagram is commutative,

$$\begin{CD} \Sigma M\otimes_R\Sigma N @>{\phi_{\Sigma M}^N}>> \Sigma(\Sigma M\otimes_R N) \\ @V{\psi_{\Sigma N}^M}VV @VV{\Sigma\psi_N^M}V \\ \Sigma(M\otimes_R\Sigma N) @>{\Sigma\phi_M^N}>> \Sigma^2(M\otimes_R N) \end{CD} \;.$$

E 4.1.7 Show that truncations are not Σ-functors.

E 4.1.8 Let $\alpha\colon M\to N$ be an isomorphism of R-complexes. Show that $1^{\operatorname{Cone}\alpha}$ is null-homotopic.

4.2 Quasi-Isomorphisms

Synopsis. Quasi-isomorphism; functor that preserves quasi-isomorphisms; semi-simple module.

More often than not, it is the homology of a complex that one is interested in; more so than the complex itself. For example, it is the homology of the singular chain complex $\mathrm{S}(X)$ that yields information about holes in the space X, and it is the homology of the Koszul complex $\mathrm{K}(x)$ that tells whether the element x is a zerodivisor.

Quasi-isomorphisms are morphisms that preserve homology.

4.2.1 Definition. A morphism $\alpha\colon M\to N$ in $\mathcal{C}(R)$ is called a *quasi-isomorphism* if the induced morphism $\mathrm{H}(\alpha)\colon\ \mathrm{H}(M)\to\mathrm{H}(N)$ is an isomorphism.

A quasi-isomorphism is marked by a '$\simeq$' next to the arrow.

Remark. Other words for quasi-isomorphism are 'homology equivalence' and 'homology isomorphism'.

Given a quasi-isomorphism of R-complexes $\alpha\colon M \to N$ there need not exist a morphism $\beta\colon N \to M$ with $\mathrm{H}(\beta) = \mathrm{H}(\alpha)^{-1}$. Moreover, for R-complexes M and N with $\mathrm{H}(M) \cong \mathrm{H}(N)$ there need not exist a quasi-isomorphism $M \to N$ or $N \to M$. Examples follow below.

4.2.2 Example. There is a quasi-isomorphism of $\mathbb{Z}$-complexes,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0\,, \end{tikzcd}$$

but there is not even a non-zero morphism in the opposite direction, as the zero map is the only homomorphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.

4.2.3 Example. Set $R = \Bbbk[x, y]$. The R-complexes

$$\begin{aligned} M &= 0 \longrightarrow R/(x) \xrightarrow{\;y\;} R/(x) \longrightarrow 0 \\ N &= 0 \longrightarrow R/(y) \xrightarrow{\;x\;} R/(y) \longrightarrow 0 \end{aligned}$$

concentrated in degrees 1 and 0 have isomorphic homology $\mathrm{H}(M) \cong \Bbbk \cong \mathrm{H}(N)$, but there are no non-zero morphisms between them and hence no quasi-isomorphism.

Quasi-isomorphisms between complexes allow for considerable leeway in the structure of the underlying graded modules. Via quasi-isomorphisms one can thus hope to replace a complex by one with better properties.

A quasi-isomorphism between complexes that originate from independent constructions may have conceptual significance. For example, De Rham's theorem asserts that for a smooth real manifold M, the embedding $\mathrm{S}^{\infty}(M) \rightarrowtail \mathrm{S}(M)$ from 2.1.34 is a quasi-isomorphism; see Massey [178, A§2]. If M is paracompact, then the morphism $\Omega(M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{S}^{\infty}(M), \mathbb{R})$ from 2.1.34 is also a quasi-isomorphism; combining these results one arrives at de Rham's theorem: The de Rham cohomology, which is defined in terms of the smooth structure on M, is isomorphic to the singular cohomology of M, whose definition involves only the structure of M as a topological space.

Soft truncations induce quasi-isomorphisms.

4.2.4 Proposition. *Let M be an R-complex and n an integer.*

(a) *If $n \geqslant \sup M$, then the surjection $\tau^{M}_{\subseteq n}\colon M \twoheadrightarrow M_{\subseteq n}$ is a quasi-isomorphism.*
(b) *If $n \leqslant \inf M$, then the embedding $\tau^{M}_{\supseteq n}\colon M_{\supseteq n} \rightarrowtail M$ is a quasi-isomorphism.*

Proof. The assertions follow immediately from 2.5.24(b) and 2.5.25(b). □

Quasi-isomorphisms have the following convenient two-out-of-three property.

4.2.5 Proposition. *Consider a commutative diagram in $\mathcal{C}(R)$ with exact rows,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0\,. \end{array}$$

If two of the morphisms φ', φ, and φ'' are quasi-isomorphisms, then so is the third.

PROOF. The assertion follows immediately from 2.2.21 and the Five Lemma 2.1.41. □

4.2.6 Proposition. *Let $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ be an exact sequence of R-complexes.*

(a) *The complex M' is acyclic if and only if α is a quasi-isomorphism.*
(b) *The complex M'' is acyclic if and only if α' is a quasi-isomorphism.*

PROOF. Both assertions are immediate from the exact sequence (2.2.20.1) and the definition, 4.2.1, of a quasi-isomorphism. □

Recall from 2.2.13 that a morphism $\alpha\colon M \to N$ of complexes restricts to a morphism on cycles, $\mathrm{Z}(M) \to \mathrm{Z}(N)$, and to a morphism on boundaries $\mathrm{B}(M) \to \mathrm{B}(N)$. Surjectivity of a quasi-isomorphism can be detected on boundaries and on cycles.

4.2.7 Lemma. *Let $\alpha\colon M \to N$ be a quasi-isomorphism of R-complexes. The following conditions are equivalent.*

(i) *α is surjective.*
(ii) *α is surjective on boundaries.*
(iii) *α is surjective on cycles.*
(iv) *α is surjective on cycles and boundaries.*

PROOF. It is immediate from 2.1.28 that a surjective morphism is surjective on boundaries, whence (i) implies (ii). An application of the Snake Lemma 2.1.45 to the diagram (2.2.14.1) shows that conditions (ii) and (iii) are equivalent and, therefore, that they both imply (iv). To prove that (iv) implies (i), apply the Snake Lemma to the following commutative diagram in $\mathcal{C}(R)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Z}(M) & \longrightarrow & M & \xrightarrow{\partial^M} & \Sigma\mathrm{B}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\Sigma\alpha} & & \\ 0 & \longrightarrow & \mathrm{Z}(N) & \longrightarrow & N & \xrightarrow{\partial^N} & \Sigma\mathrm{B}(N) & \longrightarrow & 0\,. \end{array}$$ □

REMARK. Injectivity of a quasi-isomorphism can, by a result dual to 4.2.7, be detected on boundaries and on cokernels; see E 4.2.6.

PRESERVATION OF QUASI-ISOMORPHISMS

4.2.8 Definition. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor. One says that F *preserves* quasi-isomorphisms if for every quasi-isomorphism α in $\mathcal{C}(R)$ the morphism $\mathrm{F}(\alpha)$ is $\mathcal{C}(S)$

is a quasi-isomorphism. One says that F *reflects* quasi-isomorphisms if a morphism α in $\mathcal{C}(R)$ is a quasi-isomorphism provided that $\mathrm{F}(\alpha)$ is $\mathcal{C}(S)$ is a quasi-isomorphism.

4.2.9 Example. By 2.2.15 the shift functor, Σ, preserves quasi-isomorphisms.

Recall from 2.5.24 and 2.5.25 that soft truncations are functors.

4.2.10 Proposition. *For every integer n, the functors $(-)_{\subseteq n}$ and $(-)_{\supseteq n}$ preserve quasi-isomorphisms.*

Proof. Let $\alpha\colon M \to N$ be a quasi-isomorphism. In each degree $v \geqslant n$ one has $\mathrm{H}_v(\alpha_{\supseteq n}) = \mathrm{H}_v(\alpha)$, which is an isomorphism by assumption. For $v < n$ the homomorphisms $\mathrm{H}_v(\alpha_{\supseteq n})$ are isomorphisms as the truncated complexes are zero in those degrees. A similar argument shows that $(-)_{\subseteq n}$ preserves quasi-isomorphisms. □

Several categorical constructions preserve quasi-isomorphisms.

4.2.11 Proposition. *Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{C}(R)$. If α^u is a quasi-isomorphism for every $u \in U$, then the coproduct $\coprod_{u\in U}\alpha^u$ and the product $\prod_{u\in U}\alpha^u$ are quasi-isomorphisms.*

Proof. Homology preserves coproducts and products by 3.1.10(d) and 3.1.22(d), and the assertions now follow as a (co)product of isomorphisms is an isomorphism. □

4.2.12 Proposition. *Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ be a morphism of U-direct systems in $\mathcal{C}(R)$. If U is filtered and α^u is a quasi-isomorphism for every $u \in U$, then the colimit $\operatorname{colim}_{u\in U}\alpha^u$ is a quasi-isomorphism.*

Proof. A colimit of isomorphisms is an isomorphism. Homology, as a functor, preserves filtered colimits by 3.3.15(d), and the assertion now follows from 3.2.17. □

4.2.13 Proposition. *Let $\{\kappa^u\colon M^u \to M^{u-1}\}_{u\in\mathbb{Z}}$ and $\{\lambda^u\colon N^u \to N^{u-1}\}_{u\in\mathbb{Z}}$ be towers in $\mathcal{C}(R)$ and $\{\alpha^u\colon M^u \to N^u\}_{u\in\mathbb{Z}}$ a morphism of towers. If for all $u \in \mathbb{Z}$ the morphisms κ^u, λ^u, $\mathrm{H}(\kappa^u)$, and $\mathrm{H}(\lambda^u)$ are surjective and α^u is a quasi-isomorphism, then the limit $\lim_{u\in\mathbb{Z}}\alpha^u$ is a quasi-isomorphism.*

Proof. A limit of isomorphisms in $\mathcal{C}(R)$ is an isomorphism. The canonical morphisms $\mathrm{H}(\lim_{u\in\mathbb{Z}} N^u) \to \lim_{u\in\mathbb{Z}}\mathrm{H}(N^u)$ and $\mathrm{H}(\lim_{u\in\mathbb{Z}} M^u) \to \lim_{u\in\mathbb{Z}}\mathrm{H}(M^u)$ are isomorphisms by 3.5.20, and the assertion now follows from 3.4.18. □

A functor F that commutes with homology clearly preserves quasi-isomorphisms. For ease of reference, we spell out an important special case.

4.2.14 Example. Let F be a functor on R-complexes extended from an exact functor on R-modules as in 2.1.48. By 2.2.19 one has $\mathrm{HF} = \mathrm{FH}$. From this identity it follows that F preserves quasi-isomorphisms. An example of such a functor is localization at a multiplicative subset of a commutative ring, see 2.1.50. It also follows that if F is conservative, then it reflects quasi-isomorphisms. An example of such a functor is restriction of scalars from 2.1.49.

The Hom and tensor product functors do not commute with homology, and they also fail to preserve quasi-isomorphisms. Indeed, let α be the quasi-isomorphism of $\mathbb{Z}$-complexes from 4.2.2. Not one of the induced morphisms $\operatorname{Hom}_{\mathbb{Z}}(\alpha, \mathbb{Z}/2\mathbb{Z})$, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \alpha)$, or $\alpha \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is a quasi-isomorphism.

In the next section we shall see that there are non-trivial quasi-isomorphisms α with the property that every morphism $\operatorname{Hom}(\alpha, X)$, $\operatorname{Hom}(X, \alpha)$, and $\alpha \otimes X$ is a quasi-isomorphism. Complexes X with the property that, say, $\operatorname{Hom}(\alpha, X)$ is a quasi-isomorphism for every quasi-isomorphism α also exist; they are studied in Chap. 5.

Mapping Cone of a Quasi-Isomorphism

4.2.15 Proposition. *Let $\alpha\colon M \to N$ be a morphism of R-complexes. The mapping cone sequence $0 \longrightarrow N \xrightarrow{\varepsilon} \operatorname{Cone}\alpha \xrightarrow{\varpi} \Sigma M \longrightarrow 0$ from 4.1.5 yields an exact sequence*

$$\mathrm{H}(N) \xrightarrow{\mathrm{H}(\varepsilon)} \mathrm{H}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{H}(\varpi)} \Sigma\mathrm{H}(M) \xrightarrow{\Sigma\mathrm{H}(\alpha)} \Sigma\mathrm{H}(N) \xrightarrow{\Sigma\mathrm{H}(\varepsilon)} \Sigma\mathrm{H}(\operatorname{Cone}\alpha)\ .$$

Proof. By (2.2.20.1) the mapping cone sequence induces an exact sequence,

$$\mathrm{H}(N) \xrightarrow{\mathrm{H}(\varepsilon)} \mathrm{H}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{H}(\varpi)} \mathrm{H}(\Sigma M) \xrightarrow{\eth} \Sigma\mathrm{H}(N) \xrightarrow{\Sigma\mathrm{H}(\varepsilon)} \Sigma\mathrm{H}(\operatorname{Cone}\alpha)\ ,$$

where $\eth$ is the connecting morphism in homology. Recall from 4.1.5 that one has $\varepsilon = (1^N\ 0)^{\mathsf{T}}$ and $\varpi = (0\ 1^{\Sigma M})$. For $[z]$ in $\mathrm{H}(\Sigma M)$ one has $z = (0\ 1^{\Sigma M})(0\ z)^{\mathsf{T}}$ and

$$\begin{aligned}
\varsigma_1^{\operatorname{Cone}\alpha}\partial^{\operatorname{Cone}\alpha}\begin{pmatrix}0\\ z\end{pmatrix} &= \begin{pmatrix}\varsigma_1^N & 0\\ 0 & \varsigma_1^{\Sigma M}\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma_{-1}^{\Sigma M}\\ 0 & \partial^{\Sigma M}\end{pmatrix}\begin{pmatrix}0\\ z\end{pmatrix}\\
&= \begin{pmatrix}\varsigma_1^N\alpha\varsigma_{-1}^{\Sigma M}(z)\\ 0\end{pmatrix}\\
&= \Sigma\begin{pmatrix}1^N\\ 0\end{pmatrix}((\Sigma\alpha)(z))\ ,
\end{aligned}$$

so $\eth([z]) = [(\Sigma\alpha)(z)]$ holds by 2.2.20. Thus, one has $\eth = \mathrm{H}(\Sigma\alpha) = \Sigma\mathrm{H}(\alpha)$. □

The exact sequence in 4.2.15 establishes a crucial connection between preservation of homology and vanishing of homology.

4.2.16 Theorem. *A morphism α of R-complexes is a quasi-isomorphism if and only if the complex $\operatorname{Cone}\alpha$ is acyclic.*

Proof. For a morphism $\alpha\colon M \to N$ the exact sequence in 4.2.15 shows that $\operatorname{Cone}\alpha$ is acyclic, i.e. $\mathrm{H}(\operatorname{Cone}\alpha) = 0$, if and only if $\mathrm{H}(\alpha)$ is an isomorphism. □

Semi-Simple Modules

A graded R-module is called *semi-simple* if every graded submodule is a graded direct summand.

4.2.17 Proposition. *Let M be an R-complex. If the graded R-module $M^\natural$ is semi-simple, then there is an isomorphism of R-complexes $M \cong \mathrm{H}(M) \oplus \mathrm{Cone}\, 1^{\mathrm{B}(M)}$.*

Proof. Set $Z = \mathrm{Z}(M)$, $B = \mathrm{B}(M)$, and $H = \mathrm{H}(M)$; recall from 2.2.12 the short exact sequences of R-complexes,

$$0 \longrightarrow B \xrightarrow{\iota} Z \xrightarrow{\pi} H \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Z \xrightarrow{\varepsilon} M \xrightarrow{\varsigma_1^B \partial^M} \Sigma B \longrightarrow 0\,.$$

By assumption $M^\natural$ is semi-simple, and hence so is the graded submodule $Z^\natural$. It follows that both sequences are degreewise split. In particular, there are morphisms $\sigma \colon H^\natural \to Z^\natural$ and $\tau \colon \Sigma B^\natural \to M^\natural$ with $\pi\sigma = 1^{H^\natural}$ and $\varsigma_1^B \partial^M \tau = 1^{\Sigma B^\natural}$. Thus, the map

$$H^\natural \oplus \mathrm{Cone}\, 1^{B^\natural} = H^\natural \oplus B^\natural \oplus \Sigma B^\natural \longrightarrow M^\natural$$

given by $(\varepsilon\sigma \;\, \varepsilon\iota \;\, \tau)$ is an isomorphism of graded R-modules. In fact, it is an isomorphism of complexes as one has

$$\partial^M \big(\varepsilon\sigma \;\, \varepsilon\iota \;\, \tau\big) = \big(0 \; 0 \; \partial^M\tau\big) = \big(\varepsilon\sigma \;\, \varepsilon\iota \;\, \tau\big) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varsigma_{-1}^{\Sigma B} \\ 0 & 0 & 0 \end{pmatrix} = \big(\varepsilon\sigma \;\, \varepsilon\iota \;\, \tau\big)\, \partial^{H \oplus \mathrm{Cone}\, 1^B}\,. \qquad \square$$

4.2.18 Corollary. *Let R be semi-simple. For every R-complex M there are quasi-isomorphisms $\mathrm{H}(M) \to M$ and $M \to \mathrm{H}(M)$.*

Proof. This follows from 4.2.17, as the complex $\mathrm{Cone}\, 1^{\mathrm{B}(M)}$ is acyclic by 4.2.16. $\square$

For use in Chap. 6 we record another case of complexes that can be compared to their homology via quasi-isomorphisms.

4.2.19 Lemma. *Assume that R is a principal left ideal domain. For every complex L of free R-modules there is a quasi-isomorphism $L \to \mathrm{H}(L)$.*

Proof. For every $v \in \mathbb{Z}$ let K^v be the complex $0 \to \mathrm{B}_v(L) \to \mathrm{Z}_v(L) \to 0$ concentrated in degrees $v+1$ and v. There is a quasi-isomorphism $K^v \to \Sigma^v \mathrm{H}_v(L)$ and hence by 4.2.11 a quasi-isomorphism $K \to \coprod_{v\in\mathbb{Z}} \Sigma^v \mathrm{H}_v(L) = \mathrm{H}(L)$ where $K = \coprod_{v\in\mathbb{Z}} K^v$. For every $v \in \mathbb{Z}$ the exact sequence $0 \to \mathrm{Z}_v(L) \to L_v \to \mathrm{B}_{v-1}(L) \to 0$ is split in $\mathcal{M}(R)$. Indeed, $\mathrm{B}_{v-1}(L)$ is a submodule of the free module L_{v-1}, so by 1.3.11 it is itself free and, in particular, projective. Thus, by 1.3.17 there are isomorphisms

$$L_v \longrightarrow \mathrm{Z}_v(L) \oplus \mathrm{B}_{v-1}(L)\,,$$

which yield an isomorphism of complexes $L \to K$. By composing this with the already established quasi-isomorphism $K \to \mathrm{H}(L)$ the assertion follows. $\square$

Remark. The proof of 4.2.19 only uses that submodules of free modules are free over a principal left ideal domain. In view of E 1.3.17 one can repurpose the argument to prove that there is a quasi-isomorphism $P \longrightarrow \mathrm{H}(P)$ for every complex P of projective modules over a left hereditary ring; see E 5.2.3.

EXERCISES

E 4.2.1 Show that a homomorphism of R-modules is an isomorphism if and only if it is a quasi-isomorphism when considered as a morphism of complexes.

E 4.2.2 Let α, β, and γ be morphisms of R-complexes. Show that if $\alpha\beta$ and $\beta\gamma$ are quasi-isomorphisms, then α, β, and γ are quasi-isomorphisms.

E 4.2.3 Show that a sequence of R-modules $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ with $\alpha\alpha' = 0$ is exact if one of the morphisms of complexes defined by the diagrams

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d, "\alpha'"] & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r, "\alpha"] & M'' \arrow[r] & 0 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d] & M \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\ & 0 \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

is a quasi-isomorphism and only if they both are quasi-isomorphisms.

E 4.2.4 Show that the $\mathbb{Z}/4\mathbb{Z}$-complexes

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

have isomorphic homology but that there is no quasi-isomorphism in either direction.

E 4.2.5 Show that there are surjective quasi-isomorphisms from the Koszul complex $\mathrm{K}^R(x, y)$ to each of the complexes M and N in 4.2.3. Decide if there are quasi-isomorphisms in the opposite direction $M \to \mathrm{K}^R(x, y)$ and $N \to \mathrm{K}^R(x, y)$.

E 4.2.6 Let α be a quasi-isomorphism of R-complexes. Show that the following conditions are equivalent: (*i*) α is injective; (*ii*) α is injective on boundaries; (*iii*) α is injective on cokernels; (*iv*) α is injective on boundaries and on cokernels.

E 4.2.7 Show that the Koszul complexes $\mathrm{K}^{\mathbb{Z}}(2, 3)$ and $\mathrm{K}^{\mathbb{Z}}(4, 5)$ are acyclic. Decide if there is a non-zero quasi-isomorphism $\mathrm{K}^{\mathbb{Z}}(2, 3) \to \mathrm{K}^{\mathbb{Z}}(4, 5)$ or $\mathrm{K}^{\mathbb{Z}}(4, 5) \to \mathrm{K}^{\mathbb{Z}}(2, 3)$; cf. E 2.1.11.

E 4.2.8 Assume that R is commutative. For elements $x_1, \dots, x_m$ and $y_1, \dots, y_n$ in R, show that if the Koszul complexes $\mathrm{K}^R(x_1, \dots, x_m)$ and $\mathrm{K}^R(y_1, \dots, y_n)$ have isomorphic homology, then they are acyclic or one has $(x_1, \dots, x_m) = (y_1, \dots, y_n)$ and $m = n$.

E 4.2.9 Show that a graded R-module M is semi-simple if and only if each M_v is semi-simple.

E 4.2.10 Let M and N be R-complexes such that $\mathrm{H}(M)$ is a complex of projective R-modules. Show that there is an isomorphism $\mathrm{H}(\mathrm{Hom}_R(\mathrm{H}(M), N)) \cong \mathrm{Hom}_R(\mathrm{H}(M), \mathrm{H}(N))$ given by $[\alpha] \mapsto \mathrm{H}(\alpha)$. Conclude that there is a quasi-isomorphism $\mathrm{H}(M) \to M$.

E 4.2.11 Let M and N be R-complexes such that $\mathrm{H}(N)$ is a complex of injective R-modules. Show that there is an isomorphism $\mathrm{H}(\mathrm{Hom}_R(M, \mathrm{H}(N))) \cong \mathrm{Hom}_R(\mathrm{H}(M), \mathrm{H}(N))$ given by $[\alpha] \mapsto \mathrm{H}(\alpha)$. Conclude that there is a quasi-isomorphism $N \to \mathrm{H}(N)$.

4.3 Homotopy Equivalences

SYNOPSIS. Homotopy equivalence; functor that preserves homotopy; contractible complex; mapping cylinder; ~ sequence.

The class of morphisms treated in this section sits in between isomorphisms and quasi-isomorphisms. While isomorphisms are preserved by all functors and quasi-isomorphisms only by special functors, these morphisms are preserved by the functors we care most about, among them the Hom and tensor product functors.

4.3.1 Definition. A morphism of R-complexes $\alpha\colon M \to N$ is called a *homotopy equivalence* if there is a morphism $\beta\colon N \to M$ such that $1^M - \beta\alpha$ and $1^N - \alpha\beta$ are null-homotopic, that is, $\beta\alpha \sim 1^M$ and $\alpha\beta \sim 1^N$; such a morphism β is called a *homotopy inverse* of α. A homotopy equivalence is marked by a '$\simeq$' next to the arrow.

If there exists a homotopy equivalence $\alpha\colon M \to N$, then the complexes M and N are called *homotopy equivalent*.

4.3.2 Example. Let M be an R-module and v an integer. Consider the disk complex from 2.5.29; the morphism $\mathrm{D}^v(M) \to 0$ is a homotopy equivalence by 2.2.24.

A homotopy inverse is unique up to homotopy.

4.3.3 Lemma. *Let $\alpha\colon M \to N$ and $\beta, \beta'\colon N \to M$ be morphisms in $\mathcal{C}(R)$. If $1^M \sim \beta\alpha$ and $1^N \sim \alpha\beta'$ hold, then β and β' are homotopy inverses of α and they are homotopic.*

Proof. By 2.2.25 one has $\beta = \beta 1^N \sim \beta\alpha\beta' \sim 1^M\beta' = \beta'$. Another application of 2.2.25 yields $1^N \sim \alpha\beta' \sim \alpha\beta$, so β is a homotopy inverse of α; a similar computation shows that β' is a homotopy inverse of α. $\square$

4.3.4 Lemma. *Let $\alpha\colon M \to N$ be a morphism of R-complexes.*

(a) *If α is an isomorphism, then α^{-1} is a homotopy inverse of α.*
(b) *If α is a homotopy equivalence with homotopy inverse β, then α is a quasi-isomorphism with $\mathrm{H}(\alpha)^{-1} = \mathrm{H}(\beta)$.*

Proof. Part (a) is evident as the composites $\alpha^{-1}\alpha$ and $\alpha\alpha^{-1}$ even equal the identities on M and N. Part (b) follows from 2.2.26. $\square$

Notice that the quasi-isomorphism in 4.2.2 is not a homotopy equivalence.

Homotopy equivalences constitute an important class of quasi-isomorphisms; one reason is that they are often easier to detect than general quasi-isomorphisms. Indeed, to confirm that a morphism is a quasi-isomorphism, one has to compute homology: either to check directly that the induced morphism in homology is an isomorphism or to check that the mapping cone is acyclic. To detect a homotopy equivalence, one only has to find a homotopy inverse.

A quasi-isomorphism between complexes with different levels of complication can simplify the task of computing homology. To compute the singular homology of, say, some convex subset X of $\mathbb{R}^n$ directly from the definition could be complicated. The next example, however, shows that the singular chain complexes $\mathrm{S}(pt)$ and $\mathrm{S}(X)$ are homotopy equivalent, in particular they have isomorphic homology, and the singular homology of the one-point space pt was computed with ease in 2.2.10.

4.3.5 Example. Let X be a contractible topological space. That is, there is a point x_0 in X such that the identity map $1^X\colon X \to X$ and the constant map $c\colon X \to X$ with value x_0 are homotopic, meaning that there is a continuous map $H\colon X \times [0,1] \to X$ with $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. The continuous map $\pi\colon X \to \{x_0\}$ induces a morphism of singular chain complexes, see 2.1.26 and 2.2.10,

$$\begin{tikzcd}
\mathrm{S}(X) \arrow[d, "\pi_*"] \arrow[r, phantom, "="] & \cdots \arrow[r] & \mathbb{Z}\langle \mathrm{C}(\Delta^2, X)\rangle \arrow[r, "\partial_2^X"] \arrow[d, "\pi\circ-"] & \mathbb{Z}\langle \mathrm{C}(\Delta^1, X)\rangle \arrow[r, "\partial_1^X"] \arrow[d, "\pi\circ-"] & \mathbb{Z}\langle \mathrm{C}(\Delta^0, X)\rangle \arrow[r] \arrow[d, "\pi\circ-"] & 0 \\
\mathrm{S}(\{x_0\}) \arrow[r, phantom, "="] & \cdots \arrow[r] & \mathbb{Z}\langle \sigma_2\rangle \arrow[r, "\partial_2^{\{x_0\}}"] & \mathbb{Z}\langle \sigma_1\rangle \arrow[r, "\partial_1^{\{x_0\}}"] & \mathbb{Z}\langle \sigma_0\rangle \arrow[r] & 0\,.
\end{tikzcd}$$

The morphism π_* is a homotopy equivalence with homotopy inverse ι_* induced by the embedding $\iota\colon \{x_0\} \rightarrowtail X$. In particular, the spaces X and $\{x_0\}$ have isomorphic singular homology by 4.3.4. Moreover, it follows from 2.2.10 and 4.3.2 that the complex $\mathrm{S}(\{x_0\})$, and hence also $\mathrm{S}(X)$, is homotopy equivalent to the complex with $\mathbb{Z}$ in degree 0 and zero elsewhere.

To describe the functors that preserve homotopy equivalences, we use a technical construction known as the mapping cylinder. Before starting down that road we notice that homotopy equivalences are preserved by (co)products.

4.3.6 Proposition. *Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ be a family or morphisms in $\mathcal{C}(R)$. If α^u is a homotopy equivalence for every $u \in U$, then the coproduct $\coprod_{u\in U}\alpha^u$ and the product $\prod_{u\in U}\alpha^u$ are homotopy equivalences.*

Proof. Let β^u be a homotopy inverse to α^u. It follows from 3.1.7 that the morphism $\coprod_{u\in U}(1^{M^u} - \beta^u\alpha^u)$ is null-homotopic, and by 3.1.5 one has

$$\coprod_{u\in U}(1^{M^u} - \beta^u\alpha^u) \;=\; \coprod_{u\in U} 1^{M^u} - \coprod_{u\in U}\beta^u\alpha^u \;=\; 1^{\coprod_{u\in U} M^u} - \Big(\coprod_{u\in U}\beta^u\Big)\Big(\coprod_{u\in U}\alpha^u\Big)\,.$$

By symmetry, the morphism $1^{\coprod_{u\in U} N^u} - (\coprod_{u\in U}\alpha^u)(\coprod_{u\in U}\beta^u)$ is null-homotopic, so $\coprod_{u\in U}\alpha^u$ is a homotopy equivalence with homotopy inverse $\coprod_{u\in U}\beta^u$.

A parallel argument based on 3.1.19 and 3.1.17 shows that the morphism $\prod_{u\in U}\alpha^u$ is a homotopy equivalence. $\square$

Mapping Cylinder

4.3.7 Definition. Let $\alpha\colon M \to N$ be a morphism of R-complexes. The *mapping cylinder* of α is the complex with underlying graded module

$$(\mathrm{Cyl}\,\alpha)^\natural \;=\; \begin{matrix} N^\natural \\ \oplus \\ \Sigma M^\natural \\ \oplus \\ M^\natural \end{matrix} \qquad \text{and differential} \qquad \partial^{\mathrm{Cyl}\,\alpha} \;=\; \begin{pmatrix} \partial^N & \alpha\varsigma^{\Sigma M}_{-1} & 0 \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & -\varsigma^{\Sigma M}_{-1} & \partial^M \end{pmatrix}.$$

4.3.8. A straightforward computation shows that $\partial^{\mathrm{Cyl}\,\alpha}$ is square zero,

$$\partial^{\mathrm{Cyl}\,\alpha}\partial^{\mathrm{Cyl}\,\alpha} \;=\; \begin{pmatrix} \partial^N\partial^N & \partial^N\alpha\varsigma^{\Sigma M}_{-1} + \alpha\varsigma^{\Sigma M}_{-1}\partial^{\Sigma M} & 0 \\ 0 & \partial^{\Sigma M}\partial^{\Sigma M} & 0 \\ 0 & -\varsigma^{\Sigma M}_{-1}\partial^{\Sigma M} - \partial^M\varsigma^{\Sigma M}_{-1} & \partial^M\partial^M \end{pmatrix} \;=\; 0\,;$$

it uses that α is a morphism and that $\varsigma^{\Sigma M}_{-1}$ is a degree -1 chain map.

Given a morphism $\alpha\colon M \to N$ of R-complexes, the embedding of M into $\operatorname{Cyl}\alpha$ is evidently a morphism, and an elementary computation shows that the surjection onto $\operatorname{Cone}\alpha$ is a morphism as well.

4.3.9 Definition. Let $\alpha\colon M \to N$ be a morphism of R-complexes. The degreewise split exact sequence of R-complexes,

$$0 \longrightarrow M \xrightarrow{\left(\begin{smallmatrix} 0 \\ 0 \\ 1^M \end{smallmatrix}\right)} \operatorname{Cyl}\alpha \xrightarrow{\left(\begin{smallmatrix} 1^N & 0 & 0 \\ 0 & 1^{\Sigma M} & 0 \end{smallmatrix}\right)} \operatorname{Cone}\alpha \longrightarrow 0\,,$$

is called the *mapping cylinder sequence* of α.

4.3.10 Lemma. *For every short exact sequence $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ of R-complexes, there is a commutative diagram with exact rows,*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota = \left(\begin{smallmatrix} 0 \\ 0 \\ 1^M \end{smallmatrix}\right)"] \arrow[d, equal] & \operatorname{Cyl}\alpha \arrow[r, "\pi = \left(\begin{smallmatrix} 1^N & 0 & 0 \\ 0 & 1^{\Sigma M} & 0 \end{smallmatrix}\right)"] \arrow[d, "\underline{\alpha} = (1^N\ 0\ \alpha)"] & \operatorname{Cone}\alpha \arrow[r] \arrow[d, "\underline{\beta} = (\beta\ 0)"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] & N \arrow[r, "\beta"] & X \arrow[r] & 0\,.
\end{tikzcd}$$

The morphism $\underline{\alpha}$ is a homotopy equivalence with homotopy inverse $\varepsilon = (1^N\ 0\ 0)^{\mathsf{T}}$ and $\underline{\beta}$ is a quasi-isomorphism.

Proof. The upper row is the mapping cylinder sequence of α from 4.3.9 and the lower row is exact by assumption. The next computations show that $\underline{\alpha}$ and $\underline{\beta}$ are morphisms. One has

$$\partial^N \left(1^N\ 0\ \alpha\right) \;=\; \left(\partial^N\ 0\ \partial^N\alpha\right) \;=\; \left(\partial^N\ 0\ \alpha\partial^M\right) \;=\; \left(1^N\ 0\ \alpha\right)\begin{pmatrix} \partial^N & \alpha\varsigma^{\Sigma M}_{-1} & 0 \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & -\varsigma^{\Sigma M}_{-1} & \partial^M \end{pmatrix}$$

and

$$\partial^X \left(\beta\ 0\right) \;=\; \left(\partial^X\beta\ 0\right) \;=\; \left(\beta\partial^N\ 0\right) \;=\; \left(\beta\ 0\right)\begin{pmatrix} \partial^N & \alpha\varsigma^{\Sigma M}_{-1} \\ 0 & \partial^{\Sigma M} \end{pmatrix}\,.$$

As the composite $\beta\alpha$ is zero, it is evident that the diagram is commutative. It is evident that ε is a morphism; to see that it is a homotopy inverse of $\underline{\alpha}$, consider the degree 1 homomorphism,

$$\varrho : \operatorname{Cyl}\alpha \longrightarrow \operatorname{Cyl}\alpha \qquad \text{given by} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varsigma^M_1 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

A simple calculation yields

$$\partial^{\operatorname{Cyl}\alpha}\varrho + \varrho\partial^{\operatorname{Cyl}\alpha} \;=\; \begin{pmatrix} 0 & 0 & \alpha \\ 0 & -1^{\Sigma M} & 0 \\ 0 & 0 & -1^M \end{pmatrix} \;=\; \varepsilon\underline{\alpha} - 1^{\operatorname{Cyl}\alpha}\,;$$

together with the identity $\underline{\alpha}\varepsilon = 1^N$ it shows that ε is a homotopy inverse of $\underline{\alpha}$.

Note that $\underline{\beta}$ is surjective with $(\operatorname{Ker}\underline{\beta})^\natural = (\operatorname{Im}\alpha)^\natural \oplus \Sigma M^\natural$. Let ϑ be the inverse to α considered as an isomorphism $M \longrightarrow \operatorname{Im}\alpha$. Consider the degree 1 homomorphism,

$$\sigma : \operatorname{Ker}\underline{\beta} \longrightarrow \operatorname{Ker}\underline{\beta} \qquad \text{given by} \qquad \begin{pmatrix} 0 & 0 \\ \varsigma_1^M\vartheta & 0 \end{pmatrix} .$$

There are equalities,

$$\partial^{\operatorname{Ker}\underline{\beta}}\sigma + \sigma\partial^{\operatorname{Ker}\underline{\beta}} = \begin{pmatrix} \alpha\vartheta & 0 \\ \partial^{\Sigma M}\varsigma_1^M\vartheta + \varsigma_1^M\vartheta\partial^N & \varsigma_1^M\vartheta\alpha\varsigma_{-1}^{\Sigma M} \end{pmatrix} = 1^{\operatorname{Ker}\underline{\beta}} ;$$

the last one follows as $\varsigma_1^M\vartheta$ is a chain map of degree 1 and one has $\alpha\vartheta = 1^{\operatorname{Im}\alpha}$ and $\vartheta\alpha = 1^{\Sigma M}$. Thus, $1^{\operatorname{Ker}\underline{\beta}}$ is null-homotopic; in particular $\operatorname{Ker}\underline{\beta}$ is acyclic by 2.2.26, whence it follows from 4.2.6 that $\underline{\beta}$ is a quasi-isomorphism. $\square$

4.3.11 Proposition. *For every commutative diagram of R-complexes,*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r] & 0 ,
\end{tikzcd}$$

with exact rows, there is a commutative diagram with exact rows,

$$\begin{tikzcd}
& 0 \arrow[rr] & & M \arrow[rr] \arrow[dl, "\varphi"'] \arrow[dd, equal] & & \operatorname{Cyl}\alpha \arrow[rr] \arrow[dl, "\begin{pmatrix} \psi & 0 & 0 \\ 0 & \Sigma\varphi & 0 \\ 0 & 0 & \varphi \end{pmatrix} = \kappa"'] \arrow[dd, "\underline{\alpha}", "\approxeq"'] & & \operatorname{Cone}\alpha \arrow[rr] \arrow[dl, "\begin{pmatrix} \psi & 0 \\ 0 & \Sigma\varphi \end{pmatrix} = \lambda"'] \arrow[dd, "\underline{\beta}", "\simeq"'] & & 0 \\
0 \arrow[rr] & & M' \arrow[rr] \arrow[dd, equal] & & \operatorname{Cyl}\alpha' \arrow[rr] \arrow[dd, "\underline{\alpha'}", "\approxeq"'] & & \operatorname{Cone}\alpha' \arrow[rr] \arrow[dd, "\underline{\beta'}", "\simeq"'] & & 0 & \\
& 0 \arrow[rr] & & M \arrow[rr, "\alpha"] \arrow[dl, "\varphi"'] & & N \arrow[rr, "\beta"] \arrow[dl, "\psi"'] & & X \arrow[rr] \arrow[dl, "\chi"'] & & 0 , \\
0 \arrow[rr] & & M' \arrow[rr, "\alpha'"] & & N' \arrow[rr, "\beta'"] & & X' \arrow[rr] & & 0 &
\end{tikzcd}$$

where the upper rows and vertical morphisms are as in 4.3.10.

PROOF. It is straightforward to verify that the maps κ and λ are morphisms of R-complexes, and the top is evidently a commutative diagram. The front and back are commutative diagrams by 4.3.10, and the bottom is a commutative diagram by assumption. It follows immediately from the definitions of $\underline{\alpha}$, $\underline{\alpha'}$, $\underline{\beta}$, and $\underline{\beta'}$ that the walls are commutative as well. $\square$

REMARK. If the exact sequence $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ is degreewise split, then the morphism $\underline{\beta}$ in 4.3.11 i a homotopy equivalence; see E 4.3.23.

FUNCTORS THAT PRESERVE HOMOTOPY

4.3.12 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ is said to *preserve homotopy* if for every pair of morphisms $\alpha \sim \beta$ in $\mathcal{C}(R)$ one has $\mathrm{F}(\alpha) \sim \mathrm{F}(\beta)$ in $\mathcal{C}(S)$.

Preserving the homotopy relation is equivalent to preserving homotopy equivalences, and for an additive functor it is the same as preserving null-homotopy.

4.3.13 Proposition. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a functor. The following conditions are equivalent.*

(i) F *preserves homotopy.*

(ii) *For every homotopy equivalence* α *in* $\mathcal{C}(R)$ *the morphism* $\mathrm{F}(\alpha)$ *in* $\mathcal{C}(S)$ *is a homotopy equivalence.*

Moreover, if F *is additive, then conditions* (i) *and* (ii) *are equivalent to*

(iii) *For every null-homotopic morphism* α *in* $\mathcal{C}(R)$ *the morphism* $\mathrm{F}(\alpha)$ *in* $\mathcal{C}(S)$ *is null-homotopic.*

PROOF. (i) $\Rightarrow$ (ii): Let $\alpha\colon M \to N$ be a homotopy equivalence with homotopy inverse β. The assumptions yield $\mathrm{F}(\beta)\,\mathrm{F}(\alpha) \sim 1^{\mathrm{F}(M)}$ and $\mathrm{F}(\alpha)\,\mathrm{F}(\beta) \sim 1^{\mathrm{F}(N)}$. Thus, $\mathrm{F}(\alpha)$ is a homotopy equivalence with homotopy inverse $\mathrm{F}(\beta)$.

(ii) $\Rightarrow$ (i): First, consider the following morphisms in $\mathcal{C}(R)$,

$$\begin{tikzcd} M \arrow[r, shift left, "\varepsilon = \left(\begin{smallmatrix} 1^M \\ 0 \\ 0 \end{smallmatrix}\right)"] \arrow[r, shift right, "\iota = \left(\begin{smallmatrix} 0 \\ 0 \\ 1^M \end{smallmatrix}\right)"'] & \operatorname{Cyl} 1^M \arrow[r, "\pi = (1^M\ 0\ 1^M)"] & M\,. \end{tikzcd}$$

Note that one has $\pi\varepsilon = 1^M = \pi\iota$ and, consequently, $\mathrm{F}(\pi)\,\mathrm{F}(\varepsilon) = 1^{\mathrm{F}(M)} = \mathrm{F}(\pi)\,\mathrm{F}(\iota)$ in $\mathcal{C}(S)$. The morphism π is by 4.3.10 a homotopy equivalence, and hence so is $\mathrm{F}(\pi)$ by the assumption. Now, 4.3.3 yields $\mathrm{F}(\varepsilon) \sim \mathrm{F}(\iota)$.

Let $\alpha, \beta\colon M \to N$ be homotopic morphisms of R-complexes and ϱ be a homotopy from α to β; that is, one has $\alpha - \beta = \partial^N\varrho + \varrho\partial^M$. The degree 0 homomorphism $\gamma = (\alpha\ \varrho\varsigma^{\Sigma M}_{-1}\ \beta)\colon \operatorname{Cyl} 1^M \to N$ is a morphism by the following computation,

$$\begin{aligned} \partial^N\big(\alpha\ \varrho\varsigma^{\Sigma M}_{-1}\ \beta\big) &= \big(\partial^N\alpha\ (\alpha - \beta - \varrho\partial^M)\varsigma^{\Sigma M}_{-1}\ \partial^N\beta\big) \\ &= \big(\alpha\partial^M\ \ \alpha\varsigma^{\Sigma M}_{-1} + \varrho\varsigma^{\Sigma M}_{-1}\partial^{\Sigma M} - \beta\varsigma^{\Sigma M}_{-1}\ \ \beta\partial^M\big) \\ &= \big(\alpha\ \varrho\varsigma^{\Sigma M}_{-1}\ \beta\big)\begin{pmatrix} \partial^M & \varsigma^{\Sigma M}_{-1} & 0 \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & -\varsigma^{\Sigma M}_{-1} & \partial^M \end{pmatrix}, \end{aligned}$$

which uses 2.2.5. Finally, the equalities $\alpha = \gamma\varepsilon$ and $\gamma\iota = \beta$ and 2.2.25 yield $\mathrm{F}(\alpha) = \mathrm{F}(\gamma)\,\mathrm{F}(\varepsilon) \sim \mathrm{F}(\gamma)\,\mathrm{F}(\iota) = \mathrm{F}(\beta)$.

Assuming that F is additive, it follows immediately from the definition 2.2.23 of '$\sim$' that conditions (i) and (iii) are equivalent. ◻

4.3.14 Definition. A morphism $M' \to M$ in $\mathcal{C}(R)^{\mathrm{op}}$ is called *null-homotopic* (a *quasi-isomorphism*, a *homotopy equivalence*) if the corresponding morphism $M \to M'$ in $\mathcal{C}(R)$ is null-homotopic (a quasi-isomorphism, a homotopy equivalence) as defined in 2.2.23 (in 4.2.1, in 4.3.1). Accordingly, morphisms $\alpha, \beta \colon M' \to M$ in $\mathcal{C}(R)^{\mathrm{op}}$ are called *homotopic*, and one writes $\alpha \sim \beta$, if the corresponding morphisms $M \to M'$ in $\mathcal{C}(R)$ are homotopic per 2.2.23.

A functor $\mathrm{G} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is said to *preserve homotopy* if for every pair of morphisms $\alpha \sim \beta$ in $\mathcal{C}(R)^{\mathrm{op}}$ one has $\mathrm{G}(\alpha) \sim \mathrm{G}(\beta)$ in $\mathcal{C}(S)$.

4.3.15 Proposition. *Let* $\mathrm{G} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be a functor. The following conditions are equivalent.*

(i) G *preserves homotopy.*

(ii) *For every homotopy equivalence* α *in* $\mathcal{C}(R)^{\mathrm{op}}$ *the morphism* $\mathrm{G}(\alpha)$ *in* $\mathcal{C}(S)$ *is a homotopy equivalence.*

Moreover, if G *is additive, then conditions* (i) *and* (ii) *are equivalent to*

(iii) *For every null-homotopic morphism* α *in* $\mathcal{C}(R)^{\mathrm{op}}$ *the morphism* $\mathrm{G}(\alpha)$ *in* $\mathcal{C}(S)$ *is null-homotopic.*

Proof. The assertions follow from an argument parallel to the proof of 4.3.13. □

4.3.16 Example. The shift functor, Σ, preserves homotopy.

4.3.17 Example. The homology functor, H, preserves homotopy by 2.2.26.

4.3.18 Example. A functor on R-complexes that is extended from an additive functor on R-modules, as described in 2.1.48, preserves homotopy. One example of such a functor is localization at a multiplicative subset of a commutative ring, see 2.1.50.

See 4.3.28 for examples of functors do not preserve homotopy.

4.3.19 Proposition. *Let M and N be R-complexes. The functors* $\mathrm{Hom}_R(M, –)$ *and* $\mathrm{Hom}_R(–, N)$ *preserve homotopy.*

Proof. The assertions follow immediately from 2.3.8. □

4.3.20 Proposition. *Let M be an R^{o}-complex and N an R-complex. The functors* $M \otimes_R –$ *and* $– \otimes_R N$ *preserve homotopy.*

Proof. The assertions follow immediately from 2.4.7. □

4.3.21 Proposition. *For all $n \in \mathbb{Z}$, the functors* $(–)_{\subseteq n}$ *and* $(–)_{\supseteq n}$ *preserve homotopy.*

Proof. Let $\alpha \colon M \to N$ be null-homotopic and $\sigma \colon M \to N$ a degree 1 homomorphism with $\alpha = \partial^N \sigma + \sigma \partial^M$. The degree 1 homomorphism $\widetilde{\sigma} \colon M_{\supseteq n} \to N_{\supseteq n}$ defined by $\widetilde{\sigma}_v = \sigma_v$ for $v > n$ and $\widetilde{\sigma}_n = \sigma_n|_{\mathrm{Z}_n(M)}$, and of course $\widetilde{\sigma}_v = 0$ for $v < n$, satisfies $\alpha_{\supseteq n} = \partial^{N_{\supseteq n}} \widetilde{\sigma} + \widetilde{\sigma} \partial^{M_{\supseteq n}}$; thus $\alpha_{\supseteq n}$ is null-homotopic.

A similar argument shows that $\alpha_{\subseteq n}$ is null-homotopic. □

Contractible Complexes

An acyclic complex A is characterized by the unique morphism $A \longrightarrow 0$ being a quasi-isomorphism. Next we consider complexes for which it is a homotopy equivalence.

4.3.22 Definition. An R-complex M is called *contractible* if the identity morphism 1^M is null-homotopic; a homotopy between 1^M and 0 is called a *contraction* of M.

Remark. Other words for contractible are ‘split exact’ and ‘homotopically trivial’.

4.3.23 Example. Let M be an R-module and v and integer. The disk complex $\mathrm{D}^v(M)$ is contractible; see 4.3.2.

The disk complexes are atomic contractible complexes in the sense that every such complex is a coproduct of countably many disk complexes. By 4.3.6 a coproduct of disk complexes is contractible, and the converse comes in 4.3.32; cf. 4.1.4.

4.3.24. The complex L constructed in 2.5.29 is a coproduct $\coprod_{v \in \mathbb{Z}} \mathrm{D}^v(F^v)$. Thus, for every R-complex M there is by 2.5.30 a surjective morphism $L \longrightarrow M$ where L is a contractible complex of free R-modules.

4.3.25 Example. Assume that R is commutative. Let x_1 and x_2 be elements in R with $(x_1, x_2) = R$ and choose $l_1, l_2 \in R$ with $l_1 x_1 + l_2 x_2 = 1$. Set $K = \mathrm{K}^R(x_1, x_2)$; cf. 2.1.25. The degree 1 homomorphism $\sigma\colon K \longrightarrow K$ whose non-zero components σ_0 and σ_1 are given by

$$1 \longmapsto l_1 e_1 + l_2 e_2 \qquad \text{and} \qquad \begin{aligned} e_1 &\longmapsto -l_2 e_1 \wedge e_2 \\ e_2 &\longmapsto l_1 e_1 \wedge e_2 \end{aligned}$$

satisfies $\partial^K \sigma + \sigma \partial^K = 1^K$, whence K is contractible.

4.3.26 Example. If M is a contractible R-complex, then $1^{\mathrm{H}(M)} = \mathrm{H}(1^M) = 0$ holds, see 2.2.26, so M is acyclic.

4.3.27. Every additive functor $\mathrm{F}\colon \mathcal{C}(R) \longrightarrow \mathcal{C}(S)$, or $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \longrightarrow \mathcal{C}(S)$, that preserves homotopy preserves contractability of complexes; see 4.3.13 and 4.3.15.

4.3.28 Example. The functors $\mathrm{B}, \mathrm{C}, \mathrm{Z}\colon \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$, see 2.2.13, are $\Bbbk$-linear but do not preserve homotopy. Indeed, let $M \neq 0$ be an R-module and consider the contractible complex $D = \mathrm{D}^1(M)$ from 4.3.23. The complexes $\mathrm{B}(D) = \mathrm{Z}(D) = M$ and $\mathrm{C}(D) = \Sigma M$ are not acyclic and hence not contractible by 4.3.26. The assertion now follows from 4.3.27.

The Hom functor not only preserves contractability, it detects it in a strong sense.

4.3.29 Proposition. *For an R-complex M, the following conditions are equivalent.*

(*i*) *M is contractible.*
(*ii*) $\mathrm{Hom}_R(K, M)$ *is contractible for every R-complex K.*
(*iii*) $\mathrm{Hom}_R(K, M)$ *is acyclic for every R-complex K.*

(*iv*) $\operatorname{Hom}_R(M,N)$ *is contractible for every R-complex N.*
(*v*) $\operatorname{Hom}_R(M,N)$ *is acyclic for every R-complex N.*
(*vi*) $\operatorname{Hom}_R(M,M)$ *is acyclic.*
(*vii*) $\operatorname{H}_0(\operatorname{Hom}_R(M,M)) = 0$.

PROOF. Condition (*i*) implies (*ii*), and (*iv*) by 4.3.19. The implications (*ii*) $\Rightarrow$ (*iii*) and (*iv*) $\Rightarrow$ (*v*) are evident, see 4.3.26, and (*iii*) $\Rightarrow$ (*vi*), (*v*) $\Rightarrow$ (*vi*), and (*vi*) $\Rightarrow$ (*vii*) are trivial. It follows from (*vii*) and 2.3.10 that the morphism 1^M is null-homotopic; this proves (*vii*) $\Rightarrow$ (*i*). □

REMARK. Given a contractible R-complex M it follows from 4.3.20 that $N \otimes_R M$ is contractible for every R^{o}-complex N. The converse fails: Let $0 \to M_2 \to M_1 \to M_0 \to 0$ be a sequence in $\mathcal{M}(R)$ that is pure exact but not split. Considered as a complex M of R-modules it is thus acyclic but not contractible, see 2.2.27. For every R^{o}-complex N the $\Bbbk$-complex $N \otimes_R M$ is acyclic by 5.5.14 and A.10, so if $\Bbbk$ is semi-simple, e.g. a field, then $N \otimes_R M$ is contractible; see E 4.3.3.

MAPPING CONE OF A HOMOTOPY EQUIVALENCE

Homotopy equivalences are a robust type of quasi-isomorphisms and their mapping cones are likewise acyclic for a prominent reason.

4.3.30 Theorem. *A morphism α of R-complexes is a homotopy equivalence if and only if the complex* $\operatorname{Cone}\alpha$ *is contractible.*

PROOF. Let $\alpha\colon M \to N$ be a morphism and set $C = \operatorname{Cone}\alpha$.

If α is a homotopy equivalence, then by 4.3.19 so is $\operatorname{Hom}_R(C,\alpha)$. In particular, $\operatorname{Hom}_R(C,\alpha)$ is a quasi-isomorphism by 4.3.4, and it follows from 4.2.16 that the complex $\operatorname{Cone}\operatorname{Hom}_R(C,\alpha)$ is acyclic. Since $\operatorname{Hom}_R(C,C) \cong \operatorname{Cone}\operatorname{Hom}_R(C,\alpha)$ holds by 4.1.16, the complex $\operatorname{Hom}_R(C,C)$ is acyclic; thus C is contractible by 4.3.29.

For the converse, assume that $\operatorname{Cone}\alpha$ is contractible and let $\upsilon\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha$ be a degree 1 homomorphism with $1^{\operatorname{Cone}\alpha} = \partial^{\operatorname{Cone}\alpha}\upsilon + \upsilon\partial^{\operatorname{Cone}\alpha}$. It has the form

$$\upsilon = \begin{pmatrix} \nu & \sigma \\ \tau & \Sigma\mu \end{pmatrix}$$

for homomorphisms $\nu\colon N \to N$, $\sigma\colon \Sigma M \to N$, $\tau\colon N \to \Sigma M$, and $\mu\colon M \to M$ of degree 1. There are equalities,

$$\begin{aligned}
\begin{pmatrix} 1^N & 0 \\ 0 & 1^{\Sigma M} \end{pmatrix} &= \begin{pmatrix} \partial^N & \alpha\varsigma^{\Sigma M}_{-1} \\ 0 & \partial^{\Sigma M} \end{pmatrix}\begin{pmatrix} \nu & \sigma \\ \tau & \Sigma\mu \end{pmatrix} + \begin{pmatrix} \nu & \sigma \\ \tau & \Sigma\mu \end{pmatrix}\begin{pmatrix} \partial^N & \alpha\varsigma^{\Sigma M}_{-1} \\ 0 & \partial^{\Sigma M} \end{pmatrix} \\
&= \begin{pmatrix} \partial^N\nu + \alpha\varsigma^{\Sigma M}_{-1}\tau + \nu\partial^N & \partial^N\sigma + \alpha\varsigma^{\Sigma M}_{-1}\Sigma\mu + \nu\alpha\varsigma^{\Sigma M}_{-1} + \sigma\partial^{\Sigma M} \\ \partial^{\Sigma M}\tau + \tau\partial^N & \partial^{\Sigma M}\Sigma\mu + \tau\alpha\varsigma^{\Sigma M}_{-1} + (\Sigma\mu)\partial^{\Sigma M} \end{pmatrix} .
\end{aligned}$$

Comparison of entries yields

$$\begin{aligned}
0 &= \partial^{\Sigma M}\tau + \tau\partial^N , \\
1^N &= \partial^N\nu + \alpha\varsigma^{\Sigma M}_{-1}\tau + \nu\partial^N , \quad \text{and} \\
1^{\Sigma M} &= \partial^{\Sigma M}\Sigma\mu + \tau\alpha\varsigma^{\Sigma M}_{-1} + (\Sigma\mu)\partial^{\Sigma M} .
\end{aligned}$$

The first equality shows that τ is a chain map, whence $\varsigma^{\Sigma M}_{-1}\tau\colon N \to M$ is a morphism. The second equality yields $1^N \sim \alpha\varsigma^{\Sigma M}_{-1}\tau$. An application of Σ^{-1} to the third equality yields $1^M = -\partial^M\mu + \varsigma^{\Sigma M}_{-1}\tau\alpha - \mu\partial^M$; that is, $1^M \sim \varsigma^{\Sigma M}_{-1}\tau\alpha$, whence the morphism $\varsigma^{\Sigma M}_{-1}\tau$ is a homotopy inverse of α. □

4.3.31 Corollary. *If α is an isomorphism of R-complexes, then the complex* $\operatorname{Cone}\alpha$ *is contractible.*

While a morphism with contractible mapping cone need not be an isomorphism, every contractible complex is isomorphic to the mapping cone of an isomorphism.

4.3.32 Proposition. *For an R-complex M, the following conditions are equivalent.*

(*i*) *M is contractible.*

(*ii*) *There is a graded R-module N with $M \cong \operatorname{Cone} 1^N$.*

(*iii*) *There are graded R-modules M' and M'' with $M^\natural = M' \oplus M''$ and $\partial^M|_{M'} = 0$, such that $\partial^M|_{M''}$ yields an isomorphism $M'' \cong \Sigma M'$.*

Proof. Condition (*ii*) implies (*i*) by 4.3.31.

(*i*) ⇒ (*iii*): By assumption there is a homomorphism $\sigma\colon M \to M$ of degree 1 such that $\partial^M\sigma + \sigma\partial^M = 1^M$ holds. The endomorphism $\varepsilon = \sigma\partial^M$ of $M^\natural$ satisfies

$$\varepsilon^2 = (\sigma\partial^M)(1^M - \partial^M\sigma) = \varepsilon \quad \text{and} \quad 1^M - \varepsilon = \partial^M\sigma\,,$$

whence there is an equality $M^\natural = M' \oplus M''$ with $M' = \operatorname{Im}\partial^M\sigma$ and $M'' = \operatorname{Im}\varepsilon$. Evidently, one has $\partial^M|_{M'} = 0$ and, therefore,

$$M' \subseteq \mathrm{B}(M) = \partial^M(M'') = \partial^M\sigma\partial^M(M) \subseteq \partial^M\sigma(M) = M'\,.$$

It follows that $\partial^M|_{M''}$ is a surjective homomorphism $M'' \to M'$ of degree -1. To see that $\partial^M|_{M''}$ is injective, let $m'' = \varepsilon(m)$ be an element in M'' with $0 = \partial^M(m'') = \partial^M\sigma\partial^M(m)$. Now one has $0 = \sigma\partial^M\sigma\partial^M(m) = \varepsilon^2(m) = \varepsilon(m) = m''$. Thus, $\partial^M|_{M''}\colon M'' \to \Sigma M'$ is an isomorphism of graded modules.

(*iii*) ⇒ (*ii*): It is straightforward to verify that the map given by the assignment $(m', m'') \mapsto (m', \partial^M(m''))$ is an isomorphism of R-complexes $M \to \operatorname{Cone} 1^{M'}$. □

4.3.33 Corollary. *Let M be an R-complex; it is contractible if and only if it is acyclic and the exact sequence $0 \to \mathrm{Z}_v(M) \to M_v \to \mathrm{Z}_{v-1}(M) \to 0$ is split for every $v \in \mathbb{Z}$.*

Proof. If M is contractible then, per 4.3.26, it is acyclic, so one has $\mathrm{B}(M) = \mathrm{Z}(M)$, and the sequence

$$(\dagger) \qquad 0 \longrightarrow \mathrm{Z}_v(M) \longrightarrow M_v \longrightarrow \mathrm{Z}_{v-1}(M) \longrightarrow 0$$

is the degree v part of the exact sequence 2.2.12(a). Further, M is by 4.3.32 isomorphic to $\operatorname{Cone} 1^N$ for a graded R-module N. It follows that (†) is isomorphic to the sequence $0 \to N_v \to N_v \oplus N_{v-1} \to N_{v-1} \to 0$, see 4.1.1, whence (†) is split by 2.1.47.

Conversely, if M is acyclic and the exact sequence (†) is split for every $v \in \mathbb{Z}$, then there is a homomorphism $\sigma\colon \Sigma\,\mathrm{Z}(M) \to M$ with $\sigma\partial^M = 1^{\Sigma\,\mathrm{Z}(M)}$. Thus, there is an equality $M^\natural = \mathrm{Z}(M) \oplus \sigma(\Sigma\,\mathrm{Z}(M))$, and so M is contractible by 4.3.32. □

Exercises

E 4.3.1 Let α, β, and γ be morphisms of R-complexes. Show that if $\alpha\beta$ and $\beta\gamma$ are homotopy equivalences, then α, β, and γ are homotopy equivalences.

E 4.3.2 Let α be a morphism in $\mathcal{K}(R)$. Show that if α considered in $\mathcal{K}(\Bbbk)$ is a quasi-isomorphism, then α is a quasi-isomorphism. If α considered in $\mathcal{K}(\Bbbk)$ is an isomorphism, is α then an isomorphism?

E 4.3.3 Assume that R is semi-simple. Show that every acyclic R-complex is contractible and conclude that every quasi-isomorphism of R-complexes is a homotopy equivalence.

E 4.3.4 Let $f \colon X \to Y$ be a continuous map of topological spaces. The mapping cone, $\operatorname{Cone} f$, is defined as the quotient space of $(X \times [0,1]) \uplus Y$ with respect to the equivalence relation $(x,0) \sim (x',0)$ and $(x,1) \sim f(x)$ for all $x, x' \in X$. Denote by S the singular chain complex functor, cf. 2.1.26 and E 2.1.14. Show that the complexes $\mathrm{S}(\operatorname{Cone} f)$ and $\operatorname{Cone} \mathrm{S}(f)$ are homotopy equivalent.

E 4.3.5 Show that every morphism that is homotopic to a homotopy equivalence is a homotopy equivalence.

E 4.3.6 Let α be a morphism of R-complexes; establish an isomorphism $\operatorname{Cyl}(\Sigma\alpha) \cong \Sigma\operatorname{Cyl}\alpha$.

E 4.3.7 Show that hard truncations do not preserve homotopy.

E 4.3.8 Consider a commutative diagram of R-complexes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0\,, \end{array}$$

with exact rows. Show that if two of the morphisms φ', φ, and φ'' are homotopy equivalences, the third need not be a homotopy equivalence.

E 4.3.9 (Cf. 4.3.18) Show that every functor on R-complexes that is extended from an additive functor on R-modules preserves homotopy.

E 4.3.10 Show that an R-complex may be contractible as a $\Bbbk$-complex but not as an R-complex.

E 4.3.11 Show that the $\mathbb{Z}/6\mathbb{Z}$-complex $\cdots \longrightarrow \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \cdots$ is contractible.

E 4.3.12 Assume that R is commutative. For elements $x_1, \ldots, x_n$ in R with $(x_1, \ldots, x_n) = R$, show that the Koszul complex $\mathrm{K}^R(x_1, \ldots, x_n)$ is contractible.

E 4.3.13 Let M be an R-complex; show that it is contractible if and only if the canonical sequence $0 \to \mathrm{Z}(M) \to M \to \Sigma\mathrm{Z}(M) \to 0$ is degreewise split exact.

E 4.3.14 Let M be a contractible R-complex. Show that if M is a complex of projective/injective/flat modules, then so are the complexes $\mathrm{B}(M) = \mathrm{Z}(M)$ and $\mathrm{C}(M)$.

E 4.3.15 Let M be an R-complex such that the exact sequence $0 \to \mathrm{Z}(M) \to M \to \Sigma\mathrm{B}(M) \to 0$ is degreewise split. Show that M is contractible if and only if it is acyclic. Conclude that an acyclic complex M is contractible if $\mathrm{B}(M) = \mathrm{Z}(M)$ is a complex of projective modules or a complex of injective modules.

E 4.3.16 Let R be left hereditary. (a) Show that every acyclic complex of projective R-modules is contractible. (b) Show that every acyclic complex of injective R-modules is contractible.

E 4.3.17 Let M be an R-complex. Show that if M is bounded above and $\mathrm{H}_{-v}(\operatorname{Hom}_R(M, M_v)) = 0$ holds for all $v \in \mathbb{Z}$, then M is contractible. *Hint:* E A.1

E 4.3.18 Let M be an R-complex. Show that if M is bounded below and $\mathrm{H}_v(\operatorname{Hom}_R(M_v, M)) = 0$ holds for all $v \in \mathbb{Z}$, then M is contractible. *Hint:* E A.2

E 4.3.19 Show that a bounded below complex of projective R-modules is contractible if and only if it is acyclic.

E 4.3.20 Show that a bounded above complex of injective R-modules is contractible if and only if it is acyclic.

E 4.3.21 Let M be an R-complex. Show that there is an injective morphism $M \to C$ and a surjective morphism $C' \to M$ in $\mathcal{C}(R)$ with C and C' contractible.

E 4.3.22 Let $\alpha\colon M \to N$ be a morphism of R-complexes. (a) Show that α factors as follows: $M \xrightarrow{\iota} M' \xrightarrow{\beta} N$, where ι is an injective homotopy equivalence and β is surjective. (b) Show that α factors as follows: $M \xrightarrow{\gamma} N' \xrightarrow{\pi} N$, where γ is injective and π is a surjective homotopy equivalence.

E 4.3.23 Let $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ be a degreewise split exact sequence of R-complexes. Show that the morphism $\underline{\beta}$ in 4.3.11 is a homotopy equivalence with homotopy inverse $(\sigma \;\; -\varsigma_1^M \varrho \partial^N \sigma)^{\mathrm{T}}$ where $\varrho\colon N \to M$ and $\sigma\colon X \to N$ are the splitting homomorphisms.

4.4 Standard Isomorphisms for Complexes

Synopsis. Unitor; counitor; commutativity; associativity; swap; adjunction.

Categories and functors were introduced in a 1945 paper by Eilenberg and MacLane [76]. The title "General Theory of Natural Equivalences" is suggestive and MacLane went on to write: "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation" [175, I.4]. While category theory has evolved to become much more than a language of "abstract nonsense", natural transformations have not lost standing.

The standard isomorphisms from Sect. 1.2 are natural transformations of functors on modules, in this section they are extended to complexes.

Unitor and Counitor

4.4.1. Let M be an R-complex. There is an isomorphism in $\mathcal{C}(R)$,

$$\mu_R^M : R \otimes_R M \longrightarrow M \qquad \text{given by} \qquad \mu_R^M(r \otimes m) = rm\,,$$

where $m \in M$ is a homogeneous element. It is called the *unitor*, and it is natural in M. Moreover, if M is a complex of R–S^{o}-bimodules, then it is an isomorphism in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$. Finally, it follows from 2.2.5 and 4.1.18 that μ_R as a natural transformation of functors is a Σ-transformation.

4.4.2. Let M be an R-complex. There is an isomorphism in $\mathcal{C}(R)$,

$$\epsilon_R^M : M \longrightarrow \operatorname{Hom}_R(R, M) \qquad \text{given by} \qquad \epsilon_R^M(m)(r) = rm\,,$$

where $m \in M$ is a homogeneous element. It is called the *counitor*, and it is natural in M. Moreover, if M is a complex of R–S^{o}-bimodules, then it is an isomorphism in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$. Finally, it follows from 2.2.5 and 4.1.16 that ϵ_R as a natural transformation of functors is a Σ-transformation.

Commutativity

The next construction and the proposition that follows establish a commutativity isomorphism for tensor products of complexes. It is based on, and it extends, the isomorphism from 1.2.3.

4.4.3 Construction. Let M be an R^{o}-complex and N an R-complex. The commutativity isomorphism for modules 1.2.3 induces a natural isomorphism of graded $\Bbbk$-modules, $M \otimes_R N \longrightarrow N \otimes_{R^{\mathrm{o}}} M$, with the isomorphism in degree v given by

$$(M \otimes_R N)_v = \coprod_{i\in\mathbb{Z}} M_i \otimes_R N_{v-i} \xrightarrow{\coprod_{i\in\mathbb{Z}}(-1)^{(v-i)i}\upsilon^{M_i N_{v-i}}} \coprod_{i\in\mathbb{Z}} N_{v-i} \otimes_{R^{\mathrm{o}}} M_i = (N \otimes_{R^{\mathrm{o}}} M)_v .$$

This isomorphism is also denoted υ^{MN}. For homogeneous elements $m \in M$ and $n \in N$ it is given by

$$\upsilon^{MN}(m \otimes n) \;=\; (-1)^{|n||m|} n \otimes m \;. \tag{4.4.3.1}$$

Note that (4.4.3.1) agrees with the definition in 1.2.3 for modules M and N.

4.4.4 Proposition. *Let M be an R^{o}-complex and N an R-complex. The commutativity map defined in* 4.4.3*,*

$$\upsilon^{MN} : M \otimes_R N \longrightarrow N \otimes_{R^{\mathrm{o}}} M \;,$$

is an isomorphism in $\mathcal{C}(\Bbbk)$, and it is natural in M and N. Moreover, if M is in $\mathcal{C}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$, then υ^{MN} is an isomorphism in $\mathcal{C}(Q\text{–}S^{\mathrm{o}})$. Finally, as a natural transformation of functors, υ is a Σ-transformation in each variable.

Proof. By construction, υ^{MN} is an isomorphism of graded $\Bbbk$-modules and natural in M and N. If M is in $\mathcal{C}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{C}(R\text{–}S^{\mathrm{o}})$, then υ^{MN} is a natural isomorphism of graded $Q\text{–}S^{\mathrm{o}}$-bimodules. This follows from 1.2.3 and the construction. For homogeneous elements $m \in M$ and $n \in N$ one has

$$\begin{aligned}
&\upsilon^{MN}(\partial^{M\otimes_R N}(m \otimes n))\\
&\qquad = \upsilon^{MN}(\partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n))\\
&\qquad = (-1)^{|n|(|m|-1)} n \otimes \partial^M(m) + (-1)^{|m|+(|n|-1)|m|} \partial^N(n) \otimes m\\
&\qquad = (-1)^{|n||m|}(\partial^N(n) \otimes m + (-1)^{|n|} n \otimes \partial^M(m))\\
&\qquad = (-1)^{|n||m|}(\partial^{N\otimes_{R^{\mathrm{o}}}M}(n \otimes m))\\
&\qquad = \partial^{N\otimes_{R^{\mathrm{o}}}M}(\upsilon^{MN}(m \otimes n)) \;.
\end{aligned}$$

Thus, υ^{MN} is a morphism, and hence an isomorphism, of complexes. It follows from 2.2.5 combined with 4.1.18 and 4.1.19 that υ as a natural transformation of functors is a Σ-transformation in each variable. ◻

4.4.5 Example. Assume that R is commutative. For elements x_1 and x_2 in R it is immediate from 2.4.3 and commutativity 4.4.4 that the Koszul complexes $\mathrm{K}^R(x_1, x_2)$ and $\mathrm{K}^R(x_2, x_1)$ are isomorphic.

Associativity

We extend associativity 1.2.4 to an isomorphism of complexes and apply it to describe the Koszul complex on a sequence of elements as the tensor poduct of Koszul complexes on the individual elements.

4.4.6 Construction. Let M be an R^{o}-complex, X a complex of R–S^{o}-bimodules, and N an S-complex. The associativity isomorphism for modules 1.2.4 induces a natural isomorphism $(M \otimes_R X) \otimes_S N \to M \otimes_R (X \otimes_S N)$ of graded $\Bbbk$-modules. The component in degree v is induced by $\coprod_{i\in\mathbb{Z}} \coprod_{j\in\mathbb{Z}} \omega^{M_j X_{i-j} N_{v-i}}$. Indeed, it maps

$$\begin{aligned}((M \otimes_R X) \otimes_S N)_v &= \coprod_{i\in\mathbb{Z}} \Big(\coprod_{j\in\mathbb{Z}} M_j \otimes_R X_{i-j} \Big) \otimes_S N_{v-i} \\ &\cong \coprod_{i\in\mathbb{Z}} \coprod_{j\in\mathbb{Z}} (M_j \otimes_R X_{i-j}) \otimes_S N_{v-i}\,,\end{aligned}$$

where the isomorphism follows from 3.1.12, isomorphically to

$$\begin{aligned}(M \otimes_R (X \otimes_S N))_v &= \coprod_{j\in\mathbb{Z}} M_j \otimes_R \Big(\coprod_{i\in\mathbb{Z}} X_{i-j} \otimes_S N_{v-i} \Big) \\ &\cong \coprod_{i\in\mathbb{Z}} \coprod_{j\in\mathbb{Z}} M_j \otimes_R (X_{i-j} \otimes_S N_{v-i})\,,\end{aligned}$$

where the isomorphism follows from 3.1.13. The resulting isomorphism of graded modules $(M \otimes_R X) \otimes_S N \to M \otimes_R (X \otimes_S N)$ is also denoted ω^{MXN}. On homogeneous elements $m \in M$, $x \in X$, and $n \in N$ it is given by

$$\omega^{MXN}((m \otimes x) \otimes n) = m \otimes (x \otimes n)\,. \tag{4.4.6.1}$$

Note that (4.4.6.1) agrees with the definition in 1.2.4 for modules M, X, and N.

4.4.7 Proposition. *Let M be an R^{o}-complex, X a complex of R–S^{o}-bimodules, and N an S-complex. The associativity map defined in* 4.4.6*,*

$$\omega^{MXN} : (M \otimes_R X) \otimes_S N \longrightarrow M \otimes_R (X \otimes_S N)\,,$$

is an isomorphism in $\mathcal{C}(\Bbbk)$, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{C}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$, then ω^{MXN} is an isomorphism in $\mathcal{C}(Q\text{–}T^{\mathrm{o}})$. Finally, as a natural transformation of functors, ω is a Σ-transformation in each variable.

Proof. By construction, ω^{MXN} is an isomorphism of graded $\Bbbk$-modules and natural in M, X, and N. If M is in $\mathcal{C}(Q\text{–}R^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$, then ω^{MXN} is a natural isomorphism of graded Q–T^{o}-bimodules. This follows from 1.2.4 and the construction. A straightforward computation, similar to the one in the proof of 4.4.4, shows that ω^{MXN} is a morphism, and hence an isomorphism, of complexes. It follows from 2.2.5 combined with 4.1.18 and 4.1.19 that ω as a natural transformation of functors is a Σ-transformation in each variable. □

For elements $x_1, \ldots, x_n$ in R the notation $\mathrm{K}^R(x_1) \otimes_R \cdots \otimes_R \mathrm{K}^R(x_n)$ is unambiguous by associativity 4.4.7. The next proposition generalizes 2.4.3.

4.4.8 Proposition. *Assume that R is commutative. For every sequence $x_1,\dots,x_n$ in R, there is an isomorphism of R-complexes*

$$\mathrm{K}^R(x_1,\dots,x_n) \;\cong\; \mathrm{K}^R(x_1)\otimes_R\cdots\otimes_R\mathrm{K}^R(x_n)\;.$$

Proof. By induction on n. The case $n=1$ is trivial, so it suffices for $n>1$ to prove that there is an isomorphism

$$(\diamond)\qquad \psi\colon \mathrm{K}^R(x_1,\dots,x_{n-1})\otimes_R\mathrm{K}^R(x_n) \;\longrightarrow\; \mathrm{K}^R(x_1,\dots,x_n)\;.$$

For clarity, consider $\mathrm{K}^R(x_1,\dots,x_{n-1})$ and $\mathrm{K}^R(x_n)$ to be generated by the free modules $R\langle e_1,\dots,e_{n-1}\rangle$ and $R\langle e_n\rangle$ while $\mathrm{K}^R(x_1,\dots,x_n)$ is generated by $R\langle f_1,\dots,f_n\rangle$. Let $v\in\{0,\dots,n\}$ and recall from 2.1.25 that $\mathrm{K}^R_v(x_1,\dots,x_n)$ is a free R-module of rank $\binom{n}{v}$ with basis elements $f_{h_1}\wedge\cdots\wedge f_{h_v}$. The module in degree v in the left-hand complex in $(\diamond)$ is a direct sum

$$(\mathrm{K}^R_v(x_1,\dots,x_{n-1})\otimes_R R)\oplus(\mathrm{K}^R_{v-1}(x_1,\dots,x_{n-1})\otimes_R R\langle e_n\rangle)$$

of free R-modules of rank $\binom{n-1}{v}$ and $\binom{n-1}{v-1}$, so of total rank $\binom{n}{v}$. Let ψ_v be given by

$$\begin{aligned}(e_{h_1}\wedge\cdots\wedge e_{h_v})\otimes 1 &\;\longmapsto\; f_{h_1}\wedge\cdots\wedge f_{h_v} \qquad\text{and}\\ (e_{h_1}\wedge\cdots\wedge e_{h_{v-1}})\otimes e_n &\;\longmapsto\; f_{h_1}\wedge\cdots\wedge f_{h_{v-1}}\wedge f_n\end{aligned}$$

and extended by R-linearity. It is straightforward to verify that $\psi=(\psi_v)_{0\leqslant v\leqslant n}$ is a morphism of R-complexes, and each component ψ_v has an obvious inverse, so ψ is an isomorphism. $\square$

Swap

We go on to extend swap 1.2.5 to an isomorphism of complexes.

4.4.9 Construction. Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S^{o}-complex. By 3.1.24 one has

$$\begin{aligned}\operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\mathrm{o}}}(N,X))_v &= \prod_{i\in\mathbb{Z}}\operatorname{Hom}_R\big(M_i,\prod_{j\in\mathbb{Z}}\operatorname{Hom}_{S^{\mathrm{o}}}(N_j,X_{j+i+v})\big)\\ &\cong \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}\operatorname{Hom}_R(M_i,\operatorname{Hom}_{S^{\mathrm{o}}}(N_j,X_{i+j+v}))\;,\end{aligned}$$

and similarly,

$$\operatorname{Hom}_{S^{\mathrm{o}}}(N,\operatorname{Hom}_R(M,X))_v \;\cong\; \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}\operatorname{Hom}_{S^{\mathrm{o}}}(N_j,\operatorname{Hom}_R(M_i,X_{i+j+v}))\;.$$

It follows from swap for modules 1.2.5 that the map

$$\operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\mathrm{o}}}(N,X)) \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(N,\operatorname{Hom}_R(M,X))$$

with degree v component induced by $\prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}(-1)^{ij}\zeta^{M_iX_{i+j+v}N_j}$ is a natural isomorphism of graded $\Bbbk$-modules; it is denoted by ζ^{MXN}. On homogeneous elements $\psi\in\operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\mathrm{o}}}(N,X))$, $m\in M$, and $n\in N$ it is given by

$$\zeta^{MXN}(\psi)(n)(m) \;=\; (-1)^{|m||n|}\psi(m)(n)\,. \tag{4.4.9.1}$$

Note that (4.4.9.1) agrees with the definition in 1.2.5 for modules M, X, and N.

4.4.10 Proposition. *Let M be an R-complex, X a complex of R–S°-bimodules, and N an S°-complex. The swap map defined in* 4.4.9*,*

$$\zeta^{MXN} : \operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\circ}}(N,X)) \longrightarrow \operatorname{Hom}_{S^{\circ}}(N,\operatorname{Hom}_R(M,X))\,,$$

is an isomorphism in $\mathcal{C}(\Bbbk)$, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{C}(R\text{–}Q^{\circ})$ and N is in $\mathcal{C}(T\text{–}S^{\circ})$, then ζ^{MXN} is an isomorphism in $\mathcal{C}(Q\text{–}T^{\circ})$. Finally, as a natural transformation of functors, ζ is a Σ-transformation in each variable.

Proof. By construction, ζ^{MXN} is an isomorphism of graded $\Bbbk$-modules and natural in M, X, and N. If M is in $\mathcal{C}(R\text{–}Q^{\circ})$ and N is in $\mathcal{C}(T\text{–}S^{\circ})$, then ζ^{MXN} is an isomorphism of graded Q–T°-bimodules; this follows from the construction and 1.2.5. For homogeneous elements $\psi \in \operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\circ}}(N,X))$, $m \in M$, and $n \in N$ one has

$$\begin{aligned}
&\zeta^{MXN}(\partial^{\operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\circ}}(N,X))}(\psi))(n)(m)\\
&\quad= \zeta^{MXN}(\partial^{\operatorname{Hom}_{S^{\circ}}(N,X)}\psi - (-1)^{|\psi|}\psi\partial^{M})(n)(m)\\
&\quad= (-1)^{|m||n|}(\partial^{\operatorname{Hom}_{S^{\circ}}(N,X)}(\psi(m)) - (-1)^{|\psi|}\psi(\partial^{M}(m)))(n)\\
&\quad= (-1)^{|m||n|}(\partial^{X}\psi(m) - (-1)^{|\psi(m)|}\psi(m)\partial^{N} - (-1)^{|\psi|}\psi(\partial^{M}(m)))(n)\\
&\quad= (-1)^{|m||n|}(\partial^{X}(\psi(m)(n)) - (-1)^{|\psi|+|m|}\psi(m)(\partial^{N}(n)) - (-1)^{|\psi|}\psi(\partial^{M}(m))(n))
\end{aligned}$$

and

$$\begin{aligned}
&(\partial^{\operatorname{Hom}_{S^{\circ}}(N,\operatorname{Hom}_R(M,X))}(\zeta^{MXN}(\psi)))(n)(m)\\
&\qquad= (\partial^{\operatorname{Hom}_R(M,X)}\zeta^{MXN}(\psi) - (-1)^{|\zeta^{MXN}(\psi)|}\zeta^{MXN}(\psi)\partial^{N})(n)(m)\\
&\qquad= (\partial^{\operatorname{Hom}_R(M,X)}(\zeta^{MXN}(\psi)(n)) - (-1)^{|\psi|}\zeta^{MXN}(\psi)(\partial^{N}(n)))(m)\\
&\qquad= \partial^{X}(\zeta^{MXN}(\psi)(n)(m)) - (-1)^{|\zeta^{MXN}(\psi)(n)|}\zeta^{MXN}(\psi)(n)(\partial^{M}(m))\\
&\qquad\qquad - (-1)^{|\psi|+|m|(|n|-1)}\psi(m)(\partial^{N}(n))\\
&\qquad= (-1)^{|m||n|}\partial^{X}(\psi(m)(n)) - (-1)^{|\psi|+|n|+(|m|-1)|n|}\psi(\partial^{M}(m))(n)\\
&\qquad\qquad - (-1)^{|\psi|+|m||n|-|m|}\psi(m)(\partial^{N}(n))\,.
\end{aligned}$$

Thus, ζ^{MXN} is a morphism, hence an isomorphism, of complexes.

It follows from 2.2.5 combined with 4.1.16 and 4.1.17 that ζ as a natural transformation of functors is a Σ-transformation in each variable. For the convenience of the reader we include the computation that shows that ζ is a Σ-transformationin the first variable. Fix complexes N and X. To verify that ζ^{-XN} is a Σ-transformation, recall from 4.1.17 that the natural isomorphism

$$\psi_1^M : \Sigma^{-1}\operatorname{Hom}_R(M,\operatorname{Hom}_{S^{\circ}}(N,X)) \to \operatorname{Hom}_R(\Sigma M,\operatorname{Hom}_{S^{\circ}}(N,X))\,,$$

upon suppression of the degree shifting chain maps, is given by $\vartheta \mapsto (-1)^{|\vartheta|}\vartheta$ for a homogeneous element ϑ in $\operatorname{Hom}_R(M, \operatorname{Hom}_{S^\circ}(N, X))$. Similarly, upon suppression of the degree shifting chain maps, the composite

$$\begin{aligned}\psi_2^M \colon \Sigma^{-1}\operatorname{Hom}_{S^\circ}(N, \operatorname{Hom}_R(M, X)) &\longrightarrow \operatorname{Hom}_{S^\circ}(N, \Sigma^{-1}\operatorname{Hom}_R(M, X))\\ &\longrightarrow \operatorname{Hom}_{S^\circ}(N, \operatorname{Hom}_R(\Sigma M, X))\end{aligned}$$

maps a homogeneous element φ in $\operatorname{Hom}_{S^\circ}(N, \operatorname{Hom}_R(M, X))$ to the homomorphism given by $n \mapsto (-1)^{|\varphi|+|n|}\varphi(n)$ for $n \in N$. Thus, for homogeneous elements $m \in M$, $n \in N$, and ϑ in $\operatorname{Hom}_R(M, \operatorname{Hom}_{S^\circ}(N, X))$ one has

$$\begin{aligned}(\zeta^{(\Sigma M)XN}\psi_1^M)(\vartheta)(n)(m) &= \zeta^{(\Sigma M)XN}((-1)^{|\vartheta|}\vartheta)(n)(m)\\ &= (-1)^{|\vartheta|}(-1)^{(|m|+1)|n|}\vartheta(m)(n)\\ &= (-1)^{|\zeta^{MXN}(\vartheta)|+|n|}(-1)^{|m||n|}\vartheta(m)(n)\\ &= (\psi_2^M\Sigma^{-1}\zeta^{MXN})(\vartheta)(n)(m)\ .\end{aligned}$$

Thus, ζ^{-XN} is a Σ-transformation. ☐

Adjunction

Finally, we extend adjunction 1.2.6 to an isomorphism of complexes.

4.4.11 Construction. Let M be an R-complex, X a complex of R–S°-bimodules, and N an S-complex. By 3.1.27 one has

$$\begin{aligned}\operatorname{Hom}_R(X \otimes_S N, M)_v &= \prod_{h\in\mathbb{Z}} \operatorname{Hom}_R\big(\coprod_{j\in\mathbb{Z}} X_j \otimes_S N_{h-j}, M_{h+v}\big)\\ &\cong \prod_{h\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(X_j \otimes_S N_{h-j}, M_{h+v})\\ &= \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(X_j \otimes_S N_i, M_{i+j+v})\ ,\end{aligned}$$

and 3.1.24 yields

$$\begin{aligned}\operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M))_v &= \prod_{i\in\mathbb{Z}} \operatorname{Hom}_S\big(N_i, \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(X_j, M_{j+i+v})\big)\\ &= \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} \operatorname{Hom}_S(N_i, \operatorname{Hom}_R(X_j, M_{i+j+v}))\ .\end{aligned}$$

It follows from adjunction for modules 1.2.6 that the map

$$\operatorname{Hom}_R(X \otimes_S N, M) \longrightarrow \operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M))$$

with degree v component induced by $\prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} (-1)^{ij}\rho^{M_{i+j+v}X_jN_i}$ is a natural isomorphism of graded $\Bbbk$-modules; this isomorphism is denoted by ρ^{MXN}. On homogeneous elements $\psi \in \operatorname{Hom}_R(X \otimes_S N, M)$, $n \in N$, and $x \in X$ it is given by

(4.4.11.1) $$\rho^{MXN}(\psi)(n)(x) = (-1)^{|x||n|}\psi(x \otimes n)\ .$$

Note that (4.4.11.1) agrees with the definition in 1.2.6 for modules M, X, and N.

4.4.12 Proposition. *Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S-complex. The adjunction map defined in* 4.4.11*,*

$$\rho^{MXN} : \operatorname{Hom}_R(X \otimes_S N, M) \longrightarrow \operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M)) ,$$

is an isomorphism in $\mathcal{C}(\Bbbk)$, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$, then ρ^{MXN} is an isomorphism in $\mathcal{C}(T\text{–}Q^{\mathrm{o}})$. Finally, as a natural transformation of functors, ρ is a Σ-transformation in each variable.

Proof. By construction, ρ^{MXN} is an isomorphism of graded $\Bbbk$-modules and natural in M, X, and N. If M is in $\mathcal{C}(R\text{–}T^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}Q^{\mathrm{o}})$, then ρ^{MXN} is an isomorphism of graded Q–T^{o}-bimodules; this follows from 1.2.6 and the construction. For homogeneous elements $\psi \in \operatorname{Hom}_R(X \otimes_S N, M)$, $n \in N$, and $x \in X$ one has

$$\begin{aligned}
&\rho^{MXN}(\partial^{\operatorname{Hom}_R(X\otimes_S N,M)}(\psi))(n)(x)\\
&\quad= \rho^{MXN}(\partial^M\psi - (-1)^{|\psi|}\psi\partial^{X\otimes_S N})(n)(x)\\
&\quad= (-1)^{|x||n|}(\partial^M\psi(x\otimes n) - (-1)^{|\psi|}\psi\partial^{X\otimes_S N}(x\otimes n))\\
&\quad= (-1)^{|x||n|}\partial^M\psi(x\otimes n) - (-1)^{|\psi|+|x||n|}\psi(\partial^X(x)\otimes n + (-1)^{|x|}x\otimes\partial^N(n))
\end{aligned}$$

and

$$\begin{aligned}
&(\partial^{\operatorname{Hom}_S(N,\operatorname{Hom}_R(X,M))}(\rho^{MXN}(\psi)))(n)(x)\\
&\quad= (\partial^{\operatorname{Hom}_R(X,M)}\rho^{MXN}(\psi) - (-1)^{|\rho^{MXN}(\psi)|}\rho^{MXN}(\psi)\partial^N)(n)(x)\\
&\quad= \partial^M(\rho^{MXN}(\psi)(n)(x)) - (-1)^{|\rho^{MXN}(\psi)(n)|}\rho^{MXN}(\psi)(n)(\partial^X(x))\\
&\qquad\quad - (-1)^{|\psi|}\rho^{MXN}(\psi)(\partial^N(n))(x)\\
&\quad= (-1)^{|x||n|}\partial^M\psi(x\otimes n) - (-1)^{|\psi|+|n|+(|x|-1)|n|}\psi(\partial^X(x)\otimes n)\\
&\qquad\quad - (-1)^{|\psi|+|x|(|n|-1)}\psi(x\otimes\partial^N(n))\\
&\quad= (-1)^{|x||n|}\partial^M\psi(x\otimes n) - (-1)^{|\psi|+|x||n|}\psi(\partial^X(x)\otimes n + (-1)^{|x|}x\otimes\partial^N(n)) .
\end{aligned}$$

Thus, ρ^{MXN} is a morphism, and hence an isomorphism, of complexes. It follows from 2.2.5 combined with 4.1.16, 4.1.17, 4.1.18, and 4.1.19 that ρ as a natural transformation of functors is a Σ-transformation in each variable. The detailed argument follows the template from the proof of swap 4.4.10. □

Exercises

E 4.4.1 Apply 2.4.13 and commutativity 4.4.4 to give a proof of 2.4.14.

E 4.4.2 Apply 3.2.22 and commutativity 4.4.4 to give a proof of 3.2.23.

E 4.4.3 Assume that R is commutative. Let $x_1, \dots, x_n$ be a sequence in R and σ a permutation of the set $\{1, \dots, n\}$. Show that there is an isomorphism $\mathrm{K}^R(x_1, \dots, x_n) \cong \mathrm{K}^R(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

E 4.4.4 Let M be an R-complex and consider the map $R \to \operatorname{Hom}_R(M,M)$ that maps r to multiplication by r on M. (a) Show that it is a morphism of $\Bbbk$-complexes. (b) Show that it is a morphism of $\Bbbk$-algebras.

E 4.4.5 Show that the isomorphism in 4.4.8 is an isomorphism of (differential) graded $\Bbbk$-algebras.

E 4.4.6 Let $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ be functors. Assume that F has a right adjoint F_* and that there is a natural isomorphism $\zeta : \mathrm{F}_*\mathrm{E} \to \mathrm{E}\mathrm{F}_*$. Show that there is a canonical natural transformation $\theta : \mathrm{FE} \to \mathrm{EF}$.

Assume that R is commutative. Let M and N be R-complexes and use adjunction 4.4.12 and swap 4.4.10 to show that the result above applies with $\mathrm{F} = M \otimes_R -$ and $\mathrm{E} = \operatorname{Hom}_R(N,-)$.

E 4.4.7 Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ and $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(R)$ be functors. Assume that F has a right adjoint F_* and that there is a natural isomorphism $\rho : \mathrm{G}\mathrm{F}^{\mathrm{op}} \to \mathrm{F}_*\mathrm{G}$. Show that there is a canonical natural transformation $\eta : \mathrm{FG} \to \mathrm{G}\mathrm{F}_*^{\mathrm{op}}$.

Assume that R is commutative. Let M and N be R-complexes and use adjunction 4.4.12 to show that the result above applies with $\mathrm{F} = M \otimes_R -$ and $\mathrm{G} = \operatorname{Hom}_R(-,N)$.

4.5 Evaluation Morphisms for Complexes

Synopsis. Biduality; homothety formation; (co)unit of Hom–Hom adjunction; tensor evaluation; homomorphism evaluation; (co)unit of Hom–tensor adjunction.

In this section, the evaluation homomorphisms from Sect. 1.4 are extended to morphisms of complexes. The process is the same as in the previous section, but boundedness conditions now enter the picture when we consider the question of invertibility of thse maps.

Biduality

The next construction and the results that follow it extend 1.4.2 to complexes.

4.5.1 Construction. Let M be an R-complex and X a complex of R–S^{o}-bimodules. For every $v \in \mathbb{Z}$ one has

$$\operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M,X),X)_v \;=\; \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{S^{\mathrm{o}}}\Big(\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, X_{i+j}), X_{i+v}\Big)\;.$$

To define a map from M_v to $\operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M,X),X)_v$ it suffices, in view of 3.1.15, to define, for every integer i, a map

$$M_v \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}\Big(\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, X_{i+j}), X_{i+v}\Big)\;.$$

This is achieved by postcomposing biduality 1.4.2, adjusted by a sign,

$$(-1)^{iv}\delta^{M_v}_{X_{i+v}} : M_v \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M_v, X_{i+v}), X_{i+v})\;,$$

with the map induced by the projection $\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, X_{i+j}) \twoheadrightarrow \operatorname{Hom}_R(M_v, X_{i+v})$. The map of complexes $M \to \operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M,X),X)$, defined hereby, is denoted

δ_X^M. It follows from 1.4.2 that it is a natural morphism of graded R-modules. On homogeneous elements $m \in M$ and $\psi \in \operatorname{Hom}_R(M, X)$ it is given by

$$\delta_X^M(m)(\psi) = (-1)^{|\psi||m|}\psi(m) . \tag{4.5.1.1}$$

Note that (4.5.1.1) agrees with the definition in 1.4.2 for modules M and X.

4.5.2 Proposition. *Let X be a complex of R–S^{o}-bimodules. For an R-complex M the biduality map defined in* 4.5.1*,*

$$\delta_X^M : M \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M, X), X) ,$$

is a morphism in $\mathcal{C}(R)$, and it is natural in M. Moreover, if M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$, then δ_X^M is a morphism in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$. Finally, as a natural transformation of functors, δ is a Σ-transformation.

Proof. By construction, δ_X^M is a morphism of graded R-modules and natural in M. If M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$, then δ_X^M is a morphism of graded R–Q^{o}-bimodules; this follows from 1.4.2 and the construction above. For homogeneous elements $m \in M$ and $\psi \in \operatorname{Hom}_R(M, X)$ one has

$$\begin{aligned}
&(\partial^{\operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M,X),X)}\delta_X^M(m))(\psi)\\
&\quad= (\partial^X\delta_X^M(m) - (-1)^{|\delta_X^M(m)|}\delta_X^M(m)\partial^{\operatorname{Hom}_R(M,X)})(\psi)\\
&\quad= (-1)^{|\psi||m|}\partial^X\psi(m) - (-1)^{|m|}\delta_X^M(m)(\partial^X\psi - (-1)^{|\psi|}\psi\partial^M)\\
&\quad= (-1)^{|\psi||m|}\partial^X\psi(m) - (-1)^{|m|+(|\psi|-1)|m|}(\partial^X\psi(m) - (-1)^{|\psi|}\psi\partial^M(m))\\
&\quad= (-1)^{|\psi|(|m|-1)}\psi\partial^M(m)\\
&\quad= \delta_X^M(\partial^M(m))(\psi) .
\end{aligned}$$

Thus, δ_X^M is a morphism of complexes. It follows from 2.2.5 combined with 4.1.17 that δ_X as a natural transformation of functors is a Σ-transformation. $\square$

4.5.3 Proposition. *Let M be an R-complex and X a complex of R–S^{o}-bimodules. If one module X_p is faithfully injective as an R-module, then biduality* 4.5.2*,*

$$\delta_X^M : M \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(\operatorname{Hom}_R(M, X), X) ,$$

is injective.

Proof. It is sufficient to show that δ_X^M is injective on homogeneous elements. Let $m \neq 0$ be homogeneous of degree q. By assumption, X_p is faithfully injective as an R-module, whence there is a non-zero homomorphism from the submodule $R\langle m\rangle$ of M_q to X_p. By the lifting property 1.3.26 there is then a homomorphism $\widetilde{\psi}$ in $\operatorname{Hom}_R(M_q, X_p)$ with $\widetilde{\psi}(m) \neq 0$. Let $\psi\colon M \to X$ be the degree $p-q$ homomorphism with $\psi_q = \widetilde{\psi}$ and $\psi_v = 0$ for $v \neq q$. One now has $\delta_X^M(m)(\psi) = \psi(m) = \widetilde{\psi}(m) \neq 0$, so $\delta_X^M(m)$ is non-zero. $\square$

4.5.4 Theorem. *For every complex P of finitely generated projective R-modules, $\operatorname{Hom}_R(P, R)$ is a complex of finitely generated projective R^{o}-modules, and biduality*

$$\delta_R^P : P \longrightarrow \operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(P, R), R)$$

is an isomorphism.

Proof. For $v \in \mathbb{Z}$ one has $\operatorname{Hom}_R(P, R)_v = \operatorname{Hom}_R(P_{-v}, R)$ and $(\delta_R^P)_v = (-1)^{v^2} \delta_R^{P_v}$, see 4.5.1, so the assertions follow from 1.4.3. □

Homothety Formation

Let X be a complex of R–S°-bimodules and s an element in S. It follows from the definition of an R–S°-bimodule that the homothety s^X, see 2.1.9, is R-linear. Similarly, for an element r in R the homothety r^X is S°-linear. We now consider the maps that to a ring element assigns the corresponding homothety.

4.5.5 Proposition. *For a complex X of R–S°-bimodules the homothety formation map,*

$$\chi_{S^\circ R}^X : S \longrightarrow \operatorname{Hom}_R(X, X) \qquad \textit{given by} \qquad \chi_{S^\circ R}^X(s) = s^X ,$$

is a morphism in $\mathcal{C}(S\text{–}S^\circ)$, and

$$\chi_{RS^\circ}^X : R \longrightarrow \operatorname{Hom}_{S^\circ}(X, X) \qquad \textit{given by} \qquad \chi_{RS^\circ}^X(r) = r^X ,$$

is a morphism in $\mathcal{C}(R\text{–}R^\circ)$.

Proof. Both S and $\operatorname{Hom}_R(X, X)$ are complexes of S–S°-bimodules; see 2.3.11. The computations below, where $s, u \in S$ and $x \in X$, show that the map $\chi = \chi_{S^\circ R}^X$ is both S- and S°-linear.

$$\chi(us)(x) = x(us) = (xu)s = \chi(s)(xu) = (u\chi(s))(x) \quad \text{and}$$
$$\chi(su)(x) = x(su) = (xs)u = (\chi(s)(x))u = (\chi(s)u)(x) .$$

As χ is graded of degree 0, it remains to show that it commutes with the differentials. On S the differential is zero, and S°-linearity of the differential on X yields

$$\partial^{\operatorname{Hom}(X,X)} \chi(s) = \partial^X s^X - s^X \partial^X = 0 .$$

Since a complex of R–S°-bimodules is a complex of S°–R-bimodules, and since $R = R^\circ$ as R–R°-bimodules, it follows by symmetry that $\chi_{RS^\circ}^X : R \to \operatorname{Hom}_{S^\circ}(X, X)$ is a morphism in $\mathcal{C}(R\text{–}R^\circ)$. □

Remark. Homothety formation is, in fact, a morphism of $\Bbbk$-algebras; see E 4.5.2.

4.5.6 Example. The maps $\chi_{R^\circ R}^R : R \to \operatorname{Hom}_R(R, R)$ and $\chi_{RR^\circ}^R : R \to \operatorname{Hom}_{R^\circ}(R, R)$ are isomorphisms of R–R°-bimodules. Indeed, they are ϵ_R^R and $\epsilon_{R^\circ}^R$ from 4.4.2.

Unit and Counit of Hom–Hom Adjunction

The next result is not needed before Chap. 10, but it is natural to record it here.

4.5.7 Proposition. *Let X be a complex of R–S°-bimodules; there is an adjunction,*

$$\mathcal{C}(S^{\circ}) \underset{\mathrm{Hom}_R(-,X)}{\overset{\mathrm{Hom}_{S^{\circ}}(-,X)^{\mathrm{op}}}{\rightleftarrows}} \mathcal{C}(R)^{\mathrm{op}} \;.$$

For an S°-complex N the unit of the adjunction is biduality in $\mathcal{C}(S^{\circ})$,

$$\delta_X^N : N \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{S^{\circ}}(N,X),X) \;,$$

and for R-complex M the counit, viewed as a morphism in $\mathcal{C}(R)$, is biduality

$$\delta_X^M : M \longrightarrow \mathrm{Hom}_{S^{\circ}}(\mathrm{Hom}_R(M,X),X) \;.$$

Proof. Let M be an R-complex and N an S°-complex. By 2.3.10 and swap 4.4.10 there are natural isomorphisms,

$$\begin{aligned}
\mathcal{C}(R)^{\mathrm{op}}(\mathrm{Hom}_{S^{\circ}}(N,X),M) &\cong \mathrm{Z}_0(\mathrm{Hom}_R(M,\mathrm{Hom}_{S^{\circ}}(N,X)))\\
&\cong \mathrm{Z}_0(\mathrm{Hom}_{S^{\circ}}(N,\mathrm{Hom}_R(M,X)))\\
&\cong \mathcal{C}(S^{\circ})(N,\mathrm{Hom}_R(M,X)) \;.
\end{aligned}$$

This establishes the asserted adjunction. Note that under the isomorphisms above, a morphism $\psi\colon \mathrm{Hom}_{S^{\circ}}(N,X) \to M$ in $\mathcal{C}(R)^{\mathrm{op}}$, i.e. $\psi\colon M \to \mathrm{Hom}_{S^{\circ}}(N,X)$ in $\mathcal{C}(R)$, is mapped to the morphism $\zeta^{MXN}(\psi)\colon N \to \mathrm{Hom}_R(M,X)$ in $\mathcal{C}(S^{\circ})$. Conversely, a morphism $\varphi\colon N \to \mathrm{Hom}_R(M,X)$ in $\mathcal{C}(S^{\circ})$ gets mapped to the morphism $\zeta^{NXM}(\varphi)\colon M \to \mathrm{Hom}_{S^{\circ}}(N,X)$ in $\mathcal{C}(R)$. With $M = \mathrm{Hom}_{S^{\circ}}(N,X)$ and $\psi = 1^{\mathrm{Hom}(N,X)}$ one obtains, by definition, the unit of the adjunction,

$$\alpha_X^N = \zeta^{\mathrm{Hom}(N,X)XN}(1^{\mathrm{Hom}(N,X)}) : N \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{S^{\circ}}(N,X),X) \;.$$

By (4.4.9.1) it is given by

$$\begin{aligned}
\alpha_X^N(n)(\vartheta) &= \zeta^{\mathrm{Hom}(N,X)XN}(1^{\mathrm{Hom}(N,X)})(n)(\vartheta)\\
&= (-1)^{|\vartheta||n|}1^{\mathrm{Hom}(N,X)}(\vartheta)(n)\\
&= (-1)^{|\vartheta||n|}\vartheta(n)
\end{aligned}$$

for $n \in N$ and $\vartheta \in \mathrm{Hom}_{S^{\circ}}(N,X)$, so α_X^N is the biduality morphism δ_X^N, see (4.5.1.1). A parallel argument shows that the counit of the adjunction is biduality δ_X^M. □

Remark. The biduality morphism δ_X^M respects extra ring actions on M; see 4.5.2. One can use this to extend the adjunction from 4.5.7 to an adjunction $\mathcal{C}(Q\text{–}S^{\circ}) \leftrightarrows \mathcal{C}(R\text{–}Q^{\circ})$; see E 4.5.7.

Tensor Evaluation

The next construction and the results that follow it extend 1.4.4 and 1.4.6 to complexes.

4.5.8 Construction. Let M be an R-complex, X be a complex of R–S°-bimodules, and N an S-complex. There are equalities,

$$(4.5.8.1)\qquad (\operatorname{Hom}_R(M,X)\otimes_S N)_v \;=\; \coprod_{i\in\mathbb{Z}}\Big(\prod_{h\in\mathbb{Z}}\operatorname{Hom}_R(M_h,X_{h+i})\Big)\otimes_S N_{v-i}$$

and

$$(4.5.8.2)\qquad \operatorname{Hom}_R(M,X\otimes_S N)_v \;=\; \prod_{j\in\mathbb{Z}}\operatorname{Hom}_R\Big(M_j,\coprod_{k\in\mathbb{Z}}X_k\otimes_S N_{j+v-k}\Big)\,.$$

To define a map from $(\operatorname{Hom}_R(M,X)\otimes_S N)_v$ to $\operatorname{Hom}_R(M,X\otimes_S N)_v$ it suffices, in view of 3.1.2 and 3.1.15, to define, for all integers $\imath$ and $\jmath$, a map

$$\Big(\prod_{h\in\mathbb{Z}}\operatorname{Hom}_R(M_h,X_{h+\imath})\Big)\otimes_S N_{v-\imath}\longrightarrow\operatorname{Hom}_R\Big(M_\jmath,\coprod_{k\in\mathbb{Z}}X_{\jmath+k}\otimes_S N_{v-k}\Big)\,.$$

This is done by precomposing tensor evaluation 1.4.4, adjusted by a sign,

$$\operatorname{Hom}_R(M_\jmath,X_{\jmath+\imath})\otimes_S N_{v-\imath}\xrightarrow{(-1)^{\jmath(v-\imath)}\,\theta^{M_\jmath X_{\jmath+\imath}N_{v-\imath}}}\operatorname{Hom}_R(M_\jmath,X_{\jmath+\imath}\otimes_S N_{v-\imath})\,,$$

with the map induced by the projection,

$$\prod_{h\in\mathbb{Z}}\operatorname{Hom}_R(M_h,X_{h+\imath})\;\twoheadrightarrow\;\operatorname{Hom}_R(M_\jmath,X_{\jmath+\imath})\,,$$

and postcomposing it with the map induced by the injection,

$$X_{\jmath+\imath}\otimes_S N_{v-\imath}\;\rightarrowtail\;\coprod_{k\in\mathbb{Z}}X_{\jmath+k}\otimes_S N_{v-k}\,.$$

The map of complexes defined hereby, $\operatorname{Hom}_R(M,X)\otimes_S N\to\operatorname{Hom}_R(M,X\otimes_S N)$, is denoted θ^{MXN}. Per 1.4.4 it is a natural morphism of graded $\Bbbk$-modules. On homogeneous elements $\psi\in\operatorname{Hom}_R(M,X)$, $m\in M$, and $n\in N$ it is given by

$$(4.5.8.3)\qquad \theta^{MXN}(\psi\otimes n)(m)\;=\;(-1)^{|m||n|}\psi(m)\otimes n\,.$$

Note that (4.5.8.3) agrees with the definition in 1.4.4 for modules M, X, and N.

4.5.9 Proposition. *Let M be an R-complex, X be a complex of R–S^{o}-bimodules, and N an S-complex. The tensor evaluation map defined in* 4.5.8*,*

$$\theta^{MXN}:\operatorname{Hom}_R(M,X)\otimes_S N\;\longrightarrow\;\operatorname{Hom}_R(M,X\otimes_S N)\,,$$

is a morphism in $\mathcal{C}(\Bbbk)$, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$, then θ^{MXN} is a morphism in $\mathcal{C}(Q\text{–}T^{\mathrm{o}})$. Finally, as a natural transformation of functors, θ is a Σ-transformation in each variable.

Proof. By construction, θ^{MXN} is a morphism of graded $\Bbbk$-modules and natural in M, X, and N. If M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ and N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$, then θ^{MXN} is a morphism of graded Q–T^{o}-bimodules; this follows from 1.4.4 and the construction. For homogeneous elements $\psi\in\operatorname{Hom}_R(M,X)$, $m\in M$, and $n\in N$ one has

$$(\partial^{\operatorname{Hom}_R(M,X\otimes_S N)}(\theta^{MXN}(\psi\otimes n)))(m)$$

$$
\begin{aligned}
&= (\partial^{X\otimes_S N}\theta^{MXN}(\psi\otimes n) - (-1)^{|\theta^{MXN}(\psi\otimes n)|}\theta^{MXN}(\psi\otimes n)\partial^M)(m)\\
&= (-1)^{|m||n|}\partial^{X\otimes_S N}(\psi(m)\otimes n) - (-1)^{|\psi|+|n|+(|m|-1)|n|}\psi\partial^M(m)\otimes n\\
&= (-1)^{|m||n|}(\partial^X\psi(m)\otimes n + (-1)^{|\psi(m)|}\psi(m)\otimes\partial^N(n))\\
&\qquad - (-1)^{|\psi|+|m||n|}\psi(\partial^M(m))\otimes n\\
&= (-1)^{|m||n|}\partial^X\psi(m)\otimes n + (-1)^{|m||n|+|\psi|+|m|}\psi(m)\otimes\partial^N(n)\\
&\qquad - (-1)^{|\psi|+|m||n|}\psi\partial^M(m)\otimes n
\end{aligned}
$$

and

$$
\begin{aligned}
&\theta^{MXN}(\partial^{\operatorname{Hom}_R(M,X)\otimes_S N}(\psi\otimes n))(m)\\
&\qquad = \theta^{MXN}(\partial^{\operatorname{Hom}_R(M,X)}(\psi)\otimes n + (-1)^{|\psi|}\psi\otimes\partial^N(n))(m)\\
&\qquad = \theta^{MXN}((\partial^X\psi - (-1)^{|\psi|}\psi\partial^M)\otimes n + (-1)^{|\psi|}\psi\otimes\partial^N(n))(m)\\
&\qquad = (-1)^{|m||n|}\partial^X\psi(m)\otimes n - (-1)^{|\psi|+|m||n|}\psi\partial^M(m)\otimes n\\
&\qquad\qquad + (-1)^{|\psi|+|m|(|n|-1)}\psi(m)\otimes\partial^N(n)\,.
\end{aligned}
$$

These two computations show that θ^{MXN} is a morphism of complexes. It follows from 2.2.5 combined with 4.1.16, 4.1.17, 4.1.18, and 4.1.19 that θ as a natural transformation of functors is a Σ-transformation in each variable. The detailed argument follows the template from the proof of swap 4.4.10. □

4.5.10 Theorem. *Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S-complex. Tensor evaluation* 4.5.9*,*

$$\theta^{MXN}\colon \operatorname{Hom}_R(M,X)\otimes_S N \longrightarrow \operatorname{Hom}_R(M,X\otimes_S N)\,,$$

is an isomorphism if the complexes meet one of the boundedness conditions (1)–(3) *and one of the conditions* (a)–(c) *on their modules.*

(1) *M is bounded below, and X and N are bounded above.*
(2) *M is bounded above, and X and N are bounded below.*
(3) *Two of the complexes M, X, and N are bounded.*

(a) *M or N is a complex of finitely generated projective modules.*
(b) *M is a complex of projective modules and N is degreewise finitely presented.*
(c) *M is degreewise finitely presented and N is a complex of flat modules.*

Furthermore, θ^{MXN} is an isomorphism if M or N is a bounded complex of finitely presented modules and one of the following conditions is satisfied.

(d) *M is a complex of projective modules.*
(e) *N is a complex of flat modules.*

Proof. Under any one of the conditions (a)–(c), each homomorphism $\theta^{M_h X_i N_j}$ is an isomorphism of modules by 1.4.6. To prove the first assertion, it is now sufficient to show that under each of the boundedness conditions (1)–(3), every component of θ^{MXN} is given by a direct sum of homomorphisms $\theta^{M_h X_i N_j}$.

The products and coproducts in (4.5.8.1) and (4.5.8.2) are finite under any one of the conditions (1)–(3). Indeed, under (1), assume without loss of generality that one has $M_v = 0$ for all $v < 0$ and $X_v = 0 = N_v$ for all $v > 0$; cf. 2.3.14, 2.3.16, 2.4.13, and 2.4.14. Now 2.5.12 and 2.5.18 yield $(\operatorname{Hom}_R(M,X) \otimes_S N)_v = 0 = \operatorname{Hom}_R(M, X \otimes_S N)_v$ for all $v > 0$. For $v \leqslant 0$ equation (4.5.8.1) yields

$$\begin{aligned}(\operatorname{Hom}_R(M,X) \otimes_S N)_v &= \coprod_{i \geqslant v} \Big(\prod_{j=0}^{-i} \operatorname{Hom}_R(M_j, X_{j+i}) \Big) \otimes_S N_{v-i} \\ &\cong \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} \operatorname{Hom}_R(M_j, X_{j+i}) \otimes_S N_{v-i} \,,\end{aligned}$$

and from (4.5.8.2) one gets

$$\begin{aligned}\operatorname{Hom}_R(M, X \otimes_S N)_v &= \prod_{j \geqslant 0} \operatorname{Hom}_R\Big(M_j, \coprod_{i=v}^{-j} X_{j+i} \otimes_S N_{v-i}\Big) \\ &\cong \bigoplus_{j=0}^{-v} \bigoplus_{i=v}^{-j} \operatorname{Hom}_R(M_j, X_{j+i} \otimes_S N_{v-i}) \\ &= \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} \operatorname{Hom}_R(M_j, X_{j+i} \otimes_S N_{v-i}) \,.\end{aligned}$$

In particular, the v^{th} component of the morphism θ^{MXN} is given by

$$\theta^{MXN}_v = \bigoplus_{i=v}^{0} \bigoplus_{j=0}^{-i} (-1)^{j(v-i)} \theta^{M_j X_{j+i} N_{v-i}} \,.$$

Parallel arguments apply under conditions (2) and (3). Thus, θ^{MXN} is an isomorphism when one of (1)–(3) and one of (a)–(c) holds.

If M or N is a bounded complex of finitely presented modules, then under either one of the conditions (d) and (e), each homomorphism $\theta^{M_h X_i N_j}$ is an isomorphism of modules by 1.4.6. To prove the second assertion, it is now sufficient to prove that every component of θ^{MXN} is a product or a coproduct of homomorphisms $\theta^{M_h X_i N_j}$.

First, let M be a bounded complex of finitely presented modules and assume without loss of generality that one has $M_v = 0$ for all $v < 0$ and for all $v > u$, for some $u \geqslant 0$. From (4.5.8.1) one gets

$$\begin{aligned}(\operatorname{Hom}_R(M,X) \otimes_S N)_v &= \coprod_{i \in \mathbb{Z}} \Big(\prod_{j=0}^{u} \operatorname{Hom}_R(M_j, X_{j+i}) \Big) \otimes_S N_{v-i} \\ &\cong \coprod_{i \in \mathbb{Z}} \bigoplus_{j=0}^{u} \operatorname{Hom}_R(M_j, X_{j+i}) \otimes_S N_{v-i} \,,\end{aligned}$$

and (4.5.8.2) yields

$$\begin{aligned}\operatorname{Hom}_R(M, X \otimes_S N)_v &= \prod_{j=0}^{u} \operatorname{Hom}_R\Big(M_j, \coprod_{i \in \mathbb{Z}} X_{j+i} \otimes_S N_{v-i}\Big) \\ &\cong \coprod_{i \in \mathbb{Z}} \bigoplus_{j=0}^{u} \operatorname{Hom}_R(M_j, X_{j+i} \otimes_S N_{v-i}) \,,\end{aligned}$$

where the isomorphism follows from 3.1.33, as M is bounded and the modules M_j are finitely generated. It follows that the v^{th} component of the morphism θ^{MXN} is given by

$$\theta_v^{MXN} = \coprod_{i\in\mathbb{Z}} \bigoplus_{j=0}^{u} (-1)^{j(v-i)} \theta^{M_j X_{j+i} N_{v-i}} .$$

Finally, let N be a bounded complex of finitely presented modules and assume without loss of generality that one has $N_v = 0$ for all $v < 0$ and for all $v > u$, for some $u \geqslant 0$. Now (4.5.8.1) yields

$$\begin{aligned}(\operatorname{Hom}_R(M,X) \otimes_S N)_v &= \coprod_{i=v-u}^{v} \big(\prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, X_{j+i})\big) \otimes_S N_{v-i} \\ &\cong \prod_{j\in\mathbb{Z}} \bigoplus_{i=v-u}^{v} \operatorname{Hom}_R(M_j, X_{j+i}) \otimes_S N_{v-i} ,\end{aligned}$$

where the isomorphism follows from 3.1.31, as the modules N_{v-i} are finitely presented. Further, (4.5.8.2) yields

$$\begin{aligned}\operatorname{Hom}_R(M, X \otimes_S N)_v &= \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R\big(M_j, \coprod_{i=v-u}^{v} X_{j+i} \otimes_S N_{v-i}\big) \\ &\cong \prod_{j\in\mathbb{Z}} \bigoplus_{i=v-u}^{v} \operatorname{Hom}_R(M_j, X_{j+i} \otimes_S N_{v-i}) .\end{aligned}$$

It follows that the v^{th} component of the morphism θ^{MXN} is given by

$$\theta_v^{MXN} = \prod_{j\in\mathbb{Z}} \bigoplus_{i=v-u}^{v} (-1)^{j(v-i)} \theta^{M_j X_{j+i} N_{v-i}} . \qquad \square$$

Homomorphism Evaluation

The next construction and the subsequent results extend 1.4.7 and 1.4.9 to complexes.

4.5.11 Construction. Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S^{o}-complex. There are equalities,

$$(N \otimes_S \operatorname{Hom}_R(X,M))_v = \coprod_{i\in\mathbb{Z}} N_i \otimes_S \big(\prod_{h\in\mathbb{Z}} \operatorname{Hom}_R(X_h, M_{h+v-i})\big) \tag{4.5.11.1}$$

and

$$\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N,X),M)_v = \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R\big(\coprod_{k\in\mathbb{Z}} \operatorname{Hom}_{S^{\mathrm{o}}}(N_k, X_{k+j}), M_{j+v}\big). \tag{4.5.11.2}$$

To define a map from $(N \otimes_S \operatorname{Hom}_R(X,M))_v$ to $\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N,X),M)_v$ it suffices, in view of 3.1.2 and 3.1.15, to define, for all integers $\imath$ and $\jmath$, a map

$$N_\imath \otimes_S \big(\prod_{h\in\mathbb{Z}} \operatorname{Hom}_R(X_h, M_{h+v-\imath})\big) \longrightarrow \operatorname{Hom}_R\big(\coprod_{k\in\mathbb{Z}} \operatorname{Hom}_{S^{\mathrm{o}}}(N_k, X_{k+\jmath}), M_{\jmath+v}\big) .$$

This is done by precomposing homomorphism evaluation 1.4.7, adjusted by a sign,

$$N_\imath \otimes_S \operatorname{Hom}_R(X_{\imath+\jmath}, M_{\jmath+v}) \xrightarrow{(-1)^{(v-\imath+\jmath)\imath}\eta^{M_{\jmath+v}X_{\imath+\jmath}N_\imath}} \operatorname{Hom}_R(\operatorname{Hom}_{S^\circ}(N_\imath, X_{\imath+\jmath}), M_{\jmath+v}) ,$$

with the map induced the projection,

$$\prod_{h\in\mathbb{Z}} \operatorname{Hom}_R(X_h, M_{h+v-\imath}) \twoheadrightarrow \operatorname{Hom}_R(X_{\imath+\jmath}, M_{\jmath+v}) ,$$

and postcomposing it with the map induced by

$$\prod_{k\in\mathbb{Z}} \operatorname{Hom}_{S^\circ}(N_k, X_{k+\jmath}) \twoheadrightarrow \operatorname{Hom}_{S^\circ}(N_\imath, X_{\imath+\jmath}) .$$

The map of complexes, $N \otimes_S \operatorname{Hom}_R(X, M) \to \operatorname{Hom}_R(\operatorname{Hom}_{S^\circ}(N, X), M)$, defined hereby, is denoted η^{MXN}. It follows from 1.4.7 that it is a natural morphism of graded $\Bbbk$-modules. On homogeneous elements $n \in N$, $\psi \in \operatorname{Hom}_R(X, M)$, and $\vartheta \in \operatorname{Hom}_{S^\circ}(N, X)$ it is given by

$$\eta^{MXN}(n \otimes \psi)(\vartheta) = (-1)^{(|\psi|+|\vartheta|)|n|}\psi\vartheta(n) . \tag{4.5.11.3}$$

Note that (4.5.11.3) agrees with the definition in 1.4.7 for modules M, X and N.

4.5.12 Proposition. *Let M be an R-complex, X a complex of R–S°-bimodules, and N an S°-complex. The homomorphism evaluation map defined in* 4.5.11*,*

$$\eta^{MXN} : N \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_{S^\circ}(N, X), M) ,$$

is a morphism in $\mathcal{C}(\Bbbk)$, and it is natural in M, X, and N. Moreover, if M is in $\mathcal{C}(R\text{–}Q^\circ)$ and N is in $\mathcal{C}(T\text{–}S^\circ)$, then η^{MXN} is a morphism in $\mathcal{C}(T\text{–}Q^\circ)$. Finally, as a natural transformation of functors, η is a Σ-transformation in each variable.

Proof. By construction, η^{MXN} is a morphism of graded $\Bbbk$-modules and natural in M, X, and N. If M is in $\mathcal{C}(R\text{–}Q^\circ)$ and N is in $\mathcal{C}(T\text{–}S^\circ)$, then η^{MXN} is a morphism of graded T–Q°-bimodules; this follows from 1.4.4 and the construction. For homogeneous elements $n \in N$, $\psi \in \operatorname{Hom}_R(X, M)$, and $\vartheta \in \operatorname{Hom}_{S^\circ}(N, X)$ one has

$$\begin{aligned}
&(\partial^{\operatorname{Hom}_R(\operatorname{Hom}_{S^\circ}(N,X),M)}(\eta^{MXN}(n \otimes \psi)))(\vartheta)\\
&\quad = (\partial^M \eta^{MXN}(n \otimes \psi) - (-1)^{|\eta^{MXN}(n\otimes\psi)|}\eta^{MXN}(n \otimes \psi)\partial^{\operatorname{Hom}_{S^\circ}(N,X)})(\vartheta)\\
&\quad = (-1)^{(|\psi|+|\vartheta|)|n|}\partial^M\psi\vartheta(n)\\
&\qquad\quad - (-1)^{|n|+|\psi|}\eta^{MXN}(n \otimes \psi)(\partial^X\vartheta - (-1)^{|\vartheta|}\vartheta\partial^N)\\
&\quad = (-1)^{(|\psi|+|\vartheta|)|n|}\partial^M\psi\vartheta(n)\\
&\qquad\quad - (-1)^{|n|+|\psi|+(|\psi|+|\vartheta|-1)|n|}\psi(\partial^X\vartheta(n) - (-1)^{|\vartheta|}\vartheta\partial^N(n))\\
&\quad = (-1)^{(|\psi|+|\vartheta|)|n|}(\partial^M\psi\vartheta(n) - (-1)^{|\psi|}\psi\partial^X\vartheta(n) + (-1)^{|\psi|+|\vartheta|}\psi\vartheta\partial^N(n))
\end{aligned}$$

and

$$\eta^{MXN}(\partial^{N\otimes_S\operatorname{Hom}_R(X,M)}(n \otimes \psi))(\vartheta)$$

$$= \eta^{MXN}(\partial^N(n) \otimes \psi + (-1)^{|n|} n \otimes \partial^{\operatorname{Hom}_R(X,M)}(\psi))(\vartheta)$$
$$= (-1)^{(|\psi|+|\vartheta|)(|n|-1)} \psi\vartheta\partial^N(n) + (-1)^{|n|} \eta^{MXN}(n \otimes (\partial^M \psi - (-1)^{|\psi|} \psi \partial^X))(\vartheta)$$
$$= (-1)^{(|\psi|+|\vartheta|)|n|}((-1)^{|\psi|+|\vartheta|} \psi\vartheta\partial^N(n) + \partial^M \psi\vartheta(n) - (-1)^{|\psi|} \psi\partial^X\vartheta(n)) \,.$$

These two computations show that η^{MXN} is a morphism of complexes. It follows from 2.2.5 combined with 4.1.16, 4.1.17, 4.1.18, and 4.1.19 that η as a natural transformation of functors is a Σ-transformation in each variable. The detailed argument follows the template from the proof of swap 4.4.10. ◻

4.5.13 Theorem. *Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S^{o}-complex. Homomorphism evaluation* 4.5.12*,*

$$\eta^{MXN} : N \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M) \,,$$

is an isomorphism if the complexes meet one of the boundedness conditions (1)–(3) *and condition* (a) *or* (b) *on their modules.*

(1) *M and N are bounded below, and X is bounded above.*
(2) *M and N are bounded above, and X is bounded below.*
(3) *Two of the complexes M, X, and N are bounded.*

(a) *N is a complex of finitely generated projective modules.*
(b) *N is degreewise finitely presented and M is a complex of injective modules.*

Furthermore, η^{MXN} is an isomorphism if N is a bounded complex of finitely presented modules and one of the following conditions are satisfied.

(c) *N is a complex of projective modules.*
(d) *M is a complex of injective modules.*

Proof. Under either condition (a) or (b), each homomorphism $\eta^{M_j X_h N_i}$ is an isomorphism of modules by 1.4.9. To prove the first assertion, it is now sufficient to show that under any one of the boundedness conditions (1)–(3), every component of η^{MXN} is given by a direct sum of homomorphisms $\eta^{M_j X_h N_i}$.

The products and coproducts in (4.5.11.1) and (4.5.11.2) are finite under any one of the conditions (1)–(3). Indeed, under (1), assume without loss of generality that one has $M_v = 0 = N_v$ for all $v < 0$ and $X_v = 0$ for all $v > 0$; cf. 2.3.14, 2.3.16, 2.4.13, and 2.4.14. It follows that one has $(N \otimes_S \operatorname{Hom}_R(X, M))_v = 0 = \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M)_v$ for all $v < 0$. For $v \geqslant 0$ equation (4.5.11.1) yields

$$\begin{aligned}(N \otimes_S \operatorname{Hom}_R(X, M))_v &= \coprod_{i \geqslant 0} N_i \otimes_S \Big(\prod_{j=i-v}^{0} \operatorname{Hom}_R(X_j, M_{j+v-i})\Big) \\ &\cong \bigoplus_{i=0}^{v} \bigoplus_{j=i-v}^{0} N_i \otimes_S \operatorname{Hom}_R(X_j, M_{j+v-i}) \,,\end{aligned}$$

and from (4.5.11.2) one gets

$$\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(N, X), M)_v = \prod_{h \geqslant -v} \operatorname{Hom}_R\Big(\prod_{i=0}^{-h} \operatorname{Hom}_{S^{\mathrm{o}}}(N_i, X_{i+h}), M_{h+v}\Big)$$

$$
\begin{aligned}
&\cong \bigoplus_{h=-v}^{0} \bigoplus_{i=0}^{-h} \operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N_i, X_{i+h}), M_{h+v}) \\
&\cong \bigoplus_{i=0}^{v} \bigoplus_{h=-v}^{-i} \operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N_i, X_{i+h}), M_{h+v}) \\
&\cong \bigoplus_{i=0}^{v} \bigoplus_{j=i-v}^{0} \operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N_i, X_j), M_{j+v-i}) \, .
\end{aligned}
$$

In particular, the v^{th} component of the morphism η^{MXN} is

$$
\eta_v^{MXN} = \bigoplus_{i=0}^{v} \bigoplus_{j=i-v}^{0} (-1)^{(j+v)i} \eta^{M_{j+v-i} X_j N_i} \, .
$$

Parallel arguments apply under conditions (2) and (3). Thus, η^{MXN} is an isomorphism when one of (1)–(3) and (a) or (b) hold.

If N is a bounded complex of finitely presented modules, then under either one of the conditions (c) and (d), each homomorphism $\eta^{M_p X_q N_r}$ is an isomorphism of modules by 1.4.9. To prove the second assertion, it is now sufficient to prove that every component of η^{MXN} is given by a product of homomorphisms $\eta^{M_p X_q N_r}$. Assume without loss of generality that one has $N_v = 0$ for all $v < 0$ and for all $v > u$, for some $u \geqslant 0$. From (4.5.11.1) one gets

$$
\begin{aligned}
(N \otimes_S \operatorname{Hom}_R(X, M))_v &= \prod_{i=0}^{u} \big(N_i \otimes_S \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(X_j, M_{j+v-i}) \big) \\
&\cong \bigoplus_{i=0}^{u} \prod_{j \in \mathbb{Z}} (N_i \otimes_S \operatorname{Hom}_R(X_j, M_{j+v-i})) \, ,
\end{aligned}
$$

where the isomorphism follows from 3.1.30, as N is a bounded complex of finitely presented modules. Further, (4.5.11.2) yields

$$
\begin{aligned}
\operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N, X), M)_v &= \prod_{h \in \mathbb{Z}} \operatorname{Hom}_R\big(\prod_{i=0}^{u} \operatorname{Hom}_{S^{\circ}}(N_i, X_{i+h}), M_{h+v} \big) \\
&\cong \bigoplus_{i=0}^{u} \prod_{h \in \mathbb{Z}} \operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N_i, X_{i+h}), M_{h+v}) \\
&\cong \bigoplus_{i=0}^{u} \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(\operatorname{Hom}_{S^{\circ}}(N_i, X_j), M_{j+v-i}) \, .
\end{aligned}
$$

It follows that the v^{th} component of the morphism η^{MXN} is

$$
\eta_v^{MXN} = \bigoplus_{i=0}^{u} \prod_{j \in \mathbb{Z}} (-1)^{(j+v)i} \eta^{M_{j+v-i} X_j N_i} \, . \qquad \square
$$

Unit and Counit of Hom–Tensor Adjunction

The next result is not needed before Chap. 10, but it is natural to record it here.

4.5.14 Proposition. *Let X be a complex of R–S°-bimodules. There is an adjunction,*

$$\mathcal{C}(S) \underset{\operatorname{Hom}_R(X,-)}{\overset{X\otimes_S -}{\rightleftarrows}} \mathcal{C}(R)\,.$$

For an S-complex N the unit α_X^N is the unique morphism in $\mathcal{C}(S)$ that makes the following diagram commutative,

(4.5.14.1)
$$\begin{tikzcd} S \otimes_S N \arrow[r, "\chi^{X\otimes N}"] \arrow[d, "\mu^N"', "\cong"] & \operatorname{Hom}_R(X,X) \otimes_S N \arrow[d, "\theta^{XXN}"] \\ N \arrow[r, "\alpha_X^N"] & \operatorname{Hom}_R(X, X \otimes_S N)\,. \end{tikzcd}$$

It is given by

$$\alpha_X^N(n)(x) \;=\; (-1)^{|x||n|} x \otimes n\,,$$

and if N is in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$ then α_X^N is a morphism in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$. Moreover, α_X is a Σ-transformation.

For an R-complex M the counit β_X^M is the unique morphism in $\mathcal{C}(R)$ that makes the following diagram commutative,

(4.5.14.2)
$$\begin{tikzcd} X \otimes_S \operatorname{Hom}_R(X,M) \arrow[r, "\beta_X^M"] \arrow[d, "\eta^{MXX}"'] & M \arrow[d, "\cong"', "\epsilon^M"] \\ \operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{o}}}(X,X), M) \arrow[r, "\operatorname{Hom}(\chi^X, M)"] & \operatorname{Hom}_R(R,M)\,. \end{tikzcd}$$

It is given by

$$\beta_X^M(x \otimes \phi) \;=\; (-1)^{|x||\phi|}\phi(x)\,,$$

and if M is in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$ then β_X^M is a morphism in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$. Moreover, β_X is a Σ-transformation.

Proof. Let M be an R-complex and N an S-complex. By 2.3.10 and adjunction 4.4.12 there are natural isomorphisms,

$$\begin{aligned} \mathcal{C}(R)(X \otimes_S N, M) &\cong \mathrm{Z}_0(\operatorname{Hom}_R(X \otimes_S N, M)) \\ &\cong \mathrm{Z}_0(\operatorname{Hom}_S(N, \operatorname{Hom}_R(X,M))) \\ &\cong \mathcal{C}(S)(N, \operatorname{Hom}_R(X,M))\,. \end{aligned}$$

This establishes the asserted adjunction. Under the isomorphisms above a morphism $\psi\colon X \otimes_S N \to M$ in $\mathcal{C}(R)$ is sent to the morphism $\rho^{MXN}(\psi)\colon N \to \operatorname{Hom}_R(X,M)$ in $\mathcal{C}(S)$, and a morphism $\varphi\colon N \to \operatorname{Hom}_R(X,M)$ in $\mathcal{C}(S)$ is mapped to the morphism $(\rho^{MXN})^{-1}(\varphi)\colon X \otimes_S N \to M$ in $\mathcal{C}(R)$. With $M = X \otimes_S N$ and $\psi = 1^{X\otimes N}$ one obtains, by definition, the unit of the adjunction,

$$\alpha_X^N \;=\; \rho^{(X\otimes N)XN}(1^{X\otimes N}) : N \longrightarrow \operatorname{Hom}_R(X, X \otimes_S N)\,.$$

By (4.4.11.1) it is given by

$$\alpha_X^N(n)(x) \;=\; \rho^{(X\otimes N)XN}(1^{X\otimes N})(n)(x) \;=\; (-1)^{|x||n|}1^{X\otimes N}(x \otimes n) \;=\; (-1)^{|x||n|}x \otimes n$$

for $n \in N$ and $x \in X$. Commutativity of (4.5.14.1) follows from the next computation, which uses the definition of α_X^N, homothety formation 4.5.5, and (4.5.8.3).

$$\begin{aligned}
(\alpha_X^N \circ \mu_S^N)(s \otimes n)(x) &= \alpha_X^N(sn)(x) \\
&= (-1)^{|x||n|} x \otimes sn \\
&= (-1)^{|x||n|} xs \otimes n \\
&= (-1)^{|x||n|} \chi_{S^{\circ}R}^X(s)(x) \otimes n \\
&= \theta^{XXN}(\chi_{S^{\circ}R}^X(s) \otimes n)(x) \\
&= (\theta^{XXN} \circ (\chi_{S^{\circ}R}^X \otimes_S N))(s \otimes n)(x) \,.
\end{aligned}$$

As μ^N is an isomorphism, α_X^N is the unique morphism in $\mathcal{C}(S)$ that makes the diagram (4.5.14.1) commutative.

For a complex N of S–T°-bimodules it is elementary to verify that α_X^N is a morphism in $\mathcal{C}(S\text{–}T^{\circ})$; this also follows from commutativity of (4.5.14.1) as the other morphisms in the diagram per 2.4.10, 4.4.1, and 4.5.9 are morphisms in $\mathcal{C}(S\text{–}T^{\circ})$. It follows from 4.1.16, 4.1.18, and 2.2.5 that $\operatorname{Hom}_R(X, X \otimes_S -)$ is a Σ-functor, and the action of α_X is explicitly described above. It is now straightforward to verify that α_X is a Σ-transformation, see 4.1.9.

Going back to the first part of the proof with $N = \operatorname{Hom}_R(X, M)$ and $\varphi = 1^{\operatorname{Hom}(X,M)}$ one obtains, by definition, the counit of the adjunction,

$$\beta_X^M = (\rho^{MX\operatorname{Hom}(X,M)})^{-1}(1^{\operatorname{Hom}(X,M)}) : X \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow M \,.$$

As $\rho^{MX\operatorname{Hom}(X,M)}(\beta_X^M) = 1^{\operatorname{Hom}(X,M)}$ holds one gets $\rho^{MX\operatorname{Hom}(X,M)}(\beta_X^M)(\phi) = \phi$ for every $\phi \in \operatorname{Hom}_R(X, M)$. Thus, for every $x \in X$ the second equality below holds and the first one follows from (4.4.11.1),

$$(-1)^{|x||\phi|}\beta_X^M(x \otimes \phi) = \rho^{MX\operatorname{Hom}(X,M)}(\beta_X^M)(\phi)(x) = \phi(x) \,.$$

Consequently, the counit is given by the formula $\beta_X^M(x \otimes \phi) = (-1)^{|x||\phi|}\phi(x)$. Commutativity of (4.5.14.2) now follows from the next computation, which uses homothety formation 4.5.5, the fact that r^X has degree zero, and (4.5.11.3).

$$\begin{aligned}
(\epsilon_R^M \circ \beta_X^M)(x \otimes \phi)(r) &= (-1)^{|x||\phi|} r\phi(x) \\
&= (-1)^{|x||\phi|} \phi(rx) \\
&= (-1)^{(|\phi|+|r^X|)|x|} \phi r^X(x) \\
&= \eta^{MXX}(x \otimes \phi)(r^X) \\
&= (\eta^{MXX}(x \otimes \phi) \circ \chi_{RS^{\circ}}^X)(r) \\
&= (\operatorname{Hom}(\chi_{RS^{\circ}}^X, M) \circ \eta^{MXX})(x \otimes \phi)(r) \,.
\end{aligned}$$

As ϵ_R^M is an isomorphism, β_X^M is the unique morphism in $\mathcal{C}(R)$ that makes the diagram (4.5.14.2) commutative.

For a complex M of R–Q°-bimodules it is elementary to verify that β_X^M is a morphism in $\mathcal{C}(R\text{–}Q^{\circ})$; this also follows from commutativity of (4.5.14.2) as the other morphisms in the diagram per 2.3.11, 4.4.2, and 4.5.12 are morphisms in $\mathcal{C}(R\text{–}Q^{\mathrm{o}})$. It follows from 4.1.16, 4.1.18, and 2.2.5 that $X \otimes_S \operatorname{Hom}_R(X, -)$ is a Σ-functor, and the action of β_X is explicitly described above. It is now straightforward to verify that β_X is a Σ-transformation, see 4.1.9. □

Exercises

E 4.5.1 Let L be a complex of finitely generated free R-modules. Show that $\delta_R^{\operatorname{Hom}_R(L,R)}$ is an isomorphism of R^{o}-complexes with inverse $\operatorname{Hom}_R(\delta_R^L, R)$.

E 4.5.2 Let X be a complex of R–S^{o}-bimodules. Show that the maps $\chi_{RS^{\mathrm{o}}}^X : R \to \operatorname{Hom}_{S^{\mathrm{o}}}(X, X)$ and $\chi_{S^{\mathrm{o}}R}^X : S^{\mathrm{o}} \to \operatorname{Hom}_R(X, X)$ are morphisms of $\Bbbk$-algebras.

E 4.5.3 Let P be a projective R-module, X a complex of R–S^{o}-bimodules, and N a complex of finitely presented S-modules; show that θ^{PXN} is injective.

E 4.5.4 Let N be a complex of finitely presented S^{o}-modules, X a complex of R–S^{o}-bimodules, and E an injective R-module; show that η^{EXN} is injective.

E 4.5.5 Let M be an R-complex. Show that biduality δ_R^M is an isomorphism if and only if homomorphism evaluation η^{RRM} is an isomorphism.

E 4.5.6 Let M be an R-complex. Show that if the R^{o}-complex $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ is contractible, then biduality $\delta_{\mathbb{E}}^M$ is null-homotopic.

E 4.5.7 Let X be a complex of R–S^{o}-bimodules. Show that there are adjunctions

$$\operatorname{Hom}_{S^{\mathrm{o}}}(-, X)^{\mathrm{op}} : \mathcal{C}(Q\text{–}S^{\mathrm{o}}) \rightleftarrows \mathcal{C}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} : \operatorname{Hom}_R(-, X)$$

and

$$X \otimes_S - : \mathcal{C}(S\text{–}T^{\mathrm{o}}) \rightleftarrows \mathcal{C}(R\text{–}T^{\mathrm{o}}) : \operatorname{Hom}_R(X, -)\,.$$

Hint: Zigzag identities.

E 4.5.8 Show that the natural transformation $M \otimes_R \operatorname{Hom}_R(N, -) \to \operatorname{Hom}_R(N, M \otimes_R -)$ in E 4.4.6 is tensor evaluation, up to an application of commutativity 4.4.4.

E 4.5.9 Show that the natural transformation $M \otimes_R \operatorname{Hom}_R(-, N) \to \operatorname{Hom}_R(\operatorname{Hom}_R(-, M), N)$ in E 4.4.7 is homomorphism evaluation.

Chapter 5
Resolutions

A resolution of a complex is a quasi-isomorphism between the given complex and one that is distinguished by amenable homological properties. Resolutions come in several flavors; in Chap. 7 they are used to compute derived functors, and in later chapters they are used to attach homological invariants to complexes.

For example, one can build a resolution of a module by taking a projective precover, see 1.3.12, and then a projective precover of the kernel of that homomorphism and so on. This technique can be extended to build a resolution by projective modules of a bounded below complex, and Spaltenstein [239] shows how a resolution of any complex can be obtained from resolutions of truncations through a limiting process. Here we take a different approach, developed by Avramov, Halperin, and Foxby in [25], that is based on a technique colloquially referred to as "killing cycles".

5.1 Semi-Freeness

Synopsis. Semi-basis; semi-free complex; semi-free resolution; free resolution of module.

There are several ways in which the notion of freeness for modules can be extended to complexes; the simplest of these, graded-freeness of the underlying module, was treated in Sect. 2.5. The goal of this section is to show that the homological data of any complex can be encoded into one whose underlying module is graded-free.

5.1.1 Definition. An R-complex L is called *semi-free* if the graded R-module $L^\natural$ is graded-free with a graded basis E that can be written as a disjoint union

$$E = \biguplus_{n \geqslant 0} E^n \quad \text{with} \quad E^0 \subseteq \mathrm{Z}(L) \quad \text{and} \quad \partial^L(E^n) \subseteq R\Big\langle \bigcup_{i=0}^{n-1} E^i \Big\rangle \text{ for every } n \geqslant 1\,.$$

Such a basis is called a *semi-basis* for L.

5.1.2. Let L be a graded R-module. A graded basis for L is trivially a semi-basis for the R-complex L. Thus, a graded R-module is graded-free if and only if it is semi-free as an R-complex.

5.1.3 Example. Let L be a bounded below complex of free R-modules and set $w = \inf L^\natural$. For every $n \geqslant 0$, let E^n be a basis for the free module L_{w+n}, the union $\bigcup_{n \geqslant 0} E^n$ is now a semi-basis for L. Thus L is semi-free.

5.1.4 Example. The Dold complex from 2.1.23 is a complex of free $\mathbb{Z}/4\mathbb{Z}$-modules. It has no semi-basis, as no graded basis for this complex contains a cycle.

5.1.5. Note that if an R-complex L is semi-free, then so is $\Sigma^s L$ for every integer s.

REMARK. It follows from the proof of 4.2.19 that every complex of free modules over a principal left ideal domain is isomorphic to a coproduct of bounded complexes of free modules and hence semi-free by 5.1.3 and E 5.1.4.

EXISTENCE OF SEMI-FREE RESOLUTIONS

5.1.6 Definition. A *semi-free resolution* of an R-complex M is a quasi-isomorphism $L \longrightarrow M$ of R-complexes where L is semi-free.

The next theorem achieves the goal of this section.

5.1.7 Theorem. *Every R-complex M has a semi-free resolution $\pi\colon L \xrightarrow{\simeq} M$ with $L_v = 0$ for all $v < \inf M^\natural$. Moreover, π can be chosen surjective.*

The proof relies on the next construction and follows after the proof of 5.1.9.

5.1.8 Construction. Given an R-complex $M \neq 0$, we proceed to construct a commutative diagram in $\mathcal{C}(R)$,

(5.1.8.1)
$$\begin{tikzcd}
L^0 \arrow[r, tail, "\iota^0"] \arrow[drr, "\pi^0"'] & \cdots \arrow[r, tail] & L^{n-1} \arrow[r, tail, "\iota^{n-1}"] \arrow[d, "\pi^{n-1}"'] & L^n \arrow[r, tail, "\iota^n"] \arrow[dl, "\pi^n"] & \cdots \arrow[r] & L \arrow[dlll, bend left, "\pi"] \\
 & & M & & &
\end{tikzcd}$$

For $n = 0$, choose a set Z^0 of homogeneous cycles in M whose homology classes generate $\mathrm{H}(M)$. Let $E^0 = \{e_z \mid |e_z| = |z|,\ z \in Z^0\}$ be a graded set and define an R-complex L^0 as follows:

(5.1.8.2)
$$(L^0)^\natural = R\langle E^0 \rangle \qquad \text{and} \qquad \partial^{L^0} = 0\,.$$

To see that the map $\pi^0\colon L^0 \to M$ given by the assignment $e_z \mapsto z$ is a morphism of complexes, notice that the differential on L^0 is 0 and that π^0 maps to $\mathrm{Z}(M)$, the kernel of ∂^M.

For $n \geqslant 1$ let a morphism $\pi^{n-1}\colon L^{n-1} \to M$ be given. Choose a set Z^n of homogeneous cycles in L^{n-1} whose homology classes generate the kernel of $\mathrm{H}(\pi^{n-1})$. Let $E^n = \{e_z \mid |e_z| = |z| + 1,\ z \in Z^n\}$ be a graded set and set

(5.1.8.3)
$$(L^n)^\natural = (L^{n-1})^\natural \oplus R\langle E^n \rangle \quad \text{and}$$
$$\partial^{L^n}\Big(x + \sum_{z \in Z^n} r_z e_z\Big) = \partial^{L^{n-1}}(x) + \sum_{z \in Z^n} r_z z\,.$$

This defines an R-complex. For each $z \in Z^n$ choose an element $m_z \in M$ such that $\pi^{n-1}(z) = \partial^M(m_z)$. It is elementary to verify that the map $\pi^n : L^n \longrightarrow M$ defined by

$$\pi^n\Big(x + \sum_{z \in Z^n} r_z e_z\Big) = \pi^{n-1}(x) + \sum_{z\in Z^n} r_z m_z$$

is a morphism of R-complexes. Moreover, it agrees with π^{n-1} on the subcomplex L^{n-1} of L^n. That is, there is an equality of morphisms $\pi^{n-1} = \pi^n \iota^{n-1}$, where ι^{n-1} is the embedding of L^{n-1} into L^n; cf. (5.1.8.3).

For $n < 0$ set $L^n = 0$, $\iota^n = 0$, and $\pi^n = 0$, then the family $\{\iota^n : L^n \longrightarrow L^{n+1}\}_{n\in\mathbb{Z}}$ is a telescope in $\mathcal{C}(R)$, and $\pi^n = \pi^{n+1}\iota^n$ holds for all $n \in \mathbb{Z}$. Set $L = \operatorname{colim}_{n\in\mathbb{Z}} L^n$, by 3.3.33 there is a morphism of R-complexes $\pi : L \longrightarrow M$, such that the diagram (5.1.8.1) is commutative.

5.1.9 Proposition. *Let $M \neq 0$ be an R-complex. The sets, morphisms, and complexes constructed in* 5.1.8 *have the following properties.*

(a) *Each set E^n consists of homogeneous elements of degree at least $n + \inf M^\natural$.*
(b) *Each complex L^n is semi-free with semi-basis $\biguplus_{i=0}^n E^i$.*
(c) *The complex L is semi-free with semi-basis $E = \biguplus_{n\geqslant 0} E^n$.*
(d) *The morphism $\pi : L \longrightarrow M$ is a quasi-isomorphism.*
(e) *If π^n is surjective for some $n \geqslant 0$, then π is surjective.*

Proof. Parts (a) and (b) are immediate from the definition of the sets E^n and (5.1.8.3); part (e) follows from commutativity of the diagram (5.1.8.1).

(c): The morphisms ι^n are embeddings, so $L = \operatorname{colim}_{n\in\mathbb{Z}} L^n$ is by 3.3.34 simply the union $\bigcup_{n\geqslant 0} L^n$; in particular, $\biguplus_{n\geqslant 0} E^n$ is a graded basis for $L^\natural$. By (5.1.8.3) there are inclusions $\partial^L(E^n) = \partial^{L^n}(E^n) \subseteq R\langle \bigcup_{i=0}^{n-1} E^i\rangle$ for $n \geqslant 1$, and (5.1.8.2) yields $\partial^L(E^0) = \partial^{L^0}(E^0) = 0$, so E^0 consists of cycles.

(d): For each $n \geqslant 0$ there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}(L^0) \arrow[r] \arrow[dr, "\mathrm{H}(\pi^0)"'] & \mathrm{H}(L^n) \arrow[r] \arrow[d, "\mathrm{H}(\pi^n)"] & \mathrm{H}(L) \arrow[dl, "\mathrm{H}(\pi)"] \\ & \mathrm{H}(M)\,, & \end{tikzcd}$$

induced from (5.1.8.1) By the choice of Z^0, the morphism $\mathrm{H}(\pi^0)$ is surjective and hence so is $\mathrm{H}(\pi)$. To see that $\mathrm{H}(\pi)$ is injective, let y be a cycle in L and assume that $\mathrm{H}(\pi)([y]) = 0$. Choose an integer n such that $y \in L^{n-1}$; now one has

$$0 = \mathrm{H}(\pi)([y]) = [\pi(y)] = [\pi^{n-1}(y)] = \mathrm{H}(\pi^{n-1})([y])\,,$$

so $[y]$ is in $\operatorname{Ker}\mathrm{H}(\pi^{n-1})$. By the choice of Z^n there exists a element $x \in L^{n-1}$ such that one has

$$y = \sum_{z\in Z^n} r_z z + \partial^{L^{n-1}}(x) = \partial^{L^n}\big(x + \sum_{z\in Z^n} r_z e_z\big)\,,$$

where the second equality follows from (5.1.8.3). It follows that $[y]$ is 0 in $\mathrm{H}(L^n)$ and hence also in $\mathrm{H}(L)$. Thus, $\mathrm{H}(\pi)$ is injective, and π is a quasi-isomorphism. □

PROOF OF 5.1.7. The identity morphism $0 \longrightarrow 0$ is a semi-free resolution of the zero complex. For an R-complex $M \neq 0$, apply the construction 5.1.8. It follows from parts (c) and (d) in 5.1.9 that $\pi\colon L \xrightarrow{\simeq} M$ is a semi-free resolution. Parts (a) and (c) ensure that $L_v = 0$ holds for all $v < \inf M^{\natural}$. Finally, notice that choosing Z^0 as a set of generators for $\mathrm{Z}(M)$ makes the morphism π^0 is surjective on cycles, and then π is surjective on cycles by commutativity of (5.1.8.1). As π is a quasi-isomorphism, it follows from 4.2.7 that it is surjective. □

5.1.10 Proposition. *Let L be a semi-free S-complex and X a complex of R–S^{o}-bimodules. If X is a semi-free over R, then the R-complex $X \otimes_S L$ is semi-free.*

PROOF. Let $E = \biguplus_{n \geqslant 0} E^n$ be a semi-basis for X over R and $F = \biguplus_{n \geqslant 0} F^n$ a semi-basis for L. For $n \geqslant 0$ set $G^n = \{e \otimes f \mid e \in E^i,\ f \in F^j,\ i + j = n\}$. It is elementary to verify that the graded R-module $(X \otimes_S L)^{\natural}$ is graded-free with basis $G = \biguplus_{n \geqslant 0} G^n$, and it follows from 2.4.1 that G is a semi-basis for $X \otimes_S L$. □

5.1.11 Corollary. *Let $R \to S$ be a ring homomorphism.*

(a) *For a semi-free R-complex L the S-complex $S \otimes_R L$ is semi-free.*
(b) *If S is free as an R-module, then a semi-free S-complex is semi-free over R.*

PROOF. For (b) apply 5.1.10 with $X = S$ viewed as an R–S^{o}-bimodule and note that $X \otimes_S -$ is the restriction of scalars functor $\mathcal{C}(S) \to \mathcal{C}(R)$. For (a) interchange the roles of R and S in 5.1.10 and apply it with $X = S$ viewed as an S–R^{o}-bimodule. □

BOUNDEDNESS AND FINITENESS

A suitably bounded and/or finite complex has a semi-free resolution with similar properties. The construction of such a resolution could also be performed degreewise, thus resembling the classic construction of a free resolution of a module.

5.1.12 Theorem. *Every R-complex M has a semi-free resolution $L \xrightarrow{\simeq} M$ with $L_v = 0$ for all $v < \inf M$.*

PROOF. If M is acyclic, then the morphism $0 \xrightarrow{\simeq} M$ is the desired resolution. If $\mathrm{H}(M)$ is not bounded below, then any semi-free resolution of M has the desired property. Assume now that $\mathrm{H}(M)$ is bounded below and set $w = \inf M$. By 4.2.4 there is a quasi-isomorphism $M_{\supseteq w} \to M$, and by 5.1.7 the truncated complex $M_{\supseteq w}$ has a semi-free resolution $L \xrightarrow{\simeq} M_{\supseteq w}$ with $L_v = 0$ for $v < w$. The desired semi-free resolution is the composite $L \xrightarrow{\simeq} M_{\supseteq w} \xrightarrow{\simeq} M$. □

5.1.13 Theorem. *Assume that R is left Noetherian. Every bounded below and degreewise finitely generated R-complex M has a semi-free resolution $\pi\colon L \xrightarrow{\simeq} M$ with L degreewise finitely generated and $L_v = 0$ for all $v < \inf M^{\natural}$. Moreover, π can be chosen surjective.*

Proof. The identity morphism $0 \longrightarrow 0$ is a semi-free resolution of the zero complex. Assume now that M is non-zero and set $w = \inf M^\natural$; it is an integer by the assumption on M. Apply the construction 5.1.8 to M and notice the following.

As R is left Noetherian, and M is a complex of finitely generated R-modules, the set Z^0 can be chosen such that it contains only finitely many elements of each degree. Doing so ensures that $E^0 = \{e_z \mid |e_z| = |z|,\ z \in Z^0\}$ contains only finitely many elements of each degree v and no elements of degree $v < w$; see 5.1.9(a).

As R is left Noetherian, it follows by induction that $\operatorname{Ker} \mathrm{H}(\pi^{n-1})$ is degreewise finitely generated for every $n > 1$. Choosing the set Z^n such that it has only finitely many elements of each degree ensures that the set $E^n = \{e_z \mid |e_z| = |z| + 1,\ z \in Z^n\}$ contains only finitely many elements of each degree v and no elements of degree $v < w + n$; see 5.1.9(a).

From 5.1.9 it follows that $\pi\colon L \xrightarrow{\simeq} M$ is a semi-free resolution of M, and that $E = \biguplus_{n \geqslant 0} E^n$ is a semi-basis for L. For each $v \in \mathbb{Z}$ the subset $E_v \subseteq E$ of basis elements of degree v is a basis for L_v, and it is finite as one has

$$E_v = \Big(\biguplus_{n \geqslant 0} E^n \Big)_v = \biguplus_{n=0}^{v-w} (E^n)_v \,.$$

Thus, each free module L_v is finitely generated, and $L_v = 0$ holds for all $v < w$.

Finally, as R is left Noetherian, one can choose as Z^0 a set of generators for $\mathrm{Z}(M)$ with the additional property that it contains only finitely many elements of each degree. With this choice, the quasi-isomorphism π is surjective on cycles by 5.1.9(e) and, therefore, surjective by 4.2.7. $\square$

5.1.14 Theorem. *Assume that R is left Noetherian. Every R-complex M with $\mathrm{H}(M)$ bounded below and degreewise finitely generated has a semi-free resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated and $L_v = 0$ for all $v < \inf M$.*

Proof. If M is acyclic, then the morphism $0 \xrightarrow{\simeq} M$ is the desired resolution. Assume now that M is not acyclic. Set $w = \inf M$ and apply 5.1.13 to the truncated complex $M_{\supseteq w}$ to obtain a semi-free resolution $L \xrightarrow{\simeq} M_{\supseteq w}$ with each module L_v finitely generated and $L_v = 0$ for all $v < w$. By 4.2.4 there is a quasi-isomorphism $M_{\supseteq w} \xrightarrow{\simeq} M$, and the desired resolution is the composite $L \xrightarrow{\simeq} M_{\supseteq w} \xrightarrow{\simeq} M$. $\square$

Without the assumption that $\mathrm{H}(M)$ is bounded below, a semi-free resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated may not necessarily exist; an example is provided in 20.1.20. Under the additional assumption that R is local, the semi-free resolution in 5.1.14 can be chosen to be minimal, see B.63.

The Case of Modules

5.1.15 Proposition. *Let $R \to S$ be a ring homomorphism and L an R-module.*

(a) *If L is free, then the S-module $S \otimes_R L$ is free.*
(b) *If S is free as an R-module, then a free S-module is free over R.*

Proof. The assertion follows, in view of 5.1.2, immediately from 5.1.11. $\square$

Recall from 5.1.2 that a module is free if and only if it is semi-free as a complex. Semi-free resolutions of complexes subsume the classic notion of free resolutions of modules from Cartan and Eilenberg [48].

5.1.16 Theorem. *Let M be an R-module. There is an exact sequence of R-modules,*

$$\cdots \longrightarrow L_v \longrightarrow L_{v-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0\,,$$

where each module L_v is free.

PROOF. Choose by 5.1.7 a surjective semi-free resolution $\pi\colon L \xrightarrow{\simeq} M$ with $L_v = 0$ for all $v < 0$. The displayed sequence of R-modules is the complex $\Sigma^{-1}\mathrm{Cone}\,\pi$; in particular, the map $L_0 \twoheadrightarrow M$ is the homomorphism $-\pi_0$. The cone is acyclic because π is a quasi-isomorphism; see 4.2.16. □

5.1.17 Definition. Let M be an R-module. Together, the surjective homomorphism $L_0 \to M$ and the R-complex $\cdots \to L_v \to L_{v-1} \to \cdots \to L_0 \to 0$ in 5.1.16 is called a *free resolution* of M.

5.1.18. Let M be an R-module. By 5.1.3 a free resolution of M is a semi-free resolution of M as an R-complex. Only a semi-free resolution $L \xrightarrow{\simeq} M$ with L concentrated in non-negative degrees is a free resolution of M.

5.1.19 Theorem. *Assume that R is left Noetherian. Every finitely generated R-module M has a free resolution*

$$\cdots \longrightarrow L_v \longrightarrow L_{v-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

where each module L_v is finitely generated.

PROOF. Choose by 5.1.13 a surjective semi-free resolution $\pi\colon L \xrightarrow{\simeq} M$, where each free module L_v is finitely generated and $L_v = 0$ holds for all $v < 0$. The displayed sequence is the acyclic complex $\Sigma^{-1}\mathrm{Cone}\,\pi$; cf. the proof of 5.1.16. □

EXERCISES

E 5.1.1 A *semi-free filtration* of an R-complex L is a sequence $\cdots \subseteq L^{u-1} \subseteq L^u \subseteq L^{u+1} \subseteq \cdots$ of subcomplexes, such that the graded module underlying each quotient L^u/L^{u-1} is graded-free and one has $L = \bigcup_{u\in\mathbb{Z}} L^u$, $L^{-1} = 0$, and $\partial^L(L^u) \subseteq L^{u-1}$ for all $u \in \mathbb{Z}$. Show that an R-complex is semi-free if and only if it admits a semi-free filtration.

E 5.1.2 Show that a complex L of free R-modules is semi-free if $\partial^L_v = 0$ holds for $v \ll 0$.

E 5.1.3 Let L be a semi-free R-complex; show that every subcomplex $L_{\leqslant n}$ is semi-free.

E 5.1.4 Show that a coproduct of semi-free R-complexes is semi-free.

E 5.1.5 Show that a direct summand of a semi-free R-complex need not be semi-free.

E 5.1.6 Let L be a semi-free R-complex and N a bounded and degreewise finitely generated subcomplex of L. Show that N is contained in bounded and degreewise finitely generated semi-free subcomplex of L.

E 5.1.7 Let L be a semi-free R-complex and $\alpha\colon L \to N$ a morphism in $\mathcal{C}(R)$. Show that for every surjective quasi-isomorphism $\beta\colon M \to N$ there is a morphism $\gamma\colon L \to M$ with $\alpha = \beta\gamma$. *Hint:* For each n let L^n be the semi-free subcomplex with semi-basis $\biguplus_{i=0}^{n} E^i$ and construct morphisms $\gamma^n : L^n \to M$ compatible with the embeddings $L^n \rightarrowtail L^{n+1}$.

E 5.1.8 Apply 5.1.8 to construct a free resolution of the $\mathbb{Z}/4\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$.

E 5.1.9 Construct semi-free resolutions of the complexes in 4.2.3.

E 5.1.10 Show that every morphism $\alpha\colon M \to N$ of R-complexes admits factorizations in $\mathcal{C}(R)$,

$$\begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, tail, "\iota"'] & & N \\ & X \arrow[ur, two heads, "\pi"', "\simeq"] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, tail, "\varepsilon"', "\simeq"] & & N \\ & Y \arrow[ur, two heads, "\varphi"'] & \end{tikzcd}$$

where ι and ε are injective with semi-free cokernels, π and φ are surjective, and π and ε are quasi-isomorphisms. *Hint:* Modify the first step in 5.1.8; apply 4.3.24.

E 5.1.11 Give alternative proofs of 5.1.16 and 5.1.19 based on 1.3.12.

5.2 Semi-Projectivity

Synopsis. Graded-projective module; complex of projective modules; semi-projective complex; semi-projective resolution; lifting property; projective resolution of module.

For a module P, the functor $\operatorname{Hom}(P, –)$ preserves short exact sequences of complexes if and only if it preserves acyclicity of complexes, and these properties characterize projective modules. Any complex P of projective modules has the first of these properties, see 2.3.18, but not the second; adding the requirement that the functor $\operatorname{Hom}(P, –)$ preserves acyclicity of complexes leads to the notion of semi-projectivity. We start by studying complexes of projective modules.

5.2.1. Lifting properties are a central theme in this section, and several key results can be interpreted in terms of the diagram

(5.2.1.1)
$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted] \\ M \arrow[r] & N \end{tikzcd}$$

where the solid arrows represent given maps of certain sorts, and a lifting property of P ensures the existence of a dotted map of a specific sort such that the diagram is commutative, or commutative up to homotopy.

Complexes of Projective Modules

Part (*iii*) below can be interpreted in terms of the diagram (5.2.1.1).

5.2.2 Proposition. *For an R-complex P, the following conditions are equivalent.*

(*i*) *Each R-module P_v is projective.*

(*ii*) *The functor* $\operatorname{Hom}_R(P,-)$ *is exact.*
(*iii*) *For every homomorphism* $\alpha\colon P \to N$ *and every surjective homomorphism* $\beta\colon M \to N$*, there exists a homomorphism* $\gamma\colon P \to M$ *such that* $\alpha = \beta\gamma$ *holds.*
(*iv*) *Every exact sequence* $0 \to M' \to M \to P \to 0$ *in* $\mathcal{C}(R)$ *is degreewise split.*
(*v*) *The graded module* $P^{\natural}$ *is a graded direct summand of a graded-free* R*-module.*

PROOF. The implication (*i*) $\Rightarrow$ (*ii*) is immediate from 2.3.18.

(*ii*) $\Rightarrow$ (*iii*): The homomorphism β yields a morphism $\varsigma^{N}_{-|\beta|}\beta\colon M^{\natural} \to \Sigma^{-|\beta|}N^{\natural}$ and by exactness of the functor $\operatorname{Hom}_R(P,-)$ there exists a homomorphism $\gamma \in \operatorname{Hom}_R(P,M^{\natural})$ with $\varsigma^{N}_{-|\beta|}\alpha = \varsigma^{N}_{-|\beta|}\beta\gamma$ and hence $\alpha = \beta\gamma$.

(*iii*) $\Rightarrow$ (*iv*): Let β denote the morphism $M \twoheadrightarrow P$. By (*iii*) there exists a homomorphism $\gamma\colon P \to M$ such that $1^P = \operatorname{Hom}_R(P,\beta)(\gamma) = \beta\gamma$. As β is a morphism of complexes, also the degree of γ must be 0, that is, γ is a morphism of the underlying graded modules. Hence, the sequence $0 \to M' \to M \to P \to 0$ is degreewise split.

(*iv*) $\Rightarrow$ (*v*): Choose by 5.1.7 a surjective semi-free resolution $\pi\colon L \to P$ and apply (*iv*) to the associated exact sequence $0 \to \operatorname{Ker}\pi \to L \to P \to 0$ in $\mathcal{C}(R)$. It follows that $P^{\natural}$ is a graded direct summand of the graded-free R-module $L^{\natural}$.

(*v*) $\Rightarrow$ (*i*): Each module P_v is a direct summand of a free R-module and, therefore, projective by 1.3.17. $\square$

CAVEAT. The complexes described in 5.2.2 are not the projective objects in the category $\mathcal{C}(R)$; see E 5.2.1 and E 5.2.6.

5.2.3 Corollary. *Let* $0 \to P' \to P \to P'' \to 0$ *be an exact sequence of* R*-complexes. If* P'' *is a complex of projective modules, then* P *is a complex of projective modules if and only if* P' *is a complex of projective of modules.*

PROOF. If P'' is a complex of projective modules, then $0 \to P' \to P \to P'' \to 0$ is degreewise split, and the assertion follows from 1.3.24. $\square$

5.2.4 Definition. A graded R-module P is called *graded-projective* if the R-complex P satisfies the conditions in 5.2.2.

EXISTENCE OF SEMI-PROJECTIVE RESOLUTIONS

5.2.5 Definition. An R-complex P is called *semi-projective* if $\operatorname{Hom}_R(P,\beta)$ is a surjective quasi-isomorphism for every surjective quasi-isomorphism β in $\mathcal{C}(R)$.

REMARK. Another word for semi-projective is 'DG-projective'.

5.2.6. It follows from 2.3.14 that if an R-complex P is semi-projective, then so is $\Sigma^s P$ for every integer s.

5.2.7 Example. A contractible complex of projective R-modules is semi-projective by 5.2.2 and 4.3.29.

5.2.8 Example. Let P be a bounded below complex of projective R-modules and β a surjective quasi-isomorphism. The morphism $\operatorname{Hom}_R(P,\beta)$ is surjective by 5.2.2. The complex $\operatorname{Cone}\beta$ is acyclic by 4.2.16, and hence so is $\operatorname{Hom}_R(P_v,\operatorname{Cone}\beta)$ for every $v \in \mathbb{Z}$. As P is bounded below, it follows from 4.1.16 and A.5 that $\operatorname{Cone}\operatorname{Hom}_R(P,\beta) \cong \operatorname{Hom}_R(P,\operatorname{Cone}\beta)$ is acyclic, whence β is a quasi-isomorphism. Thus, P is semi-projective.

5.2.9 Lemma. *For a semi-free R-complex L, the functor $\operatorname{Hom}_R(L,-)$ is exact and preserves acyclicity of complexes.*

Proof. Let L be a semi-free R-complex; by 2.5.27 it is a complex of free R-modules, so the functor $\operatorname{Hom}_R(L,-)$ is exact by 5.2.2. Choose a semi-basis $E = \biguplus_{i\geqslant 0} E^i$ for L. For $n < 0$ set $L^n = 0$. For $n \geqslant 0$ let L^n be the semi-free subcomplex of L with semi-basis $\biguplus_{i=0}^{n} E^i$. Now one has $L = \bigcup_{n\geqslant 0} L^n \cong \operatorname{colim}_{n\in\mathbb{Z}} L^n$, and for every $n \geqslant 0$ there is an exact sequence

$$(\dagger) \qquad 0 \longrightarrow L^{n-1} \longrightarrow L^n \longrightarrow L^n/L^{n-1} \longrightarrow 0\,.$$

The induced differential on the subquotient L^n/L^{n-1} is 0, so it is isomorphic to the graded-free R-module $R\langle E^n\rangle$. In particular, $(\dagger)$ is degreewise split by 5.2.2.

Let A be an acyclic R-complex. For every $n \geqslant 0$ the complex $\operatorname{Hom}_R(R\langle E^n\rangle, A)$ is acyclic; cf. 3.1.27. It follows by induction that $\operatorname{Hom}_R(L^n, A)$ is acyclic for all $n \geqslant 0$. The morphisms in the tower $\{\operatorname{Hom}_R(L^n, A) \to \operatorname{Hom}_R(L^{n-1}, A)\}_{n\in\mathbb{Z}}$ are surjective because the sequence $(\dagger)$ is degreewise split, and the Hom functor preserves degreewise split exact sequences; see 2.3.13. Now it follows from 3.4.29, 3.5.10, and 3.5.16 that the complex

$$\operatorname{Hom}_R(L,A) \;=\; \operatorname{Hom}_R\big(\operatorname*{colim}_{n\in\mathbb{Z}} L^n, A\big) \;\cong\; \lim_{n\in\mathbb{Z}} \operatorname{Hom}_R(L^n, A)$$

is acyclic. □

The next result offers useful characterizations of semi-projective complexes. The lifting property in part (*iii*) can be interpreted in terms of the diagram (5.2.1.1).

5.2.10 Proposition. *For an R-complex P, the following conditions are equivalent.*

(*i*) *P is semi-projective.*
(*ii*) *The functor $\operatorname{Hom}_R(P,-)$ is exact and preserves quasi-isomorphisms.*
(*iii*) *For every chain map $\alpha\colon P \to N$ and for every surjective quasi-isomorphism $\beta\colon M \to N$ there exists a chain map $\gamma\colon P \to M$ such that $\alpha = \beta\gamma$ holds.*
(*iv*) *Every exact sequence $0 \to M' \to M \to P \to 0$ in $\mathcal{C}(R)$ with M' acyclic is split.*
(*v*) *For every morphism $\alpha\colon N \to P$ and for every surjective quasi-isomorphism $\beta\colon M \to P$ there exists a morphism $\gamma\colon N \to M$ such that $\alpha = \beta\gamma$ holds.*
(*vi*) *P is a direct summand of a semi-free R-complex.*
(*vii*) *P is a complex of projective R-modules, and the functor $\operatorname{Hom}_R(P,-)$ preserves acyclicity of complexes.*

Proof. The implication $(ii) \Rightarrow (i)$ is evident.

$(i) \Rightarrow (iii)$: The morphism $\operatorname{Hom}_R(P,\beta)$ is a surjective quasi-isomorphism. In particular, it is surjective on cycles by 4.2.7. Thus, in view of 2.3.3 there exists a chain map $\gamma\colon P \to M$ such that $\alpha = \operatorname{Hom}_R(P,\beta)(\gamma) = \beta\gamma$.

$(iii) \Rightarrow (iv)$: By 4.2.6 the morphism $\beta\colon M \twoheadrightarrow P$ is a quasi-isomorphism, so there exists a chain map $\gamma\colon P \to M$ with $1^P = \beta\gamma$. As β is of degree 0, so is γ. That is, γ is a morphism in $\mathcal{C}(R)$, whence the sequence is split.

$(iv) \Rightarrow (v)$: By (iv) there is a morphism $\sigma\colon P \to M$ with $\beta\sigma = 1^P$. The desired morphism is thus $\gamma = \sigma\alpha$.

$(v) \Rightarrow (vi)$: By 5.1.7 there exists a surjective quasi-isomorphism $\beta\colon L \to P$ where L is a semi-free complex. By (v) there is a morphism $\gamma\colon P \to L$ with $1^P = \beta\gamma$, so P is a direct summand of L by 2.1.47.

$(vi) \Rightarrow (vii)$: Immediate from 5.2.2 and 5.2.9, as the Hom functor is additive.

$(vii) \Rightarrow (ii)$: The functor $\operatorname{Hom}_R(P,-)$ is exact by 5.2.2. For a quasi-isomorphism β, the complex $\operatorname{Cone}\beta$ is acyclic by 4.2.16. Hence the complex $\operatorname{Hom}_R(P,\operatorname{Cone}\beta) \cong \operatorname{Cone}\operatorname{Hom}_R(P,\beta)$ is acyclic; here the isomorphism follows from 4.1.16. Thus, the map $\operatorname{Hom}_R(P,\beta)$ is a quasi-isomorphism. □

5.2.11 Corollary. *A semi-free R-complex is semi-projective.*

Caveat. A semi-projective complex of free modules need not be semi-free; see E 5.2.4.

5.2.12 Corollary. *A graded R-module is graded-projective if and only if it is semi-projective as an R-complex.*

Proof. Let P be a graded R-module. If P is semi-projective as an R-complex, then each module P_v is projective, whence P is graded-projective.

If P is graded-projective, then it is a graded direct summand of a graded-free R-module; see 5.2.2. A graded-free R-module is semi-free as an R-complex by 5.1.2, and then P is semi-free as an R-complex by 5.2.10. □

5.2.13 Definition. A *semi-projective resolution* of an R-complex M is a quasi-isomorphism $P \to M$ of R-complexes where P is semi-projective.

Existence of semi-free resolutions implies the existence of semi-projective ones.

5.2.14 Theorem. *Every R-complex M has a semi-projective resolution $\pi\colon P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < \inf M^\natural$. Moreover, π can be chosen surjective.*

Proof. Combine 5.1.7 and 5.2.11. □

5.2.15 Theorem. *Every R-complex M has a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < \inf M$.*

Proof. Combine 5.1.12 and 5.2.11. □

Under the assumption that R is perfect, the semi-projective resolution in 5.2.15 can be chosen to be minimal, see B.60.

5.2.16 Theorem. *Assume that R is left Noetherian. Every R-complex M with* $\mathrm{H}(M)$ *bounded below and degreewise finitely generated has a semi-projective resolution* $P \xrightarrow{\simeq} M$ *with P degreewise finitely generated and $P_v = 0$ for all $v < \inf M$.*

Proof. Combine 5.1.14 and 5.2.11. □

Under the additional assumption that R is semi-perfect, the semi-projective resolution in 5.2.16 can be chosen to be minimal, see B.61.

Properties of Semi-Projective Complexes

The class of semi-projective complexes over a ring is closed under extensions, kernels of surjective morphisms, direct summands, and coproducts. Semi-projectivity is also preserved under base change.

5.2.17 Proposition. *Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of R-complexes. If P'' is semi-projective, then P' is semi-projective if and only if P is semi-projective.*

Proof. First note that since P'' is a complex of projective modules, it follows from 5.2.3 that P is a complex of projective modules if and only if P' is so. Next, let A be an acyclic R-complex. The sequence $0 \to P' \to P \to P'' \to 0$ is degreewise split by 5.2.2, and hence so is the induced sequence

$$0 \longrightarrow \operatorname{Hom}_R(P'', A) \longrightarrow \operatorname{Hom}_R(P, A) \longrightarrow \operatorname{Hom}_R(P', A) \longrightarrow 0\,;$$

see 2.3.13. As P'' is semi-projective, the complex $\operatorname{Hom}_R(P'', A)$ is acyclic by the equivalence of (*i*) and (*vi*) in 5.2.10. It follows from 2.5.6 that $\operatorname{Hom}_R(P, A)$ is acyclic if and only if $\operatorname{Hom}_R(P', A)$ is acyclic. Now 5.2.10 yields the desired conclusion. □

5.2.18 Proposition. *Let $\{P^u\}_{u\in U}$ be a family of R-complexes. The coproduct $\coprod_{u\in U} P^u$ is semi-projective if and only if each complex P^u is semi-projective.*

Proof. Let $\beta\colon M \to N$ be a surjective quasi-isomorphism. There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd}
\operatorname{Hom}_R\big(\coprod\limits_{u\in U} P^u, M\big) \arrow[r, "\operatorname{Hom}(\coprod_{u\in U} P^u{,}\beta)"] \arrow[d, "\cong"] & \operatorname{Hom}_R\big(\coprod\limits_{u\in U} P^u, N\big) \arrow[d, "\cong"'] \\
\prod\limits_{u\in U} \operatorname{Hom}_R(P^u, M) \arrow[r, "\prod_{u\in U}\operatorname{Hom}(P^u{,}\beta)"] & \prod\limits_{u\in U} \operatorname{Hom}_R(P^u, N)\,,
\end{tikzcd}$$

where the vertical maps are the canonical isomorphisms from 3.1.27. It follows that $\operatorname{Hom}_R(\coprod_{u\in U} P^u, \beta)$ is a surjective quasi-isomorphism if and only if each morphism $\operatorname{Hom}_R(P^u, \beta)$ is a surjective quasi-isomorphism. □

Also the next result can be interpreted in terms of the diagram (5.2.1.1).

5.2.19 Proposition. *Let P be a semi-projective R-complex, $\alpha\colon P \to N$ a chain map, and $\beta\colon M \to N$ a quasi-isomorphism. There exists a chain map $\gamma\colon P \to M$ such that $\alpha \sim \beta\gamma$. Moreover, γ is homotopic to any other chain map γ' with $\alpha \sim \beta\gamma'$.*

Proof. Recall from 2.3.3 the characterization of (null-homotopic) chain maps as (boundaries) cycles in Hom complexes. By 5.2.10 the induced morphism $\operatorname{Hom}_R(P,\beta)$ is a quasi-isomorphism, so there exists a $\gamma \in \mathrm{Z}(\operatorname{Hom}_R(P,M))$ such that

$$[\alpha] \;=\; \mathrm{H}(\operatorname{Hom}_R(P,\beta))([\gamma]) \;=\; [\beta\gamma]\;;$$

that is, $\alpha - \beta\gamma$ is in $\mathrm{B}(\operatorname{Hom}_R(P,N))$. Given another morphism γ' such that $\alpha \sim \beta\gamma'$, one has $[\alpha] = [\beta\gamma']$ and, therefore $0 = [\beta(\gamma-\gamma')] = \mathrm{H}(\operatorname{Hom}_R(P,\beta))([\gamma-\gamma'])$. It follows that the homology class $[\gamma-\gamma']$ is 0 as $\mathrm{H}(\operatorname{Hom}_R(P,\beta))$ is an isomorphism, so $\gamma-\gamma'$ is in $\mathrm{B}(\operatorname{Hom}_R(P,M))$. That is, γ and γ' are homotopic. □

Remark. Existence and uniqueness of lifts up to homotopy, as described in 5.2.19, is an important property of semi-projective complexes, but it does not characterize them. Complexes with this property are examined in exercises, starting with E 5.2.18.

5.2.20 Corollary. *Let P be a semi-projective R-complex and $\beta\colon M \to P$ a quasi-isomorphism. There exists a quasi-isomorphism $\gamma\colon P \to M$ with $1^P \sim \beta\gamma$.*

Proof. By 5.2.19 there is a chain map $\gamma\colon P \to M$ with $1^P \sim \beta\gamma$; comparison of degrees shows that γ is a morphism. Moreover, by 2.2.26 one has $1^{\mathrm{H}(P)} = \mathrm{H}(\beta)\,\mathrm{H}(\gamma)$, whence $\mathrm{H}(\gamma)$ is an isomorphism. □

Recall from 4.3.4 that every homotopy equivalence is a quasi-isomorphism. The next result is a partial converse.

5.2.21 Corollary. *A quasi-isomorphism of semi-projective R-complexes is a homotopy equivalence.*

Proof. Let $\beta\colon P' \to P$ be a quasi-isomorphism of semi-projective R-complexes. By 5.2.20 there are morphisms $\gamma\colon P \to P'$ and $\beta'\colon P' \to P$ such that $1^P \sim \beta\gamma$ and $1^{P'} \sim \gamma\beta'$ hold. It now follows from 4.3.3 that β is a homotopy equivalence. □

5.2.22 Proposition. *Let P be an S-complex and X a complex of R–S^{o}-bimodules. If P is semi-projective and X is semi-projective over R, then the R-complex $X \otimes_S P$ is semi-projective.*

Proof. Adjunction 4.4.12 yields a natural isomorphism,

$$\operatorname{Hom}_R(X \otimes_S P, -) \;\cong\; \operatorname{Hom}_S(P, \operatorname{Hom}_R(X,-))\;,$$

of functors from $\mathcal{C}(R)$ to $\mathcal{C}(\Bbbk)$. It follows from the assumptions on P and X that the functor $\operatorname{Hom}_S(P,\operatorname{Hom}_R(X,-))$ is exact and preserves quasi-isomorphisms. □

5.2.23 Corollary. *Let $R \to S$ be a ring homomorphism and P an R-complex.*

(a) *If P is semi-projective, then the S-complex $S \otimes_R P$ is semi-projective.*

(b) *If S is projective as an R-module, then a semi-projective S-complex is semi-projective over R.*

PROOF. For (b) apply 5.2.22 with $X = S$ viewed as an R–S^{o}-bimodule and note that $X \otimes_S -$ is the restriction of scalars functor $\mathcal{C}(S) \to \mathcal{C}(R)$. For (a) interchange the roles of R and S in 5.2.22 and apply it with $X = S$ viewed as an S–R^{o}-bimodule. □

THE CASE OF MODULES

Notice that 1.3.17 is a special case of 5.2.2. Further, since a module by 5.2.12 is projective if and only if it is semi-projective as a complex, one recovers 1.3.24 from 5.2.18 by specialization to modules.

5.2.24 Proposition. *Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of R-modules. If P'' is projective, then P' is projective if and only if P is projective.*

PROOF. The assertion follows by specialization of 5.2.3 to modules. □

5.2.25 Proposition. *Let P be an S-module and X an R–S^{o}-bimodule. If P is projective and X is projective over R, then the R-module $X \otimes_S P$ is projective.*

PROOF. The assertion follows, in view of 5.2.12, immediately from 5.2.22. □

The next result follows from 5.2.25 but is more easily recovered from 5.2.23.

5.2.26 Corollary. *Let $R \to S$ be a ring homomorphism and P an R-module.*

(a) *If P is projective, then the S-module $S \otimes_R P$ is projective.*
(b) *If S is projective as an R-module, then a projective S-module is projective over R.*

PROOF. The assertion follows, in view of 5.2.12, immediately from 5.2.23. □

Semi-projective resolutions of complexes subsume the classic notion of projective resolutions of modules.

5.2.27 Theorem. *Let M be an R-module. There is an exact sequence of R-modules,*

$$\cdots \longrightarrow P_v \longrightarrow P_{v-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0\,,$$

where each module P_v is projective.

PROOF. The assertion follows immediately from 5.1.16 and 1.3.18. □

5.2.28 Definition. Let M be an R-module. Together, the surjective homomorphism $P_0 \to M$ and the R-complex $\cdots \to P_v \to P_{v-1} \to \cdots \to P_0 \to 0$ in 5.2.27 is called a *projective resolution* of M.

5.2.29. Let M be an R-module. By 5.2.8 a projective resolution of M is a semi-projective resolution of M as an R-complex. Only a semi-projective resolution $P \xrightarrow{\simeq} M$ with P concentrated in non-negative degrees is a projective resolution.

REMARK. Let $P \xrightarrow{\simeq} M$ be a projective resolution of an R-module. It is standard to refer to the module $\operatorname{Coker}(P_{n+1} \to P_n) = \mathrm{C}_n(P)$ as an n^{th} *syzygy* of M. By Schanuel's lemma 8.1.12 it is "essentially" unique. In the case the complex P is minimal, the module $\mathrm{C}_n(P)$ may be referred to as the n^{th} syzygy of M.

EXERCISES

E 5.2.1 Show that a graded R-module is graded-projective if and only if it is a projective object in the category $\mathcal{M}_{\mathrm{gr}}(R)$.

E 5.2.2 Show that a graded R-module is graded-projective if and only if it is projective as an R-module.

E 5.2.3 Assume that R is left hereditary. Show that for every complex P of projective R-modules there is a quasi-isomorphism $P \xrightarrow{\simeq} \mathrm{H}(P)$. *Hint:* E 1.3.17.

E 5.2.4 The $\mathbb{Z}/6\mathbb{Z}$-complex $\cdots \longrightarrow \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \cdots$ is contractible; see E 4.3.11. Show that it is a semi-projective complex but not semi-free.

E 5.2.5 Let L be the $\mathbb{Z}/4\mathbb{Z}$-complex $\cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$ concentrated in non-negative degrees. Set $P = \coprod_{u<0} \Sigma^u L$, show that the complex $P_{\leqslant 0}$ is semi-projective, and compute its homology.

E 5.2.6 For an R-complex P, show that the following conditions are equivalent. (*i*) P is a projective object in the category $\mathcal{C}(R)$. (*ii*) P is a contractible complex of projective R-modules. (*iii*) P is semi-projective and acyclic. (*iv*) P is an acyclic complex of projective R-modules and $\mathrm{B}(P) = \mathrm{Z}(P)$ is a complex of projective R-modules. Conclude from 4.3.24 that the category $\mathcal{C}(R)$ has enough projectives.

E 5.2.7 Show that the Dold complex from 5.1.4 is acyclic but not contractible; conclude that it is not semi-projective.

E 5.2.8 Show that a complex P of projective R-modules is semi-projective if $\mathrm{Hom}_R(P, A)$ is acyclic for every acyclic R-complex A that is bounded above.

E 5.2.9 Assume that R is commutative and S an R-algebra that is faithfully projective as an R-module. Show that an R-complex P is semi-projective if (and only if) the S-complex $S \otimes_R P$ is semi-projective.

E 5.2.10 Show that the mapping cone of a morphism between semi-projective R-complexes is semi-projective.

E 5.2.11 Let P be a semi-projective R-complex. Show that for $v \leqslant \inf P$ the module $\mathrm{Z}_v(P)$ is projective. Conclude that if $\pi\colon P \xrightarrow{\simeq} M$ is a semi-projective resolution, then so is $\pi_{\supseteq v}$ for $v \leqslant \inf M$. Show that if M is a module, then $\pi_{\supseteq 0}$ yields a projective resolution of M.

E 5.2.12 Show that every complex over a semi-simple ring is semi-projective.

E 5.2.13 Show that a complex of projective modules over a left hereditary ring is semi-projective.

E 5.2.14 Let $0 \to P' \to P \to P'' \to 0$ be a degreewise split exact sequence of R-complexes. Show that if two of the complexes P', P, and P'' are semi-projective, then so is the third.

E 5.2.15 Show that the following conditions are equivalent for an R-complex P. (*i*) P is semi-projective. (*ii*) The complex $P_{\leqslant n}$ is semi-projective for every $n \in \mathbb{Z}$. (*iii*) P is a complex of projective R-modules and $P_{\leqslant n}$ is semi-projective for some $n \in \mathbb{Z}$.

E 5.2.16 Let L be a bounded complex of finitely generated projective S^{o}-modules and P a complex of R–S^{o}-bimodules. Show that if P is a semi-projective over R, then the R-complex $\mathrm{Hom}_{S^{\mathrm{o}}}(L, P)$ is semi-projective.

E 5.2.17 Let L be a bounded above complex of finitely generated projective S^{o}-modules and P a bounded below complex of R–S^{o}-bimodules that are projective over R. Show that the R-complex $\mathrm{Hom}_{S^{\mathrm{o}}}(L, P)$ is semi-projective.

E 5.2.18 Show that the following conditions are equivalent for an R-complex X. (*i*) For every chain map $\alpha\colon X \to N$ and every quasi-isomorphism $\beta\colon M \to N$ there exists a chain map $\gamma\colon X \to M$, unique up to homotopy, such that $\alpha \sim \beta\gamma$. (*ii*) For every quasi-isomorphism β the induced morphism $\mathrm{Hom}_R(X, \beta)$ is a quasi-isomorphism. (*iii*) For every acyclic complex A, the complex $\mathrm{Hom}_R(X, A)$ is acyclic.

A complex with these properties is called *K-projective;* Avramov, Foxby, and Halperin [25] use the term ‘homotopically projective’.

E 5.2.19 Show that a quasi-isomorphism of K-projective R-complexes is a homotopy equivalence.

E 5.2.20 Let K be an acyclic K-projective R-complex. Show that $\operatorname{Hom}_R(K, M)$ is acyclic for every R-complex M.

E 5.2.21 Show that an R-complex is semi-projective if and only if it is a complex of projective modules and K-projective. Show that a K-projective complex need not be semi-projective.

E 5.2.22 Show that a graded R-module is graded-projective if and only if it is K-projective as an R-complex.

E 5.2.23 Consider a homotopy equivalence $M \to N$ of R-complexes. Show that M is K-projective if and only if N is K-projective. Is the same true for semi-projectivity?

E 5.2.24 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{C}(R)$. (a) Show that for any two semi-projective resolutions $\pi\colon P \xrightarrow{\simeq} M$ and $\lambda\colon L \xrightarrow{\simeq} N$ there is a morphism $\tilde{\alpha}\colon P \to L$ such that the following diagram is commutative up to homotopy,

$$\begin{tikzcd} P \arrow[r, "\pi", "\simeq"'] \arrow[d, dotted, "\tilde{\alpha}"'] & M \arrow[d, "\alpha"] \\ L \arrow[r, "\lambda", "\simeq"'] & N\,. \end{tikzcd}$$

(b) Show that if λ is surjective, then $\tilde{\alpha}$ can be chosen so that the diagram is commutative.

E 5.2.25 Let x be a central element in R and $P \xrightarrow{\simeq} M$ a semi-projective resolution in $\mathcal{C}(R)$. Show that if the homothety x^M is null-homotopic, then x^P is null-homotopic.

E 5.2.26 Let $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ be an exact sequence of R-complexes. Show that there is a commutative diagram in $\mathcal{C}(R)$ in which the columns are semi-projective resolutions,

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r, "\widetilde{\alpha}'"] \arrow[d, "\pi'", "\simeq"'] & P \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi", "\simeq"'] & P'' \arrow[r] \arrow[d, "\pi''", "\simeq"'] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\alpha'"] & M \arrow[r, "\alpha"] & M'' \arrow[r] & 0\,. \end{tikzcd}$$

This is known as the Horseshoe Lemma (for semi-projective resolutions).

E 5.2.27 Let P be a complex of projective R-modules. Show that for every $n \geqslant \sup P$ the truncated complex $P_{\geqslant n}$ yields a projective resolution of the module $\mathrm{C}_n(P)$.

5.3 Semi-Injectivity

Synopsis. Character complex; graded-injective module; complex of injective modules; semi-injective complex; semi-injective resolution; lifting property; injective resolution of module.

Semi-injectivity is dual to semi-projectivity and defined in terms of the functor $\operatorname{Hom}_R(-, I)$ from $\mathcal{C}(R)^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$. The goal of this section is show that every complex has a semi-injective resolution.

Character Complex

Semi-injective resolutions come from semi-free resolutions via character complexes. Recall from 1.3.37 that $\mathbb{E}$ denotes a faithfully injective $\Bbbk$-module.

5.3.1 Definition. Let M be an R-complex. The R^{o}-complex $\operatorname{Hom}_\Bbbk(M, \mathbb{E})$ is called the *character complex* of M. The graded R-module $\operatorname{Hom}_\Bbbk(M^\natural, \mathbb{E}) = \operatorname{Hom}_\Bbbk(M, \mathbb{E})^\natural$ is called the *character module* of the graded R-module $M^\natural$.

5.3.2 Lemma. *If L is a semi-free R^{o}-complex, then* $\operatorname{Hom}_{\Bbbk}(L,\mathbb{E})$ *is a complex of injective R-modules, and the functor* $\operatorname{Hom}_R(-,\operatorname{Hom}_{\Bbbk}(L,\mathbb{E}))$ *preserves acyclicity of complexes.*

Proof. By 1.3.50 each module $\operatorname{Hom}_{\Bbbk}(L,\mathbb{E})_v = \operatorname{Hom}_{\Bbbk}(L_{-v},\mathbb{E})$ is an injective R-module. Let A be an acyclic R-complex; adjunction 4.4.12 and commutativity 4.4.4 yield isomorphisms

$$\operatorname{Hom}_R(A,\operatorname{Hom}_{\Bbbk}(L,\mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(L\otimes_R A,\mathbb{E}) \cong \operatorname{Hom}_{R^{\mathrm{o}}}(L,\operatorname{Hom}_{\Bbbk}(A,\mathbb{E}))\,,$$

and $\operatorname{Hom}_{R^{\mathrm{o}}}(L,\operatorname{Hom}_{\Bbbk}(A,\mathbb{E}))$ is acyclic by 5.2.9 and exactness of $\operatorname{Hom}_{\Bbbk}(-,\mathbb{E})$. $\square$

5.3.3 Construction. Let M be an R-complex and choose by 5.1.7 a semi-free resolution $\pi\colon L \xrightarrow{\simeq} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})$ with $L_v = 0$ for all $v < \inf \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})^{\natural}$. Precompose the induced morphism $\operatorname{Hom}_{\Bbbk}(\pi,\mathbb{E})\colon \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(M,\mathbb{E}),\mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(L,\mathbb{E})$ with biduality $\delta^M_{\mathbb{E}}\colon M \to \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(M,\mathbb{E}),\mathbb{E})$ to get a morphism of R-complexes

$$\varepsilon^M = \operatorname{Hom}_{\Bbbk}(\pi,\mathbb{E})\delta^M_{\mathbb{E}} : M \longrightarrow E\,,$$

where E is the character complex $\operatorname{Hom}_{\Bbbk}(L,\mathbb{E})$.

5.3.4 Proposition. *Let M be an R-complex. The morphisms and complexes constructed in* 5.3.3 *have the following properties.*

(a) *E is a complex of injective R-modules with $E_v = 0$ for $v > \sup M^{\natural}$, and the functor* $\operatorname{Hom}_R(-,E)$ *preserves acyclicity of complexes.*
(b) *The morphism* $\operatorname{H}(\varepsilon^M)$ *is injective.*
(c) *The morphism π can be chosen such that ε^M is injective.*

Proof. (a): By 2.5.7(b) one has $\inf \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})^{\natural} = -\sup M^{\natural}$ and, therefore, $E_v = \operatorname{Hom}_{\Bbbk}(L_{-v},\mathbb{E}) = 0$ for all $v > \sup M^{\natural}$. The other assertions follow from 5.3.2.

(b): One has $\operatorname{H}(\varepsilon^M) = \operatorname{H}(\operatorname{Hom}_{\Bbbk}(\pi,\mathbb{E}))\operatorname{H}(\delta^M_{\mathbb{E}})$, and the map $\operatorname{H}(\operatorname{Hom}_{\Bbbk}(\pi,\mathbb{E}))$ is an isomorphism by 4.2.14. It follows from 2.2.19 and 4.5.1 that the map $\operatorname{H}(\delta^M_{\mathbb{E}})$ is biduality $\delta^{\operatorname{H}(M)}_{\mathbb{E}}$, which is injective by 4.5.3. Thus $\operatorname{H}(\varepsilon^M)$ is injective.

(c): By 5.1.7 one can choose π surjective, and then it follows by exactness of $\operatorname{Hom}_{\Bbbk}(-,\mathbb{E})$ that the morphism $\operatorname{Hom}_{\Bbbk}(\pi,\mathbb{E})$ is injective. By 4.5.3 the morphism $\delta^M_{\mathbb{E}}$ is injective, and hence so is the composite ε^M. $\square$

Complexes of Injective Modules

5.3.5. Lifting properties are also central to this section; key results can be interpreted in terms of the diagram

(5.3.5.1)
$$\begin{tikzcd} K \arrow[r] \arrow[d] & M \arrow[dl, dotted] \\ I & \end{tikzcd}$$

where the solid arrows represent given maps of certain sorts, and a lifting property of I ensures the existence of a dotted map of a specific sort such that the diagram is commutative, or commutative up to homotopy.

Part (*iii*) below can be interpreted in terms of the diagram (5.3.5.1).

5.3.6 Proposition. *For an R-complex I, the following conditions are equivalent.*

(*i*) *Each R-module I_v is injective.*

(*ii*) *The functor $\operatorname{Hom}_R(-, I)$ is exact.*

(*iii*) *For every homomorphism $\alpha\colon K \to I$ and for every injective homomorphism $\beta\colon K \to M$, there exists a homomorphism $\gamma\colon M \to I$ such that $\gamma\beta = \alpha$ holds.*

(*iv*) *Every exact sequence $0 \to I \to M \to M'' \to 0$ in $\mathcal{C}(R)$ is degreewise split.*

(*v*) *The graded module $I^\natural$ is a graded direct summand of the character module $\operatorname{Hom}_\Bbbk(L, \mathbb{E})$ of a graded-free R°-module L.*

Proof. The implication (*i*) $\Rightarrow$ (*ii*) is immediate from 2.3.20.

(*ii*) $\Rightarrow$ (*iii*): The homomorphism β yields a morphism $\beta(\varsigma^K_{|\beta|})^{-1}\colon \Sigma^{|\beta|}K^\natural \to M^\natural$ and by exactness of the functor $\operatorname{Hom}_R(-, I)$ there exists a homomorphism $\gamma \in \operatorname{Hom}_R(M^\natural, I)$ with $\gamma\beta(\varsigma^K_{|\beta|})^{-1} = \alpha(\varsigma^K_{|\beta|})^{-1}$ and hence $\gamma\beta = \alpha$.

(*iii*) $\Rightarrow$ (*iv*): Let β denote the morphism $I \rightarrowtail M$. By (*iii*) there exists a homomorphism $\gamma\colon M \to I$ such that $1^I = \operatorname{Hom}_R(\beta, I)(\gamma) = \gamma\beta$ holds. As β is a morphism, also the degree of γ must be 0; that is, γ is a morphism of the underlying graded modules. Hence, the sequence $0 \to I \to M \to M'' \to 0$ is degreewise split.

(*iv*) $\Rightarrow$ (*v*): Choose by 5.3.4 an injective morphism $\varepsilon\colon I \to E$, where E is the character complex $\operatorname{Hom}_\Bbbk(L, \mathbb{E})$ of a semi-free R°-complex L. Apply (*iv*) to the exact sequence $0 \to I \to E \to \operatorname{Coker}\varepsilon \to 0$ in $\mathcal{C}(R)$. It follows that $I^\natural$ is a graded direct summand of the graded module $E^\natural = \operatorname{Hom}_\Bbbk(L^\natural, \mathbb{E})$, and $L^\natural$ is a graded-free R°-module by 2.5.27.

(*v*) $\Rightarrow$ (*i*): The character module of a free R°-module is an injective R-module by 1.3.50. A direct summand of an injective module is injective by additivity of the Hom functor. Thus, each module I_v is an injective R-module. □

Caveat. The complexes described in 5.3.6 are not the injective objects in the category $\mathcal{C}(R)$; see E 5.3.1 and E 5.3.4.

5.3.7 Corollary. *Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence of R-complexes. If I' is a complex of injective modules, then I is a complex of injective modules if and only if I'' is a complex of injective modules.*

Proof. If I' is a complex of injective modules, then $0 \to I' \to I \to I'' \to 0$ is degreewise split, and the assertion follows from 1.3.27. □

5.3.8 Definition. A graded R-module I is called *graded-injective* if the R-complex I satisfies the conditions in 5.3.6.

EXISTENCE OF SEMI-INJECTIVE RESOLUTIONS

5.3.9 Definition. An R-complex I is called *semi-injective* if $\operatorname{Hom}_R(\alpha, I)$ is an surjective quasi-isomorphism for every injective quasi-isomorphism α in $\mathcal{C}(R)$.

REMARK. Another word for semi-injective is 'DG-injective'.

5.3.10. It follows from 2.3.16 that if an R-complex I is semi-injective, then so is $\Sigma^s I$ for every integer s.

5.3.11 Example. A contractible complex of injective R-modules is semi-injective by 5.3.6 and 4.3.29.

5.3.12 Example. Let I be a bounded above complex of injective R-modules and β an injective quasi-isomorphism. The morphism $\operatorname{Hom}_R(\beta, I)$ is surjective by 5.3.6, and it follows from 4.2.16, 4.1.17, and A.2 that it is a quasi-isomorphism; cf. 5.2.8. Thus I is semi-injective.

5.3.13 Definition. A *semi-injective resolution* of an R-complex M is a quasi-isomorphism $M \to I$ of R-complexes where I is semi-injective.

REMARK. In some texts a semi-injective resolution is called a semi-injective 'coresolution'.

The goal of this section is obtained by 5.3.19; it relies on the next construction.

5.3.14 Construction. Given an R-complex M, we proceed to construct a commutative diagram in $\mathcal{C}(R)$,

(5.3.14.1)
$$\begin{tikzcd}
 & & & M \arrow[llld, bend right, "\iota"'] \arrow[ld, "\iota^n"'] \arrow[d, "\iota^{n-1}"] \arrow[rrd, "\iota^0"] & & \\
I \arrow[r] & \cdots \arrow[r, two heads] & I^n \arrow[r, two heads, "\pi^n"] & I^{n-1} \arrow[r, two heads, "\pi^{n-1}"] & \cdots \arrow[r, two heads] & I^0 .
\end{tikzcd}$$

For $n = 0$ choose by 5.3.4 an injective morphism $\iota^0 \colon M \to I^0$, where I^0 is the character complex of a semi-free R^{o}-complex.

For $n \geqslant 1$ let a morphism $\iota^{n-1} \colon M \to I^{n-1}$ be given. Choose by 5.3.4 an injective morphism $\varepsilon^n \colon \operatorname{Coker} \mathrm{H}(\iota^{n-1}) \to E^n$, where E^n is the character complex of a semi-free R^{o}-complex. The induced morphism $\mathrm{Z}(I^{n-1}) \to E^n$ is zero on boundaries and on $\iota^{n-1}(\mathrm{Z}(M))$; see (5.3.14.2) below. It extends by 5.3.6 to a homomorphism $\delta^n \colon I^{n-1} \to E^n$ with $\mathrm{B}(I^{n-1}) + \iota^{n-1}(\mathrm{Z}(M))$ contained in $\operatorname{Ker} \delta^n \cap \mathrm{Z}(I^{n-1})$.

(5.3.14.2)
$$\begin{tikzcd}
\mathrm{Z}(I^{n-1}) \arrow[r, two heads] \arrow[d, tail] & \mathrm{H}(I^{n-1}) \arrow[r, two heads] & \operatorname{Coker} \mathrm{H}(\iota^{n-1}) \arrow[d, tail, "\varepsilon^n"] \\
I^{n-1} \arrow[rr, "\delta^n"] & & E^n .
\end{tikzcd}$$

Conversely, let z be a cycle in I^{n-1}. If z is in $\operatorname{Ker} \delta^n$, then the element $[z] + \operatorname{Im} \mathrm{H}(\iota^{n-1})$ in $\operatorname{Coker} \mathrm{H}(\iota^{n-1})$ is in the kernel of ε^n, whence the homology class $[z]$ is in the image of $\mathrm{H}(\iota^{n-1})$. Thus, there is an equality

$$\mathrm{B}(I^{n-1}) + \iota^{n-1}(\mathrm{Z}(M)) \;=\; \operatorname{Ker}\delta^n \cap \mathrm{Z}(I^{n-1})\,. \tag{5.3.14.3}$$

Consider δ^n as a degree -1 homomorphism: $I^{n-1} \to \Sigma^{-1}E^n$; cf. 2.2.5. Set

$$(I^n)^\natural \;=\; (I^{n-1})^\natural \oplus (\Sigma^{-1}E^n)^\natural \quad\text{and}\quad \partial^{I^n}(i+e) \;=\; \partial^{I^{n-1}}(i) + \delta^n(i)\,. \tag{5.3.14.4}$$

This defines an R-complex, as δ^n is zero on boundaries in I^{n-1}. Notice that the projection $\pi^n\colon I^n \twoheadrightarrow I^{n-1}$ is a morphism of complexes.

For each boundary $b \in \mathrm{B}(M)$ choose a preimage m_b. The assignment

$$b \;\longmapsto\; \delta^n\iota^{n-1}(m_b) \tag{5.3.14.5}$$

is independent of the choice of preimage. Indeed, if $\widetilde{m}_b$ is another preimage of b, then $m_b - \widetilde{m}_b$ is a cycle in M, and (5.3.14.3) yields the first equality in the next computation

$$0 \;=\; \delta^n\iota^{n-1}(m_b - \widetilde{m}_b) \;=\; \delta^n\iota^{n-1}(m_b) - \delta^n\iota^{n-1}(\widetilde{m}_b)\,.$$

Thus, (5.3.14.5) defines a (degree 0) homomorphism from $\mathrm{B}(M)$ to $\Sigma^{-1}E^n$. It extends by 5.3.6 to a homomorphism $\sigma^n\colon M \to \Sigma^{-1}E^n$, and there is an equality

$$\sigma^n\partial^M \;=\; \delta^n\iota^{n-1}\,. \tag{5.3.14.6}$$

Define a map $\iota^n\colon M \to I^n$ as follows:

$$\iota^n(m) \;=\; \iota^{n-1}(m) + \sigma^n(m)\,.$$

The next computation shows that it is a morphism of R-complexes; the penultimate equality uses (5.3.14.6).

$$\partial^{I^n}\iota^n \;=\; \partial^{I^n}(\iota^{n-1} + \sigma^n) \;=\; \partial^{I^{n-1}}\iota^{n-1} + \delta^n\iota^{n-1} \;=\; \iota^{n-1}\partial^M + \sigma^n\partial^M \;=\; \iota^n\partial^M\,.$$

For $n < 0$ set $I^n = 0$, $\iota^n = 0$, and $\pi^{n+1} = 0$, then the family $\{\pi^n\colon I^n \to I^{n-1}\}_{n\in\mathbb{Z}}$ is a tower in $\mathcal{C}(R)$, and $\iota^{n-1} = \pi^n\iota^n$ holds for all $n \in \mathbb{Z}$. Set $I = \lim_{n\in\mathbb{Z}} I^n$, by 3.5.4 there is a morphism of R-complexes $\iota\colon M \to I$, given by $m \mapsto (\iota^n(m))_{n\in\mathbb{Z}}$.

5.3.15 Proposition. *Let M be an R-complex. The complexes and morphisms constructed in* 5.3.14 *have the following properties.*

(a) *Each I^n is a complex of injective R-modules with $I^n_v = 0$ for $v > \sup M^\natural$.*
(b) *I is a complex of injective R-modules with $I_v = 0$ for all $v > \sup M^\natural$, and the functor $\operatorname{Hom}_R(-, I)$ preserves acyclicity of complexes.*
(c) *The morphism $\iota\colon M \to I$ is an injective quasi-isomorphism.*

Proof. Part (a) follows from 5.3.4 and (5.3.14.4).

(b): One has $I_v = 0$ for all $v > \sup M^\natural$ by part (a) and the definition 3.4.3 of limits. Let $0 \to K \to M \to N \to 0$ be an exact sequence in $\mathcal{C}(R)$. For every $n \geqslant 0$ there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, I^n) \longrightarrow \operatorname{Hom}_R(M, I^n) \longrightarrow \operatorname{Hom}_R(K, I^n) \longrightarrow 0\,;$$

this follows from (a) and 5.3.6. Because of the degreewise split exact sequences

$$(\star) \qquad 0 \longrightarrow (\Sigma^{-1}E^n)^\natural \longrightarrow I^n \xrightarrow{\pi^n} I^{n-1} \longrightarrow 0\,,$$

the morphisms in the tower $\{\operatorname{Hom}_R(N,\pi^n)\colon \operatorname{Hom}_R(N,I^n) \to \operatorname{Hom}_R(N,I^{n-1})\}_{n\in\mathbb{Z}}$ are surjective; see 2.3.12. It now follows from 3.5.17 that the lower row in the commutative diagram below is exact.

$$\begin{tikzcd}
\operatorname{Hom}_R(M,I) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}_R(K,I) \arrow[d, "\cong"'] & \\
\lim_{n\in\mathbb{Z}} \operatorname{Hom}_R(M,I^n) \arrow[r] & \lim_{n\in\mathbb{Z}} \operatorname{Hom}_R(K,I^n) \arrow[r] & 0
\end{tikzcd}$$

The vertical maps are the isomorphisms from 3.4.23; the diagram shows that the functor $\operatorname{Hom}_R(-,I)$ is exact, whence I is a complex of injective modules by 5.3.6.

Let A be an acyclic R-complex. As above the morphisms in the induced tower $\{\operatorname{Hom}_R(A,\pi^n)\colon \operatorname{Hom}_R(A,I^n) \to \operatorname{Hom}_R(A,I^{n-1})\}_{n\in\mathbb{Z}}$ are surjective. By 5.3.4, the functors $\operatorname{Hom}_R(-,(E^n)^\natural)$ preserve acyclicity, so by induction it follows from $(\star)$ and 2.5.6 that the functors $\operatorname{Hom}_R(-,I^n)$ preserve acyclicity; in particular $\operatorname{Hom}_R(A,I^n)$ is acyclic for every n. By 3.4.23 and 3.5.16 the complex

$$\operatorname{Hom}_R(A,I) \;=\; \operatorname{Hom}_R\big(A,\lim_{n\in\mathbb{Z}} I^n\big) \;\cong\; \lim_{n\in\mathbb{Z}} \operatorname{Hom}_R(A,I^n)$$

is acyclic.

(c): As ι^0 is injective, commutativity of (5.3.14.1) shows that ι is injective as well. By 5.3.4 the morphism $\mathrm{H}(\iota^0)$ is injective, and the commutative diagram

$$(\diamond) \qquad \begin{tikzcd}
& \mathrm{H}(M) \arrow[dl, "\mathrm{H}(\iota)"'] \arrow[d, "\mathrm{H}(\iota^n)"] \arrow[dr, "\mathrm{H}(\iota^0)"] & \\
\mathrm{H}(I) \arrow[r] & \mathrm{H}(I^n) \arrow[r] & \mathrm{H}(I^0)\,,
\end{tikzcd}$$

which is induced from (5.3.14.1), shows that $\mathrm{H}(\iota)$ is injective. To see that it is surjective, let $z = (z^n)_{n\in\mathbb{Z}}$ be a cycle in I; the goal is to show that there exist elements $m \in \mathrm{Z}(M)$ and $i = (i^n)_{n\in\mathbb{Z}}$ in I with $z = \partial^I(i) + \iota(m)$. From (5.3.14.4) one gets

$$(\flat) \quad 0 \;=\; \partial^I(z) \;=\; (\partial^{I^0}(z^0),\dots,\partial^{I^{n-1}}(z^{n-1}) + \delta^n(z^{n-1}), \partial^{I^n}(z^n) + \delta^{n+1}(z^n),\dots)\,.$$

It follows for each $n \geqslant 1$ that the element z^{n-1} is a cycle in I^{n-1} with $\delta^n(z^{n-1}) = 0$, whence z^{n-1} belongs $\mathrm{B}(I^{n-1}) + \iota^{n-1}(\mathrm{Z}(M))$ by (5.3.14.3).

Choose elements j^2 in I^2 and $m \in \mathrm{Z}(M)$ such that $z^2 = \partial^{I^2}(j^2)+\iota^2(m)$ holds. The sequence $(i^n)_{n\in\mathbb{Z}}$ is constructed by induction. Set $i^1 = \pi^2(j^2)$ and $i^0 = \pi^1(i^1)$, then there are equalities $z^1 = \pi^2(z^2) = \partial^{I^1}(i^1) + \iota^1(m)$ and $z^0 = \pi^1(z^1) = \partial^{I^0}(i^0) + \iota^0(m)$. Set $i^n = 0$ for $n < 0$. Fix $n \geqslant 2$ and assume that elements $i^u \in I^u$ for $u < n$ and $j^n \in I^n$ have been constructed, such that one has

$$\begin{aligned}
z^n \;&=\; \partial^{I^n}(j^n) + \iota^n(m) \quad &\text{and}\quad z^u \;&=\; \partial^{I^u}(i^u) + \iota^u(m) \quad &\text{for } u < n\,;\\
\pi^n(j^n) \;&=\; i^{n-1} &\text{and}\quad \pi^u(i^u) \;&=\; i^{u-1} &\text{for } u < n\,.
\end{aligned}$$

Choose j' in I^{n+1} and $m' \in \mathrm{Z}(M)$ with $z^{n+1} = \partial^{I^{n+1}}(j') + \iota^{n+1}(m')$. The equalities

$$\partial^{I^n}(j^n) + \iota^n(m) \;=\; z^n \;=\; \pi^{n+1}(z^{n+1}) \;=\; \partial^{I^n}(\pi^{n+1}(j')) + \iota^n(m')$$

show that $\iota^n(m' - m)$ is a boundary in I^n. It follows from commutativity of the diagram ($\diamond$) that $\mathrm{H}(\iota^n)$ is injective, so $m' - m$ is in $\mathrm{B}(M)$ and, therefore, $\iota^{n+1}(m' - m)$ is in $\mathrm{B}(I^{n+1})$. Thus, there exists $j'' \in I^{n+1}$ with $\partial^{I^{n+1}}(j'') = \partial^{I^{n+1}}(j') + \iota^{n+1}(m' - m)$ and, therefore, $z^{n+1} = \partial^{I^{n+1}}(j'') + \iota^{n+1}(m)$. The equalities

$$\partial^{I^n}(j^n) + \iota^n(m) \;=\; z^n \;=\; \pi^{n+1}(z^{n+1}) \;=\; \partial^{I^n}(\pi^{n+1}(j'')) + \iota^n(m)$$

show that $j^n - \pi^{n+1}(j'')$ is a cycle in I^n and, therefore, $\pi^n(j^n - \pi^{n+1}(j''))$ is a cycle in I^{n-1}. Now (5.3.14.4) yields $\delta^n(\pi^n(j^n - \pi^{n+1}(j''))) = 0$, and it follows from (5.3.14.3) that there are elements $i' \in I^{n-1}$ and $c \in \mathrm{Z}(M)$ with

$$(\dagger) \qquad \pi^n(j^n - \pi^{n+1}(j'')) \;=\; \partial^{I^{n-1}}(i') + \iota^{n-1}(c)\,.$$

Choose $i'' \in I^{n+1}$ with $\pi^n\pi^{n+1}(i'') = i'$ and set $j^{n+1} = j'' + \partial^{I^{n+1}}(i'') + \iota^{n+1}(c)$. As $\partial^{I^{n+1}}(i'') + \iota^{n+1}(c)$ is a cycle in I^{n+1}, the equality

$$(\ddagger) \qquad z^{n+1} \;=\; \partial^{I^{n+1}}(j^{n+1}) + \iota^{n+1}(m)$$

holds. Set $i^n = \pi^{n+1}(j^{n+1})$; now ($\ddagger$) yields $z^n = \pi^{n+1}(z^{n+1}) = \partial^{I^n}(i^n) + \iota^n(m)$, and the third equality below follows from ($\dagger$),

$$\begin{aligned} \pi^n(i^n) &= \pi^n\pi^{n+1}(j^{n+1}) \\ &= \pi^n\pi^{n+1}(j'') + \partial^{I^{n-1}}\pi^n\pi^{n+1}(i'') + \iota^{n-1}(c) \\ &= \pi^n(j^n) \\ &= i^{n-1}\,. \end{aligned}$$

Thus, for $u < n+1$ one has

$$(\|) \qquad z^u \;=\; \partial^{I^u}(i^u) + \iota^u(m) \qquad \text{and} \qquad i^{u-1} \;=\; \pi^u(i^u)\,.$$

From ($\ddagger$) and ($\|$) it now follows that the desired element $i = (i^n)_{n\in\mathbb{Z}}$ in I with $z = \iota(m) + \partial^I(i)$ exists. ◻

The next result offers useful characterizations of semi-injective complexes. The lifting property in part (*iii*) can be interpreted in terms of the diagram (5.3.5.1).

5.3.16 Proposition. *For an R-complex I, the following conditions are equivalent.*

(*i*) *I is semi-injective.*
(*ii*) *The functor* $\mathrm{Hom}_R(-, I)$ *is exact and preserves quasi-isomorphisms.*
(*iii*) *For every chain map $\alpha\colon K \to I$ and for every injective quasi-isomorphism $\beta\colon K \to M$ there exists a chain map $\gamma\colon M \to I$ such that $\gamma\beta = \alpha$ holds.*
(*iv*) *Every exact sequence $0 \to I \to M \to M'' \to 0$ in $\mathcal{C}(R)$ with M'' acyclic is split.*
(*v*) *For every morphism $\alpha\colon I \to K$ and for every injective quasi-isomorphism $\beta\colon I \to M$ there exists a morphism $\gamma\colon M \to K$ such that $\gamma\beta = \alpha$ holds.*

(*vi*) *I is a complex of injective R-modules, and the functor* $\operatorname{Hom}_R(-,I)$ *preserves acyclicity of complexes.*

Proof. The implication (*ii*) $\Rightarrow$ (*i*) is evident.

(*i*) $\Rightarrow$ (*iii*): The morphism $\operatorname{Hom}_R(\beta,I)$ is a surjective quasi-isomorphism. In particular, it is surjective on cycles; see 4.2.7. Thus, in view of 2.3.3 there exists a chain map $\gamma\colon M \to I$ such that $\alpha = \operatorname{Hom}_R(\beta,I)(\gamma) = \gamma\beta$ holds.

(*iii*) $\Rightarrow$ (*iv*): By 4.2.6 the morphism $\beta\colon I \rightarrowtail M$ is a quasi-isomorphism, so there exists a chain map $\gamma\colon M \to I$ with $\gamma\beta = 1^I$. As β is of degree 0, so is γ. That is, γ is a morphism in $\mathcal{C}(R)$, whence the sequence is split.

(*iv*) $\Rightarrow$ (*v*): By (*iv*) there is a morphism $\varrho\colon M \to I$ with $\varrho\beta = 1^I$. The desired morphism is thus $\gamma = \alpha\varrho$.

(*v*) $\Rightarrow$ (*vi*): Chose by 5.3.15 an injective quasi-isomorphism $\beta\colon I \to I'$, where I' is a complex of injective modules such that the functor $\operatorname{Hom}_R(-,I')$ preserves acyclicity of complexes. By (*v*) there is a morphism $\gamma\colon I' \to I$ with $\gamma\beta = 1^I$. Thus I is a direct summand of I', see 2.1.47, and by additivity of the Hom functor, I is a complex of injective modules and $\operatorname{Hom}_R(-,I)$ preserves acyclicity of complexes.

(*vi*) $\Rightarrow$ (*ii*): The functor $\operatorname{Hom}_R(-,I)$ is exact by 5.3.6. For a quasi-isomorphism α, the complex $\operatorname{Cone}\alpha$ is acyclic by 4.2.16, and hence so is $\operatorname{Hom}_R(\operatorname{Cone}\alpha, I)$. By 4.1.17 the latter complex is isomorphic to $\Sigma\operatorname{Cone}\operatorname{Hom}_R(\alpha,I)$, and it follows that $\operatorname{Hom}_R(\alpha,I)$ is a quasi-isomorphism. □

5.3.17 Corollary. *Let P be an R^{o}-complex. If P is semi-projective, then the R-complex* $\operatorname{Hom}_{\Bbbk}(P,\mathbb{E})$ *is semi-injective.*

Proof. It follows from 1.3.48 that $\operatorname{Hom}_{\Bbbk}(P,\mathbb{E})$ is a complex of injective R-modules. By adjunction 4.4.12 and commutativity 4.4.4 there are natural isomorphisms

$$\operatorname{Hom}_R(-,\operatorname{Hom}_{\Bbbk}(P,\mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(P\otimes_R -,\mathbb{E}) \cong \operatorname{Hom}_{R^{\mathrm{o}}}(P,\operatorname{Hom}_{\Bbbk}(-,\mathbb{E}))$$

of functors from $\mathcal{C}(R)^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$. By assumption, $\operatorname{Hom}_{R^{\mathrm{o}}}(P,\operatorname{Hom}_{\Bbbk}(-,\mathbb{E}))$ preserves acyclicity of complexes. Thus, $\operatorname{Hom}_{\Bbbk}(P,\mathbb{E})$ is semi-injective. □

5.3.18 Corollary. *A graded R-module is graded-injective if and only if it is semi-injective as an R-complex.*

Proof. Let I be a graded R-module. If I is semi-injective as an R-complex, then each module I_v is injective and hence I is graded-injective.

If I is a graded-injective R-module, then by 5.3.6 it is a direct summand of the character module of a graded-free R^{o}-module. By 5.1.2 and 5.2.11 a graded-free module is semi-projective. Thus 5.3.17 shows that I is a direct summand of a semi-injective R-complex and hence semi-injective by additivity of the Hom functor. □

5.3.19 Theorem. *Every R-complex M has a semi-injective resolution $\iota\colon M \xrightarrow{\simeq} I$ with $I_v = 0$ for all $v > \sup M^{\natural}$. Moreover, ι can be chosen injective.*

Proof. Apply 5.3.15 to get an injective quasi-isomorphism $\iota\colon M \to I$. The complex I has $I_v = 0$ for $v > \sup M^{\natural}$, and it is semi-injective by 5.3.15 and 5.3.16. □

PROPERTIES OF SEMI-INJECTIVE COMPLEXES

The class of semi-injective complexes over a ring is closed under extensions, cokernels of injective morphisms, direct summands, and products. Semi-injectivity is also preserved under cobase change, but we postpone that to 5.4.26.

5.3.20 Proposition. *Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence of R-complexes. If I' is semi-injective, then I is semi-injective if and only if I'' is semi-injective.*

PROOF. First note that since I' is a complex of injective modules, it follows from 5.3.7 that I is a complex of injective modules if and only if I'' is so. Next, let A be an acyclic R-complex. The sequence $0 \to I' \to I \to I'' \to 0$ is degreewise split by 5.3.6, so by applying $\operatorname{Hom}_R(A,-)$ one obtains an exact sequence; see 2.3.12. As I' is semi-injective, the complex $\operatorname{Hom}_R(A, I')$ is acyclic by the equivalence of (*i*) and (*v*) in 5.3.16. It follows from 2.5.6 that $\operatorname{Hom}_R(A, I)$ is acyclic if and only if $\operatorname{Hom}_R(A, I'')$ is acyclic. Now 5.3.16 yields the desired conclusion. □

5.3.21 Proposition. *Let $\{I^u\}_{u\in U}$ be a family of R-complexes. The product $\prod_{u\in U} I^u$ is semi-injective if and only if each complex I^u is semi-injective.*

PROOF. Let $\beta\colon K \to M$ be an injective quasi-isomorphism. There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd}
\operatorname{Hom}_R\big(M, \prod_{u\in U} I^u\big) \arrow[r, "\operatorname{Hom}(\beta,\prod_{u\in U} I^u)"] \arrow[d, "\cong"] & \operatorname{Hom}_R\big(K, \prod_{u\in U} I^u\big) \arrow[d, "\cong"'] \\
\prod_{u\in U} \operatorname{Hom}_R(M, I^u) \arrow[r, "\prod_{u\in U} \operatorname{Hom}(\beta, I^u)"] & \prod_{u\in U} \operatorname{Hom}_R(M, I^u)\;,
\end{tikzcd}$$

where the vertical maps are the canonical isomorphisms from 3.1.27. It follows that $\operatorname{Hom}_R(\beta, \prod_{u\in U} I^u)$ is a surjective quasi-isomorphism if and only if each morphism $\operatorname{Hom}_R(\beta, I^u)$ is a surjective quasi-isomorphism. □

Also the next result can be interpreted in terms of the diagram (5.3.5.1).

5.3.22 Proposition. *Let I be a semi-injective R-complex, $\alpha\colon K \to I$ a chain map, and $\beta\colon K \to M$ a quasi-isomorphism. There exists a chain map $\gamma\colon M \to I$ such that $\gamma\beta \sim \alpha$. Moreover, γ is homotopic to any other chain map γ' with $\gamma'\beta \sim \alpha$.*

PROOF. Recall from 2.3.3 the characterization of (null-homotopic) chain maps as (boundaries) cycles in Hom complexes. By 5.3.16 the induced morphism $\operatorname{Hom}_R(\beta, I)$ is a quasi-isomorphism, so there exists a $\gamma \in \mathrm{Z}(\operatorname{Hom}_R(M, I))$ such that

$$[\alpha] = \mathrm{H}(\operatorname{Hom}_R(\beta, I))([\gamma]) = [\gamma\beta]\;;$$

that is, $\alpha - \gamma\beta$ is in $\mathrm{B}(\operatorname{Hom}_R(K, I))$. Given another morphism γ' such that $\gamma'\beta \sim \alpha$, one has $[\alpha] = [\gamma'\beta]$ and, therefore $0 = [(\gamma - \gamma')\beta] = \mathrm{H}(\operatorname{Hom}_R(\beta, I))([\gamma - \gamma'])$. It follows that the homology class $[\gamma - \gamma']$ is 0 as $\mathrm{H}(\mathrm{Hom}_R(\beta, I))$ is an isomorphism, so $\gamma - \gamma'$ is in $\mathrm{B}(\mathrm{Hom}_R(M, I))$. That is, γ and γ' are homotopic. □

REMARK. Existence and uniqueness of lifts up to homotopy, as described in 5.3.22, is an important property of semi-injective complexes, but it does not characterize them. Complexes with this property are examined in exercises, starting with E 5.3.19.

5.3.23 Corollary. *Let I be a semi-injective R-complex and $\beta\colon I \to M$ a quasi-isomorphism. There exists a quasi-isomorphism $\gamma\colon M \to I$ such that $\gamma\beta \sim 1^I$.*

PROOF. By 5.3.22 there exists a chain map $\gamma\colon M \to I$ with $\gamma\beta \sim 1^I$; comparison of degrees shows that γ is a morphism. Moreover, by 2.2.26 one has $\mathrm{H}(\gamma)\,\mathrm{H}(\beta) = 1^{\mathrm{H}(I)}$, whence $\mathrm{H}(\gamma)$ is an isomorphism. □

Recall from 4.3.4 that every homotopy equivalence is a quasi-isomorphism. The next result is a partial converse and akin to 5.2.21.

5.3.24 Corollary. *A quasi-isomorphism of semi-injective R-complexes is a homotopy equivalence.*

PROOF. Let $\beta\colon I \to I'$ is a quasi-isomorphism of semi-injective R-complexes. By 5.3.23 there are morphisms $\gamma\colon I' \to I$ and $\beta'\colon I \to I'$ such that $\gamma\beta \sim 1^I$ and $\beta'\gamma \sim 1^{I'}$ hold. It now follows from 4.3.3 that β is a homotopy equivalence. □

5.3.25 Proposition. *Let P be an R-complex and X a complex of R–S^{o}-bimodules. If P is semi-projective and X is semi-injective over S^{o}, then $\mathrm{Hom}_R(P, X)$ is a semi-injective S^{o}-complex.*

PROOF. By swap 4.4.10 there is a natural isomorphism

$$\mathrm{Hom}_{S^{\mathrm{o}}}(-, \mathrm{Hom}_R(P, X)) \;\cong\; \mathrm{Hom}_R(P, \mathrm{Hom}_{S^{\mathrm{o}}}(-, X))$$

of functors from $\mathcal{C}(S^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$. It follows from the assumptions on P and X that the functor $\mathrm{Hom}_R(P, \mathrm{Hom}_{S^{\mathrm{o}}}(-, X))$ is exact and preserves quasi-isomorphisms. □

BOUNDEDNESS

A complex with homology bounded above can be resolved by a bounded above semi-injective complex. Such a resolution could, in fact, be constructed degreewise, mimicking the classic construction of an injective resolution of a module.

5.3.26 Theorem. *Every R-complex M has a semi-injective resolution $M \xrightarrow{\simeq} I$ with $I_v = 0$ for all $v > \sup M$.*

PROOF. If M is acyclic, then the morphism $M \xrightarrow{\simeq} 0$ is the desired resolution. If $\mathrm{H}(M)$ is not bounded above, then any semi-injective resolution of M has the desired property. Assume now that $\mathrm{H}(M)$ is bounded above and set $u = \sup M$. By 4.2.4 there is a quasi-isomorphism $M \to M_{\subseteq u}$. By 5.3.19 the truncated complex $M_{\subseteq u}$ has a semi-injective resolution $M_{\subseteq u} \xrightarrow{\simeq} I$ with $I_v = 0$ for $v > u$. The desired semi-injective resolution is the composite $M \xrightarrow{\simeq} M_{\subseteq u} \xrightarrow{\simeq} I$. □

It is proved in B.26 that the resolution in 5.3.26 can be chosen to be minimal.

The Case of Modules

Part (*ii*) in the next result recovers the lifting property of injective modules 1.3.26; it can be interpreted in terms of the diagram (5.3.5.1). Further, it follows from 5.3.18 that a module is injective if and only if it is semi-injective as a complex. Thus one recovers 1.3.27 from 5.3.21.

5.3.27 Proposition. *For an R-module I, the following conditions are equivalent.*

(*i*) *I is injective.*

(*ii*) *For every homomorphism $\alpha\colon K \to I$ and for every injective homomorphism $\beta\colon K \to M$, there exists a homomorphism $\gamma\colon M \to I$ such that $\gamma\beta = \alpha$ holds.*

(*iii*) *Every exact sequence $0 \to I \to M \to M'' \to 0$ of R-modules is split.*

(*iv*) *I is a direct summand of the character module of a free R^{o}-module.*

Proof. Specialize 5.3.6 to modules. □

5.3.28 Proposition. *Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence of R-modules. If I' is injective, then I is injective if and only if I'' is injective.*

Proof. Specialize 5.3.7 to modules. □

5.3.29 Proposition. *Let P be an R-module and X an R–S^{o}-bimodule. If P is projective and X is injective over S^{o}, then the S^{o}-module $\operatorname{Hom}_R(P, X)$ is injective.*

Proof. The assertion follows, per 5.2.12 and 5.3.18, immediately from 5.3.25. □

Semi-injective resolutions of complexes subsume the classic notion of injective resolutions of modules. The next result is dual to 1.3.12. A homomorphism $M \rightarrowtail I$ as below is called an injective preenvelope of M; cf. D.19.

5.3.30 Proposition. *Let M be an R-module. There is an injective homomorphism of R-modules $M \to I$ where I is injective.*

Proof. The assertion is immediate from 5.3.4. □

5.3.31 Theorem. *Let M be an R-module. There is an exact sequence of R-modules,*

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_v \longrightarrow I_{v-1} \longrightarrow \cdots ,$$

where each module I_v is injective.

Proof. Choose by 5.3.19 a semi-injective resolution $\iota\colon M \xrightarrow{\simeq} I$ with $I_v = 0$ for all $v > 0$ and ι injective. The displayed sequence of R-modules is the complex $\operatorname{Cone}\iota$; in particular, the map $M_0 \rightarrowtail I_0$ is the homomorphism ι_0. The cone is acyclic because ι is a quasi-isomorphism; see 4.2.16. □

5.3.32 Definition. Let M be an R-module. Together, the injective homomorphism $M \to I_0$ and the R-complex $0 \to I_0 \to \cdots \to I_v \to I_{v-1} \to \cdots$ in 5.3.31 is called an *injective resolution* of M.

5.3.33. Let M be an R-module. By 5.3.12 an injective resolution of M is a semi-injective resolution of M as an R-complex. Only a semi-injective resolution $M \xrightarrow{\simeq} I$ with I concentrated in non-positive degrees is an injective resolution of M.

REMARK. Let $M \xrightarrow{\simeq} I$ be an injective resolution of an R-module. It is standard to refer to the module $\operatorname{Ker}(I_n \to I_{n-1}) = \mathrm{Z}_n(I)$ as an n^{th} *cosyzygy* of M. By Schanuel's lemma 8.2.13 it is "essentially" unique. In the case the complex I is minimal, the module $\mathrm{Z}_n(I)$ may be referred to as the n^{th} cosyzygy of M.

EXERCISES

E 5.3.1 Show that a graded R-module is graded-injective if and only if it is an injective object in the category $\mathcal{M}_{\mathrm{gr}}(R)$.

E 5.3.2 Show that a graded R-module is graded-injective if it is injective as an R-module. Is the converse true?

E 5.3.3 Assume that R is left hereditary. Show that for every complex I of injective R-modules there is a quasi-isomorphism $\mathrm{H}(I) \xrightarrow{\simeq} I$. *Hint:* See E 1.4.8.

E 5.3.4 For an R-complex I, show that the following conditions are equivalent. (*i*) I is an injective object in the category $\mathcal{C}(R)$. (*ii*) I is a contractible complex complex of injective R-modules. (*iii*) I is semi-injective and acyclic. (*iv*) I is an acyclic complex of injective R-modules and $\mathrm{B}(I) = \mathrm{Z}(I)$ is a complex of injective R-modules. Dualize 4.3.24 and conclude that the category $\mathcal{C}(R)$ has enough injectives.

E 5.3.5 Show that the Dold complex from 5.1.4 is an acyclic complex of injective modules. Show that it is not contractible and conclude that it is not semi-injective.

E 5.3.6 Let I be the $\mathbb{Z}/4\mathbb{Z}$-complex $0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots$ concentrated in non-positive degrees. Set $J = \prod_{u>0} \Sigma^u I$, show that the complex $J_{\geqslant 0}$ is semi-injective, and compute its homology.

E 5.3.7 Show that a complex I of injective R-modules is semi-injective if $\operatorname{Hom}_R(A, I)$ is acyclic for every acyclic R-complex A that is bounded below.

E 5.3.8 Let $R \to S$ be a ring homomorphism. Show that for a semi-injective R-complex I, the S-complex $\operatorname{Hom}_R(S, I)$ is semi-injective.

E 5.3.9 Assume that R is commutative and S an R-algebra that is faithfully projective as an R-module. Show that an R-complex I is semi-injective if (and only if) the S-complex $\operatorname{Hom}_R(S, I)$ is semi-injective.

E 5.3.10 Show that the mapping cone of a morphism between semi-injective R-complexes is semi-injective.

E 5.3.11 Let I be a semi-injective R-complex. Show that for $v \geqslant \sup M$ the module $\mathrm{C}_v(I)$ is injective. Conclude that if $\iota\colon M \xrightarrow{\simeq} I$ is a semi-injective resolution, then so is $\iota_{\subseteq v}$ for $v \geqslant \sup M$. Show that if M is a module, then $\iota_{\subseteq 0}$ yields an injective resolution of M.

E 5.3.12 Show that every complex over a semi-simple ring is semi-injective.

E 5.3.13 Show that a complex of injective modules over a left hereditary ring is semi-injective.

E 5.3.14 Let $0 \to I' \to I \to I'' \to 0$ be a degreewise split exact sequence of R-complexes. Show that if two of the complexes I', I, and I'' are semi-injective, then so is the third.

E 5.3.15 Show that the following conditions are equivalent for an R-complex I. (*i*) I is semi-injective. (*ii*) The complex $I_{\geqslant n}$ is semi-injective for every $n \in \mathbb{Z}$. (*iii*) I is a complex of injective R-modules and $I_{\geqslant n}$ is semi-injective for some $n \in \mathbb{Z}$.

E 5.3.16 Let P be a bounded complex of finitely generated projective S-modules and I a complex of R–S^{o}-bimodules. Show that if I is a semi-injective over R, then the R-complex $I \otimes_S P$ is semi-injective.

E 5.3.17 Let P be a bounded above complex of finitely generated projective S-modules and I a bounded above complex of R–S^{o}-bimodules that are injective over R. Show that the R-complex $I \otimes_S P$ is semi-injective.

E 5.3.18 Let F be a bounded above complex of flat S-modules and I a bounded above complex of R–S^{o}-bimodules that are injective over R. Show that if R is left Noetherian, then the R-complex $I \otimes_S F$ is semi-injective. *Hint:* E 1.4.11.

E 5.3.19 Show that the following conditions are equivalent for an R-complex Y. (*i*) For every chain map $\alpha\colon K \to Y$ and every quasi-isomorphism $\beta\colon K \to M$ there exists a chain map $\gamma\colon M \to Y$, unique up to homotopy, such that $\gamma\beta \sim \alpha$. (*ii*) For every quasi-isomorphism β the induced morphism $\operatorname{Hom}_R(\beta, Y)$ is a quasi-isomorphism. (*iii*) For every acyclic complex A, the complex $\operatorname{Hom}_R(A, Y)$ is acyclic.

A complex with these properties is called *K-injective;* Avramov, Foxby, and Halperin [25] use the term ‘homotopically injective’.

E 5.3.20 Show that a quasi-isomorphism of K-injective R-complexes is a homotopy equivalence.

E 5.3.21 Let K be an acyclic K-injective R-complex. Show that $\operatorname{Hom}_R(M, K)$ is acyclic for every R-complex M.

E 5.3.22 Show that an R-complex is semi-injective if and only if it is a complex of injective modules and K-injective. Show that a K-injective complex need not be semi-injective.

E 5.3.23 Show that a graded R-module is graded-injective if and only if it is K-injective as an R-complex.

E 5.3.24 Consider a homotopy equivalence $M \to N$ of R-complexes. Show that M is K-injective if and only if N is K-injective. Is the same true for semi-injectivity?

E 5.3.25 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{C}(R)$. (a) Show that for any two semi-injective resolutions $\iota\colon M \xrightarrow{\simeq} I$ and $\varepsilon\colon N \xrightarrow{\simeq} E$ there is a morphism $\tilde{\alpha}\colon I \to E$ such that the next diagram is commutative up to homotopy,

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{\iota} & I \\ {\scriptstyle\alpha}\downarrow & & \vdots\,{\scriptstyle\tilde{\alpha}} \\ N & \xrightarrow[\simeq]{\varepsilon} & E\,. \end{array}$$

(b) Show that if ι is injective, then $\tilde{\alpha}$ can be chosen so that the diagram is commutative.

E 5.3.26 Let x be a central element in R and $M \xrightarrow{\simeq} I$ a semi-injective resolution in $\mathcal{C}(R)$. Show that if the homothety x^M is null-homotopic, then x^I is null-homotopic.

E 5.3.27 Show that every morphism $\alpha\colon M \to N$ of R-complexes admits factorizations in $\mathcal{C}(R)$,

$$\begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, tail, "\varepsilon"'] & & N \\ & X \arrow[ur, two heads, "\simeq", "\varphi"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} M \arrow[rr, "\alpha"] \arrow[dr, tail, "\simeq", "\iota"'] & & N \\ & Y \arrow[ur, two heads, "\pi"'] & \end{tikzcd}$$

where ε and ι are injective, φ and π are surjective with semi-injective kernels, and φ and ι are quasi-isomorphisms. *Hint:* Apply E 5.3.4. Modify the first step in 5.3.14.

E 5.3.28 Give an alternative proof of 5.3.31 based on 5.3.30.

E 5.3.29 Let $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ be an exact sequence of R-complexes. Show that there is a commutative diagram in $\mathcal{C}(R)$ in which the columns are semi-injective resolutions,

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\alpha'"] \arrow[d, "\simeq"', "\iota'"] & M \arrow[r, "\alpha"] \arrow[d, "\simeq"', "\iota"] & M'' \arrow[r] \arrow[d, "\simeq"', "\iota''"] & 0 \\ 0 \arrow[r] & I' \arrow[r, "\tilde{\alpha}'"] & I \arrow[r, "\tilde{\alpha}"] & I'' \arrow[r] & 0\,. \end{tikzcd}$$

This is known as the Horseshoe Lemma for semi-injective resolutions.

E 5.3.30 Let I be a complex of injective R-modules. Show that for every integer $n \leqslant \inf I$ the truncated complex $I_{\leqslant n}$ yields an injective resolution of the module $\mathrm{Z}_n(I)$.

5.4 Semi-Flatness

Synopsis. Graded-flat module; complex of flat modules; semi-flat complex; semi-injective complex.

For every complex of flat modules, the functor $- \otimes F$ preserves short exact sequences of complexes, see 2.4.17, but not necessarily acyclicity of complexes. Adding this as a requirement, one arrives at the notion of semi-flatness.

Complexes of Flat Modules

5.4.1 Proposition. *For an R-complex F, the following conditions are equivalent.*

(i) *Each R-module F_v is flat.*

(ii) *The functor $- \otimes_R F$ is exact.*

(iii) *For every exact sequence $0 \to M' \to M \to F \to 0$ in $\mathcal{C}(R)$ the exact sequence $0 \to \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(M', \mathbb{E}) \to 0$ is degreewise split.*

(iv) *The character complex $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is a complex of injective R^{o}-modules; that is, the graded module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})^{\natural}$ is graded-injective.*

Proof. Conditions (i) and (iv) are equivalent by 1.3.48.

(ii) $\Leftrightarrow$ (iv): By adjunction 4.4.12 and commutativity 4.4.4 there is a natural isomorphism of functors from $\mathcal{C}(R^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$,

$$\operatorname{Hom}_{R^{\mathrm{o}}}(-, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(- \otimes_R F, \mathbb{E}) .$$

By 5.3.6 the functor on the left-hand side is exact if and only if $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is a complex of injective R^{o}-modules. As $\mathbb{E}$ is faithfully injective, the functor on the right-hand side is exact if and only if $- \otimes_R F$ is exact.

(iv) $\Rightarrow$ (iii): This implication is immediate from 5.3.6.

(iii) $\Rightarrow$ (i): Choose by 5.1.7 a surjective semi-free resolution $\pi \colon L \xrightarrow{\simeq} F$ and consider the associated short exact sequence $0 \to \operatorname{Ker} \pi \to L \to F \to 0$. By 5.3.2 the complex $\operatorname{Hom}_{\Bbbk}(L, \mathbb{E})$ consists of injective R^{o}-modules, so it follows from 5.3.6 and split exactness of the sequence

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})^{\natural} \longrightarrow \operatorname{Hom}_{\Bbbk}(L, \mathbb{E})^{\natural} \longrightarrow \operatorname{Hom}_{\Bbbk}(\operatorname{Ker} \pi, \mathbb{E})^{\natural} \longrightarrow 0$$

that each module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})_{-v} = \operatorname{Hom}_{\Bbbk}(F_v, \mathbb{E})$ is an injective R^{o}-module, whence each F_v is a flat R-module by 1.3.48. □

5.4.2 Corollary. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of R-complexes. If F'' is a complex of flat modules, then F is a complex of flat modules if and only if F' is a complex of flat modules.*

Proof. Apply 5.4.1 and 5.3.7 to the exact sequence of R°-complexes

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk}(F'', \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F', \mathbb{E}) \longrightarrow 0\,. \qquad \square$$

A short exact sequence that starts in a complex of injective modules or ends in a complex of projective modules is degreewise split, see 5.3.6 and 5.2.2. A short exact sequence that ends in a complex of flat modules exhibits a weaker form of stability.

5.4.3 Corollary. *Let $0 \to M' \to M \to F \to 0$ be an exact sequence of R-complexes and N an R°-complex. If F is a complex of flat R-modules, then the sequence $0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R F \to 0$ is exact.*

Proof. Assume that F is a complex of flat R-modules and let N be an R°-complex. As the tensor product is right exact, it is sufficient to show that the induced morphism $N \otimes_R M' \to N \otimes_R M$ is injective. There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd}
\operatorname{Hom}_{\Bbbk}(M \otimes_{R^{\circ}} N, \mathbb{E}) \arrow[r] \arrow[d, "\rho^{NM\mathbb{E}}", "\cong"'] & \operatorname{Hom}_{\Bbbk}(M' \otimes_{R^{\circ}} N, \mathbb{E}) \arrow[r] \arrow[d, "\rho^{NM'\mathbb{E}}", "\cong"'] & 0 \\
\operatorname{Hom}_{R^{\circ}}(N, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E})) \arrow[r] & \operatorname{Hom}_{R^{\circ}}(N, \operatorname{Hom}_{\Bbbk}(M', \mathbb{E})) \arrow[r] & 0\,.
\end{tikzcd}$$

The vertical maps are adjunction isomorphisms. The lower row is exact by 5.4.1 and the fact that the functor $\operatorname{Hom}_R(N, -)$ preserves degreewise split exactness of sequences; cf. 2.3.12. By commutativity of the diagram, the upper row is also exact. As $\mathbb{E}$ is faithfully injective, this implies that the sequence $0 \to M' \otimes_{R^{\circ}} N \to M \otimes_{R^{\circ}} N$ is exact, and commutativity 4.4.4 finishes the proof. $\square$

5.4.4 Definition. A graded R-module F is called *graded-flat* if the R-complex F satisfies the conditions in 5.4.1.

Characterization of Semi-Flat Complexes

5.4.5 Definition. An R-complex F is called *semi-flat* if $\beta \otimes_R F$ is an injective quasi-isomorphism for every injective quasi-isomorphism β in $\mathcal{C}(R^{\circ})$.

Remark. Another word for semi-flat is ‘DG-flat’.

5.4.6. It follows from 2.4.13 that if an R-complex F is semi-flat, then so is $\Sigma^s F$ for every integer s.

5.4.7 Example. A contractible complex of flat R-modules is semi-flat by 5.4.1, 4.3.20, and 4.3.27.

Remark. While a semi-projective/injective complex is acyclic if and only if it is contractible, see E 5.2.6/E 5.3.4, an acyclic semi-flat complex need not be contractible, see E 8.3.8. The character complex of such a complex is, however, contractible, see 5.5.22 and the subsequent Remark.

5.4.8 Example. Let F be a bounded below complex of flat R-modules and β an injective quasi-isomorphism in $\mathcal{C}(R^{\mathrm{o}})$. The morphism $\beta \otimes_R F$ is injective by 5.4.1. The complex $\operatorname{Cone}\beta$ is acyclic by 4.2.16, and hence so is $(\operatorname{Cone}\beta) \otimes_R F_v$ for every $v \in \mathbb{Z}$. As F is bounded below, it follows from 4.1.19 and A.9 that the complex $\operatorname{Cone}(\beta \otimes_R F) \cong (\operatorname{Cone}\beta) \otimes_R F$ is acyclic, whence $\beta \otimes_R F$ is a quasi-isomorphism. Thus, F is semi-flat.

Semi-projectivity and semi-injectivity are categorically dual notions. By adjointness of Hom and tensor product, semi-flatness is also, in a different sense, dual to semi-injectivity. The next result gives useful characterizations of semi-flat complexes.

5.4.9 Proposition. *For an R-complex F, the following conditions are equivalent.*

(*i*) *F is semi-flat.*
(*ii*) *The functor $- \otimes_R F$ is exact and preserves quasi-isomorphisms.*
(*iii*) *The character complex* $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ *is a semi-injective R^{o}-complex.*
(*iv*) *F is a complex of flat R-modules and the functor $- \otimes_R F$ preserves acyclicity of complexes.*

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. By commutativity 4.4.4 and adjunction 4.4.12 there is a natural isomorphism of functors from $\mathcal{C}(R^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$,

$$\operatorname{Hom}_{\Bbbk}(- \otimes_R F, \mathbb{E}) \;\cong\; \operatorname{Hom}_{R^{\mathrm{o}}}(-, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}))\;.$$

Thus it follows from 2.5.7(b) and 5.3.16 and that (ii) and (iii) are equivalent.

$(iii) \Rightarrow (iv)$: It follows from 5.3.16 that $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is a complex of injective R^{o}-modules, so F is a complex of flat R-modules by 5.4.1. Let A be an acyclic R^{o}-complex; by adjunction 4.4.12 and commutativity 4.4.4 there is an isomorphism $\operatorname{Hom}_{R^{\mathrm{o}}}(A, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(A \otimes_R F, \mathbb{E})$. The left-hand complex is acyclic by 5.3.16, so it follows by faithfulness of the functor $\operatorname{Hom}_{\Bbbk}(-, \mathbb{E})$ that $A \otimes_R F$ is acyclic.

$(iv) \Rightarrow (i)$: Let β be an injective quasi-isomorphism. The morphism $\beta \otimes_R F$ is then injective by 5.4.1. The complex $\operatorname{Cone}\beta$ is acyclic by 4.2.16, and hence so is the complex $(\operatorname{Cone}\beta) \otimes_R F \cong \operatorname{Cone}(\beta \otimes_R F)$, where the isomorphism comes from 4.1.19. It follows that $\beta \otimes_R F$ is a quasi-isomorphism. □

The characterizations of semi-projective 5.2.10 and semi-injective 5.3.16 complexes include lifting statements, but 5.4.9 does not. As far as it is possible, 5.5.3 makes up for this.

5.4.10 Corollary. *Every semi-projective R-complex is semi-flat*; *in particular, every semi-free R-complex is semi-flat.*

Proof. The assertion follows immediately from 5.3.17 and 5.2.9. □

It is proved in the next section, in 5.5.27 to be exact, that a semi-flat complex of projective modules is semi-projective.

5.4.11 Corollary. *A graded R-module is graded-flat if and only if it is semi-flat as an R-complex.*

Proof. If F is semi-flat as an R-complex, then each module F_v is flat by 5.4.9, whence F is graded-flat. If F is graded-flat, then the character module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is graded-injective by 5.4.1 and hence semi-injective as an R^{o}-complex; see 5.3.18. As an R-complex, F is then semi-flat. □

Properties of Semi-Flat Complexes

The class of semi-flat complexes over a ring is closed under extensions, kernels of surjective morphisms, direct summands, and filtered colimits. Semi-flatness is also preserved under base change.

5.4.12 Proposition. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of R-complexes. If F'' is semi-flat, then F is semi-flat if and only if F' is semi-flat.*

Proof. Apply 5.4.9 and 5.3.20 to the exact sequence of R^{o}-complexes

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk}(F'', \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F', \mathbb{E}) \longrightarrow 0\,. \qquad \square$$

5.4.13 Proposition. *Let $\{\mu^{vu} \colon F^u \to F^v\}_{u \leqslant v}$ be a U-direct system of semi-flat R-complexes. If U is filtered, then* $\operatorname{colim}_{u\in U} F^u$ *is semi-flat.*

Proof. Let $\beta \colon K \to M$ be an injective quasi-isomorphism in $\mathcal{C}(R^{\mathrm{o}})$. There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd}
\operatorname*{colim}_{u\in U}(K \otimes_R F^u) \arrow[r, "\operatorname{colim}_{u\in U}(\beta\otimes F^u)"] \arrow[d, "\cong"] & \operatorname*{colim}_{u\in U}(M \otimes_R F^u) \arrow[d, "\cong"'] \\
K \otimes_R \big(\operatorname*{colim}_{u\in U} F^u\big) \arrow[r, "\beta\otimes(\operatorname{colim}_{u\in U} F^u)"] & M \otimes_R \big(\operatorname*{colim}_{u\in U} F^u\big)\,,
\end{tikzcd}$$

where the vertical maps are the canonical isomorphisms (3.2.23.1). In view of 3.3.10 and 3.3.15(d) it follows that $\beta \otimes_R (\operatorname{colim} F^u)$ is an injective quasi-isomorphism if each map $\beta \otimes_R F^u$ is an injective quasi-isomorphism. □

Direct sums and direct summands of semi-flat complexes are semi-flat, and so are coproducts of semi-flat complexes.

5.4.14 Corollary. *Let $\{F^u\}_{u\in U}$ be a family of R-complexes. The coproduct $\coprod_{u\in U} F^u$ is semi-flat if and only if each complex F^u is semi-flat.*

Proof. It is immediate from 3.3.9 and 5.4.13 that a coproduct of semi-flat complexes is semi-flat, and it is evident from the definition that a direct summand of a semi-flat complex is semi-flat. □

Contrary to the situation for semi-projective and semi-injective complexes, a quasi-isomorphism of semi-flat R-complexes need not be a homotopy equivalence.

5.4.15 Example. It follows from 1.3.11, 1.3.12, and 5.1.18 that the $\mathbb{Z}$-module $\mathbb{Q}$ has a free resolution $\pi\colon L \xrightarrow{\simeq} \mathbb{Q}$ with $L_v = 0$ for $v \neq 0, 1$. Both $\mathbb{Z}$-complexes $\mathbb{Q}$ and L are semi-flat. Suppose $\gamma\colon \mathbb{Q} \to L$ were a homotopy inverse of π, then one would have $1^{\mathbb{Q}} \sim \pi\gamma$, and hence $1^{\mathbb{Q}} = \pi\gamma$ as $\partial^{\mathbb{Q}} = 0$. This would imply that $\mathbb{Q}$ is a direct summand of L_0 and hence a free $\mathbb{Z}$-module, but it is not.

Quasi-isomorphisms between semi-flat complexes still exhibit some robustness; two important instances are captured by the next proposition and 5.5.23.

5.4.16 Theorem. *Let $\alpha\colon F \to F'$ be a quasi-isomorphism of semi-flat R-complexes. For every R^{o}-complex M, the morphism $M \otimes_R \alpha$ is a quasi-isomorphism.*

Proof. By 5.3.16 and 5.4.9 the morphism $\operatorname{Hom}_{\Bbbk}(\alpha, \mathbb{E})$ is a quasi-isomorphism of semi-injective R^{o}-complexes and hence a homotopy equivalence by 5.3.24. Therefore, the upper horizontal map in the following commutative diagram is also a homotopy equivalence; see 4.3.19.

$$\begin{tikzcd}
\operatorname{Hom}_{R^{\mathrm{o}}}(M, \operatorname{Hom}_{\Bbbk}(F', \mathbb{E})) \arrow[rr, "\operatorname{Hom}(M, \operatorname{Hom}(\alpha, \mathbb{E}))", "\approx"'] & & \operatorname{Hom}_{R^{\mathrm{o}}}(M, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \\
\operatorname{Hom}_{\Bbbk}(F' \otimes_{R^{\mathrm{o}}} M, \mathbb{E}) \arrow[u, "\rho^{MF'\mathbb{E}}", "\cong"'] \arrow[d, "\operatorname{Hom}(\upsilon^{MF'}, \mathbb{E})"', "\cong"] & & \operatorname{Hom}_{\Bbbk}(F \otimes_{R^{\mathrm{o}}} M, \mathbb{E}) \arrow[u, "\cong", "\rho^{MF\mathbb{E}}"'] \arrow[d, "\cong"', "\operatorname{Hom}(\upsilon^{MF}, \mathbb{E})"] \\
\operatorname{Hom}_{\Bbbk}(M \otimes_R F', \mathbb{E}) \arrow[rr, "\operatorname{Hom}(M \otimes \alpha, \mathbb{E})"] & & \operatorname{Hom}_{\Bbbk}(M \otimes_R F, \mathbb{E})
\end{tikzcd}$$

The diagram shows that $\operatorname{Hom}_{\Bbbk}(M \otimes_R \alpha, \mathbb{E})$ is a quasi-isomorphism, and by faithful injectivity of $\mathbb{E}$ it follows that $M \otimes_R \alpha$ is a quasi-isomorphism; cf. 4.2.14. □

5.4.17 Proposition. *Let F be an S-complex and X a complex of R–S^{o}-bimodules. If F is semi-flat and X is semi-flat over R, then the R-complex $X \otimes_S F$ is semi-flat.*

Proof. Associativity 4.4.7 yields a natural isomorphism,

$$- \otimes_R (X \otimes_S F) \;\cong\; (- \otimes_R X) \otimes_S F\,,$$

of functors from $\mathcal{C}(R^{\mathrm{o}})$ to $\mathcal{C}(\Bbbk)$. By the assumptions on F and X, the functor $(- \otimes_R X) \otimes_S F$ is exact and preserves quasi-isomorphisms. □

5.4.18 Corollary. *Let $R \to S$ be a ring homomorphism and F an R-complex.*

(a) *If F is semi-flat, then the S-complex $S \otimes_R F$ is semi-flat.*
(b) *If S is flat as an R-module, then a semi-flat S-complex is semi-flat over R.*

Proof. For (b) apply 5.4.17 with $X = S$ viewed as an R–S^{o}-bimodule and note that $X \otimes_S -$ is the restriction of scalars functor $\mathcal{C}(S) \to \mathcal{C}(R)$. For (a) interchange the roles of R and S in 5.4.17 and apply it with $X = S$ viewed as an S–R^{o}-bimodule. □

Remark. Under additional assumptions on R, S, and F, semi-flatness of the base changed complex $S \otimes_R F$ implies semi-flatness of F; see E 5.4.4.

THE CASE OF MODULES

Part (*iii*) in the next result recovers the characterization of flat modules in terms of character modules from 1.3.48. Further, a module is by 5.4.11 flat if and only if it is semi-flat as a complex, and parallel results, 5.1.2 and 5.2.12, hold for (semi-)freeness and (semi-)projectivity. It follows that 5.4.10 packs the fact from 1.3.43 that free and projective modules are flat.

5.4.19 Proposition. *For an R-module F, the following conditions are equivalent.*

(*i*) *F is flat.*
(*ii*) *For every exact sequence $0 \to M' \to M \to F \to 0$ of R-modules, the exact sequence $0 \to \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(M', \mathbb{E}) \to 0$ is split.*
(*iii*) *The character module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is an injective R^{o}-module.*

PROOF. Specialize 5.4.1 to modules. □

5.4.20 Corollary. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of R-modules. If F'' is flat, then F' is flat if and only if F is flat.*

PROOF. Specialize 5.4.2 to modules. □

5.4.21 Proposition. *Let $\{\mu^{vu} : F^u \to F^v\}_{u \leqslant v}$ be a U-direct system of flat R-modules. If U is filtered, then $\operatorname{colim}_{u \in U} F^u$ is flat.*

PROOF. The asssertion follows, in view of 5.4.11, immediately from 5.4.13. □

REMARK. In contrast to 5.4.21, Bergman [39] shows that every module is a filtered limit of injective modules.

5.4.22 Proposition. *Let $\{F^u\}_{u \in U}$ be a family of R-modules. The coproduct $\coprod_{u \in U} F^u$ is flat if and only if each module F^u is flat.*

PROOF. The asssertion follows, in view of 5.4.11, immediately from 5.4.14. □

5.4.23 Proposition. *Let F be an S-module and X an R–S^{o}-bimodule. If F is flat and X is flat over R, then the R-module $X \otimes_S F$ is flat.*

PROOF. The assertion follows, in view of 5.4.11, immediately from 5.4.17. □

The next result follows from 5.4.23 but is more easily recovered from 5.4.18.

5.4.24 Corollary. *Let $R \to S$ be a ring homomorphism and F an R-module.*

(a) *If F is flat, then the S-module $S \otimes_R F$ is flat.*
(b) *If S is flat as an R-module, then a flat S-module is flat over R.*

PROOF. The assertion follows, in view of 5.4.11, immediately from 5.4.18. □

There is a classic notion of flat resolutions of modules; the treatment of this notion and its relation to semi-flat complexes is for technical reasons postponed to Sect. 8.3.

Semi-Injectivity Revisited

5.4.25 Proposition. *Let I be an R-complex and X a complex of R–S^{o}-bimodules. If I is semi-injective and X is semi-flat over S^{o}, then the S-complex $\operatorname{Hom}_R(X,I)$ is semi-injective.*

Proof. Adjunction 4.4.12 yields a natural isomorphism

$$\operatorname{Hom}_S(-,\operatorname{Hom}_R(X,I)) \;\cong\; \operatorname{Hom}_R(X\otimes_S -, I)$$

of functors from $\mathcal{C}(S)^{\mathrm{op}}$ to $\mathcal{C}(\Bbbk)$. It follows from the assumptions on I and X that the functor $\operatorname{Hom}_R(X\otimes_S -,I)$ is exact and preserves quasi-isomorphisms. □

5.4.26 Corollary. *Let $R\to S$ be a ring homomorphism and I an R-complex.*

(a) *If I is semi-injective, then the S-complex $\operatorname{Hom}_R(S,I)$ is semi-injective.*
(b) *If S is flat as an R^{o}-module, then a semi-injective S-complex is semi-injective over R.*

Proof. For (a) apply 5.4.25 with $X = S$ viewed as an R–S^{o}-bimodule. For (b) interchange the roles of R and S in 5.4.25 and apply it with $X = S$ viewed as an S–R^{o}-bimodule; now $\operatorname{Hom}_S(X,-)$ is the restriction of scalars functor $\mathcal{C}(S)\to\mathcal{C}(R)$. □

Injective Modules Revisited

5.4.27 Proposition. *Let I be an R-module and X an R–S^{o}-bimodule. If I is injective and X is flat over S^{o}, then the S-module $\operatorname{Hom}_R(X,I)$ is injective.*

Proof. The assertion follows, per 5.3.18 and 5.4.11, immediately from 5.4.25. □

The next result follows from 5.4.27 but is more easily recovered from 5.4.26.

5.4.28 Corollary. *Let $R\to S$ be a ring homomorphism and I an R-module.*

(a) *If I is injective, then the S-module $\operatorname{Hom}_R(S,I)$ is injective.*
(b) *If S is flat as an R^{o}-module, then an injective S-module is injective over R.*

Proof. The assertion follows per 5.3.18 immediately from 5.4.26. □

Exercises

E 5.4.1 Show that a graded R-module is graded-flat if and only if it is flat as an R-module.

E 5.4.2 Show that the Dold complex from 2.1.23 is not semi-flat.

E 5.4.3 Show that a complex of flat modules over a right hereditary ring is semi-flat.

E 5.4.4 Assume that R is commutative and S an R-algebra that is faithfully flat as an R-module. Show that an R-complex F is semi-flat if (and only if) the S-complex $S\otimes_R F$ is semi-flat.

E 5.4.5 Show that a complex F of flat R-modules is semi-flat if $A\otimes_R F$ is acyclic for every acyclic R^{o}-complex A that is bounded below.

E 5.4.6 Show that the mapping cone of a morphism between semi-flat R-complexes is semi-flat.

E 5.4.7 Let F'' be a complex of flat R-modules and $0 \to F' \to F \to F'' \to 0$ an exact sequence of R-complexes. Show that if two of the complexes F', F, and F'' are semi-flat, then so is the third.

E 5.4.8 Show that the following conditions are equivalent for an R-complex F. (*i*) F is semi-flat. (*ii*) The complex $F_{\leqslant n}$ is semi-flat for every $n \in \mathbb{Z}$. (*iii*) F is a complex of flat R-modules and $F_{\leqslant n}$ is semi-flat for some $n \in \mathbb{Z}$.

E 5.4.9 Use E 5.3.4 to give a shorter proof of 5.4.16.

E 5.4.10 Let P be a bounded complex of finitely generated projective R-modules and F a complex of R–S^{o}-bimodules. Show that if F is a semi-flat over S^{o}, then the S^{o}-complex $\operatorname{Hom}_R(P, F)$ is semi-flat.

E 5.4.11 Let P be a bounded above complex of finitely generated projective R-modules and F a bounded below complex of R–S^{o}-bimodules that are flat over S^{o}. Show that the S^{o}-complex $\operatorname{Hom}_R(P, F)$ is semi-flat.

E 5.4.12 Let P be a bounded above complex of projective R-modules and F a bounded below complex of R–S^{o}-bimodules that are flat over S^{o}. Show that if S is left coherent, then the S^{o}-complex $\operatorname{Hom}_R(P, F)$ is semi-flat. *Hint:* E 3.3.4.

E 5.4.13 Let J be a bounded below complex of injective R-modules and I a bounded above complex of R–S^{o}-bimodules that are injective over S^{o}. Show that if S is right coherent, then the S-complex $\operatorname{Hom}_R(I, J)$ is semi-flat. *Hint:* E 3.3.3.

E 5.4.14 Show that the following conditions are equivalent for an R-complex Z. (*i*) For every quasi-isomorphism β in $\mathcal{C}(R^{\mathrm{o}})$ the induced morphism $\beta \otimes_R Z$ is a quasi-isomorphism. (*ii*) For every acyclic R^{o}-complex A, the complex $A \otimes_R Z$ is acyclic.
A complex with these properties is called *K-flat;* Avramov, Foxby, and Halperin [25] use the term *homotopically flat*.

E 5.4.15 Let K be an acyclic K-flat R-complex. Show that $M \otimes_R K$ is acyclic for every R^{o}-complex M.

E 5.4.16 Show that an R-complex is semi-flat if and only if it is a complex of flat modules and K-flat. Give an example of a K-flat complex that is not semi-flat.

E 5.4.17 Show that a graded R-module is graded-flat if and only if it is K-flat as an R-complex.

E 5.4.18 Consider a homotopy equivalence $M \to N$ of R-complexes. Show that M is K-flat if and only if N is K-flat. Is the same true for semi-flatness?

E 5.4.19 Let $\alpha\colon Z \to Z'$ be a quasi-isomorphism of K-flat R-complexes. Show that for every R^{o}-complex M the morphism $M \otimes_R \alpha$ is a quasi-isomorphism.

E 5.4.20 Show that an R-complex Z is K-flat if and only if the R^{o}-complex $\operatorname{Hom}_{\Bbbk}(Z, \mathbb{E})$ is K-injective.

5.5 Structure of Semi-Flat Complexes

Synopsis. Govorov and Lazard's theorem; pure exact sequence; pure acyclc complex; categorically flat complex; perfect ring.

A semi-free complex is semi-projective and a semi-projective complex is semi-flat. The first theorem of this section "closes the circle" by showing that every semi-flat complex can be obtained as a filtered colimit of degreewise finitely generated semi-free complexes. Further, while a semi-projective complex of free modules may not be semi-free, another result, 5.5.27, clarifies the relation between semi-projectivity and semi-flatness: A semi-flat complex of projective modules is semi-projective.

Govorov and Lazard's Theorem

In the case of modules, see 5.5.7, the next theorem was obtained by Govorov [107] and Lazard [170]; the version below first appeared in [63].

5.5.1 Theorem. *For an R-complex F the following conditions are equivalent.*

(*i*) *F is semi-flat.*

(*ii*) *Every morphism of R-complexes, $\varphi\colon N \to F$, with N bounded and degreewise finitely presented admits a factorization in $\mathcal{C}(R)$,*

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "\kappa"'] & & F \\ & L \arrow[ur, "\lambda"'] & \end{tikzcd},$$

where L is a bounded complex of finitely generated free modules.

(*iii*) *F is isomorphic to a filtered colimit of bounded complexes of finitely generated free R-modules.*

Proof. To see that (*ii*) implies (*iii*), let Λ be the class of bounded complexes of finitely generated free R-modules and apply 3.3.24. The implication (*iii*) $\Rightarrow$ (*i*) is immediate by 5.4.13, which leaves us to prove that (*i*) implies (*ii*).

By 2.5.31 there is an exact sequence $L'' \xrightarrow{\psi''} L' \xrightarrow{\psi'} N \longrightarrow 0$ of R-complexes where L' and L'' are bounded complexes of finitely generated free modules. Consider the exact sequence of R^{o}-complexes,

$$(\star) \qquad 0 \longrightarrow K \xrightarrow{\iota} \mathrm{Hom}_R(L', R) \xrightarrow{\mathrm{Hom}(\psi'',R)} \mathrm{Hom}_R(L'', R)\,,$$

where K is the kernel of $\mathrm{Hom}_R(\psi'', R)$ and ι is the embedding. The functor Z_0 is by 2.2.16 left exact, so since F is a complex of flat R-modules, it follows from 5.4.1 that the functor $\mathrm{Z}_0(-\otimes_R F)$ leaves the sequence $(\star)$ exact. As L' is bounded, so is K; set $u = \inf K^\natural$. By 5.1.7 there is an exact sequence,

$$(\diamond) \qquad P \xrightarrow{\pi} K \longrightarrow 0\,,$$

where π is a quasi-isomorphism and P is a semi-free R^{o}-complex with $P_v = 0$ for all $v < u$. As F is semi-flat, $\pi \otimes_R F$ is a surjective quasi-isomorphism by 5.4.9, and it follows from 4.2.7 that the functor $\mathrm{Z}_0(-\otimes_R F)$ leaves the sequence $(\diamond)$ exact. In total, there is an exact sequence,

$$\mathrm{Z}_0(P \otimes_R F) \xrightarrow{\iota\pi\otimes F} \mathrm{Z}_0(\mathrm{Hom}_R(L', R) \otimes_R F) \xrightarrow{\mathrm{Hom}(\psi'',R)\otimes F} \mathrm{Z}_0(\mathrm{Hom}_R(L'', R) \otimes_R F)\,.$$

For every R-complex M, denote by ξ^M the composite morphism

$$\mathrm{Hom}_R(M, R) \otimes_R F \xrightarrow{\theta^{MRF}} \mathrm{Hom}_R(M, R \otimes_R F) \xrightarrow[\cong]{\mathrm{Hom}(M,\mu_R^F)} \mathrm{Hom}_R(M, F)\,,$$

where θ^{MRF} is tensor evaluation 4.5.9 and μ_R^F is the unitor 4.4.1. The morphism ξ^M is natural in M; recall from 4.5.10(3,c) that θ^{MRF}, and hence ξ^M, is an isomorphism if M is a bounded complex of finitely presented modules. From the exact sequence above, one now gets another exact sequence,

$$(\flat)\quad \mathrm{Z}_0(P \otimes_R F) \xrightarrow{\xi^{L'} \circ (\iota\pi \otimes F)} \mathrm{Z}_0(\mathrm{Hom}_R(L', F)) \xrightarrow{\mathrm{Hom}(\psi'', F)} \mathrm{Z}_0(\mathrm{Hom}_R(L'', F))\,.$$

As $\varphi\psi' \colon L' \to F$ is a morphism, it is an element in $\mathrm{Z}_0(\mathrm{Hom}_R(L', F))$; see 2.3.10. Since one has $\mathrm{Hom}_R(\psi'', F)(\varphi\psi') = \varphi\psi'\psi'' = 0$, exactness of $(\flat)$ yields an element x in $\mathrm{Z}_0(P \otimes_R F)$ with

$$(\dagger)\qquad (\xi^{L'} \circ (\iota\pi \otimes_R F))(x) \;=\; \varphi\psi'\,.$$

The graded module $P^\natural$ has a graded basis E, and x has the form $x = \sum_{i=1}^n e_i \otimes f_i$ with $e_i \in E$ and $f_i \in F$. Set $w = \max\{|e_1|, \dots, |e_n|\}$; since $P_v = 0$ holds for all $v < u$, each element e_i satisfies $u \leqslant |e_i| \leqslant w$. For $v \in \mathbb{Z}$ set $E_v = \{e \in E \mid |e| = v\}$. Next we define a bounded subcomplex P' of P such that each module P'_v is finitely generated and free: For $v \notin \{u, \dots, w\}$ set $P'_v = 0$; for $v \in \{u, \dots, w\}$ the modules P'_v are constructed recursively. Let P'_w be the finitely generated free submodule of P_w generated by the set $E'_w = \{e_1, \dots, e_n\} \cap E_w$. For $v \leqslant w$ assume that a free submodule P'_v of P_v with finite basis E'_v has been constructed. As the subset $B'_{v-1} = \{\partial^P(e) \mid e \in E'_v\}$ of P_{v-1} is finite, there is a finite subset G'_{v-1} of E_{v-1} with $B'_{v-1} \subseteq R^{\mathrm{o}}\langle G'_{v-1}\rangle$. Now let P'_{v-1} be the submodule of P_{v-1} generated by the finite set

$$E'_{v-1} \;=\; G'_{v-1} \cup (\{e_1, \dots, e_n\} \cap E_{v-1})\,.$$

By construction, $\partial^P(P'_v) \subseteq P'_{v-1}$ holds for all $v \in \mathbb{Z}$, so P' is a subcomplex of P. Note that $x = \sum_{i=1}^n e_i \otimes f_i$ belongs to $P' \otimes_R F$. As F consists of flat modules, $P' \otimes_R F$ is a subcomplex of $P \otimes_R F$; since x is in $\mathrm{Z}_0(P \otimes_R F)$, it is also in $\mathrm{Z}_0(P' \otimes_R F)$.

Set $L = \mathrm{Hom}_{R^{\mathrm{o}}}(P', R)$. As P' is a bounded complex of finitely generated free R^{o}-modules, L is a bounded complex of finitely generated free R-modules. Let $\varepsilon \colon P' \rightarrowtail P$ be the embedding and $\kappa' \colon L' \to L$ the composite morphism

$$L' \xrightarrow{\delta_R^{L'}} \mathrm{Hom}_{R^{\mathrm{o}}}(\mathrm{Hom}_R(L', R), R) \xrightarrow{\mathrm{Hom}(\iota\pi\varepsilon, R)} \mathrm{Hom}_{R^{\mathrm{o}}}(P', R) \;=\; L\,.$$

In the commutative diagram

$$\begin{tikzcd}
P' \arrow[r, "\iota\pi\varepsilon"] \arrow[d, "\delta_R^{P'}"', "\cong"] & \mathrm{Hom}_R(L', R) \arrow[d, "\cong"', "\delta_R^{\mathrm{Hom}(L',R)}"] \\
\mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{o}}}(P', R), R) \arrow[r, "\mathrm{Hom}(\mathrm{Hom}(\iota\pi\varepsilon, R), R)"] & \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{o}}}(\mathrm{Hom}_R(L', R), R), R)
\end{tikzcd}$$

the vertical morphisms are isomorphisms by 4.5.4. The zigzag identities related to the adjunction 4.5.7 yield the equality,

$$\mathrm{Hom}_R(\delta_R^{L'}, R)\, \delta_R^{\mathrm{Hom}_R(L',R)} \;=\; 1^{\mathrm{Hom}_R(L',R)}\,,$$

so $\mathrm{Hom}_R(\delta_R^{L'}, R)$ is the inverse of $\delta_R^{\mathrm{Hom}_R(L',R)}$. One now gets

$$(\ddagger)\qquad \mathrm{Hom}_R(\kappa', R)\delta_R^{P'} \;=\; \iota\pi\varepsilon\,.$$

It follows that there are equalities,

$$\operatorname{Hom}_R(\kappa'\psi'',R)\delta_R^{P'} \;=\; \operatorname{Hom}_R(\psi'',R)\iota\pi\varepsilon \;=\; 0\pi\varepsilon \;=\; 0\,,$$

and since $\delta_R^{P'}$ is an isomorphism, the morphism $\operatorname{Hom}_R(\kappa'\psi'',R)$ is zero. In particular, $\operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(\kappa'\psi'',R),R)$ is zero, and hence the commutative diagram

$$\begin{tikzcd}
L'' \arrow[rr, "\kappa'\psi''"] \arrow[d, "\delta_R^{L''}"', "\cong"] & & L \arrow[d, "\cong"', "\delta_R^{L}"] \\
\operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(L'',R),R) \arrow[rr, "\operatorname{Hom}(\operatorname{Hom}(\kappa'\psi'',R),R)"] & & \operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(L,R),R)
\end{tikzcd}$$

shows that one has $\kappa'\psi'' = 0$. Again the vertical morphisms are isomorphisms by 4.5.4. Since κ' is zero on $\operatorname{Im}\psi'' = \operatorname{Ker}\psi'$ there is a unique morphism $\kappa\colon N \to L$ with $\kappa\psi' = \kappa'$. Finally, consider the diagram,

$$(\|)\qquad \begin{tikzcd}
P'\otimes_R F \arrow[r, "\delta_R^{P'}\otimes F"] \arrow[d, "\varepsilon\otimes F"] & \operatorname{Hom}_R(L,R)\otimes_R F \arrow[r, "\xi^L"] \arrow[d, "\operatorname{Hom}(\kappa',R)\otimes F"] & \operatorname{Hom}_R(L,F) \arrow[d, "\operatorname{Hom}_R(\kappa',F)"] \\
P\otimes_R F \arrow[r, "\iota\pi\otimes F"] & \operatorname{Hom}_R(L',R)\otimes_R F \arrow[r, "\xi^{L'}"] & \operatorname{Hom}_R(L',F)\,,
\end{tikzcd}$$

where the left-hand square is commutative by (‡) and the right-hand square is commutative by naturalness of ξ. Set

$$\lambda \;=\; (\xi^L \circ (\delta_R^{P'} \otimes F))(x)\;;$$

it is an element in $\operatorname{Hom}_R(L,F)$, and since x belongs to $\mathrm{Z}_0(P'\otimes_R F)$, also λ is a cycle; that is, λ is a morphism. From (‖), from the definition of λ, and from (†) one gets $\lambda\kappa' = \varphi\psi'$. The identity $\kappa' = \kappa\psi'$ and surjectivity of ψ' now yield $\lambda\kappa = \varphi$. □

5.5.2 Corollary. *Let F be an R-complex. If every morphism $N \to F$ with N bounded and degreewise finitely presented factors through a semi-flat R-complex, then F is semi-flat.*

Proof. Let N be a bounded and degreewise finitely presented R-complex. By 5.5.1 a morphism $N \to F$ that factors through a semi-flat complex has a further factorization through af bounded complex of finitely generated free modules. The desired conclusion now follows from another application of 5.5.1. □

The next result characterizes semi-flat complexes by a lifting property akin to 5.2.10(v) for semi-projective complexes.

5.5.3 Corollary. *For an R-complex F the following conditions are equivalent.*

(i) *F is semi-flat.*

(ii) *For every morphism $\varphi\colon N \to F$ with N bounded and degreewise finitely presented and for every surjective quasi-isomorphism $\alpha\colon M \to F$ there is a morphism $\beta\colon N \to M$ with $\varphi = \alpha\beta$.*

Proof. (i) ⟹ (ii): It follows from 5.5.1 that there is a bounded complex L of finitely generated free R-modules and morphisms $\kappa\colon N \to L$ and $\lambda\colon L \to F$ with $\varphi = \lambda\kappa$.

As L is semi-projective, see 5.2.8, there exists by 5.2.10 a morphism $\gamma\colon L \to M$ with $\lambda = \alpha\gamma$, so with $\beta = \gamma\kappa$ one has $\varphi = \alpha\beta$.

(ii) $\Rightarrow$ (i): Let $\alpha\colon P \xrightarrow{\simeq} F$ be a surjective semi-projective resolution; see 5.2.14. For every morphism $\varphi\colon N \to F$ with N bounded and degreewise finitely presented, there exists by (ii) a morphism $\beta\colon N \to P$ with $\varphi = \alpha\beta$. As P is semi-flat, see 5.4.10, it follows from 5.5.2 that F is semi-flat. □

The Case of Graded Modules

5.5.4 Theorem. *For a graded R-module F the following conditions are equivalent.*

(i) *F is graded-flat.*

(ii) *Every morphism of graded R-modules, $\varphi\colon N \to F$, with N bounded and degreewise finitely presented admits a factorization in $\mathcal{M}_{\mathrm{gr}}(R)$,*

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "\kappa"'] & & F \\ & L \arrow[ur, "\lambda"'] & \end{tikzcd},$$

where L is a bounded and degreewise finitely generated graded-free module.

(iii) *F is isomorphic to a filtered colimit of bounded and degreewise finitely generated graded-free R-modules.*

Proof. (i) $\Rightarrow$ (ii): By 5.4.11 a graded-flat R-module is semi-flat as an R-complex. Thus, it follows from 5.5.1 that φ factors in $\mathcal{C}(R)$ as $N \to L \to F$, where L is a bounded complex of finitely generated free R-modules. Now $N \to L^{\natural} \to F$ is the desired factorization in $\mathcal{M}_{\mathrm{gr}}(R)$.

(ii) $\Rightarrow$ (iii): Let Λ be the class of bounded and degreewise finitely generated graded-free R-modules. By 3.2.7 it suffices to show that F is isomorphic to a filtered colimit in $\mathcal{C}(R)$ of objects from Λ. To apply 3.3.24, it must be argued that every morphism $N \to F$, where N is a bounded and degreewise finitely presented R-complex, admits a factorization in $\mathcal{C}(R)$ through an object from Λ. Since F is a graded R-module, every morphism $N \to F$ in $\mathcal{C}(R)$ factors through the morphism $N \to \mathrm{C}(N)$. The graded module $\mathrm{C}(N)$ is bounded, and by 1.3.40 it is degreewise finitely presented, so every morphism $\mathrm{C}(N) \to F$ factors by assumption through an object in Λ.

(iii) $\Rightarrow$ (i): A graded-free R-module is graded-flat by 2.5.27 and 1.3.43. A filtered colimit of graded-flat modules is graded-flat by 5.4.11, 5.4.13, and 3.2.7. □

5.5.5 Corollary. *Let F be a graded R-module. If every morphism of graded R-modules $N \to F$, with N bounded and degreewise finitely presented, factors through a graded-flat R-module, then F is graded-flat.*

Proof. Let N be a bounded and degreewise finitely presented R-module. By 5.5.4 a morphism $N \to F$ that factors through a graded-flat module has a further factorization through af bounded and degreewise finitely generated graded-free R-module. The desired conclusion now follows from another application of 5.5.4. □

The next result characterizes graded-flat modules by a lifting property akin to 5.2.2(*iii*) for graded-projective modules.

5.5.6 Corollary. *For a graded R-module F the following conditions are equivalent.*

(*i*) *F is graded-flat.*

(*ii*) *For every morphism $\varphi\colon N \to F$ of graded R-modules with N bounded and degreewise finitely presented and for every surjective morphism $\alpha\colon M \to F$ of graded R-modules there is a morphism $\beta\colon N \to M$ with $\varphi = \alpha\beta$.*

Proof. The implication (*i*) $\Rightarrow$ (*ii*) is immediate from 5.5.3.

(*ii*) $\Rightarrow$ (*i*): Let $\alpha\colon L \to F$ be a surjective morphism with L graded-free; see 2.5.28. For every morphism $\varphi\colon N \to F$ with N bounded and degreewise finitely presented, there is by (*ii*) a morphism $\beta\colon N \to L$ with $\varphi = \alpha\beta$. As L is graded-flat, see 2.5.27 and 1.3.43, it follows from 5.5.5 that F is graded-flat. ◻

The Case of Modules

5.5.7 Theorem. *For an R-module F the following conditions are equivalent.*

(*i*) *F is flat.*

(*ii*) *Every homomorphism of R-modules $\varphi\colon N \to F$ with N finitely presented admits a factorization in $\mathcal{M}(R)$,*

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "\kappa"'] & & F \\ & L \arrow[ur, "\lambda"'] & \end{tikzcd},$$

where L is a finitely generated free module.

(*iii*) *F is isomorphic to a filtered colimit of finitely generated free R-modules.*

Proof. The equivalence of (*i*)–(*iii*) is immediate from 5.5.4 in view of 3.2.7. ◻

5.5.8 Corollary. *Let F be an R-module. If every homomorphism $N \to F$ with N finitely presented factors through a flat R-module, then F is flat.*

Proof. This is a special case of 5.5.5; of course, it also follows from 5.5.7 the way 5.5.5 follows from 5.5.4. ◻

The standard characterizations of projective and injective modules include lifting properties; we can now finally provides such a characterization of flat modules.

5.5.9 Corollary. *For an R-module F the following conditions are equivalent.*

(*i*) *F is flat.*

(*ii*) *For every homomorphism $\varphi\colon N \to F$ of R-modules with N finitely presented and for every surjective homomorphism $\alpha\colon M \to F$ there is a homomorphism $\beta\colon N \to M$ with $\varphi = \alpha\beta$.*

PROOF. This is a special case of 5.5.6; of course, it also follows from 5.5.7 the way 5.5.6 follows from 5.5.4. $\square$

5.5.10. Govorov and Lazard's theorem 5.5.7 yields a short proof of 1.3.47. Indeed, if F is a finitely presented and flat R-module, then the identity 1^F factors through a finitely generated free R-module L, whence F is a direct summand of L.

PURITY

5.5.11 Definition. An exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules is called *pure* if for every finitely presented R-module N the induced sequence,

$$0 \longrightarrow \operatorname{Hom}_R(N, M') \longrightarrow \operatorname{Hom}_R(N, M) \longrightarrow \operatorname{Hom}_R(N, M'') \longrightarrow 0\,,$$

is exact.

5.5.12 Example. A split exact sequence of R-modules, $0 \to M' \to M \to M'' \to 0$, is by 2.3.12 pure exact.

The next result is known as the Homotopy Lemma.

5.5.13 Lemma. *Consider a commutative diagram in $\mathcal{M}(R)$ with exact rows,*

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' . \end{tikzcd}$$

The following conditions are equivalent.

(i) *There exists a homomorphism $\sigma\colon M'' \to N$ with $\beta\sigma = \varphi''$.*

(ii) *There exists a homomorphism $\varrho\colon M \to N'$ with $\varrho\alpha' = \varphi'$.*

PROOF. (i) $\Rightarrow$ (ii): Apply the functor $\operatorname{Hom}_R(M, –)$ to the lower row in the diagram to get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, N') \xrightarrow{\operatorname{Hom}(M,\beta')} \operatorname{Hom}_R(M, N) \xrightarrow{\operatorname{Hom}(M,\beta)} \operatorname{Hom}_R(M, N'')\,.$$

If $\beta\sigma = \varphi''$ holds, then one has $\beta(\varphi - \sigma\alpha) = \beta\varphi - \varphi''\alpha = 0$, so there is a homomorphism $\varrho\colon M \to N'$ with $\beta'\varrho = \varphi - \sigma\alpha$. Consequently, one has $\beta'\varrho\alpha' = (\varphi - \sigma\alpha)\alpha' = \varphi\alpha' = \beta'\varphi'$, and hence $\varrho\alpha' = \varphi'$, as β' is injective.

(ii) $\Rightarrow$ (i): Apply $\operatorname{Hom}_R(–, N)$ to the upper row and proceed as above. $\square$

5.5.14 Theorem. *For an exact sequence of R-modules,*

$$\eta \;=\; 0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0\,,$$

the following conditions are equivalent.

(i) *The sequence η is pure.*

(*ii*) *The sequence $K \otimes_R \eta$ is exact for every R^{o}-module K.*
(*iii*) *The sequence $\mathrm{Hom}_{\Bbbk}(\eta, E)$ is split exact every injective $\Bbbk$-module E.*
(*iv*) *The sequence $\mathrm{Hom}_{\Bbbk}(\eta, \mathbb{E})$ is split exact.*
(*v*) *For every commutative diagram in $\mathcal{M}(R)$,*

$$\begin{tikzcd} L' \arrow[r, "\varkappa"] \arrow[d, "\varphi'"'] & L \arrow[d, "\varphi"] \\ M' \arrow[r, "\alpha'"] & M \end{tikzcd},$$

where L and L' are finitely generated free R-modules, there is a homomorphism $\varrho\colon L \to M'$ with $\varrho\varkappa = \varphi'$.

Proof. (*i*) $\Leftrightarrow$ (*ii*): It follows from 3.3.22, 3.2.22, and 3.3.10 that $K \otimes_R \eta$ is exact for every R^{o}-module K if and only if it is exact for every finitely presented R^{o}-module K. For every $m \times n$ matrix α with entries in R, a finitely presented R-module N_α and a finitely presented R^{o}-module K_α are defined by the exact sequences

$$\nu_\alpha \;=\; R^m \xrightarrow{\;\cdot\alpha\;} R^n \longrightarrow N_\alpha \longrightarrow 0 \quad \text{and} \quad \kappa_\alpha \;=\; R^n \xrightarrow{\;\alpha\cdot\;} R^m \longrightarrow K_\alpha \longrightarrow 0\,.$$

Every finitely presented R-module has the form N_α, and every finitely presented R^{o}-module has the form K_α. Hence it suffices to prove, for every matrix α, that $\mathrm{Hom}_R(N_\alpha, \eta)$ is exact if and only if $K_\alpha \otimes_R \eta$ is exact. Let X be an R-module; the sequence $\mathrm{Hom}_R(\nu_\alpha, X)$ is exact and it is given by

$$0 \longrightarrow \mathrm{Hom}_R(N_\alpha, X) \longrightarrow \mathrm{Hom}_R(R^n, X) \xrightarrow{\mathrm{Hom}(\cdot\alpha, X)} \mathrm{Hom}_R(R^m, X)\,.$$

Under the counitor 1.2.2 the homomorphism of $\Bbbk$-modules $\mathrm{Hom}_R(\cdot\alpha, X)$ corresponds to $X^n \xrightarrow{\alpha\cdot} X^m$ as $\mathrm{Hom}_R(R^n, X)$ and $\mathrm{Hom}_R(R^m, X)$ get their R-module structures from the right action on R. Splicing together the exact sequences $\mathrm{Hom}_R(\nu_\alpha, X)$ and $\kappa_\alpha \otimes_R X$, one gets an exact sequence of $\Bbbk$-modules,

$$0 \longrightarrow \mathrm{Hom}_R(N_\alpha, X) \longrightarrow X^n \xrightarrow{\alpha\cdot} X^m \longrightarrow K_\alpha \otimes_R X \longrightarrow 0\,.$$

This sequence depends naturally on X, so η induces an exact sequence,

$$0 \longrightarrow \mathrm{Hom}_R(N_\alpha, \eta) \longrightarrow \eta^n \longrightarrow \eta^m \longrightarrow K_\alpha \otimes_R \eta \longrightarrow 0\,,$$

of $\Bbbk$-complexes. As η is exact, so are η^n and η^m, and it follows that $\mathrm{Hom}_R(N_\alpha, \eta)$ is exact if and only if $K_\alpha \otimes_R \eta$ is exact; cf. 2.5.6.

(*ii*) $\Rightarrow$ (*iii*): Let K be an R^{o}-module and E a $\Bbbk$-module. Adjunction 1.2.6 and commutativity 1.2.3 yield

$$(\star) \qquad \mathrm{Hom}_{R^{\mathrm{o}}}(K, \mathrm{Hom}_{\Bbbk}(\eta, E)) \;\cong\; \mathrm{Hom}_{\Bbbk}(K \otimes_R \eta, E)\,.$$

Thus, if E is injective and the sequence $K \otimes_R \eta$ is exact for every R^{o}-module K, then $\mathrm{Hom}_{R^{\mathrm{o}}}(K, \mathrm{Hom}_{\Bbbk}(\eta, E))$ is exact for every R^{o}-module K, whence the exact sequence $\mathrm{Hom}_R(\eta, E)$ of R^{o}-modules is split.

(*iii*) $\Rightarrow$ (*iv*): Evident.

$(iv) \Rightarrow (iii)$: Let K be an R°-module. If $\mathrm{Hom}_{\Bbbk}(\eta, \mathbb{E})$ is split exact, then the sequence $\mathrm{Hom}_{R^{\circ}}(K, \mathrm{Hom}_{\Bbbk}(\eta, \mathbb{E}))$ is exact, and it follows from $(\star)$ and 1.1.45 that $K \otimes_R \eta$ is exact.

$(v) \Leftrightarrow (i)$: Let N be a finitely presented R-module and consider a presentation

$$(\diamond) \qquad L' \xrightarrow{\varkappa} L \xrightarrow{\pi} N \longrightarrow 0$$

where L and L' are finitely generated free R-modules. To show that the sequence $\mathrm{Hom}_R(N, \eta)$ is exact it suffices to prove that for every homomorphism $\varphi'' \colon N \to M''$ there exists a homomorphism $\sigma \colon N \to M$ with $\alpha\sigma = \varphi''$. Given a homomorphism $\varphi'' \colon N \to M''$, the extension property 1.3.6 yields a commutative diagram in $\mathcal{M}(R)$,

$$(\flat) \qquad \begin{tikzcd} & L' \arrow[r, "\varkappa"] \arrow[d, "\varphi'"] & L \arrow[r, "\pi"] \arrow[d, "\varphi"] & N \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "\alpha'"] & M \arrow[r, "\alpha"] & M'' \arrow[r] & 0\,. \end{tikzcd}$$

By (v) there is a homomorphism $\varrho \colon L \to M'$ with $\varrho\varkappa = \varphi'$, and the existence of the desired σ now follows from 5.5.13. Conversely, consider the commutative square in (iv), define a finitely presented R-module N by exactness of $(\diamond)$, and consider the induced commutative diagram $(\flat)$. By (i) there is a homomorphism $\sigma \colon N \to M$ with $\alpha\sigma = \varphi''$, so 5.5.13 yields a homomorphism $\varrho \colon L \to M'$ with $\varrho\varkappa = \varphi'$, as desired. $\square$

Remark. The conditions in 5.5.14 are also equivalent to the exact sequence η being isomorphic to a filtered colimit of split exact sequences; see for example Jensen and Lenzing [150, Chap. 6].

5.5.15 Definition. A surjective homomorphism $\alpha \colon M \to M''$ is called a *pure epimorphism* if the associated exact sequence $0 \to \mathrm{Ker}\,\alpha \to M \to M'' \to 0$ is pure. An injective homomorphism $\alpha \colon M' \to M$ is called a *pure monomorphism* if the exact sequence $0 \to M' \to M \to \mathrm{Coker}\,\alpha \to 0$ is pure. A quotient module M'' of M is called a *pure* if the quotient map $M \twoheadrightarrow M''$ is a pure epimorphism. A submodule M' of M is called *pure* if the embedding $M' \rightarrowtail M$ is a pure monomorphism.

Remark. An R-module N with the property that the sequence $\mathrm{Hom}_R(N, \eta)$ is exact for every pure exact sequence $\eta = 0 \to M' \to M \to M'' \to 0$ is called *pure-projective*. Dually, an R-module N with the property that $\mathrm{Hom}_R(\eta, N)$ is exact for every pure exact sequence η is called *pure-injective*; they also called 'algebraically compact' modules. These notions form the basis for "pure homological algebra".

5.5.16 Definition. An R-complex M is called *pure acyclic* if it is acyclic and the exact sequence

$$0 \longrightarrow \mathrm{Z}_v(M) \longrightarrow M_v \longrightarrow \mathrm{Z}_{v-1}(M) \longrightarrow 0$$

is pure for every $v \in \mathbb{Z}$.

5.5.17 Example. A contractible R-complex is pure acyclic by 4.3.33 and 5.5.12.

The pure acyclicity property occupies a position between acyclicity and contractibility. It is a non-trivial fact, see 5.5.26, that pure acyclicity and contractibility is the same for complexes of projective modules.

Purity and Flatness

It is immediate from immediate from 5.4.3 that a short exact sequence that ends in a flat module is pure. The next result is a partial converse to this fact.

5.5.18 Proposition. *Let F be a flat R-module. An exact sequence of R-modules,*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0\,,$$

is pure if and only if F'' is flat, and in that case also F' is flat.

Proof. The character module $\operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ is by 5.4.19 injective, so it follows from 1.3.27 and 5.3.27 that the induced exact sequence of R°-modules,

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk}(F'',\mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F,\mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(F',\mathbb{E}) \longrightarrow 0\,,$$

is split if and only if $\operatorname{Hom}_{\Bbbk}(F'',\mathbb{E})$ is injective, in which case also $\operatorname{Hom}_{\Bbbk}(F',\mathbb{E})$ by another application of 1.3.27. Now invoke 5.5.14 and 5.4.19. $\square$

Categorically Flat Complexes

Acyclic semi-projective complexes and acyclic semi-injective complexes are straightforward to understand; they are dealt with in exerices E 5.2.6 and E 5.3.4. Here we give two principal descriptions—Theorems 5.5.19 and 5.5.22—of acyclic semi-flat complexes. Avramov, Foxby, and Halperin [25] use the term *categorically flat* for such complexes; the Remark after 5.5.22 suggests a justification for this terminology.

5.5.19 Theorem. *For an R-complex F the following conditions are equivalent.*

(*i*) *F is semi-flat and acyclic.*

(*ii*) *Every morphism of R-complexes $\varphi\colon N \to F$, with N bounded and degreewise finitely presented, admits a factorization $N \to L \to F$ in $\mathcal{C}(R)$, where L is a bounded and contractible complex of finitely generated free modules.*

(*iii*) *F is isomorphic to a filtered colimit of bounded and contractible complexes of finitely generated free R-modules.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let $\pi\colon P' \xrightarrow{\simeq} F$ be a surjective semi-projective resolution; see 5.2.14. It follows from 5.5.3 that φ factors through π; that is, there is a morphism $\varkappa\colon N \to P'$ with $\varphi = \pi\varkappa$. As P' is acyclic, the complex $\operatorname{Hom}_R(P',P')$ is acyclic by 5.2.10, whence P' is contractible by 4.3.29. By 4.3.32 there is a graded R-module P with $P' \cong \operatorname{Cone} 1^P = \coprod_{v\in\mathbb{Z}} \mathrm{D}^{v+1}(P_v)$. In particular, P is graded-projective, so for every v there is a module Q_v and a set E_v, such that $P_v \oplus Q_v \cong R^{(E_v)}$ holds. Set

$$L' \;=\; P' \oplus \coprod_{v\in\mathbb{Z}} \mathrm{D}^{v+1}(Q_v) \;\cong\; \coprod_{v\in\mathbb{Z}} \mathrm{D}^{v+1}(R)^{(E_v)}\;.$$

The morphism $\varkappa$ factors trivially, $\varkappa = \varpi^{P'}(\varepsilon^{P'}\varkappa)$, through the complex L' of free R-modules. Since N is bounded and degreewise finitely generated, the morphism $\varepsilon^{P'}\varkappa$, and hence φ, factors through a direct sum $L = \bigoplus_{i=1}^{n} \mathrm{D}^{v_i+1}(R)^{(E'_{v_i})}$ where each

E'_{v_i} is a finite subset of E_{v_i}; see 3.1.33. Evidently, L is a bounded and contractible complex of finitely generated free R-modules.

(ii) ⇒ (iii): Follows from 3.3.24.

(iii) ⇒ (i): By assumption there is an isomorphism $F \cong \operatorname{colim}_{u \in U} L^u$ where each L^u is a bounded contractible complex of free R-modules. In particular, each complex L^u is semi-flat and acyclic; see 5.4.7. It follows from 5.4.13 that F is semi-flat and from 3.3.16 that F is acyclic. □

5.5.20 Corollary. *Let F be an R-complex. If every morphism $N \to F$ with N bounded and degreewise finitely presented factors through a semi-flat acyclic R-complex, then F is semi-flat and acyclic.*

Proof. Let N be a bounded and degreewise finitely presented R-complex. By 5.5.1 a morphism $N \to F$ that factors through a semi-flat complex has a further factorization through af bounded complex of finitely generated free modules. The desired conclusion now follows from another application of 5.5.1. □

5.5.21 Corollary. *For an R-complex F the following conditions are equivalent.*

(i) *F is semi-flat and acyclic.*

(ii) *For every morphism $\varphi\colon N \to F$ with N bounded and degreewise finitely presented and for every surjective morphism $\alpha\colon M \to F$ there is a morphism $\beta\colon N \to M$ with $\varphi = \alpha\beta$.*

Proof. (i) ⇒ (ii): It follows from 5.5.19 that there is a contractible complex L of free R-modules and morphisms $\kappa\colon N \to L$ and $\lambda\colon L \to F$ with $\varphi = \lambda\kappa$. The map

$$\operatorname{Hom}_R(L, \alpha) : \operatorname{Hom}_R(L, M) \longrightarrow \operatorname{Hom}_R(L, F)$$

is by 5.2.2 and 4.3.29 a surjective morphism of acyclic complexes. In particular, it is a surjective quasi-isomorphism, so it is surjective on cycles by 4.2.7. In view of 2.3.10 there is a morphism $\beta' : L \to M$ with $\lambda = \alpha\beta'$; now set $\beta = \beta'\kappa$.

(ii) ⇒ (i): Let $\alpha\colon P \to F$ be a surjective morphism where P is a contractible complex of free R-modules; see 4.3.24. For every morphism $\varphi\colon N \to F$ with N bounded and degreewise finitely presented, there exists by (ii) a morphism $\beta\colon N \to P$ with $\varphi = \alpha\beta$. As P is semi-flat and acyclic, see 5.4.7, it follows from 5.5.20 that F is semi-flat and acyclic. □

5.5.22 Theorem. *For an R-complex F the following conditions are equivalent.*

(i) *F is semi-flat and acyclic.*

(ii) *F is a complex of flat R-modules and pure acyclic.*

(iii) *F is an acyclic complex, and $\mathrm{B}(F) = \mathrm{Z}(F)$ is a complex of flat R-modules.*

(iv) *F is a complex of flat R-modules, and for every finitely presented R^{o}-module M the complex $M \otimes_R F$ is acyclic.*

(v) *F is a complex of flat R-modules, and for every R^{o}-complex M the complex $M \otimes_R F$ is acyclic.*

(vi) *F is a complex of flat R-modules, and for every finitely presented R-module N the complex $\operatorname{Hom}_R(N, F)$ is acyclic.*

(*vii*) *F is a complex of flat R-modules, and for every degreewise finitely presented R-complex N the complex* $\operatorname{Hom}_R(N,F)$ *is acyclic.*

(*viii*) *The R^{o}-complex* $\operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ *is a contractible complex of injective modules.*

PROOF. The proof is cyclic and goes as follows:

$$(i) \Rightarrow (viii) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)\,.$$

$(i) \Rightarrow (viii)$: The character complex $\operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ is acyclic by 2.5.7(b), and by 5.4.9 it is semi-injective. Hence, the complex $\operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_{\Bbbk}(F,\mathbb{E}),\operatorname{Hom}_{\Bbbk}(F,\mathbb{E}))$ is acyclic, and it follows from 4.3.29 that $\operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ is contractible.

$(viii) \Rightarrow (v)$: The complex $\operatorname{Hom}_{R^{\mathrm{o}}}(M,\operatorname{Hom}_{\Bbbk}(F,\mathbb{E}))$ is acyclic by 4.3.29. Associativity 4.4.7 and commutativity 4.4.4 yield isomorphisms,

$$\operatorname{Hom}_{R^{\mathrm{o}}}(M,\operatorname{Hom}_{\Bbbk}(F,\mathbb{E})) \;\cong\; \operatorname{Hom}_{\Bbbk}(F\otimes_{R^{\mathrm{o}}} M,\mathbb{E}) \;\cong\; \operatorname{Hom}_{\Bbbk}(M\otimes_R F,\mathbb{E})\,,$$

so $M\otimes_R F$ is acyclic by 2.5.7(b).

$(v) \Rightarrow (iv)$: This implication is evident.

$(iv) \Rightarrow (vi)$: It follows from 3.3.22, 3.2.22, and 3.3.10 that $M\otimes_R F$ is acyclic for every R^{o}-module M. Tensor evaluation 4.5.10(3,c) yields an isomorphism of $\Bbbk$-complexes $\operatorname{Hom}_R(N,R)\otimes_R F \cong \operatorname{Hom}_R(N,F)$, and the left-hand complex is acyclic.

$(vi) \Rightarrow (vii)$: Since each module N_v is finitely presented, it follows from 1.3.40 that $\mathrm{C}_v(N)$ is finitely presented for every $v\in\mathbb{Z}$. Now it follows in view of A.6 that the complex $\operatorname{Hom}_R(N,F)$ is acyclic.

$(vii) \Rightarrow (iii)$: By assumption and the counitor 4.4.2 the complex $\operatorname{Hom}_R(R,F)\cong F$ is acyclic; in particular $\mathrm{B}(F)=\mathrm{Z}(F)$ holds. Let N be a finitely presented R-module and fix an integer v. A homomorphism $\varphi\colon N\to \mathrm{Z}_v(F)$ yields a chain map of R-complexes $N\to F$. As the complex $\operatorname{Hom}_R(N,F)$ is acyclic it follows from 2.3.3(b) that the homomorphism φ factors as

$$N \longrightarrow F_{v+1} \xrightarrow{\partial^F_{v+1}} \mathrm{Z}_v(F)\,.$$

As F_{v+1} is flat by assumption, it follows from 5.5.8 that $\mathrm{Z}_v(F)$ is flat.

$(iii) \Rightarrow (ii)$: For every $v\in\mathbb{Z}$ the sequence

$$0 \longrightarrow \mathrm{Z}_v(F) \longrightarrow F_v \longrightarrow \mathrm{Z}_{v-1}(F) \longrightarrow 0$$

is exact by acyclicity of F. It now follows from 5.4.20 that F_v is a flat R-module, whence the sequence is pure by 5.5.18.

$(ii) \Rightarrow (i)$: Let M be an R^{o}-module. As F is pure acyclic it follows from 5.5.14 that for every $v\in\mathbb{Z}$ the sequence

$$0 \longrightarrow M\otimes_R \mathrm{Z}_v(F) \longrightarrow M\otimes_R F_v \longrightarrow M\otimes_R \mathrm{Z}_{v-1}(F) \longrightarrow 0$$

is exact. Thus the complex $M\otimes_R F$ is acyclic, and in view of A.12 it follows that $M\otimes_R F$ is acyclic for every R^{o}-complex M. In particular, F is semi-flat. □

REMARK. The contractible complexes of injective R^{o}-modules are precisely the injective objects in $\mathcal{C}(R^{\mathrm{o}})$, see E 5.3.4, so while flatness is not a categorical notion, condition (*viii*) in Theorem 5.5.22 explains why the complexes that satisfy the conditions in that theorem get called categorically flat, cf. 5.4.1. Enochs and García Rozas [86] simply call such complexes ‘flat’.

The next corollary supplements 5.4.16.

5.5.23 Corollary. *Let $\alpha\colon F \to F'$ be a quasi-isomorphism of semi-flat R-complexes and N an R-complex. If N is degreewise finitely presented, then* $\operatorname{Hom}_R(N,\alpha)$ *is a quasi-isomorphism.*

Proof. The complex $\operatorname{Cone}\alpha$ is acyclic by 4.2.16, and it follows from 5.4.12, applied to the mapping cone sequence from 4.1.5, that it is semi-flat. If N is degreewise finitely presented, then per 5.5.22 the complex $\operatorname{Hom}_R(N,\operatorname{Cone}\alpha)$ is acyclic, so $\operatorname{Hom}_R(N,\alpha)$ is a quasi-isomorphism by 4.1.16 and another application of 4.2.16. □

The property of categorically flat complexes proved below actually characterizes such complexes and could be added to the list of equivalent conditions in 5.5.22. This was proved by Neeman [192] and generalized by Emmanouil [83]. The argument provided here follows [83].

5.5.24 Proposition. *Let F be a semi-flat R-complex and P a complex of projective R-modules. If F is acyclic, then the complex* $\operatorname{Hom}_R(P,F)$ *is acyclic.*

Proof. For every $v \in \mathbb{Z}$ there is a projective module P'_v such that $P_v \oplus P'_v$ is free. Replacing P with the direct sum of P and $\coprod_{v\in\mathbb{Z}} \Sigma^v P'_v$ one can by additivity of the Hom functor, see 2.3.10, assume that P is a complex of free R-modules. Further, since the assumptions and conclusion are invariant under shift, see 2.2.15 and 2.3.14, it suffices to show that $\mathrm{H}_0(\operatorname{Hom}_R(P,F)) = 0$ holds. That is, it suffices to show that every morphism $\alpha\colon P \to F$ is null homotopic, see 2.3.10.

For every $v \in \mathbb{Z}$ let E_v be a basis for the free R-module P_v. Let U be the set of triples (L,B,σ) where:

(1) L is a subcomplex of P such that each module L_v is free with basis $B_v \subseteq E_v$.
(2) B is the disjoint union $\biguplus_{v\in\mathbb{Z}} B_v$.
(3) $\sigma\colon L \to F$ is a degree 1 homomorphism such that $\alpha|_L = \partial^F\sigma + \sigma\partial^L$ holds.

The set U is non-empty as the zero complex is a subcomplex of P. For elements (L,B,σ) and (L',B',σ') in U declare $(L,B,\sigma) \leqslant (L',B',\sigma')$ if one has $L \subseteq L'$, $B \subseteq B'$ as graded sets, and $\sigma'|_L = \sigma$. This makes U an inductively ordered set, so by Zorn's lemma it has a maximal element (L,B,σ). We proceed to prove that $L = P$ holds, which means that σ is the desired degree 1 map with $\alpha = \partial^F\sigma + \sigma\partial^P$.

Assume towards a contradiction that L is a proper subcomplex of P. The quotient complex P/L is degreewise free, indeed the cosets of the elements in $E_v \setminus B_v$ form a basis for $(P/L)_v$. By assumption there is an integer n with $E_n \setminus B_n \neq \emptyset$; choose an element e of this set and let $E'_n = \{e\}$. Set $N_n = R\langle [e]_{L_n}\rangle$ and $N_v = 0$ for $v > n$. The image of N_n under the differential on P/L is contained in a free submodule of $(P/L)_{n-1}$ that is generated by the cosets of the elements in a finite subset E'_{n-1} of $E_{n-1} \setminus B_{n-1}$; call this module N_{n-1}. Repeating this process yields a subcomplex N of P/L such that each module N_v is a finitely generated free R-module. Let L' be the subcomplex of P with $L'_v = R\langle B_v \cup E'_v\rangle$ and notice that $L'/L \cong N$ holds. Consider the canonical exact seqeunce of R-complexes,

$$0 \longrightarrow L \longrightarrow L' \longrightarrow L'/L \longrightarrow 0\,.$$

It is degreewise split and induces per 2.3.13 an exact sequence of $\Bbbk$-complexes,

$$0 \longrightarrow \operatorname{Hom}_R(L'/L, F) \longrightarrow \operatorname{Hom}_R(L', F) \longrightarrow \operatorname{Hom}_R(L, F) \longrightarrow 0\,.$$

The homotopy σ is an element of degree 1 in $\operatorname{Hom}_R(L, F)$ and hence the image of an element σ' of degree 1 in $\operatorname{Hom}_R(L', F)$; that is, $\sigma'|_L = \sigma$ holds. The image $\partial^{\operatorname{Hom}(L,F)}(\sigma') = \partial^F\sigma' + \sigma'\partial^{L'}$ is a morphism and hence so is $\beta\colon L' \to F$ given by

$$\beta = \alpha|_{L'} - (\partial^F\sigma' + \sigma'\partial^{L'})\,.$$

As one has $\beta|_L = \alpha|_L - (\partial^F\sigma + \sigma\partial^L) = 0$, the morphism β induces a morphism $\gamma\colon L'/L \to F$. Since L'/L is degreewise finitely presented, it follows from 5.5.22 that $\operatorname{Hom}_R(L'/L, F)$ is acyclic, so γ is null-homotopic, see 2.3.3. Thus there is a degree 1 homomorphism $\tau\colon L'/L \to F$ with $\gamma = \partial^F\tau + \tau\partial^{L'/L}$. Denote by π the canonical map $L' \to L'/L$; one has $\beta = \gamma\pi$ and, therefore,

$$\begin{aligned}
\partial^F(\sigma' + \tau\pi) + (\sigma' + \tau\pi)\partial^{L'} &= \partial^F\sigma' + \sigma'\partial^{L'} + \partial^F\tau\pi + \tau\pi\partial^{L'}\\
&= \partial^F\sigma' + \sigma'\partial^{L'} + \partial^F\tau\pi + \tau\partial^{L'/L}\pi\\
&= \partial^F\sigma' + \sigma'\partial^{L'} + (\partial^F\tau + \tau\partial^{L'/L})\pi\\
&= \partial^F\sigma' + \sigma'\partial^{L'} + \gamma\pi\\
&= \partial^F\sigma' + \sigma'\partial^{L'} + \beta\\
&= \alpha|_{L'}\,.
\end{aligned}$$

With $B' = \biguplus_{v\in\mathbb{Z}}(B_v \cup E'_v)$ one now has an element $(L', B', \sigma' + \tau\pi)$ of U, and since $(\sigma' + \tau\pi)|_L = \sigma'|_L = \sigma$ holds one has $(L, B, \sigma) \leqslant (L', B', \sigma' + \tau\pi)$. The inequality is strict, which contradicts the choice of (L, B, σ). Thus one has $L = P$. $\square$

5.5.25 Corollary. *Let $\alpha\colon F \to F'$ be a quasi-isomorphism of semi-flat R-complexes and P a complex of projective R-modules. The morphism* $\operatorname{Hom}_R(P, \alpha)$ *is a quasi-isomorphism.*

Proof. The complex $\operatorname{Cone}\alpha$ is acyclic by 4.2.16, and it follows from 5.4.12, applied to the mapping cone sequence from 4.1.5, that it is semi-flat. By 5.5.24 the complex $\operatorname{Hom}_R(P, \operatorname{Cone}\alpha)$ is acyclic, so $\operatorname{Hom}_R(P, \alpha)$ is a quasi-isomorphism by 4.1.16 and another application of 4.2.16. $\square$

5.5.26 Corollary. *Let P be a complex of projective R-modules. If P is pure acyclic, then it is contractible.*

Proof. Recall from 1.3.43 that P is a complex of flat R-modules. If P is pure acyclic, then $\operatorname{Hom}_R(P, P)$ is acyclic is by 5.5.24, whence P is contractible by 4.3.29. $\square$

5.5.27 Theorem. *Let P be a complex of projective R-modules. If P is semi-flat, then it is semi-projective.*

Proof. Choose by 5.2.14 a semi-projective resolution $\pi\colon L \xrightarrow{\simeq} P$ and recall from 5.4.10 that L is semi-flat. The complex $C = \operatorname{Cone}\pi$ is acyclic by 4.2.16, and it follows from 5.4.12, applied to the mapping cone sequence from 4.1.5, that it is semi-flat.

Thus, C is pure acyclic by 5.5.22. Further C is a complex of projective R-modules, see 4.1.1 and, therefore, contractible by 5.5.26. Thus, π is by 4.3.30 a homotopy equivalence. For every acyclic R-complex M the homotopy equivalent complexes $\operatorname{Hom}_R(L, M) \cong \operatorname{Hom}_R(P, M)$, see 4.3.19, are acyclic by semi-projectivity of L, whence P is semi-projective, see 5.2.10. $\square$

Perfect Rings

We end this chapter with a homological characterization of perfect rings.

5.5.28 Lemma. *Assume that every flat R-module is projective. For every sequence $(a_i)_{i\in\mathbb{N}}$ in R there exists $n \geqslant 1$ such that for every $j > n$ there is an equality of right ideals, $(a_1 \cdots a_{j-1})R = (a_1 \cdots a_{j-1}a_j)R$.*

Proof. For $1 \leqslant i < j$ let α^{ji} be the homothety given by right multiplication on R with $a_i \cdots a_{j-1}$ and set $\alpha^{ii} = 1^R$. These maps form a direct system of R-modules; let A denote its colimit. By 5.4.21 the module A is flat; hence it is projective by the assumption on R. Set $L = R^{(\mathbb{N})}$ and let $\iota^i \colon R \rightarrowtail L$ be the embedding into the i^{th} component; note that $\{\iota^i(1)\}_{i\in\mathbb{N}}$ is a basis for L. Set $f_i = \iota^i(1) - a_i\iota^{i+1}(1)$ for every $i \in \mathbb{N}$. The elements $\{f_i\}_{i\in\mathbb{N}}$ in L are linearly independent as one has

$$\begin{aligned} &r_1 f_1 + r_2 f_2 + \cdots + r_m f_m \\ &\quad = r_1\iota^1(1) + (r_2 - r_1 a_1)\iota^2(1) + \cdots + (r_m - r_{m-1}a_{m-1})\iota^m(1) - r_m a_m \iota^{m+1}(1) \end{aligned}$$

for $r_1, r_2, \ldots, r_m$ in R. Denote by F the free submodule of L generated by $\{f_i\}_{i\in\mathbb{N}}$. Since $\iota^i(r) - \iota^j\alpha^{ji}(r) = rf_i + ra_i f_{i+1} + \cdots + ra_i \cdots a_{j-2}f_{j-1}$ holds, it follows from 3.2.3 that there is an exact sequence $0 \to F \to L \to A \to 0$. As A is projective, the embedding $F \rightarrowtail L$ has a left inverse $\pi \colon L \to F$; cf. 1.3.17. Now, with $\pi(\iota^i(1)) = \sum_{j\geqslant 1} b_{ij} f_j$ one has

$$f_i = \pi(f_i) = \pi(\iota^i(1) - a_i\iota^{i+1}(1)) = \sum_{j\geqslant 1}(b_{ij} - a_i b_{(i+1)j})f_j$$

and, therefore, $b_{ii} - a_i b_{(i+1)i} = 1$ and $b_{ij} - a_i b_{(i+1)j} = 0$ for all $j \neq i$. Thus, for every $j > 1$ there are equalities,

$$b_{1j} = a_1 b_{2j} = a_1 a_2 b_{3j} = \cdots = a_1 a_2 \cdots a_{j-1} b_{jj} = a_1 a_2 \cdots a_{j-1}(1 + a_j b_{(j+1)j}),$$

and hence $a_1 \cdots a_{j-1} = b_{1j} - a_1 \cdots a_{j-1}a_j b_{(j+1)j}$. Since there exists an n such that $b_{1j} = 0$ holds for all $j > n$, the desired assertion follows. $\square$

5.5.29 Lemma. *If the Jacobson radical of R is zero and every descending chain of principal right ideals in R becomes stationary, then R is semi-simple.*

Proof. First note that every right ideal $\mathfrak{a} \neq 0$ in R contains a minimal non-zero right ideal $\mathfrak{b}$. Indeed, by assumption one can take $\mathfrak{b}$ minimal among the non-zero principal right ideals contained in $\mathfrak{a}$. As every non-zero right ideal contains a non-zero principal right ideal, such a $\mathfrak{b}$ is minimal among all non-zero right ideals contained in $\mathfrak{a}$. Furthermore, every minimal non-zero right ideal $\mathfrak{b}$ in R has a complement. Indeed, as the Jacobson radical of R is zero, one has $\mathfrak{b} \not\subseteq \mathfrak{M}$ for some maximal right ideal $\mathfrak{M}$; since $\mathfrak{b}$ is minimal, $\mathfrak{b} \cap \mathfrak{M} = 0$ follows. Consequently, $R = \mathfrak{b} \oplus \mathfrak{M}$ holds.

Now, let $\mathfrak{b}_1$ be a minimal right ideal in R and write $R = \mathfrak{b}_1 \oplus \mathfrak{a}_1$ for some right ideal $\mathfrak{a}_1$. If $\mathfrak{a}_1 = 0$ then the R^{o}-module $R = \mathfrak{b}_1$ is simple. Otherwise, let $\mathfrak{b}_2$ be a minimal right ideal contained in $\mathfrak{a}_1$, and write $\mathfrak{a}_1 = \mathfrak{b}_2 \oplus \mathfrak{a}_2$ for some right ideal $\mathfrak{a}_2$; now one has $R = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{a}_2$. If $\mathfrak{a}_2 = 0$ then the R^{o}-module $R = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ is semi-simple. If $\mathfrak{a}_2 \neq 0$ one can continue the process, which after n iterations yields minimal right ideals $\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_n$ and right ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_n$ such that $R = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n \oplus \mathfrak{a}_n$. Each right ideal $\mathfrak{a}_n$ is principal, as it is a direct summand of R, so the process terminates with $\mathfrak{a}_n = 0$ for some n. Thus the R^{o}-module $R = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ is semi-simple. □

5.5.30 Theorem. *R is left perfect if and only if every flat R-module is projective.*

Proof. Let $\mathfrak{J}$ be the Jacobson radical of R.

"Only if": Let F be a flat R-module. By B.53 it has projective cover, so there is a projective R-module P with a superfluous submodule K such that P/K is flat. By 1.3.44 one has $\mathfrak{J}K = \mathfrak{J}P \cap K$. As K is a superfluous submodule of P it is contained in $\mathfrak{J}P$ by B.37. Thus one has $\mathfrak{J}K = K$ and, therefore, $K = 0$ by B.49.

"If": To show that $\mathfrak{J}$ is left T-nilpotent, let $(a_i)_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{J}$. By 5.5.28 there exist $n \geqslant 1$ and $r \in R$ such that one has $a_1 \cdots a_n = a_1 \cdots a_n a_{n+1} r$ and, therefore, $a_1 \cdots a_n(1 - a_{n+1}r) = 0$. Since a_{n+1} is in $\mathfrak{J}$, the element $1 - a_{n+1}r$ is a unit, and it follows that $a_1 \cdots a_n = 0$. It remains to show that the ring $\boldsymbol{k} = R/\mathfrak{J}$ is semi-simple; to this end apply 5.5.29. Clearly, the Jacobson radical of $\boldsymbol{k}$ is zero. A descending chain of principal right ideals in $\boldsymbol{k}$ has the form $(a_1)\boldsymbol{k} \supseteq (a_1a_2)\boldsymbol{k} \supseteq (a_1a_2a_3)\boldsymbol{k} \supseteq \cdots$ with $a_i \in R$. It follows from 5.5.28 that the descending chain $(a_1)R \supseteq (a_1a_2)R \supseteq (a_1a_2a_3)R \supseteq \cdots$ in R becomes stationary, and hence so does the chain in $\boldsymbol{k}$. □

Remark. The statement of the previous theorem is part of Bass' Theorem P [29] from 1960. With the existence of flat covers for all modules, which was only proved in 2001 by Bican, El Bashir, and Enochs [40], a short proof of the "if" part in 5.5.30 became available. Indeed, every R-module M has a flat cover $\pi\colon F \twoheadrightarrow M$, so if every flat R-module is projective, then π is a projective cover.

Exercises

E 5.5.1 Without recourse to 5.2.11 and 5.4.10, show that every semi-free R-complex is semi-flat. *Hint:* 5.5.1 and E 5.1.6.

E 5.5.2 Let $\eta = 0 \to F' \to F \to F'' \to 0$ be an exact sequence of R-complexes and assume that F is semi-flat. Show that if $\operatorname{Hom}_R(N, \eta)$ is exact for every bounded complex of finitely presented R-modules, then F' and F'' are semi-flat as well. *Hint:* 5.5.3.

E 5.5.3 Let $\varphi\colon R \to S$ be a ring homomorphism. (a) Show that φ is a pure monomorphism of R^{o}-modules if and only if S is faithfully flat over R^{o}. (b) Show that if S is faithfully flat over R^{o} and I is injective over R, then I is a direct summand of the R-module $\operatorname{Hom}_R(S, I)$.

E 5.5.4 Show that the sequence $0 \to M^{(U)} \to M^U \to M^U/M^{(U)} \to 0$ is pure exact for every R-module M and every set U.

E 5.5.5 Find a pure exact, but not split, sequence $0 \to M' \to M \to M'' \to 0$ of R-modules with M'' not flat.

E 5.5.6 Let M be an R-module and K an R^{o}-module. Show that if K is finitely presented, then the homomorphisms $K \otimes_R \delta_{\mathbb{E}}^{M}$ and $\delta_{\mathbb{E}}^{K\otimes_R M}$ are isomorphic in $\mathcal{M}(\Bbbk)$. Apply this to conclude that $\delta_{\mathbb{E}}^{M}$ is a pure monomorphism.

E 5.5.7 Let Λ be a complete set of representatives of isomorphism classes of finitely presented R-modules. Show that the canonical map $\coprod_{L\in\Lambda} L^{(\operatorname{Hom}_R(L,M))} \to M$ is a pure epimorphism for every R-module M.

E 5.5.8 Show that an R-module is pure-projective if and only if it is a summand of a coproduct of finitely presented R-modules.

E 5.5.9 Show that an R-module is projective if and only if it is pure-projective and flat.

E 5.5.10 Show that a pure exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules with M'' finitely presented is split.

E 5.5.11 Assume that R is left Noetherian. Show that if I is an injective R-module, then every pure exact sequence $0 \to M \to I \to N \to 0$ is split.

E 5.5.12 Show that the following conditions are equivalent. (*i*) M is pure acyclic. (*ii*) The complex $\operatorname{Hom}_R(N, M)$ is acyclic for every finitely presented R-module N. (*iii*) The complex $\operatorname{Hom}_R(N, M)$ is acyclic for every degreewise finitely presented R-complex N. (*iv*) The complex $N \otimes_R M$ is acyclic for every finitely presented R^{o}-module N. (*v*) The complex $N \otimes_R M$ is acyclic for every R^{o}-complex N. (*vi*) M is a filtered colimit of bounded and degreewise finitely presented contractible complexes. (*vii*) The character complex $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ is contractible. *Hint:* Proof of 5.5.19 and/or Emmanouil [83].

E 5.5.13 Let M be an R-complex; show that it is pure acyclic if and only if the canonical sequence $0 \to \mathrm{Z}(M) \to M \to \Sigma\mathrm{Z}(M) \to 0$ is degreewise pure exact.

E 5.5.14 Let M be an R-complex such that the exact sequence $0 \to \mathrm{Z}(M) \to M \to \Sigma\mathrm{B}(M) \to 0$ is degreewise pure. Show that M is pure acyclic if and only if it is acyclic. Conclude that an acyclic complex M is pure acyclic if $\mathrm{B}(M) = \mathrm{Z}(M)$ is a complex of flat modules.

E 5.5.15 Show that a pure acyclic degreewise pure subcomplex of a complex of flat R-modules is semi-flat.

E 5.5.16 Show that the Dold complex from 5.1.4 is acyclic but not pure acyclic; conclude that it is not semi-flat.

E 5.5.17 Show that every acyclic complex can be written as a filtered colimit of bounded below acyclic complexes, and show that not every acyclic complex can be written as a filtered colimit of bounded acyclic complexes.

E 5.5.18 Let P be an acyclic complex of projective R-modules. Show that $\mathrm{B}(P) = \mathrm{Z}(P)$ is a complex of flat R-modules if and only if it is a complex of projective R-modules.

E 5.5.19 Let $0 \to F \to P \to F \to 0$ be an exact sequence of R-modules with F flat and P projective. Show that F is projective.

The was first proved by Benson and Goodearl [37]; together with unpublished work [241] of Šťovíček it inspired a study of 'periodic modules' by Bazzoni, Cortés-Izurdiaga, and Estrada [34].

E 5.5.20 Show that the following conditions are equivalent. (*i*) R is von Neumann regular. (*ii*) Every acyclic R-complex is pure acyclic. (*iii*) Every R-complex is semi-flat. (*iv*) Every complex of pure-projective R-modules is semi-projective. (*v*) Every complex of pure-injective R-modules is semi-injective.

E 5.5.21 Let $\Bbbk$ be a field and set $R = \Bbbk^{\mathbb{N}}$. Show that $\mathfrak{a} = \Bbbk^{(\mathbb{N})}$ is and ideal in R and that the complex $F = 0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ is semi-flat; cf. 1.3.45. Show that F is not contractible.

E 5.5.22 Consider the following subsets of $\mathrm{M}_{2\times 2}(\mathbb{R})$,

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \,\middle|\, x \in \mathbb{Q} \text{ and } y \in \mathbb{R} \right\} \qquad\text{and}\qquad \mathfrak{J} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\} .$$

(a) Show that R is a commutative ring with Jacobson radical $\mathfrak{J}$. (b) Show that R is perfect but not Artinian. *Hint:* Pick an infinite descending sequence of $\mathbb{Q}$-submodules of $\mathbb{R}$.

Chapter 6
The Derived Category

From any Abelian category $\mathcal{U}$ one can obtain a triangulated category, called the derived category of $\mathcal{U}$. In this chapter, we construct the derived category of the module category $\mathcal{M}(R)$. This category, $\mathcal{D}(R)$, which is also called the derived category over R, provides an efficacious environment for homological studies of R-modules. The objects in $\mathcal{D}(R)$ are familiar, they are simply all R-complexes, but the morphisms may appear odd on first encounter: They are not maps but equivalence classes of certain diagrams. This part of the construction is technically involved but conceptually simple: the goal is to turn all quasi-isomorphisms into isomorphisms, which is achieved by a procedure that emulates the construction of a ring of fractions.

A three-step process leads from $\mathcal{M}(R)$ to $\mathcal{D}(R)$: The first step was taken in Chap. 2 with the introduction of the category of R-complexes, and what follows immediately below is the second step.

6.1 Construction of the Homotopy Category $\mathcal{K}$

SYNOPSIS. Objects and morphisms; product; coproduct; homotopy invariant functor; universal property.

Let $\mathcal{U}$ be a category and $\approx$ a congruence relation on $\mathcal{U}$. This means that for objects $M, N \in \mathcal{U}$ there is an equivalence relation, $\approx$, on the hom-set $\mathcal{U}(M, N)$ and $\approx$ is compatible with composition of morphisms in $\mathcal{U}$. The quotient category $\mathcal{U}/{\approx}$ has the same objects as $\mathcal{U}$, and for objects M, N the hom-set in $\mathcal{U}/{\approx}$ is the set $\mathcal{U}(M, N)/{\approx}$ of equivalence classes. The canonical functor $\mathrm{Q}\colon \mathcal{U} \to \mathcal{U}/{\approx}$ has the following universal property: If $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is a functor that is invariant under $\approx$, that is, $\mathrm{F}(\alpha) = \mathrm{F}(\beta)$ holds whenever $\alpha \approx \beta$, then there is a unique functor $\dot{\mathrm{F}}$ that makes the next diagram commutative,

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{Q}"'] & \mathcal{V} \\ \mathcal{U}/{\approx} \arrow[ur, dotted, "\dot{\mathrm{F}}"'] & \end{tikzcd} .$$

This section is focused on a particular quotient category: the homotopy category $\mathcal{K}(R)$. It is the quotient of $\mathcal{C}(R)$ modulo homotopy. While $\mathcal{C}(R)$ is Abelian, the category $\mathcal{K}(R)$ is, in general, not. However, it is a triangulated category; cf. Appn. E.

Objects and Morphisms

The next definition is justified by the fact that homotopy is a congruence relation on $\mathcal{C}(R)$; details are recalled in 6.1.2.

6.1.1 Definition. The *homotopy category* $\mathcal{K}(R)$ has the same objects as $\mathcal{C}(R)$, i.e. R-complexes; the morphisms in $\mathcal{K}(R)$ are homotopy classes of morphisms in $\mathcal{C}(R)$.

6.1.2. For R-complexes M and N there is, by 2.3.10, an equality of $\Bbbk$-modules $\mathcal{K}(R)(M,N) = \mathrm{H}_0(\mathrm{Hom}_R(M,N))$. Per 2.2.14 we write $[\alpha]$ for the homotopy class of a morphism α in $\mathcal{C}(R)$. If L is also an R-complex, then the composition

$$\mathcal{K}(R)(M,N) \times \mathcal{K}(R)(L,M) \longrightarrow \mathcal{K}(R)(L,N)$$

maps $([\alpha],[\beta])$ to $[\alpha\beta]$; it follows from 2.2.25 that $[\alpha\beta]$ does not depend on the choice of representatives for $[\alpha]$ and $[\beta]$.

6.1.3 Definition. Write $\mathrm{Q}_R \colon \mathcal{C}(R) \to \mathcal{K}(R)$ for the canonical functor that is the identity on objects and maps a morphism α in $\mathcal{C}(R)$ to its homotopy class $[\alpha]$. When there is no ambiguity, we write Q instead of Q_R.

By construction, the functor Q is full.

6.1.4 Lemma. *For R-complexes M and N with $M_v = 0$ for all $v < 0$ and $N_v = 0$ for all $v > 0$ the map $\mathcal{C}(R)(M,N) \to \mathcal{K}(R)(M,N)$ induced by Q is an isomorphism of $\Bbbk$-modules.*

Proof. This is a restatement of 2.5.10. □

6.1.5 Proposition. *The restriction to $\mathcal{M}(R)$ of the functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ yields an isomorphism between the module category $\mathcal{M}(R)$ and the full subcategory of $\mathcal{K}(R)$ whose objects are all R-complexes concentrated in degree 0.*

Proof. The assertion is immediate from 2.1.36 and 6.1.4. □

6.1.6 Proposition. *Let α be a morphism in $\mathcal{C}(R)$.*

(a) *$[\alpha]$ is an isomorphism in $\mathcal{K}(R)$ if and only if α is a homotopy equivalence.*
(b) *$[\alpha]$ is zero $\mathcal{K}(R)$ if and only if α is null-homotopic.*

Proof. The first assertion is immediate from 4.3.1. For the second assertion, note that the zero complex is a zero object in $\mathcal{K}(R)$. Thus, by definition, $[\alpha]$ is zero if and only if $[\alpha]$ factors through the zero complex in $\mathcal{K}(R)$, which is equivalent to saying that $[\alpha] = [0]$ holds, and by 2.2.23 this precisely means that α is null-homotopic. □

By 4.3.1 and the result above we use the symbol '$\cong$' for isomorphisms in $\mathcal{K}(R)$. An isomorphism α in $\mathcal{C}(R)$ yields an isomorphism $[\alpha]$ in $\mathcal{K}(R)$; such isomorphisms in $\mathcal{K}(R)$ may still be marked by the symbol '$\cong$'.

The next result identifies the zero objects in $\mathcal{K}(R)$. Further characterizations of these complexes are given in 4.3.29 and 4.3.32.

6.1.7 Proposition. *An R-complex is a zero object in $\mathcal{K}(R)$ if and only if it is contractible.*

Proof. Let M be an R-complex. If M is isomorphic to 0 in $\mathcal{K}(R)$, then $\mathcal{K}(R)(M,M)$ consists of a single element. In particular, $[1^M] = [0]$ holds, so 1^M is null-homotopic, i.e. M is contractible. Conversely, if M is contractible then the morphism $M \to 0$ in $\mathcal{C}(R)$ is a homotopy equivalence, whence it represents an isomorphism in $\mathcal{K}(R)$. $\square$

A conspicuous consequence of 4.3.30 and 6.1.7 is that the homotopy class $[\alpha]$ of a morphism in $\mathcal{C}(R)$ is an isomorphism in $\mathcal{K}(R)$ if and only if the complex $\operatorname{Cone}\alpha$ is isomorphic to 0 in $\mathcal{K}(R)$. In Sect. 6.2 it is proved that $\mathcal{K}(R)$ is triangulated, and thus this property of the cone follows from E.22.

Products and Coproducts

6.1.8 Lemma. *Let $\mathcal{U}$ and $\mathcal{V}$ be $\Bbbk$-prelinear categories that have the same objects and $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ a $\Bbbk$-linear functor that is the identity on objects. Let M and N be objects; if the tuple $(M \oplus N, \varpi^M, \varepsilon^M, \varpi^N, \varepsilon^N)$ is a biproduct in $\mathcal{U}$, then the tuple $(M \oplus N, \mathrm{F}(\varpi^M), \mathrm{F}(\varepsilon^M), \mathrm{F}(\varpi^N), \mathrm{F}(\varepsilon^N))$ is a biproduct in $\mathcal{V}$. In particular, if every pair of objects has a biproduct in $\mathcal{U}$ then every pair of objects has a biproduct in $\mathcal{V}$.*

Proof. All assertions follow immediately from the definitions; see 1.1.13. $\square$

Recall that a category is said to have products/coproducts if all set-indexed products/coproducts exist in the category. The next result shows that the homotopy category $\mathcal{K}(R)$ has products and coproducts. As in all other categories one uses the symbols $\prod$ and $\coprod$ for products and coproducts in $\mathcal{K}(R)$.

6.1.9 Theorem. *The homotopy category $\mathcal{K}(R)$ and the functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ are $\Bbbk$-linear. For every family $\{M^u\}_{u\in U}$ of R-complexes the next assertions hold.*

(a) *If an R-complex M with injections $\{\varepsilon^u\colon M^u \rightarrowtail M\}_{u\in U}$ is the coproduct of the family $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$, then M with the morphisms $\{[\varepsilon^u]\}_{u\in U}$ is the coproduct of $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$.*

(b) *If an R-complex M with projections $\{\varpi^u\colon M \twoheadrightarrow M^u\}_{u\in U}$ is the product of the family $\{M^u\}_{u\in U}$ in $\mathcal{C}(R)$, then M with the morphisms $\{[\varpi^u]\}_{u\in U}$ is the product of $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$.*

In particular, the homotopy category $\mathcal{K}(R)$ has products and coproducts, and the canonical functor Q preserves products and coproducts.

Proof. It is straightforward to verify that the category $\mathcal{K}(R)$ is $\Bbbk$-prelinear and that the canonical functor Q is $\Bbbk$-linear. The zero complex is a zero object in $\mathcal{K}(R)$, see 6.1.6, and $\mathcal{K}(R)$ has biproducts by 6.1.8. Thus $\mathcal{K}(R)$ is a $\Bbbk$-linear category.

(a): Let $\{[\alpha^u]\colon M^u \to N\}_{u\in U}$ be morphisms in $\mathcal{K}(R)$. The task is to show that there exists a unique morphism $[\alpha]\colon M \to N$ in $\mathcal{K}(R)$ with $[\alpha\varepsilon^u] = [\alpha^u]$ for all $u \in U$. Existence is straightforward; indeed, by the universal property of coproducts in $\mathcal{C}(R)$, there is a morphism $\alpha\colon M \to N$ with $\alpha\varepsilon^u = \alpha^u$ for all $u \in U$. Applying Q to these identities one gets $[\alpha\varepsilon^u] = [\alpha^u]$. For uniqueness, assume that α and α' are morphisms in $\mathcal{C}(R)$ with $[\alpha\varepsilon^u] = [\alpha^u] = [\alpha'\varepsilon^u]$ for every $u \in U$. It must be shown that $[\alpha] = [\alpha']$. The morphism $\beta = \alpha - \alpha'$ satisfies $[\beta\varepsilon^u] = [0]$ for every $u \in U$, so it follows from 3.1.7 that $[\beta] = [0]$, as desired.

(b): An argument similar to the proof of part (a) applies; only this time appeal to 3.1.19 instead of 3.1.7.

By construction, the canonical functor Q preserves products and coproducts. □

6.1.10. As is the case for the (co)product in any category, the (co)product in $\mathcal{K}(R)$ acts on morphisms. For a family $\{[\alpha^u]\colon M^u \to N^u\}_{u\in U}$ of morphisms in $\mathcal{K}(R)$ it follows from 6.1.9 that one has

$$\coprod_{u\in U}[\alpha^u] = \Big[\coprod_{u\in U}\alpha^u\Big] \qquad\text{and}\qquad \prod_{u\in U}[\alpha^u] = \Big[\prod_{u\in U}\alpha^u\Big],$$

where $\coprod_{u\in U}\alpha^u$ and $\prod_{u\in U}\alpha^u$ is the coproduct and the product of $\{\alpha^u\}_{u\in U}$ in $\mathcal{C}(R)$ as in 3.1.5 and 3.1.17.

6.1.11. It is immediate from 6.1.9 and 3.1.28 that the product and coproduct in $\mathcal{K}(R)$ of a finite family $\{M^u\}_{u\in U}$ of R-complexes coincide, and that this complex is the iterated biproduct $\bigoplus_{u\in U} M^u$ in $\mathcal{K}(R)$. Per 1.1.14 this complex is called the *direct sum* in $\mathcal{K}(R)$ of the family $\{M^u\}_{u\in U}$, and each M^u is called a *direct summand*.

6.1.12 Definition. A morphism $[\alpha]$ in $\mathcal{K}(R)$ is called a *quasi-isomorphism* if some, equivalently every, morphism in $\mathcal{C}(R)$ that represents the homotopy class $[\alpha]$ is a quasi-isomorphism; cf. 2.2.26. Notice that by 4.3.4 and 6.1.6 every isomorphism in $\mathcal{K}(R)$ is a quasi-isomorphism. Quasi-isomorphisms in $\mathcal{K}(R)$ are also marked by the symbol '$\simeq$'; cf. 4.2.1.

A morphism in $\mathcal{K}(R)^{\mathrm{op}}$ is called a *quasi-isomorphism* if the corresponding morphism in $\mathcal{K}(R)$ is a quasi-isomorphism as defined above; this is in line with 4.3.14.

6.1.13 Proposition. *Let $\{[\alpha^u]\colon M^u \to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{K}(R)$. If $[\alpha^u]$ is a quasi-isomorphism for every $u \in U$, then the coproduct $\coprod_{u\in U}[\alpha^u]$ and the product $\prod_{u\in U}[\alpha^u]$ are quasi-isomorphisms.*

Proof. The assertions follow immediately from 4.2.11 and 6.1.10. □

Universal Property

6.1.14 Definition. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ is called *homotopy invariant* if $\mathrm{F}(\alpha) = \mathrm{F}(\beta)$ holds for all homotopic morphisms $\alpha \sim \beta$ in $\mathcal{C}(R)$.

A functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is called *homotopy invariant* if $\mathrm{G}(\alpha) = \mathrm{G}(\beta)$ holds for all homotopic morphisms $\alpha \sim \beta$ in $\mathcal{C}(R)^{\mathrm{op}}$; cf. 4.3.14.

Notice that a functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is homotopy invariant if and only if the opposite functor $\mathcal{C}(R) \to \mathcal{V}^{\mathrm{op}}$ is homotopy invariant as defined in the first part of 6.1.14.

6.1.15 Example. Homology $\mathrm{H}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ is homotopy invariant by 2.2.26.

The canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ from 6.1.3 has the universal property described in the next theorem.

6.1.16 Theorem. *Let $\mathcal{U}$ be a category and $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ a functor. If F is homotopy invariant, then there exists a unique functor $\dot{\mathrm{F}}$ that makes the following diagram commutative,*

$$\begin{tikzcd} \mathcal{C}(R) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{Q}"'] & \mathcal{U} \\ \mathcal{K}(R) \arrow[ur, dotted, "\dot{\mathrm{F}}"'] & \end{tikzcd}$$

For every R-complex M there is an equality $\dot{\mathrm{F}}(M) = \mathrm{F}(M)$, and for every morphism $[\alpha]$ in $\mathcal{K}(R)$ one has $\dot{\mathrm{F}}([\alpha]) = \mathrm{F}(\alpha)$. Furthermore, the following assertions hold.

(a) *If $\mathcal{U}$ is $\Bbbk$-prelinear and F is $\Bbbk$-linear, then $\dot{\mathrm{F}}$ is $\Bbbk$-linear.*

(b) *If $\mathcal{U}$ has products/coproducts and F preserves products/coproducts, then $\dot{\mathrm{F}}$ preserves products/coproducts.*

Proof. Uniqueness of $\dot{\mathrm{F}}$ follows as Q is the identity on objects and full. For existence, set $\dot{\mathrm{F}}(M) = \mathrm{F}(M)$ for every R-complex M and $\dot{\mathrm{F}}([\alpha]) = \mathrm{F}(\alpha)$ for every morphism α of R-complexes; the latter assignment is well-defined as F is homotopy invariant. Evidently, $\dot{\mathrm{F}}$ is a functor with $\dot{\mathrm{F}}\mathrm{Q} = \mathrm{F}$. It remains to prove (a) and (b).

(a): If F is $\Bbbk$-linear, then so is $\dot{\mathrm{F}}$ as the equalities

$$\dot{\mathrm{F}}(x[\alpha] + [\beta]) \;=\; \dot{\mathrm{F}}([x\alpha + \beta]) \;=\; \mathrm{F}(x\alpha + \beta) \;=\; x\,\mathrm{F}(\alpha) + \mathrm{F}(\beta) \;=\; x\,\dot{\mathrm{F}}([\alpha]) + \dot{\mathrm{F}}([\beta])$$

hold for every pair α, β of parallel morphisms in $\mathcal{C}(R)$ and every element x in $\Bbbk$.

(b): Let $\{M^u\}_{u \in U}$ be a family of R-complexes and M its coproduct in $\mathcal{C}(R)$ with injections $\{\varepsilon^u\colon M^u \rightarrowtail M\}_{u \in U}$. It follows from 6.1.9 that M with injections $\{[\varepsilon^u]\}_{u \in U}$ is the coproduct in $\mathcal{K}(R)$. Thus there are commutative diagrams

$$\begin{tikzcd} \mathrm{F}(M^u) \arrow[r] \arrow[d, "\mathrm{F}(\varepsilon^u)"'] & \coprod_{u \in U} \mathrm{F}(M^u) \arrow[dl, "\varphi"] \\ \mathrm{F}(M) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \dot{\mathrm{F}}(M^u) \arrow[r] \arrow[d, "\dot{\mathrm{F}}([\varepsilon^u])"'] & \coprod_{u \in U} \dot{\mathrm{F}}(M^u) \arrow[dl, "\dot{\varphi}"] \\ \dot{\mathrm{F}}(M)\,, & \end{tikzcd}$$

where φ and $\dot{\varphi}$ are the canonical morphisms. By assumption, φ is an isomorphism, and as the two diagrams are identical, $\dot{\varphi}$ is an isomorphism as well. That is, $\dot{\mathrm{F}}$ preserves coproducts. The assertion about products is proved similarly. □

Remark. Theorem 6.1.16 shows that $\mathcal{K}(R)$ has the universal property of a quotient category discussed in the beginning of this section. Together with E 6.1.1 the theorem also shows that $\mathcal{K}(R)$ is the localization of $\mathcal{C}(R)$ with respect to the collection of homotopy equivalences. See also the introduction to Sect. 6.4 where we treat the further localization of $\mathcal{K}(R)$ with respect to the collection of quasi-isomorphisms; this leads to the derived category.

By the universal property above, certain functors on $\mathcal{C}(R)$ induce functors on $\mathcal{K}(R)$, and natural transformations follow along.

6.1.17 Proposition. *Let* $\mathrm{E}, \mathrm{F} \colon \mathcal{C}(R) \to \mathcal{U}$ *be homotopy invariant functors and consider the induced functors* $\dot{\mathrm{E}}, \dot{\mathrm{F}} \colon \mathcal{K}(R) \to \mathcal{U}$; *see* 6.1.16. *Every natural transformation* $\tau \colon \mathrm{E} \to \mathrm{F}$ *induces a natural transformation* $\dot{\tau} \colon \dot{\mathrm{E}} \to \dot{\mathrm{F}}$ *given by* $\dot{\tau}^M = \tau^M$ *for every* R*-complex* M.

Proof. For every R-complex M one has $\dot{\mathrm{E}}(M) = \mathrm{E}(M)$ and $\dot{\mathrm{F}}(M) = \mathrm{F}(M)$, whence $\dot{\tau}^M = \tau^M$ is a morphism $\dot{\mathrm{E}}(M) \to \dot{\mathrm{F}}(M)$. Let $[\alpha] \colon M \to N$ be a morphism in $\mathcal{K}(R)$. As $\tau \colon \mathrm{E} \to \mathrm{F}$ is a natural transformation of functors $\mathcal{C}(R) \to \mathcal{U}$ there are equalities,

$$\dot{\tau}^N \dot{\mathrm{E}}([\alpha]) = \tau^N \mathrm{E}(\alpha) = \mathrm{F}(\alpha)\tau^M = \dot{\mathrm{F}}([\alpha])\dot{\tau}^M ,$$

which show that $\dot{\tau} \colon \dot{\mathrm{E}} \to \dot{\mathrm{F}}$ is a natural transformation of functors $\mathcal{K}(R) \to \mathcal{U}$. $\square$

To parse and prove the next result, recall that if $\mathrm{F} \colon \mathcal{U} \to \mathcal{V}$ is a functor between categories with products (coproducts), then $\mathrm{F}^{\mathrm{op}} \colon \mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is a functor between categories with coproducts (products), and F preserves products (coproducts) if and only if F^{op} preserves coproducts (products).

6.1.18 Theorem. *Let* $\mathcal{V}$ *be a category and* $\mathrm{G} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ *a functor. If* G *is homotopy invariant, then there exists a unique functor* $\dot{\mathrm{G}}$ *that makes the following diagram commutative,*

$$\begin{tikzcd} \mathcal{C}(R)^{\mathrm{op}} \arrow[r, "\mathrm{G}"] \arrow[d, "\mathrm{Q}_R^{\mathrm{op}}"'] & \mathcal{V} \\ \mathcal{K}(R)^{\mathrm{op}} \arrow[ur, dotted, "\dot{\mathrm{G}}"'] & \end{tikzcd} .$$

For every R*-complex* M *there is an equality* $\dot{\mathrm{G}}(M) = \mathrm{G}(M)$*, and for every morphism* $[\alpha]$ *in* $\mathcal{K}(R)^{\mathrm{op}}$ *one has* $\dot{\mathrm{G}}([\alpha]) = \mathrm{G}(\alpha)$*. Furthermore, the following assertions hold.*

(a) *If* $\mathcal{V}$ *is* $\Bbbk$*-prelinear and* G *is* $\Bbbk$*-linear, then* $\dot{\mathrm{G}}$ *is* $\Bbbk$*-linear.*
(b) *If* $\mathcal{V}$ *has products/coproducts and* G *preserves products/coproducts, then* $\dot{\mathrm{G}}$ *preserves products/coproducts.*

Proof. Apply 6.1.16 to the functor $\mathrm{G}^{\mathrm{op}} \colon \mathcal{C}(R) \to \mathcal{V}^{\mathrm{op}}$. $\square$

6.1.19 Proposition. *Let* $\mathrm{G}, \mathrm{J} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ *be homotopy invariant functors and consider the induced functors* $\dot{\mathrm{G}}, \dot{\mathrm{J}} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$; *see* 6.1.18. *Every natural transformation* $\tau \colon \mathrm{G} \to \mathrm{J}$ *induces a natural transformation* $\dot{\tau} \colon \dot{\mathrm{G}} \to \dot{\mathrm{J}}$ *given by* $\dot{\tau}^M = \tau^M$ *for every* R*-complex* M.

Proof. Apply 6.1.17 to the natural transformation $\tau^{\mathrm{op}} \colon \mathrm{J}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ of functors from $\mathcal{C}(R)$ to $\mathcal{V}^{\mathrm{op}}$. $\square$

Special Case of the Universal Property

It is evident that a homotopy invariant functor $\mathcal{C}(R) \to \mathcal{C}(S)$ preserves homotopy; see 4.3.12. On the other hand, if a functor $\mathcal{C}(R) \to \mathcal{C}(S)$ preserves homotopy, then the composite $\mathcal{C}(R) \to \mathcal{C}(S) \to \mathcal{K}(S)$ is homotopy invariant. We apply the homotopy category's universal property in this important special case.

6.1.20 Theorem. *Let* $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be a functor. If* F *preserves homotopy, then there is a unique functor* $\ddot{\mathrm{F}}$ *that makes the next diagram commutative,*

$$\begin{tikzcd} \mathcal{C}(R) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{Q}_R"'] & \mathcal{C}(S) \arrow[d, "\mathrm{Q}_S"] \\ \mathcal{K}(R) \arrow[r, dotted, "\ddot{\mathrm{F}}"] & \mathcal{K}(S) \,. \end{tikzcd}$$

For every R*-complex* M *there is an equality* $\ddot{\mathrm{F}}(M) = \mathrm{F}(M)$*, and for every morphism* $[\alpha]$ *in* $\mathcal{K}(R)$ *one has* $\ddot{\mathrm{F}}([\alpha]) = [\mathrm{F}(\alpha)]$*. Furthermore, the following assertions hold.*

(a) *If* F *is* $\Bbbk$*-linear, then* $\ddot{\mathrm{F}}$ *is* $\Bbbk$*-linear.*

(b) *If* F *preserves products/coproducts, then* $\ddot{\mathrm{F}}$ *preserves products/coproducts.*

Proof. As F preserves homotopy, the composite functor $\mathrm{Q}_S\mathrm{F}$ is homotopy invariant. Thus the existence and the uniqueness of $\ddot{\mathrm{F}}$ follow from 6.1.16. In symbols one has $\ddot{\mathrm{F}} = (\mathrm{Q}_S\mathrm{F})^{\dot{}}$. The value of $\ddot{\mathrm{F}}$ on an R-complex M is $\ddot{\mathrm{F}}(M) = \mathrm{Q}_S\mathrm{F}(M) = \mathrm{F}(M)$ as Q_S is the identity on objects. The value of $\ddot{\mathrm{F}}$ on a morphism $[\alpha]$ in $\mathcal{K}(R)$ is $\ddot{\mathrm{F}}([\alpha]) = \mathrm{Q}_S\mathrm{F}(\alpha) = [\mathrm{F}(\alpha)]$.

By 6.1.9 the functor Q_S is $\Bbbk$-linear and preserves products/coproducts. Thus, if F has any of these properties, then so does $\mathrm{Q}_S\mathrm{F}$, and the assertions in parts (a) and (b) now follow from the corresponding parts in 6.1.16. □

6.1.21 Proposition. *Let* $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be functors that preserve homotopy and consider the induced functors* $\ddot{\mathrm{E}},\ddot{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$; *see* 6.1.20*. Every natural transformation* $\tau\colon \mathrm{E} \to \mathrm{F}$ *induces a natural transformation* $\ddot{\tau}\colon \ddot{\mathrm{E}} \to \ddot{\mathrm{F}}$ *given by* $\ddot{\tau}^M = [\tau^M]$ *for every* R*-complex* M*.*

Proof. Evidently, application of the canonical functor $\mathrm{Q}\colon \mathcal{C}(S) \to \mathcal{K}(S)$ to the natural transformation τ yields a natural transformation $\mathrm{Q}\tau = [\tau]\colon \mathrm{QE} \to \mathrm{QF}$ of functors $\mathcal{C}(R) \to \mathcal{K}(S)$. By definition, $\ddot{\mathrm{E}}$ and $\ddot{\mathrm{F}}$ are the functors $\mathcal{K}(R) \to \mathcal{K}(S)$ induced by QE and QF; see 6.1.16. Thus 6.1.17 gives the desired conclusion. □

We shall often abuse notation and write F for the induced functor $\ddot{\mathrm{F}}$ from 6.1.20.

6.1.22 Example. Let $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ be an additive functor. It extends by 2.1.48 to a functor $\mathcal{C}(R) \to \mathcal{C}(S)$ which by 4.3.18 preserves homotopy. Thus, it induces by 6.1.20 a functor $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$.

6.1.23 Example. The restriction of scalars functors from 1.1.12 associated to a ring homomorphism $\varphi\colon R \to S$ induce by 6.1.22 functors,

$$\mathrm{res}_R^S : \mathcal{K}(S) \longrightarrow \mathcal{K}(R) \qquad \text{and} \qquad \mathrm{res}_{R^o}^{S^o} : \mathcal{K}(S^o) \longrightarrow \mathcal{K}(R^o)\,.$$

These functors are often suppressed; even when they are not we suppress the '${}^{\mathrm{op}}$' on the opposite functors $\mathcal{K}(S)^{\mathrm{op}} \to \mathcal{K}(R)^{\mathrm{op}}$ and $\mathcal{K}(S^o)^{\mathrm{op}} \to \mathcal{K}(R^o)^{\mathrm{op}}$ to avoid clutter.

To parse the next result recall the definitions in 4.2.8.

6.1.24 Proposition. *Let $\varphi\colon R \to S$ be a ring homomorphism. The restriction of scalars functors $\mathrm{res}_R^S\colon \mathcal{K}(S) \to \mathcal{K}(R)$ and $\mathrm{res}_{R^o}^{S^o}\colon \mathcal{K}(S^o) \to \mathcal{K}(R^o)$ preserve and reflect quasi-isomorphisms.*

Proof. The assertion follows immediately from 4.2.14 and 6.1.12. □

6.1.25 Example. In contrast to the restriction of scalars functor on the category of complexes, see 2.1.49, restriction of scalars on the level of homotopy categories need not be conservative and hence not faithful, see 1.1.46. For example, consider the structure map $\Bbbk \to \Bbbk[x]$. The $\Bbbk[x]$-complex $M = 0 \longrightarrow \Bbbk[x] \xrightarrow{x} \Bbbk[x] \longrightarrow \Bbbk \longrightarrow 0$ is acyclic but not contractible, see 2.2.27. Thus, by 6.1.7, the morphism $M \to 0$ is not an isomorphism in $\mathcal{K}(\Bbbk[x])$; however, viewed as a $\Bbbk$-complex, M is contractible by 1.3.17 and hence $M \to 0$ is an isomorphism in $\mathcal{K}(\Bbbk)$.

6.1.26 Example. Let n be an integer and recall from 2.5.24 and 2.5.25 that soft truncation above $(-)_{\subseteq n}$ and soft truncation below $(-)_{\supseteq n}$ are $\Bbbk$-linear endofunctors on $\mathcal{C}(R)$. It follows from 4.3.21 that they preserve homotopy, so by 6.1.20 they yield $\Bbbk$-linear endofunctors on $\mathcal{K}(R)$, also denoted $(-)_{\subseteq n}$ and $(-)_{\supseteq n}$.

6.1.27 Theorem. *Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor. If G preserves homotopy, then there is a unique functor $\ddot{\mathrm{G}}$ that makes the next diagram commutative,*

$$\begin{tikzcd}
\mathcal{C}(R)^{\mathrm{op}} \arrow[r, "\mathrm{G}"] \arrow[d, "\mathrm{Q}_R^{\mathrm{op}}"'] & \mathcal{C}(S) \arrow[d, "\mathrm{Q}_S"] \\
\mathcal{K}(R)^{\mathrm{op}} \arrow[r, dotted, "\ddot{\mathrm{G}}"] & \mathcal{K}(S)\,.
\end{tikzcd}$$

For every R-complex M there is an equality $\ddot{\mathrm{G}}(M) = \mathrm{G}(M)$, and for every morphism $[\alpha]$ in $\mathcal{K}(R)^{\mathrm{op}}$ one has $\ddot{\mathrm{G}}([\alpha]) = [\mathrm{G}(\alpha)]$. Further, the following assertions hold.

(a) *If G is $\Bbbk$-linear, then $\ddot{\mathrm{G}}$ is $\Bbbk$-linear.*
(b) *If G preserves products/coproducts, then $\ddot{\mathrm{G}}$ preserves products/coproducts.*

Proof. Proceed as in the proof of 6.1.20, only apply 6.1.18 in place of 6.1.16. □

6.1.28 Proposition. *Let $\mathrm{G}, \mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be functors that preserve homotopy and consider the induced functors $\ddot{\mathrm{G}}, \ddot{\mathrm{J}}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$; see 6.1.27. Every natural transformation $\tau\colon \mathrm{G} \to \mathrm{J}$ induces a natural transformation $\ddot{\tau}\colon \ddot{\mathrm{G}} \to \ddot{\mathrm{J}}$ given by $\ddot{\tau}^M = [\tau^M]$ for every R-complex M.*

Proof. Proceed as in the proof of 6.1.21, only apply 6.1.19 in place of 6.1.17. □

We shall often abuse notation and write G for the induced functor $\ddot{\mathrm{G}}$ from 6.1.27.

6.1.29 Example. Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be an additive functor. It extends by 2.1.48 to a functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ which by 4.3.18 preserves homotopy. Thus, it induces by 6.1.27 a functor $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$.

6.1.30 Lemma. *Let* $\mathcal{C}(R) \xrightarrow{\mathrm{E}} \mathcal{C}(S) \xrightarrow{\mathrm{F}} \mathcal{U} \xrightarrow{\mathrm{T}} \mathcal{V}$ *be functors where* E *preserves homotopy and* F *is homotopy invariant. The functor* TFE *is then homotopy invariant, and the induced functor* $\mathcal{K}(R) \to \mathcal{V}$ *is* $\mathrm{T\dot{F}\ddot{E}}$; *in symbols,* $(\mathrm{TFE})^{\boldsymbol{\cdot}} = \mathrm{T\dot{F}\ddot{E}}$.
In particular, one has $(\mathrm{TF})^{\boldsymbol{\cdot}} = \mathrm{T\dot{F}}$ *and* $(\mathrm{FE})^{\boldsymbol{\cdot}} = \mathrm{\dot{F}\ddot{E}}$.

Proof. Evidently, TFE is homotopy invariant, and the induced functor $(\mathrm{TFE})^{\boldsymbol{\cdot}}$ is the unique functor with $(\mathrm{TFE})^{\boldsymbol{\cdot}}\,\mathrm{Q}_R = \mathrm{TFE}$. As one has $\mathrm{T\dot{F}\ddot{E}}\,\mathrm{Q}_R = \mathrm{T\dot{F}Q}_S\mathrm{E} = \mathrm{TFE}$, the assertion follows. □

6.1.31 Lemma. *Let* $\mathcal{C}(Q) \xrightarrow{\mathrm{E}} \mathcal{C}(R) \xrightarrow{\mathrm{F}} \mathcal{C}(S)$ *be functors that preserve homotopy. The functor* $\mathcal{K}(Q) \to \mathcal{K}(S)$ *induced by* FE *is* $\mathrm{\ddot{F}\ddot{E}}$; *in symbols,* $(\mathrm{FE})^{\boldsymbol{\cdot\cdot}} = \mathrm{\ddot{F}\ddot{E}}$.

Proof. The induced functor $(\mathrm{FE})^{\boldsymbol{\cdot\cdot}}$ is the unique functor with $(\mathrm{FE})^{\boldsymbol{\cdot\cdot}}\mathrm{Q}_Q = \mathrm{Q}_S\mathrm{FE}$. As one has $\mathrm{\ddot{F}\ddot{E}Q}_Q = \mathrm{\ddot{F}Q}_R\mathrm{E} = \mathrm{Q}_S\mathrm{FE}$, the assertion follows. □

Adjoint Functors

To parse the next result, recall 6.1.20 and 6.1.21.

6.1.32 Lemma. *Consider an adjunction,*

$$\mathcal{C}(S) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{C}(R)\,,$$

with unit $\alpha\colon \mathrm{Id}_{\mathcal{C}(S)} \to \mathrm{GF}$ *and counit* $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{C}(R)}$. *If* F *and* G *preserve homotopy, then the induced functors,*

$$\mathcal{K}(S) \underset{\mathrm{\ddot{G}}}{\overset{\mathrm{\ddot{F}}}{\rightleftarrows}} \mathcal{K}(R)\,,$$

is an adjunction with unit $\ddot{\alpha}\colon \mathrm{Id}_{\mathcal{K}(S)} \to \mathrm{\ddot{G}\ddot{F}}$ *and counit* $\ddot{\beta}\colon \mathrm{\ddot{F}\ddot{G}} \to \mathrm{Id}_{\mathcal{K}(R)}$.

Proof. For the unit and counit of the given adjunction one has the zigzag identities $\mathrm{G}\beta \circ \alpha\mathrm{G} = 1_{\mathrm{G}}$ and $\beta\mathrm{F} \circ \mathrm{F}\alpha = 1_{\mathrm{F}}$. It follows, cf. 6.1.31, that $\mathrm{\ddot{G}}\ddot{\beta} \circ \ddot{\alpha}\mathrm{\ddot{G}} = 1_{\mathrm{\ddot{G}}}$ and $\ddot{\beta}\mathrm{\ddot{F}} \circ \mathrm{\ddot{F}}\ddot{\alpha} = 1_{\mathrm{\ddot{F}}}$ hold, which is equivalent to saying that $(\mathrm{\ddot{F}}, \mathrm{\ddot{G}})$ is an adjunction with unit $\ddot{\alpha}$ and counit $\ddot{\beta}$. □

To parse the next result, recall 6.1.27 and 6.1.28.

6.1.33 Lemma. *Consider an adjunction,*

$$\mathcal{C}(S) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{C}(R)^{\mathrm{op}}\,,$$

with unit $\alpha\colon \mathrm{Id}_{\mathcal{C}(S)} \to \mathrm{GF}$ *and counit* $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{C}(R)^{\mathrm{op}}}$. *If* F *and* G *preserve homotopy, then the induced functors,*

$$\mathcal{K}(S) \underset{\ddot{\mathrm{G}}}{\overset{\ddot{\mathrm{F}}}{\rightleftarrows}} \mathcal{K}(R)^{\mathrm{op}}\,,$$

is an adjunction with unit $\ddot{\alpha}\colon \mathrm{Id}_{\mathcal{K}(S)} \to \ddot{\mathrm{G}}\ddot{\mathrm{F}}$ *and counit* $\ddot{\beta}\colon \ddot{\mathrm{F}}\ddot{\mathrm{G}} \to \mathrm{Id}_{\mathcal{K}(R)^{\mathrm{op}}}$.

PROOF. The assertions follow from an argument similar to the proof of 6.1.32. □

6.1.34. The opposite of the counit $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{C}(R)^{\mathrm{op}}}$ from 6.1.33 is a natural transformation $\beta^{\mathrm{op}}\colon \mathrm{Id}_{\mathcal{C}(R)} \to \mathrm{F}^{\mathrm{op}}\mathrm{G}^{\mathrm{op}}$. In concrete settings it is often more natural to consider β^{op} instead of β. Note that one has $(\beta^{\mathrm{op}})^{\ddot{}} = (\ddot{\beta})^{\mathrm{op}}$.

EXERCISES

E 6.1.1 Show that a functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ is homotopy invariant if and only if $\mathrm{F}(\alpha)$ is an isomorphism in $\mathcal{U}$ for every homotopy equivalence α in $\mathcal{C}(R)$. *Hint:* Proof of 4.3.13.

E 6.1.2 Let P be a semi-projective R-complex and I a semi-injective R-complex. Show that if α is a quasi-isomorphism, then $\mathcal{K}(R)(P,\alpha)$ and $\mathcal{K}(R)(\alpha, I)$ are isomorphisms.

E 6.1.3 Show that the homotopy category $\mathcal{K}(\mathbb{Z})$ is not Abelian.

E 6.1.4 Show that homology, H, and shift, Σ, induce commuting functors on $\mathcal{K}(R)$.

E 6.1.5 Assume that R is semi-simple. Show that the categories $\mathcal{K}(R)$ and $\mathcal{M}_{\mathrm{gr}}(R)$ are equivalent and conclude that $\mathcal{K}(R)$ is Abelian.

E 6.1.6 Show that $\mathcal{M}_{\mathrm{gr}}(R)$ is isomorphic to a full subcategory of $\mathcal{K}(R)$.

E 6.1.7 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{K}(R)$. Show that for any two semi-projective resolutions $\pi\colon P \xrightarrow{\simeq} M$ and $\lambda\colon L \xrightarrow{\simeq} N$ there is a unique morphism $\tilde{\alpha}\colon P \to L$ in $\mathcal{K}(R)$ with $\alpha\pi = \lambda\tilde{\alpha}$.

E 6.1.8 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{K}(R)$. Show that for any two semi-injective resolutions $\iota\colon M \xrightarrow{\simeq} I$ and $\varepsilon\colon N \xrightarrow{\simeq} E$ there is a unique morphism $\tilde{\alpha}\colon I \to E$ in $\mathcal{K}(R)$ with $\varepsilon\alpha = \tilde{\alpha}\iota$.

E 6.1.9 Consider the full subcategories of $\mathcal{K}(R)$ defined by specifying their objects as follows,

$$\mathcal{K}(\mathrm{Prj}\,R) = \{P \in \mathcal{K}(R) \mid P \text{ is a complex of projective modules}\} \quad \text{and}$$
$$\mathcal{K}_{\mathrm{prj}}(R) = \{P \in \mathcal{K}(R) \mid P \text{ is semi-projective}\}\,.$$

Show that both of these categories have coproducts and that $\mathcal{K}_{\mathrm{prj}}(R)$ has products.

E 6.1.10 Consider the full subcategories of $\mathcal{K}(R)$ defined by specifying their objects as follows,

$$\mathcal{K}(\mathrm{Inj}\,R) = \{I \in \mathcal{K}(R) \mid I \text{ is a complex of injective modules}\} \quad \text{and}$$
$$\mathcal{K}_{\mathrm{inj}}(R) = \{I \in \mathcal{K}(R) \mid I \text{ is semi-injective}\}\,.$$

Show that both of these categories have products and that $\mathcal{K}_{\mathrm{inj}}(R)$ has coproducts.

6.2 Triangulation of $\mathcal{K}$

SYNOPSIS. Strict triangle; distinguished triangle; quasi-triangulated functor/transformation; universal property of $\mathcal{K}$ revisited; homology.

The mapping cone construction is the key to the triangulated structure on the homotopy category. The definition of a triangulated category is recalled in Appn. E

6.2.1. By 4.3.16 and 6.1.20 there is a unique $\Bbbk$-linear endofunctor $\ddot{\Sigma}$ on $\mathcal{K}(R)$ that makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{C}(R) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{Q}"'] & \mathcal{C}(R) \arrow[d, "\mathrm{Q}"] \\ \mathcal{K}(R) \arrow[r, "\ddot{\Sigma}"] & \mathcal{K}(R)\,. \end{tikzcd}$$

By 6.1.31 it is an isomorphism with inverse induced by $\Sigma^{-1}\colon \mathcal{C}(R) \to \mathcal{C}(R)$. By the usual abuse of notation, the functor $\ddot{\Sigma}$ is written Σ or, occasionally, $\Sigma_{\mathcal{K}}$. For a morphism $[\alpha]$ in $\mathcal{K}(R)$ one has $\Sigma_{\mathcal{K}}[\alpha] = [\Sigma\,\alpha]$.

Consider the $\Bbbk$-linear category $\mathcal{K}(R)$, see 6.1.9, equipped with the $\Bbbk$-linear automorphism $\Sigma = \Sigma_{\mathcal{K}}$. One may now speak of candidate triangles in $\mathcal{K}(R)$; cf. E.1.

6.2.2 Lemma. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{C}(R)$. The image of the diagram*

$$M \xrightarrow{\;\alpha\;} N \xrightarrow{\left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(\,0\;\,1^{\Sigma M}\,)} \Sigma M$$

under the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ *is a candidate triangle in* $\mathcal{K}(R)$.

Proof. It must be proved that the three composites in $\mathcal{C}(R)$,

$$\lambda = \begin{pmatrix} 1^N \\ 0 \end{pmatrix}\alpha \;=\; \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \mu \;=\; \begin{pmatrix} 0 & 1^{\Sigma M} \end{pmatrix}\begin{pmatrix} 1^N \\ 0 \end{pmatrix} = 0\,, \text{ and } \quad \nu \;=\; (\Sigma\alpha)\begin{pmatrix} 0 & 1^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 0 & \Sigma\alpha \end{pmatrix}$$

are null-homotopic. Since μ is even zero in $\mathcal{C}(R)$, one is left to consider λ and ν. Define degree 1 homomorphisms $\varrho\colon M \to \operatorname{Cone}\alpha$ and $\tau\colon \operatorname{Cone}\alpha \to \Sigma N$ by

$$\varrho \;=\; \begin{pmatrix} 0 \\ \varsigma_1^M \end{pmatrix} \qquad \text{and} \qquad \tau \;=\; \begin{pmatrix} \varsigma_1^N & 0 \end{pmatrix}\;.$$

As ς_1 is a degree 1 chain map and the diagram (2.2.5.1) is commutative, it follows that there are equalities $\partial^{\operatorname{Cone}\alpha}\varrho + \varrho\partial^M = \lambda$ and $\partial^{\Sigma N}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \nu$. Indeed, one has

$$\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix}\begin{pmatrix} 0 \\ \varsigma_1^M \end{pmatrix} + \begin{pmatrix} 0 \\ \varsigma_1^M \end{pmatrix}\partial^M \;=\; \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

and

$$\partial^{\Sigma N}\begin{pmatrix} \varsigma_1^N & 0 \end{pmatrix} + \begin{pmatrix} \varsigma_1^N & 0 \end{pmatrix}\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix} \;=\; \begin{pmatrix} 0 & \Sigma\alpha \end{pmatrix}\;. \qquad \square$$

6.2.3 Definition. A candidate triangle in $\mathcal{K}(R)$ of the form considered in 6.2.2 is called a *strict triangle*. A candidate triangle in $\mathcal{K}(R)$ that is isomorphic, in the sense of E.1, to a strict triangle is called a *distinguished triangle*.

6.2.4 Theorem. *The homotopy category $\mathcal{K}(R)$, equipped with the automorphism Σ and the collection of distinguished triangles defined in* 6.2.3*, is triangulated.*

Proof. We verify the axioms in E.2/E.3.

(TR0): Evidently, the collection of distinguished triangles is closed under isomorphisms. Furthermore, it follows from 4.3.31 and 6.1.7 that application of the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ to the diagram in $\mathcal{C}(R)$,

$$M \xrightarrow{1^M} M \xrightarrow{\left(\begin{smallmatrix}1^M\\0\end{smallmatrix}\right)} \operatorname{Cone} 1^M \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M\,,$$

yields, up to isomorphism in $\mathcal{K}(R)$, the candidate triangle $M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M$ which, therefore, is distinguished.

(TR1): By the definition of morphisms in $\mathcal{K}(R)$, every morphism in this category fits into a strict, and hence distinguished, triangle; see 6.2.2.

(TR2′): By E.5 it it sufficient to verify (TR2). Thus, let

$$\Delta \;=\; M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} \Sigma M'$$

be a distinguished triangle in $\mathcal{K}(R)$. It must be shown that the candidate triangles

$$\Delta^+ \;=\; N' \xrightarrow{\beta'} X \xrightarrow{\gamma'} \Sigma M' \xrightarrow{-\Sigma\alpha'} \Sigma N'$$

and

$$\Delta^- \;=\; \Sigma^{-1}X' \xrightarrow{-\Sigma^{-1}\gamma'} M' \xrightarrow{\alpha'} N' \xrightarrow{\beta'} X'$$

are distinguished. Up to isomorphism, Δ is given by application of the canonical functor Q to a diagram in $\mathcal{C}(R)$ of the form,

$$M \xrightarrow{\alpha} N \xrightarrow{\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M\,.$$

Thus, the candidate triangles Δ^+ and Δ^- are, up to isomorphism, given by application of Q to the following diagrams in $\mathcal{C}(R)$,

$$N \xrightarrow{\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{(0\ 1^{\Sigma M})} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N\,,$$

and

$$\Sigma^{-1}\operatorname{Cone}\alpha \xrightarrow{(0\ -1^M)} M \xrightarrow{\alpha} N \xrightarrow{\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha\,.$$

These two diagrams in $\mathcal{C}(R)$ are the top rows in ($\star$) and ($\diamond$) below. By definition, the bottom rows in ($\star$) and ($\diamond$) give strict triangles in $\mathcal{K}(R)$ when the functor Q is applied; see 6.2.3. Thus, to show that Δ^+ and Δ^- are distinguished triangles in $\mathcal{K}(R)$, it suffices to argue that ($\star$) and ($\diamond$) are commutative up to homotopy, and that the vertical morphisms in both diagrams are homotopy equivalences.

$$(\star)\qquad \begin{tikzcd}
N \arrow[r, "\varepsilon = \left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)"] \arrow[d, equal] & \operatorname{Cone}\alpha \arrow[r, "(0\ 1^{\Sigma M})"] \arrow[d, equal] & \Sigma M \arrow[r, "-\Sigma\alpha"] \arrow[d, "\varphi = \left(\begin{smallmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{smallmatrix}\right)"] & \Sigma N \arrow[d, equal] \\
N \arrow[r, "\varepsilon = \left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)"'] & \operatorname{Cone}\alpha \arrow[r, "\left(\begin{smallmatrix}1^N & 0\\0 & 1^{\Sigma M}\\0 & 0\end{smallmatrix}\right)"] & \operatorname{Cone}\varepsilon \arrow[r, "(0\ 0\ 1^{\Sigma N})"] & \Sigma N
\end{tikzcd}$$

$$
(\diamond)\qquad
\begin{tikzcd}
\Sigma^{-1}\operatorname{Cone}\alpha \arrow[r, "\pi = (0\ -1^M)"] \arrow[d, equal] & M \arrow[r, "\alpha"] \arrow[d, equal] & N \arrow[r, "\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)"] \arrow[d, "\psi = \left(\begin{smallmatrix}0\\1^N\\0\end{smallmatrix}\right)"] & \operatorname{Cone}\alpha \arrow[d, equal] \\
\Sigma^{-1}\operatorname{Cone}\alpha \arrow[r, "\pi = (0\ -1^M)"] & M \arrow[r, "\left(\begin{smallmatrix}1^M\\0\\0\end{smallmatrix}\right)"] & \operatorname{Cone}\pi \arrow[r, "\left(\begin{smallmatrix}0 & 1^N & 0\\0 & 0 & 1^{\Sigma M}\end{smallmatrix}\right)"] & \operatorname{Cone}\alpha
\end{tikzcd}
$$

First consider the diagram $(\star)$ and let $\vartheta\colon \operatorname{Cone}\varepsilon \to \Sigma M$ be the map $(0\ 1^{\Sigma M}\ 0)$. Note that φ and ϑ are morphisms, as one has

$$
\partial^{\operatorname{Cone}\varepsilon}\varphi = \begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{pmatrix} = \begin{pmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{pmatrix}\partial^{\Sigma M} = \varphi\partial^{\Sigma M}
$$

and

$$
\partial^{\Sigma M}\vartheta = \begin{pmatrix}0 & \partial^{\Sigma M} & 0\end{pmatrix} = \begin{pmatrix}0 & 1^{\Sigma M} & 0\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix} = \vartheta\partial^{\operatorname{Cone}\varepsilon}.
$$

Next we argue that φ is a homotopy equivalence with homotopy inverse ϑ. Evidently one has $\vartheta\varphi = 1^{\Sigma M}$, so it remains to show that the morphism

$$
1^{\operatorname{Cone}\varepsilon} - \varphi\vartheta = \begin{pmatrix}1^N & 0 & 0\\ 0 & 1^{\Sigma M} & 0\\ 0 & 0 & 1^{\Sigma N}\end{pmatrix} - \begin{pmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{pmatrix}\begin{pmatrix}0 & 1^{\Sigma M} & 0\end{pmatrix} = \begin{pmatrix}1^N & 0 & 0\\ 0 & 0 & 0\\ 0 & \Sigma\alpha & 1^{\Sigma N}\end{pmatrix}
$$

is null-homotopic. The degree 1 homomorphism

$$
\sigma : \operatorname{Cone}\varepsilon \longrightarrow \operatorname{Cone}\varepsilon \qquad \text{given by} \qquad \sigma = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix}
$$

is the desired homotopy. Indeed, one has

$$
\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \varsigma^N_1 & 0 & 0\end{pmatrix}\begin{pmatrix}\partial^N & \alpha\varsigma^{\Sigma M}_{-1} & \varsigma^{\Sigma N}_{-1}\\ 0 & \partial^{\Sigma M} & 0\\ 0 & 0 & \partial^{\Sigma N}\end{pmatrix} = \begin{pmatrix}1^N & 0 & 0\\ 0 & 0 & 0\\ 0 & \Sigma\alpha & 1^{\Sigma N}\end{pmatrix};
$$

that is, $\partial^{\operatorname{Cone}\varepsilon}\sigma + \sigma\partial^{\operatorname{Cone}\varepsilon} = 1^{\operatorname{Cone}\varepsilon} - \varphi\vartheta$ holds. Thus ϑ is a homotopy inverse of φ.

Now we turn to the issue of commutativity of $(\star)$. The left- and right-hand squares in $(\star)$ are even commutative in $\mathcal{C}(R)$. For the commutativity, up to homotopy, of the middle square, it must be proved that the morphism $\beta\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\varepsilon$, given by

$$
\beta = \begin{pmatrix}1^N & 0\\ 0 & 1^{\Sigma M}\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0\\1^{\Sigma M}\\-\Sigma\alpha\end{pmatrix}\begin{pmatrix}0 & 1^{\Sigma M}\end{pmatrix} = \begin{pmatrix}1^N & 0\\ 0 & 0\\ 0 & \Sigma\alpha\end{pmatrix},
$$

is null-homotopic. Consider the degree 1 homomorphism,

$$\tau : \operatorname{Cone}\alpha \longrightarrow \operatorname{Cone}\varepsilon \qquad \text{given by} \qquad \tau = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix}.$$

It is straightforward to verify the equality

$$\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} & \varsigma_{-1}^{\Sigma N} \\ 0 & \partial^{\Sigma M} & 0 \\ 0 & 0 & \partial^{\Sigma N} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \varsigma_1^N & 0 \end{pmatrix}\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \begin{pmatrix} 1^N & 0 \\ 0 & 0 \\ 0 & \Sigma\alpha \end{pmatrix};$$

that is, $\partial^{\operatorname{Cone}\varepsilon}\tau + \tau\partial^{\operatorname{Cone}\alpha} = \beta$ holds, and hence β is null-homotopic.

Similar arguments show that the diagram $(\diamond)$ is commutative up to homotopy and that ψ is a homotopy equivalence with homotopy inverse $\xi = (\alpha \;\, 1^N \;\, 0)$.

(TR4′): Consider a diagram in $\mathcal{C}(R)$, commutative up to homotopy,

$$(\flat) \qquad \begin{array}{ccccccc} M & \xrightarrow{\alpha} & N & \xrightarrow{\binom{1^N}{0}} & \operatorname{Cone}\alpha & \xrightarrow{(0\;1^{\Sigma M})} & \Sigma M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle\Sigma\varphi} \\ M' & \xrightarrow{\alpha'} & N' & \xrightarrow{\binom{1^{N'}}{0}} & \operatorname{Cone}\alpha' & \xrightarrow{(0\;1^{\Sigma M'})} & \Sigma M'\,. \end{array}$$

To verify (TR4′) one has, in view of the definition of distinguished triangles in $\mathcal{K}(R)$, to establish a morphism $\chi\colon \operatorname{Cone}\alpha \to \operatorname{Cone}\alpha'$ with the following properties. In the first place, χ makes the diagram $(\flat)$ commutative up to homotopy; observe that $(\mathrm{Q}(\varphi), \mathrm{Q}(\psi), \mathrm{Q}(\chi))$ is then a morphism of distinguished triangles in $\mathcal{K}(R)$. Secondly, the mapping cone candidate triangle of $(\mathrm{Q}(\varphi), \mathrm{Q}(\psi), \mathrm{Q}(\chi))$, must be a distinguished triangle in $\mathcal{K}(R)$; it is given by application of Q to the diagram in $\mathcal{C}(R)$,

$$(\dagger) \qquad \begin{matrix} M' \\ \oplus \\ N \end{matrix} \xrightarrow{\begin{pmatrix} \alpha' & \psi \\ 0 & -1^N \\ 0 & 0 \end{pmatrix}} \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} \xrightarrow{\begin{pmatrix} 1^{N'} & \chi^{11} & \chi^{12} \\ 0 & \chi^{21} & \chi^{22} \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}} \begin{matrix} \operatorname{Cone}\alpha' \\ \oplus \\ \Sigma M \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix}} \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix},$$

where the morphisms χ^{ij} are the entries in χ considered as a 2×2 matrix. First we construct a morphism χ that makes $(\flat)$ commutative up to homotopy. By assumption, there is a degree 1 homomorphism $\sigma\colon M \to N'$ such that the equality $\psi\alpha - \alpha'\varphi = \partial^{N'}\sigma + \sigma\partial^M$ holds. Consider the degree 0 homomorphism,

$$\chi : \operatorname{Cone}\alpha \longrightarrow \operatorname{Cone}\alpha' \qquad \text{given by} \qquad \chi = \begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix}.$$

It is straightforward verify that it is a morphism; that is, one has

$$\partial^{\operatorname{Cone}\alpha'}\chi = \begin{pmatrix} \partial^{N'} & \alpha'\varsigma_{-1}^{\Sigma M'} \\ 0 & \partial^{\Sigma M'} \end{pmatrix}\begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix} = \begin{pmatrix} \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & \Sigma\varphi \end{pmatrix}\begin{pmatrix} \partial^N & \alpha\varsigma_{-1}^{\Sigma M} \\ 0 & \partial^{\Sigma M} \end{pmatrix} = \chi\partial^{\operatorname{Cone}\alpha}.$$

Notice that χ makes the middle and right-hand squares in $(\flat)$ commutative.

Finally, to see that application of the functor Q to $(\dagger)$ yields a distinguished triangle in $\mathcal{K}(R)$, note that $(\dagger)$ is the top row in the following diagram, and that the bottom row yields a strict triangle in $\mathcal{K}(R)$ when Q is applied. Thus, it suffices to argue that the diagram below is commutative up to homotopy, and that the vertical morphisms are homotopy equivalences.

$$
\begin{tikzcd}[column sep=huge]
\begin{matrix} M' \\ \oplus \\ N \end{matrix} \arrow[r, "\theta = \begin{pmatrix} \alpha' & \psi \\ 0 & -1^N \\ 0 & 0 \end{pmatrix}"] \arrow[d, equal] & \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} \arrow[r, "\begin{pmatrix} 1^{N'} & \psi & \sigma\varsigma^{\Sigma M}_{-1} \\ 0 & 0 & \Sigma\varphi \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix}"] \arrow[d, equal] & \begin{matrix} \operatorname{Cone}\alpha' \\ \oplus \\ \Sigma M \end{matrix} \arrow[r, "\begin{pmatrix} 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix}"] \arrow[d, "\xi = \begin{pmatrix} 1^{N'} & 0 & \sigma\varsigma^{\Sigma M}_{-1} \\ 0 & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \\ 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix}"] & \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix} \arrow[d, equal] \\
\begin{matrix} M' \\ \oplus \\ N \end{matrix} \arrow[r, "\theta = \begin{pmatrix} \alpha' & \psi \\ 0 & -1^N \\ 0 & 0 \end{pmatrix}"] & \begin{matrix} N' \\ \oplus \\ \operatorname{Cone}\alpha \end{matrix} \arrow[r, "\begin{pmatrix} 1^{N'} & 0 & 0 \\ 0 & 1^N & 0 \\ 0 & 0 & 1^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] & \operatorname{Cone}\theta \arrow[r, swap, "\begin{pmatrix} 0 & 0 & 0 & 1^{\Sigma M'} & 0 \\ 0 & 0 & 0 & 0 & 1^{\Sigma N} \end{pmatrix}"] & \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix}
\end{tikzcd}
$$

The differentials on the complexes $\operatorname{Cone}\alpha' \oplus \Sigma M$ and $\operatorname{Cone}\theta$ are given by

$$
\begin{pmatrix} \partial^{N'} & \alpha'\varsigma^{\Sigma M'}_{-1} & 0 \\ 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & \partial^{\Sigma M} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \partial^{N'} & 0 & 0 & \alpha'\varsigma^{\Sigma M'}_{-1} & \psi\varsigma^{\Sigma N}_{-1} \\ 0 & \partial^{N} & \alpha\varsigma^{\Sigma M}_{-1} & 0 & -\varsigma^{\Sigma N}_{-1} \\ 0 & 0 & \partial^{\Sigma M} & 0 & 0 \\ 0 & 0 & 0 & \partial^{\Sigma M'} & 0 \\ 0 & 0 & 0 & 0 & \partial^{\Sigma N} \end{pmatrix}.
$$

Consider the degree 0 homomorphism,

$$
\eta \colon \operatorname{Cone}\theta \longrightarrow \operatorname{Cone}\alpha' \oplus \Sigma M \qquad \text{given by} \qquad \eta = \begin{pmatrix} 1^{N'} & \psi & \sigma\varsigma^{\Sigma M}_{-1} & 0 & 0 \\ 0 & 0 & \Sigma\varphi & 1^{\Sigma M'} & 0 \\ 0 & 0 & -1^{\Sigma M} & 0 & 0 \end{pmatrix}.
$$

It is straightforward to verify that ξ and η are morphisms. Evidently there is an equality $\eta\xi = 1^{\operatorname{Cone}\alpha' \oplus \Sigma M}$. Furthermore, the morphism $\xi\eta - 1^{\operatorname{Cone}\theta}$ is null-homotopic, as the degree 1 homomorphism

$$
\tau \colon \operatorname{Cone}\theta \longrightarrow \operatorname{Cone}\theta \qquad \text{given by} \qquad \tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varsigma^{N}_{1} & 0 & 0 & 0 \end{pmatrix}
$$

satisfies the identity $\partial^{\operatorname{Cone}\theta}\tau + \tau\partial^{\operatorname{Cone}\theta} = \xi\eta - 1^{\operatorname{Cone}\theta}$. Hence ξ is a homotopy equivalence with homotopy inverse η.

The left-hand and right-hand squares in the diagram are commutative. The diagram's middle square is commutative up to homotopy; indeed, the difference morphism $\gamma \colon N' \oplus \operatorname{Cone}\alpha \to \operatorname{Cone}\theta$, given by

$$\gamma = \begin{pmatrix} 1^{N'} & 0 & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & -1^{\Sigma M} \\ 0 & 1^{\Sigma M'} & \Sigma\varphi \\ 0 & 0 & -\Sigma\alpha \end{pmatrix} \begin{pmatrix} 1^{N'} & \psi & \sigma\varsigma_{-1}^{\Sigma M} \\ 0 & 0 & \Sigma\varphi \\ 0 & 0 & -1^{\Sigma M} \end{pmatrix} - \begin{pmatrix} 1^{N'} & 0 & 0 \\ 0 & 1^{N} & 0 \\ 0 & 0 & 1^{\Sigma M} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \psi & 0 \\ 0 & -1^{N} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma\alpha \end{pmatrix},$$

is null-homotopic. This follows as

$$\varrho : N' \oplus \operatorname{Cone}\alpha \longrightarrow \operatorname{Cone}\theta \qquad \text{given by} \qquad \varrho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varsigma_1^{N} & 0 \end{pmatrix}$$

is a degree 1 homomorphism with $\partial^{\operatorname{Cone}\theta}\varrho + \varrho\partial^{N'\oplus\operatorname{Cone}\alpha} = \gamma$. □

The triangulated structure on $\mathcal{K}(R)$ is by 6.2.3 based on triangles that come from mapping cone sequences; the next lemma shows that starting from mapping cylinder sequences, see 4.3.9, would yield the same structure. This is used later in the chapter to show that every short exact sequence of complexes can be completed to a triangle in the derived category.

6.2.5 Lemma. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{C}(R)$. The image of the diagram*

$$\begin{tikzcd}[column sep=huge, row sep=large]
M \arrow[r, "\iota = \left(\begin{smallmatrix} 0 \\ 0 \\ 1^M \end{smallmatrix}\right)"] \arrow[d, equal] & \operatorname{Cyl}\alpha \arrow[r, "\pi = \left(\begin{smallmatrix} 1^N & 0 & 0 \\ 0 & 1^{\Sigma M} & 0 \end{smallmatrix}\right)"] \arrow[d, "\cong"', "\underline{\alpha} = \left(\begin{smallmatrix} 1^N & 0 & \alpha \end{smallmatrix}\right)"] & \operatorname{Cone}\alpha \arrow[r, "\varpi = \left(\begin{smallmatrix} 0 & 1^{\Sigma M} \end{smallmatrix}\right)"] \arrow[d, equal] & \Sigma M \arrow[d, equal] \\
M \arrow[r, "\alpha"'] & N \arrow[r, "\varepsilon = \left(\begin{smallmatrix} 1^N \\ 0 \end{smallmatrix}\right)"'] & \operatorname{Cone}\alpha \arrow[r, "\varpi = \left(\begin{smallmatrix} 0 & 1^{\Sigma M} \end{smallmatrix}\right)"'] & \Sigma M
\end{tikzcd}$$

under the canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ is an isomorphism of candidate triangles in $\mathcal{K}(R)$. In particular, the image of the top row is a distinguished triangle.

Proof. The image of the bottom row is by definition a distinguished triangle in $\mathcal{K}(R)$. The diagram's left- and right-hand squares are commutative. Consider

$$\sigma : \operatorname{Cyl}\alpha \longrightarrow \operatorname{Cone}\alpha \qquad \text{given by} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varsigma_1^{M} \end{pmatrix}.$$

It is a degree 1 homomorphism, and one has

$$\partial^{\operatorname{Cone}\alpha}\sigma + \sigma\partial^{\operatorname{Cyl}\alpha} = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 1^{\Sigma M} & 0 \end{pmatrix} = \pi - \varepsilon\underline{\alpha}\,,$$

so the middle square is commutative up to homotopy. Finally, $\underline{\alpha}$ is a homotopy equivalence by 4.3.10, so $\mathrm{Q}(\underline{\alpha})$ is an isomorphism. □

QUASI-TRIANGULATED FUNCTORS

We introduce a condition that ensures that a functor from $\mathcal{C}(R)$ to a triangulated category induces a triangulated functor on $\mathcal{K}(R)$.

6.2.6 Definition. Let $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be a triangulated category. A functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ is called *quasi-triangulated* if it is additive and there exists a natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$ such for every morphism $\alpha\colon M \to N$ of R-complexes the diagram

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{F}(\operatorname{Cone}\alpha) \xrightarrow{\phi^M \mathrm{F}(0\ 1^{\Sigma M})} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$.

6.2.7 Example. The canonical functor $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ is quasi-triangulated with the identity transformation playing the role of ϕ; see 6.2.1, 6.1.9, and 6.2.3.

6.2.8 Theorem. *Let $\mathcal{U}$ be a triangulated category and $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ a homotopy invariant functor. If* F *is quasi-triangulated with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$, then the induced functor $\dot{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{U}$, see* 6.1.16*, is triangulated with associated natural isomorphism $\dot{\phi}\colon \dot{\mathrm{F}}\Sigma \to \Sigma_{\mathcal{U}}\dot{\mathrm{F}}$.*

PROOF. As F is additive, so if $\dot{\mathrm{F}}$ by 6.1.16. It follows from 6.1.17 that ϕ induces a natural isomorphism $\dot{\phi}\colon \dot{\mathrm{F}}\Sigma = (\mathrm{F}\Sigma)^{\cdot} \to (\Sigma_{\mathcal{U}}\mathrm{F})^{\cdot} = \Sigma_{\mathcal{U}}\dot{\mathrm{F}}$, where the equalities follows from 6.1.30 and 6.2.1. By 6.2.3 every distinguished triangle in $\mathcal{K}(R)$ is isomorphic to a strict triangle, that is, to a candidate triangle of the form

$$M \xrightarrow{\mathrm{Q}(\alpha)} N \xrightarrow{\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \operatorname{Cone}\alpha \xrightarrow{\mathrm{Q}(0\ 1^{\Sigma M})} \Sigma M\,,$$

where $\alpha\colon M \to N$ is a morphism in $\mathcal{C}(R)$. Thus, it suffices to show that the following candidate triangle in $\mathcal{U}$ is distinguished,

$$\dot{\mathrm{F}}(M) \xrightarrow{\dot{\mathrm{F}}\mathrm{Q}(\alpha)} \dot{\mathrm{F}}(N) \xrightarrow{\dot{\mathrm{F}}\mathrm{Q}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \dot{\mathrm{F}}(\operatorname{Cone}\alpha) \xrightarrow{\dot{\phi}^M \dot{\mathrm{F}}\mathrm{Q}(0\ 1^{\Sigma M})} \Sigma_{\mathcal{U}}\dot{\mathrm{F}}(M)\,.$$

However, this diagram is identical to the one in 6.2.6, which is a distinguished triangle in $\mathcal{U}$ as F is assumed to be quasi-triangulated. $\square$

6.2.9 Definition. Let $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be a triangulated category and $\mathrm{E}, \mathrm{F}\colon \mathcal{C}(R) \to \mathcal{U}$ quasi-triangulated functors with associated natural isomorphisms $\psi\colon \mathrm{E}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{E}$ and $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$. A natural transformation $\tau\colon \mathrm{E} \to \mathrm{F}$ is called *quasi-triangulated* if the following diagram is commutative for every R-complex M,

$$\begin{tikzcd}
\mathrm{E}(\Sigma M) \arrow[r, "\psi^M", "\cong"'] \arrow[d, "\tau^{\Sigma M}"'] & \Sigma_{\mathcal{U}}\mathrm{E}(M) \arrow[d, "\Sigma_{\mathcal{U}}\tau^M"] \\
\mathrm{F}(\Sigma M) \arrow[r, "\phi^M", "\cong"'] & \Sigma_{\mathcal{U}}\mathrm{F}(M)\,.
\end{tikzcd}$$

Thus, $\tau^{\Sigma M}$ and $\Sigma_{\mathcal{U}}\tau^M$ are isomorphic, and the isomorphism is natural in M.

For the next statement, recall from E.12 the notion of a triangulated transformation of triangulated functors.

6.2.10 Proposition. *Let* $\mathcal{U}$ *be a triangulated category. Let* $\mathrm{E}, \mathrm{F} \colon \mathcal{C}(R) \to \mathcal{U}$ *be quasi-triangulated and homotopy invariant functors and* $\tau \colon \mathrm{E} \to \mathrm{F}$ *a natural transformation. If* τ *is quasi-triangulated, then the induced natural transformation* $\dot{\tau} \colon \dot{\mathrm{E}} \to \dot{\mathrm{F}}$ *of triangulated functors, see* 6.1.17 *and* 6.2.8*, is triangulated.*

Proof. By assumption the functors E and F are quasi-triangulated; denote the associated natural isomorphisms by ψ and ϕ, respectively. By 6.2.8 the induced functors $\dot{\mathrm{E}}$ and $\dot{\mathrm{F}}$ are triangulated with associated natural isomorphisms $\dot{\psi}$ and $\dot{\phi}$. For an R-complex M, the equality $\dot{\phi}^M \dot{\tau}^{\Sigma M} = (\Sigma_{\mathcal{U}} \dot{\tau}^M)\dot{\psi}^M$ holds as the left-hand side is $\phi^M \tau^{\Sigma M}$, the right-hand side $(\Sigma_{\mathcal{U}} \tau^M)\psi^M$, and those two composites agree by assumption. □

The next definition ensures that also $\mathrm{Q}^{\mathrm{op}} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{K}(R)^{\mathrm{op}}$ is quasi-triangulated.

6.2.11 Definition. Let $(\mathcal{V}, \Sigma_{\mathcal{V}})$ be a triangulated category. A functor $\mathrm{G} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ is called *quasi-triangulated* if the functor G^{op} from $\mathcal{C}(R)$ to the triangulated category $(\mathcal{V}^{\mathrm{op}}, \Sigma_{\mathcal{V}}^{-1})$, see E.6, is quasi-triangulated in the sense of 6.2.6. Explicitly, this means that G is an additive functor with a natural isomorphism $\psi \colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ of functors $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$, such that the diagram

$$\Sigma_{\mathcal{V}}^{-1}\mathrm{G}(M) \xrightarrow{\mathrm{G}(0\ 1^{\Sigma M})\psi^M} \mathrm{G}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{G}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{G}(N) \xrightarrow{\mathrm{G}(\alpha)} \mathrm{G}(M)$$

is a distinguished triangle in $\mathcal{V}$ for every morphism $\alpha \colon M \to N$ of R-complexes.

6.2.12 Theorem. *Let* $\mathcal{V}$ *be a triangulated category and* $\mathrm{G} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ *a homotopy invariant functor. If* G *is quasi-triangulated with associated natural isomorphism* $\psi \colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$*, then the functor* $\dot{\mathrm{G}} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$*, see* 6.1.18*, is triangulated with associated natural isomorphism* $\dot{\psi} \colon \Sigma_{\mathcal{V}}^{-1}\dot{\mathrm{G}} \to \dot{\mathrm{G}}\Sigma$.

Proof. Apply 6.2.8 to the functor $\mathrm{G}^{\mathrm{op}} \colon \mathcal{C}(R) \to \mathcal{V}^{\mathrm{op}}$. □

6.2.13 Definition. Let $(\mathcal{V}, \Sigma_{\mathcal{V}})$ be a triangulated category and $\mathrm{G}, \mathrm{J} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ quasi-triangulated functors with associated natural isomorphisms $\psi \colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$ and $\phi \colon \Sigma_{\mathcal{V}}^{-1}\mathrm{J} \to \mathrm{J}\Sigma$. A natural transformation $\tau \colon \mathrm{G} \to \mathrm{J}$ is called *quasi-triangulated* if the following diagram is commutative for every R-complex M,

$$\begin{tikzcd}
\Sigma_{\mathcal{V}}^{-1}\mathrm{G}(M) \arrow[r, "\psi^M", "\cong"'] \arrow[d, "\Sigma_{\mathcal{V}}^{-1}\tau^M"'] & \mathrm{G}(\Sigma M) \arrow[d, "\tau^{\Sigma M}"] \\
\Sigma_{\mathcal{V}}^{-1}\mathrm{J}(M) \arrow[r, "\phi^M", "\cong"'] & \mathrm{J}(\Sigma M) .
\end{tikzcd}$$

Thus, $\tau^{\Sigma M}$ and $\Sigma_{\mathcal{V}}^{-1}\tau^M$ are isomorphic, and the isomorphism is natural in M.

6.2.14 Proposition. *Let $\mathcal{V}$ be a triangulated category. Let $\mathrm{G}, \mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{V}$ be quasi-triangulated and homotopy invariant functors and $\tau\colon \mathrm{G} \to \mathrm{J}$ a natural transformation. If τ is quasi-triangulated, then the induced natural transformation $\dot\tau\colon \dot{\mathrm{G}} \to \dot{\mathrm{J}}$ of triangulated functors, see* 6.1.19 *and* 6.2.12*, is triangulated.*

Proof. Apply 6.2.10 to the natural transformation $\tau^{\mathrm{op}}\colon \mathrm{J}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ of functors from $\mathcal{C}(R)$ to $\mathcal{V}^{\mathrm{op}}$. □

Universal Property Revisited

We begin with a result that extends 6.2.7.

6.2.15 Lemma. *Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor. If F is a Σ-functor with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$, then the functor $\mathrm{Q}_S\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{K}(S)$ is quasi-triangulated with associated natural isomorphism $\mathrm{Q}_S\phi$.*

Proof. As F is additive, so is $\mathrm{Q}_S\mathrm{F}$, and evidently $\mathrm{Q}_S\phi$ is a natural isomorphism from $(\mathrm{Q}_S\mathrm{F})\Sigma$ to $\mathrm{Q}_S\Sigma\mathrm{F} = \Sigma(\mathrm{Q}_S\mathrm{F})$, see 6.2.1. It remains to argue that for every morphism $\alpha\colon M \to N$ of R-complexes, the diagram

$$\mathrm{Q}_S\mathrm{F}(M) \xrightarrow{\mathrm{Q}_S\mathrm{F}(\alpha)} \mathrm{Q}_S\mathrm{F}(N) \xrightarrow{\mathrm{Q}_S\mathrm{F}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)} \mathrm{Q}_S\mathrm{F}(\operatorname{Cone}\alpha) \xrightarrow{\mathrm{Q}_S(\phi^M)\mathrm{Q}_S\mathrm{F}(0\ 1^{\Sigma M})} \Sigma\mathrm{Q}_S\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{K}(S)$. By assumption, see 4.1.8, there exists an isomorphism $\breve\alpha$ in $\mathcal{C}(S)$ that makes the following diagram commutative,

$$(\star)\qquad \begin{tikzcd} \mathrm{F}(M) \arrow[r, "\mathrm{F}(\alpha)"] \arrow[d, equal] & \mathrm{F}(N) \arrow[r, "\mathrm{F}\left(\begin{smallmatrix}1^N\\0\end{smallmatrix}\right)"] \arrow[d, equal] & \mathrm{F}(\operatorname{Cone}\alpha) \arrow[r, "\phi^M\mathrm{F}(0\ 1^{\Sigma M})"] \arrow[d, dotted, "\breve\alpha", "\cong"'] & \Sigma\mathrm{F}(M) \arrow[d, equal] \\ \mathrm{F}(M) \arrow[r, "\mathrm{F}(\alpha)"] & \mathrm{F}(N) \arrow[r, "\left(\begin{smallmatrix}1^{\mathrm{F}(N)}\\0\end{smallmatrix}\right)"] & \operatorname{Cone}\mathrm{F}(\alpha) \arrow[r, "(0\ 1^{\Sigma\mathrm{F}(M)})"] & \Sigma\mathrm{F}(M)\,. \end{tikzcd}$$

By application of the functor Q_S to $(\star)$, the top row becomes the diagram in question and the bottom row becomes a strict triangle by definition; see 6.2.3. It follows that the relevant diagram is a distinguished triangle. □

The next theorem justifies the notion of Σ-functors.

6.2.16 Theorem. *Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that preserves homotopy. If F is a Σ-functor with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$, then the induced functor $\ddot{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$, see* 6.1.20*, is triangulated with associated natural isomorphism $\ddot\phi\colon \ddot{\mathrm{F}}\Sigma \to \Sigma\ddot{\mathrm{F}}$.*

Proof. It follows from 6.2.15 that $\mathrm{Q}_S\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{K}(S)$ is a quasi-triangulated functor with associated natural isomorphism $\mathrm{Q}_S\phi$. Thus 6.2.8 implies that the functor $(\mathrm{Q}_S\mathrm{F})^{\cdot} = \ddot{\mathrm{F}}$ is triangulated with associated natural isomorphism $(\mathrm{Q}_S\phi)^{\cdot} = \ddot\phi$ where the equalities follow from 6.1.20 and 6.1.21. □

6.2.17 Proposition. *Let* $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be functors that preserve homotopy and* $\tau\colon \mathrm{E} \to \mathrm{F}$ *a natural transformation. If* E *and* F *are* Σ*-functors and* τ *is a* Σ*-transformation, then the induced natural transformation* $\ddot{\tau}\colon \ddot{\mathrm{E}} \to \ddot{\mathrm{F}}$ *of triangulated functors, see* 6.1.21 *and* 6.2.16*, is triangulated.*

Proof. Let E and F be Σ-functors with associated natural isomorphisms ψ and ϕ. By 6.2.15 the functors $\mathrm{Q}_S\mathrm{E}, \mathrm{Q}_S\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{K}(S)$ are quasi-triangulated with associated natural isomorphisms $\mathrm{Q}_S\psi$ and $\mathrm{Q}_S\phi$. The transformation $\mathrm{Q}_S\tau\colon \mathrm{Q}_S\mathrm{E} \to \mathrm{Q}_S\mathrm{F}$ is quasi-triangulated, indeed, if one applies Q_S to the commutative diagram in 4.1.9, then by 6.2.1 the outcome is the required commutative diagram; see 6.2.9. Now 6.2.10 yields that the natural transformation $(\mathrm{Q}_S\tau)^{\cdot} = \ddot{\tau}$ from $(\mathrm{Q}_S\mathrm{E})^{\cdot} = \ddot{\mathrm{E}}$ to $(\mathrm{Q}_S\mathrm{F})^{\cdot} = \ddot{\mathrm{F}}$ is triangulated; the equalities follow from 6.1.20 and 6.1.21. □

6.2.18 Theorem. *Let* $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be a functor that preserves homotopy. If* G *is a* Σ*-functor with associated natural isomorphism* $\psi\colon \Sigma^{-1}\mathrm{G} \to \mathrm{G}\Sigma$*, then the induced functor* $\ddot{\mathrm{G}}$*, see* 6.1.27*, is triangulated with associated natural isomorphism* $\ddot{\psi}\colon \Sigma^{-1}\ddot{\mathrm{G}} \to \ddot{\mathrm{G}}\Sigma$*.*

Proof. One proceeds as in the proof of 6.2.16, only one applies 6.1.27, 6.1.28, and 6.2.12 in place of 6.1.20, 6.1.21, and 6.2.8. □

6.2.19 Proposition. *Let* $\mathrm{G},\mathrm{J}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be functors that preserve homotopy and* $\tau\colon \mathrm{G} \to \mathrm{J}$ *a natural transformation. If* G *and* J *are* Σ*-functors and* τ *is a* Σ*-transformation, then the induced transformation* $\ddot{\tau}\colon \ddot{\mathrm{G}} \to \ddot{\mathrm{J}}$ *of triangulated functors, see* 6.1.28 *and* 6.2.18*, is triangulated.*

Proof. One proceeds as in the proof of 6.2.17, only one applies 6.1.27, 6.1.28, and 6.2.14 in place of 6.1.20, 6.1.21, and 6.2.10. □

Homology

Homology induces a functor on the homotopy category. It is a primary example of a homological functor in the sense of E.15.

6.2.20 Proposition. *The homology functor* $\mathrm{H}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ *maps homotopy equivalences to isomorphisms. The induced functor* $\mathrm{H}\colon \mathcal{K}(R) \to \mathcal{C}(R)$ *from* 6.1.16 *is* $\Bbbk$*-linear and it preserves products and coproducts. Moreover, one has* $\mathrm{H}\Sigma = \Sigma\mathrm{H}$*.*

Proof. The first assertion follows from 4.3.4 and thus $\mathrm{H}\colon \mathcal{C}(R) \to \mathcal{C}(R)$ induces by 6.1.16 a functor $\dot{\mathrm{H}}\colon \mathcal{K}(R) \to \mathcal{C}(R)$, which outside of this proof is denoted H. By 2.2.15, 3.1.10(d), and 3.1.22(d), the functor H is $\Bbbk$-linear and it preserves products and coproducts. It follows from 6.1.16 that the induced functor $\dot{\mathrm{H}}$ has the same properties. Now 6.2.1, the definition of $\dot{\mathrm{H}}$, and 2.2.15 yield equalities $\dot{\mathrm{H}}\Sigma\mathrm{Q} = \dot{\mathrm{H}}\mathrm{Q}\Sigma = \mathrm{H}\Sigma = \Sigma\mathrm{H} = \Sigma\dot{\mathrm{H}}\mathrm{Q}$, where $\mathrm{Q}\colon \mathcal{C}(R) \to \mathcal{K}(R)$ is the canonical functor from 6.1.3. It now follows from the uniqueness assertion in 6.1.16 that $\dot{\mathrm{H}}\Sigma = \Sigma\dot{\mathrm{H}}$ holds. □

In the balance of this chapter, we use Greek letters for morphisms in $\mathcal{K}(R)$, i.e. homotopy classes of morphisms in $\mathcal{C}(R)$.

The last assertion in the next theorem is akin to 4.2.5 and the Five Lemma E.18 in $\mathcal{K}(R)$. The conclusion is weaker than the Five Lemma's, so is the hypothesis.

6.2.21 Theorem. *For every morphism of distinguished triangles in $\mathcal{K}(R)$,*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\ M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' \end{tikzcd},$$

there is a commutative diagram in $\mathcal{C}(R)$ with exact rows,

$$(6.2.21.1)\quad \begin{tikzcd} \mathrm{H}(M) \arrow[r, "\mathrm{H}(\alpha)"] \arrow[d, "\mathrm{H}(\varphi)"] & \mathrm{H}(N) \arrow[r, "\mathrm{H}(\beta)"] \arrow[d, "\mathrm{H}(\psi)"] & \mathrm{H}(X) \arrow[r, "\mathrm{H}(\gamma)"] \arrow[d, "\mathrm{H}(\chi)"] & \Sigma\mathrm{H}(M) \arrow[r, "\Sigma\mathrm{H}(\alpha)"] \arrow[d, "\Sigma\mathrm{H}(\varphi)"] & \Sigma\mathrm{H}(N) \arrow[d, "\Sigma\mathrm{H}(\psi)"] \\ \mathrm{H}(M') \arrow[r, "\mathrm{H}(\alpha')"] & \mathrm{H}(N') \arrow[r, "\mathrm{H}(\beta')"] & \mathrm{H}(X') \arrow[r, "\mathrm{H}(\gamma')"] & \Sigma\mathrm{H}(M') \arrow[r, "\Sigma\mathrm{H}(\alpha')"] & \Sigma\mathrm{H}(N') \end{tikzcd}.$$

In particular, if two of the morphisms φ, ψ, and χ in $\mathcal{K}(R)$ are quasi-isomorphisms, then so is the third.

Proof. By 6.2.20 the functor $\mathrm{H}\colon \mathcal{K}(R) \to \mathcal{C}(R)$ satisfies the identity $\mathrm{H}\Sigma = \Sigma\mathrm{H}$; it is, therefore, evident that (6.2.21.1) is commutative. We argue that the upper row is exact; a parallel argument shows that the lower row is exact as well. By the definition of distinguished triangles 6.2.3, there is a morphism $\widetilde{\alpha}\colon \widetilde{M} \to \widetilde{N}$ in $\mathcal{C}(R)$ and an isomorphism of candidate triangles in $\mathcal{K}(R)$,

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\cong"] & N \arrow[r, "\beta"] \arrow[d, "\cong"] & X \arrow[r, "\gamma"] \arrow[d, "\cong"] & \Sigma M \arrow[d, "\cong"] \\ \widetilde{M} \arrow[r, "\mathrm{Q}(\widetilde{\alpha})"] & \widetilde{N} \arrow[r, "\mathrm{Q}(\varepsilon)"] & \operatorname{Cone}\widetilde{\alpha} \arrow[r, "\mathrm{Q}(\varpi)"] & \Sigma\widetilde{M} \end{tikzcd},$$

where ε and ϖ denote $(1^{\widetilde{N}}\ 0)^{\mathsf{T}}$ and $(0\ 1^{\Sigma\widetilde{M}})$. Thus, the upper row in (6.2.21.1) is isomorphic to the sequence

$$\mathrm{H}(\widetilde{M}) \xrightarrow{\mathrm{HQ}(\widetilde{\alpha})} \mathrm{H}(\widetilde{N}) \xrightarrow{\mathrm{HQ}(\varepsilon)} \mathrm{H}(\operatorname{Cone}\widetilde{\alpha}) \xrightarrow{\mathrm{HQ}(\varpi)} \Sigma\mathrm{H}(\widetilde{M}) \xrightarrow{\Sigma\mathrm{HQ}(\widetilde{\alpha})} \Sigma\mathrm{H}(\widetilde{N})\,.$$

This sequence is nothing but

$$\mathrm{H}(\widetilde{M}) \xrightarrow{\mathrm{H}(\widetilde{\alpha})} \mathrm{H}(\widetilde{N}) \xrightarrow{\mathrm{H}(\varepsilon)} \mathrm{H}(\operatorname{Cone}\widetilde{\alpha}) \xrightarrow{\mathrm{H}(\varpi)} \Sigma\mathrm{H}(\widetilde{M}) \xrightarrow{\Sigma\mathrm{H}(\widetilde{\alpha})} \Sigma\mathrm{H}(\widetilde{N})\,,$$

which is exact by 4.2.15.

The final assertion about quasi-isomorphisms follows from commutativity of the diagram (6.2.21.1) and the Five Lemma 2.1.41. □

Remark. For every integer m, the homology functor H_m on $\mathcal{K}(R)$ is naturally isomorphic to the functor $\mathcal{K}(R)(\Sigma^m R, -)$. In combination with E.16 and E.17, this can be used to give different proof of 6.2.21.

Exercises

E 6.2.1 Show that for every split exact sequence $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ in $\mathcal{C}(R)$, the diagram
$$M \xrightarrow{[\alpha]} N \xrightarrow{[\beta]} X \xrightarrow{0} \Sigma M$$
is a distinguished triangle in $\mathcal{K}(R)$.

E 6.2.2 Show that an exact sequence $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ of R-modules is split if and only if there exists a distinguished triangle $M \xrightarrow{[\alpha]} N \xrightarrow{[\beta]} X \longrightarrow \Sigma M$ in $\mathcal{K}(R)$. *Hint:* E.22.

E 6.2.3 Show that every degreewise split exact sequence $0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow 0$ in $\mathcal{C}(R)$ can be completed to a distinguished triangle in $\mathcal{K}(R)$,
$$M \xrightarrow{[\alpha]} N \xrightarrow{[\beta]} X \xrightarrow{\gamma} \Sigma M \;,$$
with $\gamma = [-\varsigma_1^M \varrho \partial^N \sigma]$, where $\varrho\colon N \to M$ and $\sigma\colon X \to N$ are the splitting homomorphisms. Compare to E 4.3.8. *Hint:* See E 4.3.23.

E 6.2.4 Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $\mathcal{S}$ a subcategory of $\mathcal{T}$ that is closed under isomorphisms. Show that $\mathcal{S}$ is a triangulated subcategory if and only if $(\mathcal{S}, \Sigma)$ is a triangulated category and the embedding functor $\mathcal{S} \to \mathcal{T}$ is full and triangulated.

E 6.2.5 Let $\mathcal{S}$ be a triangulated subcategory of $(\mathcal{T}, \Sigma)$ and $M \to N \to X \to \Sigma M$ a distinguished triangle in $\mathcal{T}$. Show that if two of the objects M, N, and X are in $\mathcal{S}$, then the third object is also in $\mathcal{S}$.

E 6.2.6 Show that the full subcategory of $\mathcal{K}(R)$ whose objects are all acyclic R-complexes is triangulated.

E 6.2.7 Show that the full subcategories of $\mathcal{K}(R)$ defined by specifying their objects as follows:
$$\mathcal{K}_{\sqsubset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded above complex } M' \text{ with } M \cong M'\}\;,$$
$$\mathcal{K}_{\square}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded complex } M' \text{ with } M \cong M'\}\;, \quad \text{and}$$
$$\mathcal{K}_{\sqsupset}(R) = \{M \in \mathcal{K}(R) \mid \text{there is a bounded below complex } M' \text{ with } M \cong M'\}\;.$$
are triangulated subcategories of $\mathcal{K}(R)$.

E 6.2.8 Show that the functors $(-)_{\subseteq n}, (-)_{\supseteq n}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ preserve products and coproducts but are not triangulated.

E 6.2.9 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T}, \Sigma)$. A morphism $\alpha\colon M \to N$ is called $\mathcal{S}$-*trivial* if in some, equivalently in every, distinguished triangle,
$$M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M \;,$$
the object X belongs to $\mathcal{S}$. Describe the $\mathcal{S}$-trivial morphisms in the category $\mathcal{K}(R)$ if $\mathcal{S}$ consists of all acyclic R-complexes; cf. E 6.2.6.

E 6.2.10 Show that $\mathcal{K}(\operatorname{Prj} R)$, see E 6.1.9, is a triangulated category but not a triangulated subcategory of $\mathcal{K}(R)$. Show that the inclusion functor $\mathcal{K}(\operatorname{Prj} R) \to \mathcal{K}(R)$ is triangulated.

E 6.2.11 Show that $\mathcal{K}_{\mathrm{prj}}(R)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{Prj} R)$; see E 6.1.9.

E 6.2.12 Show that $\mathcal{K}(\operatorname{Inj} R)$, see E 6.1.10, is a triangulated category but not a triangulated subcategory of $\mathcal{K}(R)$. Show that the inclusion functor $\mathcal{K}(\operatorname{Inj} R) \to \mathcal{K}(R)$ is triangulated.

E 6.2.13 Show that $\mathcal{K}_{\mathrm{inj}}(R)$ is a triangulated subcategory of $\mathcal{K}(\operatorname{Inj} R)$; see E 6.1.10.

E 6.2.14 Give a proof of the Five Lemma in $\mathcal{K}(R)$ without using E.18.

E 6.2.15 Give a proof of 6.2.21 using the ideas in the subsequent Remark.

6.3 Resolutions

Synopsis. Unique lifting property; functoriality of resolution.

To all intents and purposes, the non-uniqueness inherent in the resolutions of complexes and liftings of morphisms disappears in the homotopy category.

Recall that Greek letters, unless otherwise specified, denote morphisms in $\mathcal{K}(R)$.

Uniqueness of Liftings

We rephrase 5.2.19–5.2.21 in the language of the homotopy category.

6.3.1 Proposition. *Let P be a semi-projective R-complex, $\alpha\colon P \to N$ a morphism, and $\beta\colon M \to N$ a quasi-isomorphism in $\mathcal{K}(R)$. There exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\gamma"'] \arrow[d, "\alpha"] \\ M \arrow[r, "\simeq", "\beta"'] & N \,. \end{tikzcd}$$

Proof. This is a special case of 5.2.19 for chain maps of degree zero. □

A surjective quasi-isomorphism to a semi-projective complex has a right inverse in the category of complexes, see 5.2.10. In the homotopy category, every quasi-isomorphism to a semi-projective complex has a right inverse. See also B.56.

6.3.2 Corollary. *Let $\beta\colon M \to P$ be a quasi-isomorphism in $\mathcal{K}(R)$. If P is semi-projective, then β has a right inverse in $\mathcal{K}(R)$ which is also a quasi-isomorphism.*

Proof. This is a reformulation of 5.2.20. □

6.3.3 Corollary. *Let $\beta\colon P \to P'$ be a quasi-isomorphism in $\mathcal{K}(R)$. If P and P' are semi-projective, then β is an isomorphism in $\mathcal{K}(R)$.*

Proof. This is a reformulation of 5.2.21. □

6.3.4 Corollary. *Let P be a semi-projective R-complex. Consider morphisms*

$$P \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M \xrightarrow[\simeq]{\varphi} N$$

in $\mathcal{K}(R)$ where φ is a quasi-isomorphism. If $\varphi\alpha = \varphi\beta$ holds, then one has $\alpha = \beta$.

Proof. This is an immediate consequence of 6.3.1. □

We also recast 5.3.22–5.3.24 in the language of the homotopy category.

6.3.5 Proposition. *Let I be a semi-injective R-complex, $\alpha\colon M \to I$ a morphism, and $\beta\colon M \to N$ a quasi-isomorphism in $\mathcal{K}(R)$. There exists a unique morphism γ that makes the following diagram in $\mathcal{K}(R)$ commutative,*

$$\begin{tikzcd} M \arrow[r, "\beta", "\simeq"'] \arrow[d, "\alpha"'] & N \arrow[dl, dotted, "\gamma"] \\ I\,. \end{tikzcd}$$

Proof. This is a special case of 5.3.22 for chain maps of degree zero. □

An injective quasi-isomorphism from a semi-injective complex has a left inverse in the category of complexes, see 5.3.16. In the homotopy category, every quasi-isomorphism from a semi-injective complex has a left inverse. See also B.23.

6.3.6 Corollary. *Let $\beta\colon I \to M$ be a quasi-isomorphism in $\mathcal{K}(R)$. If I is semi-injective, then β has a left inverse in $\mathcal{K}(R)$ which is also a quasi-isomorphism.*

Proof. This is a reformulation of 5.3.23. □

6.3.7 Corollary. *Let $\beta\colon I \to I'$ be a quasi-isomorphism in $\mathcal{K}(R)$. If I and I' are semi-injective, then β is an isomorphism in $\mathcal{K}(R)$.*

Proof. This is a reformulation of 5.3.24. □

The next lemma is dual to 6.3.4.

6.3.8 Corollary. *Let I be a semi-injective R-complex. Consider morphisms*

$$\begin{tikzcd} M \arrow[r, "\varphi", "\simeq"'] & N \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & I \end{tikzcd}$$

in $\mathcal{K}(R)$ where φ is a quasi-isomorphism. If $\alpha\varphi = \beta\varphi$ holds, then one has $\alpha = \beta$.

Proof. This is an immediate consequence of 6.3.5. □

Functoriality of Resolutions

In the homotopy category, all semi-projective resolutions of a given complex are isomorphic, and a morphism of complexes lifts uniquely to their resolutions. This is sufficient to make the process of taking semi-projective resolutions functorial.

We apply the terminology from 5.2.13 and 5.3.13 to quasi-isomorphisms in the homotopy category. That is, a quasi-isomorphism $P \to M$ in $\mathcal{K}(R)$, where P is semi-projective is called a semi-projective resolution of M; similarly a quasi-isomorphism $M \to I$, where I is semi-injective, is called a semi-injective resolution of M.

6.3.9 Construction. For every R-complex M, choose by 5.2.15 a semi-projective resolution $\pi_R^M\colon \mathrm{P}_R(M) \xrightarrow{\simeq} M$ in $\mathcal{K}(R)$. For every morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ there is by 6.3.1 a unique morphism in $\mathcal{K}(R)$, denoted $\mathrm{P}_R(\alpha)$, such that the diagram

$$
(6.3.9.1) \qquad \begin{tikzcd} \mathrm{P}_R(M) \arrow[r, "\pi_R^M", "\simeq"'] \arrow[d, "\mathrm{P}_R(\alpha)"'] & M \arrow[d, "\alpha"] \\ \mathrm{P}_R(N) \arrow[r, "\pi_R^N", "\simeq"'] & N \end{tikzcd}
$$

is commutative. Similarly, there is a unique morphism in $\mathcal{K}(R)$, which we denote ϕ^M, such that the following diagram is commutative,

$$
(6.3.9.2) \qquad \begin{tikzcd} & \mathrm{P}_R(\Sigma M) \arrow[dl, "\phi^M"'] \arrow[d, "\pi_R^{\Sigma M}", "\simeq"'] \\ \Sigma \mathrm{P}_R(M) \arrow[r, "\simeq", "\Sigma\, \pi_R^M"'] & \Sigma M\,. \end{tikzcd}
$$

6.3.10 Definition. The functor P_R established in the next theorem is called the *semi-projective resolution functor*. When there is no ambiguity, we write P instead of P_R and, likewise, we drop the subscript from the associated natural transformation π_R.

6.3.11 Theorem. *The assignments $M \mapsto \mathrm{P}_R(M)$ and $\alpha \mapsto \mathrm{P}_R(\alpha)$ from* 6.3.9 *define an endofunctor on $\mathcal{K}(R)$. This functor P_R is $\Bbbk$-linear, it preserves coproducts, it maps quasi-isomorphisms to isomorphisms, and ϕ is a natural isomorphism $\mathrm{P}_R\Sigma \to \Sigma\mathrm{P}_R$ such that P_R is triangulated. Furthermore, π_R is a triangulated natural transformation $\mathrm{P}_R \to \mathrm{Id}_{\mathcal{K}(R)}$ and π_R^M is a quasi-isomorphism for every R-complex M.*

Proof. Per 6.3.10 we drop the subscript from the functor P_R and the transformation π_R. For morphisms $\alpha\colon M \to N$ and $\beta\colon L \to M$ in $\mathcal{K}(R)$, the composite $\mathrm{P}(\alpha)\,\mathrm{P}(\beta)$ makes the following diagram commutative,

$$
\begin{tikzcd} \mathrm{P}(L) \arrow[r, "\pi^L", "\simeq"'] \arrow[d, dotted] & L \arrow[d, "\alpha\beta"] \\ \mathrm{P}(N) \arrow[r, "\pi^N", "\simeq"'] & N\,, \end{tikzcd}
$$

commutative, and hence $\mathrm{P}(\alpha\beta) = \mathrm{P}(\alpha)\,\mathrm{P}(\beta)$ holds by the definition of $\mathrm{P}(\alpha\beta)$. Similarly one finds that the equality $\mathrm{P}(1^M) = 1^{\mathrm{P}(M)}$ holds for every R-complex M and that $\mathrm{P}(x\alpha + \beta) = x\,\mathrm{P}(\alpha) + \mathrm{P}(\beta)$ holds for every pair α, β of parallel morphisms in $\mathcal{K}(R)$ and every element x in $\Bbbk$. Thus, P is a $\Bbbk$-linear functor.

Commutativity of (6.3.9.1) implies that if α is a quasi-isomorphism in $\mathcal{K}(R)$, then so is $\mathrm{P}(\alpha)$. It now follows from 6.3.3 that $\mathrm{P}(\alpha)$ is an isomorphism; thus P maps quasi-isomorphisms to isomorphisms. Commutativity of (6.3.9.1) also shows that π is a natural transformation from P to $\mathrm{Id}_{\mathcal{K}(R)}$.

To see that P preserves coproducts, let $\{M^u\}_{u\in U}$ be a family of R-complexes and notice that there is a commutative diagram in $\mathcal{K}(R)$,

$$\begin{tikzcd} \coprod_{u\in U} \mathrm{P}(M^u) \arrow[rr, "\varphi"] \arrow[dr, "\coprod_{u\in U}\pi^{M^u}"', "\simeq"] & & \mathrm{P}\big(\coprod_{u\in U} M^u\big) \arrow[dl, "\pi^{\coprod_{u\in U} M^u}", "\simeq"'] \\ & \coprod_{u\in U} M^u\,, & \end{tikzcd}$$

where φ is the canonical morphism; cf. (3.1.8.1). The morphism $\pi^{\coprod M^u}$ is a quasi-isomorphism by construction and $\coprod_{u\in U}\pi^{M^u}$ is a quasi-isomorphism by 6.1.13; therefore, φ is a quasi-isomorphism. By 5.2.18 the complex $\coprod_{u\in U}\mathrm{P}(M^u)$ is semi-projective, so it follows from 6.3.3 that φ is an isomorphism.

It remains to show that ϕ is a natural isomorphism such that P is triangulated. Once that has been established, commutativity of (6.3.9.2) shows that π is triangulated; further π^M is by construction a quasi-isomorphism for every R-complex M.

Commutativity of (6.3.9.2) shows that ϕ^M is a quasi-isomorphism, and since its domain and codomain are semi-projective R-complexes, 6.3.3 implies that ϕ^M is an isomorphism. To show that ϕ is natural, it must be argued that $\phi^N\mathrm{P}(\Sigma\alpha) = (\Sigma\mathrm{P}(\alpha))\phi^M$ holds for every morphism $\alpha\colon M\to N$ in $\mathcal{K}(R)$. Since $\mathrm{P}(\Sigma M)$ is semi-projective and $\Sigma\pi^N$ is a quasi-isomorphism, it suffices by 6.3.4 to argue that $(\Sigma\pi^N)\phi^N\mathrm{P}(\Sigma\alpha) = (\Sigma\pi^N)(\Sigma\mathrm{P}(\alpha))\phi^M$ holds. And that is a straightforward computation using the commutativity of (6.3.9.1) and (6.3.9.2),

$$\begin{aligned}(\Sigma\pi^N)\phi^N\mathrm{P}(\Sigma\alpha) &= \pi^{\Sigma N}\mathrm{P}(\Sigma\alpha)\\ &= (\Sigma\alpha)\pi^{\Sigma M}\\ &= (\Sigma\alpha)(\Sigma\pi^M)\phi^M\\ &= (\Sigma\pi^N)(\Sigma\mathrm{P}(\alpha))\phi^M\,.\end{aligned}$$

Finally, we argue that P with the natural isomorphism $\phi\colon \mathrm{P}\Sigma\to\Sigma\mathrm{P}$ is triangulated. By the definition 6.2.3 of distinguished triangles in $\mathcal{K}(R)$, it is enough to argue that P maps every strict triangle in $\mathcal{K}(R)$ to a distinguished one. Consider a morphism $\alpha\colon M\to N$ in $\mathcal{C}(R)$ and the associated strict triangle in $\mathcal{K}(R)$,

$$M \xrightarrow{[\alpha]} N \xrightarrow{\varepsilon} \operatorname{Cone}\alpha \xrightarrow{\varpi} \Sigma M\,,$$

where ε and ϖ are the homotopy classes of the morphisms $(1^N\ 0)^{\mathrm{T}}$ and $(0\ 1^{\Sigma M})$ in $\mathcal{C}(R)$. It must be shown that the upper candidate triangle in the following diagram is distinguished,

(b)
$$\begin{tikzcd} \mathrm{P}(M) \arrow[rr, "\mathrm{P}([\alpha])"] \arrow[dr, "\pi^M", "\simeq"'] \arrow[dd, equal] & & \mathrm{P}(N) \arrow[rr, "\mathrm{P}(\varepsilon)"] \arrow[dr, "\pi^N", "\simeq"'] \arrow[dd, equal] & & \mathrm{P}(\operatorname{Cone}\alpha) \arrow[rr, "\phi^M\mathrm{P}(\varpi)"] \arrow[dr, "\pi^{\operatorname{Cone}\alpha}", "\simeq"'] \arrow[dd, "\chi'"'] & & \Sigma\mathrm{P}(M) \arrow[dr, "\Sigma\pi^M", "\simeq"'] \arrow[dd, equal] & \\ & M \arrow[rr, "{[\alpha]}"] & & N \arrow[rr, "\varepsilon"] & & \operatorname{Cone}\alpha \arrow[rr, "\varpi"] & & \Sigma M \\ \mathrm{P}(M) \arrow[rr, "{[\widetilde{\alpha}]}"] \arrow[ur, "\pi^M"', "\simeq"] & & \mathrm{P}(N) \arrow[rr, "\widetilde{\varepsilon}"] \arrow[ur, "\pi^N"', "\simeq"] & & \operatorname{Cone}\widetilde{\alpha} \arrow[rr, "\widetilde{\varpi}"] \arrow[ur, "\chi"', "\simeq"] & & \Sigma\mathrm{P}(M)\,. \arrow[ur, "\Sigma\pi^M"', "\simeq"] & \end{tikzcd}$$

Commutativity of (6.3.9.1) and (6.3.9.2) shows that the upper face in ($\flat$) is commutative. Let $\widetilde{\alpha}$ be a morphism in $\mathcal{C}(R)$ that represents the homotopy class $\mathrm{P}([\alpha])$, i.e. one has $\mathrm{P}([\alpha]) = [\widetilde{\alpha}]$. The lower row in ($\flat$) is the strict triangle in $\mathcal{K}(R)$ associated to $\widetilde{\alpha}$. As $\mathcal{K}(R)$ is triangulated category there exists a morphism χ that makes the lower face in ($\flat$) commutative; see E.2. As π^M and π^N are quasi-isomorphisms, so is χ by 6.2.21. By 6.3.1 there exists a morphism χ' in $\mathcal{K}(R)$ that makes the third vertical wall in ($\flat$) commutative; that is, one has $\chi\chi' = \pi^{\operatorname{Cone}\alpha}$. Note that χ' is a quasi-isomorphism as χ and $\pi^{\operatorname{Cone}\alpha}$ are so. We now argue that the back face in ($\flat$) is commutative, i.e. that the equalities hold,

$$(\diamond) \qquad \widetilde{\varepsilon} = \chi'\mathrm{P}(\varepsilon) \qquad \text{and} \qquad \widetilde{\varpi}\chi' = \phi^M\mathrm{P}(\varpi)\,.$$

As the domains of these morphisms are semi-projective complexes, and since χ and $\Sigma\pi^M$ are quasi-isomorphisms, it follows from 6.3.4 that the equalities in ($\diamond$) hold if and only if the next equalities hold,

$$(\star) \qquad \chi\widetilde{\varepsilon} = \chi\chi'\mathrm{P}(\varepsilon) \qquad \text{and} \qquad (\Sigma\pi^M)\widetilde{\varpi}\chi' = (\Sigma\pi^M)\phi^M\mathrm{P}(\varpi)\,.$$

That these equalities hold follows from the parts of the diagram ($\flat$) that are already known to be commutative. Indeed, one has $\chi\widetilde{\varepsilon} = \varepsilon\pi^N = \pi^{\operatorname{Cone}\alpha}\mathrm{P}(\varepsilon) = \chi\chi'\mathrm{P}(\varepsilon)$ and $(\Sigma\pi^M)\widetilde{\varpi}\chi' = \varpi\chi\chi' = \varpi\pi^{\operatorname{Cone}\alpha} = (\Sigma\pi^M)\phi^M\mathrm{P}(\varpi)$.

Application of 5.2.17 to the exact sequence $0 \to \mathrm{P}(N) \to \operatorname{Cone}\widetilde{\alpha} \to \Sigma\mathrm{P}(M) \to 0$ from 4.1.5 shows that the complex $\operatorname{Cone}\widetilde{\alpha}$ is semi-projective. Therefore χ' is a quasi-isomorphism whose domain and codomain are semi-projective complexes. It follows from 6.3.3 that χ' is an isomorphism in $\mathcal{K}(R)$, and hence the back face in ($\flat$) is an isomorphism of candidate triangles in $\mathcal{K}(R)$. Since the lower candidate triangle is strict, the upper candidate triangle is distinguished, as desired. □

6.3.12 Proposition. *Let* P *and* π *be as in* 6.3.11. *For every* R*-complex* M *there is an equality* $\mathrm{P}(\pi^M) = \pi^{\mathrm{P}(M)}$ *of morphisms* $\mathrm{P}(\mathrm{P}(M)) \to \mathrm{P}(M)$.

Proof. Replacing M and α with $\mathrm{P}(M)$ and π^M in (6.3.9.1) one gets the commutative diagram below, and the asserted equality follows from 6.3.4.

$$\begin{tikzcd}
\mathrm{P}(\mathrm{P}(M)) \arrow[r, "\pi^{\mathrm{P}(M)}", "\simeq"'] \arrow[d, "\mathrm{P}(\pi^M)"'] & \mathrm{P}(M) \arrow[d, "\pi^M", "\simeq"'] \\
\mathrm{P}(M) \arrow[r, "\pi^M", "\simeq"'] & M\,.
\end{tikzcd}$$
□

6.3.13 Proposition. *Let* P *and* $\widetilde{\mathrm{P}}$ *be endofunctors on* $\mathcal{K}(R)$ *defined per* 6.3.9 *by choosing, for every* R*-complex* M*, semi-projective resolutions,*

$$\pi^M : \mathrm{P}(M) \xrightarrow{\simeq} M \qquad \textit{and} \qquad \widetilde{\pi}^M : \widetilde{\mathrm{P}}(M) \xrightarrow{\simeq} M\,.$$

There exists a unique natural isomorphism $\varphi\colon \mathrm{P} \to \widetilde{\mathrm{P}}$ *with* $\widetilde{\pi}\varphi = \pi$.

Proof. By 6.3.1 there exists for every R-complex M a unique morphism φ^M in $\mathcal{K}(R)$ such that the following diagram is commutative,

$$\begin{tikzcd} & \mathrm{P}(M) \arrow[dl, "\varphi^M"'] \arrow[d, "\simeq"', "\pi^M"] \\ \widetilde{\mathrm{P}}(M) \arrow[r, "\simeq", "\widetilde{\pi}^M"'] & M\,. \end{tikzcd}$$

Since π^M and $\widetilde{\pi}^M$ are quasi-isomorphisms, so is φ^M, and hence 6.3.3 yields that φ^M is an isomorphism in $\mathcal{K}(R)$. In fact, $\varphi\colon \mathrm{P} \to \widetilde{\mathrm{P}}$ is a natural isomorphism. Indeed, for a morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$, both $\varphi^N \mathrm{P}(\alpha)$ and $\widetilde{\mathrm{P}}(\alpha)\varphi^M$ make the diagram

$$\begin{tikzcd} \mathrm{P}(M) \arrow[r, "\pi^M", "\simeq"'] \arrow[d, dotted] & M \arrow[d, "\alpha"] \\ \widetilde{\mathrm{P}}(N) \arrow[r, "\widetilde{\pi}^N", "\simeq"'] & N \end{tikzcd}$$

commutative, whence $\varphi^N \mathrm{P}(\alpha) = \widetilde{\mathrm{P}}(\alpha)\varphi^M$ holds by 6.3.4. □

The homology functor on $\mathcal{K}(R)$ plays a key role in the next result.

6.3.14 Proposition. *Let* P *and* π *be as in* 6.3.11. *For R-modules M and N, the map*

$$\mathcal{K}(R)(M,N) \longrightarrow \mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \quad \text{given by} \quad \alpha \longmapsto \mathrm{P}(\alpha)$$

is an isomorphism of $\Bbbk$-modules with inverse given by $\beta \mapsto \mathrm{H}(\pi^N)\,\mathrm{H}(\beta)\,\mathrm{H}(\pi^M)^{-1}$.

Proof. For α in $\mathcal{K}(R)(M,N)$ and β in $\mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N))$ set $\Phi(\alpha) = \mathrm{P}(\alpha)$ and $\Psi(\beta) = \mathrm{H}(\pi^N)\,\mathrm{H}(\beta)\,\mathrm{H}(\pi^M)^{-1}$; note that Φ and Ψ are $\Bbbk$-linear maps as P and H are $\Bbbk$-linear functors. Apply the functor H to the identity $\pi^N \mathrm{P}(\alpha) = \alpha\pi^M$ from (6.3.9.1) to get $\mathrm{H}(\pi^N)\,\mathrm{H}(\mathrm{P}(\alpha)) = \mathrm{H}(\alpha)\,\mathrm{H}(\pi^M) = \alpha\,\mathrm{H}(\pi^M)$ and, consequently, $\Psi\Phi(\alpha) = \alpha$. By construction, see 6.3.9, the map $\Phi\Psi(\beta) = \mathrm{P}(\Psi(\beta))$ is the unique morphism that makes the following diagram commutative,

$$\begin{tikzcd} \mathrm{P}(M) \arrow[r, "\pi^M", "\simeq"'] \arrow[d, dotted] & M \arrow[d, "\Psi(\beta)"] \\ \mathrm{P}(N) \arrow[r, "\pi^N", "\simeq"'] & N\,. \end{tikzcd}$$

Thus, to show that $\Phi\Psi(\beta) = \beta$ holds, it suffices to argue that $\pi^N\beta = \Psi(\beta)\pi^M$ holds. As $\mathrm{H}(\Psi(\beta)) = \Psi(\beta) = \mathrm{H}(\pi^N)\,\mathrm{H}(\beta)\,\mathrm{H}(\pi^M)^{-1}$ holds, one has $\mathrm{H}(\pi^N\beta) = \mathrm{H}(\Psi(\beta)\pi^M)$; in particular, $\mathrm{H}_0(\pi^N\beta) = \mathrm{H}_0(\Psi(\beta)\pi^M)$. As M is a module one can by 5.2.14 and 6.3.13 assume that $\mathrm{P}(M)_v = 0$ holds for $v < 0$, whence 2.5.10 yields $\pi^N\beta = \Psi(\beta)\pi^M$. □

There is a parallel story to tell about semi-injective resolutions.

6.3.15 Construction. For every R-complex M, choose an injective semi-injective resolution $\iota_R^M\colon M \xrightarrow{\simeq} \mathrm{I}_R(M)$ in $\mathcal{K}(R)$. For every morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ there is by 6.3.5 a unique morphism in $\mathcal{K}(R)$, denoted $\mathrm{I}_R(\alpha)$, such that the diagram

(6.3.15.1)
$$\begin{tikzcd} M \arrow[r, "\iota_R^M", "\simeq"'] \arrow[d, "\alpha"'] & \mathrm{I}_R(M) \arrow[d, "\mathrm{I}_R(\alpha)"] \\ N \arrow[r, "\iota_R^N", "\simeq"'] & \mathrm{I}_R(N) \end{tikzcd}$$

is commutative. Similarly, there is a unique morphism in $\mathcal{K}(R)$, which we denote by ϕ^M, such that the following diagram commutative,

(6.3.15.2)
$$\begin{tikzcd} \Sigma M \arrow[r, "\iota_R^{\Sigma M}", "\simeq"'] \arrow[d, "\Sigma\iota_R^M"', "\simeq"] & \mathrm{I}_R(\Sigma M) \arrow[dl, "\phi^M"] \\ \Sigma \mathrm{I}_R(M) \,. & \end{tikzcd}$$

6.3.16 Definition. The functor I_R established in the next theorem is called the *semi-injective resolution functor*. When there is no ambiguity, we write I instead of I_R and, likewise, we drop the subscript from the associated natural transformation ι_R.

6.3.17 Theorem. *The assignments $M \mapsto \mathrm{I}_R(M)$ and $\alpha \mapsto \mathrm{I}_R(\alpha)$ from* 6.3.15 *define an endofunctor on $\mathcal{K}(R)$. This functor I_R is $\Bbbk$-linear, it preserves products, it maps quasi-isomorphisms to isomorphisms, and ϕ is a natural isomorphism $\mathrm{I}_R\Sigma \to \Sigma\mathrm{I}_R$ such that I_R is triangulated. Furthermore, ι_R is a triangulated natural transformation $\mathrm{Id}_{\mathcal{K}(R)} \to \mathrm{I}_R$, and ι_R^M is a quasi-isomorphism for every R-complex M.*

Proof. An argument analogous to the proof of 6.3.11 applies. □

6.3.18 Proposition. *Let* I *and ι be as in* 6.3.17. *For every R-complex M there is an equality $\mathrm{I}(\iota^M) = \iota^{\mathrm{I}(M)}$ of morphisms $\mathrm{I}(M) \to \mathrm{I}(\mathrm{I}(M))$.*

Proof. An argument analogous to the proof of 6.3.12 applies. □

6.3.19 Proposition. *Let* I *and $\widetilde{\mathrm{I}}$ be endofunctors on $\mathcal{K}(R)$ defined per* 6.3.15 *by choosing, for every R-complex M, semi-injective resolutions,*

$$\iota^M : M \xrightarrow{\simeq} \mathrm{I}(M) \qquad \textit{and} \qquad \widetilde{\iota}^M : M \xrightarrow{\simeq} \widetilde{\mathrm{I}}(M)\,.$$

There exists a unique natural isomorphism $\varphi\colon \widetilde{\mathrm{I}} \to \mathrm{I}$ with $\varphi\widetilde{\iota} = \iota$.

Proof. An argument analogous to the proof of 6.3.13 applies. □

There is, of course, also a result parallel to 6.3.14 for the semi-injective resolution functor. We will not need to refer to it, so we leave it as an exercise.

The equalities in 6.3.12 and 6.3.18 are special cases of a category theoretical phenomenon captured by the next lemma, which we record for later use.

6.3.20 Lemma. *Let $\mathcal{U}$ be a category.*

(a) *Let* F *be an endofunctor on $\mathcal{U}$ and $\varphi\colon \mathrm{Id}_{\mathcal{U}} \to \mathrm{F}$ a natural transformation. If $\varphi^{\mathrm{F}(X)}$ and $\mathrm{F}(\varphi^X)$ are isomorphisms for every $X \in \mathcal{U}$, then $\varphi^{\mathrm{F}(X)} = \mathrm{F}(\varphi^X)$ holds for every $X \in \mathcal{U}$.*

(b) *Let* G *be an endofunctor on* $\mathcal{U}$ *and* $\psi\colon \mathrm{G} \to \mathrm{Id}_{\mathcal{U}}$ *a natural transformation. If* $\psi^{\mathrm{G}(X)}$ *and* $\mathrm{G}(\psi^X)$ *are isomorphisms for every* $X \in \mathcal{U}$*, then* $\psi^{\mathrm{G}(X)} = \mathrm{G}(\psi^X)$ *holds for every* $X \in \mathcal{U}$.

Proof. (a): First note that since $\varphi\colon \mathrm{Id}_{\mathcal{U}} \to \mathrm{F}$ is a natural transformation, there is for every morphism $\alpha\colon M \to N$ in $\mathcal{U}$ an equality,

$$\varphi^N \alpha = \mathrm{F}(\alpha)\varphi^M . \tag{$\flat$}$$

We start by showing that if X and Y are objects in $\mathcal{U}$ such that φ^Y is an isomorphism, and $\beta, \gamma\colon \mathrm{F}(X) \to Y$ are morphisms, then $\beta\varphi^X = \gamma\varphi^X$ implies $\beta = \gamma$. Indeed, if one has $\beta\varphi^X = \gamma\varphi^X$, then $\mathrm{F}(\beta)\,\mathrm{F}(\varphi^X) = \mathrm{F}(\gamma)\,\mathrm{F}(\varphi^X)$ holds, and thus $\mathrm{F}(\beta) = \mathrm{F}(\gamma)$ since $\mathrm{F}(\varphi^X)$ is assumed to be an isomorphism. Using ($\flat$) one now gets

$$\varphi^Y \beta = \mathrm{F}(\beta)\varphi^{\mathrm{F}(X)} = \mathrm{F}(\gamma)\varphi^{\mathrm{F}(X)} = \varphi^Y \gamma$$

and, therefore, $\beta = \gamma$ as φ^Y is assumed to be an isomorphism.

Now, let $X \in \mathcal{U}$ be given. The result above applies to the object $Y = \mathrm{F}(\mathrm{F}(X))$ and the morphisms $\beta = \varphi^{\mathrm{F}(X)}$ and $\gamma = \mathrm{F}(\varphi^X)$ from $\mathrm{F}(X)$ to $\mathrm{F}(\mathrm{F}(X))$. Applying ($\flat$) to the morphism $\varphi^X\colon X \to \mathrm{F}(X)$ one gets $\varphi^{\mathrm{F}(X)}\varphi^X = \mathrm{F}(\varphi^X)\varphi^X$, and consequently one has $\varphi^{\mathrm{F}(X)} = \mathrm{F}(\varphi^X)$, as claimed.

(b): The proof is dual to that of part (a). □

Resolutions and Restrictions of Scalars

We close this section with two technical results that are used repeatedly in Chap. 7.

6.3.21 Proposition. *Let* $R \to S$ *be a ring homomorphism. There is a unique natural transformation* ϱ_R^S *of functors* $\mathcal{K}(S) \to \mathcal{K}(R)$ *that makes the diagram,*

$$\begin{tikzcd} & \mathrm{P}_R \operatorname{res}_R^S \arrow[dl, dotted, "\varrho_R^S"'] \arrow[d, "\pi_R \operatorname{res}_R^S", "\simeq"'] \\ \operatorname{res}_R^S \mathrm{P}_S \arrow[r, "\simeq", "\operatorname{res}_R^S \pi_S"'] & \operatorname{res}_R^S \end{tikzcd} ,$$

commutative. For every S*-complex* N *the morphism* $(\varrho_R^S)^N$ *is a quasi-isomorphism, and if* S *is projective as an* R*-module, then it is an isomorphism in* $\mathcal{K}(R)$.

Proof. To ease the notation, set $\operatorname{res} = \operatorname{res}_R^S$. Let N in $\mathcal{K}(S)$ be given. In the following diagram in $\mathcal{K}(R)$, the horizontal and vertical morphisms are quasi-isomorphisms by 6.3.11 and the fact that the functor res preserves quasi-isomorphisms,

$$\begin{tikzcd} & \mathrm{P}_R(\operatorname{res}(N)) \arrow[dl, dotted, "\varrho^N"'] \arrow[d, "\pi_R^{\operatorname{res}(N)}", "\simeq"'] \\ \operatorname{res}(\mathrm{P}_S(N)) \arrow[r, "\simeq", "\operatorname{res}(\pi_S^N)"'] & \operatorname{res}(N) \end{tikzcd} . \tag{$\diamond$}$$

By 6.3.1 there is a unique morphism ϱ^N that makes the diagram ($\diamond$) commutative; as the other two morphisms are quasi-isomorphisms, so is ϱ^N. If S is projective as an R-module, then $\mathrm{P}_S(N)$ is a semi-projective R-complex by 5.2.23(b). Now it follows from 5.2.21 that ϱ^N is a homotopy equivalence, i.e. an isomorphism in $\mathcal{K}(R)$.

It remains to argue that ϱ is a natural transformation of functors. Let $\alpha\colon M \to N$ be a morphism in $\mathcal{K}(S)$. It must be shown that $\varrho^N \mathrm{P}_R(\operatorname{res}(\alpha)) = \operatorname{res}(\mathrm{P}_S(\alpha))\varrho^M$ holds. In the computation below, the 1st and 3rd equalities hold by commutativity of ($\diamond$), the 2nd and 5th equalities hold as π is a natural transformation, and the 4th and 6th equalities hold as res is a functor.

$$\begin{aligned}
\operatorname{res}(\pi_S^N)\varrho^N \mathrm{P}_R(\operatorname{res}(\alpha)) &= \pi_R^{\operatorname{res}(N)} \mathrm{P}_R(\operatorname{res}(\alpha))\\
&= \operatorname{res}(\alpha)\pi_R^{\operatorname{res}(M)}\\
&= \operatorname{res}(\alpha)\operatorname{res}(\pi_S^M)\varrho^M\\
&= \operatorname{res}(\alpha\pi_S^M)\varrho^M\\
&= \operatorname{res}(\pi_S^N \mathrm{P}(\alpha))\varrho^M\\
&= \operatorname{res}(\pi_S^N)\operatorname{res}(\mathrm{P}_S(\alpha))\varrho^M\ .
\end{aligned}$$

The desired equality $\varrho^N \mathrm{P}_R(\operatorname{res}(\alpha)) = \operatorname{res}(\mathrm{P}_S(\alpha))\varrho^M$ now follows from 6.3.4. □

6.3.22 Proposition. *Let $R \to S$ be a ring homomorphism. There is a unique natural transformation ε_R^S of functors $\mathcal{K}(S) \to \mathcal{K}(R)$ that makes the diagram,*

$$\begin{tikzcd}
\operatorname{res}_R^S \arrow[r, "\operatorname{res}_R^S \iota_S", "\simeq"'] \arrow[d, "\iota_R \operatorname{res}_R^S"', "\simeq"] & \operatorname{res}_R^S \mathrm{I}_S \arrow[dl, dotted, "\varepsilon_R^S"] \\
\mathrm{I}_R \operatorname{res}_R^S\ , &
\end{tikzcd}$$

commutative. For every S-complex N the morphism $(\varepsilon_R^S)^N$ is a quasi-isomorphism, and if S is flat as an R°-module, then it is an isomorphism in $\mathcal{K}(R)$.

Proof. This follows from an argument similar to the proof of 6.3.21, but based on 6.3.17, 5.4.26(b), and 5.3.24. □

Exercises

E 6.3.1 Give a proof of 6.3.17.

E 6.3.2 Let α be a morphism in $\mathcal{K}(R)$. Show that there exists a quasi-isomorphism φ with $\varphi\alpha = 0$ if and only if there exists a quasi-isomorphism ψ with $\alpha\psi = 0$.

E 6.3.3 Let α be a morphism in $\mathcal{K}(R)$. (a) Show that α is a quasi-isomorphism only if $\mathrm{P}(\alpha)$ is an isomorphism. (b) Show that α is a quasi-isomorphism only if $\mathrm{I}(\alpha)$ is an isomorphism.

E 6.3.4 Let M, N be R-complexes. Show that the map $\mathcal{K}(R)(M,N) \to \mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N))$ given by $\alpha \mapsto \mathrm{P}(\alpha)$ needs neither be injective nor surjective.

E 6.3.5 Let M, N be R-modules. Show that the map $\mathcal{K}(R)(M,N) \to \mathcal{K}(R)(\mathrm{I}(M),\mathrm{I}(N))$ given by $\alpha \mapsto \mathrm{I}(\alpha)$ is an isomorphism.

E 6.3.6 Let M, N be R-complexes. Show that the map $\mathcal{K}(R)(M,N) \to \mathcal{K}(R)(\mathrm{I}(M),\mathrm{I}(N))$ given by $\alpha \mapsto \mathrm{I}(\alpha)$ needs neither be injective nor surjective.

E 6.3.7 Let M, N be R-complexes. Establish isomorphisms,
$$\mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \xrightarrow{\cong} \mathcal{K}(R)(\mathrm{P}(M),\mathrm{I}(N)) \xleftarrow{\cong} \mathcal{K}(R)(\mathrm{I}(M),\mathrm{I}(N))\,,$$
and describe their inverses.

E 6.3.8 Show that the endofunctor P on $\mathcal{K}(R)$ is left adjoint for I.

6.4 Construction of the Derived Category $\mathfrak{D}$

Synopsis. Fraction; objects and morphisms; derived category; product; coproduct; universal property.

Let $\mathcal{U}$ be a category and $\mathcal{X}$ a collection of morphisms in $\mathcal{U}$. One may seek a category $\mathcal{X}^{-1}\mathcal{U}$—called the localization of $\mathcal{U}$ with respect to $\mathcal{X}$—together with a functor $\mathrm{V}\colon \mathcal{U} \to \mathcal{X}^{-1}\mathcal{U}$ that has the following universal property: The functor V maps elements in $\mathcal{X}$ to isomorphisms in $\mathcal{X}^{-1}\mathcal{U}$, and for every functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ that maps elements in $\mathcal{X}$ to isomorphisms in $\mathcal{V}$ there exists a unique functor $\acute{\mathrm{F}}$ that makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{V}"'] & \mathcal{V} \\ \mathcal{X}^{-1}\mathcal{U}\,. \arrow[ur, dotted, "\acute{\mathrm{F}}"'] & \end{tikzcd}$$

There is a formal way to construct $\mathcal{X}^{-1}\mathcal{U}$; however, it may yield a “category” in which the hom-sets are not sets but proper classes. Thus, the localization of $\mathcal{U}$ with respect to $\mathcal{X}$ may not exist. An early motivation for the development of the theory of model categories was to circumvent such set theoretic problems.

We proceed to localize the homotopy category $\mathcal{K}(R)$ with respect to the collection of quasi-isomorphisms. The outcome is a category $\mathcal{D}(R)$—semi-projective resolutions can be harnessed to show that the hom-sets in $\mathcal{D}(R)$ are actual sets—called the derived category over R; it inherits a triangulated structure from $\mathcal{K}(R)$.

Fractions of Morphisms

Recall that we use Greek letters for morphisms in the homotopy category.

6.4.1 Definition. Let M and N be R-complexes. A *left prefraction* from M to N is a pair (α,φ) of morphisms in $\mathcal{K}(R)$ such that α and φ have the same domain, the codomain of φ is M, the codomain of α is N, and φ is a quasi-isomorphism:
$$M \xleftarrow[\simeq]{\varphi} U \xrightarrow{\alpha} N\,.$$

Two left prefractions (α^1, φ^1) and (α^2, φ^2) from M to N are equivalent, in symbols $(\alpha^1, \varphi^1) \equiv (\alpha^2, \varphi^2)$, if there exist a left prefraction (α, φ) from M to N and morphisms μ^1 and μ^2 that make the following diagram in $\mathcal{K}(R)$ commutative,

(6.4.1.1)
$$\begin{tikzcd} & U^1 \arrow[dl, "\varphi^1"'] \arrow[dr, "\alpha^1"] & \\ M & U \arrow[l, dotted, "\varphi"] \arrow[r, dotted, "\alpha"] \arrow[u, dotted, "\mu^1"'] \arrow[d, dotted, "\mu^2"] & N \, . \\ & U^2 \arrow[ul, "\varphi^2"] \arrow[ur, "\alpha^2"'] & \end{tikzcd}$$

6.4.2. Notice that the morphisms μ^1 and μ^2 in (6.4.1.1) are quasi-isomorphisms. Consequently, α^1 is a quasi-isomorphism if and only if α^2 is a quasi-isomorphism.

The relation on left prefractions defined above is an equivalence relation, and it can be described conveniently by way of the resolution functors from Sect. 6.3.

6.4.3 Proposition. *Let M and N be R-complexes. For left prefractions (α^1, φ^1) and (α^2, φ^2) from M to N the following conditions are equivalent.*

(*i*) *One has* $(\alpha^1, \varphi^1) \equiv (\alpha^2, \varphi^2)$.
(*ii*) $\mathrm{P}(\alpha^1)\,\mathrm{P}(\varphi^1)^{-1}$ *and* $\mathrm{P}(\alpha^2)\,\mathrm{P}(\varphi^2)^{-1}$ *are identical morphisms* $\mathrm{P}(M) \to \mathrm{P}(N)$.
(*iii*) $\mathrm{I}(\alpha^1)\,\mathrm{I}(\varphi^1)^{-1}$ *and* $\mathrm{I}(\alpha^2)\,\mathrm{I}(\varphi^2)^{-1}$ *are identical morphisms* $\mathrm{I}(M) \to \mathrm{I}(N)$.

In particular, the relation $\equiv$ is an equivalence relation on the class of left prefractions from M to N.

Proof. Recall from 6.3.11 and 6.3.17 that P and I map quasi-isomorphisms to isomorphisms; this is already implicit in the statements.

(*i*) $\Rightarrow$ (*iii*): If one has $(\alpha^1, \varphi^1) \equiv (\alpha^2, \varphi^2)$, then there exists a commutative diagram of the form (6.4.1.1). Application of I yields a commutative diagram,

$$\begin{tikzcd} & \mathrm{I}(U^1) \arrow[dl, "\mathrm{I}(\varphi^1)"'] \arrow[dr, "\mathrm{I}(\alpha^1)"] & \\ \mathrm{I}(M) & \mathrm{I}(U) \arrow[l, "\mathrm{I}(\varphi)"] \arrow[r, "\mathrm{I}(\alpha)"] \arrow[u, "\mathrm{I}(\mu^1)"'] \arrow[d, "\mathrm{I}(\mu^2)"] & \mathrm{I}(N) \, , \\ & \mathrm{I}(U^2) \arrow[ul, "\mathrm{I}(\varphi^2)"] \arrow[ur, "\mathrm{I}(\alpha^2)"'] & \end{tikzcd}$$

from which it follows that $\mathrm{I}(\alpha^1)\,\mathrm{I}(\varphi^1)^{-1} = \mathrm{I}(\alpha^2)\,\mathrm{I}(\varphi^2)^{-1}$ holds.

(*iii*) $\Rightarrow$ (*ii*): Let $\beta\colon X \to Y$ be a morphism in $\mathcal{K}(R)$. One has $\iota^Y\beta = \mathrm{I}(\beta)\iota^X$ by 6.3.15, and an application of P yields $\mathrm{P}(\beta) = \mathrm{P}(\iota^Y)^{-1}\mathrm{P}\mathrm{I}(\beta)\,\mathrm{P}(\iota^X)$. Thus one has

$$\begin{aligned} \mathrm{P}(\alpha^1)\,\mathrm{P}(\varphi^1)^{-1} &= (\mathrm{P}(\iota^N)^{-1}\mathrm{P}\mathrm{I}(\alpha^1)\,\mathrm{P}(\iota^{U^1}))(\mathrm{P}(\iota^M)^{-1}\mathrm{P}\mathrm{I}(\varphi^1)\,\mathrm{P}(\iota^{U^1}))^{-1} \\ &= \mathrm{P}(\iota^N)^{-1}\mathrm{P}(\mathrm{I}(\alpha^1)\,\mathrm{I}(\varphi^1)^{-1})\,\mathrm{P}(\iota^M)\,, \end{aligned}$$

and the equality $\mathrm{P}(\alpha^2)\,\mathrm{P}(\varphi^2)^{-1} = \mathrm{P}(\iota^N)^{-1}\mathrm{P}(\mathrm{I}(\alpha^2)\,\mathrm{I}(\varphi^2)^{-1})\,\mathrm{P}(\iota^M)$ is proved similarly. Thus $\mathrm{I}(\alpha^1)\,\mathrm{I}(\varphi^1)^{-1} = \mathrm{I}(\alpha^2)\,\mathrm{I}(\varphi^2)^{-1}$ implies $\mathrm{P}(\alpha^1)\,\mathrm{P}(\varphi^1)^{-1} = \mathrm{P}(\alpha^2)\,\mathrm{P}(\varphi^2)^{-1}$.

(ii) $\Rightarrow$ *(i)*: Assume that $\mathrm{P}(\alpha^1)\,\mathrm{P}(\varphi^1)^{-1}$ and $\mathrm{P}(\alpha^2)\,\mathrm{P}(\varphi^2)^{-1}$ are identical, and denote this morphism by α. Set $\mu^i = \pi^{U^i}\,\mathrm{P}(\varphi^i)^{-1}$ for $i = 1, 2$; the commutative diagram

$$\begin{tikzcd}
& U^1 \arrow[dl, "\varphi^1"'] \arrow[dr, "\alpha^1"] & \\
M & \mathrm{P}(M) \arrow[l, "\simeq", "\pi^M"'] \arrow[r, "\pi^N \alpha"] \arrow[u, "\mu^1"'] \arrow[d, "\mu^2"] & N \\
& U^2 \arrow[ul, "\varphi^2"] \arrow[ur, "\alpha^2"'] &
\end{tikzcd}$$

shows that (α^1, φ^1) and (α^2, φ^2) are equivalent. □

6.4.4 Definition. Let M and N be R-complexes. For a left prefraction (α, φ) from M to N, denote by ${}^{\alpha}\!/_{\varphi}$ the equivalence class containing (α, φ). The class ${}^{\alpha}\!/_{\varphi}$ is called a *left fraction* from M to N, and the collection of all such is denoted $\mathcal{D}(R)(M, N)$.

Remark. Some authors—among them Kashiwara and Shapira [156]—refer to the equivalence classes defined above as 'right fractions'. We follow those—among them Gabriel and Zisman [103] and Weibel [253]—who opt for 'left fractions'; see also the Remark after 6.4.26.

Objects and Morphisms

The notation introduced in 6.4.4 is suggestive and, indeed, we are poised to prove that there is a category $\mathcal{D}(R)$ whose objects are all R-complexes and in which the hom-set $\mathcal{D}(R)(M, N)$ is the collection of all left fractions from M to N.

6.4.5 Lemma. *Consider a diagram in $\mathcal{K}(R)$,*

$$\begin{tikzcd}
& V \arrow[d, "\psi", "\simeq"'] & \\
M & U \arrow[l, "\varphi"', "\simeq"] \arrow[r, "\alpha"] & N\,,
\end{tikzcd}$$

where φ and ψ are quasi-isomorphisms. There is an equality ${}^{(\alpha\psi)}\!/_{(\varphi\psi)} = {}^{\alpha}\!/_{\varphi}$.

Proof. The assertion follows from the commutative diagram

$$\begin{tikzcd}
& V \arrow[dl, "\varphi\psi"'] \arrow[dr, "\alpha\psi"] \arrow[d, equal] & \\
M & V \arrow[l, "\varphi\psi"] \arrow[r, "\alpha\psi"] \arrow[d, "\psi"] & N \\
& U\,, \arrow[ul, "\varphi"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

which shows that the left prefractions $(\alpha\psi, \varphi\psi)$ and (α, φ) are equivalent. $\square$

REMARK. Another way to prove 6.4.5 is to notice that $\mathrm{P}(\alpha\psi)\,\mathrm{P}(\varphi\psi)^{-1} = \mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1}$ holds and apply 6.4.3.

The next result shows that any two left fractions with the same domain and codomain have a common denominator.

6.4.6 Lemma. *Let M and N be R-complexes and $\alpha^1/_{\varphi^1}$ and $\alpha^2/_{\varphi^2}$ be left fractions from M to N. There exist morphisms β^1, β^2 and a quasi-isomorphism φ in $\mathcal{K}(R)$ such that the equalities $\alpha^1/_{\varphi^1} = \beta^1/_{\varphi}$ and $\alpha^2/_{\varphi^2} = \beta^2/_{\varphi}$ hold.*

PROOF. Consider the diagrams

$$\begin{tikzcd} & \mathrm{P}(M) \arrow[d, "\pi^{U^1}\mathrm{P}(\varphi^1)^{-1}", "\simeq"'] & \\ M & U^1 \arrow[l, "\varphi^1"', "\simeq"] \arrow[r, "\alpha^1"] & N \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & \mathrm{P}(M) \arrow[d, "\pi^{U^2}\mathrm{P}(\varphi^2)^{-1}", "\simeq"'] & \\ M & U^2 \arrow[l, "\varphi^2"', "\simeq"] \arrow[r, "\alpha^2"] & N\,. \end{tikzcd}$$

For $i = 1,2$ one has $\varphi^i\pi^{U^i}\mathrm{P}(\varphi^i)^{-1} = \pi^M$ by 6.3.9. Hence 6.4.5 implies that with $\beta^i = \alpha^i\pi^{U^i}\mathrm{P}(\varphi^i)^{-1}$ and $\varphi = \pi^M$ the equality $\alpha^i/_{\varphi^i} = \beta^i/_{\varphi}$ holds. $\square$

The collection of all left prefractions from M to N is a proper class (i.e. not a set); the collection of equivalence classes of these left prefractions is, however, a set.

6.4.7 Proposition. *Let M and N be R-complexes. The map*

$$\mathcal{K}(R)(\mathrm{P}(M), \mathrm{P}(N)) \longrightarrow \mathcal{D}(R)(M,N) \quad \text{given by} \quad \beta \longmapsto (\pi^N\beta)/_{\pi^M}$$

is a bijection with inverse given by $\alpha/_{\varphi} \mapsto \mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1}$. In particular, the collection $\mathcal{D}(R)(M,N)$ of left fractions from M to N is a set.

PROOF. For β in $\mathcal{K}(R)(\mathrm{P}(M), \mathrm{P}(N))$ and $\alpha/_{\varphi}$ in $\mathcal{D}(R)(M,N)$ set $\Phi(\beta) = (\pi^N\beta)/_{\pi^M}$ and $\Psi(\alpha/_{\varphi}) = \mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1}$; notice that the latter is well-defined by 6.4.3. By 6.4.5 and commutativity of the diagram (6.3.9.1) one has

$$\Phi\Psi(\alpha/_{\varphi}) = (\pi^N\mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1})/_{\pi^M} = (\pi^N\mathrm{P}(\alpha))/_{(\pi^M\mathrm{P}(\varphi))} = (\alpha\pi^U)/_{(\varphi\pi^U)} = \alpha/_{\varphi}\,,$$

where U is the common domain of α and φ. Similarly, 6.3.12 and (6.3.9.1) yield

$$\Psi\Phi(\beta) = \mathrm{P}(\pi^N\beta)\,\mathrm{P}(\pi^M)^{-1} = \mathrm{P}(\pi^N)\,\mathrm{P}(\beta)\,\mathrm{P}(\pi^M)^{-1} = \pi^{\mathrm{P}(N)}\mathrm{P}(\beta)(\pi^{\mathrm{P}(M)})^{-1} = \beta\,.$$

Thus Φ is bijective with inverse Ψ. In particular, the class $\mathcal{D}(R)(M,N)$ is in one-to-one correspondence with the set $\mathcal{K}(R)(\mathrm{P}(M), \mathrm{P}(N))$, so $\mathcal{D}(R)(M,N)$ is a set. $\square$

6.4.8 Proposition. *Let M and N be R-complexes. The set $\mathcal{D}(R)(M,N)$ of left fractions from M to N is a $\Bbbk$-module with the following operations.*

- *For $\alpha^1/_{\varphi^1}$ and $\alpha^2/_{\varphi^2}$ in $\mathcal{D}(R)(M,N)$ set*

$$\alpha^1/_{\varphi^1} + \alpha^2/_{\varphi^2} = (\beta^1 + \beta^2)/_{\varphi}$$

  for any choice of left prefractions (β^i, φ) with $\alpha^i/_{\varphi^i} = \beta^i/_{\varphi}$ for $i = 1,2$, cf. 6.4.6.

- *For x in $\Bbbk$ and ${}^{\alpha}/_{\varphi}$ in $\mathcal{D}(R)(M,N)$ set*

$$x({}^{\alpha}/_{\varphi}) \;=\; {}^{(x\alpha)}/_{\varphi}\,.$$

The equivalence class ${}^{0}/_{1^M}$ containing the left prefraction $M \xleftarrow{1^M} M \xrightarrow{0} N$ is the zero element in the $\Bbbk$-module $\mathcal{D}(R)(M,N)$. Finally, with this $\Bbbk$-module structure, the map $\mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \to \mathcal{D}(R)(M,N)$ from 6.4.7 *is an isomorphism of $\Bbbk$-modules.*

Proof. Let Φ denote the bijection $\mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \to \mathcal{D}(R)(M,N)$ from 6.4.7. Since $\mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N))$ is a $\Bbbk$-module, see 6.1.9, one turns the set $\mathcal{D}(R)(M,N)$ into a $\Bbbk$-module, and Φ into an isomorphism, by defining the operations as follows,

$${}^{\alpha^1}/_{\varphi^1} + {}^{\alpha^2}/_{\varphi^2} \;=\; \Phi\big(\Phi^{-1}({}^{\alpha^1}/_{\varphi^1}) + \Phi^{-1}({}^{\alpha^2}/_{\varphi^2})\big) \quad\text{and}\quad x({}^{\alpha}/_{\varphi}) \;=\; \Phi\big(x\,\Phi^{-1}({}^{\alpha}/_{\varphi})\big)\,.$$

Evidently, with this $\Bbbk$-module structure on $\mathcal{D}(R)(M,N)$, the zero element is

$$\Phi(0) \;=\; {}^{(\pi^N 0)}/_{\pi^M} \;=\; {}^{(0\pi^M)}/_{(1^M\pi^M)} \;=\; {}^{0}/_{1^M}\,,$$

where the last equality holds by 6.4.5. We argue that these operations on $\mathcal{D}(R)(M,N)$ agree with the asserted ones; in particular, these operations are well-defined.

For the addition operation, assume that one has ${}^{\alpha^1}/_{\varphi^1} = {}^{\beta^1}/_{\varphi}$ and ${}^{\alpha^2}/_{\varphi^2} = {}^{\beta^2}/_{\varphi}$, cf. 6.4.6. It must be argued that the equality

$$(\star)\qquad \Phi^{-1}({}^{\alpha^1}/_{\varphi^1}) + \Phi^{-1}({}^{\alpha^2}/_{\varphi^2}) \;=\; \Phi^{-1}\big({}^{(\beta^1+\beta^2)}/_{\varphi}\big)$$

holds. Additivity of the functor P and 6.4.3 yield

$$\begin{aligned}\mathrm{P}(\alpha^1)\,\mathrm{P}(\varphi^1)^{-1} + \mathrm{P}(\alpha^2)\,\mathrm{P}(\varphi^2)^{-1} \;&=\; \mathrm{P}(\beta^1)\,\mathrm{P}(\varphi)^{-1} + \mathrm{P}(\beta^2)\,\mathrm{P}(\varphi)^{-1}\\ &=\; \mathrm{P}(\beta^1+\beta^2)\,\mathrm{P}(\varphi)^{-1}\,,\end{aligned}$$

which shows $(\star)$. A similar argument takes care of $\Bbbk$-multiplication. □

Composition of Fractions

Our aim is to make fractions the morphisms in a category, namely the derived category. To this end, it must be defined how fractions are composed.

6.4.9 Lemma. *Let $\beta\colon M \to V$ be a morphism and $\psi\colon N \to V$ a quasi-isomorphism in $\mathcal{K}(R)$. There exist a morphism α and a quasi-isomorphism φ such that the following diagram in $\mathcal{K}(R)$ is commutative,*

$$(6.4.9.1)\qquad \begin{array}{ccc} U & \xrightarrow{\ \alpha\ } & N \\ {\scriptstyle\varphi}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle\psi} \\ M & \xrightarrow{\ \beta\ } & V\,. \end{array}$$

Conversely, given a morphism α and a quasi-isomorphism φ, there exist β and ψ such that (6.4.9.1) *is commutative.*

PROOF. Choose by 5.2.15 a semi-projective resolution $\varphi\colon U \xrightarrow{\simeq} M$ and apply 6.3.1 to get a morphism α such that (6.4.9.1) is commutative. Conversely, given α and φ, choose by 5.3.26 a semi-injective resolution $\psi\colon N \xrightarrow{\simeq} V$ and apply 6.3.5 to get the morphism β such that (6.4.9.1) is commutative. □

REMARK. One does not need semi-projective and semi-injective resolutions to prove 6.4.9; in fact, they may be proved using only that the homotopy category is triangulated; see E 6.5.9.

6.4.10 Proposition. *Let L, M, and N be R-complexes. There is a composition rule,*

$$\mathcal{D}(R)(M,N) \times \mathcal{D}(R)(L,M) \longrightarrow \mathcal{D}(R)(L,N)\,,$$

given by

$$({}^{\alpha}/_{\varphi},\, {}^{\beta}/_{\psi}) \longmapsto ({}^{\alpha}/_{\varphi})({}^{\beta}/_{\psi}) = {}^{(\alpha\gamma)}/_{(\psi\chi)}\,,$$

where ${}^{\gamma}/_{\chi}$ is any left fraction that makes the diagram

(6.4.10.1)
$$\begin{tikzcd}[column sep=small] & & W \arrow[dl, dotted, "\chi"', "\simeq"] \arrow[dr, dotted, "\gamma"] & & \\ & V \arrow[dl, "\psi"', "\simeq"] \arrow[dr, "\beta"] & & U \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\alpha"] & \\ L & & M & & N \end{tikzcd}$$

in $\mathcal{K}(R)$ commutative, cf. 6.4.9. *This composition rule is $\Bbbk$-bilinear and associative. Moreover, one has the following special cases,*

(6.4.10.2)
$$({}^{\alpha}/_{1^M})({}^{\beta}/_{\psi}) = {}^{(\alpha\beta)}/_{\psi} \qquad \textit{and} \qquad ({}^{\alpha}/_{\varphi})({}^{1^M}/_{\psi}) = {}^{\alpha}/_{(\psi\varphi)}\,.$$

PROOF. Consider the isomorphism $\Phi_{MN}\colon \mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \to \mathcal{D}(R)(M,N)$ of $\Bbbk$-modules from 6.4.8. As composition of morphisms in $\mathcal{K}(R)$ is $\Bbbk$-bilinear and associative, one evidently obtains a $\Bbbk$-bilinear and associative composition rule

$$\mathcal{D}(R)(M,N) \times \mathcal{D}(R)(L,M) \longrightarrow \mathcal{D}(R)(L,N)\,,$$

by the assignment

$$({}^{\alpha}/_{\varphi},\, {}^{\beta}/_{\psi}) \longmapsto \Phi_{LN}\big(\Phi_{MN}^{-1}({}^{\alpha}/_{\varphi})\,\Phi_{LM}^{-1}({}^{\beta}/_{\psi})\big)\,.$$

We argue that this composition of left fractions agrees with the asserted one; in particular, the latter is well-defined. Let ${}^{\gamma}/_{\chi}$ be any left fraction that makes the diagram (6.4.10.1) commutative. It must be verified that the equality

$$\Phi_{LN}^{-1}\big({}^{(\alpha\gamma)}/_{(\psi\chi)}\big) = \Phi_{MN}^{-1}({}^{\alpha}/_{\varphi})\,\Phi_{LM}^{-1}({}^{\beta}/_{\psi})$$

holds, and that follows from the commutativity of (6.4.10.1), indeed, one has

$$\mathrm{P}(\alpha\gamma)\,\mathrm{P}(\psi\chi)^{-1} = \mathrm{P}(\alpha)\,\mathrm{P}(\gamma)\,\mathrm{P}(\chi)^{-1}\,\mathrm{P}(\psi)^{-1} = \mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1}\,\mathrm{P}(\beta)\,\mathrm{P}(\psi)^{-1}\,.$$

The special cases in (6.4.10.2) are evident. □

The following definition is justified by the subsequent proposition.

6.4.11 Definition. The *derived category* $\mathcal{D}(R)$ has the same objects as $\mathcal{C}(R)$ and $\mathcal{K}(R)$, that is, R-complexes. For R-complexes M and N, the hom-set $\mathcal{D}(R)(M,N)$ is the set of all left fractions from M to N; see 6.4.4. Composition in $\mathcal{D}(R)$ is given by the rule in 6.4.10.

6.4.12 Proposition. *The derived category $\mathcal{D}(R)$ is a category. For an R-complex M, the identity morphism in $\mathcal{D}(R)$ is ${}^{1^M}/_{1^M}$.*

Proof. It follows from 6.4.7 that $\mathcal{D}(R)(M,N)$ is a set for every pair M,N of R-complexes. Composition in $\mathcal{D}(R)$ is associative by 6.4.10. That ${}^{1^M}/_{1^M}$ is the identity for M with respect to the composition in $\mathcal{D}(R)$ is a consequence of (6.4.10.2). $\square$

In 6.4.25 it is shown that the category $\mathcal{D}(R)$ is $\Bbbk$-linear. For morphisms $\alpha\colon M \to N$ and $\beta\colon L \to M$ in $\mathcal{K}(R)$ one has ${}^{(\alpha\beta)}/_{1^L} = ({}^{\alpha}/_{1^M})({}^{\beta}/_{1^L})$ by (6.4.10.2); this justifies the next definition.

6.4.13 Definition. Write $\mathrm{V}_R\colon \mathcal{K}(R) \to \mathcal{D}(R)$ for the canonical functor that is the identity on objects and maps a morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ to ${}^{\alpha}/_{1^M}$. When there is no ambiguity, we write V instead of V_R.

For a morphism α in $\mathcal{K}(R)$ one conveniently writes $\mathrm{V}(\alpha) = {}^{\alpha}/_{1}$; that is, the (unspecified) domain of the morphism is omitted from the symbol.

The localization process that leads to the derived category adds just enough morphisms to make all quasi-isomorphisms invertible. A homomorphism of modules is a quasi-isomorphism only if it is an isomorphism, so in the case of modules, no new morphisms are needed. The next results make this precise.

6.4.14 Proposition. *For R-modules M and N the map $\mathcal{K}(R)(M,N) \to \mathcal{D}(R)(M,N)$ induced by* V *is an isomorphism of $\Bbbk$-modules.*

Proof. The map in question is the composite of the $\Bbbk$-module isomorphisms

$$\mathcal{K}(R)(M,N) \longrightarrow \mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(N)) \longrightarrow \mathcal{D}(R)(M,N)$$

from 6.3.14 and 6.4.8. Indeed, for a morphism α in $\mathcal{K}(R)$ one has ${}^{(\pi^N\mathrm{P}(\alpha))}/_{\pi^M} = {}^{(\alpha\pi^M)}/_{\pi^M} = {}^{\alpha}/_{1^M}$, where the last equality follows from 6.4.5. $\square$

6.4.15 Theorem. *The restriction to $\mathcal{M}(R)$ of the functor* $\mathrm{VQ}\colon \mathcal{C}(R) \to \mathcal{D}(R)$ *yields an isomorphism between the module category $\mathcal{M}(R)$ and the full subcategory of $\mathcal{D}(R)$ whose objects are all R-complexes concentrated in degree* 0*.*

Proof. By 6.1.4 and 6.4.14 the functor $\mathrm{VQ}\colon \mathcal{M}(R) \to \mathcal{D}(R)$ is full and faithful, and hence it yields an isomorphism from $\mathcal{M}(R)$ to its image in $\mathcal{D}(R)$. $\square$

Remark. The full subcategory $\mathcal{M}(R)$ of $\mathcal{D}(R)$ is not closed under isomorphisms. The smallest full subcategory of $\mathcal{D}(R)$ that contains $\mathcal{M}(R)$ and is closed under isomorphisms is the one whose objects are complexes with homology concentrated in degree 0, and it is equivalent to $\mathcal{M}(R)$; see E 6.4.5 and see also E 7.6.1.

ISOMORPHISMS

6.4.16 Lemma. *Let α, β, and γ be morphisms in $\mathcal{K}(R)$. If $\alpha\beta$ and $\beta\gamma$ are quasi-isomorphisms then α, β, and γ are quasi-isomorphisms.*

PROOF. Since $\mathrm{H}(\alpha)\,\mathrm{H}(\beta) = \mathrm{H}(\alpha\beta)$ is an isomorphism, $\mathrm{H}(\beta)$ has a left-inverse, and as $\mathrm{H}(\beta)\,\mathrm{H}(\gamma) = \mathrm{H}(\beta\gamma)$ is an isomorphism, $\mathrm{H}(\beta)$ has a right inverse. Consequently, $\mathrm{H}(\beta)$ is an isomorphism. It follows that also $\mathrm{H}(\alpha)$ and $\mathrm{H}(\gamma)$ are isomorphisms. □

6.4.17 Proposition. *A morphism α/φ in $\mathcal{D}(R)$ is an isomorphism if and only if α is a quasi-isomorphism, in which case one has $(\alpha/\varphi)^{-1} = \varphi/\alpha$.*

PROOF. Let α/φ be a morphism in $\mathcal{D}(R)$ from M to N.

If α is a quasi-isomorphism, then φ/α is a morphism from N to M in $\mathcal{D}(R)$. The first equality in the computation $(\varphi/\alpha)(\alpha/\varphi) = \varphi/\varphi = (1^M\varphi)/(1^M\varphi) = 1^M/1^M$ is elementary to verify; the last one follows from 6.4.5. Similarly, one has $(\alpha/\varphi)(\varphi/\alpha) = \alpha/\alpha = (1^N\alpha)/(1^N\alpha) = 1^N/1^N$, whence α/φ is an isomorphism with inverse φ/α.

Conversely, assume that α/φ is an isomorphism. Denote by U the common domain of α and φ. By the arguments above, $1^U/\varphi$ is an isomorphism, and since one has $\alpha/\varphi = (\alpha/1^U)(1^U/\varphi)$ by (6.4.10.2), it follows that $\alpha/1^U$ is an isomorphism; denote by β/ψ its inverse. From 6.4.2 and the equalities $1^N/1^N = (\alpha/1^U)(\beta/\psi) = (\alpha\beta)/\psi$ it follows that $\alpha\beta$ is a quasi-isomorphism. Furthermore, one has $1^U/1^U = (\beta/\psi)(\alpha/1^U) = (\beta\gamma)/\chi$ for some morphism γ and quasi-isomorphism χ. Another application of 6.4.2 gives that $\beta\gamma$ is a quasi-isomorphism, and hence α is a quasi-isomorphism by 6.4.16. □

6.4.18 Corollary. *A morphism $\alpha\colon M \to N$ in $\mathcal{K}(R)$ is a quasi-isomorphism if and only if $\mathrm{V}(\alpha) = \alpha/1^M$ is an isomorphism in $\mathcal{D}(R)$, in which case the inverse is $1^M/\alpha$.*

PROOF. This is an immediate consequence of 6.4.17. □

6.4.19. In view of the preceding results, it is natural to mark isomorphisms in $\mathcal{D}(R)$ by the symbol '$\simeq$', which is used for quasi-isomorphisms in $\mathcal{C}(R)$ and $\mathcal{K}(R)$.

In particular, an isomorphism or a homotopy equivalence α in $\mathcal{C}(R)$ yields an isomorphism $[\alpha]/1$ in $\mathcal{D}(R)$.

For complexes with certain lifting properties there is a conceptual converse to 6.4.18. That is, isomorphisms in $\mathcal{D}(R)$ yield quasi-isomorphisms of complexes.

6.4.20 Proposition. *Let P and M be R-complexes. If P is semi-projective and M and P are isomorphic in $\mathcal{D}(R)$, then there is a quasi-isomorphism $P \to M$.*

PROOF. If P and M are isomorphic in $\mathcal{D}(R)$, then by 6.4.17 there are quasi-isomorphisms $P \leftarrow U \to M$. By 6.3.2 there is a quasi-isomorphism $P \to U$ which composed with $U \to M$ yields the desired quasi-isomorphism. □

6.4.21 Proposition. *Let I and M be R-complexes. If I is semi-injective and M and I are isomorphic in $\mathcal{D}(R)$, then there is a quasi-isomorphism $M \to I$.*

PROOF. If M and I are isomorphic, then there are quasi-isomorphisms $M \leftarrow U \to I$; see 6.4.17. The lifting property 6.3.5 now yields a quasi-isomorphism $M \to I$. □

Remark. Existence of an isomorphism $M \to N$ in $\mathcal{D}(R)$ does not imply that there is even a non-zero morphism $M \to N$ in $\mathcal{C}(R)$; see E 6.4.6.

It follows from 6.4.17 that complexes that are isomorphic in the derived category have isomorphic homology. As the next example shows, the converse is not true.

6.4.22 Example. Over the ring $\mathbb{Z}/4\mathbb{Z}$, consider the complexes

$$P = 0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0 \quad \text{and}$$
$$M = 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

concentrated in degrees 1 and 0. Evidently, one has $\mathrm{H}(P) \cong M \cong \mathrm{H}(M)$. The complex P is semi-projective by 5.2.8, so if P and M were isomorphic in $\mathcal{D}(\mathbb{Z}/4\mathbb{Z})$, then there would by 6.4.20 exist a quasi-isomorphism $P \to M$. However, it is evident that every morphism $\alpha \colon P \to M$ in $\mathcal{C}(\mathbb{Z}/4\mathbb{Z})$ has $\mathrm{H}_1(\alpha) = 0$.

Remark. The take-away from the example above is that there is more to a complex than its homology, even from the point of view of the derived category. There is a way, in the derived category, to build a complex from its homology—see Dwyer, Greenleese, and Iyengar [74]—but in 6.4.22 the homology of P equally builds P and M.

6.4.23 Proposition. *Assume that R is semi-simple or a principal left ideal domain. For every R-complex M there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$.*

Proof. If R is semi-simple, then the assertion follows from 4.2.18 and 6.4.18. If R is a principal left ideal domain, pick by 5.1.7 a semi-free resolution $\pi \colon L \xrightarrow{\simeq} M$. It follows from 4.2.19 that there there is a quasi-isomorphism $\alpha \colon L \to \mathrm{H}(L)$, and now $(\mathrm{H}(\pi)\alpha)/_{\pi}$ is an isomorphism in $\mathcal{D}(R)$ from M to $\mathrm{H}(M)$ by 6.4.17. $\square$

Remark. Semi-simple rings and principal left ideal domains are examples of left hereditary rings; thus 6.4.23 records two special cases of E 6.4.10.

Zero Objects

The complexes that are isomorphic in the homotopy category to the complex 0 are precisely the contractible complexes. The next result explains what is means for a complex to be isomorphic to 0 in the derived category.

6.4.24 Proposition. *An R-complex is a zero object in $\mathcal{D}(R)$ if and only if it is acyclic.*

Proof. In the homotopy category one has $\mathrm{P}(0) \approx 0$. Thus for every R-complex P, each of the sets $\mathcal{K}(R)(\mathrm{P}(0), P)$ and $\mathcal{K}(R)(P, \mathrm{P}(0))$ consists of a single element. It follows from 6.4.7 that for every R-complex M, each of the sets $\mathcal{D}(R)(0, M)$ and $\mathcal{D}(R)(M, 0)$ consists of a single element. Thus the complex 0 is both an initial and a terminal object in $\mathcal{D}(R)$, i.e. a zero object.

If M is acyclic, then $M \to 0$ is a quasi-isomorphism in $\mathcal{K}(R)$, and it follows from 6.4.18 that M and 0 are isomorphic in $\mathcal{D}(R)$. Conversely, if M is isomorphic to 0 in $\mathcal{D}(R)$, then there exist by 6.4.17 quasi-isomorphisms $M \xleftarrow{\simeq} U \xrightarrow{\simeq} 0$ in $\mathcal{K}(R)$, and hence M is acyclic. $\square$

For a morphism α in $\mathcal{C}(R)$, it follows from 6.4.17, 4.2.16, and 6.4.24 that $\mathrm{V}\mathrm{Q}(\alpha)$ is an isomorphism in $\mathcal{D}(R)$ if and only if $\operatorname{Cone}\alpha$ is isomorphic to 0 in $\mathcal{D}(R)$. In Sect. 6.5 it is proved that $\mathcal{D}(R)$ is triangulated, and thus this property of the cone follows from E.22.

Products and Coproducts

Next we show that the category $\mathcal{D}(R)$ has products and coproducts. As in all other categories one uses the symbols $\prod$ and $\coprod$ for products and coproducts in $\mathcal{D}(R)$.

6.4.25 Theorem. *The derived category $\mathcal{D}(R)$ and the functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ are $\Bbbk$-linear. For every family $\{M^u\}_{u\in U}$ of R-complexes the next assertions hold.*

(a) *If an R-complex M with injections $\{\varepsilon^u\colon M^u \to M\}_{u\in U}$ is the coproduct of the family $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$, then M with the morphisms $\{\varepsilon^u/1^{M^u}\}_{u\in U}$ is the coproduct of $\{M^u\}_{u\in U}$ in $\mathcal{D}(R)$.*

(b) *If an R-complex M with projections $\{\varpi^u\colon M \to M^u\}_{u\in U}$ is the product of the family $\{M^u\}_{u\in U}$ in $\mathcal{K}(R)$, then M with the morphisms $\{\varpi^u/1^M\}_{u\in U}$ is the product of $\{M^u\}_{u\in U}$ in $\mathcal{D}(R)$.*

In particular, the derived category $\mathcal{D}(R)$ has products and coproducts, and the canonical functor V preserves products and coproducts.

Proof. By 6.4.8 the hom-sets in $\mathcal{D}(R)$ are $\Bbbk$-modules, and composition of morphisms in $\mathcal{D}(R)$ is $\Bbbk$-bilinear by 6.4.10. Thus the category $\mathcal{D}(R)$ is $\Bbbk$-prelinear. The functor V is $\Bbbk$-linear. Indeed, for parallel morphisms α, β in $\mathcal{K}(R)$ and x in $\Bbbk$, it follows from 6.4.8 that one has

$$\mathrm{V}(x\alpha+\beta) \;=\; (x\alpha+\beta)/1 \;=\; x(\alpha/1) + (\beta/1) \;=\; x\,\mathrm{V}(\alpha) + \mathrm{V}(\beta)\,.$$

To show that $\mathcal{D}(R)$ is $\Bbbk$-linear, it must be argued that it has biproducts and a zero object. That $\mathcal{D}(R)$ has a zero object follows from 6.4.24. As $\mathcal{K}(R)$ has biproducts, see 6.1.9, it follows from 6.1.8, applied to the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$, that $\mathcal{D}(R)$ has biproducts as well.

(a): Let $\{\alpha^u/\varphi^u\colon M^u \to N\}_{u\in U}$ be morphisms in $\mathcal{D}(R)$. It must be shown that there is a unique morphism $\alpha/\varphi\colon M \to N$ in $\mathcal{D}(R)$ with $(\alpha/\varphi)(\varepsilon^u/1^{M^u}) = \alpha^u/\varphi^u$ for all $u \in U$. The functor $\mathrm{P}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ preserves coproducts, see 6.3.11; hence the coproduct $\coprod_{v\in U} \mathrm{P}(M^v)$ and the u^{th} injection $\mathrm{P}(M^u) \to \coprod_{v\in U} \mathrm{P}(M^v)$ are naturally identified with the complex $\mathrm{P}(M)$ and the morphism $\mathrm{P}(\varepsilon^u)\colon \mathrm{P}(M^u) \to \mathrm{P}(M)$. For existence, consider the family $\{\mathrm{P}(\alpha^u)\,\mathrm{P}(\varphi^u)^{-1}\colon \mathrm{P}(M^u) \to \mathrm{P}(N)\}_{u\in U}$ of morphisms in $\mathcal{K}(R)$. By the universal property of coproducts in $\mathcal{K}(R)$ there is a morphism $\psi\colon \mathrm{P}(M) \to \mathrm{P}(N)$ with $\psi\,\mathrm{P}(\varepsilon^u) = \mathrm{P}(\alpha^u)\,\mathrm{P}(\varphi^u)^{-1}$ for all $u \in U$. Denoting by X^u the common domain of φ^u and α^u one has

$$\begin{aligned}
\big((\pi^N\psi)/_{\pi^M}\big)\big(\varepsilon^u/1^{M^u}\big) \;&=\; (\pi^N\psi\,\mathrm{P}(\varepsilon^u))/_{\pi^{M^u}}\\
&=\; (\pi^N\mathrm{P}(\alpha^u)\,\mathrm{P}(\varphi^u)^{-1})/_{\pi^{M^u}}\\
&=\; (\pi^N\mathrm{P}(\alpha^u))/_{(\pi^{M^u}\mathrm{P}(\varphi^u))}
\end{aligned}$$

$$
\begin{aligned}
&= (\alpha^u \pi^{X^u})/_{(\varphi^u \pi^{X^u})} \\
&= \alpha^u/_{\varphi^u} \,.
\end{aligned}
$$

In this computation, the 1st equality follows from commutativity of the diagram,

$$
\begin{tikzcd}[column sep=small]
& & \mathrm{P}(M^u) \arrow[dl, "\pi^{M^u}"', "\simeq"] \arrow[dr, "\mathrm{P}(\varepsilon^u)"] & & \\
& M^u \arrow[dl, "1^{M^u}"'] \arrow[dr, "\varepsilon^u"] & & \mathrm{P}(M) \arrow[dl, "\pi^M"', "\simeq"] \arrow[dr, "\pi^N \psi"] & \\
M^u & & M & & N \,,
\end{tikzcd}
$$

the 2nd equality holds by the defining property of ψ, the 3rd holds by 6.4.5, the 4th follows from (6.3.9.1), and the 5th holds by another application of 6.4.5. Consequently, the morphism $\alpha/_\varphi = (\pi^N\psi)/_{\pi^M}$ has the desired property.

For uniqueness, assume that $(\alpha/_\varphi)(\varepsilon^u/_{1^{M^u}})$ is zero in $\mathcal{D}(R)$ for all $u \in U$; it must be shown that $\alpha/_\varphi$ is zero. By 6.4.10 the composite $(\alpha/_\varphi)(\varepsilon^u/_{1^{M^u}})$ has the form $(\alpha\gamma^u)/_{\chi^u}$ where γ^u is a morphism and χ^u is a quasi-isomorphism with $\varepsilon^u\chi^u = \varphi\gamma^u$. As each $(\alpha\gamma^u)/_{\chi^u}$ is assumed to be zero, it follows that for every $u \in U$ one has

$$0 \;=\; \mathrm{P}(\alpha\gamma^u)\,\mathrm{P}(\chi^u)^{-1} \;=\; \mathrm{P}(\alpha)\,\mathrm{P}(\gamma^u)\,\mathrm{P}(\chi^u)^{-1} \;=\; \mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1}\,\mathrm{P}(\varepsilon^u)\,.$$

By the universal property of coproducts in $\mathcal{K}(R)$, it follows that $\mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1} = 0$ holds, and hence $\alpha/_\varphi$ is the zero morphism by 6.4.3.

(b): Let $\{\alpha^u/_{\varphi^u} \colon N \to M^u\}_{u\in U}$ be morphisms in $\mathcal{D}(R)$. It must be shown that there is a unique morphism $\alpha/_\varphi \colon N \to M$ in $\mathcal{D}(R)$ with $(\varpi^u/_{1^M})(\alpha/_\varphi) = \alpha^u/_{\varphi^u}$ for all $u \in U$. The functor $\mathrm{I}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ preserves products, see 6.3.17, and hence the product $\prod_{v\in U}\mathrm{I}(M^v)$ and the uth projection $\prod_{v\in U}\mathrm{I}(M^v) \to \mathrm{I}(M^u)$ are naturally identified with the complex $\mathrm{I}(M)$ and the morphism $\mathrm{I}(\varpi^u)\colon \mathrm{I}(M) \to \mathrm{I}(M^u)$.

For existence, consider the family $\{\mathrm{I}(\alpha^u)\,\mathrm{I}(\varphi^u)^{-1} \colon \mathrm{I}(N) \to \mathrm{I}(M^u)\}_{u\in U}$ of morphisms in $\mathcal{K}(R)$. By the universal property of products in $\mathcal{K}(R)$ there is a morphism $\psi\colon \mathrm{I}(N) \to \mathrm{I}(M)$ with $\mathrm{I}(\varpi^u)\psi = \mathrm{I}(\alpha^u)\,\mathrm{I}(\varphi^u)^{-1}$ for all $u \in U$. Consider the morphism $\alpha = \pi^M\,\mathrm{P}(\iota^M)^{-1}\,\mathrm{P}(\psi)\,\mathrm{P}(\iota^N)$ from $\mathrm{P}(N)$ to M. By (6.4.10.2) there is an equality $(\varpi^u/_{1^M})(\alpha/_{\pi^N}) = (\varpi^u\alpha)/_{\pi^N}$; we argue that this left fraction is equal to $\alpha^u/_{\varphi^u}$. By 6.4.3 this is equivalent to showing that one has $\mathrm{I}(\varpi^u\alpha)\,\mathrm{I}(\pi^N)^{-1} = \mathrm{I}(\alpha^u)\,\mathrm{I}(\varphi^u)^{-1}$, i.e.

$$(\dagger)\qquad \mathrm{I}(\varpi^u)\,\mathrm{I}(\pi^M)\,\mathrm{IP}(\iota^M)^{-1}\,\mathrm{IP}(\psi)\,\mathrm{IP}(\iota^N)\,\mathrm{I}(\pi^N)^{-1} \;=\; \mathrm{I}(\alpha^u)\,\mathrm{I}(\varphi^u)^{-1}\,.$$

As ψ satisfies $\mathrm{I}(\varpi^u)\psi = \mathrm{I}(\alpha^u)\,\mathrm{I}(\varphi^u)^{-1}$, the equality ($\dagger$) will follow if the identity

$$\mathrm{I}(\pi^M)\,\mathrm{IP}(\iota^M)^{-1}\,\mathrm{IP}(\psi)\,\mathrm{IP}(\iota^N)\,\mathrm{I}(\pi^N)^{-1} \;=\; \psi$$

holds, which by 6.3.8 happens if and only if one has

$$(\ddagger)\qquad \mathrm{I}(\pi^M)\,\mathrm{IP}(\iota^M)^{-1}\,\mathrm{IP}(\psi)\,\mathrm{IP}(\iota^N)\,\mathrm{I}(\pi^N)^{-1}\iota^N\pi^N \;=\; \psi\iota^N\pi^N\,.$$

The commutative diagram, cf. (6.3.15.1),

$$
\begin{tikzcd}
N \arrow[d, "\simeq"', "\iota^N"] & \mathrm{P}(N) \arrow[l, "\pi^N"'] \arrow[r, "\mathrm{P}(\iota^N)"] \arrow[d, "\simeq"', "\iota^{\mathrm{P}(N)}"] & \mathrm{PI}(N) \arrow[r, "\mathrm{P}(\psi)"] \arrow[d, "\simeq"', "\iota^{\mathrm{PI}(N)}"] & \mathrm{PI}(M) \arrow[d, "\simeq"', "\iota^{\mathrm{PI}(M)}"] & \mathrm{P}(M) \arrow[l, "\mathrm{P}(\iota^M)"'] \arrow[r, "\pi^M"] \arrow[d, "\simeq"', "\iota^{\mathrm{P}(M)}"] & M \arrow[d, "\simeq"', "\iota^M"] \\
\mathrm{I}(N) & \mathrm{IP}(N) \arrow[l, "\mathrm{I}(\pi^N)"'] \arrow[r, "\mathrm{IP}(\iota^N)"] & \mathrm{IPI}(N) \arrow[r, "\mathrm{IP}(\psi)"] & \mathrm{IPI}(M) & \mathrm{IP}(M) \arrow[l, "\mathrm{IP}(\iota^M)"'] \arrow[r, "\mathrm{I}(\pi^M)"] & \mathrm{I}(M) \, ,
\end{tikzcd}
$$

shows that the left-hand side of (‡) is equal to $\iota^M \pi^M \mathrm{P}(\iota^M)^{-1} \mathrm{P}(\psi) \mathrm{P}(\iota^N)$, and thus it must be argued that there is an equality $\iota^M \pi^M \mathrm{P}(\iota^M)^{-1} \mathrm{P}(\psi) \mathrm{P}(\iota^N) = \psi \iota^N \pi^N$. However, this follows from the commutative diagram, cf. (6.3.9.1),

$$
\begin{tikzcd}
\mathrm{P}(N) \arrow[r, "\mathrm{P}(\iota^N)"] \arrow[d, "\simeq"', "\pi^N"] & \mathrm{PI}(N) \arrow[r, "\mathrm{P}(\psi)"] \arrow[d, "\simeq"', "\pi^{\mathrm{I}(N)}"] & \mathrm{PI}(M) \arrow[d, "\simeq"', "\pi^{\mathrm{I}(M)}"] & \mathrm{P}(M) \arrow[l, "\mathrm{P}(\iota^M)"'] \arrow[d, "\simeq"', "\pi^M"] \\
N \arrow[r, "\iota^N"] & \mathrm{I}(N) \arrow[r, "\psi"] & \mathrm{I}(M) & M \, . \arrow[l, "\iota^M"']
\end{tikzcd}
$$

For uniqueness, assume that the fraction $(\varpi^u/1^M)(\alpha/\varphi) = (\varpi^u\alpha)/\varphi$ is zero for all $u \in U$; it must be shown that α/φ is zero. As $\mathrm{I}(\varpi^u)\,\mathrm{I}(\alpha)\,\mathrm{I}(\varphi)^{-1} = \mathrm{I}(\varpi^u\alpha)\,\mathrm{I}(\varphi)^{-1} = 0$ holds for every $u \in U$, see 6.4.3, it follows from the universal property of products in $\mathcal{K}(R)$ that $\mathrm{I}(\alpha)\,\mathrm{I}(\varphi)^{-1} = 0$ holds. Evidently, one also has $\mathrm{I}(0)\,\mathrm{I}(1^N)^{-1} = 0$, so α/φ is the zero morphism by another application of 6.4.3.

By construction, the canonical functor V preserves products and coproducts. □

6.4.26. It is immediate from 6.4.25 and 6.1.11 that the product and coproduct in $\mathcal{D}(R)$ of a finite family $\{M^u\}_{u\in U}$ of R-complexes coincide, and that this complex is the iterated biproduct $\bigoplus_{u\in U} M^u$ in $\mathcal{D}(R)$. Per 1.1.14 this complex is called the *direct sum* in $\mathcal{D}(R)$ of the family $\{M^u\}_{u\in U}$, and each M^u is called a *direct summand*.

Remark. Let M and N be R-complexes. A *right prefraction* from M to N is a diagram in $\mathcal{K}(R)$,

$$M \xrightarrow{\beta} V \xleftarrow[\simeq]{\psi} N \, , \tag{$*$}$$

where ψ is a quasi-isomorphism. Dually to 6.4.1 one can define an equivalence relation on the collection of right prefractions from M to N; the equivalence class containing the right prefraction $(*)$ is denoted $\psi\backslash\beta$ and called a *right fraction*. Like $\mathcal{D}(R)(M, N)$, the collection $\mathcal{D}'(R)(M, N)$ of all right fractions from M to N is a set. The collection of all such sets provide the hom-sets for a category $\mathcal{D}'(R)$ whose objects are all R-complexes. There is a functor $\mathcal{D}'(R) \to \mathcal{D}(R)$; it is the identity on objects and it maps a right fraction $\psi\backslash\beta$ to the left fraction α/φ for any choice of morphism α and quasi-isomorphism φ such that the diagram (6.4.9.1) in $\mathcal{K}(R)$ is commutative. The functor $\mathcal{D}'(R) \to \mathcal{D}(R)$ yields an equivalence and, consequently, the derived category may just as well be constructed using right fractions. We soon prove that $\mathcal{D}(R)$ is a triangulated category; similarly so is $\mathcal{D}'(R)$, and the equivalence between $\mathcal{D}'(R)$ and $\mathcal{D}(R)$ is actually an equivalence of triangulated categories.

We do not pursue the right fraction point of view beyond this Remark, even though it does have certain advantages. For example, as the proof of 6.4.25 reveals, the argument for existence of coproducts in $\mathcal{D}(R)$ is more straightforward than the one proving existence of products. This is because left fractions mesh better with coproducts than with products. Dually, it is straightforward to show existence of products in $\mathcal{D}'(R)$, but slightly more involved to establish the existence of coproducts. Had we proved the equivalence between $\mathcal{D}'(R)$ and $\mathcal{D}(R)$, existence of products in $\mathcal{D}(R)$ would follow immediately from the existence of products in $\mathcal{D}'(R)$.

Universal Property

The canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ from 6.4.13 has a universal property described in the next theorem.

6.4.27 Theorem. *Let $\mathcal{U}$ be a category and $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{U}$ a functor. If F maps quasi-isomorphisms to isomorphisms, then there exists a unique functor $\acute{\mathrm{F}}$ that makes the following diagram commutative,*

$$\begin{tikzcd} \mathcal{K}(R) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{V}"'] & \mathcal{U} \\ \mathcal{D}(R) \arrow[ur, dotted, "\acute{\mathrm{F}}"'] & \end{tikzcd} .$$

For every R-complex M there is an equality $\acute{\mathrm{F}}(M) = \mathrm{F}(M)$, and for every morphism α/φ in $\mathcal{D}(R)$ one has $\acute{\mathrm{F}}(\alpha/\varphi) = \mathrm{F}(\alpha)\,\mathrm{F}(\varphi)^{-1}$. Furthermore, the next assertions hold.

(a) *If $\mathcal{U}$ is $\Bbbk$-prelinear and F is $\Bbbk$-linear, then $\acute{\mathrm{F}}$ is $\Bbbk$-linear.*

(b) *If $\mathcal{U}$ has products/coproducts and F preserves products/coproducts, then $\acute{\mathrm{F}}$ preserves products/coproducts.*

Proof. Assume that there is a functor $\acute{\mathrm{F}}$ with $\acute{\mathrm{F}}\mathrm{V} = \mathrm{F}$. As V is the identity on objects, one has $\acute{\mathrm{F}}(M) = \mathrm{F}(M)$ for every R-complex M. A morphism α/φ in $\mathcal{D}(R)$ can by (6.4.10.2) and 6.4.17 be written $\alpha/\varphi = (\alpha/1)(1/\varphi) = (\alpha/1)(\varphi/1)^{-1} = \mathrm{V}(\alpha)\,\mathrm{V}(\varphi)^{-1}$, and thus there are equalities,

$$\acute{\mathrm{F}}(\alpha/\varphi) = \acute{\mathrm{F}}(\mathrm{V}(\alpha)\,\mathrm{V}(\varphi)^{-1}) = (\acute{\mathrm{F}}\mathrm{V}(\alpha))(\acute{\mathrm{F}}\mathrm{V}(\varphi))^{-1} = \mathrm{F}(\alpha)\,\mathrm{F}(\varphi)^{-1} .$$

Consequently, the functor $\acute{\mathrm{F}}$ is uniquely determined by F.

For existence, notice that if $\alpha^1/\varphi^1 = \alpha^2/\varphi^2$ holds in $\mathcal{D}(R)$, then there is an equality $\mathrm{F}(\alpha^1)\,\mathrm{F}(\varphi^1)^{-1} = \mathrm{F}(\alpha^2)\,\mathrm{F}(\varphi^2)^{-1}$ in $\mathcal{U}$. Indeed, this follows by applying F to the diagram (6.4.1.1); cf. the proof of 6.4.3. Thus, one can set $\acute{\mathrm{F}}(M) = \mathrm{F}(M)$ for R-complexes M and $\acute{\mathrm{F}}(\alpha/\varphi) = \mathrm{F}(\alpha)\,\mathrm{F}(\varphi)^{-1}$ for morphisms α/φ in $\mathcal{D}(R)$. With this definition, one evidently has $\acute{\mathrm{F}}\mathrm{V} = \mathrm{F}$.

In order for $\acute{\mathrm{F}}$ to be a functor, it must preserve identity morphisms and respect composition. By definition, $\acute{\mathrm{F}}(1^M/1^M) = \mathrm{F}(1^M)\,\mathrm{F}(1^M)^{-1} = 1^{\mathrm{F}(M)} = 1^{\acute{\mathrm{F}}(M)}$ holds for every R-complex M. Let α/φ and β/ψ be composable morphisms in $\mathcal{D}(R)$. By 6.4.10 the composition $(\alpha/\varphi)(\beta/\psi)$ is $(\alpha\gamma)/(\psi\chi)$ for any choice of morphism γ and quasi-isomorphism χ in $\mathcal{K}(R)$ with $\beta\chi = \varphi\gamma$. Thus there are equalities,

$$\begin{aligned} \acute{\mathrm{F}}((\alpha/\varphi)(\beta/\psi)) &= \acute{\mathrm{F}}((\alpha\gamma)/(\psi\chi)) \\ &= \mathrm{F}(\alpha\gamma)\,\mathrm{F}(\psi\chi)^{-1} \\ &= \mathrm{F}(\alpha)\,\mathrm{F}(\gamma)\,\mathrm{F}(\chi)^{-1}\mathrm{F}(\psi)^{-1} \\ &= \mathrm{F}(\alpha)\,\mathrm{F}(\varphi)^{-1}\mathrm{F}(\beta)\,\mathrm{F}(\psi)^{-1} \\ &= \acute{\mathrm{F}}(\alpha/\varphi)\acute{\mathrm{F}}(\beta/\psi) . \end{aligned}$$

(a): Assume that F is $\Bbbk$-linear, let $\alpha^1/_{\varphi^1}$ and $\alpha^2/_{\varphi^2}$ be parallel morphisms and x an element in $\Bbbk$. Write $\alpha^1/_{\varphi^1} = \beta^1/_{\varphi}$ and $\alpha^2/_{\varphi^2} = \beta^2/_{\varphi}$ for morphisms β^1, β^2 and a quasi-isomorphism φ, see 6.4.6. Now 6.4.8 yields

$$\begin{aligned}\acute{\mathrm{F}}\big(x\big(\alpha^1/_{\varphi^1}\big) + \alpha^2/_{\varphi^2}\big) &= \acute{\mathrm{F}}\big((x\beta^1 + \beta^2)/_{\varphi}\big)\\ &= \mathrm{F}(x\beta^1 + \beta^2)\,\mathrm{F}(\varphi)^{-1}\\ &= (x\,\mathrm{F}(\beta^1) + \mathrm{F}(\beta^2))\,\mathrm{F}(\varphi)^{-1}\\ &= x\,\mathrm{F}(\beta^1)\,\mathrm{F}(\varphi)^{-1} + \mathrm{F}(\beta^2)\,\mathrm{F}(\varphi)^{-1}\\ &= x\,\acute{\mathrm{F}}\big(\beta^1/_{\varphi}\big) + \acute{\mathrm{F}}\big(\beta^2/_{\varphi}\big)\\ &= x\,\acute{\mathrm{F}}\big(\alpha^1/_{\varphi^1}\big) + \acute{\mathrm{F}}\big(\alpha^2/_{\varphi^2}\big)\,,\end{aligned}$$

and hence $\acute{\mathrm{F}}$ is $\Bbbk$-linear.

(b): The proof of 6.1.16(b) applies to prove part (b) in this theorem; only one has to replace the functor Q by V and the reference to 6.1.9 by one to 6.4.25. □

By the universal property above, certain types of functors on $\mathcal{K}(R)$ induce functors on $\mathcal{D}(R)$ and natural transformations follow along.

6.4.28 Proposition. *Let* $\mathrm{E}, \mathrm{F}\colon \mathcal{K}(R) \to \mathcal{U}$ *be functors that map quasi-isomorphisms to isomorphisms and consider the induced functors* $\acute{\mathrm{E}}, \acute{\mathrm{F}}\colon \mathcal{D}(R) \to \mathcal{U}$ *from* 6.4.27. *Every natural transformation* $\tau\colon \mathrm{E} \to \mathrm{F}$ *induces a natural transformation* $\acute{\tau}\colon \acute{\mathrm{E}} \to \acute{\mathrm{F}}$ *given by* $\acute{\tau}^M = \tau^M$ *for every* R*-complex* M.

Proof. For every R-complex M one has $\acute{\mathrm{E}}(M) = \mathrm{E}(M)$ and $\acute{\mathrm{F}}(M) = \mathrm{F}(M)$ by 6.4.27, whence $\acute{\tau}^M = \tau^M$ is a morphism $\acute{\mathrm{E}}(M) \to \acute{\mathrm{F}}(M)$. Let $\alpha/_{\varphi}\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and denote by U the common domain of α and φ. Since $\tau\colon \mathrm{E} \to \mathrm{F}$ is a natural transformation of functors $\mathcal{K}(R) \to \mathcal{U}$ there are equalities,

$$\acute{\tau}^N \acute{\mathrm{E}}(\alpha/_{\varphi}) = \tau^N \mathrm{E}(\alpha)\,\mathrm{E}(\varphi)^{-1} = \mathrm{F}(\alpha)\tau^U\,\mathrm{E}(\varphi)^{-1} = \mathrm{F}(\alpha)\,\mathrm{F}(\varphi)^{-1}\tau^M = \acute{\mathrm{F}}(\alpha/_{\varphi})\acute{\tau}^M\,,$$

which show that $\acute{\tau}\colon \acute{\mathrm{E}} \to \acute{\mathrm{F}}$ is a natural transformation of functors $\mathcal{D}(R) \to \mathcal{U}$. □

6.4.29 Theorem. *Let* $\mathcal{V}$ *be a category and* $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ *a functor. If* G *maps quasi-isomorphisms to isomorphisms, then there exists a unique functor* $\acute{\mathrm{G}}$ *that makes the following diagram commutative,*

$$\begin{tikzcd} \mathcal{K}(R)^{\mathrm{op}} \arrow[r, "\mathrm{G}"] \arrow[d, "\mathrm{V}^{\mathrm{op}}"'] & \mathcal{V} \\ \mathcal{D}(R)^{\mathrm{op}} \arrow[ur, dotted, "\acute{\mathrm{G}}"'] & \end{tikzcd}\;.$$

For every R*-complex* M *there is an equality* $\acute{\mathrm{G}}(M) = \mathrm{G}(M)$*, and for every morphism* $\alpha/_{\varphi}$ *in* $\mathcal{D}(R)^{\mathrm{op}}$ *one has* $\acute{\mathrm{G}}(\alpha/_{\varphi}) = \mathrm{G}(\varphi)^{-1}\mathrm{G}(\alpha)$*. Further, the next assertions hold.*

(a) *If* $\mathcal{V}$ *is* $\Bbbk$*-prelinear and* G *is* $\Bbbk$*-linear, then* $\acute{\mathrm{G}}$ *is* $\Bbbk$*-linear.*

(b) *If* $\mathcal{V}$ *has products/coproducts and* G *preserves products/coproducts, then* $\acute{\mathrm{G}}$ *preserves products/coproducts.*

Proof. Apply 6.4.27 to the functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{K}(R) \to \mathcal{V}^{\mathrm{op}}$. □

6.4.30 Proposition. *Let* $\mathrm{G}, \mathrm{J}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ *be functors that map quasi-isomorphisms to isomorphisms and consider the induced functors* $\acute{\mathrm{G}}, \acute{\mathrm{J}}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{V}$ *from* 6.4.29. *Every natural transformation* $\tau\colon \mathrm{G} \to \mathrm{J}$ *induces a natural transformation* $\acute{\tau}\colon \acute{\mathrm{G}} \to \acute{\mathrm{J}}$ *given by* $\acute{\tau}^M = \tau^M$ *for every R-complex M.*

Proof. Apply 6.4.28 to the natural transformation $\tau^{\mathrm{op}}\colon \mathrm{J}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ of functors from $\mathcal{K}(R)$ to $\mathcal{V}^{\mathrm{op}}$. □

Special Case of the Universal Property

We apply the derived category's universal property in an important special case.

6.4.31 Theorem. *Let* $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *be a functor. If* F *preserves quasi-isomorphisms, then there is a unique functor* $\H{\mathrm{F}}$ *that makes the next diagram commutative,*

$$\begin{tikzcd} \mathcal{K}(R) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{V}_R"'] & \mathcal{K}(S) \arrow[d, "\mathrm{V}_S"] \\ \mathcal{D}(R) \arrow[r, dotted, "\H{\mathrm{F}}"] & \mathcal{D}(S) \end{tikzcd} .$$

For every R-complex M there is an equality $\H{\mathrm{F}}(M) = \mathrm{F}(M)$, *and for every morphism* α/φ *in* $\mathcal{D}(R)$ *one has* $\H{\mathrm{F}}(\alpha/\varphi) = \mathrm{F}(\alpha)/\mathrm{F}(\varphi)$. *Further, the following assertions hold.*

(a) *If* F *is* $\Bbbk$*-linear, then* $\H{\mathrm{F}}$ *is* $\Bbbk$*-linear.*
(b) *If* F *preserves products/coproducts, then* $\H{\mathrm{F}}$ *preserves products/coproducts.*

Proof. As F preserves quasi-isomorphisms, the functor $\mathrm{V}_S\mathrm{F}$ maps quasi-isomorphisms in $\mathcal{K}(R)$ to isomorphisms in $\mathcal{D}(S)$; see 6.4.18. Hence, the existence and uniqueness of $\H{\mathrm{F}}$ follow from 6.4.27. In symbols one has $\H{\mathrm{F}} = (\mathrm{V}_S\mathrm{F})\acute{}$. The value of $\H{\mathrm{F}}$ on an R-complex M is $\H{\mathrm{F}}(M) = \mathrm{V}_S\mathrm{F}(M) = \mathrm{F}(M)$ since V_S is the identity on objects. By 6.4.27, 6.4.17, and (6.4.10.2) the value of $\H{\mathrm{F}}$ on a morphism α/φ is

$$\H{\mathrm{F}}(\alpha/\varphi) \;=\; (\mathrm{V}_S\mathrm{F}(\alpha))(\mathrm{V}_S\mathrm{F}(\varphi))^{-1} \;=\; (\mathrm{F}(\alpha)/1)(1/\mathrm{F}(\varphi)) \;=\; \mathrm{F}(\alpha)/\mathrm{F}(\varphi)\,.$$

By 6.4.25 the functor V_S is $\Bbbk$-linear and preserves products/coproducts. Thus, if F has one or more of these properties, then so has $\mathrm{V}_S\mathrm{F}$, and the assertions in parts (a) and (b) now follow from the corresponding parts in 6.4.27. □

6.4.32 Example. Assume that R is commutative and let U be a multiplicative subset of R. The localization functor $U^{-1}\colon \mathcal{C}(R) \to \mathcal{C}(U^{-1}R)$ preserves homotopy by 4.3.18 and quasi-isomorphisms by 4.2.14, and hence so does the naturally isomorphic functor $U^{-1}R \otimes_R -$ from 2.1.50. Thus, it follows from 6.1.20 and 6.4.31 that they induce naturally isomorphic functors $U^{-1} \simeq U^{-1}R \otimes_R -\colon \mathcal{D}(R) \to \mathcal{D}(U^{-1}R)$.

6.4.33 Proposition. *Let* $\mathrm{E}, \mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *be functors that preserve quasi-isomorphisms and consider the induced functors* $\H{\mathrm{E}}, \H{\mathrm{F}}\colon \mathcal{K}(R) \to \mathcal{K}(S)$; *see* 6.4.31.

Every natural transformation $\tau\colon \mathrm{E} \to \mathrm{F}$ induces a natural transformation $\tilde{\tau}\colon \tilde{\mathrm{E}} \to \tilde{\mathrm{F}}$ given by $\tilde{\tau}^M = \tau^M/1$ for every R-complex M.

Proof. Evidently, application of the canonical functor $\mathrm{V}\colon \mathcal{K}(S) \to \mathcal{D}(S)$ to the natural transformation τ yields a natural transformation $\mathrm{V}\tau\colon \mathrm{VE} \to \mathrm{VF}$ of functors from $\mathcal{K}(R)$ to $\mathcal{D}(S)$. By definition, $\tilde{\mathrm{E}}$ and $\tilde{\mathrm{F}}$ are the functors from $\mathcal{D}(R)$ to $\mathcal{D}(S)$ induced by VE and VF; see 6.4.27. Thus 6.4.28 gives the desired conclusion. □

6.4.34 Theorem. *Let $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be a functor. If G preserves quasi-isomorphisms, then there is a unique functor $\tilde{\mathrm{G}}$ that makes the next diagram commutative,*

$$\begin{tikzcd}
\mathcal{K}(R)^{\mathrm{op}} \arrow[r, "\mathrm{G}"] \arrow[d, "\mathrm{V}_R^{\mathrm{op}}"'] & \mathcal{K}(S) \arrow[d, "\mathrm{V}_S"] \\
\mathcal{D}(R)^{\mathrm{op}} \arrow[r, dotted, "\tilde{\mathrm{G}}"] & \mathcal{D}(S)\,.
\end{tikzcd}$$

For every R-complex M one has $\tilde{\mathrm{G}}(M) = \mathrm{G}(M)$, and for every morphism α/φ in $\mathcal{D}(R)^{\mathrm{op}}$ one has $\tilde{\mathrm{G}}(\alpha/\varphi) = (1/\mathrm{G}(\varphi))(\mathrm{G}(\alpha)/1)$. Furthermore, the next assertions hold.

(a) *If G is $\Bbbk$-linear, then $\tilde{\mathrm{G}}$ is $\Bbbk$-linear.*
(b) *If G preserves products/coproducts, then $\tilde{\mathrm{G}}$ preserves products/coproducts.*

Proof. Proceed as in the proof of 6.4.31, only apply 6.4.29 in place of 6.4.27. □

6.4.35 Proposition. *Let $\mathrm{G},\mathrm{J}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be functors that preserve quasi-isomorphisms and consider the induced functors $\tilde{\mathrm{G}}, \tilde{\mathrm{J}}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$; see 6.4.34. Every natural transformation $\tau\colon \mathrm{G} \to \mathrm{J}$ induces a natural transformation $\tilde{\tau}\colon \tilde{\mathrm{G}} \to \tilde{\mathrm{J}}$ given by $\tilde{\tau}^M = \tau^M/1$ for every R-complex M.*

Proof. Proceed as in the proof of 6.4.33, only apply 6.4.29 and 6.4.30 in place of 6.4.27 and 6.4.28. □

We occasionally abuse notation and write F and G for the induced functors $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ from 6.4.31 and 6.4.34.

6.4.36 Example. The restrictions of scalars functors from 6.1.23, associated to a ring homomorphism $\varphi\colon R \to S$, preserve quasi-isomorphisms by 6.1.24. The functors

$$\mathrm{res}^S_R : \mathcal{D}(S) \longrightarrow \mathcal{D}(R) \qquad \text{and} \qquad \mathrm{res}^{S^{\mathrm{o}}}_{R^{\mathrm{o}}} : \mathcal{D}(S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R^{\mathrm{o}})\,,$$

induced per 6.4.31, are usually suppressed, and even when they are not, we suppress the '$^{\mathrm{op}}$' on the opposite functors $\mathcal{D}(S)^{\mathrm{op}} \to \mathcal{D}(R)^{\mathrm{op}}$ and $\mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}} \to \mathcal{D}(R^{\mathrm{o}})^{\mathrm{op}}$.

Recall from 1.1.46 that a faithful functor is conservative. Next we prove that the functors from 6.4.36 are conservative; however, they may not be faithful, see 7.3.37.

6.4.37 Proposition. *Let $\varphi\colon R \to S$ be a ring homomorphism. The restriction of scalars functors $\mathrm{res}^S_R\colon \mathcal{D}(S) \to \mathcal{D}(R)$ and $\mathrm{res}^{S^{\mathrm{o}}}_{R^{\mathrm{o}}}\colon \mathcal{D}(S^{\mathrm{o}}) \to \mathcal{D}(R^{\mathrm{o}})$ are conservative.*

Proof. By 6.1.24 the functors $\mathrm{res}^S_R\colon \mathcal{K}(S) \to \mathcal{K}(R)$ and $\mathrm{res}^{S^{\mathrm{o}}}_{R^{\mathrm{o}}}\colon \mathcal{K}(S^{\mathrm{o}}) \to \mathcal{K}(R^{\mathrm{o}})$ preserve and reflect quasi-isomorphisms, so the assertion follows from 6.4.17. □

6.4.38 Example. Let n be an integer and recall from 6.1.26 that soft truncation above $(-)_{\subseteq n}$ and soft truncation below $(-)_{\supseteq n}$ are $\Bbbk$-linear endofunctors on $\mathcal{K}(R)$. It follows from 4.2.10 that they preserve quasi-isomorphisms, so by 6.4.31 they yield $\Bbbk$-linear endofunctors on $\mathcal{D}(R)$, also denoted $(-)_{\subseteq n}$ and $(-)_{\supseteq n}$.

6.4.39 Lemma. *Let* $\mathcal{K}(R) \xrightarrow{\mathrm{E}} \mathcal{K}(S) \xrightarrow{\mathrm{F}} \mathcal{U} \xrightarrow{\mathrm{T}} \mathcal{V}$ *be functors where* E *preserves quasi-isomorphisms and* F *maps quasi-isomorphisms to isomorphisms. The functor* $\mathcal{D}(R) \to \mathcal{V}$ *induced by* TFE *is* $\mathrm{T}\acute{\mathrm{F}}\acute{\acute{\mathrm{E}}}$; *in symbols,* $(\mathrm{TFE})\acute{} = \mathrm{T}\acute{\mathrm{F}}\acute{\acute{\mathrm{E}}}$.

In particular, one has $(\mathrm{TF})\acute{} = \mathrm{T}\acute{\mathrm{F}}$ *and* $(\mathrm{FE})\acute{} = \acute{\mathrm{F}}\acute{\acute{\mathrm{E}}}$.

Proof. The induced functor $(\mathrm{TFE})\acute{}$ is the unique functor with $(\mathrm{TFE})\acute{}\,\mathrm{V}_R = \mathrm{TFE}$. As one has $\mathrm{T}\acute{\mathrm{F}}\acute{\acute{\mathrm{E}}}\mathrm{V}_R = \mathrm{T}\acute{\mathrm{F}}\mathrm{V}_S\mathrm{E} = \mathrm{TFE}$, the assertion follows. ◻

6.4.40 Lemma. *Let* $\mathcal{K}(Q) \xrightarrow{\mathrm{E}} \mathcal{K}(R) \xrightarrow{\mathrm{F}} \mathcal{K}(S)$ *be functors that preserve quasi-isomorphisms. The functor* $\mathcal{D}(Q) \to \mathcal{D}(S)$ *induced by* FE *is* $\acute{\acute{\mathrm{F}}}\acute{\acute{\mathrm{E}}}$; *i.e.* $(\mathrm{FE})\acute{\acute{}} = \acute{\acute{\mathrm{F}}}\acute{\acute{\mathrm{E}}}$.

Proof. The induced functor $(\mathrm{FE})\acute{\acute{}}$ is the unique functor with $(\mathrm{FE})\acute{\acute{}}\,\mathrm{V}_Q = \mathrm{V}_S\mathrm{FE}$. As one has $\acute{\acute{\mathrm{F}}}\acute{\acute{\mathrm{E}}}\mathrm{V}_Q = \acute{\acute{\mathrm{F}}}\mathrm{V}_R\mathrm{E} = \mathrm{V}_S\mathrm{FE}$, the assertion follows. ◻

Adjoint Functors

An adjunction of appropriate functors between categories of complexes induces an adjunction between homotopy categories, see 6.1.32. We close this section by recording how an adjunction on the level of homotopy categories induces an adjunction on the derived category level.

To parse the next result, recall 6.4.31 and 6.4.33.

6.4.41 Lemma. *Consider an adjunction,*

$$\mathcal{K}(S) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{K}(R)\,,$$

with unit $\alpha\colon \mathrm{Id}_{\mathcal{K}(S)} \to \mathrm{GF}$ *and counit* $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{K}(R)}$. *If* F *and* G *preserve quasi-isomorphisms, then the induced functors,*

$$\mathcal{D}(S) \underset{\acute{\acute{\mathrm{G}}}}{\overset{\acute{\acute{\mathrm{F}}}}{\rightleftarrows}} \mathcal{D}(R)\,,$$

is an adjunction with unit $\acute{\acute{\alpha}}\colon \mathrm{Id}_{\mathcal{D}(S)} \to \acute{\acute{\mathrm{G}}}\acute{\acute{\mathrm{F}}}$ *and counit* $\acute{\acute{\beta}}\colon \acute{\acute{\mathrm{F}}}\acute{\acute{\mathrm{G}}} \to \mathrm{Id}_{\mathcal{D}(R)}$.

Proof. For the unit and counit of the given adjunction one has the zigzag identities $\mathrm{G}\beta \circ \alpha\mathrm{G} = 1_{\mathrm{G}}$ and $\beta\mathrm{F} \circ \mathrm{F}\alpha = 1_{\mathrm{F}}$. It follows, cf. 6.4.40, that $\acute{\acute{\mathrm{G}}}\acute{\acute{\beta}} \circ \acute{\acute{\alpha}}\acute{\acute{\mathrm{G}}} = 1_{\acute{\acute{\mathrm{G}}}}$ and $\acute{\acute{\beta}}\acute{\acute{\mathrm{F}}} \circ \acute{\acute{\mathrm{F}}}\acute{\acute{\alpha}} = 1_{\acute{\acute{\mathrm{F}}}}$ hold, which is equivalent to saying that $(\acute{\acute{\mathrm{F}}}, \acute{\acute{\mathrm{G}}})$ is an adjunction with unit $\acute{\acute{\alpha}}$ and counit $\acute{\acute{\beta}}$. ◻

To parse the next result, recall 6.4.34 and 6.4.35.

6.4.42 Lemma. *Consider an adjunction,*

$$\mathcal{K}(S) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{K}(R)^{\mathrm{op}} ,$$

with unit $\alpha\colon \mathrm{Id}_{\mathcal{K}(S)} \to \mathrm{GF}$ *and counit* $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{K}(R)^{\mathrm{op}}}$. *If* F *and* G *preserve quasi-isomorphisms, then the induced functors,*

$$\mathcal{D}(S) \underset{\H{\mathrm{G}}}{\overset{\H{\mathrm{F}}}{\rightleftarrows}} \mathcal{D}(R)^{\mathrm{op}} ,$$

is an adjunction with unit $\H{\alpha}\colon \mathrm{Id}_{\mathcal{D}(S)} \to \H{\mathrm{G}}\H{\mathrm{F}}$ *and counit* $\H{\beta}\colon \H{\mathrm{F}}\H{\mathrm{G}} \to \mathrm{Id}_{\mathcal{D}(R)^{\mathrm{op}}}$.

Proof. This follows from an argument parallel to the proof of 6.4.41. □

6.4.43. The opposite of the counit $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{K}(R)^{\mathrm{op}}}$ from 6.4.42 is a natural transformation $\beta^{\mathrm{op}}\colon \mathrm{Id}_{\mathcal{K}(R)} \to \mathrm{F}^{\mathrm{op}}\mathrm{G}^{\mathrm{op}}$. In concrete settings it is often more natural to consider β^{op} instead of β. Note that one has $(\beta^{\mathrm{op}})^{\H{}} = (\H{\beta})^{\mathrm{op}}$.

Exercises

E 6.4.1 Let M and N be R-complexes. Show that the collection of left prefractions from M to N is a proper class; that is, not a set.

E 6.4.2 Show that the isomorphism in 6.4.7 is a natural isomorphism of $\Bbbk$-modules.

E 6.4.3 For R-complexes M, N show that there are natural isomorphisms $\mathcal{K}(R)(\mathrm{P}(M), N) \cong \mathcal{D}(R)(M, N)$ and $\mathcal{K}(R)(M, \mathrm{I}(N)) \cong \mathcal{D}(R)(M, N)$.

E 6.4.4 Write down explicitly the inverse of the functor $\mathcal{M}(R) \to \mathrm{VQ}(\mathcal{M}(R))$ from 6.4.15.

E 6.4.5 Let $\mathcal{D}_0(R)$ be the full subcategory of $\mathcal{D}(R)$ whose objects are all complexes that are isomorphic in $\mathcal{D}(R)$ to one in the full subcategory $\mathcal{M}(R)$. (a) Show that a complex M is in $\mathcal{D}_0(R)$ if and only if $\mathrm{H}(M)$ is concentrated in degree 0. (b) Show that $\mathcal{D}_0(R)$ is equivalent to $\mathcal{M}(R)$.

E 6.4.6 Show that the complexes in 4.2.3 are isomorphic in $\mathcal{D}(R)$. *Hint:* E 5.1.9.

E 6.4.7 Show that a left fraction α/φ is zero if and only if there exists a quasi-isomorphism μ with $\alpha\mu = 0$; see also E 6.3.2.

E 6.4.8 Consider $\Bbbk$ as a module over the polynomial algebra $\Bbbk[x]$ with the trivial x-action. Show that there is a non-zero morphism $\varkappa\colon \Bbbk \to \Sigma\Bbbk$ in $\mathcal{D}(\Bbbk[x])$ with $\mathrm{H}(\varkappa) = 0$.

E 6.4.9 Show that $\mathcal{M}_{\mathrm{gr}}(R)$ is isomorphic to a full subcategory of $\mathcal{D}(R)$.

E 6.4.10 Assume that R is left hereditary. Show that there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$ for every R-complex M. *Hint:* E 5.2.3.

6.5 Triangulation of $\mathcal{D}$

Synopsis. Distinguished triangle; triangulated functor; universal property of $\mathcal{D}$ revisited; homology; Five Lemma; distinguished triangle from short exact sequence.

The homotopy category $\mathcal{K}(R)$ is triangulated but rarely Abelian, and same holds for the derived category $\mathcal{D}(R)$. While the triangulated structure on $\mathcal{D}(R)$ is induced from $\mathcal{K}(R)$, it is closer to the Abelian structure on $\mathcal{C}(R)$ in the crucial sense that every short exact sequence in $\mathcal{C}(R)$ yields a distinguished triangle in $\mathcal{D}(R)$.

An important proof-technical tool in our approach to the triangulation on $\mathcal{D}(R)$ is the semi-projective resolution functor on $\mathcal{K}(R)$ and the one it induces on $\mathcal{D}(R)$.

6.5.1 Lemma. *The resolution functor* $\mathrm{P}\colon \mathcal{K}(R) \to \mathcal{K}(R)$ *from* 6.3.11 *induces a functor* $\acute{\mathrm{P}}\colon \mathcal{D}(R) \to \mathcal{K}(R)$ *with* $\acute{\mathrm{P}}\mathrm{V} = \mathrm{P}$; *it is* $\Bbbk$*-linear and preserves coproducts.*

Proof. By 6.3.11 the functor P maps quasi-isomorphisms to isomorphisms, and it is $\Bbbk$-linear and preserves coproducts. The assertions now follow from 6.4.27. □

When there is no risk of ambiguity, we write P instead of $\acute{\mathrm{P}}$.

6.5.2 Lemma. *Let* $\xi\colon M \to N$ *be a morphism in* $\mathcal{D}(R)$*. The morphism* $\mathrm{P}(\xi)$ *in* $\mathcal{K}(R)$ *fits into the following commutative diagram in* $\mathcal{D}(R)$*,*

$$\begin{tikzcd}
\mathrm{P}(M) \arrow[r, "\pi^M/1", "\simeq"'] \arrow[d, "\mathrm{P}(\xi)/1"'] & M \arrow[d, "\xi"] \\
\mathrm{P}(N) \arrow[r, "\pi^N/1", "\simeq"'] & N\,.
\end{tikzcd}$$

Proof. With $\xi = \alpha/\varphi$ one has $\mathrm{P}(\xi)/1 = (\mathrm{P}(\alpha)\,\mathrm{P}(\varphi)^{-1})/1 = \mathrm{P}(\alpha)/\mathrm{P}(\varphi)$ by 6.4.27 and 6.4.5. Now (6.4.10.2) and (6.3.9.1) yield

$$\big(\pi^N/1\big)\big(\mathrm{P}(\xi)/1\big) \;=\; \big(\pi^N/1\big)\big(\mathrm{P}(\alpha)/\mathrm{P}(\varphi)\big) \;=\; (\pi^N\mathrm{P}(\alpha))/\mathrm{P}(\varphi) \;=\; (\alpha\pi^U)/\mathrm{P}(\varphi)\,,$$

where U is the common domain of α and φ. The commutative diagram $\mathcal{K}(R)$,

$$\begin{tikzcd}
 & & \mathrm{P}(U) \arrow[dl, "\mathrm{P}(\varphi)"', "\simeq"] \arrow[dr, "\pi^U"] & & \\
 & \mathrm{P}(M) \arrow[dl, "1"'] \arrow[dr, "\pi^M"] & & U \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\alpha"] & \\
\mathrm{P}(M) & & M & & N\,,
\end{tikzcd}$$

shows that the composite $(\alpha/\varphi)(\pi^M/1)$ is also equal to $(\alpha\pi^U)/\mathrm{P}(\varphi)$. □

6.5.3. By 4.2.9 and 6.4.31 there is a unique $\Bbbk$-linear endofunctor $\acute{\acute{\Sigma}}$ on $\mathcal{D}(R)$ that makes the following diagram commutative,

$$\begin{tikzcd}
\mathcal{K}(R) \arrow[r, "\Sigma"] \arrow[d, "\mathrm{V}"'] & \mathcal{K}(R) \arrow[d, "\mathrm{V}"] \\
\mathcal{D}(R) \arrow[r, "\acute{\acute{\Sigma}}"] & \mathcal{D}(R)\,.
\end{tikzcd}$$

By 6.4.40 it is an isomorphism with inverse induced by $\Sigma^{-1}\colon \mathcal{K}(R) \to \mathcal{K}(R)$. By the usual abuse of notation, the functor $\acute{\acute{\Sigma}}$ is written Σ or, occasionally, $\Sigma_{\mathcal{D}}$. For a morphism α/φ in $\mathcal{D}(R)$ one has $\Sigma_{\mathcal{D}}(\alpha/\varphi) = (\Sigma\alpha)/(\Sigma\varphi)$.

6.5.4. By 6.3.11 there is a natural isomorphism $\phi\colon \mathrm{P}\Sigma_{\mathcal{K}} \to \Sigma_{\mathcal{K}}\mathrm{P}$ of endofunctors on $\mathcal{K}(R)$ that makes the functor P triangulated. It follows from 6.4.28, 6.4.39, and 6.5.3 that there is an induced natural isomorphsim of functors $\mathcal{D}(R) \to \mathcal{K}(R)$,

$$\acute{\mathrm{P}}\Sigma_{\mathcal{D}} \;=\; (\mathrm{P}\Sigma_{\mathcal{K}})\acute{} \xrightarrow{\acute{\phi}} (\Sigma_{\mathcal{K}}\mathrm{P})\acute{} \;=\; \Sigma_{\mathcal{K}}\acute{\mathrm{P}}\,.$$

We often simplify the notation and write $\phi\colon \mathrm{P}\Sigma \to \Sigma\mathrm{P}$ for the isomorphism above.

Consider the $\Bbbk$-linear category $\mathcal{D}(R)$, see 6.4.25, equipped with the $\Bbbk$-linear automorphism $\Sigma = \Sigma_{\mathcal{D}}$. One may now speak of candidate triangles in $\mathcal{D}(R)$; cf. E.1.

6.5.5 Definition. A candidate triangle in $\mathcal{D}(R)$ is called a *distinguished triangle* if it is isomorphic to the image of a distinguished triangle in $\mathcal{K}(R)$ under the canonical functor $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$.

Next we show that the semi-projective resolution functor $\mathrm{P}\colon \mathcal{D}(R) \to \mathcal{K}(R)$ preserves distinguished triangles. Together with the already established fact that $\mathcal{K}(R)$ is triangulated, this yields a short pass to the fact that $\mathcal{D}(R)$ is triangulated.

6.5.6 Proposition. *Consider the functor* $\mathrm{P}\colon \mathcal{D}(R) \to \mathcal{K}(R)$ *from* 6.5.1 *and the natural isomorphism* $\phi\colon \mathrm{P}\Sigma \to \Sigma\mathrm{P}$ *from* 6.5.4. *For every distinguished triangle in* $\mathcal{D}(R)$,

$$M \xrightarrow{\xi} N \xrightarrow{\vartheta} X \xrightarrow{\varkappa} \Sigma M\,,$$

the candidate triangle

$$\mathrm{P}(M) \xrightarrow{\mathrm{P}(\xi)} \mathrm{P}(N) \xrightarrow{\mathrm{P}(\vartheta)} \mathrm{P}(X) \xrightarrow{\phi^M \mathrm{P}(\varkappa)} \Sigma\mathrm{P}(M)$$

is distinguished in $\mathcal{K}(R)$.

Proof. As in 6.5.1, write P for the resolution functor on $\mathcal{K}(R)$ and $\acute{\mathrm{P}}$ for the induced functor on $\mathcal{D}(R)$. By definition, the given distinguished triangle in $\mathcal{D}(R)$ is isomorphic to one of the form

$$\widetilde{M} \xrightarrow{\mathrm{V}(\alpha)} \widetilde{N} \xrightarrow{\mathrm{V}(\beta)} \widetilde{X} \xrightarrow{\mathrm{V}(\gamma)} \Sigma\widetilde{M}\,,$$

where $\widetilde{M} \xrightarrow{\alpha} \widetilde{N} \xrightarrow{\beta} \widetilde{X} \xrightarrow{\gamma} \Sigma\widetilde{M}$ is a distinguished triangle in $\mathcal{K}(R)$. Thus it suffices to argue that the candidate triangle

$$\acute{\mathrm{P}}(\widetilde{M}) \xrightarrow{\acute{\mathrm{P}}\mathrm{V}(\alpha)} \acute{\mathrm{P}}(\widetilde{N}) \xrightarrow{\acute{\mathrm{P}}\mathrm{V}(\beta)} \acute{\mathrm{P}}(\widetilde{X}) \xrightarrow{\acute{\phi}^{\widetilde{M}}\acute{\mathrm{P}}\mathrm{V}(\gamma)} \Sigma\acute{\mathrm{P}}(\widetilde{M})\,,$$

is distinguished. However, this candidate triangle is nothing but

$$\mathrm{P}(\widetilde{M}) \xrightarrow{\mathrm{P}(\alpha)} \mathrm{P}(\widetilde{N}) \xrightarrow{\mathrm{P}(\beta)} \mathrm{P}(\widetilde{X}) \xrightarrow{\phi^{\widetilde{M}}\mathrm{P}(\gamma)} \Sigma\mathrm{P}(\widetilde{M})\,,$$

which is distinguished by 6.3.11. $\square$

6.5.7 Theorem. *The derived category* $\mathcal{D}(R)$, *equipped with the automorphism* Σ *and the collection of distinguished triangles defined in* 6.5.5, *is triangulated. Moreover, the canonical functor* $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ *is triangulated.*

Proof. By 6.5.3 one has $\mathrm{V}\Sigma = \Sigma\mathrm{V}$. Thus, once it has been established that the category $\mathcal{D}(R)$ is triangulated, the functor V is triangulated by 6.5.5.

(TR0): Follows immediately from the definition of distinguished triangles in $\mathcal{D}(R)$ and the fact that $(\mathcal{K}(R), \Sigma)$ satisfies (TR0).

(TR1): Let $\alpha/\varphi \colon M \to N$ be a morphism in $\mathcal{D}(R)$ and denote by U the common domain of α and φ. As $(\mathcal{K}(R), \Sigma)$ satisfies (TR1), the morphism α fits in a distinguished triangle in $\mathcal{K}(R)$, say,

$$U \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma U\,.$$

Applying V to this diagram, one gets by 6.5.5 a distinguished triangle in $\mathcal{D}(R)$, namely the upper row in the following commutative diagram,

$$\begin{tikzcd} U \arrow[r, "\mathrm{V}(\alpha)"] \arrow[d, "\mathrm{V}(\varphi)"'] & N \arrow[r, "\mathrm{V}(\beta)"] \arrow[d, equal] & X \arrow[r, "\mathrm{V}(\gamma)"] \arrow[d, equal] & \Sigma U \arrow[d, "\Sigma\mathrm{V}(\varphi)"] \\ M \arrow[r, "\alpha/\varphi"] & N \arrow[r, "\mathrm{V}(\beta)"] & X \arrow[r, "(\Sigma\mathrm{V}(\varphi))\,\mathrm{V}(\gamma)"] & \Sigma M\,. \end{tikzcd}$$

As $\mathrm{V}(\varphi)$ is an isomorphism by 6.4.18, the lower row is a distinguished triangle.

(TR2′): Follows immediately from the fact that $(\mathcal{K}(R), \Sigma)$ satisfies (TR2′).

(TR4′): Consider a commutative diagram in $\mathcal{D}(R)$,

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\ M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M'\,, \end{tikzcd} \tag{$\flat$}$$

where the rows are distinguished triangles. We construct a morphism $\chi \colon X \to X'$ in $\mathcal{D}(R)$ that makes ($\flat$) commutative, and such that the mapping cone candidate triangle of (φ, ψ, χ) in $\mathcal{D}(R)$ is distinguished.

Consider the functor $\mathrm{P} \colon \mathcal{D}(R) \to \mathcal{K}(R)$ from 6.5.1 and the natural isomorphism $\phi \colon \mathrm{P}\Sigma \to \Sigma\mathrm{P}$ from 6.5.4. In the following commutative diagram in $\mathcal{K}(R)$, the rows are distinguished triangles by 6.5.6,

$$\begin{tikzcd} \mathrm{P}(M) \arrow[r, "\mathrm{P}(\alpha)"] \arrow[d, "\mathrm{P}(\varphi)"] & \mathrm{P}(N) \arrow[r, "\mathrm{P}(\beta)"] \arrow[d, "\mathrm{P}(\psi)"] & \mathrm{P}(X) \arrow[r, "\phi^M \mathrm{P}(\gamma)"] & \Sigma\mathrm{P}(M) \arrow[d, "\Sigma\mathrm{P}(\varphi)"] \\ \mathrm{P}(M') \arrow[r, "\mathrm{P}(\alpha')"] & \mathrm{P}(N') \arrow[r, "\mathrm{P}(\beta')"] & \mathrm{P}(X') \arrow[r, "\phi^{M'} \mathrm{P}(\gamma')"] & \Sigma\mathrm{P}(M')\,. \end{tikzcd} \tag{$\diamond$}$$

As the category $\mathcal{K}(R)$ is triangulated, there exists a morphism $\vartheta \colon \mathrm{P}(X) \to \mathrm{P}(X')$ that makes ($\diamond$) commutative and such that the mapping cone candidate triangle of $(\mathrm{P}(\varphi), \mathrm{P}(\psi), \vartheta)$ in $\mathcal{K}(R)$ is distinguished. The front in the next diagram in $\mathcal{D}(R)$ is commutative, indeed, it is the image of ($\diamond$) under the functor $\mathrm{V} \colon \mathcal{K}(R) \to \mathcal{D}(R)$,

$$(\star)\qquad
\begin{tikzcd}
& M \arrow[rr, "\alpha"] \arrow[dd, "\varphi"' near start] & & N \arrow[rr, "\beta"] \arrow[dd, "\psi"' near start] & & X \arrow[rr, "\gamma"] & & \Sigma M \arrow[dd, "\Sigma\varphi"] \\
\mathrm{P}(M) \arrow[ur, "\pi^M/_1", "\simeq"'] \arrow[rr, "\mathrm{P}(\alpha)/_1" near end] \arrow[dd, "\mathrm{P}(\varphi)/_1"'] & & \mathrm{P}(N) \arrow[ur, "\pi^N/_1", "\simeq"'] \arrow[rr, "\mathrm{P}(\beta)/_1" near end] \arrow[dd, "\mathrm{P}(\psi)/_1"' near start] & & \mathrm{P}(X) \arrow[ur, "\pi^X/_1", "\simeq"'] \arrow[rr, "(\phi^M\,\mathrm{P}(\gamma))/_1"] \arrow[dd, "\vartheta/_1"' near start] & & \Sigma\mathrm{P}(M) \arrow[ur, "\Sigma(\pi^M/_1)", "\simeq"'] \arrow[dd, "\Sigma(\mathrm{P}(\varphi)/_1)"' near start] & \\
& M' \arrow[rr, "\alpha'" near start] & & N' \arrow[rr, "\beta'" near start] & & X' \arrow[rr, "\gamma'" near start] & & \Sigma M' \,. \\
\mathrm{P}(M') \arrow[ur, "\simeq", "\pi^{M'}/_1"'] \arrow[rr, "\mathrm{P}(\alpha')/_1"'] & & \mathrm{P}(N') \arrow[ur, "\simeq", "\pi^{N'}/_1"'] \arrow[rr, "\mathrm{P}(\beta')/_1"'] & & \mathrm{P}(X') \arrow[ur, "\simeq", "\pi^{X'}/_1"'] \arrow[rr, "(\phi^{M'}\,\mathrm{P}(\gamma'))/_1"'] & & \Sigma\mathrm{P}(M') \arrow[ur, "\simeq", "\Sigma(\pi^{M'}/_1)"'] &
\end{tikzcd}$$

The back in $(\star)$ is the commutative diagram $(\flat)$. The three complete vertical walls are commutative by 6.5.2, and so are the left-hand and middle squares in both the top and the bottom. The right-hand square in the top is also commutative; indeed, one has

$$\begin{aligned}
\big(\Sigma(\pi^M/_1)\big)\big((\phi^M\,\mathrm{P}(\gamma))/_1\big) &= ((\Sigma\pi^M)\phi^M\,\mathrm{P}(\gamma))/_1 \\
&= (\pi^{\Sigma M}\,\mathrm{P}(\gamma))/_1 \\
&= (\pi^{\Sigma M}/_1)(\mathrm{P}(\gamma)/_1) \\
&= \gamma(\pi^X/_1)\,,
\end{aligned}$$

where the 1st and 3rd equalities are by (6.4.10.2), the 2nd equality holds by (6.3.9.2), and the 4th equality holds by 6.5.2. A similar computation shows that the right-hand square in the bottom of $(\star)$ is commutative as well, and hence the entire diagram is commutative.

Now define $\chi\colon X \longrightarrow X'$ to be the composite of morphisms

$$X \xrightarrow{(\pi^X/_1)^{-1}} \mathrm{P}(X) \xrightarrow{\vartheta/_1} \mathrm{P}(X') \xrightarrow{\pi^{X'}/_1} X'\,.$$

A straightforward diagram chase shows that χ makes the back in $(\star)$ commutative, and hence (φ, ψ, χ) is a morphism of distinguished triangles. It remains to see that the mapping cone candidate triangle of (φ, ψ, χ) is distinguished. Since by E.23 it is isomorphic to the mapping cone candidate triangle, Δ, of $(\mathrm{P}(\varphi)/_1, \mathrm{P}(\psi)/_1, \vartheta/_1)$, it suffices to argue that Δ is distinguished. Evidently, Δ is isomorphic to the image of the mapping cone candidate triangle, Δ_0, of $(\mathrm{P}(\varphi), \mathrm{P}(\psi), \vartheta)$ under the functor V. By assumption, Δ_0 is distinguished in $\mathcal{K}(R)$, and hence its image Δ is distinguished in $\mathcal{D}(R)$ by definition; see 6.5.5. $\square$

Remark. Let $\mathcal{K}_{\mathrm{prj}}(R)$ denote the full subcategory of $\mathcal{K}(R)$ whose objects are the semi-projective R-complexes. The proof of 6.2.4 shows that $(\mathcal{K}_{\mathrm{prj}}(R), \Sigma)$ is a triangulated category, albeit not a triangulated subcategory of $\mathcal{K}(R)$; see E 6.2.11. The composite functor $\mathcal{K}_{\mathrm{prj}}(R) \longrightarrow \mathcal{K}(R) \longrightarrow \mathcal{D}(R)$ is a triangulated equivalence; see E 6.5.4.

TRIANGULATED FUNCTORS

We revisit the derived category's universal property and show that triangulated functors on $\mathcal{K}(R)$ induce triangulated functors on $\mathcal{D}(R)$.

6.5.8 Theorem. *Let $\mathcal{U}$ be a triangulated category and $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{U}$ a functor that maps quasi-isomorphisms to isomorphisms. If F is triangulated with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$, then the induced functor $\acute{\mathrm{F}}\colon \mathcal{D}(R) \to \mathcal{U}$, see* 6.4.27*, is triangulated with associated natural isomorphism $\acute{\phi}\colon \acute{\mathrm{F}}\Sigma \to \Sigma_{\mathcal{U}}\acute{\mathrm{F}}$.*

PROOF. It follows from 6.4.28 that the given natural isomorphism ϕ induces a natural isomorphism $\acute{\phi}\colon \acute{\mathrm{F}}\Sigma = (\mathrm{F}\Sigma)\acute{} \to (\Sigma_{\mathcal{U}}\mathrm{F})\acute{} = \Sigma_{\mathcal{U}}\acute{\mathrm{F}}$ of functors $\mathcal{D}(R) \to \mathcal{U}$ where the equalities follow from 6.4.39 and 6.5.3. For every distinguished triangle

$$(\diamond) \qquad M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$$

in $\mathcal{K}(R)$, the diagram

$$(\star) \qquad \mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\phi^M \mathrm{F}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$. By definition, every distinguished triangle in $\mathcal{D}(R)$ has, up to isomorphism, the form

$$M \xrightarrow{\mathrm{V}(\alpha)} N \xrightarrow{\mathrm{V}(\beta)} X \xrightarrow{\mathrm{V}(\gamma)} \Sigma M$$

for some distinguished triangle $(\diamond)$ in $\mathcal{K}(R)$. Hence, it must be verified that

$$(\dagger\dagger) \qquad \acute{\mathrm{F}}(M) \xrightarrow{\acute{\mathrm{F}}\mathrm{V}(\alpha)} \acute{\mathrm{F}}(N) \xrightarrow{\acute{\mathrm{F}}\mathrm{V}(\beta)} \acute{\mathrm{F}}(X) \xrightarrow{\acute{\phi}^M \acute{\mathrm{F}}\mathrm{V}(\gamma)} \Sigma_{\mathcal{U}}\acute{\mathrm{F}}(M)$$

is a distinguished triangle, which is evident since $(\dagger\dagger)$ is nothing but $(\star)$. □

6.5.9 Proposition. *Let $\mathcal{U}$ be a triangulated category. Let $\mathrm{E}, \mathrm{F}\colon \mathcal{K}(R) \to \mathcal{U}$ be triangulated functors that map quasi-isomorphisms to isomorphisms and $\tau\colon \mathrm{E} \to \mathrm{F}$ be a natural transformation. If τ is triangulated, then the induced natural transformation $\acute{\tau}\colon \acute{\mathrm{E}} \to \acute{\mathrm{F}}$ of triangulated functors, see* 6.4.28 *and* 6.5.8*, is triangulated.*

PROOF. By assumption the functors E and F are triangulated; denote the associated natural isomorphisms by ϕ and ψ, respectively. By 6.5.8 the induced functors $\acute{\mathrm{E}}$ and $\acute{\mathrm{F}}$ are triangulated with associated natural isomorphisms $\acute{\phi}$ and $\acute{\psi}$. For an R-complex M, the equality $\acute{\psi}^M \acute{\tau}^{\Sigma M} = (\Sigma\acute{\tau}^M)\acute{\phi}^M$ holds as the left-hand side is $\psi^M \tau^{\Sigma M}$, the right-hand side $(\Sigma\tau^M)\phi^M$, and those two composites agree by assumption. □

6.5.10 Theorem. *Let $\mathcal{V}$ be a triangulated category and $\mathrm{G}\colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ a functor that maps quasi-isomorphisms to isomorphisms. If G is triangulated with associated natural isomorphism $\psi\colon \Sigma_{\mathcal{V}}^{-1}\mathrm{G} \to \mathrm{G}\Sigma$, then the functor $\acute{\mathrm{G}}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{V}$, see* 6.4.29*, is triangulated with associated natural isomorphism $\acute{\psi}\colon \Sigma_{\mathcal{V}}^{-1}\acute{\mathrm{G}} \to \acute{\mathrm{G}}\Sigma$.*

PROOF. Apply 6.5.8 to the functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{K}(R) \to \mathcal{V}^{\mathrm{op}}$. □

6.5.11 Proposition. *Let $\mathcal{V}$ be a triangulated category. Let $\mathrm{G}, \mathrm{J} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{V}$ be triangulated functors that map quasi-isomorphisms to isomorphisms and $\tau \colon \mathrm{G} \to \mathrm{J}$ a natural transformation. If τ is triangulated, then the induced natural transformation $\acute{\tau} \colon \acute{\mathrm{G}} \to \acute{\mathrm{J}}$ of triangulated functors, see* 6.4.30 *and* 6.5.10*, is triangulated.*

Proof. Apply 6.5.9 to the natural transformation $\tau^{\mathrm{op}} \colon \mathrm{J}^{\mathrm{op}} \to \mathrm{G}^{\mathrm{op}}$ of functors from $\mathcal{K}(R)$ to $\mathcal{V}^{\mathrm{op}}$. □

Universal Property Revisited

6.5.12 Lemma. *Let $\mathrm{F} \colon \mathcal{K}(R) \to \mathcal{K}(S)$ be a functor. If F is triangulated with associated natural isomorphism ϕ, then the functor $\mathrm{V}_S\,\mathrm{F}$ is triangulated with associated natural isomorphism $\mathrm{V}_S\,\phi$.*

Proof. By 6.5.7 and 6.5.3 the functor V_S is triangulated with $\mathrm{V}_S\,\Sigma = \Sigma\,\mathrm{V}_S$. The assertion now follows from E.9. □

6.5.13 Theorem. *Let $\mathrm{F} \colon \mathcal{K}(R) \to \mathcal{K}(S)$ be a functor that preserves quasi-isomorphisms. If F is triangulated with associated natural isomorphism $\phi \colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$, then the induced functor $\acute{\acute{\mathrm{F}}} \colon \mathcal{D}(R) \to \mathcal{D}(S)$, see* 6.4.31*, is triangulated with associated natural isomorphism $\acute{\acute{\phi}} \colon \acute{\acute{\mathrm{F}}}\Sigma \to \Sigma\acute{\acute{\mathrm{F}}}$.*

Proof. By 6.5.12 the functor $\mathrm{V}_S\,\mathrm{F}$ is triangulated with associated natural isomorphism $\mathrm{V}_S\,\phi$. Now 6.5.8 implies that $(\mathrm{V}_S\,\mathrm{F})\acute{} = \acute{\acute{\mathrm{F}}}$ is triangulated with associated natural isomorphism $(\mathrm{V}_S\,\phi)\acute{} = \acute{\acute{\phi}}$ where the equalities hold by 6.4.31 and 6.4.33. □

6.5.14 Proposition. *Let $\mathrm{E}, \mathrm{F} \colon \mathcal{K}(R) \to \mathcal{K}(S)$ be functors that preserve quasi-isomorphisms and $\tau \colon \mathrm{E} \to \mathrm{F}$ a natural transformation. If E and F are triangulated and τ is triangulated, then the induced natural transformation $\acute{\acute{\tau}} \colon \acute{\acute{\mathrm{E}}} \to \acute{\acute{\mathrm{F}}}$ of triangulated functors, see* 6.4.33 *and* 6.5.13*, is triangulated.*

Proof. Let E and F be triangulated with associated natural isomorphisms ϕ and ψ. By 6.5.12 the functors $\mathrm{V}_S\,\mathrm{E}, \mathrm{V}_S\,\mathrm{F} \colon \mathcal{K}(R) \to \mathcal{D}(S)$ are triangulated with natural isomorphisms $\mathrm{V}_S\,\phi$ and $\mathrm{V}_S\,\psi$. As the natural transformation $\tau \colon \mathrm{E} \to \mathrm{F}$ is triangulated and $\mathrm{V}_S\,\Sigma = \Sigma\,\mathrm{V}_S$, the natural transformation $\mathrm{V}_S\,\tau \colon \mathrm{V}_S\,\mathrm{E} \to \mathrm{V}_S\,\mathrm{F}$ is triangulated as well. By 6.5.9 the natural transformation $(\mathrm{V}_S\,\tau)\acute{} = \acute{\acute{\tau}}$ from $(\mathrm{V}_S\,\mathrm{E})\acute{} = \acute{\acute{\mathrm{E}}}$ to $(\mathrm{V}_S\,\mathrm{F})\acute{} = \acute{\acute{\mathrm{F}}}$ is triangulated, where the equalities hold by 6.4.31 and 6.4.33. □

6.5.15 Theorem. *Let $\mathrm{G} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be a functor that preserves quasi-isomorphisms. If G is triangulated with associated natural isomorphism $\psi \colon \Sigma^{-1}\mathrm{G} \to \mathrm{G}\Sigma$, then the functor $\acute{\acute{\mathrm{G}}}$, see* 6.5.15*, is triangulated with associated natural isomorphism $\acute{\acute{\psi}} \colon \Sigma^{-1}\acute{\acute{\mathrm{G}}} \to \acute{\acute{\mathrm{G}}}\Sigma$.*

Proof. Proceed as in the proof of 6.4.31, only apply 6.4.29 in place of 6.4.27. □

6.5.16 Proposition. *Let $\mathrm{G}, \mathrm{J} \colon \mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(S)$ be functors that preserve quasi-isomorphisms and $\tau \colon \mathrm{G} \to \mathrm{J}$ a natural transformation. If G and J are triangulated and τ is triangulated, then the induced natural transformation $\acute{\acute{\tau}} \colon \acute{\acute{\mathrm{G}}} \to \acute{\acute{\mathrm{J}}}$ of triangulated functors, see* 6.4.35 *and* 6.5.15*, is triangulated.*

PROOF. Proceed as in the proof of 6.4.33, only apply 6.4.29 and 6.4.30 in place of 6.4.27 and 6.4.28. □

HOMOLOGY

Homology induces a functor on the derived category. It is a primary example of a homological functor in the sense of E.15.

6.5.17 Proposition. *The homology functor* $\mathrm{H}\colon \mathcal{K}(R) \to \mathcal{C}(R)$ *from* 6.2.20 *maps quasi-isomorphisms to isomorphisms. The induced functor* $\mathrm{H}\colon \mathcal{D}(R) \to \mathcal{C}(R)$ *from* 6.4.27 *is* $\Bbbk$*-linear, it preserves products and coproducts, and one has* $\mathrm{H}\Sigma = \Sigma\mathrm{H}$.

A morphism ξ *in* $\mathcal{D}(R)$ *is an isomorphism if and only if* $\mathrm{H}(\xi)$ *is an isomorphism.*

PROOF. The first assertion follows from the definition of quasi-isomorphisms, and thus $\mathrm{H}\colon \mathcal{K}(R) \to \mathcal{C}(R)$ induces by 6.4.27 a functor $\acute{\mathrm{H}}\colon \mathcal{D}(R) \to \mathcal{C}(R)$, which outside of this proof is denoted H. By 6.2.21, the functor H is $\Bbbk$-linear and preserves products and coproducts. It follows from 6.4.27 that the induced functor $\acute{\mathrm{H}}$ has the same properties. As $\mathrm{H}\Sigma = \Sigma\mathrm{H}$ holds, the definition of $\acute{\mathrm{H}}$ yields equalities $\acute{\mathrm{H}}\Sigma\mathrm{V} = \acute{\mathrm{H}}\mathrm{V}\Sigma = \mathrm{H}\Sigma = \Sigma\mathrm{H} = \Sigma\acute{\mathrm{H}}\mathrm{V}$, where $\mathrm{V}\colon \mathcal{K}(R) \to \mathcal{D}(R)$ is the canonical functor. It now follows from the uniqueness assertion in 6.4.27 that $\acute{\mathrm{H}}\Sigma = \Sigma\acute{\mathrm{H}}$ holds.

For a morphism α/φ in $\mathcal{D}(R)$ one has $\acute{\mathrm{H}}(\alpha/\varphi) = \mathrm{H}(\alpha)\,\mathrm{H}(\varphi)^{-1}$. It follows from 6.4.17 that α/φ is an isomorphism if and only if $\acute{\mathrm{H}}(\alpha/\varphi)$ is an isomorphism. □

6.5.18. Note from 6.5.17 that the supremum, infimum, and amplitude of complexes, defined in 2.5.4, are invariants of objects in the derived category. That is, for complexes $M \simeq M'$ one has $\sup M = \sup M'$, $\inf M = \inf M'$, and $\operatorname{amp} M = \operatorname{amp} M'$.

REMARK. While zero objects and isomorphisms in the derived category can be recognized in homology, see 6.4.24 and 6.5.17, zero morphisms can not; see E 6.4.8.

The last assertion in the next theorem is the Five Lemma in $\mathcal{D}(R)$; cf. E.18.

6.5.19 Theorem. *For every morphism of distinguished triangles in* $\mathcal{D}(R)$*,*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\ M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' , \end{tikzcd}$$

there is a commutative diagram in $\mathcal{C}(R)$*,*

$$\begin{tikzcd} \mathrm{H}(M) \arrow[r, "\mathrm{H}(\alpha)"] \arrow[d, "\mathrm{H}(\varphi)"] & \mathrm{H}(N) \arrow[r, "\mathrm{H}(\beta)"] \arrow[d, "\mathrm{H}(\psi)"] & \mathrm{H}(X) \arrow[r, "\mathrm{H}(\gamma)"] \arrow[d, "\mathrm{H}(\chi)"] & \Sigma\mathrm{H}(M) \arrow[r, "\Sigma\mathrm{H}(\alpha)"] \arrow[d, "\Sigma\mathrm{H}(\varphi)"] & \Sigma\mathrm{H}(N) \arrow[d, "\mathrm{H}(\psi)"] \\ \mathrm{H}(M') \arrow[r, "\mathrm{H}(\alpha')"] & \mathrm{H}(N') \arrow[r, "\mathrm{H}(\beta')"] & \mathrm{H}(X') \arrow[r, "\mathrm{H}(\gamma')"] & \Sigma\mathrm{H}(M') \arrow[r, "\Sigma\mathrm{H}(\alpha')"] & \Sigma\mathrm{H}(N') , \end{tikzcd} \tag{6.5.19.1}$$

with exact rows. In particular, if two of the morphisms φ, ψ, *and* χ *are isomorphisms, then so is the third.*

Proof. By 6.5.17 the functor $\mathrm{H}\colon \mathcal{D}(R) \to \mathcal{C}(R)$ satisfies the identity $\mathrm{H}\Sigma = \Sigma\mathrm{H}$; it follows that (6.5.19.1) is commutative. We argue that the upper row is exact; a parallel argument shows that the lower row is exact. By the definition of distinguished triangles 6.5.5, there exists an isomorphism of candidate triangles in $\mathcal{D}(R)$,

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\simeq"'] & N \arrow[r, "\beta"] \arrow[d, "\simeq"'] & X \arrow[r, "\gamma"] \arrow[d, "\simeq"'] & \Sigma M \arrow[d, "\simeq"'] \\ \widetilde{M} \arrow[r, "\mathrm{V}(\widetilde{\alpha})"] & \widetilde{N} \arrow[r, "\mathrm{V}(\widetilde{\beta})"] & \widetilde{X} \arrow[r, "\mathrm{V}(\widetilde{\gamma})"] & \Sigma\widetilde{M}\,, \end{tikzcd}$$

where $\widetilde{M} \xrightarrow{\widetilde{\alpha}} \widetilde{N} \xrightarrow{\widetilde{\beta}} \widetilde{X} \xrightarrow{\widetilde{\gamma}} \Sigma\widetilde{M}$ is a distinguished triangle in $\mathcal{K}(R)$. It follows that the upper row in (6.5.19.1) is isomorphic to the sequence

$$\mathrm{H}(\widetilde{M}) \xrightarrow{\mathrm{HV}(\widetilde{\alpha})} \mathrm{H}(\widetilde{N}) \xrightarrow{\mathrm{HV}(\widetilde{\beta})} \mathrm{H}(\widetilde{X}) \xrightarrow{\mathrm{HV}(\widetilde{\gamma})} \Sigma\mathrm{H}(\widetilde{M}) \xrightarrow{\Sigma\mathrm{HV}(\widetilde{\alpha})} \Sigma\mathrm{H}(\widetilde{N})\,.$$

This sequence is nothing but

$$\mathrm{H}(\widetilde{M}) \xrightarrow{\mathrm{H}(\widetilde{\alpha})} \mathrm{H}(\widetilde{N}) \xrightarrow{\mathrm{H}(\widetilde{\beta})} \mathrm{H}(\widetilde{X}) \xrightarrow{\mathrm{H}(\widetilde{\gamma})} \Sigma\mathrm{H}(\widetilde{M}) \xrightarrow{\Sigma\mathrm{H}(\widetilde{\alpha})} \Sigma\mathrm{H}(\widetilde{N})\,,$$

which is exact by 6.2.21.

The last assertion follows, in view of the final assertion in 6.5.17, from the Five Lemma 2.1.41 applied to the diagram (6.5.19.1). ◻

Remark. For every integer m, the homology functor H_m on $\mathcal{D}(R)$ is naturally isomorphic to the functor $\mathcal{D}(R)(\Sigma^m R, -)$. In combination with E.16 and E.17, this can be used to give different proof of 6.5.19.

The last three assertions in the result below are, in view of 6.4.24, special cases of E.22, which holds in any triangulated category.

6.5.20 Corollary. *Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a distinguished triangle in $\mathcal{D}(R)$. The following inequalities hold.*

$$\begin{aligned} \sup M &\leqslant \max\{\sup N, \sup X - 1\}\,, & \inf M &\geqslant \min\{\inf N, \inf X - 1\}\,, \\ \sup N &\leqslant \max\{\sup M, \sup X\}\,, & \inf N &\geqslant \min\{\inf M, \inf X\}\,, \\ \sup X &\leqslant \max\{\sup M + 1, \sup N\}\,, \quad \textit{and} & \inf X &\geqslant \min\{\inf M + 1, \inf N\}\,. \end{aligned}$$

In particular, if two of the complexes M, N, and X are acyclic, then so is the third.

Furthermore, the following assertions hold.

(a) *M is acyclic if and only if β is an isomorphism.*
(b) *N is acyclic if and only if γ is an isomorphism.*
(c) *X is acyclic if and only if α is an isomorphism.*

Proof. The assertions are immediate from 6.5.19 in view of 6.5.17. ◻

6.5.21 Corollary. *Let $\mathrm{F}\colon \mathcal{D}(R) \to \mathcal{D}(S)$ be a triangulated functor with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma\mathrm{F}$. For every morphism,*

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\
M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' ,
\end{tikzcd}$$

of distinguished triangles in $\mathcal{D}(R)$ there is a commutative diagram in $\mathcal{M}(S)$,

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{H}_{v+1}(\delta)"] & \mathrm{H}_v(\mathrm{F}(M)) \arrow[r, "\mathrm{H}_v(\mathrm{F}(\alpha))"] \arrow[d, "\mathrm{H}_v(\mathrm{F}(\varphi))"] & \mathrm{H}_v(\mathrm{F}(N)) \arrow[r, "\mathrm{H}_v(\mathrm{F}(\beta))"] \arrow[d, "\mathrm{H}_v(\mathrm{F}(\psi))"] & \mathrm{H}_v(\mathrm{F}(X)) \arrow[r, "\mathrm{H}_v(\delta)"] \arrow[d, "\mathrm{H}_v(\mathrm{F}(\chi))"] & \mathrm{H}_{v-1}(\mathrm{F}(M)) \arrow[r, "\mathrm{H}_{v-1}(\mathrm{F}(\alpha))"] \arrow[d, "\mathrm{H}_{v-1}(\mathrm{F}(\varphi))"] & \cdots \\
\cdots \arrow[r, "\mathrm{H}_{v+1}(\delta')"'] & \mathrm{H}_v(\mathrm{F}(M')) \arrow[r, "\mathrm{H}_v(\mathrm{F}(\alpha'))"'] & \mathrm{H}_v(\mathrm{F}(N')) \arrow[r, "\mathrm{H}_v(\mathrm{F}(\beta'))"'] & \mathrm{H}_v(\mathrm{F}(X')) \arrow[r, "\mathrm{H}_v(\delta')"'] & \mathrm{H}_{v-1}(\mathrm{F}(M')) \arrow[r, "\mathrm{H}_{v-1}(\mathrm{F}(\alpha'))"'] & \cdots ,
\end{tikzcd}$$

with exact rows. Here δ and δ' are the composites $\phi^M\mathrm{F}(\gamma)$ and $\phi^{M'}\mathrm{F}(\gamma')$.

Proof. The triangulated functor F maps the given morphism to a morphism,

$$\begin{tikzcd}
\mathrm{F}(M) \arrow[r, "\mathrm{F}(\alpha)"] \arrow[d, "\mathrm{F}(\varphi)"] & N \arrow[r, "\mathrm{F}(\beta)"] \arrow[d, "\mathrm{F}(\psi)"] & X \arrow[r, "\phi^M\mathrm{F}(\gamma)"] \arrow[d, "\mathrm{F}(\chi)"] & \Sigma\mathrm{F}(M) \arrow[d, "\Sigma\mathrm{F}(\varphi)"] \\
\mathrm{F}(M') \arrow[r, "\mathrm{F}(\alpha')"] & \mathrm{F}(N') \arrow[r, "\mathrm{F}(\beta')"] & \mathrm{F}(X') \arrow[r, "\phi^{M'}\mathrm{F}(\gamma')"] & \Sigma\mathrm{F}(M') ,
\end{tikzcd}$$

of distinguished triangles in $\mathcal{D}(S)$. Now 6.5.19 yields a commutative diagram in $\mathcal{C}(S)$ which, written out degreewise, is the asserted diagram in $\mathcal{M}(S)$. $\square$

6.5.22 Corollary. *Let $\mathrm{G}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ be a triangulated functor with associated natural isomorphism $\phi\colon \Sigma^{-1}\mathrm{G} \to \mathrm{G}\Sigma$. For every morphism,*

$$\begin{tikzcd}
M & N \arrow[l, "\alpha"'] & X \arrow[l, "\beta"'] & \Sigma M \arrow[l, "\gamma"'] \\
M' \arrow[u, "\varphi"] & N' \arrow[l, "\alpha'"] \arrow[u, "\psi"] & X' \arrow[l, "\beta'"] \arrow[u, "\chi"] & \Sigma M' \arrow[l, "\gamma'"] \arrow[u, "\Sigma\varphi"'] ,
\end{tikzcd}$$

of distinguished triangles in $\mathcal{D}(R)^{\mathrm{op}}$ there is a commutative diagram in $\mathcal{M}(S)$,

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{H}_{v+1}(\mathrm{G}(\alpha'))"] & \mathrm{H}_{v+1}(\mathrm{G}(M')) \arrow[r, "\mathrm{H}_v(\delta')"] \arrow[d, "\mathrm{H}_{v+1}(\mathrm{G}(\varphi))"] & \mathrm{H}_v(\mathrm{G}(X')) \arrow[r, "\mathrm{H}_v(\mathrm{G}(\beta'))"] \arrow[d, "\mathrm{H}_v(\mathrm{G}(\chi))"] & \mathrm{H}_v(\mathrm{G}(N')) \arrow[r, "\mathrm{H}_v(\mathrm{G}(\alpha'))"] \arrow[d, "\mathrm{H}_v(\mathrm{G}(\psi))"] & \mathrm{H}_v(\mathrm{G}(M')) \arrow[r, "\mathrm{H}_{v-1}(\delta')"] \arrow[d, "\mathrm{H}_v(\mathrm{G}(\varphi))"] & \cdots \\
\cdots \arrow[r, "\mathrm{H}_{v+1}(\mathrm{G}(\alpha))"'] & \mathrm{H}_{v+1}(\mathrm{G}(M)) \arrow[r, "\mathrm{H}_v(\delta)"'] & \mathrm{H}_v(\mathrm{G}(X)) \arrow[r, "\mathrm{H}_v(\mathrm{G}(\beta))"'] & \mathrm{H}_v(\mathrm{G}(N)) \arrow[r, "\mathrm{H}_v(\mathrm{G}(\alpha))"'] & \mathrm{H}_v(\mathrm{G}(M)) \arrow[r, "\mathrm{H}_{v-1}(\delta)"'] & \cdots
\end{tikzcd}$$

with exact rows. Here δ and δ' are the composites $\mathrm{G}(\gamma)\phi^M$ and $\mathrm{G}(\gamma')\phi^{M'}$.

Proof. The triangulated functor G maps the given morphism to a morphism of distinguished triangles in $\mathcal{D}(S)$,

$$\begin{tikzcd}
\Sigma^{-1}\mathrm{G}(M') \arrow[r, "\mathrm{G}(\gamma')\phi^{M'}"] \arrow[d, "\Sigma^{-1}\mathrm{G}(\varphi)"] & \mathrm{G}(X') \arrow[r, "\mathrm{G}(\beta')"] \arrow[d, "\mathrm{G}(\chi)"] & \mathrm{G}(N') \arrow[r, "\mathrm{G}(\alpha')"] \arrow[d, "\mathrm{G}(\psi)"] & \mathrm{G}(M') \arrow[d, "\mathrm{G}(\varphi)"] \\
\Sigma^{-1}\mathrm{G}(M) \arrow[r, "\mathrm{G}(\gamma)\phi^{M}"] & \mathrm{G}(X) \arrow[r, "\mathrm{G}(\beta)"] & \mathrm{G}(N) \arrow[r, "\mathrm{G}(\alpha)"] & \mathrm{G}(M)\,,
\end{tikzcd}$$

and 6.5.19 applies to yield a commutative diagram in $\mathcal{C}(S)$ which, written out degreewise, is the asserted diagram in $\mathcal{M}(S)$. ◻

The next example shows that not every morphism in $\mathcal{D}(R)$ has a kernel; in particular $\mathcal{D}(R)$ is not Abelian.

6.5.23 Example. First recall that a kernel of a morphism α in an additive category is a morphism ι such that $\alpha\iota = 0$ and ι has the universal property that every morphism ι' satisfying $\alpha\iota' = 0$ factors uniquely through ι. Every kernel is a monomorphism.

We argue that a morphism $\alpha\colon M \to N$ in $\mathcal{D}(R)$ that satisfies the next conditions does not have a kernel in $\mathcal{D}(R)$. For a concrete example, take $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ in $\mathcal{D}(\mathbb{Z})$.

(1) The graded R-module $\mathrm{H}(M)$ is indecomposable.

(2) The morphism $\mathrm{H}(\alpha)\colon \mathrm{H}(M) \to \mathrm{H}(N)$ does not have a left inverse.

Indeed, by 6.5.7 the category $\mathcal{D}(R)$ is triangulated, so it follows from E.24 that if α has a kernel $\iota\colon K \to M$, then ι has a left inverse. Now (TR1) in E.2 and E.22 imply that there is an isomorphism $M \simeq K \oplus X$ in $\mathcal{D}(R)$ for some R-complex X. It follows that $\mathrm{H}(M) \cong \mathrm{H}(K) \oplus \mathrm{H}(X)$, so (1) implies that $\mathrm{H}(K) = 0$ or $\mathrm{H}(X) = 0$ holds, which by 6.4.24 means that $K \simeq 0$ or $X \simeq 0$ in $\mathcal{D}(R)$. If $K \simeq 0$, then α is a monomorphism, and hence it has a left inverse, again by E.24, which contradicts (2). If $X \simeq 0$, then ι is an isomorphism and hence $\alpha = 0$, which also contradicts (2).

Distinguished Triangles from Short Exact Sequences

Every short exact sequence of complexes induces a distinguished triangle in the derived category; in applications that is a natural source of distinguished triangles. To avoid clutter, it is common practice to denote the morphism $\mathrm{VQ}(\alpha)$ in $\mathcal{D}(R)$ induced by a morphism α in $\mathcal{C}(R)$ by the very same symbol, α, rather than the fraction $[\alpha]/1$. We apply this practice frequently in the balance of book, it premiers here:

6.5.24 Theorem. *For every commutative diagram in $\mathcal{C}(R)$ with exact rows,*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r] & 0\,,
\end{tikzcd}$$

there is a morphism of distinguished triangles in $\mathcal{D}(R)$,

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\
M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' ,
\end{tikzcd}$$

with $\gamma = [\varpi]/[\underline{\beta}]$ *and* $\gamma' = [\varpi']/[\underline{\beta}']$ *where* $\underline{\beta}$, $\underline{\beta}'$ *are the quasi-isomorphisms from* 4.3.10 *and* ϖ, ϖ' *are the canonical morphisms* $\operatorname{Cone}\alpha \twoheadrightarrow \Sigma M$ *and* $\operatorname{Cone}\alpha' \twoheadrightarrow \Sigma M'$.

Proof. In the following diagram in $\mathcal{D}(R)$, the middle and left-hand boxes are commutative by 4.3.11. From the definition of λ, it follows that $\varpi'\lambda = (\Sigma\varphi)\varpi$ holds, and hence the top of the right-hand box is commutative. The front and back walls of the right-hand box are evidently commutative, and so is the right wall. A diagram chase now yields

$$\mathrm{VQ}(\varpi')\,\mathrm{VQ}(\underline{\beta}')^{-1} \circ \mathrm{VQ}(\chi) \circ \mathrm{VQ}(\underline{\beta}) \;=\; \Sigma\mathrm{VQ}(\varphi) \circ \mathrm{VQ}(\varpi)\,\mathrm{VQ}(\underline{\beta})^{-1} \circ \mathrm{VQ}(\underline{\beta})\,,$$

and since $\mathrm{VQ}(\underline{\beta})$ is an isomorphism, it follows that the bottom of the right-hand box is commutative as well.

$$\begin{tikzcd}
 & M \arrow[rr, "\mathrm{VQ}(\iota)"] \arrow[dl, "\mathrm{VQ}(\varphi)"'] \arrow[dd, equal] & & \operatorname{Cyl}\alpha \arrow[rr, "\mathrm{VQ}(\pi)"] \arrow[dl, "\mathrm{VQ}(\kappa)"', "\simeq"] \arrow[dd, "\mathrm{VQ}(\underline{\alpha})", "\simeq"'] & & \operatorname{Cone}\alpha \arrow[rr, "\mathrm{VQ}(\varpi)"] \arrow[dl, "\mathrm{VQ}(\lambda)"', "\simeq"] \arrow[dd, "\mathrm{VQ}(\underline{\beta})", "\simeq"'] & & \Sigma M \arrow[dl, "\Sigma\mathrm{VQ}(\varphi)"'] \arrow[dd, equal] \\
M' \arrow[rr, "\mathrm{VQ}(\iota')"'] \arrow[dd, equal] & & \operatorname{Cyl}\alpha' \arrow[rr, "\mathrm{VQ}(\pi')"'] \arrow[dd, "\mathrm{VQ}(\underline{\alpha}')", "\simeq"'] & & \operatorname{Cone}\alpha' \arrow[rr, "\mathrm{VQ}(\varpi')"'] \arrow[dd, "\mathrm{VQ}(\underline{\beta}')", "\simeq"'] & & \Sigma M' \arrow[dd, equal] & \\
 & M \arrow[rr, "\mathrm{VQ}(\alpha)"'] \arrow[dl, "\mathrm{VQ}(\varphi)"] & & N \arrow[rr, "\mathrm{VQ}(\beta)"'] \arrow[dl, "\mathrm{VQ}(\psi)"] & & X \arrow[rr, "\mathrm{VQ}(\varpi)\mathrm{VQ}(\underline{\beta})^{-1}"'] \arrow[dl, "\mathrm{VQ}(\chi)"] & & \Sigma M\,. \arrow[dl, "\Sigma\mathrm{VQ}(\varphi)"] \\
M' \arrow[rr, "\mathrm{VQ}(\alpha')"'] & & N' \arrow[rr, "\mathrm{VQ}(\beta')"'] & & X' \arrow[rr, "\mathrm{VQ}(\varpi')\mathrm{VQ}(\underline{\beta}')^{-1}"'] & & \Sigma M' &
\end{tikzcd}$$

The upper rows are by 6.2.5 images under V of distinguished triangles in $\mathcal{K}(R)$ and hence distinguished in $\mathcal{D}(R)$; thus the top is a morphism of distinguished triangles in $\mathcal{D}(R)$. As all vertical morphisms are isomorphisms, the bottom is a morphism of distinguished triangles as well. Finally, by 6.4.18 and (6.4.10.2) one has $\mathrm{VQ}(\varpi)\,\mathrm{VQ}(\underline{\beta})^{-1} = [\varpi]/[\underline{\beta}]$ and $\mathrm{VQ}(\varpi')\,\mathrm{VQ}(\underline{\beta}')^{-1} = [\varpi']/[\underline{\beta}']$. With the simplified notation for the remaining morphisms—α for $\mathrm{VQ}(\alpha)$ etc.—the bottom of the diagram is the asserted morphism of distinguished triangles. □

Remark. A general short exact sequence $0 \to M \to N \to X \to 0$ of complexes does not induce a distinguished triangle $M \to N \to X \to \Sigma M$ in the homotopy category; see E 6.2.2.

Exercises

E 6.5.1 Assume that R is semi-simple. Show that the categories $\mathcal{D}(R)$ and $\mathcal{M}_{\mathrm{gr}}(R)$ are equivalent and conclude that $\mathcal{D}(R)$ is Abelian.

E 6.5.2 Assume that R is left hereditary. Show that $\mathcal{D}(R)$ and $\mathcal{K}(\operatorname{Prj} R)$ are equivalent as triangulated categories; see E 6.1.9.

E 6.5.3 Assume that R is left hereditary. Show that $\mathcal{D}(R)$ and $\mathcal{K}(\operatorname{Inj} R)$ are equivalent as triangulated categories; see E 6.1.10.

E 6.5.4 Show that $\mathcal{K}_{\mathrm{prj}}(R)$ and $\mathcal{D}(R)$ are equivalent as triangulated categories; see E 6.2.11.

E 6.5.5 Show that $\mathcal{K}_{\mathrm{inj}}(R)$ and $\mathcal{D}(R)$ are equivalent as triangulated categories; see E 6.2.13.

E 6.5.6 Let $(\mathcal{T}, \Sigma)$ be a triangulated category. A commutative square in $\mathcal{T}$,

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & N \arrow[d, "\psi"] \\ M \arrow[r, "\beta"] & V \end{tikzcd}$$

is called *homotopy Cartesian* if there exists a distinguished triangle of the form

$$U \xrightarrow{\left(\begin{smallmatrix} \varphi \\ -\alpha \end{smallmatrix}\right)} \begin{matrix} M \\ \oplus \\ N \end{matrix} \xrightarrow{(\beta \;\; \psi)} V \xrightarrow{\gamma} \Sigma U .$$

The pair (φ, α) is called a *homotopy pullback* of (β, ψ), and (β, ψ) is called a *homotopy pushout* of (φ, α). Show that homotopy pushouts and pullbacks always exist.

E 6.5.7 Show that the functors $(-)_{\subseteq n}, (-)_{\supseteq n} \colon \mathcal{D}(R) \to \mathcal{D}(R)$ preserve products and coproducts but are not triangulated.

E 6.5.8 Let M be an R-complex with $\mathrm{H}_v(M) = 0$ for $v \neq 0$. (a) Show that if $M_v = 0$ holds for $v < 0$, then there is a quasi-isomorphism $M \to \mathrm{H}(M)$ in $\mathcal{C}(R)$. (b) Show that if $M_v = 0$ holds for $v > 0$, then there is a quasi-isomorphism $\mathrm{H}(M) \xrightarrow{\simeq} M$ in $\mathcal{C}(R)$. (c) Conclude that for a complex M' with $\operatorname{amp} M' = 0$ there is an isomorphism $M' \simeq \mathrm{H}(M')$ in $\mathcal{D}(R)$.

E 6.5.9 Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $(\mathcal{T}, \Sigma)$ and consider the homotopy Cartesian square in $\mathcal{T}$ from E 6.5.6. Show that the morphism α is $\mathcal{S}$-trivial if and only if β is $\mathcal{S}$-trivial in the sense of E 6.2.9. *Hint:* Neeman [191, 1.5].

E 6.5.10 Verify the inequalities in 6.5.20.

E 6.5.11 Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow \Sigma M$ be a distinguished triangle in $\mathcal{D}(R)$; show that the following conditions are equivalent. (*i*) $\mathrm{H}(\alpha)$ is injective. (*ii*) $\mathrm{H}(\beta)$ is surjective. (*iii*) The sequence $0 \longrightarrow \mathrm{H}(M) \xrightarrow{\mathrm{H}(\alpha)} \mathrm{H}(N) \xrightarrow{\mathrm{H}(\beta)} \mathrm{H}(X) \longrightarrow 0$ is exact.

E 6.5.12 Show that the full subcategories of $\mathcal{D}(R)$ defined by specifying their objects as follows:

$$\begin{aligned} \mathcal{D}_{\sqsubset}(R) &= \{M \in \mathcal{D}(R) \mid \text{there is a bounded above complex } M' \text{ with } M \simeq M'\}\,, \\ \mathcal{D}_{\square}(R) &= \{M \in \mathcal{D}(R) \mid \text{there is a bounded complex } M' \text{ with } M \simeq M'\}\,, \quad \text{and} \\ \mathcal{D}_{\sqsupset}(R) &= \{M \in \mathcal{D}(R) \mid \text{there is a bounded below complex } M' \text{ with } M \simeq M'\} \end{aligned}$$

are triangulated subcategories of $\mathcal{D}(R)$.

E 6.5.13 Give a proof of 6.5.19 using the ideas in the subsequent Remark.

Part II
Tools

THE DERIVED CATEGORY is now at hand, and it is time to start using it. While the ultime purpose of the book is to apply the methods of derived categories to study communative Noetherian rings, the methods are useful without the full force of those assumptions. For example, the arguably most basic homological invariants, the homological dimensions, are meaningful invariants in the derived category over any ring. In the five chapters that make up this part of the book, we develop the basic theory of invariants and functors on the derived category that play central roles in the homological study of rings. For parts of the theory, the rings need to be Noetherian; this assumption is imposed in all of Chap. 10 and sparsely in Chaps. 8, 9, and 11. An assumption of commutativity is only introduced in Chap. 11.

Chapter 7
Derived Functors

Exact functors are the structure preserving mappings between Abelian categories. In the world of triangulated categories, they are replaced by triangulated functors. In the context of this book, the utility of derived categories stems predominantly from the fact that well-behaved, though not necessarily exact, functors $\mathcal{C}(R) \to \mathcal{C}(S)$ yield triangulated functors $\mathcal{D}(R) \to \mathcal{D}(S)$ through a process known as derivation.

7.1 Induced Functors on the Homotopy Category

SYNOPSIS. Functor that preserves homotopy; the functor Hom; homotopy category of complexes of bimodules; the functor $\otimes$; unitor; counitor; commutativity; associativity; swap; adjunction; biduality; tensor evaluation; homomorphism evaluation.

We start by merging and expanding some key results from Chap. 6.

7.1.1 Definition. In the product category

$$\mathcal{C}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{C}(R_m)^{\mathrm{op}} \times \mathcal{C}(S_1) \times \cdots \times \mathcal{C}(S_n) ,$$

parallel morphisms $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ and $(\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_n)$ are said to be *homotopic* if one has $\alpha_i \sim \alpha'_i$ in $\mathcal{C}(R_i)^{\mathrm{op}}$ and $\beta_j \sim \beta'_j$ in $\mathcal{C}(S_j)$ for all indices i and j. A functor

$$\mathcal{C}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{C}(R_m)^{\mathrm{op}} \times \mathcal{C}(S_1) \times \cdots \times \mathcal{C}(S_n) \longrightarrow \mathcal{C}(T)$$

is said to *preserve homotopy* if it maps homotopic morphisms to homotopic morphisms; cf. 4.3.12 and 4.3.14.

7.1.2 Theorem. *Let* $\mathrm{F}\colon \mathcal{C}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{C}(R_m)^{\mathrm{op}} \times \mathcal{C}(S_1) \times \cdots \times \mathcal{C}(S_n) \to \mathcal{C}(T)$ *be a functor. If* F *preserves homotopy, then there exists a unique functor* $\ddot{\mathrm{F}}$ *that makes the following diagram commutative,*

$$
\begin{tikzcd}
\mathcal{C}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{C}(R_m)^{\mathrm{op}} \times \mathcal{C}(S_1) \times \cdots \times \mathcal{C}(S_n) \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{Q}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{Q}_{R_m}^{\mathrm{op}} \times \mathrm{Q}_{S_1} \times \cdots \times \mathrm{Q}_{S_n}"'] & \mathcal{C}(T) \arrow[d, "\mathrm{Q}_T"] \\
\mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \arrow[r, dotted, "\ddot{\mathrm{F}}"] & \mathcal{K}(T)\,.
\end{tikzcd}
$$

The functor $\ddot{\mathrm{F}}$ *acts on objects and morphisms as follows*:

$$
\ddot{\mathrm{F}}(M_1,\dots,M_m,N_1,\dots,N_n) \;=\; \mathrm{F}(M_1,\dots,M_m,N_1,\dots,N_n) \quad \textit{and}
$$
$$
\ddot{\mathrm{F}}([\alpha_1],\dots,[\alpha_m],[\beta_1],\dots,[\beta_n]) \;=\; [\mathrm{F}(\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n)]\,.
$$

Furthermore, the following assertions hold.

(a) *If* F *is* $\Bbbk$*-linear in one of the variables, then* $\ddot{\mathrm{F}}$ *is* $\Bbbk$*-linear in the corresponding variable*; *in particular, if* F *is* $\Bbbk$*-multilinear, then* $\ddot{\mathrm{F}}$ *is* $\Bbbk$*-multilinear.*

(b) *If* F *preserves products/coproducts in one of the variables, then* $\ddot{\mathrm{F}}$ *preserves products/coproducts in the corresponding variable.*

(c) *If* F *is a* Σ*-functor in one of the variables, then* $\ddot{\mathrm{F}}$ *is triangulated in the corresponding variable.*

Proof. The arguments in the proofs of 6.1.20/6.1.27 and 6.2.16/6.2.18 apply mutatis mutandis to establish the claims. □

Remark. For every Abelian category $\mathcal{A}$, one can construct the category $\mathcal{C}(\mathcal{A})$ of complexes and the homotopy category $\mathcal{K}(\mathcal{A})$, and there is a canonical functor $\mathrm{Q}_{\mathcal{A}} : \mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ with the expected universal property. In Chaps. 2 and 6 we carried out these constructions explicitly for the category $\mathcal{A} = \mathcal{M}(R)$. The general constructions of $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ respect the formation of products and opposites of categories; for example, there are isomorphisms

$$
\mathcal{C}(\mathcal{A}_1 \times \mathcal{A}_2) \;\cong\; \mathcal{C}(\mathcal{A}_1) \times \mathcal{C}(\mathcal{A}_2) \qquad \text{and} \qquad \mathcal{C}(\mathcal{A}^{\mathrm{op}}) \;\cong\; \mathcal{C}(\mathcal{A})^{\mathrm{op}}\,.
$$

In particular, for the Abelian category $\mathcal{A} = \mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R)$ there are identifications,

$$
\mathcal{C}(\mathcal{A}) \;\cong\; \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \qquad \text{and} \qquad \mathcal{K}(\mathcal{A}) \;\cong\; \mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R)\,.
$$

Thus, the universal property of the canonical functor $\mathrm{Q}_{\mathcal{A}}$ applies to show e.g. that the functor

$$
\mathrm{Hom}_R(-,-) : \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \longrightarrow \mathcal{C}(\Bbbk)\,,
$$

induces a functor $\mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R) \to \mathcal{K}(\Bbbk)$; this indicates another way to prove 7.1.2.

7.1.3 Proposition. *Let*

$$
\mathrm{F}, \mathrm{G} : \mathcal{C}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{C}(R_m)^{\mathrm{op}} \times \mathcal{C}(S_1) \times \cdots \times \mathcal{C}(S_n) \;\longrightarrow\; \mathcal{C}(T)
$$

be functors that preserve homotopy. Consider the induced functors from 7.1.2,

$$
\ddot{\mathrm{F}}, \ddot{\mathrm{G}} : \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \;\longrightarrow\; \mathcal{K}(T)\,.
$$

Every natural transformation $\tau : \mathrm{F} \to \mathrm{G}$ *induces a natural transformation* $\ddot{\tau} : \ddot{\mathrm{F}} \to \ddot{\mathrm{G}}$ *given by* $\ddot{\tau}^X = [\tau^X]$ *for every object* $X = (M_1,\dots,M_m,N_1,\dots,N_n)$*. Moreover, if* F *and* G *are* Σ*-functors in the same variable and if* τ *is a* Σ*-transformation in that variable, then the natural transformation* $\ddot{\tau}$ *is triangulated in the corresponding variable.*

Proof. The arguments in the proofs of 6.1.21/6.1.28 and 6.2.17/6.2.19 apply mutatis mutandis to establish the claims. □

In the balance of this section we apply the machinery developed above to the Hom and tensor product functors and to the menagerie of natural transformations that compare composites of these functors.

HOM FUNCTOR

7.1.4 Example. By 2.3.9 the functor $\operatorname{Hom}_R(-,-)\colon \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ preserves homotopy. Thus, by 7.1.2 there exists a unique functor $\operatorname{H\ddot{o}m}_R(-,-)$ that makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \arrow[r, "\operatorname{Hom}_R(-,-)"] \arrow[d, "\mathrm{Q}_R^{\mathrm{op}} \times \mathrm{Q}_R"'] & \mathcal{C}(\Bbbk) \arrow[d, "\mathrm{Q}_\Bbbk"] \\ \mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R) \arrow[r, "\operatorname{H\ddot{o}m}_R(-,-)"] & \mathcal{K}(\Bbbk)\,. \end{tikzcd}$$

It acts as follows on objects and morphisms:

$$\operatorname{H\ddot{o}m}_R(M,N) = \operatorname{Hom}_R(M,N) \quad \text{and} \quad \operatorname{H\ddot{o}m}_R([\alpha],[\beta]) = [\operatorname{Hom}_R(\alpha,\beta)]\,.$$

Moreover, $\operatorname{H\ddot{o}m}_R(-,-)$ is $\Bbbk$-bilinear, see 2.3.10, it preserves products in both variables, see 3.1.24 and 3.1.27, and it is triangulated in both variables, see 4.1.16 and 4.1.17. By usual abuse of notation we denote the functor $\operatorname{H\ddot{o}m}_R$ by Hom_R.

7.1.5 Definition. Let $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ denote the category $\mathcal{K}(R \otimes_\Bbbk S^{\mathrm{o}})$.

7.1.6. In view of 2.1.38 the category $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ is naturally identified with the category whose objects are complexes of R–S^{o}-bimodules, and whose hom-sets consist of homotopy classes of R- and S^{o}-linear chain maps of degree 0. The homotopy relation used here is defined as in 2.2.23, only now homotopies are required to be both R- and S^{o}-linear.

REMARK. Per the Remark after 7.1.2, one can construct the homotopy category $\mathcal{K}(\mathcal{A})$ over any Abelian category $\mathcal{A}$. The content of 7.1.6 is that $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ is naturally equivalent to $\mathcal{K}(\mathcal{M}(R\text{–}S^{\mathrm{o}}))$.

7.1.7 Addendum (to 7.1.4). By 2.3.11 there is a functor Hom_R from the category $\mathcal{C}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{C}(R\text{–}S^{\mathrm{o}})$ to $\mathcal{C}(Q\text{–}S^{\mathrm{o}})$, and by 7.1.2 it induces a $\Bbbk$-bilinear functor,

$$\operatorname{Hom}_R(-,-) : \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})\,,$$

which preserves products and is triangulated in both variables.

TENSOR PRODUCT FUNCTOR

7.1.8 Example. By 2.4.8 the functor $-\otimes_R -\colon \mathcal{C}(R^{\mathrm{o}}) \times \mathcal{C}(R) \to \mathcal{C}(\Bbbk)$ preserves homotopy. Thus, 7.1.2 yields a unique functor $-\ddot{\otimes}_R-$ that makes the diagram

$$\begin{tikzcd}
\mathcal{C}(R^{\mathrm{o}}) \times \mathcal{C}(R) \arrow[r, "- \otimes_R -"] \arrow[d, "\mathrm{Q}_{R^{\mathrm{o}}} \times \mathrm{Q}_R"'] & \mathcal{C}(\Bbbk) \arrow[d, "\mathrm{Q}_{\Bbbk}"] \\
\mathcal{K}(R^{\mathrm{o}}) \times \mathcal{K}(R) \arrow[r, "- \ddot{\otimes}_R -"] & \mathcal{K}(\Bbbk)
\end{tikzcd}$$

commutative. It acts as follows on objects and morphisms:

$$M \ddot{\otimes}_R N \;=\; M \otimes_R N \qquad \text{and} \qquad [\alpha] \ddot{\otimes}_R [\beta] \;=\; [\alpha \otimes_R \beta] \;.$$

Moreover, $- \ddot{\otimes}_R -$ is $\Bbbk$-bilinear, see 2.4.9, it preserves coproducts in both variables, see 3.1.12 and 3.1.13, and it is triangulated in both variables, see 4.1.18 and 4.1.19. By habitual abuse of notation we generally denote the functor $\ddot{\otimes}_R$ by $\otimes_R$.

7.1.9 Addendum (to 7.1.8). By 2.4.10 there is a functor $- \otimes_R -$ from the category $\mathcal{C}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{C}(R\text{–}S^{\mathrm{o}})$ to $\mathcal{C}(Q\text{–}S^{\mathrm{o}})$, and by 7.1.2 it induces a $\Bbbk$-bilinear functor,

$$- \otimes_R - : \mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \;\longrightarrow\; \mathcal{K}(Q\text{–}S^{\mathrm{o}}) \;,$$

which preserves coproducts and is triangulated in both variables.

Standard Isomorphisms in the Homotopy Category

Having established Hom and $\otimes$ as functors on the level of homotopy categories, our next objective is to extend the natural transformations and isomorphisms from Sects. 4.4 and 4.5 to that setting.

7.1.10. Consider the functors

$$\mathrm{F}, \mathrm{G} : \mathcal{C}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{C}(R\text{–}S^{\mathrm{o}}) \times \mathcal{C}(S\text{–}T^{\mathrm{o}}) \;\longrightarrow\; \mathcal{C}(Q\text{–}T^{\mathrm{o}})$$

given by

$$\mathrm{F}(M, X, N) \;=\; (M \otimes_R X) \otimes_S N \qquad \text{and} \qquad \mathrm{G}(M, X, N) \;=\; M \otimes_R (X \otimes_S N) \;,$$

and consider the associativity isomorphism $\omega \colon \mathrm{F} \to \mathrm{G}$ from 4.4.7. By 2.4.8 the functors F and G preserve homotopy, so 7.1.2 yields functors

$$\ddot{\mathrm{F}}, \ddot{\mathrm{G}} : \mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \times \mathcal{K}(S\text{–}T^{\mathrm{o}}) \;\longrightarrow\; \mathcal{K}(Q\text{–}T^{\mathrm{o}})$$

given by

$$\ddot{\mathrm{F}}(M, X, N) \;=\; (M \ddot{\otimes}_R X) \ddot{\otimes}_S N \qquad \text{and} \qquad \ddot{\mathrm{G}}(M, X, N) \;=\; M \ddot{\otimes}_R (X \ddot{\otimes}_S N) \;.$$

Moreover, by 7.1.3 there is an induced natural isomorphism $\ddot{\omega} \colon \ddot{\mathrm{F}} \to \ddot{\mathrm{G}}$, and it is triangulated in each variable, as ω is a Σ-transformationin each variable. Combined with 2.4.10 these arguments prove 7.1.14 below. The other results 7.1.11–7.1.19 are proved similarly.

7.1.11 Proposition. *For M in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ there is an isomorphism in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ induced by the unitor* 4.4.1*,*

$$\mu_R^M : R \otimes_R M \;\longrightarrow\; M \;.$$

It is natural in M, and as a natural transformation of functors, μ_R is triangulated.

Proof. See 7.1.10. □

7.1.12 Proposition. *For M in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ there is an isomorphism in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ induced by the counitor* 4.4.2*,*

$$\epsilon_R^M : M \longrightarrow \operatorname{Hom}_R(R, M) .$$

It is natural in M, and as a natural transformation of functors, ϵ_R is triangulated.

Proof. See 7.1.10. □

7.1.13 Proposition. *For M in $\mathcal{K}(Q\text{–}R^{\mathrm{o}})$ and N in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ there is an isomorphism in $\mathcal{K}(Q\text{–}S^{\mathrm{o}})$ induced by commutativity* 4.4.4*,*

$$\upsilon^{MN} : M \otimes_R N \longrightarrow N \otimes_{R^{\mathrm{o}}} M ,$$

and it is natural in M and N. Moreover, as a natural transformation of functors, υ is triangulated in each variable.

Proof. See 7.1.10. □

7.1.14 Proposition. *For M in $\mathcal{K}(Q\text{–}R^{\mathrm{o}})$, X in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{K}(S\text{–}T^{\mathrm{o}})$ there is an isomorphism in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ induced by associativity* 4.4.7*,*

$$\omega^{MXN} : (M \otimes_R X) \otimes_S N \longrightarrow M \otimes_R (X \otimes_S N) ,$$

and it is natural in M, X, and N. Moreover, as a natural transformation of functors, ω is triangulated in each variable.

Proof. See 7.1.10. □

7.1.15 Proposition. *For M in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}}$, X in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{K}(T\text{–}S^{\mathrm{o}})^{\mathrm{op}}$ there is an isomorphism in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ induced by swap* 4.4.10*,*

$$\zeta^{MXN} : \operatorname{Hom}_R(M, \operatorname{Hom}_{S^{\mathrm{o}}}(N, X)) \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(N, \operatorname{Hom}_R(M, X)) ,$$

and it is natural in M, X, and N. Moreover, as a natural transformation of functors, ζ is triangulated in each variable.

Proof. See 7.1.10. □

7.1.16 Proposition. *For M in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})$, X in $\mathcal{K}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}$, and N in $\mathcal{K}(S\text{–}T^{\mathrm{o}})^{\mathrm{op}}$ there is an isomorphism in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$ induced by adjunction* 4.4.12*,*

$$\rho^{MXN} : \operatorname{Hom}_R(X \otimes_S N, M) \longrightarrow \operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M)) ,$$

and it is natural in M, X, and N. Moreover, as a natural transformation of functors, ρ is triangulated in each variable.

Proof. See 7.1.10. □

Evaluation Morphisms in the Homotopy Category

7.1.17 Proposition. *Let X be in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$. For M in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})$ there is a morphism in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})$ induced by biduality* 4.5.2*,*

$$\delta_X^M : M \longrightarrow \mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{Hom}_R(M,X),X)\,,$$

and it is natural in M. As a natural transformation of functors, δ_X is triangulated.

Proof. See 7.1.10. □

7.1.18 Proposition. *For M in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}}$, X in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{K}(S\text{–}T^{\mathrm{o}})$ there is a morphism in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ induced by tensor evaluation* 4.5.9*,*

$$\theta^{MXN} : \mathrm{Hom}_R(M,X)\otimes_S N \longrightarrow \mathrm{Hom}_R(M,X\otimes_S N)\,,$$

and it is natural in M, X, and N. Moreover, as a natural transformation of functors, θ is triangulated in each variable.

Proof. See 7.1.10. □

7.1.19 Proposition. *For M in $\mathcal{K}(R\text{–}Q^{\mathrm{o}})$, X in $\mathcal{K}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}$, and N in $\mathcal{K}(T\text{–}S^{\mathrm{o}})$ there is a morphism in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$ induced by homomorphism evaluation* 4.5.12*,*

$$\eta^{MXN} : N\otimes_S \mathrm{Hom}_R(X,M) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{S^{\mathrm{o}}}(N,X),M)\,,$$

and it is natural in M, X, and N. Moreover, as a natural transformation of functors, η is triangulated in each variable.

Proof. See 7.1.10. □

The evaluation morphisms in the homotopy category, 7.1.17–7.1.19, are isomorphisms under the hypotheses in 4.5.4, 4.5.10, and 4.5.13.

Exercises

E 7.1.1 Let K be an R^{o}-complex and set $\mathrm{F}(-) = K\otimes_R -$. Show that for every distinguished triangle $M \to N \to X \to \Sigma M$ in $\mathcal{K}(R)$ there is an exact sequence in $\mathcal{C}(\Bbbk)$,

$$\mathrm{H}(\mathrm{F}(M)) \longrightarrow \mathrm{H}(\mathrm{F}(N)) \longrightarrow \mathrm{H}(\mathrm{F}(X)) \longrightarrow \Sigma\mathrm{H}(\mathrm{F}(M)) \longrightarrow \Sigma\mathrm{H}(\mathrm{F}(N))\,.$$

E 7.1.2 Let K be an R-complex. (a) Let $\mathrm{F}(-)$ be the functor $\mathrm{Hom}_R(K,-)$ and show that for every distinguished triangle $M \to N \to X \to \Sigma M$ in $\mathcal{K}(R)$ there is an exact sequence in $\mathcal{C}(\Bbbk)$,

$$\mathrm{H}(\mathrm{F}(M)) \longrightarrow \mathrm{H}(\mathrm{F}(N)) \longrightarrow \mathrm{H}(\mathrm{F}(X)) \longrightarrow \Sigma\mathrm{H}(\mathrm{F}(M)) \longrightarrow \Sigma\mathrm{H}(\mathrm{F}(N))\,.$$

(b) Denote by $\mathrm{G}(-)$ the functor $\mathrm{Hom}_R(-,K)$ and show that for every distinguished triangle $M \leftarrow N \leftarrow X \leftarrow \Sigma M$ in $\mathcal{K}(R)^{\mathrm{op}}$ there is an exact sequence in $\mathcal{C}(\Bbbk)$,

$$\Sigma^{-1}\mathrm{H}(\mathrm{G}(N)) \longrightarrow \Sigma^{-1}\mathrm{H}(\mathrm{G}(M)) \longrightarrow \mathrm{H}(\mathrm{G}(X)) \longrightarrow \mathrm{H}(\mathrm{G}(N)) \longrightarrow \mathrm{H}(\mathrm{G}(M))\,.$$

7.2 Induced Functors on the Derived Category

SYNOPSIS. Functor that preserves quasi-isomorphisms; left/right derived functor; left/right derived natural transformation; universal property of left/right derived functor.

In the balance of this chapter, we use Greek letters for morphisms in $\mathcal{K}(R)$, that is, homotopy classes of morphisms in $\mathcal{C}(R)$. The next definition is in line with 7.1.1.

7.2.1 Definition. A morphism $(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$ in the product category $\mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n)$ is called a *quasi-isomorphism* if all components α_i and β_j are quasi-isomorphisms as defined in 6.1.12.

7.2.2 Theorem. *Let* $\mathrm{F}\colon \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \to \mathcal{K}(T)$ *be a functor. If* F *preserves quasi-isomorphisms, then there exists a unique functor* $\acute{\acute{\mathrm{F}}}$ *that makes the following diagram commutative,*

$$\begin{array}{ccc}
\mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) & \xrightarrow{\ \mathrm{F}\ } & \mathcal{K}(T) \\
{\scriptstyle \mathrm{V}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{V}_{R_m}^{\mathrm{op}} \times \mathrm{V}_{S_1} \times \cdots \times \mathrm{V}_{S_n}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{V}_T} \\
\mathcal{D}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{D}(R_m)^{\mathrm{op}} \times \mathcal{D}(S_1) \times \cdots \times \mathcal{D}(S_n) & \xrightarrow{\ \acute{\acute{\mathrm{F}}}\ } & \mathcal{D}(T)\,.
\end{array}$$

The functor $\acute{\acute{\mathrm{F}}}$ *acts as follows on objects and morphisms*:

$$\acute{\acute{\mathrm{F}}}(M_1, \dots, M_m, N_1, \dots, N_n) \;=\; \mathrm{F}(M_1, \dots, M_m, N_1, \dots, N_n) \quad \textit{and}$$

$$\acute{\acute{\mathrm{F}}}\big({}^{\alpha_1}\!/\!_{\varphi_1}, \dots, {}^{\alpha_m}\!/\!_{\varphi_m}, {}^{\beta_1}\!/\!_{\psi_1}, \dots, {}^{\beta_n}\!/\!_{\psi_n}\big) \;=$$
$$\frac{\mathrm{F}(M_1', \dots, M_m', \beta_1, \dots, \beta_n)}{1^{\mathrm{F}(M_1', \dots, M_m', Y_1, \dots, Y_n)}} \circ \frac{1^{\mathrm{F}(M_1', \dots, M_m', Y_1, \dots, Y_n)}}{\mathrm{F}(\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n)} \circ \frac{\mathrm{F}(\alpha_1, \dots, \alpha_m, N_1, \dots, N_n)}{1^{\mathrm{F}(M_1, \dots, M_m, N_1, \dots, N_n)}}\,.$$

Here ${}^{\alpha_i}\!/\!_{\varphi_i}\colon M_i \to M_i'$ *are morphisms in* $\mathcal{D}(R_i)^{\mathrm{op}}$ *and* ${}^{\beta_j}\!/\!_{\psi_j}\colon N_j \to N_j'$ *are morphisms in* $\mathcal{D}(S_j)$; *the common domain of* β_i *and* ψ_i *is* Y_i. *Moreover, the next assertions hold.*

(a) *If* F *is* $\Bbbk$*-linear in one of the variables, then* $\acute{\acute{\mathrm{F}}}$ *is* $\Bbbk$*-linear in the corresponding variable*; *in particular, if* F *is* $\Bbbk$*-multilinear, then* $\acute{\acute{\mathrm{F}}}$ *is* $\Bbbk$*-multilinear.*
(b) *If* F *preserves products/coproducts in one of the variables, then* $\acute{\acute{\mathrm{F}}}$ *preserves products/coproducts in the corresponding variable.*
(c) *If* F *is triangulated in one of the variables, then* $\acute{\acute{\mathrm{F}}}$ *is triangulated in the corresponding variable.*

PROOF. The arguments in the proofs of 6.4.31/6.4.34 and 6.5.13/6.5.15 apply mutatis mutandis to establish the claims. For example, to show that $\acute{\acute{\mathrm{F}}}$ is uniquely determined one can argue as follows. Set

$$\mathrm{V} \;=\; \mathrm{V}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{V}_{R_m}^{\mathrm{op}} \times \mathrm{V}_{S_1} \times \cdots \times \mathrm{V}_{S_n}\;.$$

Note that ${}^{\alpha_i}\!/\!_{\varphi_i}$ is a morphism $M_i' \to M_i$ in $\mathcal{D}(R_i)$ and let X_i be the common domain of α_i and φ_i. Consider in $\mathcal{K}(R_1) \times \cdots \times \mathcal{K}(R_m)$ the objects $M = (M_1, \dots, M_m)$, $M' = (M_1', \dots, M_m')$, and $X = (X_1, \dots, X_m)$ together with the morphisms

$$\alpha = (\alpha_1, \dots, \alpha_m) : X \longrightarrow M \qquad \text{and} \qquad \varphi = (\varphi_1, \dots, \varphi_m) : X \longrightarrow M' .$$

We use the shorthand notation α/φ for the morphism $(\alpha_1/\varphi_1, \dots, \alpha_m/\varphi_m)$ from M' to M in $\mathcal{D}(R_1) \times \cdots \times \mathcal{D}(R_m)$. The symbols N, N', Y, β, ψ, and β/ψ have similar meanings. By (6.4.10.2) one has $\alpha/\varphi = (\alpha/1^X)(1^X/\varphi)$ in $\mathcal{D}(R_1) \times \cdots \times \mathcal{D}(R_m)$ and thus $\alpha/\varphi = (1^X/\varphi)(\alpha/1^X)$ holds in $\mathcal{D}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{D}(R_m)^{\mathrm{op}}$. Similarly, one has $\beta/\psi = (\beta/1^Y)(1^Y/\psi)$ in $\mathcal{D}(S_1) \times \cdots \times \mathcal{D}(S_n)$. It follows that there are equalities,

$$\begin{aligned}
(\alpha/\varphi, \beta/\psi) &= \big(1^{M'}/1^{M'}, \beta/1^Y\big)\big(1^X/\varphi, 1^Y/\psi\big)\big(\alpha/1^X, 1^N/1^N\big) \\
&= \mathrm{V}(1^{M'}, \beta)\, \mathrm{V}(\varphi, \psi)^{-1}\, \mathrm{V}(\alpha, 1^N) .
\end{aligned}$$

Consequently, for a functor $\tilde{\tilde{\mathrm{F}}}$ that satisfies $\tilde{\tilde{\mathrm{F}}}\,\mathrm{V} = \mathrm{V}_T\,\mathrm{F}$, one has

$$\begin{aligned}
\tilde{\tilde{\mathrm{F}}}(\alpha/\varphi, \beta/\psi) &= \tilde{\tilde{\mathrm{F}}}\,\mathrm{V}(1^{M'}, \beta) \circ (\tilde{\tilde{\mathrm{F}}}\,\mathrm{V}(\varphi, \psi))^{-1} \circ \tilde{\tilde{\mathrm{F}}}\,\mathrm{V}(\alpha, 1^N) \\
&= \mathrm{V}_T\,\mathrm{F}(1^{M'}, \beta) \circ (\mathrm{V}_T\,\mathrm{F}(\varphi, \psi))^{-1} \circ \mathrm{V}_T\,\mathrm{F}(\alpha, 1^N) \\
&= \mathrm{V}_T\,\mathrm{F}(M', \beta) \circ (\mathrm{V}_T\,\mathrm{F}(\varphi, \psi))^{-1} \circ \mathrm{V}_T\,\mathrm{F}(\alpha, N) \\
&= \frac{\mathrm{F}(M', \beta)}{1^{\mathrm{F}(M', Y)}} \circ \frac{1^{\mathrm{F}(M', Y)}}{\mathrm{F}(\varphi, \psi)} \circ \frac{\mathrm{F}(\alpha, N)}{1^{\mathrm{F}(M, N)}} ,
\end{aligned}$$

as asserted. □

7.2.3. The expression in 7.2.2 for the action of $\tilde{\tilde{\mathrm{F}}}$ on morphisms can by (6.4.10.2) be simplified as follows:

$$\tilde{\tilde{\mathrm{F}}}\big(\alpha_1/\varphi_1, \dots, \alpha_m/\varphi_m, \beta_1/\psi_1, \dots, \beta_n/\psi_n\big) = \frac{\mathrm{F}(M'_1, \dots, M'_m, \beta_1, \dots, \beta_n)}{\mathrm{F}(\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n)} \circ \frac{\mathrm{F}(\alpha_1, \dots, \alpha_m, N_1, \dots, N_n)}{1^{\mathrm{F}(M_1, \dots, M_m, N_1, \dots, N_n)}} .$$

If $m = 0$, i.e. there are no categories $\mathcal{K}(R_i)^{\mathrm{op}}$, the expression simplifies even further:

$$\tilde{\tilde{\mathrm{F}}}\big(\beta_1/\psi_1, \dots, \beta_n/\psi_n\big) = \frac{\mathrm{F}(\beta_1, \dots, \beta_n)}{\mathrm{F}(\psi_1, \dots, \psi_n)} .$$

7.2.4. In 7.2.2 is considered a functor from a product of homotopy categories, and opposite categories of such, to a single homotopy category. Occasionally we need to consider functors from a product of homotopy categories, and opposite categories of such, to another product of homotopy categories, and opposite categories of such. It is straightforward to establish a version of 7.2.2 for such functors. That is, if

$$\begin{array}{c}
\mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \\
\Big\downarrow \mathrm{F} \\
\mathcal{K}(Q_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(Q_k)^{\mathrm{op}} \times \mathcal{K}(T_1) \times \cdots \times \mathcal{K}(T_l)
\end{array}$$

is a functor that preserves quasi-isomorphisms, see 7.2.1, then there exists a unique functor $\tilde{\tilde{\mathrm{F}}}$ between the corresponding products of derived categories, and opposite categories of such, satisfying the identity

$$\tilde{\tilde{\mathrm{F}}} \circ (\mathrm{V}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{V}_{R_m}^{\mathrm{op}} \times \mathrm{V}_{S_1} \times \cdots \times \mathrm{V}_{S_n}) = (\mathrm{V}_{Q_1}^{\mathrm{op}} \times \cdots \times \mathrm{V}_{Q_k}^{\mathrm{op}} \times \mathrm{V}_{T_1} \times \cdots \times \mathrm{V}_{T_l}) \circ \mathrm{F} .$$

Moreover, let G be a functor whose domain is the codomain of F and whose codomain is a product of homotopy categories, and opposite categories of such. The proof of 6.4.40 shows that if G preserves quasi-isomorphisms, then $(\mathrm{GF})\tilde{} = \tilde{\mathrm{G}}\tilde{\mathrm{F}}$.

7.2.5 Proposition. *Let*

$$\mathrm{F}, \mathrm{G} : \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \longrightarrow \mathcal{K}(T)$$

be functors that preserve quasi-isomorphisms and consider the induced functors from 7.2.2,

$$\tilde{\mathrm{F}}, \tilde{\mathrm{G}} : \mathcal{D}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{D}(R_m)^{\mathrm{op}} \times \mathcal{D}(S_1) \times \cdots \times \mathcal{D}(S_n) \longrightarrow \mathcal{D}(T)\,.$$

Every natural transformation $\tau \colon \mathrm{F} \to \mathrm{G}$ *induces a natural transformation* $\tilde{\tau} \colon \tilde{\mathrm{F}} \to \tilde{\mathrm{G}}$ *given by* $\tilde{\tau}^X = \tau^X/1$ *for every object* $X = (M_1, \ldots, M_m, N_1, \ldots, N_n)$. *Moreover, if the functors* F *and* G *are triangulated in the same variable and if the transformation* τ *is triangulated in that variable, then* $\tilde{\tau}$ *is triangulated in the corresponding variable.*

PROOF. The arguments in the proofs of 6.4.33/6.4.35 and 6.5.14/6.5.16 apply mutatis mutandis to establish the claims. ◻

DERIVED FUNCTORS

To facilitate further discussion, we introduce some shorthand notation for use, exclusively, in this section.

7.2.6 Definition. For rings $R_1, \ldots, R_m$ and $S_1, \ldots, S_n$ set

$$\begin{aligned} \mathcal{K} &= \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n)\,, \\ \mathcal{D} &= \mathcal{D}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{D}(R_m)^{\mathrm{op}} \times \mathcal{D}(S_1) \times \cdots \times \mathcal{D}(S_n)\,, \quad \text{and} \\ \mathrm{V} &= \mathrm{V}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{V}_{R_m}^{\mathrm{op}} \times \mathrm{V}_{S_1} \times \cdots \times \mathrm{V}_{S_n}\,. \end{aligned}$$

For clarity, we do not in this section suppress the ‘op’ on the resolution functors $\mathrm{P}_R, \mathrm{I}_R \colon \mathcal{K}(R) \to \mathcal{K}(R)$ when they are considered as functors $\mathcal{K}(R)^{\mathrm{op}} \to \mathcal{K}(R)^{\mathrm{op}}$.

7.2.7 Construction. Adopt the notation from 7.2.6. Let $\tau \colon \mathrm{F} \to \mathrm{G}$ be a natural transformation of functors $\mathcal{K} \to \mathcal{K}(T)$. We proceed to construct functors and natural transformations $\mathbf{L}\tau \colon \mathbf{L}\mathrm{F} \to \mathbf{L}\mathrm{G}$ and $\mathbf{R}\tau \colon \mathbf{R}\mathrm{F} \to \mathbf{R}\mathrm{G}$.

Consider the endofunctor $\mathrm{A} = \mathrm{I}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{I}_{R_m}^{\mathrm{op}} \times \mathrm{P}_{S_1} \times \cdots \times \mathrm{P}_{S_n}$ on $\mathcal{K}$. Since A maps quasi-isomorphisms to isomorphisms, see 6.3.11 and 6.3.17, the same is true for the composite functor FA. Therefore 7.2.2 implies the existence of a unique functor $\mathbf{L}\mathrm{F} = (\mathrm{FA})\tilde{}$ that makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\mathrm{FA}"] \arrow[d, "\mathrm{V}"'] & \mathcal{K}(T) \arrow[d, "\mathrm{V}_T"] \\ \mathcal{D} \arrow[r, "\mathbf{L}\mathrm{F}"] & \mathcal{D}(T)\,. \end{tikzcd}$$

Further, 7.2.5 yields a unique transformation $\mathbf{L}\tau = (\tau\mathrm{A})^{\acute{}\acute{}}\colon \mathbf{L}\mathrm{F} \to \mathbf{L}\mathrm{G}$.

Similarly, we consider the endofunctor $\mathrm{B} = \mathrm{P}_{R_1}^{\mathrm{op}} \times \cdots \times \mathrm{P}_{R_m}^{\mathrm{op}} \times \mathrm{I}_{S_1} \times \cdots \times \mathrm{I}_{S_n}$ on $\mathcal{K}$. As the composite FB maps quasi-isomorphisms to isomorphisms, there is by 7.2.2 a unique functor $\mathbf{R}\mathrm{F} = (\mathrm{FB})^{\acute{}\acute{}}$ that makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\mathrm{FB}"] \arrow[d, "\mathrm{V}"'] & \mathcal{K}(T) \arrow[d, "\mathrm{V}_T"] \\ \mathcal{D} \arrow[r, "\mathbf{R}\mathrm{F}"] & \mathcal{D}(T) . \end{tikzcd}$$

Further, 7.2.5 yields a unique transformation $\mathbf{R}\tau = (\tau\mathrm{B})^{\acute{}\acute{}}\colon \mathbf{R}\mathrm{F} \to \mathbf{R}\mathrm{G}$.

7.2.8 Definition. Let

$$\mathrm{F} : \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \longrightarrow \mathcal{K}(T)$$

be a functor. Every functor that is naturally isomorphic to $\mathbf{L}\mathrm{F}$ from 7.2.7 is denoted $\mathsf{L}\mathrm{F}$ and called the *left derived functor* of F. Similarly, every functor that is naturally isomorphic to $\mathbf{R}\mathrm{F}$ is denoted $\mathsf{R}\mathrm{F}$ and called the *right derived functor* of F.

Let G be another functor with the same domain and codomain as F and $\tau\colon \mathrm{F} \to \mathrm{G}$ a natural transformation. Every natural transformation that is naturally isomorphic to $\mathbf{L}\tau$ from 7.2.7 is denoted $\mathsf{L}\tau$ and called the *left derived transformation* of τ. Similarly, every transformation that is naturally isomorphic to $\mathbf{R}\tau$ is denoted $\mathsf{R}\tau$ and called the *right derived transformation* of τ.

REMARK. Weibel [253] calls the functors defined in 7.2.8 the '(total) derived functors', and Hovey [136] goes for 'total derived functors'.

At first sight, the definitions of derived functors in 7.2.8 may seem kind of random. They do, however, have universal properties as explained in 7.2.19 and 7.2.20.

7.2.9 Example. Consider for $n \geqslant 1$ the endofunctor $\mathrm{F} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ on $\mathcal{K}(\mathbb{Z})$. For $m \geqslant 1$ the complex $P = 0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow 0$, concentrated in degrees 1 and 0, yields a projective resolution of $\mathbb{Z}/m\mathbb{Z}$, so there are isomorphisms in $\mathcal{D}(\mathbb{Z})$,

$$\mathsf{L}\mathrm{F}(\mathbb{Z}/m\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, P) \simeq 0 .$$

In comparison, the module $\mathrm{F}(\mathbb{Z}/m\mathbb{Z})$ is not necessarily zero; cf. 1.1.8.

The complex $I = 0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, concentrated in degrees 0 and -1, yields an injective resolution of $\mathbb{Z}$, and hence there are isomorphisms in $\mathcal{D}(\mathbb{Z})$,

$$\mathsf{R}\mathrm{F}(\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, I) \simeq \Sigma^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \Sigma^{-1}(\mathbb{Z}/n\mathbb{Z}) .$$

In comparison, the module $\mathrm{F}(\mathbb{Z})$ is zero.

7.2.10 Lemma. *Consider the endofunctors* A *and* B *on* $\mathcal{K}$ *from* 7.2.7 *and the natural transformations*

$$\mathrm{A} \xrightarrow{\lambda_0} \mathrm{Id}_{\mathcal{K}} \xrightarrow{\rho_0} \mathrm{B} \quad \textit{given by} \quad \begin{aligned} \lambda_0 &= \iota_{R_1}^{\mathrm{op}} \times \cdots \times \iota_{R_m}^{\mathrm{op}} \times \pi_{S_1} \times \cdots \times \pi_{S_n} \quad \textit{and} \\ \rho_0 &= \pi_{R_1}^{\mathrm{op}} \times \cdots \times \pi_{R_m}^{\mathrm{op}} \times \iota_{S_1} \times \cdots \times \iota_{S_n} , \end{aligned}$$

where π and ι are the natural transformations from 6.3.11 *and* 6.3.17*. For every X in $\mathcal{K}$ the morphisms λ_0^X and ρ_0^X are quasi-isomorphisms. Moreover, there are identities* $\mathrm{A}\lambda_0 = \lambda_0\mathrm{A}$ *and* $\mathrm{B}\rho_0 = \rho_0\mathrm{B}$ *of natural isomorphisms* $\mathrm{A}^2 \to \mathrm{A}$ *and* $\mathrm{B} \to \mathrm{B}^2$.

Proof. It follows from 6.3.11 and 6.3.17, in view of 7.2.1, that λ_0^X and ρ_0^X are quasi-isomorphisms. It follows from 6.3.12 and 6.3.18 that $\mathrm{A}\lambda_0 = \lambda_0\mathrm{A}$ and $\mathrm{B}\rho_0 = \rho_0\mathrm{B}$ hold, and by another application of 6.3.11 and 6.3.17 these natural transformations are isomorphisms. □

A functor that preserves quasi-isomorphisms is its own derived functor.

7.2.11 Example. Adopt the notation from 7.2.6 and let $\mathrm{F}\colon \mathcal{K} \to \mathcal{K}(T)$ be a functor that preserves quasi-isomorphisms. The functor $\H{\mathrm{F}}$ from 7.2.2 is both a left derived and a right derived functor of F, that is,

$$\mathsf{LF} = \H{\mathrm{F}} = \mathsf{RF}\ .$$

Indeed, application of F to $\lambda_0\colon \mathrm{A} \to \mathrm{Id}_{\mathcal{K}}$ from 7.2.10 yields a natural transformation $\mathrm{F}\lambda_0\colon \mathrm{FA} \to \mathrm{F}$. Since $(\mathrm{F}\lambda_0)^X = \mathrm{F}(\lambda_0^X)$ is a quasi-isomorphism for every $X \in \mathcal{K}$, there is by 7.2.5 a natural isomorphism $(\mathrm{F}\lambda_0)^{\H{}}\colon \mathbf{L}\mathrm{F} = (\mathrm{FA})^{\H{}} \to \H{\mathrm{F}}$. Similarly, one sees that $\mathbf{R}\mathrm{F}$ is naturally isomorphic to $\H{\mathrm{F}}$.

Specific examples of functors that that preserve quasi-isomorphisms include the identity functor $\mathrm{Id}_{\mathcal{K}(R)}$, the resolution functors P_R and I_R, the functors $\mathrm{Hom}_R(P,-)$ and $\mathrm{Hom}_R(-,I)$ where P is a semi-projective R-complex and I is a semi-injective R-complex, and $F \otimes_R -$ where F is a semi-flat R^{o}-complex; see 6.3.11, 6.3.17, 5.2.10, 5.3.16, and 5.4.9.

Quasi-isomorphic functors have isomorphic derived functors.

7.2.12 Example. Adopt the notation from 7.2.6. Let $\mathrm{F}, \mathrm{G}\colon \mathcal{K} \to \mathcal{K}(T)$ be functors and $\tau\colon \mathrm{F} \to \mathrm{G}$ a natural transformation. If τ^X is a quasi-isomorphism for every $X \in \mathcal{K}$, then LF and RF are naturally isomorphic to LG and RG, respectively. Indeed, in this case $\mathbf{L}\tau = (\tau\mathrm{A})^{\H{}}$ and $\mathbf{R}\tau = (\tau\mathrm{B})^{\H{}}$ are natural isomorphisms.

In particular, the natural transformation $\pi\colon \mathrm{P}_R \to \mathrm{Id}_{\mathcal{K}(R)}$ from 6.3.11 shows that the derived functors LP_R and RP_R are both naturally isomorphic to $\mathrm{Id}_{\mathcal{D}(R)}$. Similarly, the derived functors LI_R and RI_R of the semi-injective resolution functor, see 6.3.17, are both naturally isomorphic to $\mathrm{Id}_{\mathcal{D}(R)}$.

Before we move on to more substantial examples of derived functors in Sects. 7.3 and 7.4, we explore some general properties of derived functors.

7.2.13 Lemma. *Let* $\mathrm{F}, \mathrm{G}\colon \mathcal{U} \to \mathcal{V}$ *be naturally isomorphic functors.*

(a) *Assume that $\mathcal{U}$ and $\mathcal{V}$ are $\Bbbk$-prelinear categories. If* F *is $\Bbbk$-linear, then so is* G.
(b) *Assume that the categories $\mathcal{U}$ and $\mathcal{V}$ have products/coproducts. If* F *preserves products/coproducts, then so does* G.
(c) *Assume that $\mathcal{U}$ and $\mathcal{V}$ are triangulated and let $\tau\colon \mathrm{F} \to \mathrm{G}$ be a natural isomorphism. If* F *is triangulated with natural isomorphism $\phi\colon \mathrm{F}\Sigma_{\mathcal{U}} \to \Sigma_{\mathcal{V}}\mathrm{F}$, then*

the natural isomorphism $\psi\colon \mathrm{G}\Sigma_{\mathcal{U}} \to \Sigma_{\mathcal{V}}\mathrm{G}$ *given by* $\psi^M = (\Sigma\tau^M)\phi^M(\tau^{\Sigma M})^{-1}$ *makes the functor* G *and the natural transformation* τ *triangulated.*

Proof. All three assertions follow directly from the relevant definitions. ◻

7.2.14 Theorem. *Consider a functor,*

$$\mathrm{F} : \mathcal{K}(R_1)^{\mathrm{op}} \times \cdots \times \mathcal{K}(R_m)^{\mathrm{op}} \times \mathcal{K}(S_1) \times \cdots \times \mathcal{K}(S_n) \longrightarrow \mathcal{K}(T)\ .$$

(a) *If* F *is* $\Bbbk$*-linear in one of its variables, then its derived functors* LF *and* RF *are* $\Bbbk$*-linear in the corresponding variable. In particular, if* F *is* $\Bbbk$*-multilinear, then* LF *and* RF *are* $\Bbbk$*-multilinear.*

(b) *If* F *preserves coproducts in one of its variables, then* LF *preserves coproducts in the corresponding variable.*

(c) *If* F *preserves products in one of its variables, then* RF *preserves products in the corresponding variable.*

(d) *If* F *is triangulated in one of its variables, then* LF *and* RF *are triangulated in the corresponding variable.*

Proof. By 7.2.8 and 7.2.13 it is sufficient to verify the assertions for the functors $\mathbf{L}\mathrm{F} = (\mathrm{FA})^{''}$ and $\mathbf{R}\mathrm{F} = (\mathrm{FB})^{''}$ from 7.2.7. By 6.3.11 and 6.3.17 the resolution functors P and I are $\Bbbk$-linear. Thus, if F is $\Bbbk$-linear in one of its variables, then FA and FB are $\Bbbk$-linear in the corresponding variable. Part (a) now follows from 7.2.2(a).

The remaining three parts follow from 7.2.2 by similar arguments. To this end recall that P is triangulated and preserves coproducts, whence P^{op} is triangulated and preserves products, and that I is triangulated and preserves products, whence I^{op} is triangulated and preserves coproducts; see 6.3.11, 6.3.17, and E.10. ◻

Recall from 6.1.22 that a functor between module categories induces a functor between homotopy categories.

7.2.15 Proposition. *Let* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ *be an additive functor and let* M *be an* R*-complex. The derived functors of the induced functor* $\mathrm{F}\colon \mathcal{K}(R) \to \mathcal{K}(S)$ *satisfy*

$$\inf \mathsf{L}\mathrm{F}(M) \geqslant \inf M \qquad \textit{and} \qquad -\sup \mathsf{R}\mathrm{F}(M) \geqslant -\sup M\ .$$

Proof. The inequalities follow from the definitions, 7.2.8, of the derived functors combined with 5.2.15 and 5.3.26. ◻

Universal Properties of Derived Functors

So far, our approach to derived functors has been constructive in the sense that the very definition of a derived functor tells how to compute it. There is, of course, also an abstract approach, which we now proceed to explore.

Adopt the notation from 7.2.6. Let $\mathrm{F}\colon \mathcal{K} \to \mathcal{K}(T)$ be a functor. One can, in general, not expect to find a functor F' that makes the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{V}"'] & \mathcal{K}(T) \arrow[d, "\mathrm{V}_T"] \\ \mathcal{D} \arrow[r, "\mathrm{F}'"] & \mathcal{D}(T) \end{tikzcd}$$

commutative. Indeed, it follows from 6.4.17 that a necessary condition for the existence of such a functor F' is that F preserves quasi-isomorphisms. On the other hand, 7.2.2 shows that this condition is sufficient to guarantee the existence of F'.

7.2.16 Definition. Adopt the notation from 7.2.6. Let $\mathrm{F}\colon \mathcal{K} \to \mathcal{K}(T)$ be a functor. To F we associate two categories $\mathcal{L}_\mathrm{F}$ and $\mathcal{R}_\mathrm{F}$ defined as follows.

The objects in $\mathcal{L}_\mathrm{F}$ are pairs (F', ξ') where $\mathrm{F}'\colon \mathcal{D} \to \mathcal{D}(T)$ is a functor and $\xi'\colon \mathrm{F}'\mathrm{V} \to \mathrm{V}_T\mathrm{F}$ is a natural transformation. A morphism $\zeta\colon (\mathrm{F}', \xi') \to (\mathrm{F}'', \xi'')$ in $\mathcal{L}_\mathrm{F}$ is a natural transformation $\zeta\colon \mathrm{F}' \to \mathrm{F}''$ that makes the next diagram commutative,

$$\begin{tikzcd} & \mathrm{F}'\mathrm{V} \arrow[dl, "\zeta\mathrm{V}"'] \arrow[d, "\xi'"] \\ \mathrm{F}''\mathrm{V} \arrow[r, "\xi''"'] & \mathrm{V}_T\mathrm{F} \end{tikzcd} .$$

The objects in $\mathcal{R}_\mathrm{F}$ are pairs (F', ξ') where $\mathrm{F}'\colon \mathcal{D} \to \mathcal{D}(T)$ is a functor and $\xi'\colon \mathrm{V}_T\mathrm{F} \to \mathrm{F}'\mathrm{V}$ is a natural transformation. A morphism $\zeta\colon (\mathrm{F}', \xi') \to (\mathrm{F}'', \xi'')$ in $\mathcal{R}_\mathrm{F}$ is a natural transformation $\zeta\colon \mathrm{F}' \to \mathrm{F}''$ that makes the next diagram commutative,

$$\begin{tikzcd} \mathrm{V}_T\mathrm{F} \arrow[r, "\xi'"] \arrow[d, "\xi''"'] & \mathrm{F}'\mathrm{V} \arrow[dl, "\zeta\mathrm{V}"] \\ \mathrm{F}''\mathrm{V} & \end{tikzcd} .$$

The category $\mathcal{L}_\mathrm{F}$ has an initial object, namely the pair $(\mathrm{O}, 0)$ where $\mathrm{O}\colon \mathcal{D} \to \mathcal{D}(T)$ is the zero functor and $0\colon \mathrm{OV} \to \mathrm{V}_T\mathrm{F}$ is the zero transformation. Similarly, the pair $(\mathrm{O}, 0)$, where this time 0 denotes the zero transformation $\mathrm{V}_T\mathrm{F} \to \mathrm{OV}$, is a terminal object in $\mathcal{R}_\mathrm{F}$. A less trivial fact is that $\mathcal{L}_\mathrm{F}$ has a terminal object and that $\mathcal{R}_\mathrm{F}$ has an initial object; this is proved in 7.2.19 below.

7.2.17 Construction. Adopt the notation from 7.2.6. Let $\mathrm{F}\colon \mathcal{K} \to \mathcal{K}(T)$ be a functor. We construct objects $(\mathbf{L}\mathrm{F}, \lambda)$ in $\mathcal{L}_\mathrm{F}$ and $(\mathbf{R}\mathrm{F}, \rho)$ in $\mathcal{R}_\mathrm{F}$, where $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{F}$ are the functors from 7.2.7.

Recall the natural transformation $\lambda_0\colon \mathrm{A} \to \mathrm{Id}_\mathcal{K}$ from 7.2.10. A natural transformation λ is defined as follows:

$$(\mathbf{L}\mathrm{F})\,\mathrm{V} = \mathrm{V}_T\,\mathrm{F}\mathrm{A} \xrightarrow{\mathrm{V}_T\mathrm{F}\lambda_0} \mathrm{V}_T\,\mathrm{F}\,\mathrm{Id}_\mathcal{K} = \mathrm{V}_T\,\mathrm{F} \ .$$

Similarly, with $\rho_0\colon \mathrm{Id}_\mathcal{K} \to \mathrm{B}$ from 7.2.10 a natural transformation ρ is given by

$$\mathrm{V}_T\,\mathrm{F} = \mathrm{V}_T\,\mathrm{F}\,\mathrm{Id}_\mathcal{K} \xrightarrow{\mathrm{V}_T\mathrm{F}\rho_0} \mathrm{V}_T\,\mathrm{F}\mathrm{B} = (\mathbf{R}\mathrm{F})\,\mathrm{V} \ .$$

7.2.18 Lemma. *Adopt the notation from* 7.2.6. *Let* $\mathrm{F}, \mathrm{G} \colon \mathcal{K} \to \mathcal{D}(T)$ *be functors and* $\tau_1, \tau_2 \colon \mathrm{F} \to \mathrm{G}$ *natural transformations.*

(a) *Assume that* F *maps quasi-isomorphisms in* $\mathcal{K}$ *to isomorphisms in* $\mathcal{D}(T)$*. If* $\tau_1 \mathrm{A} = \tau_2 \mathrm{A}$ *holds, then one has* $\tau_1 = \tau_2$*.*

(b) *Assume that* G *maps quasi-isomorphisms in* $\mathcal{K}$ *to isomorphisms in* $\mathcal{D}(T)$*. If* $\tau_1 \mathrm{B} = \tau_2 \mathrm{B}$ *holds, then one has* $\tau_1 = \tau_2$*.*

Proof. (a): Consider the natural transformation $\lambda_0 \colon \mathrm{A} \to \mathrm{Id}_{\mathcal{K}}$ from 7.2.10. As τ_i is a natural transformation, there is a commutative diagram of natural transformations,

$$\begin{tikzcd} \mathrm{F}\mathrm{A} \arrow[r, "\mathrm{F}\lambda_0"] \arrow[d, "\tau_i \mathrm{A}"'] & \mathrm{F} \arrow[d, "\tau_i"] \\ \mathrm{G}\mathrm{A} \arrow[r, "\mathrm{G}\lambda_0"] & \mathrm{G} \end{tikzcd} .$$

As λ_0^X is a quasi-isomorphism for every X in $\mathcal{K}$, the assumption on F implies that $\mathrm{F}\lambda_0$ is a natural isomorphism. It follows that $\tau_i = (\mathrm{G}\lambda_0)(\tau_i \mathrm{A})(\mathrm{F}\lambda_0)^{-1}$ holds. Consequently, if one has $\tau_1 \mathrm{A} = \tau_2 \mathrm{A}$, then $\tau_1 = \tau_2$ holds.

(b): An argument similar to the proof of part (a) applies. □

7.2.19 Theorem. *Adopt the notation from* 7.2.6. *Let* $\mathrm{F} \colon \mathcal{K} \to \mathcal{K}(T)$ *be a functor.*

(a) *The object* $(\mathbf{L}\mathrm{F}, \lambda)$ *constructed in* 7.2.17 *is terminal in the category* $\mathcal{L}_{\mathrm{F}}$*.*

(b) *The object* $(\mathbf{R}\mathrm{F}, \rho)$ *constructed in* 7.2.17 *is initial in the category* $\mathcal{R}_{\mathrm{F}}$*.*

Proof. (a): It must be argued that for every object (F', ξ') in $\mathcal{L}_{\mathrm{F}}$ there exists a unique morphism $(\mathrm{F}', \xi') \to (\mathbf{L}\mathrm{F}, \lambda)$. Let ζ_1 and ζ_2 be any two such morphisms, that is, each $\zeta_i \colon \mathrm{F}' \to \mathbf{L}\mathrm{F}$ is a natural transformation such that the diagram

$$\begin{tikzcd} & \mathrm{F}'\mathrm{V} \arrow[dl, "\zeta_i \mathrm{V}"'] \arrow[d, "\xi'"] \\ (\mathbf{L}\mathrm{F})\,\mathrm{V} \arrow[r, "\lambda"'] & \mathrm{V}_T\,\mathrm{F} \end{tikzcd}$$

is commutative. As the functor $\mathrm{V} \colon \mathcal{K} \to \mathcal{D}$ is the identity on objects, one has $\zeta_1 = \zeta_2$ if and only if $\zeta_1 \mathrm{V} = \zeta_2 \mathrm{V}$, and by 7.2.18 this holds if and only if $\zeta_1 \mathrm{V}\mathrm{A} = \zeta_2 \mathrm{V}\mathrm{A}$. To verify the latter, note that the diagram above yields the commutative diagram

$$\begin{tikzcd} & \mathrm{F}'\mathrm{V}\mathrm{A} \arrow[dl, "\zeta_i \mathrm{V}\mathrm{A}"'] \arrow[d, "\xi'\mathrm{A}"] \\ (\mathbf{L}\mathrm{F})\,\mathrm{V}\mathrm{A} \arrow[r, "\lambda\mathrm{A}"'] & \mathrm{V}_T\,\mathrm{F}\mathrm{A} \end{tikzcd} .$$

By definition, one has $\lambda \mathrm{A} = \mathrm{V}_T\,\mathrm{F}\lambda_0\mathrm{A}$, which is a natural isomorphism by 7.2.10. It follows that $\zeta_i \mathrm{V}\mathrm{A} = (\lambda \mathrm{A})^{-1}(\xi'\mathrm{A})$ for $i = 1, 2$, and hence $\zeta_1 \mathrm{V}\mathrm{A} = \zeta_2 \mathrm{V}\mathrm{A}$.

It remains to prove the existence of a morphism $(\mathrm{F}', \xi') \to (\mathbf{L}\mathrm{F}, \lambda)$ in $\mathcal{L}_{\mathrm{F}}$. There are natural transformations,

$$\mathrm{F}'\mathrm{V} \xleftarrow[\simeq]{\mathrm{F}'\mathrm{V}\lambda_0} \mathrm{F}'\mathrm{V}\mathrm{A} \xrightarrow{\xi'\mathrm{A}} \mathrm{V}_T\mathrm{F}\mathrm{A} = (\mathbf{L}\mathrm{F})\,\mathrm{V}\,,$$

and thus a natural transformation $\widetilde{\zeta}\colon \mathrm{F}'\mathrm{V} \to (\mathbf{L}\mathrm{F})\,\mathrm{V}$ given by $\widetilde{\zeta} = (\xi'\mathrm{A})(\mathrm{F}'\mathrm{V}\lambda_0)^{-1}$. It follows from 6.4.28 that there is a natural transformation $\zeta\colon \mathrm{F}' \to \mathbf{L}\mathrm{F}$ with $\zeta\mathrm{V} = \widetilde{\zeta}$. To verify the equality $\lambda(\zeta\mathrm{V}) = \xi'$, that is, $\lambda\widetilde{\zeta} = \xi'$, amounts by the definitions of λ and $\widetilde{\zeta}$ to showing that $(\mathrm{V}_T\mathrm{F}\lambda_0)(\xi'\mathrm{A}) = \xi'(\mathrm{F}'\mathrm{V}\lambda_0)$ holds, i.e. that the diagram

$$\begin{tikzcd}
\mathrm{F}'\mathrm{V}\mathrm{A} \arrow[r, "\mathrm{F}'\mathrm{V}\lambda_0"] \arrow[d, "\xi'\mathrm{A}"'] & \mathrm{F}'\mathrm{V} \arrow[d, "\xi'"] \\
\mathrm{V}_T\mathrm{F}\mathrm{A} \arrow[r, "\mathrm{V}_T\mathrm{F}\lambda_0"] & \mathrm{V}_T\mathrm{F}
\end{tikzcd}$$

is commutative, which it is as $\xi'\colon \mathrm{F}'\mathrm{V} \to \mathrm{V}_T\mathrm{F}$ is a natural transformation.

(b): An argument similar to the proof of part (a) applies. □

7.2.20. All terminal objects in a given category are isomorphic. In particular, every terminal object in $\mathcal{L}_\mathrm{F}$ is isomorphic to $(\mathbf{L}\mathrm{F},\lambda)$. If (F',ξ') and $(\mathbf{L}\mathrm{F},\lambda)$ are isomorphic in $\mathcal{L}_\mathrm{F}$ then, in particular, F' and $\mathbf{L}\mathrm{F}$ are naturally isomorphic functors, i.e. F' is $\mathsf{L}\mathrm{F}$. On the other hand, if $\varphi\colon \mathrm{F}' \to \mathbf{L}\mathrm{F}$ is a natural isomorphism—that is, F' is $\mathsf{L}\mathrm{F}$—then there is a unique natural transformation $\xi'\colon \mathrm{F}'\mathrm{V} \to \mathrm{V}_T\mathrm{F}$, namely $\xi' = \lambda \circ (\varphi\mathrm{V})$, such that (F',ξ') is isomorphic to $(\mathbf{L}\mathrm{F},\lambda)$ in $\mathcal{L}_\mathrm{F}$. Thus, for most purposes one can suppress the natural transformation and identify a terminal object in $\mathcal{L}_\mathrm{F}$ with a left derived functor of F in the sense of 7.2.8.

Similarly, an initial object in $\mathcal{R}_\mathrm{F}$ is nothing but a right derived functor of F in the sense of 7.2.8.

Every natural transformation $\tau\colon \mathrm{F} \to \mathrm{G}$ of functors from $\mathcal{K}$ to $\mathcal{K}(T)$ induces a natural transformation $\mathbf{L}\tau\colon\ \mathbf{L}\mathrm{F} \to \mathbf{L}\mathrm{G}$; see 7.2.7. In the abstract approach to derived functors explained above, this natural transformation is obtained from part (a) in the next result applied to $\mathrm{F}' = \mathbf{L}\mathrm{F}$ and $\mathrm{G}' = \mathbf{L}\mathrm{G}$.

7.2.21 Proposition. *Adopt the notation from* 7.2.6. *Let* $\mathrm{F},\mathrm{G}\colon \mathcal{K} \to \mathcal{K}(T)$ *be functors and* $\tau\colon \mathrm{F} \to \mathrm{G}$ *a natural transformation.*

(a) *Let* $(\mathrm{F}',\xi') \in \mathcal{L}_\mathrm{F}$ *and* $(\mathrm{G}',\eta') \in \mathcal{L}_\mathrm{G}$. *If* (G',η') *is terminal, then there is a unique natural transformation* $\tau'\colon \mathrm{F}' \to \mathrm{G}'$ *that makes the next diagram commutative,*

$$\begin{tikzcd}
\mathrm{F}'\mathrm{V} \arrow[r, "\tau'\mathrm{V}"] \arrow[d, "\xi'"'] & \mathrm{G}'\mathrm{V} \arrow[d, "\eta'"] \\
\mathrm{V}_T\mathrm{F} \arrow[r, "\mathrm{V}_T\tau"] & \mathrm{V}_T\mathrm{G}\ .
\end{tikzcd}$$

(b) *Let* $(\mathrm{F}',\xi') \in \mathcal{R}_\mathrm{F}$ *and* $(\mathrm{G}',\eta') \in \mathcal{R}_\mathrm{G}$. *If* (F',ξ') *is initial, then there is a unique natural transformation* $\tau'\colon \mathrm{F}' \to \mathrm{G}'$ *that makes the next diagram commutative,*

$$\begin{tikzcd} \mathrm{V}_T\,\mathrm{F} \arrow[r, "\mathrm{V}_T\,\tau"] \arrow[d, "\xi'"'] & \mathrm{V}_T\,\mathrm{G} \arrow[d, "\eta'"] \\ \mathrm{F}'\,\mathrm{V} \arrow[r, "\tau'\,\mathrm{V}"] & \mathrm{G}'\,\mathrm{V}\,. \end{tikzcd}$$

PROOF. (a): Note that $(\mathrm{F}', (\mathrm{V}_T\,\tau)\circ\xi')$ is an object in $\mathcal{L}_{\mathrm{G}}$. A natural transformation τ' that makes the diagram commutative is a morphism $\tau'\colon (\mathrm{F}', (\mathrm{V}_T\,\tau)\circ\xi') \to (\mathrm{G}', \eta')$ in $\mathcal{L}_{\mathrm{G}}$, and there exists a unique such morphism as (G', η') is terminal in $\mathcal{L}_{\mathrm{G}}$.

(b): An argument similar to the proof of part (a) applies. ◻

EXERCISES

E 7.2.1 Assume that R is left hereditary and let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor that preserves homotopy. Show that for every R-module M one has $\mathrm{H}_v(\mathsf{LF}(M)) = 0 = \mathrm{H}_{-v}(\mathsf{RF}(M))$ for $v \neq 0, 1$. *Hint:* E 1.3.17 and E 1.4.8.

E 7.2.2 Assume that R is left hereditary and let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor that preserves homotopy. Show that for every R-module M one has $\mathrm{H}_v(\mathsf{LG}(M)) = 0 = \mathrm{H}_{-v}(\mathsf{RG}(M))$ for $v \neq 0, 1$. *Hint:* E 1.3.17 and E 1.4.8.

E 7.2.3 Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be an additive functor and M an R-complex. (a) Let X be a K-projective R-complex isomorphic to M in $\mathcal{D}(R)$; show that there is an isomorphism $\mathsf{LF}(M) \simeq \mathrm{F}(X)$ in $\mathcal{D}(S)$. (b) Let Y be a K-injective R-complex isomorphic to M in $\mathcal{D}(R)$; show that there is an isomorphism $\mathsf{RF}(M) \simeq \mathrm{F}(Y)$ in $\mathcal{D}(S)$.

E 7.2.4 Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be an additive functor and M an R-complex. (a) Let X be a K-projective R-complex isomorphic to M in $\mathcal{D}(R)$; show that there is an isomorphism $\mathsf{RG}(M) \simeq \mathrm{G}(X)$ in $\mathcal{D}(S)$. (b) Let Y be a K-injective R-complex isomorphic to M in $\mathcal{D}(R)$; show that there is an isomorphism $\mathsf{LG}(M) \simeq \mathrm{G}(Y)$ in $\mathcal{D}(S)$.

E 7.2.5 (Cf. 7.2.9) Let $n \in \mathbb{N}$ and show that one has $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ in $\mathcal{M}(\mathbb{Z})$.

E 7.2.6 Let $n > 0$ be an integer and set $\mathrm{F} = -\otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Compute the groups $\mathrm{H}_v(\mathsf{LF}(M))$ and $\mathrm{H}_{-v}(\mathsf{RF}(M))$ for $v = 0, 1$ and $M = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, $M = \mathbb{Q}$, and $M = \mathbb{Q}/\mathbb{Z}$.

E 7.2.7 Let $n > 0$ be an integer and set $\mathrm{G} = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/n\mathbb{Z})$. Compute the groups $\mathrm{H}_v(\mathsf{LG}(M))$ and $\mathrm{H}_{-v}(\mathsf{RG}(M))$ for $v = 0, 1$ and $M = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, $M = \mathbb{Q}$, and $M = \mathbb{Q}/\mathbb{Z}$.

E 7.2.8 Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be an additive functor and M an R-complex. Show:

$$-\sup \mathsf{RG}(M) \geqslant \inf M \qquad \text{and} \qquad \inf \mathsf{LG}(M) \geqslant -\sup M\,.$$

E 7.2.9 Let $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(S)$ be an additive functor and M an R-module. (a) Show that there is a canonical homomorphism $\mathrm{H}_0\,\mathsf{LF}(M) \to \mathrm{F}(M)$ which is natural in M and an isomorphism if F is right exact. (b) Show that there is a canonical homomophism $\mathrm{F}(M) \to \mathrm{H}_0\,\mathsf{RF}(M)$ which is natural in M and an isomorphism if F is left exact.

E 7.2.10 Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be an additive functor and M an R-module. (a) Show that there is a canonical homomorphism $\mathrm{G}(M) \to \mathrm{H}_0\,\mathsf{RG}(M)$ which is natural in M and an isomorphism if G is left exact. (b) Show that there is a canonical homomophism $\mathrm{H}_0\,\mathsf{LG}(M) \to \mathrm{G}(M)$ which is natural in M and an isomorphism if G is right exact.

E 7.2.11 (a) Give an example of a functor F, such that F preserves products but LF does not. (b) Give an example of a functor F, such that LF preserves products but F does not.

E 7.2.12 (a) Give an example of a functor F, such that F preserves coproducts but RF does not. (b) Give an example of a functor F, such that RF preserves coproducts but F does not.

7.3 Derived Hom Functor

SYNOPSIS. The functor RHom; derived category of complexes of bimodules; augmentation of RHom; semi-projective replacement; semi-injective replacement; Ext functors; exact Ext sequence.

In this section and the next we apply the machinery from the previous section to the Hom and tensor product functors.

7.3.1 Definition. The functor $\operatorname{Hom}_R(-,-)\colon \mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R) \to \mathcal{K}(\Bbbk)$ from 7.1.4 has by 7.2.8 a right derived functor, written

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(\Bbbk)\;;$$

it is per 7.2.7 induced by the functor $\operatorname{Hom}_R(\mathrm{P}_R(-), \mathrm{I}_R(-))$.

Additional ring actions on R-complexes M and N yield additional actions on the complex $\operatorname{Hom}_R(M,N)$; see 7.1.7. To what extent such additional actions carry over to the complex $\mathsf{RHom}_R(M,N)$ is a central topic in this section. To address it, we introduce an extra layer of rings, as we consider ring homomorphisms, say, $\varphi\colon R \otimes_\Bbbk S^{\mathrm{o}} \to B$. By restriction of scalars every B-module is an R–S^{o}-bimodule, see 1.1.12. Moreover, B—and every B–B^{o}-bimodule—is an $(R \otimes_\Bbbk S^{\mathrm{o}})$–$B^{\mathrm{o}}$-bimodule as well as a B–$(S \otimes_\Bbbk R^{\mathrm{o}})$-bimodule via φ; see 1.1.29. Following 1.1.12 the restiction of scalars functors res_R^B and $\operatorname{res}_{R^{\mathrm{o}}}^{B^{\mathrm{o}}}$ are mostly suppressed, but when we refer to B "as an R-module" or "as an R^{o}-module" it means that one of these functors is applied.

AUGMENTATION

7.3.2 Definition. Let $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ denote the category $\mathcal{D}(R \otimes_\Bbbk S^{\mathrm{o}})$.

7.3.3. In view of 7.1.6 the category $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ is naturally identified with the category whose objects are complexes of R–S^{o}-bimodules, and whose hom-sets consist of left fractions, i.e. equivalence classes of diagrams

$$M \xleftarrow[\simeq]{\varphi} U \xrightarrow{\alpha} N$$

where α and φ are homotopy classes of R- and S^{o}-linear chain maps of degree 0, and φ is a quasi-isomorphism. In the definition of the equivalence relation, see 6.4.1, the morphisms μ^1 and μ^2 are, of course, also R- and S^{o}-linear.

7.3.4 Setup. Consider ring homomorphisms

$$R \otimes_\Bbbk Q^{\mathrm{o}} \longrightarrow A \qquad \text{and} \qquad R \otimes_\Bbbk S^{\mathrm{o}} \longrightarrow B\,.$$

A functor $\mathrm{E}\colon \mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \to \mathcal{D}(Q\text{–}S^{\mathrm{o}})$ can be compared to the functor RHom_R from 7.3.1 via the diagram,

$$
\begin{tikzcd}
\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \arrow[r, "\mathrm{E}"] \arrow[d, "\mathrm{res}_R^A \times \mathrm{res}_R^B"'] & \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \arrow[d, "\mathrm{res}_{\Bbbk}^{Q\otimes S^{\mathrm{o}}}"] \\
\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R) \arrow[r, "\mathsf{RHom}_R(-,-)"] & \mathcal{D}(\Bbbk)\,.
\end{tikzcd}
\tag{7.3.4.1}
$$

7.3.5 Definition. Adopt the setup 7.3.4. If there exists a natural isomorphism,

$$\varphi :\ \mathrm{res}_{\Bbbk}^{Q\otimes S^{\mathrm{o}}} \mathrm{E} \longrightarrow \mathsf{RHom}_R(\mathrm{res}_R^A, \mathrm{res}_R^B)\,,$$

i.e. (7.3.4.1) is commutative up to natural isomorphism, then the functor E is called an *augmentation* of RHom_R and denoted by the same symbol. That is, one writes

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}})$$

and says that RHom_R is *augmented* to a functor from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$.

CAVEAT. Adopt the setup 7.3.4 and consider restriction of scalars followed by Hom from 7.1.7,

$$\mathrm{D}\colon \mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \xrightarrow{\mathrm{res}^A_{R\otimes Q^{\mathrm{o}}} \times\, \mathrm{res}^B_{R\otimes S^{\mathrm{o}}}} \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \xrightarrow{\mathrm{Hom}_R(-,-)} \mathcal{K}(Q\text{–}S^{\mathrm{o}})\,.$$

This functor has a right derived functor, $\mathrm{E} = \mathsf{R}\mathrm{D}$, given by $\mathrm{E} = \mathrm{Hom}_R(\mathrm{P}_A(-), \mathrm{I}_B(-))^{\widetilde{}}$; see 7.2.7 and 7.2.8. For this functor E the diagram (7.3.4.1) need not be commutative up to natural isomorphism; see for example 7.3.10. Thus, even though RHom_R according to 7.2.8 could be the notation for the derived functor E, we *only* use it in situations where the diagram (7.3.4.1) is commutative up to natural isomorphism. This causes no ambiguity—these are the only situations in which we are interested in the derived functor E—and the dissonance with 7.2.8 is, in fact, limited. Indeed, the meaning of the symbol RHom_R depends on which category the functor Hom_R is defined on. That information is not encoded in the symbol $\mathsf{RHom}_R(-,-)$, it only gives away that both arguments must be R-complexes. Thus, without further context the only certain interpretation of the symbol is the one from 7.3.1, and 7.3.5 ensures that this interpretation always is valid.

Theorem 7.3.6 is our most general statement about augmentations of RHom_R; commonly used special cases are found in 7.3.8–7.3.9 and 7.3.12–7.3.13. The question of how to evaluate the functor is addressed in 7.3.7 and 7.3.20.

7.3.6 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$ and $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$ be ring homomorphisms. If condition* (a) *or* (b) *below is satisfied, then RHom_R is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}})\,. \tag{7.3.6.1}$$

This functor is $\Bbbk$-bilinear, it preserves products in both variables, and it is triangulated in both variables.

(a) *A is projective as an R-module.*
(b) *B is flat as an R^{o}-module.*

Further, (7.3.6.1) *is induced by functors $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$ as follows*:

(a′) *If* (a) *is satisfied, then* $\mathsf{RHom}_R(-,-)$ *is induced by* $\mathrm{Hom}_R(\mathrm{P}_A(-),-)$.
(b′) *If* (b) *is satisfied, then* $\mathsf{RHom}_R(-,-)$ *is induced by* $\mathrm{Hom}_R(-,\mathrm{I}_B(-))$.

PROOF. For clarity the restriction of scalars functors, see 6.1.23 and 6.4.36, are not suppressed in this proof. Consider the functor

$$\mathrm{D} = \operatorname{Hom}_R(-,-) \circ (\operatorname{res}^A_{R\otimes Q^\circ} \times \operatorname{res}^B_{R\otimes S^\circ}) = \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ}, \operatorname{res}^B_{R\otimes S^\circ}) .$$

Recall the natural transformations ϱ and ε from 6.3.21 and 6.3.22. The next diagram defines a natural transformation τ of functors from $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B)$ to $\mathcal{K}(\Bbbk)$,

$$\begin{tikzcd}
\operatorname{res}^{Q\otimes S^\circ}_{\Bbbk} \circ \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^\circ} \mathrm{I}_B) \arrow[r, equal] \arrow[dd, "\tau"'] & \operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \operatorname{res}^B_R \mathrm{I}_B) \arrow[d, "\operatorname{Hom}(1,\varepsilon^B_R)"] \\
& \operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \mathrm{I}_R \operatorname{res}^B_R) \arrow[d, "\operatorname{Hom}(\varrho^A_R,1)"] \\
\operatorname{Hom}_R(\mathrm{P}_R, \mathrm{I}_R) \circ (\operatorname{res}^A_R \times \operatorname{res}^B_R) \arrow[r, equal] & \operatorname{Hom}_R(\mathrm{P}_R \operatorname{res}^A_R, \mathrm{I}_R \operatorname{res}^B_R) .
\end{tikzcd}$$

Set $\mathrm{E} = \mathsf{R}\mathrm{D}$. Now 7.2.4 and 7.2.5, per 7.2.8, yield a natural transformation

$$\acute{\tau} : \operatorname{res}^{Q\otimes S^\circ}_{\Bbbk} \circ \mathrm{E} \longrightarrow \mathsf{R}\!\operatorname{Hom}_R(-,-) \circ (\operatorname{res}^A_R \times \operatorname{res}^B_R) .$$

To prove that E is an augmentation of $\mathsf{R}\!\operatorname{Hom}_R$, it suffices by 7.3.5 to show that $\varphi = \acute{\tau}$ is a natural isomorphism, i.e. that τ^{MN} is a quasi-isomorphism for every A-complex M and B-complex N. In the following, let M and N denote such complexes.

Recall from 6.3.21 and 6.3.22 that $(\varrho^A_R)^M$ and $(\varepsilon^B_R)^N$ are quasi-isomorphisms. The complex $\mathrm{I}_R(\operatorname{res}^B_R(N))$ is semi-injective, whence $\operatorname{Hom}_R((\varrho^A_R)^M, \mathrm{I}_R(\operatorname{res}^B_R(N)))$ is a quasi-isomorphism.

Assume first that condition (a) is satisfied. It follows from 5.2.23(b) that the R-complex $\operatorname{res}^A_R(\mathrm{P}_A(M))$ is semi-projective, so $\operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A(M), (\varepsilon^B_R)^N)$ is a quasi-isomorphism. Thus, τ^{MN} is a quasi-isomorphism as desired.

Assume now that condition (b) is satisfied. It follows from 6.3.22 that $(\varepsilon^B_R)^N$ is an isomorphism in $\mathcal{K}(R)$, whence τ^{MN} is a quasi-isomorphism.

Thus if (a) or (b) is satisfied, then one writes $\mathsf{R}\!\operatorname{Hom}_R$ for the functor E; see 7.3.5. It follows from 7.1.7 and 7.2.14 that $\mathrm{E} = \mathsf{R}\mathrm{D}$ is $\Bbbk$-bilinear, preserves products in both variables, and is triangulated in both variables.

(a′): With the restriction of scalars functors included, the claim is that the functors

$$\mathrm{E} = \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^\circ} \mathrm{I}_B)^{\prime\prime} \quad \text{and} \quad \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^\circ})^{\prime\prime}$$

from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^\circ)$ are naturally isomorphic; indeed, as already established the functor E is (7.3.6.1). Consider the natural transformation

$$\operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^\circ}) \xrightarrow{\sigma = \operatorname{Hom}(1, \operatorname{res}^B_{R\otimes S^\circ} \iota_B)} \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^\circ} \mathrm{I}_B)$$

of functors $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \to \mathcal{K}(Q\text{–}S^\circ)$. To see that the induced transformation $\acute{\sigma}$ is a natural isomorphism, let M and N be complexes in $\mathcal{K}(A)$ and $\mathcal{K}(B)$ respectively; it must be verified that σ^{MN} is a quasi-isomorphism. This follows as the complex $\operatorname{res}^A_{R\otimes Q^\circ}(\mathrm{P}_A(M))$ under condition (a) is semi-projective over R by 5.2.23(b), whence the functor $\operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^\circ}(\mathrm{P}_A(M)), -)$ preserves the quasi-isomorphism $\operatorname{res}^B_{R\otimes S^\circ}(\iota^N_B)$.

(b′): Proceeding as in part (a′), the natural transformation

$$\operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}}, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \mathrm{I}_B) \xrightarrow{\varkappa = \operatorname{Hom}(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}} \pi_A, 1)} \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \mathrm{I}_B)$$

compares the relevant functors from $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B)$ to $\mathcal{K}(Q\text{–}S^{\mathrm{o}})$. Under condition (b) the R-complex $\operatorname{res}^B_R(\mathrm{I}_B(N))$ is by 5.4.26(b) semi-injective for every B-complex N. Thus, for every A-complex M the functor $\operatorname{Hom}_R(-, \operatorname{res}^B_R(\mathrm{I}_B(N)))$ preserves the quasi-isomorphism $\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}}(\pi^M_A)$, whence $\varkappa^{MN}$ is a quasi-isomorphism. It follows that $\tilde{\varkappa}$ is a natural isomorphism of functors from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$. ◻

7.3.7. Consider the functor (7.3.6.1) and morphisms $\alpha/\varphi \colon M' \to M$ in $\mathcal{D}(A)$ and $\beta/\psi \colon N \to N'$ in $\mathcal{D}(B)$. If condition (a) in 7.3.6 is satisfied, then by 7.2.3 one has

$$\mathsf{RHom}_R(M,N) = \operatorname{Hom}_R(\mathrm{P}_A(M), N)$$

and

$$\mathsf{RHom}_R(\alpha/\varphi, \beta/\psi) = \frac{\operatorname{Hom}(\mathrm{P}_A(M'),\beta)}{\operatorname{Hom}(\mathrm{P}_A(\varphi),\psi)} \circ \frac{\operatorname{Hom}(\mathrm{P}_A(\alpha),N)}{1^{\operatorname{Hom}(\mathrm{P}_A(M),N)}}\ .$$

One can verify that the composite reduces to $\operatorname{Hom}(\mathrm{P}_A(\alpha)\,\mathrm{P}_A(\varphi)^{-1},\beta)/\operatorname{Hom}(\mathrm{P}_A(M),\psi)$.

Similarly, if condition (b) in 7.3.6 is satisfied, then one has

$$\mathsf{RHom}_R(M,N) = \operatorname{Hom}_R(M, \mathrm{I}_B(N))$$

and

$$\mathsf{RHom}_R(\alpha/\varphi, \beta/\psi) = \frac{\operatorname{Hom}(M',\mathrm{I}_B(\beta))}{\operatorname{Hom}(\varphi,\mathrm{I}_B(\psi))} \circ \frac{\operatorname{Hom}(\alpha,\mathrm{I}_B(N))}{1^{\operatorname{Hom}(M,\mathrm{I}_B(N))}}\ .$$

Applied with $S = \Bbbk$, the next result recovers the functor RHom_R from 7.3.1, now induced by a functor that only involves a resolution of the first variable.

7.3.8 Corollary. *The functor* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(S^{\mathrm{o}})\ ,$$

and this functor is induced by $\operatorname{Hom}_R(\mathrm{P}_R(-),-) : \mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(S^{\mathrm{o}})$.

Proof. Apply 7.3.6 with $Q = \Bbbk$, $A = R$, and $B = R \otimes_\Bbbk S^{\mathrm{o}}$. Condition (a) is now trivially satisfied; in particular, the last assertion follows from 7.3.6(a′). ◻

Applied with $Q = \Bbbk$, the next result recovers the functor RHom_R from 7.3.1, now induced by a functor that only involves a resolution of the second variable.

7.3.9 Corollary. *The functor* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}Q^{\mathrm{o}}) \times \mathcal{D}(R)^{\mathrm{op}} \longrightarrow \mathcal{D}(Q)\ ,$$

and this functor is induced by $\operatorname{Hom}_R(-,\mathrm{I}_R(-)) : \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(R) \longrightarrow \mathcal{K}(Q)$.

Proof. Apply 7.3.6 with $A = R \otimes_\Bbbk Q^{\mathrm{o}}$, $S = \Bbbk$, and $B = R$. Condition (b) is now trivially satisfied; in particular, the last assertion follows from 7.3.6(b′). ◻

The next example shows that the diagram (7.3.4.1) need not be commutative up to natural isomorphism.

7.3.10 Example. Set $\Bbbk = \mathbb{Z} = R$ and $A = Q = \mathbb{Z}/2\mathbb{Z} = S = B$; one then has

$$R \otimes_{\Bbbk} Q^{\mathrm{o}} = A = \mathbb{Z}/2\mathbb{Z} = B = R \otimes_{\Bbbk} S^{\mathrm{o}} = Q \otimes_{\Bbbk} S^{\mathrm{o}} .$$

The functor $\operatorname{Hom}_{\mathbb{Z}}(-,-)\colon \mathcal{K}(\mathbb{Z}/2\mathbb{Z})^{\mathrm{op}} \times \mathcal{K}(\mathbb{Z}/2\mathbb{Z}) \to \mathcal{K}(\mathbb{Z}/2\mathbb{Z})$ has a right derived functor $\mathrm{E}\colon \mathcal{D}(\mathbb{Z}/2\mathbb{Z})^{\mathrm{op}} \times \mathcal{D}(\mathbb{Z}/2\mathbb{Z}) \to \mathcal{D}(\mathbb{Z}/2\mathbb{Z})$ which, since $\mathbb{Z}/2\mathbb{Z}$ is a field is given by $\mathrm{E} = \operatorname{Hom}_{\mathbb{Z}}(-,-)$. Thus, for $M = \mathbb{Z}/2\mathbb{Z} = N$ one has

$$\mathrm{E}(M,N) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} .$$

By 7.3.8 the complex $\mathsf{RHom}_{\mathbb{Z}}(M,N)$ can be computed as $\operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}/2\mathbb{Z})$, where P is the semi-projective $\mathbb{Z}$-complex $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0$ concentrated in degrees 1 and 0. Evidently, the complex

$$\mathsf{RHom}_{\mathbb{Z}}(M,N) = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}/2\mathbb{Z}) = 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

has non-zero homology in degrees 0 and -1, so it is not isomorphic to $\mathrm{E}(M,N)$ in the derived category $\mathcal{D}(\mathbb{Z})$.

To identify non-trivial circumstances under which condition (a) or (b) in 7.3.6 is satisfied, we record a few special cases of results from Chap. 5.

7.3.11 Lemma. *The following assertions hold.*

(a) *If Q is projective as a $\Bbbk$-module, then $R \otimes_{\Bbbk} Q^{\mathrm{o}}$ is projective as an R-module and every semi-projective $R \otimes_{\Bbbk} Q^{\mathrm{o}}$-complex is semi-projective as an R-complex.*

(b) *If Q is flat as a $\Bbbk$-module, then $R \otimes_{\Bbbk} Q^{\mathrm{o}}$ is flat as an R-module and every semi-flat $R \otimes_{\Bbbk} Q^{\mathrm{o}}$-complex is semi-flat as an R-complex.*

(c) *If S is flat as a $\Bbbk$-module, then $R \otimes_{\Bbbk} S^{\mathrm{o}}$ is flat as an R^{o}-module and every semi-injective $R \otimes_{\Bbbk} S^{\mathrm{o}}$-complex is semi-injective as an R-complex.*

Proof. The assertions in (a) follow from 5.2.25 and 5.2.23(b), those in (b) follow from 5.4.23 and 5.4.18(b), and those in (c) follow from 5.4.23 and 5.4.26(b). □

7.3.12 Proposition. *Assume that Q is projective as a $\Bbbk$-module. The functor* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) ;$$

it is induced by $\operatorname{Hom}_R(\mathrm{P}_{R\otimes_{\Bbbk}Q^{\mathrm{o}}}(-),-) : \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$.

Proof. Apply 7.3.6 with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$ and $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$; condition (a) is satisfied per 7.3.11(a), so the conclusion follows from 7.3.6(a′). □

7.3.13 Proposition. *Assume that S is flat as a $\Bbbk$-module. The functor* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) ;$$

it is induced by $\operatorname{Hom}_R(-,\mathrm{I}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(-)) : \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$.

Proof. Apply 7.3.6 with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$ and $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$; condition (b) is satisfied per 7.3.11(c), so the conclusion follows from 7.3.6(b′). □

Note that 7.3.8 and 7.3.9 are special cases of 7.3.12 and 7.3.13 with $Q = \Bbbk$ and $S = \Bbbk$, respectively. Further applications of 7.3.12 and 7.3.13 can be obtained from the following example.

7.3.14 Example. Examples of $\Bbbk$-algebras that are free as $\Bbbk$-modules are matrix algebras $\mathrm{M}_{n\times n}(\Bbbk)$ as well as polynomial algebras $\Bbbk[x_1,\dots,x_n]$ and their truncations $\Bbbk[x_1,\dots,x_n]/(x_1,\dots,x_n)^p$; in particular, these algebras are faithfully flat as $\Bbbk$-modules by 1.3.43. If $\Bbbk$ is Noetherian, then power series algebras $\Bbbk[\![x_1,\dots,x_n]\!]$ are flat as $\Bbbk$-modules, see 8.3.26; per 1.3.43 they are even faithfully flat as $\Bbbk$-modules as they have $\Bbbk$ as a direct summand.

If $\Bbbk$ is a field—or for that matter a von Neumann regular ring, see 8.5.8—then every $\Bbbk$-algebra is flat as a $\Bbbk$-module, so RHom_R is invariably augmented to a functor $\mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}})$.

In the literature on derived Hom functors over non-commutative rings, it is fairly standard to make the blanket assumption that all rings are algebras over some fixed field; the example above makes it clear why that is convenient. For commutative rings there is, however, no need for that assumption. Together with the one that follows, the example demonstrates how the setup with the extra layer of rings, A and B, allows us to treat the commutative and non-commutative cases together.

7.3.15 Example. Assume that R is commutative and recall that R-modules are considered to be symmetric R–R-bimodules, in particular, they are modules over the enveloping algebra R^{e}; see 1.1.28. It is elementary to verify that the multiplication map $R^{\mathrm{e}} \to R$ given by $x \otimes y \mapsto xy$ is a ring homomorphism, and that the composite $R \to R^{\mathrm{e}} \to R$ is the identity. Thus, with $Q = R = A$ condition (a) in 7.3.6 is satisfied and RHom_R is augmented to a functor $\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R\text{–}S^{\mathrm{o}})$. Similarly, with $S = R = B$ condition (b) in 7.3.6 is satisfied and RHom_R is augmented to a functor $\mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(Q\text{–}R)$.

Semi-Projective and Semi-Injective Replacements

7.3.16 Definition. Let M be an R-complex. A semi-projective R-complex that is isomorphic to M in $\mathcal{D}(R)$ is called a *semi-projective replacement* of M.

The distinction between a replacement and a resolution is subtle but convenient, one is a complex the other a morphism. They are, though, two sides of a coin. A semi-projective resolution evidently yields a semi-projective replacement, and the converse is also true:

7.3.17 Proposition. *Let M, P, and P' be R-complexes. If P and P' are semi-projective replacements of M, then there is a homotopy equivalence $P' \longrightarrow P$ and a quasi-isomorphism $P \longrightarrow M$, i.e. a semi-projective resolution.*

Proof. By assumption there are isomorphisms $P' \simeq P \simeq M$ in $\mathcal{D}(R)$, so 6.4.20 yields quasi-isomorphisms $P' \to P \to M$. By 5.2.21 the first of these is a homotopy equivalence. $\square$

7.3.18 Definition. Let M be an R-complex. A semi-injective R-complex that is isomorphic to M in $\mathcal{D}(R)$ is called a *semi-injective replacement* of M.

7.3.19 Proposition. *Let M, I, and I' be R-complexes. If I and I' are semi-injective replacements of M, then there is a homotopy equivalence $I \longrightarrow I'$ and a quasi-isomorphism $M \longrightarrow I$, i.e. a semi-injective resolution.*

Proof. By assumption there are isomorphisms $M \simeq I \simeq I'$ in $\mathcal{D}(R)$, so 6.4.21 yields quasi-isomorphisms $M \to I \to I'$. By 5.3.24 the first of these is a homotopy equivalence. □

7.3.20 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$ and $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$ be ring homomorphisms.*

(a) *Assume that A is projective as an R-module. For an A-complex M with a semi-projective replacement P, the functor from* 7.3.6

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(M,-) : \mathcal{D}(B) &\longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \quad \textit{is induced by} \\ \mathrm{Hom}_R(P,-) : \mathcal{K}(B) &\longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}}) \,. \end{aligned}$$

(b) *Assume that B is flat as an R^{o}-module. For a B-complex N with a semi-injective replacement I, the functor from* 7.3.6

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(-,N) : \mathcal{D}(A)^{\mathrm{op}} &\longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \quad \textit{is induced by} \\ \mathrm{Hom}_R(-,I) : \mathcal{K}(A)^{\mathrm{op}} &\longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}}) \,. \end{aligned}$$

Proof. (a): The functor $\mathsf{R}\mathrm{Hom}_R(M,-)$ is by 7.3.6(a′) induced by $\mathrm{Hom}_R(\mathrm{P}_A(M),-)$. By 7.3.17 and 6.1.6 there is an isomorphism $\varphi\colon \mathrm{P}_A(M) \to P$ in $\mathcal{K}(A)$, and hence a natural isomorphism $\tau = \mathrm{Hom}_R(\varphi,-)\colon \mathrm{Hom}_R(P,-) \to \mathrm{Hom}_R(\mathrm{P}_A(M),-)$ of functors from $\mathcal{K}(B)$ to $\mathcal{K}(Q\text{–}S^{\mathrm{o}})$. In particular, the functor $\mathrm{Hom}_R(P,-)$ preserves quasi-isomorphisms. Thus $\tau''\colon \mathrm{Hom}_R(P,-)'' \to \mathsf{R}\mathrm{Hom}_R(M,-)$ is a natural isomorphism of functors, see 7.2.5.

(b): A parallel argument applies. □

The next two corollaries compare to 7.3.8 and 7.3.9.

7.3.21 Corollary. *For an R-complex M with a semi-projective replacement P, the functor*

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(M,-) : \mathcal{D}(R\text{–}S^{\mathrm{o}}) &\longrightarrow \mathcal{D}(S^{\mathrm{o}}) \quad \textit{is induced by} \\ \mathrm{Hom}_R(P,-) : \mathcal{K}(R\text{–}S^{\mathrm{o}}) &\longrightarrow \mathcal{K}(S^{\mathrm{o}}) \,. \end{aligned}$$

Proof. Apply 7.3.20(a) with $Q = \Bbbk$, $A = R$, and $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$. □

7.3.22 Corollary. *For an R-complex N with a semi-injective replacement I, the functor*

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(-,N) : \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} &\longrightarrow \mathcal{D}(Q) \quad \textit{is induced by} \\ \mathrm{Hom}_R(-,I) : \mathcal{K}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} &\longrightarrow \mathcal{K}(Q) \,. \end{aligned}$$

Proof. Apply 7.3.20(b) with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$, $S = \Bbbk$, and $B = R$. □

EXT FUNCTORS

For the next definition, recall from 6.5.17 that complexes that are isomorphic in the derived category have isomorphic homology. Recall also that 7.3.6 offer conditions under wich the opening assumption in the definition below is satisfied.

7.3.23 Definition. Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$ and $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$ be ring homomorphisms and assume that RHom_R per 7.3.5 is augmented to a functor,

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) .$$

For every $m \in \mathbb{Z}$ denote by $\mathrm{Ext}_R^m(-,-)$ the functor

$$\mathrm{H}_{-m}(\mathsf{RHom}_R(-,-)) : \mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \longrightarrow \mathcal{M}(Q\text{–}S^{\mathrm{o}}) .$$

It follows from 7.3.6 and 6.5.17 that the functors Ext_R^m are $\Bbbk$-bilinear. The choice of the symbol ‘Ext’ is justified in the Remark after 7.3.30.

7.3.24. Let M and N be R-complexes. Per 6.5.18 there are, with the usual conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, equalities,

$$\begin{aligned} \inf\{m \in \mathbb{Z} \mid \mathrm{Ext}_R^m(M,N) \neq 0\} &= -\sup \mathsf{RHom}_R(M,N) \quad \text{and} \\ \sup\{m \in \mathbb{Z} \mid \mathrm{Ext}_R^m(M,N) \neq 0\} &= -\inf \mathsf{RHom}_R(M,N) . \end{aligned}$$

7.3.25. In view of 7.3.8 and 7.3.9 there are Ext functors,

$$\begin{aligned} \mathrm{Ext}_R^m(-,-) &: \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{M}(S^{\mathrm{o}}) \quad \text{and} \\ \mathrm{Ext}_R^m(-,-) &: \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{M}(Q) . \end{aligned}$$

The Ext functors are closely related to hom-sets in the derived category.

7.3.26 Proposition. *Let M and N be R-complexes. For every integer m there is an isomorphism of $\Bbbk$-modules,*

$$\mathrm{Ext}_R^m(M,N) \cong \mathcal{D}(R)(M, \Sigma^m N) ,$$

and it is natural in M and N.

PROOF. There are isomorphisms of $\Bbbk$-modules, which are all natural in M and N,

$$\begin{aligned} \mathrm{Ext}_R^m(M,N) &= \mathrm{H}_0(\Sigma^m \mathsf{RHom}_R(M,N)) \\ &\cong \mathrm{H}_0(\mathsf{RHom}_R(M,\Sigma^m N)) \\ &\cong \mathrm{H}_0(\mathrm{Hom}_R(\mathrm{P}(M),\mathrm{P}(\Sigma^m N))) \\ &\cong \mathcal{K}(R)(\mathrm{P}(M),\mathrm{P}(\Sigma^m N)) \\ &\cong \mathcal{D}(R)(M,\Sigma^m N) . \end{aligned}$$

The equality holds by 6.5.17 and the definition of Ext. The 1st isomorphism follows as $\mathsf{RHom}_R(M,-)$ is triangulated; see 7.3.6. The 2nd isomorphism follows from 7.3.8 and the isomorphism $\Sigma^m N \simeq \mathrm{P}(\Sigma^m N)$ in $\mathcal{D}(R)$, the 3rd holds by 6.1.2, and the 4th holds by 6.4.8. □

For modules there is an even simpler description of Ext^0.

7.3.27 Corollary. *Let M and N be R-modules. There is an isomorphism of $\Bbbk$-modules,*

$$\mathrm{Ext}_R^0(M,N) \;\cong\; \mathrm{Hom}_R(M,N)\,,$$

and it is natural in M and N. Furthermore, $\mathrm{Ext}_R^m(M,N) = 0$ holds for all $m < 0$.

Proof. The isomorphism follows from 7.3.26 and 6.4.15. A projective resolution $P \xrightarrow{\simeq} M$ has $P_m = 0$ for $m < 0$, see 5.2.28. Thus $\mathrm{Hom}_R(P,N)_{-m} = 0$ holds for all $m < 0$, and since the complex $\mathrm{Hom}_R(P,N)$ is $\mathsf{R}\mathrm{Hom}_R(M,N)$ by 7.3.21, it follows that $\mathrm{Ext}_R^m(M,N) = 0$ holds for all $m < 0$. □

Remark. In classic homological algebra it is standard to refer to Ext_R^m as the 'm^{th} right derived functor' of Hom_R considered as a functor on R-modules. This explains why $\mathsf{R}\mathrm{Hom}_R$ in some places is called the 'total right derived functor' of Hom_R; see also the Remark after 7.2.8.

7.3.28 Example. The complex $0 \longrightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$, concentrated in degrees 0 and -1, yields an injective resolution of $\mathbb{Z}$. By 7.3.27 one has $\mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Q},\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$; that is, the induced map,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}) \xrightarrow{\mathrm{Hom}(\mathbb{Q},\pi)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})\,,$$

is injective. Recall from C.13 that $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}(2^\infty)$ is a direct summand of $\mathbb{Q}/\mathbb{Z}$. The composite $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \twoheadrightarrow \mathbb{Z}(2^\infty) \rightarrowtail \mathbb{Q}/\mathbb{Z}$ is not in the image of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\pi)$; indeed, no homomorphism $\mathbb{Q} \to \mathbb{Q}$ maps, say, every fraction $\frac{1}{3^n}$ to an integer. Thus one has $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Q},\mathbb{Z}) \neq 0$.

7.3.29 Lemma. *Let M be an R-complex and n an integer. If $\mathrm{H}_v(M) = 0$ holds for all $v \neq n$, then there is an isomorphism $M \simeq \Sigma^n \mathrm{H}_n(M)$ in $\mathcal{D}(R)$.*

Proof. There are quasi-isomorphisms $M \xleftarrow{\tau_{\supseteq n}^M} M_{\supseteq n} \xrightarrow{\tau_{\subseteq n}^{M_{\supseteq n}}} (M_{\supseteq n})_{\subseteq n} = \Sigma^n \mathrm{H}_n(M)$ by 4.2.4. The assertion now follows from 6.4.18. □

7.3.30 Proposition. *Let M and N be R-modules. If $\mathrm{Ext}_R^m(M,N) = 0$ holds for all $m > 0$, then there is an isomorphism $\mathsf{R}\mathrm{Hom}_R(M,N) \simeq \mathrm{Hom}_R(M,N)$ in $\mathcal{D}(\Bbbk)$.*

Proof. The assertion follows immediately from 7.3.27 and 7.3.29. □

Remark. Let M and N be R-modules. Two extensions of M by N,

$$\Xi = 0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0 \qquad\text{and}\qquad \Xi' = 0 \longrightarrow N \longrightarrow X' \longrightarrow M \longrightarrow 0\,,$$

are equivalent, in symbols $\Xi \sim \Xi'$, if there is a homomorphism $X \to X'$ that makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & X' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

commutative. It follows from the Five Lemma 1.1.2 that the map $X \to X'$ is an isomorphism and, consequently, $\sim$ is an equivalence relation. Because it was first studied by Yoneda [260], the set of equivalence classes of extensions of M by N is called *Yoneda Ext* and denoted by $\mathrm{YExt}_R^1(M,N)$. It can be shown that $\mathrm{YExt}_R^1(M,N)$ is an Abelian group under the so-called Baer sum, defined via pullbacks. It follows from the proof of 2.1.47 that the set of all trivial extensions of M by N constitutes an equivalence class under $\sim$; this is the zero element in the group $\mathrm{YExt}^1_R(M,N)$. It can also be shown that YExt^1_R yields a functor from $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R)$ to $\mathcal{M}(\mathbb{Z})$, and that there is an isomorphism of Abelian groups,

$$\mathrm{YExt}^1_R(M,N) \cong \mathrm{Ext}^1_R(M,N)\,,$$

which is natural in M and N. One consequence of this isomorphism is that $\mathrm{Ext}^1_R(M,N) = 0$ holds if and only if every extension of M by N is trivial. For a proof of this last assertion that works directly with the definition of Ext, see 7.3.36.

7.3.31 Proposition. *Let M and N be R-complexes and m and s be integers. There are isomorphisms in $\mathcal{M}(\Bbbk)$,*

$$\mathrm{Ext}^m_R(\Sigma^{-s}M,N) \cong \mathrm{Ext}^{m+s}_R(M,N) \cong \mathrm{Ext}^m_R(M,\Sigma^s N)\,,$$

and they are natural in M and N.

Proof. The functor $\mathsf{RHom}_R(-,-)$ is triangulated in both variables by 7.3.6, so there are isomorphisms in $\mathcal{D}(\Bbbk)$,

$$\mathsf{RHom}_R(\Sigma^{-s}M,N) \simeq \Sigma^s\,\mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_R(M,\Sigma^s N)\,,$$

which are natural in M and N. Apply the functor H_{-m} to these isomorphisms and recall from 6.5.17 that one has $\mathrm{H}_{-m}\,\Sigma^s = \mathrm{H}_{-(m+s)}$. The assertion now follows from the definition, 7.3.23, of Ext. $\square$

7.3.32 Proposition. *Let N be an R-complex and $\{M^u\}_{u\in U}$ a family of R-complexes. For every $m \in \mathbb{Z}$ there is an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\mathrm{Ext}^m_R\Big(\coprod_{u\in U} M^u, N\Big) \cong \prod_{u\in U} \mathrm{Ext}^m_R(M^u,N)\,.$$

Proof. As the functors $\mathsf{RHom}_R(-,N)\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(\Bbbk)$ and $\mathrm{H}_{-m}\colon \mathcal{D}(\Bbbk) \to \mathcal{M}(\Bbbk)$ preserve products, see 7.3.6 and 6.5.17, so does the composite functor $\mathrm{Ext}^m_R(-,N)$, see 7.3.23. $\square$

7.3.33 Proposition. *Let M be an R-complex and $\{N^u\}_{u\in U}$ a family of R-complexes. For every $m \in \mathbb{Z}$ there is an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\mathrm{Ext}^m_R\Big(M, \prod_{u\in U} N^u\Big) \cong \prod_{u\in U} \mathrm{Ext}^m_R(M,N^u)\,.$$

Proof. As the functors $\mathsf{RHom}_R(M,-)\colon \mathcal{D}(R) \to \mathcal{D}(\Bbbk)$ and $\mathrm{H}_{-m}\colon \mathcal{D}(\Bbbk) \to \mathcal{M}(\Bbbk)$ preserve products, see 7.3.6 and 6.5.17, so does the composite functor $\mathrm{Ext}^m_R(M,-)$, see 7.3.23. $\square$

7.3.34 Proposition. *Assume that R is left Noetherian. Let M be an R-module and $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{M}(R)$. If M is finitely generated and U is filtered, then there is for every $m \in \mathbb{Z}$ an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\operatorname*{colim}_{u\in U} \mathrm{Ext}^m_R(M,N^u) \cong \mathrm{Ext}^m_R\big(M, \operatorname*{colim}_{u\in U} N^u\big)\,.$$

PROOF. Choose per 5.1.19 a degreewise finitely generated free resolution $L \xrightarrow{\simeq} M$. By 3.3.15(d) and degreewise application of 3.3.17 one gets isomorphisms,

$$\begin{aligned}\operatorname*{colim}_{u\in U} \mathrm{H}_{-m}(\mathrm{Hom}_R(L, N^u)) &\cong \mathrm{H}_{-m}\big(\operatorname*{colim}_{u\in U} \mathrm{Hom}_R(L, N^u)\big)\\ &\cong \mathrm{H}_{-m}\big(\,\mathrm{Hom}_R\big(L, \operatorname*{colim}_{u\in U} N^u\big)\big)\,.\end{aligned}$$

The assertion now follows from 7.3.21 and 7.3.23. □

Per 6.5.24 the next result applies, in particular, to a commutative diagram in $\mathcal{C}(R)$ whose rows are short exact sequences. It shows that, viewed as functors from $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R)$ to $\mathcal{M}(\Bbbk)$, cf. 6.4.15, the Ext functors are half exact in either variable.

7.3.35 Theorem. *Consider a morphism of distinguished triangles in* $\mathcal{D}(R)$,

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r, "\alpha''"] \arrow[d, "\varphi''"] & \Sigma M' \arrow[d, "\Sigma\varphi'"] \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \arrow[r, "\beta''"] & \Sigma N'\,. \end{tikzcd}$$

For every R-complex X there is a commutative diagram in $\mathcal{M}(\Bbbk)$ *with exact rows,*

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Ext}_R^m(X,M') \arrow[r, "\mathrm{Ext}_R^m(X,\alpha')"] \arrow[d, "\mathrm{Ext}_R^m(X,\varphi')"] & \mathrm{Ext}_R^m(X,M) \arrow[r, "\mathrm{Ext}_R^m(X,\alpha)"] \arrow[d, "\mathrm{Ext}_R^m(X,\varphi)"] & \mathrm{Ext}_R^m(X,M'') \arrow[r, "\mathrm{Ext}_R^m(X,\alpha'')"] \arrow[d, "\mathrm{Ext}_R^m(X,\varphi'')"] & \mathrm{Ext}_R^{m+1}(X,M') \arrow[r, "\mathrm{Ext}_R^{m+1}(X,\alpha')"] \arrow[d, "\mathrm{Ext}_R^{m+1}(X,\varphi')"] & \cdots \\ \cdots \arrow[r] & \mathrm{Ext}_R^m(X,N') \arrow[r, "\mathrm{Ext}_R^m(X,\beta')"'] & \mathrm{Ext}_R^m(X,N) \arrow[r, "\mathrm{Ext}_R^m(X,\beta)"'] & \mathrm{Ext}_R^m(X,N'') \arrow[r, "\mathrm{Ext}_R^m(X,\beta'')"'] & \mathrm{Ext}_R^{m+1}(X,N') \arrow[r, "\mathrm{Ext}_R^{m+1}(X,\beta')"'] & \cdots \end{tikzcd}$$

where the isomorphisms $\mathrm{Ext}_R^m(X,\Sigma-) \cong \mathrm{Ext}_R^{m+1}(X,-)$ *from* 7.3.31 *are supressed.*

For every R-complex Y there is a commutative diagram in $\mathcal{M}(\Bbbk)$ *with exact rows,*

$$\begin{tikzcd} \cdots \arrow[r, "\mathrm{Ext}_R^{m-1}(\beta',Y)"] & \mathrm{Ext}_R^{m-1}(N',Y) \arrow[r, "\mathrm{Ext}_R^m(\beta'',Y)"] \arrow[d, "\mathrm{Ext}_R^{m-1}(\varphi',Y)"] & \mathrm{Ext}_R^m(N'',Y) \arrow[r, "\mathrm{Ext}_R^m(\beta,Y)"] \arrow[d, "\mathrm{Ext}_R^m(\varphi'',Y)"] & \mathrm{Ext}_R^m(N,Y) \arrow[r, "\mathrm{Ext}_R^m(\beta',Y)"] \arrow[d, "\mathrm{Ext}_R^m(\varphi,Y)"] & \mathrm{Ext}_R^m(N',Y) \arrow[r] \arrow[d, "\mathrm{Ext}_R^m(\varphi',Y)"] & \cdots \\ \cdots \arrow[r, "\mathrm{Ext}_R^{m-1}(\alpha',Y)"'] & \mathrm{Ext}_R^{m-1}(M',Y) \arrow[r, "\mathrm{Ext}_R^m(\alpha'',Y)"'] & \mathrm{Ext}_R^m(M'',Y) \arrow[r, "\mathrm{Ext}_R^m(\alpha,Y)"'] & \mathrm{Ext}_R^m(M,Y) \arrow[r, "\mathrm{Ext}_R^m(\alpha',Y)"'] & \mathrm{Ext}_R^m(M',Y) \arrow[r] & \cdots \end{tikzcd}$$

where the isomorphisms $\mathrm{Ext}_R^m(\Sigma-,Y) \cong \mathrm{Ext}_R^{m-1}(-,Y)$ *from* 7.3.31 *are supressed.*

PROOF. The functor $\mathsf{R}\mathrm{Hom}_R(-,-)$ is triangulated in both variables by 7.3.6, so the assertions follow from 6.5.21 and 6.5.22. □

Ext vanishing has a very tangible interpretation; see also the Remark after 7.3.30.

7.3.36 Proposition. *Let M and N be R-modules. One has $\operatorname{Ext}^1_R(M,N) = 0$ if and only if every short exact sequence of the form $0 \to N \to X \to M \to 0$ is split.*

Proof. "Only if": Let $0 \longrightarrow N \longrightarrow X \xrightarrow{\pi} M \longrightarrow 0$ be an exact sequence of R-modules and apply 7.3.35 and 7.3.27 to get an exact sequence of $\Bbbk$-modules,

$$\operatorname{Hom}_R(M,X) \xrightarrow{\operatorname{Hom}(M,\pi)} \operatorname{Hom}_R(M,M) \longrightarrow \operatorname{Ext}^1_R(M,N)\,.$$

Since one has $\operatorname{Ext}^1_R(M,N) = 0$ the map $\operatorname{Hom}_R(M,\pi)$ is surjective, so there exists a homomorphism $\sigma\colon M \to X$ with $1^M = \operatorname{Hom}_R(M,\pi)(\sigma) = \pi\sigma$.

"If": By 1.3.12 there is an exact sequence $0 \longrightarrow K \xrightarrow{\iota} L \longrightarrow M \longrightarrow 0$ of R-modules with L free; by 7.3.35 and 7.3.27 yields an exact sequence of $\Bbbk$-modules,

$$\operatorname{Hom}_R(L,N) \xrightarrow{\operatorname{Hom}(\iota,N)} \operatorname{Hom}_R(K,N) \longrightarrow \operatorname{Ext}^1_R(M,N) \longrightarrow \operatorname{Ext}^1_R(L,N)\,.$$

As L by 1.3.18 is a semi-projective replacement of itself, it follows from 7.3.21 and the definition, 7.3.23, of Ext that one has $\operatorname{Ext}^1_R(L,N) = 0$. Thus, to prove that $\operatorname{Ext}^1_R(M,N) = 0$ holds, it suffices to see that $\operatorname{Hom}_R(\iota,N)$ is surjective. Let $\alpha\colon K \to N$ be a homomorphism. By 3.2.28 there is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, "\alpha"] & L \arrow[r] \arrow[d, "\alpha'"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota'"] & N \sqcup_K L \arrow[r] & M \arrow[r] & 0\,.
\end{tikzcd}$$

By assumption, the lower exact sequece is split, so there exists a homomorphism $\varrho\colon N \sqcup_K L \to N$ with $\varrho\iota' = 1^N$. Thus $\operatorname{Hom}_R(\iota,N)(\varrho\alpha') = \varrho\alpha'\iota = \varrho\iota'\alpha = \alpha$. $\square$

7.3.37 Example. The restriction of scalars functor $\mathcal{D}(\Bbbk[x]) \to \mathcal{D}(\Bbbk)$ induced by the structure map $\Bbbk \to \Bbbk[x]$ is not faithful. By 7.3.26 one has

$$\mathcal{D}(\Bbbk[x])(\Bbbk, \Sigma\Bbbk[x]) \cong \operatorname{Ext}^1_{\Bbbk[x]}(\Bbbk,\Bbbk[x]) \quad\text{and}\quad \mathcal{D}(\Bbbk)(\Bbbk,\Sigma\Bbbk[x]) \cong \operatorname{Ext}^1_{\Bbbk}(\Bbbk,\Bbbk[x])\,.$$

From the definition of Ext, 7.3.23, it is immediate that $\operatorname{Ext}^1_{\Bbbk}(\Bbbk,\Bbbk[x]) = 0$ holds. On the other hand, one has $\operatorname{Ext}^1_{\Bbbk[x]}(\Bbbk,\Bbbk[x]) \neq 0$ by 7.3.36 as the exact sequence $0 \longrightarrow \Bbbk[x] \xrightarrow{x} \Bbbk[x] \longrightarrow \Bbbk \longrightarrow 0$ does not split in $\mathcal{M}(\Bbbk[x])$.

The final two results of this section are not needed before Chap. 9, but they are standard applications of 7.3.35 and natural to record here.

7.3.38 Proposition. *Let M be an R-complex and $0 \to N' \to N \to N'' \to 0$ an exact sequence in $\mathcal{C}(R)$. If $\operatorname{Ext}^1_R(M_v,N'_i) = 0$ holds for all $v,i \in \mathbb{Z}$, then the sequence of complexes $0 \to \operatorname{Hom}_R(M,N') \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N'') \to 0$ is exact.*

Proof. For every $i \in \mathbb{Z}$ there is an exact sequence $0 \to N'_i \to N_i \to N''_i \to 0$ of R-modules. By 7.3.35 and 7.3.27 it yields for every $v \in \mathbb{Z}$ an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M_v,N'_i) \longrightarrow \operatorname{Hom}_R(M_v,N_i) \longrightarrow \operatorname{Hom}_R(M_v,N''_i) \longrightarrow \operatorname{Ext}^1_R(M_v,N'_i)\,.$$

The assertion now follows from 2.3.17. $\square$

7.3.39 Proposition. *Let N be an R-complex and $0 \to M' \to M \to M'' \to 0$ an exact sequence in $\mathcal{C}(R)$. If $\operatorname{Ext}^1_R(M''_v, N_i) = 0$ holds for all $v, i \in \mathbb{Z}$, then the sequence of complexes $0 \to \operatorname{Hom}_R(M'', N) \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N) \to 0$ is exact.*

Proof. An argument parallel to the proof of 7.3.38 applies, only one uses 2.3.19 instead of 2.3.17. □

Exercises

E 7.3.1 Let P be a faithfully projective R-module and ξ be a morphism in $\mathcal{D}(R)$. Show that if $\mathsf{R}\!\operatorname{Hom}_R(P, \xi)$ is an isomorphism, then ξ is an isomorphism.

E 7.3.2 Let I be a faithfully injective R-module and ξ be a morphism in $\mathcal{D}(R)$. Show that if $\mathsf{R}\!\operatorname{Hom}_R(\xi, I)$ is an isomorphism, then ξ is an isomorphism.

E 7.3.3 Show that the $\Bbbk$-linear map $R \otimes_{\Bbbk} R^{\mathrm{o}} \to R$ given by $x \otimes y \mapsto xy$ is a ring homomorphism if and only if R is commutative.

E 7.3.4 Assume that R is semi-simple. Show that for R-complexes M and N there is an isomorphism $\mathsf{R}\!\operatorname{Hom}_R(M, N) \simeq \operatorname{Hom}_R(\mathrm{H}(M), \mathrm{H}(N))$ in $\mathcal{D}(\Bbbk)$.

E 7.3.5 Let M be an R-module and $P \xrightarrow{\simeq} M$ a projective resolution. Set $Z = \mathrm{Z}_0(P)$ and let ι denote the embedding $Z \rightarrowtail P_0$. Show that for every R-module N there is an isomorphism $\operatorname{Ext}^1_R(M, N) \cong \operatorname{Hom}_R(Z, N)/\operatorname{Im}\operatorname{Hom}_R(\iota, N)$. Conclude that M is projective if and only if $\operatorname{Ext}^1_R(M, -) = 0$ if and only if $\operatorname{Ext}^1_R(M, Z) = 0$.

E 7.3.6 Use Ext to solve E 1.4.7.

E 7.3.7 Assume that R is a principal left ideal domain. Show that every finitely generated R-module M with $\operatorname{Ext}^1_R(M, R) = 0$ is free.

E 7.3.8 Show that for every torsion $\mathbb{Z}$-module M one has $\operatorname{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

E 7.3.9 Let M be a $\mathbb{Z}$-module and $n \in \mathbb{N}$. Show that one has $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) \cong M/nM$.

E 7.3.10 Show that every finite $\mathbb{Z}$-module has the form $\operatorname{Ext}^1_{\mathbb{Z}}(M, N)$ for suitable $\mathbb{Z}$-modules M and N. *Hint:* E 7.3.9.

E 7.3.11 Let M be a $\mathbb{Z}$-module. Show that $\operatorname{Ext}^m_{\mathbb{Z}}(M, -) = 0 = \operatorname{Ext}^m_{\mathbb{Z}}(-, M)$ holds for $m \geqslant 2$. Conclude that the functors $\operatorname{Ext}^1_{\mathbb{Z}}(M, -)$ and $\operatorname{Ext}^1_{\mathbb{Z}}(-, M)$ are right exact.

E 7.3.12 An R-module E is called *fp-injective* if $\operatorname{Ext}^1_R(M, E) = 0$ holds for every finitely presented R-module M. Show that for an R^{o}-module F the next conditions are equivalent: (*i*) F is flat; (*ii*) $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is injective; (*iii*) $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is fp-injective.

E 7.3.13 Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of R-modules. Show that if E' and E'' are fp-injective, then E is fp-injective.

E 7.3.14 Show that an R-module E is fp-injective if and only if every short exact sequence $0 \to E \to M \to N \to 0$ is pure.

E 7.3.15 Let E be an fp-injective R-module. Show that an exact sequence $0 \to M \to E \to N \to 0$ is pure if and only if M is fp-injective.

E 7.3.16 Show that an R-module is injective if and only if it is pure-injective and fp-injective.

E 7.3.17 Show that ring homomorphisms $R \otimes_{\Bbbk} S \to T$ are in one-to-one correspondence with pairs of ring homomorphisms $R \to T$ and $S \to T$ with commuting images.

E 7.3.18 Let $R \to S$ be a ring homomorphism. Show that S is projective as an R-module if and only if every semi-projective S-complex is semi-projective over R.

E 7.3.19 Let $R \to S$ be a ring homomorphism. Show that S is flat as an R^{o}-module if and only if every semi-injective S-complex is semi-injective over R.

7.4 Derived Tensor Product Functor

SYNOPSIS. The functor $\otimes^{\mathsf{L}}$; augmentation of $\otimes^{\mathsf{L}}$; semi-flat replacement; Tor functors; exact Tor sequence.

For tensor product functors it is standard to write $- \otimes^{\mathsf{L}} -$ rather than $\mathsf{L}(- \otimes -)$.

7.4.1 Definition. The functor $-\otimes_R -\colon \mathcal{K}(R^{\mathrm{o}}) \times \mathcal{K}(R) \to \mathcal{K}(\Bbbk)$ from 7.1.8 has by 7.2.8 a left derived functor, written

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R) \longrightarrow \mathcal{D}(\Bbbk) \; ;$$

it is per 7.2.7 induced by the functor $\mathrm{P}_{R^{\mathrm{o}}}(-) \otimes_R \mathrm{P}_R(-)$.

Additional ring actions on an R^{o}-complex M or an R-complex N yield additional actions on the tensor product complex $M \otimes_R N$; see 7.1.9. To what extent such additional actions carry over to the derived tensor product $M \otimes^{\mathsf{L}}_R N$ is the first question addressed in this section. See also the discussion after 7.3.1.

AUGMENTATION

7.4.2 Setup. Consider ring homomorphisms

$$Q \otimes_{\Bbbk} R^{\mathrm{o}} \longrightarrow A \qquad \text{and} \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B \,.$$

A functor $\mathrm{E}\colon \mathcal{D}(A) \times \mathcal{D}(B) \to \mathcal{D}(Q\text{–}S^{\mathrm{o}})$ can be compared to the functor $\otimes^{\mathsf{L}}_R$ from 7.4.1 via the diagram,

(7.4.2.1)
$$\begin{tikzcd}
\mathcal{D}(A) \times \mathcal{D}(B) \arrow[r, "\mathrm{E}"] \arrow[d, "\mathrm{res}^A_{R^{\mathrm{o}}} \times \mathrm{res}^B_R"'] & \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \arrow[d, "\mathrm{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk}"] \\
\mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R) \arrow[r, "-\otimes^{\mathsf{L}}_R-"] & \mathcal{D}(\Bbbk) \,.
\end{tikzcd}$$

7.4.3 Definition. Adopt the setup 7.4.2. If there exists a natural isomorphism,

$$\varphi : \mathrm{res}^A_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_R \mathrm{res}^B_R \longrightarrow \mathrm{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \mathrm{E} \,,$$

i.e. (7.4.2.1) is commutative up to natural isomorphism, then the functor E is called an *augmentation* $\otimes^{\mathsf{L}}_R$ and denoted by the same symbol. That is, one writes

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(A) \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}})$$

and says that $\otimes^{\mathsf{L}}_R$ is *augmented* to a functor from $\mathcal{D}(A) \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$.

CAVEAT. Adopt the setup 7.4.2 and consider restriction of scalars followed by the tensor product,

$$\mathrm{D}\colon \mathcal{K}(A) \times \mathcal{K}(B) \xrightarrow{\mathrm{res}^A_{Q\otimes R^{\mathrm{o}}} \times \mathrm{res}^B_{R\otimes S^{\mathrm{o}}}} \mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \xrightarrow{-\otimes_R-} \mathcal{K}(Q\text{–}S^{\mathrm{o}}) \; ;$$

cf. 7.1.9. This functor has a left derived functor $\mathrm{E} = \mathsf{L}\mathrm{D}$, which per 7.2.7 and 7.2.8 is given by $\mathrm{E} = (\mathrm{P}_A(-) \otimes_R \mathrm{P}_B(-))^{\sim}$. The next example shows that for this functor E the diagram (7.4.2.1) need not be commutative up to natural isomorphism. Thus, even though E *is* the left derived functor of a tensor product, we *only* denote it by the symbol $\otimes^{\mathsf{L}}_R$ in situations where the diagram (7.4.2.1) is commutative up to natural isomorphism. Not only are these, anyway, the only situations in which we are interested in the derived functor E, but the dissonance with 7.2.8 is limited. Indeed, the meaning of the symbol $\otimes^{\mathsf{L}}_R$ depends on which category the tensor product $\otimes_R$ is defined on. That information is not encoded in the symbol $- \otimes^{\mathsf{L}}_R -$, it only gives away that both arguments must be R^{o}- and R-complexes. Without further context the only certain interpretation of the symbol is, therefore, the one from 7.4.1, and 7.4.3 ensures that this interpretation always is valid.

The next example shows that the diagram (7.4.2.1) need not be commutative up to natural isomorphism.

7.4.4 Example. With $\Bbbk = \mathbb{Z} = R$ and $A = Q = \mathbb{Z}/2\mathbb{Z} = S = B$ one has

$$Q \otimes_{\Bbbk} R^{\mathrm{o}} \;=\; A \;=\; \mathbb{Z}/2\mathbb{Z} \;=\; B \;=\; R \otimes_{\Bbbk} S^{\mathrm{o}} \;=\; Q \otimes_{\Bbbk} S^{\mathrm{o}} \,.$$

The functor $- \otimes_{\mathbb{Z}} - : \mathcal{K}(\mathbb{Z}/2\mathbb{Z}) \times \mathcal{K}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathcal{K}(\mathbb{Z}/2\mathbb{Z})$ has a left derived functor $\mathrm{E} : \mathcal{D}(\mathbb{Z}/2\mathbb{Z}) \times \mathcal{D}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathcal{D}(\mathbb{Z}/2\mathbb{Z})$ which, since $\mathbb{Z}/2\mathbb{Z}$ is a field is given by $\mathrm{E} = - \otimes_{\mathbb{Z}} -$. Thus, for $M = \mathbb{Z}/2\mathbb{Z} = N$ one has

$$\mathrm{E}(M, N) \;=\; \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \;=\; \mathbb{Z}/2\mathbb{Z} \,.$$

By 7.4.1 one can compute $M \otimes^{\mathsf{L}}_{\mathbb{Z}} N$ as $P \otimes_{\mathbb{Z}} P$, where P is the semi-projective $\mathbb{Z}$-complex $0 \longrightarrow \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \longrightarrow 0$ concentrated in degrees 1 and 0. Suppressing the isomorphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$ one gets the complex

$$M \otimes^{\mathsf{L}}_{\mathbb{Z}} N \;=\; P \otimes_{\mathbb{Z}} P \;=\; 0 \longrightarrow \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right)} \begin{matrix} \mathbb{Z} \\ \oplus \\ \mathbb{Z} \end{matrix} \xrightarrow{(2\;2)} \mathbb{Z} \longrightarrow 0 \,.$$

It is concentrated in degrees 2, 1, and 0, and it has homology in degree 1 as, say, $(-1, 1)$ is a cycle but not a boundary. Thus the complexes $M \otimes^{\mathsf{L}}_{\mathbb{Z}} N$ and $\mathrm{E}(M, N)$ are not isomorphic in the derived category $\mathcal{D}(\mathbb{Z})$.

Theorem 7.4.5 is our most general statement about augmentations of $\otimes^{\mathsf{L}}_R$; commonly used special cases are recorded in 7.4.8–7.4.11. The question of how to evaluate the functor is addressed in 7.4.7 and 7.4.15.

7.4.5 Theorem. *Let $Q \otimes_{\Bbbk} R^{\mathrm{o}} \to A$ and $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$ be ring homomorphisms. If condition* (a) *or* (b) *below is satisfied, then $\otimes^{\mathsf{L}}_R$ is augmented as follows*:

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(A) \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \,. \tag{7.4.5.1}$$

This functor is $\Bbbk$-bilinear, it preserves coproducts in both variables, and it is triangulated in both variables.

(a) *A is flat as an R^{o}-module.*
(b) *B is flat as an R-module.*

Further, (7.4.5.1) *is induced by functors $\mathcal{K}(A) \times \mathcal{K}(B) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$ as follows*:

(a′) *If* (a) *is satisfied, then $- \otimes^{\mathsf{L}}_R -$ is induced by $\mathrm{P}_A(-) \otimes_R -$.*
(b′) *If* (b) *is satisfied, then $- \otimes^{\mathsf{L}}_R -$ is induced by $- \otimes_R \mathrm{P}_B(-)$.*

Proof. For clarity the restriction of scalars functors, see 6.1.23 and 6.4.36, are not suppressed in this proof. Consider the functor

$$\mathrm{D} \;=\; (-\otimes_R -)\circ(\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\times\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \;=\; \operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\;.$$

Recall the natural transformation ϱ from 6.3.21. The next diagram defines a natural transformation τ of functors from $\mathcal{K}(A)\times\mathcal{K}(B)$ to $\mathcal{K}(\Bbbk)$,

$$\begin{tikzcd}
(\mathrm{P}_{R^{\mathrm{o}}}\otimes_R\mathrm{P}_R)\circ(\operatorname{res}^A_{R^{\mathrm{o}}}\times\operatorname{res}^B_R) \arrow[r, equal] \arrow[dd, "\tau"'] & \mathrm{P}_{R^{\mathrm{o}}}\operatorname{res}^A_{R^{\mathrm{o}}}\otimes_R\mathrm{P}_R\operatorname{res}^B_R \arrow[d, "\varrho^A_{R^{\mathrm{o}}}\otimes 1"] \\
& \operatorname{res}^A_{R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\mathrm{P}_R\operatorname{res}^B_R \arrow[d, "1\otimes\varrho^B_R"] \\
\operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk}\circ(\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\mathrm{P}_B) \arrow[r, equal] & \operatorname{res}^A_{R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_R\mathrm{P}_B\;.
\end{tikzcd}$$

Set E = LD. Now 7.2.5 and 6.4.40, per 7.2.7, yield a natural transformation

$$\tilde{\tau}\colon\;(-\otimes^{\mathsf{L}}_R -)\circ(\operatorname{res}^A_{R^{\mathrm{o}}}\times\operatorname{res}^B_R)\;\longrightarrow\;\operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk}\circ\mathrm{E}\;.$$

To prove that E is an augmentation of $\otimes^{\mathsf{L}}_R$, it suffices by 7.4.3 to show that $\varphi=\tilde{\tau}$ is a natural isomorphism, i.e. that τ^{MN} is a quasi-isomorphism for every A-complex M and every B-complex N; in the following M and N denote such complexes.

Recall from 6.3.21 that $(\varrho^A_{R^{\mathrm{o}}})^M$ is a quasi-isomorphism. Since the R-complex $\mathrm{P}_R(\operatorname{res}^B_R(N))$ is semi-projective, in particular semi-flat, $(\varrho^A_{R^{\mathrm{o}}})^M\otimes_R\mathrm{P}_R(\operatorname{res}^B_R(N))$ is a quasi-isomorphism.

If condition (a) is satisfied, then the R^{o}-complex $\operatorname{res}^A_{R^{\mathrm{o}}}(\mathrm{P}_A(M))$ is semi-flat by 5.4.18(b). It follows that $\operatorname{res}^A_{R^{\mathrm{o}}}(\mathrm{P}_A(M))\otimes_R(\varrho^B_R)^N$ and, therefore, τ^{MN} is a quasi-isomorphism. If condition (b) is satisfied, then the R-complex $\operatorname{res}^B_R(\mathrm{P}_B(N))$ is semi-flat by 5.4.18(b), so $(\varrho^B_R)^N$ is a quasi-isomorphism of semi-flat complexes. It follows from 5.4.16 that $\operatorname{res}^A_{R^{\mathrm{o}}}(\mathrm{P}_A(M))\otimes_R(\varrho^B_R)^N$ and, therefore, τ^{MN} is a quasi-isomorphism.

Thus if (a) or (b) is satisfied, then one writes $\otimes^{\mathsf{L}}_R$ for the functor E; see 7.4.3. It follows from 7.1.8 and 7.2.14 that the functor E = LD is $\Bbbk$-bilinear, preserves coproducts in both variables, and is triangulated in both variables.

(a′): With the restriction of scalars functors included, the claim is that the functors

$$\mathrm{E}\;=\;(\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\mathrm{P}_B)^{\tilde{}}\quad\text{ and }\quad(\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}})^{\tilde{}}$$

from $\mathcal{D}(A)\times\mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$ are naturally isomorphic; indeed, as already established the functor E is (7.4.5.1). Consider the natural transformation

$$\sigma\;=\;1\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\pi_B\,:\,\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\mathrm{P}_B\;\longrightarrow\;\operatorname{res}^A_{Q\otimes R^{\mathrm{o}}}\mathrm{P}_A\otimes_R\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}$$

of functors $\mathcal{K}(A)\times\mathcal{K}(B)\to\mathcal{K}(Q\text{–}S^{\mathrm{o}})$. To see that the induced natural transformation $\tilde{\sigma}$ is an isomorphism, let M and N be complexes in $\mathcal{K}(A)$ and $\mathcal{K}(B)$ respectively; it must be verified that σ^{MN} is a quasi-isomorphism. This follows as the complex

$\operatorname{res}^A_{Q\otimes R^o}(\mathrm{P}_A(M))$ under condition (a) is semi-flat over R^o by 5.4.18(b), whence the functor $\operatorname{res}^A_{Q\otimes R^o}(\mathrm{P}_A(M)\otimes_R -)$ preserves the quasi-isomorphism $\operatorname{res}^B_{R\otimes S^o}(\pi^N_B)$.

(b′): By symmetry the argument above applies with the obvious substitutions. □

7.4.6 Scholium. (a) The proof of 7.4.5 shows in view of 6.3.21 that if condition (a) in 7.4.5 is satisfied, then the transformation

$$\varrho^A_{R^o}\otimes_R \pi_R \operatorname{res}^B_R : \mathrm{P}_{R^o}\operatorname{res}^A_{R^o}\otimes_R \mathrm{P}_R\operatorname{res}^B_R \longrightarrow \operatorname{res}^A_{R^o}\mathrm{P}_A\otimes_R \operatorname{res}^B_R$$

induces the natural isomorphism that per 7.4.3 establishes the functor induced by $\mathrm{P}_A(-)\otimes_R - = \operatorname{res}^A_{Q\otimes R^o}\mathrm{P}_A\otimes_R \operatorname{res}^B_{R\otimes S^o}$ as an augmentation of $\otimes^{\mathsf{L}}_R$.

(b) Similarly, if condition (b) in 7.4.5 is satisfied, then the transformation

$$\pi_{R^o}\operatorname{res}^A_{R^o}\otimes_R \varrho^B_R : \mathrm{P}_{R^o}\operatorname{res}^A_{R^o}\otimes_R \mathrm{P}_R\operatorname{res}^B_R \longrightarrow \operatorname{res}^A_{R^o}\otimes_R \operatorname{res}^B_R\mathrm{P}_B$$

induces the natural isomorphism that per 7.4.3 establishes the functor induced by $-\otimes_R \mathrm{P}_B(-) = \operatorname{res}^A_{Q\otimes R^o}\otimes_R \operatorname{res}^B_{R\otimes S^o}\mathrm{P}_B$ as an augmentation of $\otimes^{\mathsf{L}}_R$.

7.4.7. Consider the functor (7.4.5.1) and morphisms $\alpha/\varphi\colon M\to M'$ in $\mathcal{D}(A)$ and $\beta/\psi\colon N\to N'$ in $\mathcal{D}(B)$. If condition (a) in 7.4.5 is satisfied, then by 7.2.3 one has

$$M\otimes^{\mathsf{L}}_R N = \mathrm{P}_A(M)\otimes_R N \quad\text{and}\quad \alpha/\varphi\otimes^{\mathsf{L}}_R\beta/\psi = (\mathrm{P}_A(\alpha)\otimes\beta)/(\mathrm{P}_A(\varphi)\otimes\psi)\,.$$

Similarly, if condition (b) in 7.4.5 is satisfied, then one has

$$M\otimes^{\mathsf{L}}_R N = M\otimes_R \mathrm{P}_B(N) \quad\text{and}\quad \alpha/\varphi\otimes^{\mathsf{L}}_R\beta/\psi = (\alpha\otimes\mathrm{P}_B(\beta))/(\varphi\otimes\mathrm{P}_B(\psi))\,.$$

Applied with $S=\Bbbk$, the next result recovers the functor $\otimes^{\mathsf{L}}_R$ from 7.4.1, now induced by a functor that only involves a resolution of the first variable.

7.4.8 Corollary. *The functor* $\otimes^{\mathsf{L}}_R$ *is augmented as follows*:

$$-\otimes^{\mathsf{L}}_R - : \mathcal{D}(R^o)\times\mathcal{D}(R\text{–}S^o) \longrightarrow \mathcal{D}(S^o)\,.$$

Further, this functor is induced by $\mathrm{P}_{R^o}(-)\otimes_R - : \mathcal{K}(R\text{–}S^o)\longrightarrow\mathcal{K}(S^o)$.

Proof. Apply 7.4.5 with $Q=\Bbbk$, $A=R^o$, and $B=R\otimes_{\Bbbk}S^o$. Condition (a) is now trivially satisfied; in particular, the last assertion follows from 7.4.5(a′). □

Applied with $Q=\Bbbk$, the next result recovers the functor $\otimes^{\mathsf{L}}_R$ from 7.4.1, now induced by a functor that only involves a resolution of the second variable.

7.4.9 Corollary. *The functor* $\otimes^{\mathsf{L}}_R$ *is augmented as follows*:

$$-\otimes^{\mathsf{L}}_R - : \mathcal{D}(Q\text{–}R^o)\times\mathcal{D}(R) \longrightarrow \mathcal{D}(Q)\,.$$

Further, this functor is induced by $-\otimes_R\mathrm{P}_R(-) : \mathcal{K}(Q\text{–}R^o)\times\mathcal{K}(R)\longrightarrow\mathcal{K}(Q)$.

Proof. Apply 7.4.5 with $A=Q\otimes_{\Bbbk}R^o$, $S=\Bbbk$, and $B=R$. Condition (b) is now trivially satisfied; in particular, the last assertion follows from 7.4.5(b′). □

7.4.10 Proposition. *Assume that Q is flat as a $\Bbbk$-module. The functor $\otimes^{\mathsf{L}}_R$ is augmented as follows*:

$$-\otimes^{\mathsf{L}}_R - : \mathcal{D}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) ;$$

it is induced by $\mathrm{P}_{Q\otimes_{\Bbbk}R^{\mathrm{o}}}(-)\otimes_R - : \mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$.

Proof. Apply 7.4.5 with $A = Q\otimes_{\Bbbk} R^{\mathrm{o}}$ and $B = R\otimes_{\Bbbk} S^{\mathrm{o}}$; condition (a) is satisfied per 7.3.11(b), so the conclusion follows from 7.4.5(a′). □

7.4.11 Proposition. *Assume that S is flat as a $\Bbbk$-module. The functor $\otimes^{\mathsf{L}}_R$ is augmented as follows*:

$$-\otimes^{\mathsf{L}}_R - : \mathcal{D}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) ;$$

it is induced by $-\otimes_R \mathrm{P}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(-) : \mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{K}(Q\text{–}S^{\mathrm{o}})$.

Proof. Apply 7.4.5 with $A = Q\otimes_{\Bbbk} R^{\mathrm{o}}$ and $B = R\otimes_{\Bbbk} S^{\mathrm{o}}$; condition (b) is satisfied per 7.3.11(b), so the conclusion follows from 7.4.5(b′). □

Note that 7.4.8 and 7.4.9 are special cases of 7.4.10 and 7.4.11 with $Q = \Bbbk$ and $S = \Bbbk$, respectively. Further applications of 7.4.10 and 7.4.11 can be obtained from 7.3.14. In particular, if $\Bbbk$ is a field—or for that matter a von Neumann regular ring, see 8.5.8—then every $\Bbbk$-algebra is flat as a $\Bbbk$-module, so $\otimes^{\mathsf{L}}_R$ is augmented to a functor $\mathcal{D}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}})$.

In the literature on derived tensor product functors over non-commutative rings it is common to impose a blanket assumption—for example that the rings are algebras over some fixed field—to ensure that the derived functor upholds additional ring actions; see also the discussion after 7.3.14. For commutative rings there is no need for such an assumption, and the next example shows how the setup with the rings A and B also accommodates the commutative case.

7.4.12 Example. Assume that R is commutative and recall that R-modules are considered to be symmetric R–R-bimodules, in particular, they are modules over the enveloping algebra R^{e}. Thus, with $Q = R = A$ condition (a) in 7.4.5 is satisfied, see 7.3.15, and the derived tensor product $\otimes^{\mathsf{L}}_R$ is augmented to a functor $\mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \to \mathcal{D}(R\text{–}S^{\mathrm{o}})$. Similarly, with $S = R = B$ condition (b) is satisfied and $\otimes^{\mathsf{L}}_R$ is augmented to a functor $\mathcal{D}(Q\text{–}R) \times \mathcal{D}(R) \to \mathcal{D}(Q\text{–}R)$.

Semi-Flat Replacments

7.4.13 Definition. Let M be an R-complex. A semi-flat R-complex that is isomorphic to M in $\mathcal{D}(R)$ is called a *semi-flat replacement* of M.

7.4.14 Example. For every semi-projective resolution $P \xrightarrow{\simeq} M$ the complex P is a semi-flat replacement of M.

The $\mathbb{Z}$-complex $0 \to \mathbb{Z} \to \mathbb{Q} \to 0$, concentrated in degrees 1 and 0, is a semi-flat replacement of $\mathbb{Q}/\mathbb{Z}$.

7.4.15 Theorem. *Let $Q \otimes_{\Bbbk} R^{\circ} \to A$ and $R \otimes_{\Bbbk} S^{\circ} \to B$ be ring homomorphisms.*

(a) *Assume that A is flat as an R°-module. For an A-complex M with a semi-flat replacement F, the functor from* 7.4.5

$$\begin{aligned} M \otimes_R^{\mathsf{L}} - : \mathcal{D}(B) &\longrightarrow \mathcal{D}(Q\text{–}S^{\circ}) \quad \textit{is induced by} \\ F \otimes_R - : \mathcal{K}(B) &\longrightarrow \mathcal{K}(Q\text{–}S^{\circ})\,. \end{aligned}$$

(b) *Assume that B is flat as an R-module. For a B-complex N with a semi-flat replacement G, the functor from* 7.4.5

$$\begin{aligned} - \otimes_R^{\mathsf{L}} N : \mathcal{D}(A) &\longrightarrow \mathcal{D}(Q\text{–}S^{\circ}) \quad \textit{is induced by} \\ - \otimes_R G : \mathcal{K}(A) &\longrightarrow \mathcal{K}(Q\text{–}S^{\circ})\,. \end{aligned}$$

Proof. (a): The functor $M \otimes_R^{\mathsf{L}} -$ is by 7.4.5(a′) induced by $\mathrm{P}_A(M) \otimes_R -$. By 5.2.20 there is a quasi-isomorphism $\varphi\colon \mathrm{P}_A(M) \to F$ in $\mathcal{K}(A)$, and hence a natural transformation $\tau = \varphi \otimes_R -\colon \mathrm{P}_A(M) \otimes_R - \to F \otimes_R -$ of functors from $\mathcal{K}(B)$ to $\mathcal{K}(Q\text{–}S^{\circ})$. The assumption that A is flat over R° implies by 5.4.18(b) that F is semi-flat over R°, so the functor $F \otimes_R -$ preserves quasi-isomorphisms. By the same argument, $\mathrm{P}_A(M)$ is semi-flat over R°, so it follows from 5.4.16 that the morphism $\tau^N = \varphi \otimes_R N$ is a quasi-isomorphism for every B-complex N. Thus $\acute{\tau}\colon M \otimes_R^{\mathsf{L}} - \to (F \otimes_R -)^{\acute{}\acute{}}$ is a natural isomorphism of functors, see 7.2.5.

(b): By symmetry the argument above applies with the obvious substitutions. □

The next two corollaries compare to 7.4.8 and 7.4.9.

7.4.16 Corollary. *For an R°-complex M with a semi-flat replacement F, the functor*

$$M \otimes_R^{\mathsf{L}} - : \mathcal{D}(R\text{–}S^{\circ}) \longrightarrow \mathcal{D}(S^{\circ}) \quad \textit{is induced by} \quad F \otimes_R - : \mathcal{K}(R\text{–}S^{\circ}) \longrightarrow \mathcal{K}(S^{\circ})\,.$$

Proof. Apply 7.4.15 with $Q = \Bbbk$, $A = R^{\circ}$, and $B = R \otimes_{\Bbbk} S^{\circ}$. □

7.4.17 Corollary. *For an R-complex N with a semi-flat replacement G, the functor*

$$- \otimes_R^{\mathsf{L}} N : \mathcal{D}(Q\text{–}R^{\circ}) \longrightarrow \mathcal{D}(Q) \quad \textit{is induced by} \quad - \otimes_R G : \mathcal{K}(Q\text{–}R^{\circ}) \longrightarrow \mathcal{K}(Q)\,.$$

Proof. Apply 7.4.15 with $A = Q \otimes_{\Bbbk} R^{\circ}$, $S = \Bbbk$, and $B = R$. □

Tor Functors

For the next definition, recall from 6.5.17 that complexes that are isomorphic in the derived category have isomorphic homology. Recall also that 7.4.5 offers conditions under which the opening assumption in the definition below is satisfied.

7.4.18 Definition. Let $Q \otimes_{\Bbbk} R^{\circ} \to A$ and $R \otimes_{\Bbbk} S^{\circ} \to B$ be ring homomorphisms and assume that $\otimes_R^{\mathsf{L}}$ per 7.4.3 is augmented to a functor,

$$- \otimes_R^{\mathsf{L}} - : \mathcal{D}(A) \times \mathcal{D}(B) \longrightarrow \mathcal{D}(Q\text{–}S^{\circ})\,.$$

For every $m \in \mathbb{Z}$ denote by $\operatorname{Tor}_m^R(-,-)$ the functor

$$\mathrm{H}_m(- \otimes_R^{\mathsf{L}} -) : \mathcal{D}(A) \times \mathcal{D}(B) \longrightarrow \mathcal{M}(Q\text{–}S^{\circ})\,.$$

It follows from 7.4.5 and 6.5.17 that the functors Tor_m^R are $\Bbbk$-bilinear.

Remark. The symbol 'Tor' reflects fact that these functors measure torsion: If R is an integral domain with field of fractions Q, then for every R-module M there is an isomorphism $\operatorname{Tor}_1^R(M, Q/R) \cong M_{\mathrm{T}}$, where M_{T} is the torsion submodule of M. See E 11.2.17 and E 11.2.19.

7.4.19. Let M be an R^{o}-complex and N an R-complex. Per 6.5.18 there are, with the usual conventions $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$, equalities,

$$\begin{aligned} \inf\{m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(M,N) \neq 0\} &= \inf(M \otimes_R^{\mathsf{L}} N) \quad \text{and} \\ \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(M,N) \neq 0\} &= \sup(M \otimes_R^{\mathsf{L}} N)\,. \end{aligned}$$

7.4.20. In view of 7.4.8 and 7.4.9 there are Tor functors,

$$\begin{aligned} \operatorname{Tor}_m^R(-,-) &: \mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{M}(S^{\mathrm{o}}) \quad \text{and} \\ \operatorname{Tor}_m^R(-,-) &: \mathcal{D}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R) \longrightarrow \mathcal{M}(Q)\,. \end{aligned}$$

For modules there is a simple description of Tor_0.

7.4.21 Proposition. *Let M be an R^{o}-module and N an R-module. There is an isomorphism of $\Bbbk$-modules,*

$$\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N\,,$$

and it is natural in M and N. Furthermore, $\operatorname{Tor}_m^R(M,N) = 0$ *holds for all $m < 0$.*

Proof. By 5.2.14 one can assume that the semi-projective R-complex $\mathrm{P}(N)$ satisfies $\mathrm{P}(N)_m = 0$ for all $m < 0$. In the chain,

$$\operatorname{Tor}_0^R(M,N) = \mathrm{H}_0(M \otimes_R \mathrm{P}(N)) \cong M \otimes_R \mathrm{H}_0(\mathrm{P}(N)) = M \otimes_R N\,,$$

the left-hand equality follows from 7.4.9 and the definition 7.4.18 of Tor, and the middle isomorphism comes from 2.5.16; it is natural in M and $\mathrm{P}(N)$. As H_0 and P are functors, the resulting isomorphism $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$ is natural in M and N. Finally, for $m < 0$ one has $(M \otimes_R \mathrm{P}(N))_m = M \otimes_R \mathrm{P}(N)_m = 0$ and hence also $\operatorname{Tor}_m^R(M,N) = \mathrm{H}_m(M \otimes_R \mathrm{P}(N)) = 0$, as claimed. □

Remark. In classic homological algebra it is standard to refer to Tor_m^R as the 'm^{th} left derived functor' of the tensor product $\otimes_R$ on R-modules. This explains why $\otimes_R^{\mathsf{L}}$ in some places is called the 'total left derived functor' of $\otimes_R$; see also the Remark after 7.2.8.

7.4.22 Proposition. *Let M be an R^{o}-module and N an R-module. If* $\operatorname{Tor}_m^R(M,N) = 0$ *holds for all $m > 0$, then there is an isomorphism $M \otimes_R^{\mathsf{L}} N \simeq M \otimes_R N$ in $\mathcal{D}(\Bbbk)$.*

Proof. The assertion follows immediately from 7.4.21 and 7.3.29. □

7.4.23 Proposition. *Let M be an R^{o}-complex, N an R-complex. For every $m \in \mathbb{Z}$ there is an isomorphism,* $\operatorname{Tor}_m^R(M,N) \cong \operatorname{Tor}_m^{R^{\mathrm{o}}}(N,M)$*, and it is natural in M and N.*

Proof. The isomorphisms follow by the definition, 7.4.18, of Tor from commutativity 4.4.4 of the tensor product, 7.4.9, and 7.4.8.

$$\operatorname{Tor}_m^R(M,N) = \mathrm{H}_m(M \otimes_R \mathrm{P}(N)) \cong \mathrm{H}_m(\mathrm{P}(N) \otimes_{R^{\mathrm{o}}} M) \cong \operatorname{Tor}_m^{R^{\mathrm{o}}}(N,M)\,. \quad □$$

7.4.24 Proposition. *Let M be an R°-complex, N an R-complex and m and s be integers. There are isomorphisms in $\mathcal{M}(\Bbbk)$,*

$$\operatorname{Tor}^R_m(\Sigma^s M, N) \;\cong\; \operatorname{Tor}^R_{m-s}(M, N) \;\cong\; \operatorname{Tor}^R_m(M, \Sigma^s N)\,,$$

and they are natural in M and N.

PROOF. The functor $-\otimes^{\mathsf{L}}_R -$ is triangulated in both variables by 7.4.5, so there are isomorphisms in $\mathcal{D}(\Bbbk)$,

$$(\Sigma^s M)\otimes^{\mathsf{L}}_R N \;\simeq\; \Sigma^s(M\otimes^{\mathsf{L}}_R N) \;\simeq\; M\otimes^{\mathsf{L}}_R (\Sigma^s N)\,,$$

which are natural in M and N. Apply the functor H_m to these isomorphisms and recall from 6.5.17 that one has $\mathrm{H}_m\,\Sigma^s = \mathrm{H}_{m-s}$. The assertion now follows from the definition, 7.4.18, of Tor. $\square$

7.4.25 Proposition. *Let N be an R-complex and $\{\mu^{vu} : M^u \to M^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R^{\circ})$. If U is filtered there is for every $m \in \mathbb{Z}$ an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\operatorname*{colim}_{u\in U} \operatorname{Tor}^R_m(M^u, N) \;\cong\; \operatorname{Tor}^R_m\big(\operatorname*{colim}_{u\in U} M^u, N\big)\,.$$

PROOF. Let G be a semi-flat replacement of N. There are isomorphisms,

$$\operatorname*{colim}_{u\in U} \mathrm{H}_m(M^u\otimes_R G) \;\cong\; \mathrm{H}_m\big(\operatorname*{colim}_{u\in U}(M^u\otimes_R G)\big) \;\cong\; \mathrm{H}_m\big(\big(\operatorname*{colim}_{u\in U} M^u\big)\otimes_R G\big)\,,$$

by 3.3.15(d) and 3.2.22, so the assertion follows from 7.4.17 and 7.4.18. $\square$

7.4.26 Proposition. *Let M be an R°-complex and $\{\nu^{vu} : N^u \to N^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R)$. If U is filtered there is for every $m \in \mathbb{Z}$ an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\operatorname*{colim}_{u\in U} \operatorname{Tor}^R_m(M, N^u) \;\cong\; \operatorname{Tor}^R_m\big(M, \operatorname*{colim}_{u\in U} N^u\big)\,.$$

PROOF. Proceed as in the proof of 7.4.25, but use 3.2.23 in place of 3.2.22, and 7.4.16 instead of 7.4.17. $\square$

7.4.27 Proposition. *Assume that R is left Noetherian. Let N be an R-module and $\{M^u\}_{u\in U}$ a family of R°-modules. If N is finitely generated, then there is for every $m \in \mathbb{Z}$ an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\operatorname{Tor}^R_m\Big(\prod_{u\in U} M^u, N\Big) \;\cong\; \prod_{u\in U} \operatorname{Tor}^R_m(M^u, N)\,.$$

PROOF. Choose per 5.1.19 a degreewise finitely generated free resolution $L \xrightarrow{\simeq} N$. By degreewise application of 3.1.31 and by 3.1.22(d) one gets isomorphisms,

$$\mathrm{H}_m\big(\big(\prod_{u\in U} M^u\big)\otimes_R L\big) \;\cong\; \mathrm{H}_m\big(\prod_{u\in U}(M^u\otimes_R L)\big) \;\cong\; \prod_{u\in U} \mathrm{H}_m(M^u\otimes_R L)\,.$$

The assertion now follows from 7.4.17 and 7.4.18. $\square$

7.4.28 Proposition. *Assume that R is right Noetherian. Let M be an R°-module and $\{N^u\}_{u\in U}$ a family of R-modules. If M is finitely generated, then there is for every $m \in \mathbb{Z}$ an isomorphism in $\mathcal{M}(\Bbbk)$,*

$$\operatorname{Tor}_m^R\Big(M, \prod_{u\in U} N^u\Big) \cong \prod_{u\in U} \operatorname{Tor}_m^R(M, N^u)\,.$$

PROOF. Proceed as in the proof of 7.4.27, but use 3.1.30 in place of 3.1.31, and 7.4.16 instead of 7.4.17. □

Per 6.5.24 the next result applies, in particular, to a commutative diagram in $\mathcal{C}(R)$ whose rows are short exact sequences. It shows that, viewed as functors from $\mathcal{M}(R^{\mathrm{o}})\times\mathcal{M}(R)$ to $\mathcal{M}(\Bbbk)$, see 6.4.15, the Tor functors are half exact in either variable.

7.4.29 Theorem. *Consider a morphism of distinguished triangles in $\mathcal{D}(R)$,*

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r, "\alpha''"] \arrow[d, "\varphi''"] & \Sigma M' \arrow[d, "\Sigma\varphi'"] \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \arrow[r, "\beta''"] & \Sigma N'\,. \end{tikzcd}$$

For every R^{o}-complex X there is a commutative diagram in $\mathcal{M}(\Bbbk)$ with exact rows.

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Tor}_m^R(X,M') \arrow[r, "\operatorname{Tor}_m^R(X,\alpha')"] \arrow[d, "\operatorname{Tor}_m^R(X,\varphi')"] & \operatorname{Tor}_m^R(X,M) \arrow[r, "\operatorname{Tor}_m^R(X,\alpha)"] \arrow[d, "\operatorname{Tor}_m^R(X,\varphi)"] & \operatorname{Tor}_m^R(X,M'') \arrow[r, "\operatorname{Tor}_m^R(X,\alpha'')"] \arrow[d, "\operatorname{Tor}_m^R(X,\varphi'')"] & \operatorname{Tor}_{m-1}^R(X,M') \arrow[r, "\operatorname{Tor}_{m-1}^R(X,\alpha')"] \arrow[d, "\operatorname{Tor}_{m-1}^R(X,\varphi')"] & \cdots \\ \cdots \arrow[r] & \operatorname{Tor}_m^R(X,N') \arrow[r, "\operatorname{Tor}_m^R(X,\beta')"'] & \operatorname{Tor}_m^R(X,N) \arrow[r, "\operatorname{Tor}_m^R(X,\beta)"'] & \operatorname{Tor}_m^R(X,N'') \arrow[r, "\operatorname{Tor}_m^R(X,\beta'')"'] & \operatorname{Tor}_{m-1}^R(X,N') \arrow[r, "\operatorname{Tor}_{m-1}^R(X,\beta')"'] & \cdots \end{tikzcd}$$

where the isomorphisms $\operatorname{Tor}_m^R(X,\Sigma-) \cong \operatorname{Tor}_{m-1}^R(X,-)$ from 7.4.24 are supressed.

If instead $(\varphi',\varphi,\varphi'')$ is a morphism of distinguished triangles in $\mathcal{D}(R^{\mathrm{o}})$, then for every R-complex Y there is a commutative diagram in $\mathcal{M}(\Bbbk)$ with exact rows,

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Tor}_m^R(M',Y) \arrow[r, "\operatorname{Tor}_m^R(\alpha',Y)"] \arrow[d, "\operatorname{Tor}_m^R(\varphi',Y)"] & \operatorname{Tor}_m^R(M,Y) \arrow[r, "\operatorname{Tor}_m^R(\alpha,Y)"] \arrow[d, "\operatorname{Tor}_m^R(\varphi,Y)"] & \operatorname{Tor}_m^R(M'',Y) \arrow[r, "\operatorname{Tor}_m^R(\alpha'',Y)"] \arrow[d, "\operatorname{Tor}_m^R(\varphi'',Y)"] & \operatorname{Tor}_{m-1}^R(M',Y) \arrow[r, "\operatorname{Tor}_{m-1}^R(\alpha',Y)"] \arrow[d, "\operatorname{Tor}_{m-1}^R(\varphi',Y)"] & \cdots \\ \cdots \arrow[r] & \operatorname{Tor}_m^R(N',Y) \arrow[r, "\operatorname{Tor}_m^R(\beta',Y)"'] & \operatorname{Tor}_m^R(N,Y) \arrow[r, "\operatorname{Tor}_m^R(\beta,Y)"'] & \operatorname{Tor}_m^R(N'',Y) \arrow[r, "\operatorname{Tor}_m^R(\beta'',Y)"'] & \operatorname{Tor}_{m-1}^R(N',Y) \arrow[r, "\operatorname{Tor}_{m-1}^R(\beta',Y)"'] & \cdots \end{tikzcd}$$

where the isomorphisms $\operatorname{Tor}_m^R(\Sigma-,Y) \cong \operatorname{Tor}_{m-1}^R(-,Y)$ from 7.4.24 are supressed.

PROOF. The functor $-\otimes_R^{\mathsf{L}}-$ is triangulated in both variables by 7.4.5, so the assertions follow from 6.5.21. □

The two results below are not needed before Chap. 9, but they are standard applications of 7.4.29 and natural to record here.

7.4.30 Proposition. *Let M be an R°-complex and $0 \to N' \to N \to N'' \to 0$ an exact sequence in $\mathcal{C}(R)$. If $\operatorname{Tor}_1^R(M_v, N_i'') = 0$ holds for all $v, i \in \mathbb{Z}$, then the sequence of complexes $0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$ is exact.*

Proof. For every $i \in \mathbb{Z}$ there is an exact sequence $0 \to N_i' \to N_i \to N_i'' \to 0$ of R-modules. By 7.4.29 and 7.4.21 it yields for every $v \in \mathbb{Z}$ an exact sequence

$$\operatorname{Tor}_1^R(M_v, N_i'') \longrightarrow M_v \otimes_R N_i' \longrightarrow M_v \otimes_R N_i \longrightarrow M_v \otimes_R N_i'' \longrightarrow 0 .$$

The assertion now follows from 2.4.15. □

7.4.31 Proposition. *Let N be an R-complex and $0 \to M' \to M \to M'' \to 0$ an exact sequence in $\mathcal{C}(R^{\circ})$. If $\operatorname{Tor}_1^R(M_v'', N_i) = 0$ holds for all $v, i \in \mathbb{Z}$, then the sequence of complexes $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ is exact.*

Proof. An argument parallel to the proof of 7.4.30 applies, only one uses 2.4.16 instead of 2.4.15. □

Exercises

E 7.4.1 Let F be a faithfully flat R°-module and ξ a morphism in $\mathcal{D}(R)$. Show that if $F \otimes_R^{\mathsf{L}} \xi$ is an isomorphism, then ξ is an isomorphism.

E 7.4.2 (Cf. 7.3.7) Adopt the setup from 7.3.7 and verify the identity

$$\frac{\operatorname{Hom}(\mathrm{P}_A(M'), \beta)}{\operatorname{Hom}(\mathrm{P}_A(\varphi), \psi)} \circ \frac{\operatorname{Hom}(\mathrm{P}_A(\alpha), N)}{1^{\operatorname{Hom}(\mathrm{P}_A(M), N)}} = \frac{\operatorname{Hom}(\mathrm{P}_A(\alpha)\,\mathrm{P}_A(\varphi)^{-1}, \beta)}{\operatorname{Hom}(\mathrm{P}_A(M), \psi)} .$$

E 7.4.3 Assume that R is semi-simple. Show that for R-complexes M and N there is an isomorphism $M \otimes_R^{\mathsf{L}} N \simeq \mathrm{H}(M) \otimes_R \mathrm{H}(N)$ in $\mathcal{D}(\Bbbk)$.

E 7.4.4 Note that $\mathbb{Q}$ is a semi-flat replacement of the $\mathbb{Z}$-complex L from 5.4.15 and show that there does not exist a quasi-isomorphism $\mathbb{Q} \to L$ in $\mathcal{C}(\mathbb{Z})$.

E 7.4.5 Let M, P, and F be R-complexes. Show that if P is a semi-projective replacement of M and F a semi-flat replacement of M, then there is a quasi-isomorphism $\alpha\colon P \to F$ such that $X \otimes_R \alpha$ is a quasi-isomorphism for every R°-complex X.

E 7.4.6 Let M be an R°-complex, N an R-complex, and Z a K-flat complex that is isomorphic to N in $\mathcal{D}(R)$. Show that there is an isomorphism $M \otimes_R Z \simeq M \otimes_R^{\mathsf{L}} N$ in $\mathcal{D}(\Bbbk)$.

E 7.4.7 Let $R \to S$ be a ring homomorphism. Show that S is flat as an R-module if and only if every semi-flat S-complex is semi-flat over R.

E 7.4.8 Let M be an R-complex and N an R°-complex. Show that there is an isomorphism of $\Bbbk$-modules $\operatorname{Ext}_{R^{\circ}}^m(N, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_m^R(N, M), \mathbb{E})$ for every $m \in \mathbb{Z}$.

E 7.4.9 Let M be a $\mathbb{Z}$-module and $n \in \mathbb{N}$. Show that one has $\operatorname{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/n\mathbb{Z}) \cong (0 :_M n\mathbb{Z})$.

E 7.4.10 Let M be a $\mathbb{Z}$-module. Show that $\operatorname{Tor}_m^{\mathbb{Z}}(M, -) = 0$ holds for $m \geqslant 2$ and conclude that the functor $\operatorname{Tor}_1^{\mathbb{Z}}(M, -)$ is left exact.

E 7.4.11 Let $\mathfrak{a}$ be a left ideal and $\mathfrak{b}$ a right ideal in R. Show that there is an isomorphism of $\Bbbk$-modules $\operatorname{Tor}_1^R(R/\mathfrak{b}, R/\mathfrak{a}) \cong (\mathfrak{b} \cap \mathfrak{a})/\mathfrak{b}\mathfrak{a}$.

E 7.4.12 Show that for finite $\mathbb{Z}$-modules M and N there are isomorphisms

$$\operatorname{Ext}_R^1(M, N) \;\cong\; \operatorname{Hom}_R(M, N) \;\cong\; M \otimes_R N \;\cong\; \operatorname{Tor}_1^R(M, N) ,$$

which are not natural in M and N.

E 7.4.13 Let M, X, and N be $\mathbb{Z}$-modules. Show that there is an isomorphism of $\mathbb{Z}$-modules $\operatorname{Ext}_{\mathbb{Z}}^1(\operatorname{Tor}_1^{\mathbb{Z}}(M, X), N) \cong \operatorname{Ext}_{\mathbb{Z}}^1(M, \operatorname{Ext}_{\mathbb{Z}}^1(X, N))$. *Hint:* 2.5.8, 2.2.19, and E 7.3.11.

7.5 Standard Isomorphisms in the Derived Category

SYNOPSIS. Unitor; counitor; commutativity; associativity; swap; adjunction.

The standard isomorphisms 7.1.11–7.1.16 in the homotopy category induce isomorphisms in the derived category. The induced isomorphisms compare composites of derived Hom and tensor product functors; by default they are isomorphisms in the derived category $\mathcal{D}(\Bbbk)$, but in situations where the functors are appropriately augmented, the standard isomorphisms, too, become augmented.

7.5.1 Lemma. *Assume that R or S is flat as a $\Bbbk$-module. The functor $\otimes^{\mathsf{L}}_R$ is augmented as follows*:

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(R\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R\text{–}S^{\mathrm{o}})$$

and the endofunctor $R \otimes^{\mathsf{L}}_R -$ on $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ is induced by the endofunctor $R \otimes_R -$ on $\mathcal{K}(R\text{–}S^{\mathrm{o}})$.

PROOF. First assume that R is flat as a $\Bbbk$-module. By 7.4.10 the functor $\otimes^{\mathsf{L}}_R$ is augmented as claimed and induced by $\mathrm{P}_{R\otimes_{\Bbbk}R^{\mathrm{o}}}(-) \otimes_R -$. The quasi-isomorphism

$$\pi = \pi^{R}_{R\otimes_{\Bbbk}R^{\mathrm{o}}} : \mathrm{P}_{R\otimes_{\Bbbk}R^{\mathrm{o}}}(R) \longrightarrow R$$

in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$ is by 7.3.11(b) a quasi-isomorphism of semi-flat R^{o}-complexes. It thus follows from 5.4.16 that the natural transformation $\pi \otimes_R -$ evaluated at any complex of $R\text{–}S^{\mathrm{o}}$-bimodules is a quasi-isomorphism. That is, it yields a natural isomorphism of functors $R \otimes^{\mathsf{L}}_R - = (\mathrm{P}_{R\otimes_{\Bbbk}R^{\mathrm{o}}}(R) \otimes_R -)^{\text{\'\'}} \to (R \otimes_R -)^{\text{\'\'}}$; see 6.4.33.

Now assume that S is flat as a $\Bbbk$-module. By 7.4.11 the functor $\otimes^{\mathsf{L}}_R$ is augmented as claimed and induced by $- \otimes_R \mathrm{P}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(-)$. The natural transformation

$$R \otimes_R \pi_{R\otimes_{\Bbbk}S^{\mathrm{o}}} : R \otimes_R \mathrm{P}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(-) \longrightarrow R \otimes_R -$$

evaluated at any complex of $R\text{–}S^{\mathrm{o}}$-bimodules is a quasi-isomorphism. Thus it yields a natural isomorphism of functors $R \otimes^{\mathsf{L}}_R - = (R \otimes_R \mathrm{P}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(-))^{\text{\'\'}} \to (R \otimes_R -)^{\text{\'\'}}$. □

UNITOR

The treatment of the unitor and counitor is slightly different from the treatment of commutativity, associativity, swap, and adjunction for the following reason: It is unavoidable that, say, the commutativity isomorphism by default is only $\Bbbk$-linear, but application of the unitor or counitor should not result in loss of structure. However, if M is an R-complex, then saying that $R \otimes^{\mathsf{L}}_R M$ is an R-complex already implies that $\otimes^{\mathsf{L}}_R$ is augmented to a functor from $\mathcal{D}(R\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R)$ to $\mathcal{D}(R)$; saying that $\mathsf{R}\mathrm{Hom}_R(R, M)$ is an R-complex carries similar implications.

7.5.2 Construction. Assume that R or S is flat as a $\Bbbk$-module. By 7.5.1 there is a functor

$$R \otimes^{\mathsf{L}}_R - : \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R\text{–}S^{\mathrm{o}}) \quad \text{induced by} \quad R \otimes_R - .$$

By 7.1.11 there is a natural isomorphism of endofunctors on $\mathcal{K}(R\text{–}S^{\mathrm{o}})$,

$$\mu_R : R \otimes_R - \longrightarrow \mathrm{Id}_{\mathcal{K}(R\text{–}S^{\mathrm{o}})} .$$

It induces by 7.2.5 a natural isomorphism of endofunctors on $\mathcal{D}(R\text{–}S^{\mathrm{o}})$,

$$(7.5.2.1) \qquad \boldsymbol{\mu}_R = \H{\mu}_R : R \otimes^{\mathsf{L}}_R - \longrightarrow \mathrm{Id}_{\mathcal{D}(R\text{–}S^{\mathrm{o}})} .$$

7.5.3 Definition. The natural isomorphism (7.5.2.1) is called *unitor*.

7.5.4 Proposition. *Assume that R or S is flat as a $\Bbbk$-module. For M in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ the unitor is an isomorphism in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$,*

$$\boldsymbol{\mu}_R^M : R \otimes^{\mathsf{L}}_R M \longrightarrow M ,$$

and natural in M; it is induced by the isomorphism μ_R^M in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$. As a natural transformation, $\boldsymbol{\mu}_R$ is triangulated.

Proof. The natural isomorphism $\boldsymbol{\mu}_R = \H{\mu}_R$ from 7.5.2 is triangulated by 7.2.5 as μ_R is triangulated by 7.1.11. □

Counitor

7.5.5 Lemma. *Assume that R is projective or S is flat as a $\Bbbk$-module. The functor $\mathsf{R}\mathrm{Hom}_R$ is augmented as follows*:

$$\mathsf{R}\mathrm{Hom}_R(-,-) : \mathcal{D}(R\text{–}R^{\mathrm{o}}) \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R\text{–}S^{\mathrm{o}})$$

and the endofunctor $\mathsf{R}\mathrm{Hom}_R(R,-)$ on $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ is induced by the endofunctor $\mathrm{Hom}_R(R,-)$ on $\mathcal{K}(R\text{–}S^{\mathrm{o}})$.

Proof. First assume that R is projective as a $\Bbbk$-module. By 7.3.12 the functor $\mathsf{R}\mathrm{Hom}_R(-,-)$ is augmented as claimed and induced by $\mathrm{Hom}_R(\mathrm{P}_{R\otimes_\Bbbk R^{\mathrm{o}}}(-),-)$. The quasi-isomorphism

$$\pi = \pi^R_{R\otimes_\Bbbk R^{\mathrm{o}}} : \mathrm{P}_{R\otimes_\Bbbk R^{\mathrm{o}}}(R) \longrightarrow R$$

in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$ is by 7.3.11(a) a quasi-isomorphism of semi-projective R-complexes. It thus follows from 5.2.21 and 4.3.19 that the natural transformation $\mathrm{Hom}_R(\pi,-)$ evaluated at any complex of R–S^{o}-bimodules is a homotopy equivalence, in particular a quasi-isomorphism. Therefore, it yields a natural isomorphism of functors $\mathrm{Hom}_R(R,-)^{\H{}} \to \mathrm{Hom}_R(\mathrm{P}_{R\otimes_\Bbbk R^{\mathrm{o}}}(R),-)^{\H{}} = \mathsf{R}\mathrm{Hom}_R(R,-)$; see 6.4.33.

Now assume that S is flat as a $\Bbbk$-module. By 7.3.13 the functor $\mathsf{R}\mathrm{Hom}_R$ is augmented as claimed and induced by $\mathrm{Hom}_R(-,\mathrm{I}_{R\otimes_\Bbbk S^{\mathrm{o}}}(-))$. The natural transformation

$$\mathrm{Hom}_R(R,\iota_{R\otimes_\Bbbk S^{\mathrm{o}}}) : \mathrm{Hom}_R(R,-) \longrightarrow \mathrm{Hom}_R(R,\mathrm{I}_{R\otimes_\Bbbk S^{\mathrm{o}}}(-))$$

evaluated at any complex of R–S^{o}-bimodules is a quasi-isomorphism. Thus it yields a natural isomorphism $\mathrm{Hom}_R(R,-)^{\H{}} \to \mathrm{Hom}_R(R,\mathrm{I}_{R\otimes_\Bbbk S^{\mathrm{o}}}(-))^{\H{}} = \mathsf{R}\mathrm{Hom}_R(R,-)$. □

7.5.6 Construction. Assume that R is projective or S is flat as a $\Bbbk$-module. By 7.5.5 there is a functor

$$\mathsf{R}\mathrm{Hom}_R(R,-) : \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(R\text{–}S^{\mathrm{o}}) \quad \text{induced by} \quad \mathrm{Hom}_R(R,-) .$$

By 7.1.12 there is a natural isomorphism of endofunctors on $\mathcal{K}(R\text{–}S^{\mathrm{o}})$,

$$\epsilon_R : \mathrm{Id}_{\mathcal{K}(R\text{–}S^{\mathrm{o}})} \longrightarrow \mathrm{Hom}_R(R, -) .$$

It induces by 7.2.5 a natural isomorphism of endofunctors on $\mathcal{D}(R\text{–}S^{\mathrm{o}})$,

$$\text{(7.5.6.1)} \qquad \boldsymbol{\epsilon}_R = \epsilon_R'' : \mathrm{Id}_{\mathcal{D}(R\text{–}S^{\mathrm{o}})} \longrightarrow \mathsf{R}\mathrm{Hom}_R(R, -) .$$

7.5.7 Definition. The natural isomorphism (7.5.6.1) is called *counitor*.

7.5.8 Proposition. *Assume that R is projective or S is flat as a $\Bbbk$-module. For M in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ the counitor is an isomorphism in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$,*

$$\boldsymbol{\epsilon}_R^M : M \longrightarrow \mathsf{R}\mathrm{Hom}_R(R, M) ,$$

and natural in M; it is induced by the isomorphism ϵ_R^M in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$. As a natural transformation, $\boldsymbol{\epsilon}_R$ is triangulated.

Proof. The natural isomorphism $\boldsymbol{\epsilon}_R = \epsilon_R''$ from 7.5.6 is triangulated by 7.2.5 as ϵ_R is triangulated by 7.1.12. □

The remaining standard isomorphisms—commutativity, associativity, swap, and adjunction—can be treated along the same lines as the derived functors in Sects. 7.3 and 7.4. Our most general results about augmented standard isomorphsms are 7.5.13 (commutativity), 7.5.20 (associativity), 7.5.27 (swap), and 7.5.33 (adjunction); commonly used special cases are recorded in 7.5.14, 7.5.21, 7.5.28, and 7.5.34.

Commutativity

7.5.9 Construction. Recall from 7.4.1 that there are functors

$$- \otimes_R^{\mathsf{L}} - : \mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R) \longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad \mathrm{P}_{R^{\mathrm{o}}}(-) \otimes_R \mathrm{P}_R(-) \quad \text{and}$$
$$- \otimes_{R^{\mathrm{o}}}^{\mathsf{L}} - : \mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R) \longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad \mathrm{P}_R(-) \otimes_{R^{\mathrm{o}}} \mathrm{P}_{R^{\mathrm{o}}}(-) .$$

To be clear, the latter functor maps (M, N) in $\mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R)$ to $N \otimes_{R^{\mathrm{o}}}^{\mathsf{L}} M$. Consider the natural isomorphism,

$$\upsilon^{\mathrm{P}(-)\,\mathrm{P}(-)} : \mathrm{P}_{R^{\mathrm{o}}}(-) \otimes_R \mathrm{P}_R(-) \longrightarrow \mathrm{P}_R(-) \otimes_{R^{\mathrm{o}}} \mathrm{P}_{R^{\mathrm{o}}}(-) ,$$

of functors from $\mathcal{K}(R^{\mathrm{o}}) \times \mathcal{K}(R)$ to $\mathcal{K}(\Bbbk)$ induced by commutativity 7.1.13. There is a natural isomorphism of functors from $\mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R)$ to $\mathcal{D}(\Bbbk)$ induced by 7.2.5,

$$\text{(7.5.9.1)} \qquad \boldsymbol{\upsilon} = (\upsilon^{\mathrm{P}(-)\,\mathrm{P}(-)})'' : - \otimes_R^{\mathsf{L}} - \longrightarrow - \otimes_{R^{\mathrm{o}}}^{\mathsf{L}} - .$$

7.5.10 Definition. The natural isomorphism (7.5.9.1) is called *commutativity*.

Commutativity, $\boldsymbol{\upsilon}$, is by construction a natural isomorphism of functors from $\mathcal{D}(R^{\mathrm{o}}) \times \mathcal{D}(R)$ to $\mathcal{D}(\Bbbk)$. In some cases, $\boldsymbol{\upsilon}$ can be augmented to a natural isomorphism of functors on derived categories of complexes with additional ring actions.

7.5.11 Setup. Consider ring homomorphisms,

$$Q \otimes_{\Bbbk} R^{\mathrm{o}} \longrightarrow A \qquad \text{and} \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B\,.$$

Let $\mathrm{E_I}$ and $\mathrm{E_{II}}$ be functors from $\mathcal{D}(A) \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\begin{aligned} \varphi_{\mathrm{I}} &: \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} \operatorname{res}^{B}_{R} \longrightarrow \operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \mathrm{E_I} \quad \text{and} \\ \varphi_{\mathrm{II}} &: \operatorname{res}^{B}_{R} \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}} \longrightarrow \operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \mathrm{E_{II}}\,, \end{aligned}$$

where the functors on the left are those from 7.5.9 precomposed with $\operatorname{res}^{A}_{R^{\mathrm{o}}} \times \operatorname{res}^{B}_{R}$. Notice that the functors $\mathrm{E_I}$ and $\mathrm{E_I}$ per 7.4.3 are augmentations of $\otimes^{\mathsf{L}}_{R}$ and $\otimes^{\mathsf{L}}_{R^{\mathrm{o}}}$.

7.5.12 Definition. Adopt the setup 7.5.11. A natural transformation $\boldsymbol{\upsilon}_0 \colon \mathrm{E_I} \to \mathrm{E_{II}}$ is called an *augmentation* of commutativity if the next diagram is commutative,

$$(7.5.12.1) \qquad \begin{tikzcd} \operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \mathrm{E_I} \arrow[r, "\operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \boldsymbol{\upsilon}_0", "\simeq"'] & \operatorname{res}^{Q\otimes S^{\mathrm{o}}}_{\Bbbk} \mathrm{E_{II}} \\ \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} \operatorname{res}^{B}_{R} \arrow[u, "\varphi_{\mathrm{I}}", "\simeq"'] \arrow[r, "\boldsymbol{\upsilon}(\operatorname{res}^{A}_{R^{\mathrm{o}}} \times \operatorname{res}^{B}_{R})"] & \operatorname{res}^{B}_{R} \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}}\,; \arrow[u, "\simeq", "\varphi_{\mathrm{II}}"'] \end{tikzcd}$$

here $\boldsymbol{\upsilon}$ on the lower horizontal arrow is (7.5.9.1). In this case, $\boldsymbol{\upsilon}_0 \colon \mathrm{E_I} \to \mathrm{E_{II}}$ is written $\boldsymbol{\upsilon} \colon - \otimes^{\mathsf{L}}_{R} - \to - \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} -$, and one says that commutativity is *augmented* to a natural isomorphism of functors from $\mathcal{D}(A) \times \mathcal{D}(B)$ to $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$.

7.5.13 Theorem. *Let $Q \otimes_{\Bbbk} R^{\mathrm{o}} \to A$ and $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$ be ring homomorphisms. If condition* (a) *or* (b) *below is satisfied, then commutativity is augmented to a natural isomorphism of functors $\mathcal{D}(A) \times \mathcal{D}(B) \to \mathcal{D}(Q\text{–}S^{\mathrm{o}})$. That is, for M in $\mathcal{D}(A)$ and N in $\mathcal{D}(B)$ there is an isomorphism in $\mathcal{D}(Q\text{–}S^{\mathrm{o}})$,*

$$(7.5.13.1) \qquad \boldsymbol{\upsilon}^{MN} : M \otimes^{\mathsf{L}}_{R} N \longrightarrow N \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} M\,,$$

which is natural in M and N. As a natural transformation of functors, this augmented $\boldsymbol{\upsilon}$ is triangulated in each variable.

(a) *A is flat as an R^{o}-module.*
(b) *B is flat as an R-module.*

Moreover, (7.5.13.1) *is induced by isomorphisms in $\mathcal{K}(Q\text{–}S^{\mathrm{o}})$ as follows*:

(a′) *If* (a) *is satisfied, then $\boldsymbol{\upsilon}^{MN}$ is induced by $\upsilon^{\mathrm{P}_A(M)N}$.*
(a′) *If* (b) *is satisfied, then $\boldsymbol{\upsilon}^{MN}$ is induced by $\upsilon^{M\,\mathrm{P}_B(N)}$.*

Proof. Let $\mathrm{U_I} = - \otimes_R -$ and $\mathrm{U_{II}} = - \otimes_{R^{\mathrm{o}}} -$ be the tensor product functors from $\mathcal{K}(R^{\mathrm{o}}) \times \mathcal{K}(R)$ to $\mathcal{K}(\Bbbk)$ and $\mathrm{V_I}$ and $\mathrm{V_{II}}$ the tensor product functors from $\mathcal{K}(Q\text{–}R^{\mathrm{o}}) \times \mathcal{K}(R\text{–}S^{\mathrm{o}})$ to $\mathcal{K}(Q\text{–}S^{\mathrm{o}})$. Set $\mathrm{F} = \mathrm{P}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}} \times \mathrm{P}_{R} \operatorname{res}^{B}_{R}$ and consider the functors from $\mathcal{K}(A) \times \mathcal{K}(B)$ to $\mathcal{K}(\Bbbk)$,

$$\mathrm{U_I\,F} = \mathrm{P}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes_R \mathrm{P}_R \operatorname{res}^{B}_{R} \qquad \text{and} \qquad \mathrm{U_{II}\,F} = \mathrm{P}_R \operatorname{res}^{B}_{R} \otimes_{R^{\mathrm{o}}} \mathrm{P}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}}\,.$$

They induce the derived functors $\operatorname{res}^A_{R^o} \otimes^{\mathsf{L}}_R \operatorname{res}^B_R$ and $\operatorname{res}^B_R \otimes^{\mathsf{L}}_{R^o} \operatorname{res}^A_{R^o}$, which are the functors from 7.5.9 precomposed with $\operatorname{res}^A_{R^o} \times \operatorname{res}^B_R$.

Assume that (a) is satisfied and set

$$\mathrm{F}_0 \;=\; \operatorname{res}^A_{Q\otimes R^o} \mathrm{P}_A \times \operatorname{res}^B_{R\otimes S^o} : \mathcal{K}(A)\times\mathcal{K}(B) \;\longrightarrow\; \mathcal{K}(Q\text{–}R^o)\times\mathcal{K}(R\text{–}S^o)$$

Let E_{I} and E_{II} be the functors $\mathcal{D}(A)\times\mathcal{D}(B)\to\mathcal{D}(Q\text{–}S^o)$ induced per 7.4.5(a′) by

$$\mathrm{V}_{\mathrm{I}}\,\mathrm{F}_0 \;=\; \operatorname{res}^A_{Q\otimes R^o} \mathrm{P}_A \otimes_R \operatorname{res}^B_{R\otimes S^o} \quad\text{and}\quad \mathrm{V}_{\mathrm{II}}\,\mathrm{F}_0 \;=\; \operatorname{res}^B_{R\otimes S^o} \otimes_{R^o} \operatorname{res}^A_{Q\otimes R^o} \mathrm{P}_A\,.$$

Consider the natural transformation of functors,

$$\mathrm{F} = \mathrm{P}_{R^o}\operatorname{res}^A_{R^o}\times \mathrm{P}_R\operatorname{res}^B_R \xrightarrow{\tau\,=\,\varrho^A_{R^o}\times\pi_R\operatorname{res}^B_R} \operatorname{res}^A_{R^o}\mathrm{P}_A\times\operatorname{res}^B_R = (\operatorname{res}^{Q\otimes R^o}_{R^o}\times\operatorname{res}^{R\otimes S^o}_R)\,\mathrm{F}_0\,.$$

It follows from 7.4.6 that the natural transformation

$$\mathrm{U}_{\mathrm{I}}\,\tau :\; \mathrm{U}_{\mathrm{I}}\,\mathrm{F} \;\longrightarrow\; \mathrm{U}_{\mathrm{I}}(\operatorname{res}^{Q\otimes R^o}_{R^o}\times\operatorname{res}^{R\otimes S^o}_R)\,\mathrm{F}_0 \;=\; \operatorname{res}^{Q\otimes S^o}_{\Bbbk}\mathrm{V}_{\mathrm{I}}\,\mathrm{F}_0$$

induces a natural isomorphism φ_{I}, as stipulated in 7.5.11; similarly a natural isomorphism φ_{II} is induced by $\mathrm{U}_{\mathrm{II}}\,\tau$. Commutativity 7.1.13 is a natural isomorphism $\upsilon\colon \mathrm{U}_{\mathrm{I}}\to\mathrm{U}_{\mathrm{II}}$ and $\upsilon\colon\mathrm{V}_{\mathrm{I}}\to\mathrm{V}_{\mathrm{II}}$. The commutative diagram

$$\begin{tikzcd}
\operatorname{res}^{Q\otimes S^o}_{\Bbbk}\mathrm{V}_{\mathrm{I}}\,\mathrm{F}_0 \arrow[r, "\operatorname{res}^{Q\otimes S^o}_{\Bbbk}\upsilon\,\mathrm{F}_0"] & \operatorname{res}^{Q\otimes S^o}_{\Bbbk}\mathrm{V}_{\mathrm{II}}\,\mathrm{F}_0 \\
\mathrm{U}_{\mathrm{I}}\,\mathrm{F} \arrow[u, "\mathrm{U}_{\mathrm{I}}\,\tau"] \arrow[r, "\upsilon\,\mathrm{F}"] & \mathrm{U}_{\mathrm{II}}\,\mathrm{F} \arrow[u, "\mathrm{U}_{\mathrm{II}}\,\tau"']
\end{tikzcd}$$

induces the diagram (7.5.12.1); in particular, $\upsilon\,\mathrm{F}_0\colon \mathrm{V}_{\mathrm{I}}\,\mathrm{F}_0\to\mathrm{V}_{\mathrm{II}}\,\mathrm{F}_0$ induces a natural isomorphism $\boldsymbol{\upsilon}_0\colon\mathrm{E}_{\mathrm{I}}\to\mathrm{E}_{\mathrm{II}}$. Thus if condition (a) is satisfied, then commutativity is augmented to a natural isomorphism of functors $\mathcal{D}(A)\times\mathcal{D}(B)\to\mathcal{D}(Q\text{–}S^o)$ which is induced by $\upsilon\,\mathrm{F}_0$ as claimed in (a′). The assertion about triangulation follows from 7.2.5 and 7.1.13.

The remaining assertions are proved similarly. □

Notice that condition (a) in the next corollary is satisfied if $Q = \Bbbk$ and (b) is satisfied if $S = \Bbbk$. Either condition is satisfied if $\Bbbk$ is a field.

7.5.14 Corollary. *For complexes M in $\mathcal{D}(Q\text{–}R^o)$ and N in $\mathcal{D}(R\text{–}S^o)$ commutativity,*

$$\boldsymbol{\upsilon}^{MN} : M\otimes^{\mathsf{L}}_R N \;\longrightarrow\; N\otimes^{\mathsf{L}}_{R^o} M\,,$$

is an isomorphism in $\mathcal{D}(Q\text{–}S^o)$ under either of the following conditions:

(a) *Q is flat as a $\Bbbk$-module.*
(b) *S is flat as a $\Bbbk$-module.*

Proof. Apply 7.5.13 with $A = Q\otimes_{\Bbbk} R^o$ and $B = R\otimes_{\Bbbk} S^o$ and invoke 7.3.11(b). □

7.5.15 Example. Let $Q\otimes_{\Bbbk} R^o \to A$ and $R\otimes_{\Bbbk} S^o\to B$ be ring homomorphisms and assume that B is flat as an R-module. Let M be an A-complex and N a B-complex with a semi-flat replacement G. By 6.4.20 there is a quasi-isomorphism $\vartheta\colon \mathrm{P}_B(N)\to G$ in $\mathcal{K}(B)$ which yields a commutative diagram in $\mathcal{K}(Q\text{–}S^o)$,

$$\begin{tikzcd}
M \otimes_R \mathrm{P}_B(N) \arrow[r, "\upsilon^{M\,\mathrm{P}(N)}"] \arrow[d, "M \otimes \vartheta"', "\simeq"] & \mathrm{P}_B(N) \otimes_{R^\mathrm{o}} M \arrow[d, "\simeq"', "\vartheta \otimes M"] \\
M \otimes_R G \arrow[r, "\upsilon^{MG}"] & G \otimes_{R^\mathrm{o}} M \,.
\end{tikzcd}$$

The vertical morphisms are quasi-isomorphisms by 5.4.16 and semi-flatness of $\mathrm{P}_B(N)$ and G as R-complexes; see 5.4.18(b). It follows that υ^{MG} induces an isomorphism in $\mathcal{D}(Q\text{–}S^\mathrm{o})$ that is isomorphic to the augmented commutativity isomorphism $\boldsymbol{\upsilon}^{MN}$ from 7.5.13(b′).

Associativity

7.5.16 Construction. Recall from 7.4.8 and 7.4.9 that there are functors,

$$\begin{aligned}
-\otimes^{\mathsf{L}}_R - : \mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) &\longrightarrow \mathcal{D}(S^\mathrm{o}) \quad \text{induced by} \quad \mathrm{P}_{R^\mathrm{o}}(-) \otimes_R - \quad \text{and}\\
-\otimes^{\mathsf{L}}_S - : \mathcal{D}(S^\mathrm{o}) \times \mathcal{D}(S) &\longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad -\otimes_S \mathrm{P}_S(-)\,.
\end{aligned}$$

As the functor $(-\otimes^{\mathsf{L}}_R -) \otimes^{\mathsf{L}}_S -$ is the composite of the functors

$$\mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S) \xrightarrow{(-\otimes^{\mathsf{L}}_R -) \times \mathrm{Id}_{\mathcal{D}(S)}} \mathcal{D}(S^\mathrm{o}) \times \mathcal{D}(S) \xrightarrow{-\otimes^{\mathsf{L}}_S -} \mathcal{D}(\Bbbk)\,,$$

it follows from 7.2.4 that the functor

$$\begin{aligned}
(-\otimes^{\mathsf{L}}_R -) \otimes^{\mathsf{L}}_S - : \mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S) &\longrightarrow \mathcal{D}(\Bbbk)\\
\text{is induced by } (\mathrm{P}_{R^\mathrm{o}}(-) \otimes_R -) \otimes_S \mathrm{P}_S(-)&\,.
\end{aligned} \tag{7.5.16.1}$$

Similarly, the composite

$$\begin{aligned}
-\otimes^{\mathsf{L}}_R (-\otimes^{\mathsf{L}}_S -) : \mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S) &\longrightarrow \mathcal{D}(\Bbbk)\\
\text{is induced by } \mathrm{P}_{R^\mathrm{o}}(-) \otimes_R (-\otimes_S \mathrm{P}_S(-))&\,.
\end{aligned} \tag{7.5.16.2}$$

Now, consider the natural isomorphism,

$$\omega^{\mathrm{P}(-)-\mathrm{P}(-)} : (\mathrm{P}_{R^\mathrm{o}}(-) \otimes_R -) \otimes_S \mathrm{P}_S(-) \longrightarrow \mathrm{P}_{R^\mathrm{o}}(-) \otimes_R (-\otimes_S \mathrm{P}_S(-))\,,$$

of functors $\mathcal{K}(R^\mathrm{o}) \times \mathcal{K}(R\text{–}S^\mathrm{o}) \times \mathcal{K}(S) \to \mathcal{K}(\Bbbk)$ induced by associativity 7.1.14. By 7.2.5 there is a natural isomorphism of functors $\mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S) \to \mathcal{D}(\Bbbk)$,

$$\boldsymbol{\omega} = (\omega^{\mathrm{P}(-)-\mathrm{P}(-)})^{\sim} : (-\otimes^{\mathsf{L}}_R -) \otimes^{\mathsf{L}}_S - \longrightarrow -\otimes^{\mathsf{L}}_R (-\otimes^{\mathsf{L}}_S -)\,. \tag{7.5.16.3}$$

7.5.17 Definition. The natural isomorphism (7.5.16.3) is called *associativity*.

Associativity, $\boldsymbol{\omega}$, is by construction a natural isomorphism of functors from $\mathcal{D}(R^\mathrm{o}) \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S)$ to $\mathcal{D}(\Bbbk)$. In some cases, $\boldsymbol{\omega}$ can be augmented to a natural isomorphism of functors on derived categories of complexes with additional ring actions.

7.5.18 Setup. Consider ring homomorphisms,

$$Q \otimes_{\Bbbk} R^{\mathrm{o}} \longrightarrow A\,, \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B\,, \qquad \text{and} \qquad S \otimes_{\Bbbk} T^{\mathrm{o}} \longrightarrow C\,.$$

Let E_{I} and E_{II} be functors from $\mathcal{D}(A) \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\begin{aligned}
\varphi_{\mathrm{I}} &: (\operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} \operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}}) \otimes^{\mathsf{L}}_{S} \operatorname{res}^{C}_{S} \longrightarrow \operatorname{res}^{Q \otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \quad \text{and}\\
\varphi_{\mathrm{II}} &: \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} (\operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_{S} \operatorname{res}^{C}_{S}) \longrightarrow \operatorname{res}^{Q \otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}}\,,
\end{aligned}$$

where the functors on the left are (7.5.16.1) and (7.5.16.2) both precomposed with $\operatorname{res}^{A}_{R^{\mathrm{o}}} \times \operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}} \times \operatorname{res}^{C}_{S}$.

7.5.19 Definition. Adopt the setup 7.5.18. A natural transformation $\boldsymbol{\omega}_0 \colon \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is called an *augmentation* of associativity if the next diagram is commutative,

$$\begin{tikzcd}
\operatorname{res}^{Q \otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \arrow[r, "\operatorname{res}^{Q \otimes T^{\mathrm{o}}}_{\Bbbk} \omega_0"] & \operatorname{res}^{Q \otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \\
(\operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} \operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}}) \otimes^{\mathsf{L}}_{S} \operatorname{res}^{C}_{S} \arrow[u, "\varphi_{\mathrm{I}}", "\simeq"'] \arrow[r, "\omega\,(\operatorname{res}^{A}_{R^{\mathrm{o}}} \times \operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}} \times \operatorname{res}^{C}_{S})", "\simeq"'] & \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes^{\mathsf{L}}_{R} (\operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_{S} \operatorname{res}^{C}_{S}) \arrow[u, "\simeq", "\varphi_{\mathrm{II}}"'] \,;
\end{tikzcd}$$

here ω on the lower horizontal arrow is (7.5.16.3). In this case, $\boldsymbol{\omega}_0 \colon \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is written $\boldsymbol{\omega} \colon (- \otimes^{\mathsf{L}}_{R} -) \otimes^{\mathsf{L}}_{S} - \to - \otimes^{\mathsf{L}}_{R} (- \otimes^{\mathsf{L}}_{S} -)$, and one says that associativity is *augmented* to a natural isomorphism of functors from $\mathcal{D}(A) \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$.

7.5.20 Theorem. *Let $Q \otimes_{\Bbbk} R^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $S \otimes_{\Bbbk} T^{\mathrm{o}} \to C$ be ring homomorphisms. If condition* (a), (b)*, or* (c) *below is met, then associtivity is augmented to a natural isomorphism of functors $\mathcal{D}(A) \times \mathcal{D}(B) \times \mathcal{D}(C) \to \mathcal{D}(Q\text{–}T^{\mathrm{o}})$. That is, for M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$ there is an isomorphism,*

$$\boldsymbol{\omega}^{MXN} : (M \otimes^{\mathsf{L}}_{R} X) \otimes^{\mathsf{L}}_{S} N \longrightarrow M \otimes^{\mathsf{L}}_{R} (X \otimes^{\mathsf{L}}_{S} N)\,, \tag{7.5.20.1}$$

in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ which is natural in M, X, and N. As a natural transformation of functors, this augmented $\boldsymbol{\omega}$ is triangulated in each variable.

(a) *A is flat as an R^{o}-module and C is flat as an S-module.*
(b) *A is flat as an R^{o}-module and B is flat as an S^{o}-module.*
(c) *B is flat as an R-module and C is flat as an S-module.*

Moreover, (7.5.20.1) *is induced by isomorphisms in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ as follows*:

(a′) *If* (a) *is satisfied, then $\boldsymbol{\omega}^{MXN}$ is induced by $\omega^{\mathrm{P}_A(M) X \mathrm{P}_C(N)}$.*
(b′) *If* (b) *is satisfied, then $\boldsymbol{\omega}^{MXN}$ is induced by $\omega^{\mathrm{P}_A(M)\, \mathrm{P}_B(X) N}$.*
(c′) *If* (c) *is satisfied, then $\boldsymbol{\omega}^{MXN}$ is induced by $\omega^{M\, \mathrm{P}_B(X)\, \mathrm{P}_C(N)}$.*

Proof. Consider the functors from $\mathcal{K}(A) \times \mathcal{K}(B) \times \mathcal{K}(C)$ to $\mathcal{K}(\Bbbk)$,

$$\begin{aligned}
\mathrm{C}_{\mathrm{I}} &= (\mathrm{P}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes_{R} \operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}}) \otimes_{S} \mathrm{P}_{S} \operatorname{res}^{C}_{S}\,;\\
\mathrm{C}_{\mathrm{II}} &= \mathrm{P}_{R^{\mathrm{o}}} \operatorname{res}^{A}_{R^{\mathrm{o}}} \otimes_{R} (\operatorname{res}^{B}_{R \otimes S^{\mathrm{o}}} \otimes_{S} \mathrm{P}_{S} \operatorname{res}^{C}_{S})\,.
\end{aligned}$$

They induce the functors $(\operatorname{res}^A_{R^\circ} \otimes^{\mathsf{L}}_R \operatorname{res}^B_{R\otimes S^\circ}) \otimes^{\mathsf{L}}_S \operatorname{res}^C_S$ and $\operatorname{res}^A_{R^\circ} \otimes^{\mathsf{L}}_R (\operatorname{res}^B_{R\otimes S^\circ} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S)$, which are (7.5.16.1) and (7.5.16.2) precomposed with $\operatorname{res}^A_{R^\circ} \times \operatorname{res}^B_{R\otimes S^\circ} \times \operatorname{res}^C_S$.

Assume that condition (b) is satisfied and consider the functors

$$\begin{aligned} \mathrm{D_I} &= (\operatorname{res}^A_{Q\otimes R^\circ} \mathrm{P}_A \otimes_R \operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B) \otimes_S \operatorname{res}^C_{S\otimes T^\circ}\,, \\ \mathrm{D_{II}} &= \operatorname{res}^A_{Q\otimes R^\circ} \mathrm{P}_A \otimes_R (\operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B \otimes_S \operatorname{res}^C_{S\otimes T^\circ}) \end{aligned}$$

from $\mathcal{K}(A) \times \mathcal{K}(B) \times \mathcal{K}(C)$ to $\mathcal{K}(Q\text{–}T^\circ)$. The functor $\operatorname{res}^{Q\otimes T^\circ}_{\Bbbk} \mathrm{D_I}$ compares to $\mathrm{C_I}$ via the following natural transformations,

$$(\dagger)\qquad \begin{array}{c} \operatorname{res}^{Q\otimes T^\circ}_{\Bbbk} \mathrm{D_I} \;=\; (\operatorname{res}^A_{R^\circ} \mathrm{P}_A \otimes_R \operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B) \otimes_S \operatorname{res}^C_S \\ \Big\uparrow \sigma_1 = (\varrho^A_{R^\circ} \otimes 1)\otimes 1 \\ (\mathrm{P}_{R^\circ} \operatorname{res}^A_{R^\circ} \otimes_R \operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B) \otimes_S \operatorname{res}^C_S \\ \Big\uparrow \sigma_2 = (1\otimes 1)\otimes \pi_S \operatorname{res}^C_S \\ (\mathrm{P}_{R^\circ} \operatorname{res}^A_{R^\circ} \otimes_R \operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B) \otimes_S \mathrm{P}_S \operatorname{res}^C_S \\ \Big\downarrow \sigma_3 = (1\otimes \operatorname{res}^B_{R\otimes S^\circ} \pi_B)\otimes 1 \\ \mathrm{C_I} \;=\; (\mathrm{P}_{R^\circ} \operatorname{res}^A_{R^\circ} \otimes_R \operatorname{res}^B_{R\otimes S^\circ}) \otimes_S \mathrm{P}_S \operatorname{res}^C_S\,. \end{array}$$

Let (M, X, N) be an object in $\mathcal{K}(A) \times \mathcal{K}(B) \times \mathcal{K}(C)$; we argue that the morphisms σ_i^{MXN} are quasi-isomorphisms. As (b) is satisfied, 5.4.18(b) implies that the complex $\mathrm{P}_A(M)$ is semi-flat over R° and $\mathrm{P}_B(X)$ is semi-flat over S°. With $\varrho = \varrho^A_{R^\circ}$ from 6.3.21 one has $\sigma_1^{MXN} = (\varrho^M \otimes_R \mathrm{P}_B(X)) \otimes_S N$. As ϱ^M is a quasi-isomorphism of semi-flat R°-complexes, its tensor product with $\mathrm{P}_B(X)$ is a quasi-isomorphism of semi-flat S°-complexes, see 5.4.16 and 5.4.17, so σ_1^{MXN} is a quasi-isomorphism, again by 5.4.16. As just noted, the S°-complex $\mathrm{P}_{R^\circ}(M) \otimes_R \mathrm{P}_B(X)$ is semi-flat and thus $\sigma_2^{MXN} = (\mathrm{P}_{R^\circ}(M) \otimes_R \mathrm{P}_B(X)) \otimes_S \pi^N_S$ is a quasi-isomorphism. Finally, the morphism $\sigma_3^{MXN} = (\mathrm{P}_{R^\circ}(M) \otimes_R \pi^X_B) \otimes_S \mathrm{P}_S(N)$ is a quasi-isomorphism, as π^X_B is a quasi-isomorphism and the complexes $\mathrm{P}_{R^\circ}(M)$ and $\mathrm{P}_S(N)$ are semi-flat.

The functor $\mathrm{C_I}$ preserves quasi-isomorphisms, this is implicit in 7.5.16, so by the argument above, $\operatorname{res}^{Q\otimes T^\circ}_{\Bbbk} \mathrm{D_I}$ and hence $\mathrm{D_I}$ preserves quasi-isomorphisms. Let $\mathrm{E_I}$ denote the functor from $\mathcal{D}(A) \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^\circ)$ induced per 7.2.2 by $\mathrm{D_I}$.

The diagram (†) induces a natural isomorphism of functors,

$$(\ddagger)\qquad \varphi_{\mathrm{I}} : (\operatorname{res}^A_{R^\circ} \otimes^{\mathsf{L}}_R \operatorname{res}^B_{R\otimes S^\circ}) \otimes^{\mathsf{L}}_S \operatorname{res}^C_S \longrightarrow \operatorname{res}^{Q\otimes T^\circ}_{\Bbbk} \mathrm{E_I}\,;$$

we proceed to give a more compact description of it. Consider the functor

$$\mathrm{F}_0 : \mathcal{K}(A) \times \mathcal{K}(B) \times \mathcal{K}(C) \longrightarrow \mathcal{K}(Q\text{–}R^\circ) \times \mathcal{K}(R\text{–}S^\circ) \times \mathcal{K}(S\text{–}T^\circ)$$

given by $\mathrm{F}_0 = \operatorname{res}^A_{Q\otimes R^\circ} \mathrm{P}_A \times \operatorname{res}^B_{R\otimes S^\circ} \mathrm{P}_B \times \operatorname{res}^C_{S\otimes T^\circ}$. With the abbreviated notation $\mathrm{U_I}$ for the functor

$$(-\otimes_R -)\otimes_S - : \mathcal{K}(Q\text{–}R^\circ) \times \mathcal{K}(R\text{–}S^\circ) \times \mathcal{K}(S\text{–}T^\circ) \longrightarrow \mathcal{K}(Q\text{–}T^\circ)$$

one has $\mathrm{D_I} = \mathrm{U_I}\,\mathrm{F}_0$. Further (†) can be written

$$\mathrm{C_I} \;=\; \mathrm{U_I}\,\mathrm{F}_4 \xleftarrow{\mathrm{U_I}\,\tau_3} \mathrm{U_I}\,\mathrm{F}_3 \xrightarrow{\mathrm{U_I}\,\tau_2} \mathrm{U_I}\,\mathrm{F}_2 \xrightarrow{\mathrm{U_I}\,\tau_1} \mathrm{U_I}\,\mathrm{F}_1 \;=\; \mathrm{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}}\,\mathrm{D_I}$$

where the functors

$$\mathrm{F}_i : \mathcal{K}(A)\times\mathcal{K}(B)\times\mathcal{K}(C) \;\longrightarrow\; \mathcal{K}(R^{\mathrm{o}})\times\mathcal{K}(R\text{–}S^{\mathrm{o}})\times\mathcal{K}(S)$$

and transformations τ_i between them are implicitly defined by (†). The natural isomorphism (‡) can now be written as $\varphi_{\mathrm{I}} = (\mathrm{U_I}\,\tau_1)^{''}(\mathrm{U_I}\,\tau_2)^{''}(\mathrm{U_I}\,\tau_3)^{''-1}$; see 7.2.5.

With the notation $\mathrm{U_{II}}$ for the functor $-\otimes_R(-\otimes_S -)$ one has $\mathrm{D_{II}} = \mathrm{U_{II}}\,\mathrm{F}_0$. An argument parallel to the one above shows that each transformation $\mathrm{U_{II}}\,\tau_i$ evaluated at an object (M,X,N) is a quasi-isomorphism. It follows that $\mathrm{D_{II}}$ preserves quasi-isomorphisms; the induced functor from $\mathcal{D}(A)\times\mathcal{D}(B)\times\mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ is denoted $\mathrm{E_{II}}$; see 7.2.2. Finally, $\varphi_{\mathrm{II}} = (\mathrm{U_{II}}\,\tau_1)^{''}(\mathrm{U_{II}}\,\tau_2)^{''}(\mathrm{U_I}\,\tau_3)^{''-1}$ is a natural isomorphism of functors,

$$\varphi_{\mathrm{II}} : \mathrm{res}_{R^{\mathrm{o}}}^{A} \otimes_R^{\mathsf{L}} (\mathrm{res}_{R\otimes S^{\mathrm{o}}}^{B} \otimes_S^{\mathsf{L}} \mathrm{res}_S^{C}) \;\longrightarrow\; \mathrm{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}}\,\mathrm{E_{II}}\;.$$

Associativity 7.1.14 yields a natural isomorphism $\omega\colon \mathrm{U_I}\to\mathrm{U_{II}}$. The composite $\omega\,\mathrm{F}_0\colon \mathrm{D_I}\to\mathrm{D_{II}}$ induces per 7.2.5 a natural isomorphism $\omega_0\colon \mathrm{E_I}\to\mathrm{E_{II}}$. Now the commutative diagram

$$\begin{tikzcd}
\mathrm{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}}\,\mathrm{D_I} \arrow[r, "\mathrm{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}}\,\omega\,\mathrm{F}_0"] \arrow[d, equal] & \mathrm{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}}\,\mathrm{D_{II}} \arrow[d, equal] \\
\mathrm{U_I}\,\mathrm{F}_1 \arrow[r, "\omega\,\mathrm{F}_1"] & \mathrm{U_{II}}\,\mathrm{F}_1 \\
\mathrm{U_I}\,\mathrm{F}_2 \arrow[r, "\omega\,\mathrm{F}_2"] \arrow[u, "\mathrm{U_I}\,\tau_1"] & \mathrm{U_{II}}\,\mathrm{F}_2 \arrow[u, "\mathrm{U_{II}}\,\tau_1"'] \\
\mathrm{U_I}\,\mathrm{F}_3 \arrow[r, "\omega\,\mathrm{F}_3"] \arrow[u, "\mathrm{U_I}\,\tau_2"] \arrow[d, "\mathrm{U_I}\,\tau_3"'] & \mathrm{U_{II}}\,\mathrm{F}_3 \arrow[u, "\mathrm{U_{II}}\,\tau_2"'] \arrow[d, "\mathrm{U_{II}}\,\tau_3"] \\
\mathrm{C_I} = \mathrm{U_I}\,\mathrm{F}_4 \arrow[r, "\omega\,\mathrm{F}_4"] & \mathrm{U_{II}}\,\mathrm{F}_4 = \mathrm{C_{II}}
\end{tikzcd}$$

induces the diagram in 7.5.19. Thus if condition (b) is satisfied, then associativity is augmented to a natural isomorphism of functors $\mathcal{D}(A)\times\mathcal{D}(B)\times\mathcal{D}(C)\to\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ which is induced by $\omega\,\mathrm{F}_0$ as claimed in (b). The assertion about triangulation follows from 7.2.5 and 7.1.14.

A parallel argument shows that associativity is augmented if condition (c) is satisfied and represented as stated in (c′). A similar, but simpler, argument shows that associativity is augmented under condition (a) and represented as stated in (a′). □

Notice that any of the conditions in the next corollary is satisfied if $\Bbbk$ is a field.

7.5.21 Corollary. *For complexes M in $\mathcal{D}(Q\text{–}R^{\mathrm{o}})$, X in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{D}(S\text{–}T^{\mathrm{o}})$ associativity*

$$\omega^{MXN} : (M\otimes_R^{\mathsf{L}} X)\otimes_S^{\mathsf{L}} N \;\longrightarrow\; M\otimes_R^{\mathsf{L}}(X\otimes_S^{\mathsf{L}} N)\,,$$

is an isomorphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ under each of the following conditions:

(a) *Q and T are flat as $\Bbbk$-modules.*
(b) *Q and R are flat as $\Bbbk$-modules.*
(c) *S and T are flat as $\Bbbk$-modules.*

PROOF. Apply 7.5.20 with $A = Q \otimes_{\Bbbk} R^{\mathrm{o}}$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = S \otimes_{\Bbbk} T^{\mathrm{o}}$ and invoke 7.3.11(b). □

7.5.22 Example. Adopt the setup from 7.5.18 and let M be an A-complex, X a B-complex, and N a C-complex. Let F be a semi-flat replacement of M and F' a semi-flat replacement of N. By 6.4.20 there are quasi-isomorphisms $\vartheta\colon \mathrm{P}_A(M) \to F$ in $\mathcal{K}(A)$ and $\vartheta'\colon \mathrm{P}_C(N) \to F'$ in $\mathcal{K}(C)$ which give a commutative diagram in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$,

$$\begin{tikzcd}
(\mathrm{P}_A(M) \otimes_R X) \otimes_S \mathrm{P}_C(N) \arrow[r, "\omega^{\mathrm{P}(M)X\mathrm{P}(N)}"] \arrow[d, "(\vartheta\otimes X)\otimes\vartheta'"'] & \mathrm{P}_A(M) \otimes_R (X \otimes_S \mathrm{P}_C(N)) \arrow[d, "\vartheta\otimes(X\otimes\vartheta')"] \\
(F \otimes_R X) \otimes_S F' \arrow[r, "\omega^{FXF'}"] & F \otimes_R (X \otimes_S F')\,.
\end{tikzcd}$$

Assume that A is flat as an R^{o}-module and C is flat as an S-module. It follows from 5.4.18(b) that F and $\mathrm{P}_A(M)$ are semi-flat R^{o}-complexes and that F' and $\mathrm{P}_C(M)$ are semi-flat S-complexes. Thus the vertical maps in the diagram are quasi-isomorphisms by 5.4.16. It follows that $\omega^{FXF'}$ induces a morphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ which is isomorphic to the augmented associativity isomorphism $\boldsymbol{\omega}^{MXN}$ from 7.5.20(a′).

SWAP

7.5.23 Construction. Recall from 7.3.8 that there are functors,

$$\begin{aligned}
\mathsf{RHom}_{S^{\mathrm{o}}}(-,-) : \mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) &\longrightarrow \mathcal{D}(R) \quad \text{induced by} \quad \mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),-)\,,\\
\mathsf{RHom}_R(-,-) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R) &\longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad \mathrm{Hom}_R(\mathrm{P}_R(-),-)\,.
\end{aligned}$$

It follows from 7.2.4, cf. the argument in 7.5.16, that the composite functor

(7.5.23.1)
$$\begin{gathered}
\mathsf{RHom}_R(-,\mathsf{RHom}_{S^{\mathrm{o}}}(-,-)) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}} \longrightarrow \mathcal{D}(\Bbbk)\\
\text{is induced by } \mathrm{Hom}_R(\mathrm{P}_R(-),\mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),-))\,.
\end{gathered}$$

Applied to an object (M,X,N) this functor yields $\mathsf{RHom}_R(M,\mathsf{RHom}_{S^{\mathrm{o}}}(N,X))$.

Similarly, the composite

(7.5.23.2)
$$\begin{gathered}
\mathsf{RHom}_{S^{\mathrm{o}}}(-,\mathsf{RHom}_R(-,-)) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}} \longrightarrow \mathcal{D}(\Bbbk)\\
\text{is induced by } \mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),\mathrm{Hom}_R(\mathrm{P}_R(-),-))\,.
\end{gathered}$$

Applied to an object (M,X,N) this functor yields $\mathsf{RHom}_{S^{\mathrm{o}}}(N,\mathsf{RHom}_R(M,X))$.

Now, consider the natural isomorphism,

$$\mathrm{Hom}_R(\mathrm{P}_R(-),\mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),-)) \xrightarrow{\zeta^{\mathrm{P}(-)-\mathrm{P}(-)}} \mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),\mathrm{Hom}_R(\mathrm{P}_R(-),-))\,,$$

of functors from $\mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \times \mathcal{K}(S^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{K}(\Bbbk)$ induced by swap 7.1.15. There is a natural isomorphism of functors $\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}} \to \mathcal{D}(\Bbbk)$ induced by 7.2.5,

(7.5.23.3)
$$\mathsf{RHom}_R(-, \mathsf{RHom}_{S^{\mathrm{o}}}(-,-)) \xrightarrow{\;\zeta = (\zeta^{\mathrm{P}(-)-\mathrm{P}(-)})''\;} \mathsf{RHom}_{S^{\mathrm{o}}}(-, \mathsf{RHom}_R(-,-))\,.$$

7.5.24 Definition. The natural isomorphism (7.5.23.3) is called *swap*.

Swap, ζ, is by construction a natural isomorphism of functors from the category $\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S^{\mathrm{o}})^{\mathrm{op}}$ to $\mathcal{D}(\Bbbk)$. In some cases, ζ can be augmented to a natural isomorphism of functors on derived categories of complexes with additional ring actions.

7.5.25 Setup. Consider ring homomorphisms,

$$R \otimes_{\Bbbk} Q^{\mathrm{o}} \longrightarrow A\,, \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B\,, \qquad \text{and} \qquad T \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow C\,.$$

Let E_{I} and E_{II} be functors from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)^{\mathrm{op}}$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\begin{aligned} \varphi_{\mathrm{I}} &: \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \longrightarrow \mathsf{RHom}_R(\operatorname{res}^A_R, \mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^C_{S^{\mathrm{o}}}, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}})) \quad \text{and} \\ \varphi_{\mathrm{II}} &: \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \longrightarrow \mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^C_{S^{\mathrm{o}}}, \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}))\,, \end{aligned}$$

where the functors on the right are (7.5.23.1) and (7.5.23.2) both precomposed with $\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{oo}}} \times \operatorname{res}^C_{S^{\mathrm{o}}}$.

7.5.26 Definition. Adopt the setup 7.5.25. A natural transformation $\zeta_0 \colon \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is called an *augmentation* of swap if the next diagram is commutative,

$$\begin{tikzcd}
\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \arrow[r, "\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \zeta_0"] \arrow[d, "\varphi_{\mathrm{I}}"', "\simeq"] & \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \arrow[d, "\simeq"', "\varphi_{\mathrm{II}}"] \\
\mathsf{RHom}_R(\operatorname{res}^A_R, \mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^C_{S^{\mathrm{o}}}, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}})) \arrow[r, "\zeta\,(\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{oo}}} \times \operatorname{res}^C_{S^{\mathrm{o}}})", "\simeq"'] & \mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^C_{S^{\mathrm{o}}}, \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}))
\end{tikzcd}$$

where ζ on the lower horizontal map is (7.5.23.3). In this case, $\zeta_0 \colon \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is written $\zeta \colon \mathsf{RHom}_R(-, \mathsf{RHom}_{S^{\mathrm{o}}}(-,-)) \to \mathsf{RHom}_{S^{\mathrm{o}}}(-, \mathsf{RHom}_R(-,-))$, and swap is said to be *augmented* to an isomorphism of functors $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)^{\mathrm{op}} \to \mathcal{D}(Q\text{–}T^{\mathrm{o}})$.

7.5.27 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $T \otimes_{\Bbbk} S^{\mathrm{o}} \to C$ be ring homomorphisms. If condition* (a), (b), *or* (c) *below is met, then swap is augmented to a natural isomorphism of functors $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)^{\mathrm{op}} \to \mathcal{D}(Q\text{–}T^{\mathrm{o}})$. That is, for M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$ there is an isomorphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$,*

(7.5.27.1)
$$\zeta^{MXN} \colon \mathsf{RHom}_R(M, \mathsf{RHom}_{S^{\mathrm{o}}}(N, X)) \longrightarrow \mathsf{RHom}_{S^{\mathrm{o}}}(N, \mathsf{RHom}_R(M, X))\,,$$

which is natural in M, X, and N. As a natural transformation of functors, this augmented ζ is triangulated in each variable.

(a) *A is projective as an R-module and C is projective as an S°-module.*
(b) *A is projective as an R-module and B is flat as an S-module.*
(c) *B is flat as an R°-module and C is projective as an S°-module.*

Moreover, (7.5.27.1) *is induced by isomorphisms in $\mathcal{K}(Q\text{–}T^{\circ})$ as follows*:
(a′) *If* (a) *is satisfied, then $\boldsymbol{\zeta}^{MXN}$ is induced by $\zeta^{\mathrm{P}_A(M)X\mathrm{P}_C(N)}$.*
(b′) *If* (b) *is satisfied, then $\boldsymbol{\zeta}^{MXN}$ is induced by $\zeta^{\mathrm{P}_A(M)\,\mathrm{I}_B(X)N}$.*
(c′) *If* (c) *is satisfied, then $\boldsymbol{\zeta}^{MXN}$ is induced by $\zeta^{M\,\mathrm{I}_B(X)\,\mathrm{P}_C(N)}$.*

Proof. The proof of 7.5.20 provides a template. Let M be an A-complex, X a B-complex, and N a C-complex. If condition (a) is satisfied, then the crucial input is that $\mathrm{P}_A(M)$ is a semi-projective R-complex and $\mathrm{P}_C(N)$ is a semi-projective S°-complex, both by 5.2.23(b). If condition (b) is satisfied, then $\mathrm{P}_A(M)$ is a semi-projective R-complex, $\mathrm{I}_B(X)$ is a semi-injective S°-complex by 5.4.26(b), and as an S°-complex $\operatorname{Hom}_R(\mathrm{P}_A(M), \mathrm{I}_B(X))$ is semi-injective by 5.3.25. Dually, if condition (c) is satisfied, then $\mathrm{P}_C(M)$ is a semi-projective S°-complex, $\mathrm{I}_B(X)$ is a semi-injective R-complex by 5.4.26(b), and as an R-complex $\operatorname{Hom}_{S^{\circ}}(\mathrm{P}_C(M), \mathrm{I}_B(X))$ is semi-injective by 5.3.25. □

Notice that any of the conditions in the next corollary is satisfied if $\Bbbk$ is a field.

7.5.28 Corollary. *For complexes M in $\mathcal{D}(R\text{–}Q^{\circ})$, X in $\mathcal{D}(R\text{–}S^{\circ})$, and N in $\mathcal{D}(T\text{–}S^{\circ})$ swap,*

$$\boldsymbol{\zeta}^{MXN} : \mathsf{RHom}_R(M, \mathsf{RHom}_{S^{\circ}}(N, X)) \longrightarrow \mathsf{RHom}_{S^{\circ}}(N, \mathsf{RHom}_R(M, X)),$$

is an isomorphism in $\mathcal{D}(Q\text{–}T^{\circ})$ under each of the following conditions:
(a) *Q and T are projective as $\Bbbk$-modules.*
(b) *Q is projective and R is flat as $\Bbbk$-modules.*
(c) *S is flat and T is projective as $\Bbbk$-modules.*

Proof. Apply 7.5.27 with $A = R \otimes_{\Bbbk} Q^{\circ}$, $B = R \otimes_{\Bbbk} S^{\circ}$, and $C = T \otimes_{\Bbbk} S^{\circ}$ and invoke 7.3.11(a,c). □

Adjunction

7.5.29 Construction. Recall from 7.4.9 and 7.3.9 that there are functors,

$$\begin{aligned} -\otimes^{\mathsf{L}}_S - &: \mathcal{D}(R\text{–}S^{\circ}) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(R) \quad \text{induced by} \quad -\otimes_S \mathrm{P}_S(-)\,, \\ \mathsf{RHom}_R(-,-) &: \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad \operatorname{Hom}_R(-, \mathrm{I}_R(-))\,. \end{aligned}$$

It follows from 7.2.4, cf. the argument in 7.5.16, that the composite functor

$$\begin{aligned} \mathsf{RHom}_R(-\otimes^{\mathsf{L}}_S -, -) &: \mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\circ})^{\mathrm{op}} \times \mathcal{D}(S)^{\mathrm{op}} \longrightarrow \mathcal{D}(\Bbbk) \\ &\text{is induced by } \operatorname{Hom}_R(-\otimes_S \mathrm{P}_S(-), \mathrm{I}_R(-))\,. \end{aligned} \tag{7.5.29.1}$$

Applied to an object (M, X, N) this functor yields $\mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S N, M)$.

Similarly, the composite

$$\begin{aligned}\mathsf{RHom}_S(-,\mathsf{RHom}_R(-,-)) : \mathcal{D}(R)\times\mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}\times\mathcal{D}(S)^{\mathrm{op}} &\longrightarrow \mathcal{D}(\Bbbk)\\ \text{is induced by } \operatorname{Hom}_S(\mathrm{P}_S(-),\operatorname{Hom}_R(-,\mathrm{I}_R(-)))\,.\end{aligned} \tag{7.5.29.2}$$

Applied to an object (M,X,N) this functor yields $\mathsf{RHom}_S(N,\mathsf{RHom}_R(X,M))$.

Now, consider the natural isomorphism,

$$\rho^{\mathrm{I}(-)-\mathrm{P}(-)} : \operatorname{Hom}_R(-\otimes_S \mathrm{P}_S(-),\mathrm{I}_R(-)) \longrightarrow \operatorname{Hom}_S(\mathrm{P}_S(-),\operatorname{Hom}_R(-,\mathrm{I}_R(-)))\,,$$

of functors from $\mathcal{K}(R)\times\mathcal{K}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}\times\mathcal{K}(S)^{\mathrm{op}}$ to $\mathcal{K}(\Bbbk)$ induced by adjunction 7.1.16. There is a natural isomorphism of functors from $\mathcal{D}(R)\times\mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}\times\mathcal{D}(S)^{\mathrm{op}}$ to $\mathcal{D}(\Bbbk)$ induced by 7.2.5,

$$\mathsf{RHom}_R(-\otimes^{\mathsf{L}}_S -,-) \xrightarrow{\ \boldsymbol{\rho}=(\rho^{\mathrm{I}(-)-\mathrm{P}(-)})^{''}\ } \mathsf{RHom}_S(-,\mathsf{RHom}_R(-,-))\,. \tag{7.5.29.3}$$

7.5.30 Definition. The natural isomorphism (7.5.29.3) is called *adjunction*.

Adjunction, $\boldsymbol{\rho}$, is by construction a natural isomorphism of functors from the category $\mathcal{D}(R)\times\mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}}\times\mathcal{D}(S)^{\mathrm{op}}$ to $\mathcal{D}(\Bbbk)$. In some cases, $\boldsymbol{\rho}$ can be augmented to a natural isomorphism of functors on derived categories of complexes with additional ring actions.

7.5.31 Setup. Consider ring homomorphisms,

$$R\otimes_\Bbbk Q^{\mathrm{o}} \longrightarrow A\,,\qquad R\otimes_\Bbbk S^{\mathrm{o}} \longrightarrow B\,,\qquad \text{and}\qquad S\otimes_\Bbbk T^{\mathrm{o}} \longrightarrow C\,.$$

Let E_{I} and E_{II} be functors from $\mathcal{D}(A)\times\mathcal{D}(B)^{\mathrm{op}}\times\mathcal{D}(C)^{\mathrm{op}}$ to $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\varphi_{\mathrm{I}} : \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \longrightarrow \mathsf{RHom}_R(\operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S, \operatorname{res}^A_R)$$

and

$$\varphi_{\mathrm{II}} : \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \longrightarrow \mathsf{RHom}_S(\operatorname{res}^C_S, \mathsf{RHom}_R(\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}, \operatorname{res}^A_R))\,,$$

where the functors on the right are (7.5.29.1) and (7.5.29.2) both precomposed with $\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_S$.

7.5.32 Definition. Adopt the setup 7.5.31. A natural transformation $\boldsymbol{\rho}_0 : \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is called an *augmentation* of adjunction if the next diagram is commutative,

$$\begin{tikzcd}
\operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \arrow[r, "\operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \boldsymbol{\rho}_0"] \arrow[d, "\varphi_{\mathrm{I}}"', "\simeq"] & \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \arrow[d, "\simeq"', "\varphi_{\mathrm{II}}"] \\
\mathsf{RHom}_R(\operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S, \operatorname{res}^A_R) \arrow[r, "\boldsymbol{\rho}(\operatorname{res}^A_R\times\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}\times\operatorname{res}^C_S)", "\simeq"'] & \mathsf{RHom}_S(\operatorname{res}^C_S, \mathsf{RHom}_R(\operatorname{res}^B_{R\otimes S^{\mathrm{o}}}, \operatorname{res}^A_R))\,;
\end{tikzcd}$$

here $\boldsymbol{\rho}$ on the lower horizontal arrow (7.5.29.3). In this case, $\boldsymbol{\rho}_0 : \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is written $\boldsymbol{\rho} : \mathsf{RHom}_R(-\otimes^{\mathsf{L}}_S -,-) \to \mathsf{RHom}_S(-,\mathsf{RHom}_R(-,-))$, and adjunction is said to be *augmented* to an isomorphism of functors $\mathcal{D}(A)\times\mathcal{D}(B)^{\mathrm{op}}\times\mathcal{D}(C)^{\mathrm{op}} \to \mathcal{D}(T\text{–}Q^{\mathrm{o}})$.

7.5.33 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $S \otimes_{\Bbbk} T^{\mathrm{o}} \to C$ be ring homomorphisms. If condition* (a), (b), *or* (c) *below is met, then adjunction is augmented to a natural isomorphism of functors $\mathcal{D}(A) \times \mathcal{D}(B)^{\mathrm{op}} \times \mathcal{D}(C)^{\mathrm{op}} \to \mathcal{D}(T\text{–}Q^{\mathrm{o}})$. That is, for M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$ there is an isomorphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$,*

$$(7.5.33.1) \qquad \boldsymbol{\rho}^{MXN} : \mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S N, M) \longrightarrow \mathsf{RHom}_S(N, \mathsf{RHom}_R(X, M)) ,$$

which is natural in M, X, and N. As a natural transformation of functors, this augmented $\boldsymbol{\rho}$ is triangulated in each variable.

(a) *A is flat as an R^{o}-module and C is projective as an S-module.*
(b) *A is flat as an R^{o}-module and B is flat as an S^{o}-module.*
(c) *B is projective as an R-module and C is projective as an S-module.*

Moreover, (7.5.33.1) *is induced by isomorphisms in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ as follows*:

(a′) *If* (a) *is satisfied, then $\boldsymbol{\rho}^{MXN}$ is induced by $\rho^{\mathrm{I}_A(M) X \mathrm{P}_C(N)}$.*
(b′) *If* (b) *is satisfied, then $\boldsymbol{\rho}^{MXN}$ is induced by $\rho^{\mathrm{I}_A(M)\, \mathrm{P}_B(X) N}$.*
(c′) *If* (c) *is satisfied, then $\boldsymbol{\rho}^{MXN}$ is induced by $\rho^{M\, \mathrm{P}_B(X)\, \mathrm{P}_C(N)}$.*

Proof. The proof of 7.5.20 provides a template. Let M be an A-complex, X a B-complex, and N a C-complex. Under condition (a), the crucial input is that $\mathrm{I}_A(M)$ is a semi-injective R-complex, see 5.4.26(b), and $\mathrm{P}_C(N)$ is a semi-projective S-complex by 5.2.23(b). If (b) is satisfied, then $\mathrm{I}_A(M)$ is a semi-injective R-complex, $\mathrm{P}_B(X)$ is a semi-flat S^{o}-complex by 5.4.18(b), and as an S-complex $\mathrm{Hom}_R(\mathrm{P}_B(X), \mathrm{I}_A(M))$ is semi-injective by 5.4.25. If condition (c) is satisfied, then $\mathrm{P}_C(N)$ is a semi-projective S-complex and $\mathrm{P}_B(X)$ is a semi-projective R-complex, both by 5.2.23(b), and as an R-complex $\mathrm{P}_B(X) \otimes_S \mathrm{P}_C(N)$ is semi-projective by 5.2.22. □

Notice that any of the conditions in the next corollary is satisfied if $\Bbbk$ is a field.

7.5.34 Corollary. *For complexes M in $\mathcal{D}(R\text{–}Q^{\mathrm{o}})$, X in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{D}(S\text{–}T^{\mathrm{o}})$ adjunction,*

$$\boldsymbol{\rho}^{MXN} : \mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S N, M) \longrightarrow \mathsf{RHom}_S(N, \mathsf{RHom}_R(X, M)) ,$$

is an isomorphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$ under each of the following conditions:

(a) *Q is flat and T is projective as $\Bbbk$-modules.*
(b) *Q and R are flat as $\Bbbk$-modules.*
(c) *S and T are projective as $\Bbbk$-modules.*

Proof. Apply 7.5.33 with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = S \otimes_{\Bbbk} T^{\mathrm{o}}$ and invoke 7.3.11. □

7.5.35 Example. Adopt the setup from 7.5.31 and let M be an A-complex, X a B-complex, and N a C-complex. Let F be a semi-flat replacement of X. By 6.4.20 there is a quasi-isomorphism $\vartheta \colon \mathrm{P}_B(X) \to F$ in $\mathcal{K}(B)$ which yields a commutative diagram in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$,

$$\begin{tikzcd}
\operatorname{Hom}_R(F \otimes_S N, \mathrm{I}_A(M)) \arrow[r, "\rho^{\mathrm{I}(M)FN}"] \arrow[d, "\operatorname{Hom}(\vartheta\otimes N, \mathrm{I}(M))"'] & \operatorname{Hom}_S(N, \operatorname{Hom}_R(F, \mathrm{I}_A(M))) \arrow[d, "\operatorname{Hom}(N, \operatorname{Hom}(\vartheta, \mathrm{I}(M)))"] \\
\operatorname{Hom}_R(\mathrm{P}_B(X) \otimes_S N, \mathrm{I}_A(M)) \arrow[r, "\rho^{\mathrm{I}(M)\mathrm{P}(X)N}"] & \operatorname{Hom}_S(N, \operatorname{Hom}_R(\mathrm{P}_B(X), \mathrm{I}_A(M))) \, .
\end{tikzcd}$$

Assume that A is flat as an R^{o}-module and B is flat as an S^{o}-module. It follows from 5.4.18(b) that F and $\mathrm{P}_B(X)$ are semi-flat S^{o}-complexes and from 5.4.26(b) that $\mathrm{I}_A(M)$ is a semi-injective R-complex. Thus the left-hand vertical map is a quasi-isomorphism by 5.4.16. From 5.4.25 it follows that $\operatorname{Hom}_R(\mathrm{P}_B(X), \mathrm{I}_A(M))$ and $\operatorname{Hom}_R(F, \mathrm{I}_A(M))$ are semi-injective S-complexes, so the right-hand vertical map is a homotopy equivalence by 5.3.24. It follows that $\rho^{\mathrm{I}(M)FN}$ induces a morphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$ which is isomorphic to the augmented adjunction isomorphism $\boldsymbol{\rho}^{MXN}$ from 7.5.33(b′).

Exercises

E 7.5.1 Let M be an R^{o}-module and N an R-module. Show that for every $m \in \mathbb{Z}$ there is an isomorphism $\operatorname{Tor}_m^R(M, N) \cong \operatorname{Tor}_m^{R^{\mathrm{o}}}(N, M)$.

E 7.5.2 Let M be an R^{o}-module, X an R–S^{o}-bimodule, and F a flat S-module. Show that there is an isomorphism $\operatorname{Tor}_m^R(M, X) \otimes_S F \cong \operatorname{Tor}_m^R(M, X \otimes_S F)$ for every $m \in \mathbb{Z}$.

E 7.5.3 Let P be a projective R-module, X an R–S^{o}-bimodule, and N an S^{o}-module. Show that there is an isomorphism $\operatorname{Hom}_R(P, \operatorname{Ext}_{S^{\mathrm{o}}}^m(N, X)) \cong \operatorname{Ext}_{S^{\mathrm{o}}}^m(N, \operatorname{Hom}_R(P, X))$ for every $m \in \mathbb{Z}$.

E 7.5.4 Let I be an injective R-module, X an R–S^{o}-bimodule, and N an S-module. Show that there is an isomorphism, $\operatorname{Hom}_R(\operatorname{Tor}_m^S(X, N), I) \cong \operatorname{Ext}_S^m(N, \operatorname{Hom}_R(X, I))$ for every $m \in \mathbb{Z}$.

E 7.5.5 Let X be a complex of R–S^{o}-bimodules. Show that there is an adjunction,
$$X \otimes_S^{\mathsf{L}} - : \mathcal{D}(S) \rightleftarrows \mathcal{D}(R) : \mathsf{R}\mathrm{Hom}_R(X, -) \, .$$

E 7.5.6 Show that the restriction of scalars functor $\operatorname{res}_R^S : \mathcal{D}(S) \to \mathcal{D}(R)$ has a left adjoint and a right adjoint. *Hint:* E 7.5.5.

7.6 Boundedness and Finiteness

Synopsis. Bounded (above/below) derived category; preservation of boundedness and finiteness by the functors $\mathsf{R}\mathrm{Hom}$ and $\otimes^{\mathsf{L}}$.

Conditions of vanishing and finiteness of homology modules identify important subcategories of $\mathcal{D}(R)$.

Bounded Subcategories of the Derived Category

Recall from 6.5.17 that homology is a functor on $\mathcal{D}(R)$.

7.6.1 Definition. The full subcategories $\mathcal{D}_{\sqsubset}(R)$, $\mathcal{D}_{\sqsupset}(R)$, and $\mathcal{D}_{\square}(R)$ of $\mathcal{D}(R)$ are defined by specifying their objects as follows,

$$\begin{aligned}
\mathcal{D}_{\sqsubset}(R) &= \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is bounded above}\}\,, \\
\mathcal{D}_{\sqsupset}(R) &= \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is bounded below}\}\,, \quad \text{and} \\
\mathcal{D}_{\square}(R) &= \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is bounded}\}\,.
\end{aligned}$$

7.6.2. Note that $\mathcal{D}_{\square}(R)$ is the intersection of the subcategories $\mathcal{D}_{\sqsubset}(R)$ and $\mathcal{D}_{\sqsupset}(R)$. In terms of the invariants from 2.5.4 these categories can be described as follows,

$$\begin{aligned}
\mathcal{D}_{\sqsubset}(R) &= \{M \in \mathcal{D}(R) \mid \sup M < \infty\}\,, \\
\mathcal{D}_{\sqsupset}(R) &= \{M \in \mathcal{D}(R) \mid \inf M > -\infty\}\,, \quad \text{and} \\
\mathcal{D}_{\square}(R) &= \{M \in \mathcal{D}(R) \mid \operatorname{amp} M < \infty\}\,.
\end{aligned}$$

7.6.3 Proposition. *The categories $\mathcal{D}_{\sqsubset}(R)$, $\mathcal{D}_{\sqsupset}(R)$, and $\mathcal{D}_{\square}(R)$ are triangulated subcategories of $\mathcal{D}(R)$, and they are closed under soft truncations.*

Proof. We verify the axioms from E.14. By definition, all three subcategories of $\mathcal{D}(R)$ are full; evidently they are $\Bbbk$-linear and closed under isomorphisms and shifts. It follows from 7.6.2 and 6.5.20 that they are closed under distinguished triangles. Closure under soft truncations is evident. $\square$

7.6.4. For an R-complex M it follows from 4.2.4 and 6.4.18 that in $\mathcal{D}(R)$ there are isomorphisms $M \simeq M_{\subseteq n}$ for every $n \geqslant \sup M$ and $M \simeq M_{\supseteq n}$ for every $n \leqslant \inf M$.

7.6.5 Proposition. *There are equalities of full subcategories of $\mathcal{D}(R)$:*

$$\begin{aligned}
\mathcal{D}_{\sqsubset}(R) &= \{M \in \mathcal{D}(R) \mid M \text{ is isomorphic in } \mathcal{D}(R) \text{ to a bounded above complex}\}\,, \\
\mathcal{D}_{\sqsupset}(R) &= \{M \in \mathcal{D}(R) \mid M \text{ is isomorphic in } \mathcal{D}(R) \text{ to a bounded below complex}\}\,, \\
\mathcal{D}_{\square}(R) &= \{M \in \mathcal{D}(R) \mid M \text{ is isomorphic in } \mathcal{D}(R) \text{ to a bounded complex}\}\,.
\end{aligned}$$

Proof. The inclusions "$\supseteq$" are trivial; the inclusions "$\subseteq$" follow from 7.6.4. $\square$

7.6.6 Proposition. *Let M be an R-complex and n an integer. One has the following distinguished triangles in $\mathcal{D}(R)$.*

(a) $\Sigma^n \mathrm{H}_n(M) \longrightarrow M_{\subseteq n} \longrightarrow M_{\subseteq n-1} \longrightarrow \Sigma(\Sigma^n \mathrm{H}_n(M))\,.$

(b) $M_{\supseteq n+1} \longrightarrow M_{\supseteq n} \longrightarrow \Sigma^n \mathrm{H}_n(M) \longrightarrow \Sigma(M_{\supseteq n+1})\,.$

(c) $M_{\supseteq n} \longrightarrow M \longrightarrow M_{\subseteq n-1} \longrightarrow \Sigma(M_{\supseteq n})\,.$

Proof. To construct (a), note that there is a short exact sequence in $\mathcal{C}(R)$,

$$0 \longrightarrow H \longrightarrow M_{\subseteq n} \longrightarrow M_{\subseteq n-1} \longrightarrow 0\,,$$

and hence by 6.5.24 a distinguished triangle $H \to M_{\subseteq n} \to M_{\subseteq n-1} \to \Sigma H$ in $\mathcal{D}(R)$. The complex

$$H = 0 \longrightarrow \mathrm{C}_n(M) \xrightarrow{\bar{\partial}^M_n} \mathrm{B}_{n-1}(M) \longrightarrow 0$$

is concentrated in degrees n and $n-1$, and the embedding $\Sigma^n \mathrm{H}_n(M) \to H$ is evidently a quasi-isomorphism; in particular it yields an isomorphism in $\mathcal{D}(R)$.

A dual argument establishes (b). To construct (c), consider the short exact sequence in $\mathcal{C}(R)$,

$$0 \longrightarrow M_{\supseteq n} \longrightarrow M \longrightarrow C \longrightarrow 0\,,$$

where the cokernel C is the complex $0 \to \mathrm{B}_{n-1}(M) \to M_{n-1} \to M_{n-2} \to \cdots$. The surjection $C \to M_{\subseteq n-1}$ is a quasi-isomorphism; in particular it yields an isomorphism in $\mathcal{D}(R)$, and the existence of the desired triangle follows from 6.5.24. □

REMARK. A *t-structure* on a triangulated category $(\mathcal{T}, \Sigma)$ is a pair $(\mathcal{A}, \mathcal{B})$ of full subcategories that are closed under isomorphisms, direct sums, and direct summands such that (1) $\Sigma\mathcal{A} \subseteq \mathcal{A}$ and $\Sigma^{-1}\mathcal{B} \subseteq \mathcal{B}$; (2) $\mathcal{T}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$; (3) for every $M \in \mathcal{T}$ there is a distinguished triangle $A \to M \to B \to \Sigma A$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The triangle 7.6.6(c) together with the fact that there are no non-zero morphisms $M_{\supseteq 1} \to M_{\subseteq 0}$ in $\mathcal{D}(R)$, see E 7.6.3, shows that truncations give rise to a t-structure on $\mathcal{D}(R)$.

With appropriate boundedness conditions on the arguments, one can excert some control over the boundedness of derived Hom and tensor product complexes. This theme is comprehensively revisited in A.23–A.34.

7.6.7 Proposition. *Let M and N be R-complexes that are not acyclic. One has*

$$-\sup \mathsf{RHom}_R(M,N) \;\geqslant\; \inf M - \sup N\,.$$

Further, if M is in $\mathcal{D}_{\sqsupset}(R)$ with $u = \inf M$ and N is in $\mathcal{D}_{\sqsubset}(R)$ with $s = \sup N$, then $\mathsf{RHom}_R(M,N)$ belongs to $\mathcal{D}_{\sqsubset}(\Bbbk)$, and there is an isomorphism of $\Bbbk$-modules,

$$\mathrm{H}_{-(u-s)}(\mathsf{RHom}_R(M,N)) \;\cong\; \mathrm{Hom}_R(\mathrm{H}_u(M), \mathrm{H}_s(N))\,.$$

PROOF. The inequality is trivial if $\inf M = -\infty$ or $\sup N = \infty$, so one can assume that M is in $\mathcal{D}_{\sqsupset}(R)$ and N is in $\mathcal{D}_{\sqsubset}(R)$. Set $u = \inf M$ and $s = \sup N$. By 5.2.15 there exists a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < u$. One has $\mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_R(M, N_{\subseteq w}) = \mathrm{Hom}_R(P, N_{\subseteq w})$ in $\mathcal{D}(\Bbbk)$ by 7.6.4 and 7.3.21, and the assertions follow from 2.5.12. □

7.6.8 Proposition. *Let M and N be R-complexes that are not acyclic. One has*

$$\inf\,(M \otimes^{\mathsf{L}}_R N) \;\geqslant\; \inf M + \inf N\,.$$

Further, if M and N belong to $\mathcal{D}_{\sqsupset}(R)$ with $u = \inf M$ and $w = \inf N$, then $M \otimes^{\mathsf{L}}_R N$ belongs to $\mathcal{D}_{\sqsupset}(\Bbbk)$, and there is an isomorphism of $\Bbbk$-modules,

$$\mathrm{H}_{u+w}(M \otimes^{\mathsf{L}}_R N) \;\cong\; \mathrm{H}_u(M) \otimes_R \mathrm{H}_w(N)\,.$$

PROOF. The inequality is trivial if $\inf M$ or $\inf N$ equals $-\infty$, so one can assume that M and N belong to $\mathcal{D}_{\sqsupset}(R)$. Set $u = \inf M$ and $w = \inf N$. By 5.2.15 there exists a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < u$. There are isomorphisms $M \otimes^{\mathsf{L}}_R N \simeq M \otimes^{\mathsf{L}}_R N_{\supseteq w} = P \otimes_R N_{\supseteq w}$ in $\mathcal{D}(\Bbbk)$ by 7.6.4 and 7.4.8, and the assertions follow from 2.5.18. □

Under additional assumptions, the inequalities in the next two result are equalities; see 17.6.14 and 17.6.17.

7.6.9 Proposition. *Let M and N be R-complexes. If M is in $\mathcal{D}_{\sqsubset}(R)$, then one has*

$$-\sup \mathsf{RHom}_R(M,N) \;\geqslant\; \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(M),N)+v \mid v\in\mathbb{Z}\}\,.$$

Proof. The inequality is trivial if M is acyclic, so assume that M is not acyclic, set $u = \inf M$, and proceed by induction on $\operatorname{amp} M$. If $\operatorname{amp} M = 0$, i.e. $\mathrm{H}(M)$ is concentrated in degree u, then 7.3.29 and the fact that $\mathsf{RHom}_R(-,N)$ is triangulated, see 7.3.6, imply that there are isomorphisms in $\mathcal{D}(\Bbbk)$,

$$\mathsf{RHom}_R(M,N) \;\simeq\; \mathsf{RHom}_R(\Sigma^u \mathrm{H}_u(M),N) \;\simeq\; \Sigma^{-u}\mathsf{RHom}_R(\mathrm{H}_u(M),N)\,.$$

Thus 2.5.5 yields $-\sup \mathsf{RHom}_R(M,N) = -\sup \mathsf{RHom}_R(\mathrm{H}_u(M),N)+u$, so the inequality is even an equality in this case.

Now let $w-u>0$ and assume that the inequality holds for complexes of amplitude less than $w-u$. If $\operatorname{amp} M = w-u$ holds, then $\mathrm{H}(M)$ is concentrated in degrees $w,\dots,u$, and there is a distinguished triangle $M_{\supseteq u+1} \to M \to M_{\subseteq u} \to \Sigma(M_{\supseteq u+1})$ in $\mathcal{D}(R)$; see 7.6.6(c). Application of $\mathsf{RHom}_R(-,N)$ yields a distinguished triangle,

$$\begin{aligned}\mathsf{RHom}_R(M_{\subseteq u},N) \longrightarrow \mathsf{RHom}_R(M,N) \longrightarrow&\\ \mathsf{RHom}_R(M_{\supseteq u+1},N) \longrightarrow \Sigma\,&\mathsf{RHom}_R(M_{\subseteq u},N)\,.\end{aligned}$$

The complex $M_{\subseteq u}$ has amplitude 0 and is concentrated in degree u, so one has

$$-\sup \mathsf{RHom}_R(M_{\subseteq u},N) \;=\; -\sup \mathsf{RHom}_R(\mathrm{H}_u(M),N)+u\,.$$

By the induction hypothesis one has

$$\begin{aligned}-\sup \mathsf{RHom}_R(M_{\supseteq u+1},N) \;&\geqslant\; \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(M_{\supseteq u+1}),N)+v \mid v\in\mathbb{Z}\}\\ &=\; \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(M),N)+v \mid u+1\leqslant v\leqslant w\}\,.\end{aligned}$$

The desired inequality is now immediate from 6.5.20 applied to the distinguished triangle displayed above. □

7.6.10 Proposition. *Let M be an R°-complex and N an R-complex. If M is in $\mathcal{D}_{\sqsubset}(R^{\circ})$, then one has*

$$\inf\,(M\otimes^{\mathsf{L}}_R N) \;\geqslant\; \inf\{\inf\,(\mathrm{H}_v(M)\otimes^{\mathsf{L}}_R N)+v \mid v\in\mathbb{Z}\}\,.$$

Proof. In the computation below, the first and last equalities follow from 2.5.7(b) and commutativity 7.5.10. The second equality holds by adjunction 7.5.30, and 7.6.9 yields the inequality.

$$\begin{aligned}\inf\,(M\otimes^{\mathsf{L}}_R N) \;&=\; -\sup \mathsf{RHom}_{\Bbbk}(N\otimes^{\mathsf{L}}_{R^{\circ}} M,\mathbb{E})\\ &=\; -\sup \mathsf{RHom}_{R^{\circ}}(M,\mathsf{RHom}_{\Bbbk}(N,\mathbb{E}))\\ &\geqslant\; \inf\{-\sup \mathsf{RHom}_{\Bbbk}(N\otimes^{\mathsf{L}}_{R^{\circ}} \mathrm{H}_v(M),\mathbb{E})+v \mid v\in\mathbb{Z}\}\\ &=\; \inf\{\inf\,(\mathrm{H}_v(K)\otimes^{\mathsf{L}}_R M)+v \mid v\in\mathbb{Z}\}\,.\end{aligned}$$ □

Accounting Principles

A complex of vector spaces and its homology are isomorphic in the derived category, see e.g. 6.4.23. Moreover, it is a fact—perhaps already known from exercises in Chap. 5 and in any event a special case of 7.6.11(c) below—that a complex of vector spaces is both semi-injective and semi-flat. It follows that the suprema and infima of derived Hom and tensor product complexes over a field can be computed exactly in terms of the suprema and infima of the arguments, see 7.6.12. In [96] these formulas were dubbed Accounting Principles, see also the Remark after 7.6.12.

7.6.11 Lemma. *Let R be an integral domain with field of fractions Q and M an R-complex. If the homothety $r^{\mathrm{H}(M)}$ is an isomorphism for every $r \neq 0$, then the following assertions hold.*

(a) *There is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathcal{D}(R)$.*

(b) *$\mathrm{H}(M)$ is a Q-complex and in $\mathcal{C}(Q)$ there are isomorphisms,*

$$\mathrm{H}(M) \;\cong\; Q \otimes_R \mathrm{H}(M) \;\cong\; \mathrm{H}(Q \otimes_R M)\,.$$

(c) *As an R-complex, $\mathrm{H}(M)$ is a semi-flat and a semi-injective replacement of M.*

Proof. It follows from the assumption that the embedding $\iota\colon R \to Q$ induces an isomorphism $\iota \otimes_R \mathrm{H}(M)\colon \mathrm{H}(M) \to Q \otimes_R \mathrm{H}(M)$ of R-complexes; in particular, $\mathrm{H}(M)$ is a Q-complex and $\iota \otimes_R \mathrm{H}(M)$ is an isomorphism in $\mathcal{C}(Q)$. Further, 2.2.19 yields $Q \otimes_R \mathrm{H}(M) \cong \mathrm{H}(Q \otimes_R M)$ as Q is flat as an R-module, see 1.3.42. This proves part (b) and shows that the canonical map $M \to Q \otimes_R M$ is a quasi-isomorphism. In $\mathcal{D}(R)$ one now has $M \simeq Q \otimes_R M \simeq \mathrm{H}(Q \otimes_R M) \simeq \mathrm{H}(M)$, where the middle isomorphism holds by 6.4.23. This proves part (a). As an R-module, a Q-vector space is flat by 1.3.42 and 5.4.22 and injective by 1.3.33. Since $Q \otimes_R \mathrm{H}(M)$ is a graded Q-vector space it is, as an R-complex, semi-flat by 5.4.11 and semi-injective by 5.3.18. In view of (a), this proves part (c). □

7.6.12 Proposition. *Let Q be a field. For Q-complexes K and L there are isomorphisms in $\mathcal{D}(Q)$,*

$$\mathsf{RHom}_Q(K,L) \simeq \mathrm{Hom}_Q(\mathrm{H}(K),\mathrm{H}(L)) \quad \textit{and} \quad K \otimes^{\mathsf{L}}_Q L \simeq \mathrm{H}(K) \otimes_Q \mathrm{H}(L)\,.$$

Moreover, if one has $\mathrm{H}(K) \neq 0 \neq \mathrm{H}(L)$, then the following equalities hold:

$$-\sup \mathsf{RHom}_Q(K,L) = \inf K - \sup L\,, \qquad \inf\,(K \otimes^{\mathsf{L}}_Q L) = \inf K + \inf L\,,$$
$$-\inf \mathsf{RHom}_Q(K,L) = \sup K - \inf L\,, \quad \textit{and} \quad \sup\,(K \otimes^{\mathsf{L}}_Q L) = \sup K + \sup L\,.$$

Proof. For every Q-complex M the graded Q-vector space $\mathrm{H}(M)$ is by 7.6.11(c) a semi-flat and a semi-injective replacement of M; the asserted isomorphisms in $\mathcal{D}(Q)$ follow. Assuming that $\mathrm{H}(K)$ and $\mathrm{H}(L)$ are non-zero, the four equalities follow from 2.1.4 and 2.1.14. □

Remark. The isomorphisms in 7.6.12 are commonly known as Künneth Formulas, and indeed they can be derived from 2.5.8 and 2.5.14; see E 2.5.3 and E 2.5.4.

Finite Homology and Noetherian Rings

7.6.13 Definition. The full subcategory $\mathcal{D}^{\mathrm{f}}(R)$ of $\mathcal{D}(R)$ is defined by specifying its objects as follows,

$$\mathcal{D}^{\mathrm{f}}(R) = \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is degreewise finitely generated}\} .$$

The full subcategory $\mathcal{D}^{\mathrm{f}}(R) \cap \mathcal{D}_{\sqsubset}(R)$ is denoted by $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. Similarly, one defines the subcategories $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.

7.6.14 Proposition. *If R is left Noetherian, then $\mathcal{D}^{\mathrm{f}}(R)$ is a triangulated subcategory of $\mathcal{D}(R)$ and closed under soft truncations.*

Proof. We verify the axioms from E.14. By definition $\mathcal{D}^{\mathrm{f}}(R)$ is full, and evidently it is additive and closed under isomorphisms and shifts; see 6.5.17. Since R is left Noetherian, it follows from 6.5.19 that $\mathcal{D}^{\mathrm{f}}(R)$ is closed under distinguished triangles. Closure under soft truncations is evident. ◻

From 7.6.14 and 7.6.3 it follows that if R is left Noetherian, then $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, and $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ are triangulated subcategories of $\mathcal{D}(R)$ and closed under soft truncations.

7.6.15 Proposition. *If R is left Noetherian, then the following hold*:

$$\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R) = \left\{ M \in \mathcal{D}(R) \;\middle|\; \begin{array}{c} M \text{ is isomorphic in } \mathcal{D}(R) \text{ to a bounded below} \\ \text{and degreewise finitely generated complex} \end{array} \right\}$$

and

$$\mathcal{D}^{\mathrm{f}}_{\square}(R) = \left\{ M \in \mathcal{D}(R) \;\middle|\; \begin{array}{c} M \text{ is isomorphic in } \mathcal{D}(R) \text{ to a bounded} \\ \text{and degreewise finitely generated complex} \end{array} \right\} .$$

Proof. The inclusions "$\supseteq$" are trivial. For a complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, choose $u \in \mathbb{Z}$ with $u \leqslant \inf M$. By 5.1.14 there is a semi-free resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated and $L_v = 0$ for all $v < u$. This proves the first equality.

For M in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ there exists by the argument above a bounded below and degreewise finitely generated complex L with $M \simeq L$ in $\mathcal{D}(R)$. Choose $w \in \mathbb{Z}$ with $w \geqslant \sup M$, by 7.6.4 and 6.4.38 there are isomorphisms $M \simeq M_{\subseteq w} \simeq L_{\subseteq w}$ in $\mathcal{D}(R)$, and $L_{\subseteq w}$ is bounded and degreewise finitely generated. ◻

Remark. Under extra assumptions on R one can prove that a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ is isomorphic in $\mathcal{D}(R)$ to a bounded above complex of finitely generated R-modules; see E 18.2.12.

7.6.16 Proposition. *Assume that R is left Noetherian and S right Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and X a complex in $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$. If $\mathrm{H}(X)$ is degreewise finitely generated over S^{o}, then the complex $\mathsf{RHom}_R(M,X)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S^{\mathrm{o}})$.*

Proof. Since M is in $\mathcal{D}_{\sqsupset}(R)$, the functor $\mathrm{F} = \mathsf{RHom}_R(M,-)\colon \mathcal{D}(R\text{–}S^{\mathrm{o}}) \to \mathcal{D}(S^{\mathrm{o}})$ is bounded above, see A.26(a). As F by 7.3.6 is triangulated, it maps $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$ to $\mathcal{D}_{\sqsubset}(S^{\mathrm{o}})$, see A.25(a). To show that $\mathrm{F}(X)$ belongs to $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})$, one can by 7.6.14 apply A.29(c) with

$$\mathcal{U} = \{Y \in \mathcal{D}(R\text{–}S^{\mathrm{o}}) \mid \mathrm{res}^{R\otimes S^{\mathrm{o}}}_{S^{\mathrm{o}}}(Y) \in \mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})\} .$$

It thus suffices to argue that $\mathrm{F}(Y)$ is in $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})$ for every R–S^{o}-bimodule Y that is finitely generated over S^{o}, so let Y be such a module. Choose by 5.2.16 a semi-projective resolution $P \xrightarrow{\simeq} M$ with P degreewise finitely generated and $P_v = 0$ for $v < \inf M$. The S^{o}-complex $\mathrm{F}(Y) = \operatorname{Hom}_R(P, Y)$, see 7.3.21, consists by 2.5.13 of finitely generated modules, and hence its homology is degreewise finitely generated as S is right Noetherian. □

7.6.17 Corollary. *Assume that R is Noetherian. For every complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ the complex $\mathsf{RHom}_R(M, R)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R^{\mathrm{o}})$.*

Proof. The assertion is immediate from 7.6.16 applied with $R = S$. □

7.6.18 Proposition. *Assume that R and S are left Noetherian. Let N be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$ and X a complex in $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$. If $\mathrm{H}(X)$ is degreewise finitely generated over R, then the complex $X \otimes^{\mathsf{L}}_S N$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$.*

Proof. As N is in $\mathcal{D}_{\sqsupset}(R)$, the functor $\mathrm{F} = - \otimes^{\mathsf{L}}_S N \colon \mathcal{D}(R\text{–}S^{\mathrm{o}}) \to \mathcal{D}(R)$ is bounded below by A.27(a). As F by 7.4.5 is triangulated, it maps $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$ to $\mathcal{D}_{\sqsupset}(R)$, see A.25(b). To see that $\mathrm{F}(X)$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$, one can by 7.6.14 apply A.29(b) with

$$\mathcal{U} = \{Y \in \mathcal{D}(R\text{–}S^{\mathrm{o}}) \mid \operatorname{res}^{R\otimes S^{\mathrm{o}}}_R(Y) \in \mathcal{D}^{\mathrm{f}}(R)\}\,.$$

It thus suffices to argue that $\mathrm{F}(Y)$ is in $\mathcal{D}^{\mathrm{f}}(R)$ for every R–S^{o}-bimodule Y that is finitely generated over R. To this end choose by 5.2.16 a semi-projective resolution $P \xrightarrow{\simeq} N$ with P degreewise finitely generated and $P_v = 0$ for $v < \inf N$. The R-complex $\mathrm{F}(Y) = Y \otimes_S P$, see 7.4.17, consists by 2.5.19 of finitely generated modules, so its homology is degreewise finitely generated as R is left Noetherian. □

Exercises

E 7.6.1 Show that $\mathcal{D}_{\square}(R)$ is the smallest triangulated subcategory of $\mathcal{D}(R)$ that contains the full subcategory $\mathcal{M}(R)$.

E 7.6.2 Let $\mathcal{U}$ be a triangulated subcategory of $\mathcal{D}(R)$ that is closed under soft truncations above or below. Show that for every M in $\mathcal{U}$ and every $v \in \mathbb{Z}$ the module $\mathrm{H}_v(M)$ is in $\mathcal{U}$.

E 7.6.3 Let M be an R-complex; show that there is no non-zero morphism $M_{\supseteq 1} \to M_{\subseteq 0}$ in $\mathcal{D}(R)$.

E 7.6.4 Exhibit complexes $M \in \mathcal{D}_{\sqsupset}(\mathbb{Z})$ and $N \in \mathcal{D}_{\sqsubset}(\mathbb{Z})$ such that $-\sup \mathsf{RHom}_{\mathbb{Z}}(M, N)$ is strictly greater than $\inf M - \sup N$.

E 7.6.5 Exhibit complexes $M \in \mathcal{D}_{\sqsupset}(\mathbb{Z})$ and $N \in \mathcal{D}_{\sqsupset}(\mathbb{Z})$ such that $\inf(M \otimes^{\mathsf{L}}_{\mathbb{Z}} N)$ is strictly greater than $\inf M + \inf N$.

E 7.6.6 Prove 7.6.10 without recourse to 7.6.9.

E 7.6.7 Let M and N be R-complexes. Show that if N is in $\mathcal{D}_{\square}(R)$, then one has
$$-\sup \mathsf{RHom}_R(M, N) \geqslant \inf\{-\sup \mathsf{RHom}_R(M, \mathrm{H}_v(N)) - v \mid v \in \mathbb{Z}\}\,.$$

E 7.6.8 Assume that $\Bbbk$ is Noetherian and R left Noetherian. Let X be a complex in $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$ and N a complex in $\mathcal{D}_{\sqsubset}(R)$. Show that if $\mathrm{H}(X)$ is degreewise finitely generated over R and $\mathrm{H}(N)$ degreewise finitely generated over $\Bbbk$, then $\mathsf{RHom}_R(X, N)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$.

Chapter 8
Homological Dimensions

The construction of a semi-free resolution, or a semi-injective resolution, of an R-complex yields a complex that, though it has remarkable properties, is isomorphic in the derived category to the original complex. What one may have to concede in this exchange is boundedness: Even if one starts with a module, the resolutions may not be bounded. Homological dimensions are measures of the size of resolution.

Avramov and Foxby outline the theory of homological dimensions for complexes in [21]. We follow that outline which adheres to the blueprint set by Cartan and Eilenberg's treatment of homological dimensions of modules in [48].

8.1 Projective Dimension

Synopsis. Vanishing of Ext; projective dimension; Schanuel's lemma; Horseshoe Lemma; projective dimension over Noetherian/semi-perfect/perfect ring; projective dimension of module.

First we connect projectivity of modules to vanishing of Ext.

8.1.1 Lemma. *For an R-module P, the following conditions are equivalent.*

(i) *P is projective.*

(ii) $-\inf \mathsf{RHom}_R(P,N) \leqslant -\inf N$ *holds for every R-complex N.*

(iii) *There exists a surjective homomorphism $\pi\colon L \to P$ with L projective and with* $\operatorname{Ext}_R^1(P, \operatorname{Ker}\pi) = 0$.

Proof. If P is projective, then 1^P is a projective resolution of P, and one has $\mathsf{RHom}_R(P,N) = \operatorname{Hom}_R(P,N)$ in $\mathcal{D}(\Bbbk)$. Now 2.5.7(a) yields $-\inf \mathsf{RHom}_R(P,N) \leqslant -\inf N$. Thus, ($i$) implies ($ii$) which, in particular, implies that the functor $\operatorname{Ext}_R^1(P,-)$ is zero on modules, whence (iii) follows. To see that (iii) implies (i), consider the short exact sequence $0 \to \operatorname{Ker}\pi \to L \to P \to 0$. It is split by 7.3.36, so P is a direct summand of a projective module and hence projective by 1.3.24. $\square$

Semi-Projective Replacements and Projective Dimension

Recall from 5.2.15 that every complex is isomorphic in the derived category to a semi-projective complex; by 7.3.16 such a complex is called a semi-projective replacement. A complex of finite projective dimension is one that is isomorphic to a bounded above semi-projective complex.

8.1.2 Definition. Let M be an R-complex. The *projective dimension* of M, written $\operatorname{pd}_R M$, is defined as

$$\operatorname{pd}_R M \;=\; \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c}\text{There is a semi-projective replacement}\\ P \text{ of } M \text{ with } P_v = 0 \text{ for all } v > n\end{array}\right\}\,,$$

with the convention $\inf \varnothing = \infty$. One says that $\operatorname{pd}_R M$ is *finite* if $\operatorname{pd}_R M < \infty$ holds.

The convention that a complex of projective dimension $-\infty$ has finite projective dimension may appear odd, but as we notice below this only happens for acyclic complexes. Saying that a complex has finite projective dimension thus conveys that it has a bounded above semi-projective replacement.

8.1.3. Let M be an R-complex. For every semi-projective replacement P of M one has $\mathrm{H}(P) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\operatorname{pd}_R M \;\geqslant\; \sup M \qquad\text{and}\qquad \operatorname{pd}_R \Sigma^s M \;=\; \operatorname{pd}_R M + s \quad\text{for every integer } s\,.$$

Moreover, one has $\operatorname{pd}_R M = -\infty$ if and only if M is acyclic.

Note that the definition of projective dimension could also be written

$$\operatorname{pd}_R M \;=\; \inf\{\sup P^\natural \mid P \text{ is a semi-projective replacement of } M\}\,.$$

8.1.4 Proposition. *Let $R \to S$ be a ring homomorphism and M be an R-complex. There is an inequality,*

$$\operatorname{pd}_S(S \otimes^{\mathsf{L}}_R M) \;\leqslant\; \operatorname{pd}_R M\,.$$

Proof. For every semi-projective replacement P of the R-complex M, the S-complex $S \otimes_R P$ is a semi-projective replacement of $S \otimes^{\mathsf{L}}_R M$ by 5.2.23(a). As $P_v = 0$ implies $(S \otimes_R P)_v = 0$, the desired inequality follows from 8.1.2. □

Even if S is flat as an R-module, the inequality in 8.1.4 may be strict; but see also the Remark after 15.4.19.

8.1.5 Example. It follows from 5.4.15 that one has $\operatorname{pd}_{\mathbb{Z}} \mathbb{Q} \leqslant 1$. On the other hand, if there were a semi-projective replacement P of $\mathbb{Q}$ in $\mathcal{D}(\mathbb{Z})$ with $P_1 = 0$, then one would have $\mathbb{Q} \cong \mathrm{H}_0(P) \cong \mathrm{Z}_0(P)$, but $\mathrm{Z}_0(P)$ would be a submodule of a free $\mathbb{Z}$-module and hence free by 1.3.11; a contradiction. Thus $\operatorname{pd}_{\mathbb{Z}} \mathbb{Q} = 1$ holds, and for the base changed module $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ one has $\operatorname{pd}_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ by 8.1.2 and 8.1.3.

We now aim for a theorem that asserts, in summary form, that complexes of finite projective dimension are characterized by vanishing of Ext and that any semi-projective replacement of such a complex can be truncated to yield one of minimal length. We need the next two lemmas to prove it. The isomorphism in the first lemma is colloquially referred to as "dimension shifting".

8.1.6 Lemma. *Let M be an R-complex, P a semi-projective replacement of M, and N an R-module. For all integers $m > 0$ and $n \geqslant \sup M$ one has*

$$\operatorname{Ext}_R^{n+m}(M,N) \;\cong\; \operatorname{Ext}_R^m(\mathrm{C}_n(P),N)\;.$$

PROOF. Recall from 5.2.8 that $P_{\geqslant n}$ is a semi-projective R-complex. As one has $n \geqslant \sup M = \sup P$, the canonical morphism $\Sigma^{-n}P_{\geqslant n} \twoheadrightarrow \mathrm{C}_n(P)$ is a projective resolution. In the next computation, the 1st, 2nd, and 5th identities follow from the definitions of RHom and Hom; the 4th follows from 2.3.14.

$$\begin{aligned}
\mathrm{H}_{-(n+m)}(\mathsf{RHom}_R(M,N)) &\cong \mathrm{H}_{-(n+m)}(\operatorname{Hom}_R(P,N))\\
&= \mathrm{H}_{-(n+m)}(\operatorname{Hom}_R(P_{\geqslant n},N))\\
&= \mathrm{H}_{-m}(\Sigma^n \operatorname{Hom}_R(P_{\geqslant n},N))\\
&\cong \mathrm{H}_{-m}(\operatorname{Hom}_R(\Sigma^{-n}P_{\geqslant n},N))\\
&\cong \mathrm{H}_{-m}(\mathsf{RHom}_R(\mathrm{C}_n(P),N))\;;
\end{aligned}$$

the definition of Ext, 7.3.23, now yields the asserted isomorphism. □

8.1.7 Lemma. *Let P be a semi-projective R-complex and v an integer. The complex $P_{\subseteq v}$ is semi-projective if and only if the module $\mathrm{C}_v(P)$ is projective.*

PROOF. If the complex $P_{\subseteq v}$ is semi-projective, then the module $\mathrm{C}_v(P) = (P_{\subseteq v})_v$ is projective. If $\mathrm{C}_v(P)$ is a projective, then it is semi-projective as a complex by 5.2.12. There is an exact sequence $0 \to P_{\leqslant v-1} \to P_{\subseteq v} \to \Sigma^v \mathrm{C}_v(P) \to 0$, so by 5.2.17 the complex $P_{\subseteq v}$ is semi-projective if and only if $P_{\leqslant v-1}$ is semi-projective, which it is by 5.2.8 and 5.2.17 applied to the exact sequence $0 \to P_{\leqslant v-1} \to P \to P_{\geqslant v} \to 0$. □

8.1.8 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.
(*ii*) $-\inf \mathsf{RHom}_R(M,N) \leqslant n - \inf N$ *holds for every R-complex N.*
(*iii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^{n+1}(M,N) = 0$ *holds for every R-module N.*
(*iv*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^1(\mathrm{C}_n(P),\mathrm{C}_{n+1}(P)) = 0$ *holds for some, equivalently every, semi-projective replacement P of M.*
(*v*) $n \geqslant \sup M$ *and for some, equivalently every, semi-projective replacement P of M, the module $\mathrm{C}_n(P)$ is projective.*
(*vi*) $n \geqslant \sup M$ *and for every semi-projective replacement P of M, there is a semi-projective resolution $P_{\subseteq n} \xrightarrow{\simeq} M$.*
(*vii*) *There is a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v > n$ and for all $v < \inf M$.*

In particular, there are equalities,

$$\begin{aligned}
\operatorname{pd}_R M &= \sup\{-\inf \mathsf{RHom}_R(M,N) \mid N \text{ is an } R\text{-module}\}\\
&= \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(M,N) \neq 0 \text{ for some } R\text{-module } N\}\;.
\end{aligned}$$

PROOF. In addition to the seven conditions in the statement we consider

(ii') $-\inf \mathsf{RHom}_R(M,N) \leqslant n$ holds for every R-module N,

and prove that all eight conditions are equivalent. The implications (ii) $\Rightarrow$ (ii') and (vii) $\Rightarrow$ (i) are trivial.

(i) $\Rightarrow$ (ii): One can assume that N is in $\mathcal{D}_{\sqsupset}(R)$ and not acyclic; otherwise the inequality is trivial. Set $u = \inf N$; it is an integer and there is an isomorphism $\mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_R(M,N_{\supseteq u})$; see 4.2.4. By assumption there exists a semi-projective replacement P of M with $P_v = 0$ for all $v > n$, and by 7.3.17 it yields a semi-projective resolution $P \xrightarrow{\simeq} M$. Thus, one has $\inf \mathsf{RHom}_R(M,N) = \inf \operatorname{Hom}_R(P,N_{\supseteq u})$. For $v < u - n$ and $p \in \mathbb{Z}$, one of the inequalities $p > n$ or $p + v \leqslant n + v < u$ holds, so the module

$$\operatorname{Hom}_R(P,N_{\supseteq u})_v = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(P_p,(N_{\supseteq u})_{p+v})$$

is zero. In particular, $\mathrm{H}_v(\operatorname{Hom}_R(P,N_{\supseteq u})) = 0$ holds for $v < u - n$, so the inequality $\inf \mathsf{RHom}_R(M,N) \geqslant u - n$ holds as desired.

(ii') $\Rightarrow$ (iii): The second assertion is immediate from 7.3.24. Let E be a faithfully injective R-module; such a module exists by 1.3.38. Per 2.5.7(b) one now has $n \geqslant -\inf \mathsf{RHom}_R(M,E) = -\inf \operatorname{Hom}_R(M,E) = \sup M$.

(iii) $\Rightarrow$ (iv): Let P be a semi-projective replacement of M. It follows from 8.1.6 and the assumptions that $\operatorname{Ext}^1_R(\mathrm{C}_n(P),\mathrm{C}_{n+1}(P)) = 0$ holds.

(iv) $\Rightarrow$ (v): Let P be a semi-projective replacement of M. As $n \geqslant \sup M = \sup P$ holds, the sequence $0 \to \mathrm{C}_{n+1}(P) \to P_n \to \mathrm{C}_n(P) \to 0$ is exact, and the assertion follows from 8.1.1.

(v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (vi): Let P be a semi-projective replacement of M. As one has $n \geqslant \sup M = \sup P$, the morphism $\tau^P_{\subseteq n}\colon P \twoheadrightarrow P_{\subseteq n}$ is a quasi-isomorphism. Since the module $\mathrm{C}_n(P)$ is projective, the complex $P_{\subseteq n}$ is semi-projective by 8.1.7. Finally, it follows from the isomorphisms $P_{\subseteq n} \simeq P \simeq M$ in $\mathcal{D}(R)$ and 7.3.17 that there is a quasi-isomorphism $P_{\subseteq n} \to M$.

This argument shows that the "some" part of (v) implies (i), and thus (i)–(v) are equivalent. The argument also shows that the "every" part of (v) implies (vi).

(vi) $\Rightarrow$ (vii): Choose by 5.2.15 a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < \inf M$. By (vi) there is a quasi-isomorphism $P_{\subseteq n} \to M$, which is the desired resolution.

In the last assertion, the first equality follows from the equivalence of (i) and (ii') while the second holds by 7.3.24. □

The next corollary applies, in particular, to a short exact sequence of complexes, see 6.5.24.

8.1.9 Corollary. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $p' = \operatorname{pd}_R M'$, $p = \operatorname{pd}_R M$, and $p'' = \operatorname{pd}_R M''$ there are inequalities,*

$$p' \leqslant \max\{p, p''-1\}, \quad p \leqslant \max\{p',p''\}, \quad \textit{and} \quad p'' \leqslant \max\{p'+1,p\}.$$

In particular, if two of the complexes M', M, and M'' have finite projective dimension, then so has the third.

PROOF. For every R-module N there is a distinguished triangle,

$$\Sigma^{-1}\mathsf{RHom}_R(M',N) \longrightarrow \mathsf{RHom}_R(M'',N) \longrightarrow \mathsf{RHom}_R(M,N) \longrightarrow \mathsf{RHom}_R(M',N)\,.$$

The inequalities now follow from 8.1.8 and 6.5.20. □

REMARK. Corollary 8.1.9 basically shows that the complexes of finite projective dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 8.1.13 and also 10.1.21.

Every vector space has projective dimension 0 as a module over the base field. It follows from 1.3.11 and 1.3.21 that a module over a principal left ideal domain has projective dimension at most 1. It is not, though, hard to showcase a ring with modules of infinite projective dimension.

8.1.10 Example. The $\mathbb{Z}/4\mathbb{Z}$-complex

$$P \;=\; \cdots \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0\,,$$

concentrated in non-negative degrees, is a semi-projective replacement of the module $(\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/4\mathbb{Z}$ is indecomposable as a $\mathbb{Z}/4\mathbb{Z}$-module, no cokernel $\mathrm{C}_n(P) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 0$ is projective, so one has $\mathrm{pd}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \infty$.

8.1.11 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes; there is an equality,*

$$\mathrm{pd}_R\Big(\coprod_{u\in U} M^u\Big) \;=\; \sup_{u\in U}\{\mathrm{pd}_R\, M^u\}\,.$$

PROOF. Let N be an R-module. The functor $\mathsf{RHom}_R(-,N)$ preserves products by 3.1.27 and 7.2.14(c), so in view of 3.1.23 there are equalities,

$$\begin{aligned} -\inf\big(\mathsf{RHom}_R\big(\coprod_{u\in U} M^u, N\big)\big) \;&=\; -\inf\big(\prod_{u\in U}\mathsf{RHom}_R(M^u,N)\big) \\ &=\; \sup_{u\in U}\{-\inf \mathsf{RHom}_R(M^u,N)\}\,. \end{aligned}$$

The desired equality now follows from 8.1.8. □

SCHANUEL'S LEMMA AND THE HORSESHOE LEMMA

The next two results are known as Schanuel's lemma and the Horseshoe Lemma (for semi-projective complexes) the former can be seen as a refinement of 8.1.8(v).

8.1.12 Lemma. *Let M be an R-complex and P and P' be semi-projective replacements of M. For every $v \in \mathbb{Z}$ there exist projective R-modules L and L' with $\mathrm{C}_v(P) \oplus L \cong \mathrm{C}_v(P') \oplus L'$.*

PROOF. By 7.3.17 there is a homotopy equivalence $\alpha\colon P \to P'$, and by 4.3.30 the complex $\mathrm{Cone}\,\alpha$ is contractible. Furthermore, it consists of projective modules, so 4.3.33 shows that every module $\mathrm{Z}_n(\mathrm{Cone}\,\alpha)$ is projective. By 4.3.21 the morphism $\alpha_{\subseteq v}\colon P_{\subseteq v} \to P'_{\subseteq v}$ is also a homotopy equivalence so $\mathrm{Cone}(\alpha_{\subseteq v})$, that is,

$$0 \longrightarrow \mathrm{C}_v(P) \longrightarrow \mathrm{C}_v(P') \oplus P_{v-1} \longrightarrow P'_{v-1} \oplus P_{v-2} \xrightarrow{\partial^{\mathrm{Cone}\,\alpha}_{v-1}} P'_{v-2} \oplus P_{v-3} \longrightarrow \cdots ,$$

is contractible. Hence one has $\mathrm{C}_v(P) \oplus \mathrm{Z}_{v-1}(\mathrm{Cone}\,\alpha) \cong \mathrm{C}_v(P') \oplus P_{v-1}$. □

The next lemma applies, in particular, to a short exact sequence of complexes, see 6.5.24.

8.1.13 Lemma. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. If P' and P'' are semi-projective replacements of M' and M'', then there is an exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{C}(R)$ with P a semi-projective replacement of M.*

Proof. Rotation, (TR2) in E.2, of the given triangle yields a distinguished triangle, $\Sigma^{-1}M'' \to M' \to M \to M''$. As there are isomorphisms $M' \simeq P'$ and $M'' \simeq P''$ in $\mathcal{D}(R)$ there is also a distinguished triangle $\Sigma^{-1}P'' \xrightarrow{\gamma} P' \longrightarrow M \longrightarrow P''$. Since the complexes P' and P'' are semi-projective, the morphism γ in $\mathcal{D}(R)$ is induced by a morphism $\alpha\colon \Sigma^{-1}P'' \to P'$ in $\mathcal{C}(R)$; see 6.1.1 and 6.4.7. The diagram $\Sigma^{-1}P'' \xrightarrow{\alpha} P' \longrightarrow \mathrm{Cone}\,\alpha \longrightarrow P''$ in $\mathcal{C}(R)$ is a distinguished triangle when viewed as a diagram in $\mathcal{D}(R)$, see 6.2.3 and 6.5.5. As $\mathcal{D}(R)$ is triangulated, (TR3) in E.2 yields a morphism $\chi\colon \mathrm{Cone}\,\alpha \to M$ such that the diagram in $\mathcal{D}(R)$,

$$\begin{tikzcd}
\Sigma^{-1}P'' \arrow[r, "\gamma"] \arrow[d, equal] & P' \arrow[r] \arrow[d, equal] & \mathrm{Cone}\,\alpha \arrow[r] \arrow[d, "\chi"] & P'' \arrow[d, equal] \\
\Sigma^{-1}P'' \arrow[r, "\gamma"] & P' \arrow[r] & M \arrow[r] & P''
\end{tikzcd},$$

is commutative. It follows from 6.5.19 that χ is an isomorphism. Set $P = \mathrm{Cone}\,\alpha$; by 4.1.5 there is a short exact sequence $0 \to P' \to P \to P'' \to 0$, whence it follows from 5.2.17 that the complex P is semi-projective. □

Noetherian Rings and Homological Finiteness

Over a Noetherian ring, the projective dimension of a finitely generated module can be detected by vanishing of homology with coefficients in cyclic modules.

8.1.14 Theorem. *Assume that R is left Noetherian, let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and n an integer. The following conditions are equivalent.*

- (i) $\mathrm{pd}_R\, M \leqslant n$.
- (ii) $n \geqslant \sup M$ *and* $\mathrm{Ext}^{n+1}_R(M, R/\mathfrak{a}) = 0$ *for every left ideal* $\mathfrak{a}$ *in* R.
- (iii) *There is a semi-projective resolution $P \xrightarrow{\simeq} M$ with P degreewise finitely generated and $P_v = 0$ for all $v > n$ and for all $v < \inf M$.*

In particular, there are equalities,

$$\begin{aligned}
\mathrm{pd}_R\, M &= \sup\{-\inf \mathsf{RHom}_R(M, R/\mathfrak{a}) \mid \mathfrak{a} \text{ is a left ideal in } R\} \\
&= \sup\{m \in \mathbb{Z} \mid \mathrm{Ext}^m_R(M, R/\mathfrak{a}) \neq 0 \text{ for some left ideal } \mathfrak{a} \text{ in } R\}\,.
\end{aligned}$$

Proof. It follows from the definition of projective dimension that (*iii*) implies (*i*), and the implication (*i*) $\Rightarrow$ (*ii*) holds by 8.1.8. To see that (*ii*) implies (*iii*), notice first that by (*ii*) and 1.3.51 one has $\mathrm{Ext}_R^{n+1}(M,N) = 0$ for every finitely generated R-module N. Choose by 5.2.16 a semi-projective resolution $P \xrightarrow{\simeq} M$ with P degreewise finitely generated and $P_v = 0$ for all $v < \inf M$. As $n \geqslant \sup M = \sup P$ holds, the sequence $0 \to \mathrm{C}_{n+1}(P) \to P_n \to \mathrm{C}_n(P) \to 0$ is exact; moreover, the modules $\mathrm{C}_v(P)$ are finitely generated. By 8.1.6 there is an isomorphism

$$\mathrm{Ext}_R^1(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P)) \cong \mathrm{Ext}_R^{n+1}(M, \mathrm{C}_{n+1}(P)) = 0\,,$$

whence $\mathrm{C}_n(P)$ is projective by 8.1.1. Now 8.1.8 finishes the proof. □

8.1.15 Corollary. *Assume that R is left Noetherian let and M be a complex in* $\mathcal{D}_{\square}^{\mathrm{f}}(R)$. *If M has finite projective dimension, then the following equalities hold*:

$$\mathrm{pd}_R\, M = -\inf \mathsf{R}\mathrm{Hom}_R(M,R) = \sup\{m \in \mathbb{Z} \mid \mathrm{Ext}_R^m(M,R) \neq 0\}\,.$$

Proof. The equalities are trivial if M is acyclic, so assume that it is not and let $n \in \mathbb{Z}$ be the projective dimension of M. By 8.1.14 there is a left ideal $\mathfrak{a}$ in R with $\mathrm{Ext}_R^n(M, R/\mathfrak{a}) \neq 0$. The exact sequence $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ induces per 7.3.35 an exact sequence

$$(\dagger) \qquad \cdots \longrightarrow \mathrm{Ext}_R^n(M,R) \longrightarrow \mathrm{Ext}_R^n(M,R/\mathfrak{a}) \longrightarrow \mathrm{Ext}_R^{n+1}(M,\mathfrak{a})\,.$$

By 8.1.8 one has $\mathrm{Ext}_R^{n+1}(M,\mathfrak{a}) = 0$, so exactness of (†) implies $\mathrm{Ext}_R^n(M,R) \neq 0$. This yields $n \leqslant -\inf \mathsf{R}\mathrm{Hom}_R(M,R)$ and the opposite inequality holds by 8.1.8. □

Remark. The assumption of finite projective dimension in 8.1.15 is necessary; see E 8.1.11.

Perfect and Semi-Perfect Rings

Existence of minimal semi-projective resolutions is treated in B.60 and B.61.

8.1.16 Proposition. *Let M be an R-complex and n an integer. If* $P \xrightarrow{\simeq} M$ *is a minimal semi-projective resolution, then the following conditions are equivalent.*

(*i*) $\mathrm{pd}_R\, M \leqslant n$.

(*ii*) $n \geqslant \sup M$ *and* $P_{n+1} = 0$.

(*iii*) $P_v = 0$ *for all* $v > n$.

In particular, one has $\mathrm{pd}_R\, M = \sup P^{\natural}$.

Proof. If $\mathrm{pd}_R\, M \leqslant n$ holds, then there is a semi-projective replacement P' of M with $P'_v = 0$ for all $v > n$. It follows from 6.4.20 that there is a quasi-isomorphism $P' \to P$, and by B.56 it has a right inverse, whence $P_v = 0$ holds for $v > n$. Thus (*i*) implies (*iii*) which, in turn, implies (*ii*). If one has $n \geqslant \sup M$ and $P_{n+1} = 0$, then the module $\mathrm{C}_n(P) = P_n$ is projective, whence $\mathrm{pd}_R\, M \leqslant n$ holds by 8.1.8. □

Recall from B.51 that every left or right Artinian ring is left perfect, whence the next theorem holds for such ring.

8.1.17 Theorem. *Assume that R is left perfect with Jacobson radical $\mathfrak{J}$ and set $\boldsymbol{k} = R/\mathfrak{J}$. Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.
(*ii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^{n+1}(M,\boldsymbol{k}) = 0$.
(*iii*) $n \geqslant \sup M$ *and* $\operatorname{Tor}^R_{n+1}(\boldsymbol{k},M) = 0$.

In particular, one has

$$\operatorname{pd}_R M \;=\; -\inf \mathsf{R}\mathrm{Hom}_R(M,\boldsymbol{k}) \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\;.$$

Proof. Choose by B.60 a semi-projective resolution $P \xrightarrow{\simeq} M$ with P minimal. The graded module $P^\natural$ is semi-perfect by B.53, so $\partial^P(P) \subseteq \mathfrak{J}P$ holds by B.55. It follows that the complex $\boldsymbol{k} \otimes_R P$ has zero differential, so one has

$$\operatorname{Tor}^R_v(\boldsymbol{k},M) \;=\; \mathrm{H}_v(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; (\boldsymbol{k} \otimes_R P)_v \;=\; \boldsymbol{k} \otimes_R P_v\;.$$

By B.40(a) the module $\boldsymbol{k} \otimes_R P_v \cong P_v/\mathfrak{J}P_v$ is non-zero for every $P_v \neq 0$, so one has $\operatorname{Tor}^R_v(\boldsymbol{k},M) = 0$ if and only if $P_v = 0$. The equivalence of conditions (*i*) and (*iii*) now follows from 8.1.16.

It also follows from the inclusion $\partial^P(P) \subseteq \mathfrak{J}P$ that the complex $\operatorname{Hom}_R(P,\boldsymbol{k})$ has zero differential, whence one has

$$\operatorname{Ext}^v_R(M,\boldsymbol{k}) \;=\; \mathrm{H}_{-v}(\mathsf{R}\mathrm{Hom}_R(M,\boldsymbol{k})) \;=\; \operatorname{Hom}_R(P,\boldsymbol{k})_{-v} \;=\; \operatorname{Hom}_R(P_v,\boldsymbol{k})\;.$$

For every $v \in \mathbb{Z}$ with $P_v \neq 0$ the $\boldsymbol{k}$-module $P_v/\mathfrak{J}P_v$ is non-zero. As $\boldsymbol{k}$ is semi-simple, it follows that there is a direct summand of $P_v/\mathfrak{J}P_v$ which is isomorphic to a non-zero ideal in $\boldsymbol{k}$. Thus, there is a non-zero homomorphism $P_v \twoheadrightarrow P_v/\mathfrak{J}P_v \to \boldsymbol{k}$. In particular, one has $\operatorname{Ext}^v_R(M,\boldsymbol{k}) = 0$ if and only if $P_v = 0$. The equivalence of (*i*) and (*ii*) now follows from 8.1.16. The final equalities follow from the equivalence of (*i*)–(*iii*). □

Recall from B.44 that every local ring is semi-perfect, whence the next theorem holds for Noetherian local rings.

8.1.18 Theorem. *Assume that R is left Noetherian and semi-perfect with Jacobson radical $\mathfrak{J}$ and set $\boldsymbol{k} = R/\mathfrak{J}$. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.
(*ii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^{n+1}(M,\boldsymbol{k}) = 0$.
(*iii*) $n \geqslant \sup M$ *and* $\operatorname{Tor}^R_{n+1}(\boldsymbol{k},M) = 0$.

In particular, there are equalities,

$$\operatorname{pd}_R M \;=\; -\inf \mathsf{R}\mathrm{Hom}_R(M,\boldsymbol{k}) \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\;.$$

Proof. Choose by B.61 a semi-projective resolution $P \xrightarrow{\simeq} M$ with P minimal and degreewise finitely generated. The graded module $P^\natural$ is semi-perfect by B.46, and from this point the proof of 8.1.17 applies verbatim. □

THE CASE OF MODULES

8.1.19. Notice from 8.1.8 that a non-zero R-module is projective if and only if it has projective dimension 0 as an R-complex.

8.1.20 Theorem. *Let M be an R-module and $n \geqslant 0$ an integer. The following conditions are equivalent.*

(*i*) $\mathrm{pd}_R\, M \leqslant n$.

(*ii*) *One has* $\mathrm{Ext}_R^m(M,N) = 0$ *for every R-module N and every integer $m > n$.*

(*iii*) *One has* $\mathrm{Ext}_R^{n+1}(M,N) = 0$ *for every R-module N.*

(*iv*) *There is a projective resolution* $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$.

(*v*) *In every projective resolution* $\cdots \to P_v \to P_{v-1} \to \cdots \to P_0 \to M \to 0$ *the module* $\mathrm{Coker}(P_{n+1} \to P_n)$ *is projective.*

In particular, there is an equality,

$$\mathrm{pd}_R\, M \;=\; \sup\{m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(M,N) \neq 0 \text{ for some } R\text{-module } N\}\,.$$

PROOF. By 5.2.27 every R-module M has a projective resolution

$$\cdots \longrightarrow P_v \longrightarrow P_{v-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0\,.$$

In every such resolution, the surjective homomorphism $P_0 \to M$ yields a semi-projective resolution of M, considered as a complex; cf. 5.2.29. Thus the complex $\cdots \to P_v \to P_{v-1} \to \cdots \to P_0 \to 0$ is a semi-projective replacement of M. The equivalence of the five conditions now follows from the equivalence of (*i*)–(*iii*), (*v*), and (*vii*) in 8.1.8. The asserted equality holds by 8.1.8 in view of 7.3.27. □

REMARK. Let $A \neq 0$ be a $\mathbb{Z}$-module; it follows from 1.3.11 that $\mathrm{pd}_{\mathbb{Z}}\, A$ is 0 or 1. If $\mathrm{pd}_{\mathbb{Z}}\, A = 0$, which by 1.3.21 means that A is free, then $\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Z}) = 0$ holds. By E 7.3.7 the converse is true if A is finitely generated and, more generally, Stein [240] has shown that every countably generated $\mathbb{Z}$-module A with $\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Z}) = 0$ is free. The question whether every $\mathbb{Z}$-module A with $\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Z}) = 0$ is free is known as Whitehead's problem; Shelah [231] has shown that it is undecidable within ZFC.

8.1.21 Theorem. *Assume that R is left Noetherian, let M be a finitely generated R-module and $n \geqslant 0$ an integer. The following conditions are equivalent.*

(*i*) $\mathrm{pd}_R\, M \leqslant n$.

(*ii*) *One has* $\mathrm{Ext}_R^{n+1}(M,R/\mathfrak{a}) = 0$ *for every left ideal $\mathfrak{a}$ in R.*

(*iii*) *There is a projective resolution* $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *with each module P_v finitely generated.*

In particular, there is an equality,

$$\mathrm{pd}_R\, M \;=\; \sup\{m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(M,R/\mathfrak{a}) \neq 0 \text{ for some left ideal } \mathfrak{a} \text{ in } R\}\,.$$

PROOF. By 5.1.19 every finitely generated R-module M has a projective resolution

$$\cdots \longrightarrow P_v \longrightarrow P_{v-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each module P_v finitely generated. An argument parallel to the proof of 8.1.20 shows that the equivalence of the three conditions follows from 8.1.14 and that the asserted equality holds. □

Exercises

E 8.1.1 Let M be an R-complex of finite projective dimension n. Show that M has a semi-free replacement L with $L_v = 0$ for $v > n + 1$.

E 8.1.2 Let $R \to S$ be a ring homomorphism. Show that $\operatorname{pd}_R N \leqslant \operatorname{pd}_S N + \operatorname{pd}_R S$ holds for every S-complex N with $\mathrm{H}(N) \neq 0$.

E 8.1.3 Let R be semi-simple. Show that $\operatorname{pd}_R M = \sup M$ holds for every R-complex M.

E 8.1.4 Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $w = \sup M$. Show that for every semi-projective replacement P of M one has $\operatorname{pd}_R M = w + \operatorname{pd}_R \mathrm{C}_w(P)$.

E 8.1.5 Let M be an R-complex. Show that $\operatorname{pd}_R M$ is finite if and only if $\mathrm{H}(\mathsf{RHom}_R(M, N))$ is bounded below for every R-module N.

E 8.1.6 Let M be an R-complex and assume that it is isomorphic in $\mathcal{D}(R)$ to a K-projective complex X with $X_v = 0$ for all $v > n$. Show that $\operatorname{pd}_R M$ is at most n, and conclude that one could use K-projective replacements in 8.1.2.

E 8.1.7 Let R be left hereditary. Show that $\operatorname{pd}_R M \leqslant \sup M + 1$ holds for every R-complex M.

E 8.1.8 Show that every finitely generated R-module has finite projective dimension if and only if every left ideal in R has finite projective dimension. *Hint:* Proof of 1.3.51.

E 8.1.9 Let M be an R-complex of finite projective dimension n. Show that there exists a projective R-module P with $\operatorname{Ext}_R^n(M, P) \neq 0$.

E 8.1.10 Let $0 \to K \to L \to M \to 0$ and $0 \to K' \to L' \to M \to 0$ be exact sequences of R-modules. Show that if $\operatorname{Ext}_R^1(L, K') = 0 = \operatorname{Ext}_R^1(L', K)$ holds, then there is an isomorphism $L \oplus K' \cong L' \oplus K$.

E 8.1.11 Show that $\operatorname{Ext}_{\mathbb{Z}/4\mathbb{Z}}^m(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) = 0$ holds for all $m \geqslant 1$ and compare to 8.1.10.

E 8.1.12 Show that a finitely generated $\mathbb{Z}/4\mathbb{Z}$-module M is projective if and only if one has $\operatorname{Ext}_{\mathbb{Z}/4\mathbb{Z}}^1(M, \mathbb{Z}/2\mathbb{Z}) = 0$.

E 8.1.13 Show that the full subcategory of R-complexes of finite projective dimension is a triangulated subcategory of $\mathcal{D}_{\sqsubset}(R)$.

8.2 Injective Dimension

Synopsis. Vanishing of Ext; injective dimension; Schanuel's lemma; Horseshoe Lemma; minimal semi-injective resolution; injective dimension over Artinian ring; injective dimension of module; coproduct of injective modules.

This section develops, initially, in close parallel to the previous one.

8.2.1 Lemma. *For an R-module I the following conditions are equivalent.*

(*i*) *I is injective.*

(*ii*) $-\inf \mathsf{RHom}_R(M, I) \leqslant \sup M$ *holds for every R-complex M.*

(*iii*) *One has $\operatorname{Ext}_R^1(R/\mathfrak{a}, I) = 0$ for every left ideal $\mathfrak{a}$ in R.*

(*iv*) *There exists an injective homomorphism $\iota\colon I \to E$ with E injective and with $\operatorname{Ext}_R^1(\operatorname{Coker}\iota, I) = 0$.*

Proof. If the module I is injective, then 1^I is a injective resolution of I, and one has $\mathsf{RHom}_R(M,I) = \operatorname{Hom}_R(M,I)$ in $\mathcal{D}(\Bbbk)$. Now 2.5.7(b) yields $-\inf \mathsf{RHom}_R(M,I) \leqslant \sup M$. Thus, (*i*) implies (*ii*) which, in particular, implies that the functor $\operatorname{Ext}^1_R(-,I)$ is zero on modules. Hence (*ii*) implies (*iii*) and, in view of 5.3.4, also (*iv*).

(*iii*) $\Rightarrow$ (*i*): Consider the short exact sequence $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$. It yields an exact sequence $\operatorname{Hom}_R(R,I) \to \operatorname{Hom}_R(\mathfrak{a},I) \to \operatorname{Ext}^1_R(R/\mathfrak{a},I) = 0$ by 7.3.35 7.3.27. Thus, every homomorphism $\mathfrak{a} \to I$ is the restriction of a homomorphism $R \to I$, whence I is injective by Baer's criterion 1.3.30.

(*iv*) $\Rightarrow$ (*i*): The exact sequence $0 \to I \to E \to \operatorname{Coker}\iota \to 0$ is split by 7.3.36, so I is a direct summand of an injective module and hence injective by 1.3.27. □

Remark. Part (*iii*) in 8.2.1 is the homological formulation of Baer's criterion, it has been sharpened by Vamos [245]; see E 8.2.1.

Semi-Injective Replacements and Injective Dimension

Recall from 7.3.18 that a semi-injective replacement of an R-complex M is a semi-injective R-complex that is isomorphic to M in $\mathcal{D}(R)$.

8.2.2 Definition. Let M be an R-complex. The *injective dimension* of M, written $\operatorname{id}_R M$, is defined as

$$\operatorname{id}_R M = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{matrix}\text{There is a semi-injective replacement}\\ I \text{ of } M \text{ with } I_{-v} = 0 \text{ for all } v > n\end{matrix} \right\},$$

with the convention $\inf \varnothing = \infty$. One says that $\operatorname{id}_R M$ is *finite* if $\operatorname{id}_R M < \infty$ holds.

A comment similar to the one after 8.1.2 justifies the last convention in 8.2.2.

Remark. The minus sign in the definition above appears because we use homological notation; if one is concerned, primarily, with semi-injective replacements, then cohomological notation is a more natural choice. See also the Remark after 2.1.23.

8.2.3. Let M be an R-complex. For every semi-injective replacement I of M one has $\mathrm{H}(I) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\operatorname{id}_R M \geqslant -\inf M \quad \text{and} \quad \operatorname{id}_R \Sigma^s M = \operatorname{id}_R M - s \quad \text{for every integer } s\,.$$

Moreover, one has $\operatorname{id}_R M = -\infty$ if and only if M is acyclic.

Note that the definition of injective dimension could also be written

$$\operatorname{id}_R M = \inf\{-\inf I^\natural \mid I \text{ is a semi-injective replacement of } M\}\,.$$

8.2.4 Proposition. *Let $R \to S$ be a ring homomorphism and M be an R-complex. There is an inequality,*

$$\operatorname{id}_S \mathsf{RHom}_R(S,M) \leqslant \operatorname{id}_R M\,.$$

Proof. For every semi-injective replacement I of the R-complex M, the S-complex $\operatorname{Hom}_R(S,I)$ is a semi-injective replacement of $\mathsf{RHom}_R(S,M)$ by 5.4.26(a). As one has $I_{-v} = 0$ implies $\operatorname{Hom}_R(S,I)_{-v} = 0$, the desired inequality follows from 8.2.2. □

Even if S is flat over R, the inequality in 8.2.4 may be strict, but see also 17.3.16.

8.2.5 Example. Let R be the integral domain $\mathbb{Q}[x,y]$ and S its field of fractions, which is flat as an R-module by 1.3.42. The cardinality of R, and hence of S, is $\aleph_0$, so D.9 yields $\operatorname{pd}_R S \leqslant 1$. Consequently, for every R-module M, the homology of the complex $\mathsf{RHom}_R(S,M)$ is concentrated in degrees -1 and 0; see 8.1.8. As S is a field, 6.4.23 yields an isomorphism $\mathsf{RHom}_R(S,M) \simeq (\Sigma^{-1}V^1) \oplus V^0$ in $\mathcal{D}(S)$ where V^n is the S-vector space $\operatorname{Ext}^n_R(S,M)$ for $n = 0,1$, in particular $\operatorname{id}_S \mathsf{RHom}_R(S,M) \leqslant 1$ by 1.3.28. On the other hand, $\operatorname{id}_R \mathbb{Q} \geqslant 2$. Indeed, as the Koszul complex $\mathrm{K}^R(x,y)$ yields a projective resolution of $\mathbb{Q}$, see 2.2.9, there are isomorphisms in $\mathcal{D}(R)$,

$$\mathsf{RHom}_R(\mathbb{Q},\mathbb{Q}) \;\simeq\; \operatorname{Hom}_R(\mathrm{K}^R(x,y),\mathbb{Q}) \;\simeq\; \mathbb{Q} \oplus (\Sigma^{-1}\mathbb{Q}^2) \oplus (\Sigma^{-2}\mathbb{Q})\;.$$

In particular, $\operatorname{Ext}^2_R(\mathbb{Q},\mathbb{Q}) = \mathbb{Q} \neq 0$ holds, whence one has $\operatorname{id}_R \mathbb{Q} \geqslant 2$ by 8.2.2; see also 8.2.19.

The isomorphism below is colloquially referred to as “dimension shifting”.

8.2.6 Lemma. *Let M be an R-complex, I a semi-injective replacement of M, and N an R-module. For all integers $m > 0$ and $n \geqslant -\inf M$ one has*

$$\operatorname{Ext}^{n+m}_R(N,M) \;\cong\; \operatorname{Ext}^m_R(N,\mathrm{Z}_{-n}(I))\;.$$

Proof. Recall from 5.3.12 that $I_{\leqslant -n}$ is a semi-injective R-complex. As one has $n \geqslant -\inf M = -\inf I$, the canonical morphism $\mathrm{Z}_{-n}(I) \rightarrowtail \Sigma^n I_{\leqslant -n}$ is an injective resolution. In the next computation, the 1st, 2nd, and 5th identities follow from the definitions of RHom and Hom; the 4th follows from 2.3.16.

$$\begin{aligned}
\mathrm{H}_{-(n+m)}(\mathsf{RHom}_R(N,M)) &\cong \mathrm{H}_{-(n+m)}(\operatorname{Hom}_R(N,I))\\
&= \mathrm{H}_{-(n+m)}(\operatorname{Hom}_R(N,I_{\leqslant -n}))\\
&= \mathrm{H}_{-m}(\Sigma^n \operatorname{Hom}_R(N,I_{\leqslant -n}))\\
&\cong \mathrm{H}_{-m}(\operatorname{Hom}_R(N,\Sigma^n I_{\leqslant -n}))\\
&\cong \mathrm{H}_{-m}(\mathsf{RHom}_R(N,\mathrm{Z}_{-n}(I)))\;;
\end{aligned}$$

the definition of Ext, 7.3.23, now yields the asserted isomorphism. □

8.2.7 Lemma. *Let I be a semi-injective R-complex and v an integer. The complex $I_{\supseteq v}$ is semi-injective if and only if the module $\mathrm{Z}_v(I)$ is injective.*

Proof. If the complex $I_{\supseteq v}$ is semi-injective, then the module $\mathrm{Z}_v(I) = (I_{\supseteq v})_v$ is injective. If $\mathrm{Z}_v(I)$ is an injective module, then it is semi-injective as a complex by 5.3.18. There is an exact sequence $0 \to \Sigma^v \mathrm{Z}_v(I) \to I_{\supseteq v} \to I_{\geqslant v+1} \to 0$, so by 5.3.20 the complex $I_{\supseteq v}$ is semi-injective if and only if $I_{\geqslant v+1}$ is semi-injective, which it is by 5.3.12 and 5.3.20 applied to the exact sequence $0 \to I_{\leqslant v} \to I \to I_{\geqslant v+1} \to 0$. □

8.2.8 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{id}_R M \leqslant n$.
(*ii*) $-\inf \mathsf{RHom}_R(N,M) \leqslant n + \sup N$ *holds for every R-complex N.*

(*iii*) $n \geqslant -\inf M$ *and* $\operatorname{Ext}_R^{n+1}(R/\mathfrak{a}, M) = 0$ *holds for every left ideal* $\mathfrak{a}$ *in* R.

(*iv*) $n \geqslant -\inf M$ *and one has* $\operatorname{Ext}_R^1(\mathrm{Z}_{-(n+1)}(I), \mathrm{Z}_{-n}(I)) = 0$ *for some, equivalently every, semi-injective replacement* I *of* M.

(*v*) $n \geqslant -\inf M$ *and for some, equivalently every, semi-injective replacement* I *of* M*, the module* $\mathrm{Z}_{-n}(I)$ *is injective.*

(*vi*) $n \geqslant -\inf M$ *and for every semi-injective replacement* I *of* M*, there is a semi-injective resolution* $M \xrightarrow{\simeq} I_{\supseteq -n}$.

(*vii*) *There is a semi-injective resolution* $M \xrightarrow{\simeq} I$ *with* $I_{-v} = 0$ *for all* $v > n$ *and for all* $v < -\sup M$.

In particular, there are equalities,

$$\begin{aligned}\operatorname{id}_R M &= \sup\{-\inf \mathsf{RHom}_R(R/\mathfrak{a}, M) \mid \mathfrak{a} \text{ is a left ideal in } R\}\\ &= \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(R/\mathfrak{a}, M) \neq 0 \text{ for some left ideal } \mathfrak{a} \text{ in } R\}\,.\end{aligned}$$

Proof. In addition to the seven conditions in the statement we consider

(*ii′*) $-\inf \mathsf{RHom}_R(N, M) \leqslant n$ holds for every R-module N,

and prove that all eight conditions are equivalent. The implications (*ii*) ⇒ (*ii′*) and (*vii*) ⇒ (*i*) are trivial.

(*i*) ⇒ (*ii*): One can assume that N is in $\mathcal{D}_{\sqsubset}(R)$ and not acyclic; otherwise the inequality is trivial. Set $w = \sup N$; it is an integer and there is an isomorphism $\mathsf{RHom}_R(N, M) \simeq \mathsf{RHom}_R(N_{\subseteq w}, M)$; see 4.2.4. Choose a semi-injective replacement I of M with $I_{-v} = 0$ for all $v > n$; by 7.3.19 it yields a semi-injective resolution $M \xrightarrow{\simeq} I$. Thus, one has $\inf \mathsf{RHom}_R(N, M) = \inf \operatorname{Hom}_R(N_{\subseteq w}, I)$. For $v > n + w$ and $p \in \mathbb{Z}$, one of the inequalities $p > w$ or $v - p \geqslant v - w > n$ holds, so the module

$$\operatorname{Hom}_R(N_{\subseteq w}, I)_{-v} = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R((N_{\subseteq w})_p, I_{-(v-p)})$$

is zero. In particular, $\mathrm{H}_{-v}(\operatorname{Hom}_R(N_{\subseteq w}, I)) = 0$ holds for $v > n + w$, so the inequality $-\inf \mathsf{RHom}_R(M, N) \leqslant n + w$ holds as desired.

(*ii′*) ⇒ (*iii*): The second assertion is immediate from 7.3.24, and an application of (*ii′*) with $N = R$ yields $-\inf M = -\inf \mathsf{RHom}_R(R, M) \leqslant n$.

(*iii*) ⇒ (*v*): Let I be a semi-injective replacement of M. It follows from 8.2.6 that $\operatorname{Ext}_R^1(R/\mathfrak{a}, \mathrm{Z}_{-n}(I)) = 0$ holds for every left ideal $\mathfrak{a}$ in R, whence $\mathrm{Z}_{-n}(I)$ is an injective module by 8.2.1.

(*v*) ⇒ (*iv*): This implication is immediate from 8.2.1.

(*iv*) ⇒ (*i*) and (*iv*) ⇒ (*vi*): Let I be a semi-injective replacement of M. As one has $-n \leqslant \inf M = \inf I$, the sequence $0 \to \mathrm{Z}_{-n}(I) \to I_{-n} \to \mathrm{Z}_{-(n+1)}(I) \to 0$ is exact, and it follows from 8.2.1 that the module $\mathrm{Z}_{-n}(I)$ is injective. Further, the canonical morphism $\tau^{I}_{\supseteq -n} \colon I_{\supseteq -n} \rightarrowtail I$ is a quasi-isomorphism, so in $\mathcal{D}(R)$ there is an isomorphism $M \simeq I_{\supseteq -n}$, and the right-hand complex is semi-injective by 8.2.7. Now it follows from 7.3.19 that there is quasi-isomorphism $M \to I_{\supseteq -n}$.

This argument shows that the "some" part of (*v*) implies (*i*), and thus (*i*)–(*v*) are equivalent. The argument also shows that the "every" part of (*v*) implies (*vi*).

$(vi) \Rightarrow (vii)$: Choose by 5.3.26 a semi-injective resolution $M \xrightarrow{\simeq} I$ with $I_{-v} = 0$ for all $v < -\sup M$. In particular, I is a semi-injective replacement of M, so by (vi) there is a quasi-isomorphism $M \longrightarrow I_{\supseteq -n}$, which is the desired resolution.

In the last assertion, the first equality follows from the equivalence of (i) and (ii') while the second holds by 7.3.24. □

The next corollary applies, in particular, to a short exact sequence of complexes, see 6.5.24.

8.2.9 Corollary. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $\imath' = \mathrm{id}_R\, M'$, $\imath = \mathrm{id}_R\, M$, and $\imath'' = \mathrm{id}_R\, M''$ there are inequalities,*

$$\imath' \leqslant \max\{\imath, \imath'' + 1\}\,, \quad \imath \leqslant \max\{\imath', \imath''\}\,, \quad \textit{and} \quad \imath'' \leqslant \max\{\imath' - 1, \imath\}\,.$$

In particular, if two of the complexes M', M, and M'' have finite injective dimension, then so has the third.

Proof. For every R-module N there is a distinguished triangle,

$$\mathsf{RHom}_R(N, M') \longrightarrow \mathsf{RHom}_R(N, M) \longrightarrow \\ \mathsf{RHom}_R(N, M'') \to \Sigma\mathsf{RHom}_R(N, M')\,.$$

The inequalities now follow from 8.2.8 and 6.5.20. □

Remark. Corollary 8.2.9 basically shows that the complexes of finite injective dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 8.2.17 and also 10.1.21.

Over a semi-simple ring every module is injective, see 1.3.28; in particular, such a ring is self-injective. As the next example illustrates, modules over a self-injective ring can have infinite injective dimension.

8.2.10 Proposition. *Assume that R is a principal ideal domain. For every ideal $\mathfrak{a} \neq 0$ in R, the quotient ring $R/\mathfrak{a}$ is self-injective.*

Proof. We use Baer's criterion 1.3.30. Choose an element $a \neq 0$ in R with $\mathfrak{a} = (a)$. Every ideal in $R/(a)$ has the form $(b)/(a)$ with $a = bc$ for some $c \in R$. Given a homomorphism $\varphi\colon (b)/(a) \to R/(a)$ of $R/(a)$-modules, one has $\varphi([b]_{(a)}) = [x]_{(a)}$ for some $x \in R$. As $bc = a$ one has $cx \in (a)$, that is, $cx = ay$ for some $y \in R$. Hence $ax = bcx = bay$ and thus $x = by$. So $\varphi'\colon R/(a) \to R/(a)$ given by $[r]_{(a)} \mapsto [ry]_{(a)}$ is a homomorphism of $R/(a)$-modules whose restriction to $(b)/(a)$ is φ. □

8.2.11 Example. The ring $\mathbb{Z}/4\mathbb{Z}$ is self-injective by 8.2.10. Hence the complex

$$I = 0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots\,,$$

concentrated in non-positive degrees, is a semi-injective replacement of the $\mathbb{Z}/4\mathbb{Z}$-module $2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. As $\mathbb{Z}/4\mathbb{Z}$ is indecomposable as a $\mathbb{Z}/4\mathbb{Z}$-module, no kernel $\mathrm{Z}_{-n}(I) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geqslant 0$ is injective, so one has $\mathrm{id}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \infty$.

8.2.12 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes; there is an equality,*

$$\operatorname{id}_R\Bigl(\prod_{u\in U} M^u\Bigr) = \sup_{u\in U}\{\operatorname{id}_R M^u\}\,.$$

Proof. Let N be an R-module. The functor $\mathsf{RHom}_R(N,-)$ preserves products by 3.1.24 and 7.2.14, so in view of 3.1.23 there are equalities,

$$\begin{aligned}-\inf \mathsf{RHom}_R\bigl(N,\prod_{u\in U} M^u\bigr) &= -\inf\bigl(\prod_{u\in U}\mathsf{RHom}_R(N,M^u)\bigr)\\ &= \sup_{u\in U}\{-\inf \mathsf{RHom}_R(N,M^u)\}\,.\end{aligned}$$

The desired equality now follows from 8.2.8. □

Schanuel's Lemma and the Horseshoe Lemma

The next two results are known as Schanuel's lemma and the Horseshoe Lemma (for semi-injective complexes) the former can be seen as a refinement of 8.2.8(v).

8.2.13 Lemma. *Let M be an R-complex and I and I' be semi-injective replacements of M. For every $v\in\mathbb{Z}$ there exist injective R-modules E and E' with $\mathrm{Z}_v(I)\oplus E\cong \mathrm{Z}_v(I')\oplus E'$.*

Proof. By 7.3.19 there is a homotopy equivalence $\alpha\colon I\to I'$, and by 4.3.30 the complex $\operatorname{Cone}\alpha$ is contractible. Furthermore, it consists of injective modules, so 4.3.33 shows that every module $\mathrm{C}_n(\operatorname{Cone}\alpha)$ is injective. By 4.3.21 the morphism $\alpha_{\supseteq v}\colon I_{\supseteq v}\to I'_{\supseteq v}$ is also a homotopy equivalence so $\operatorname{Cone}(\alpha_{\supseteq v})$, that is,

$$\cdots\longrightarrow I'_{v+3}\oplus I_{v+2}\xrightarrow{\partial^{\operatorname{Cone}\alpha}_{v+3}} I'_{v+2}\oplus I_{v+1}\longrightarrow I'_{v+1}\oplus \mathrm{Z}_v(I)\longrightarrow \mathrm{Z}_v(I')\longrightarrow 0\,,$$

is contractible. Hence one has $\mathrm{Z}_v(I')\oplus \mathrm{C}_{v+2}(\operatorname{Cone}\alpha)\cong I'_{v+1}\oplus \mathrm{Z}_v(I)$. □

The next lemma applies, in particular, to a short exact sequence of complexes, see 6.5.24.

8.2.14 Lemma. *Let $M'\to M\to M''\to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. If I' and I'' are semi-injective replacements of M' and M'', then there is an exact sequence $0\to I'\to I\to I''\to 0$ in $\mathcal{C}(R)$ with I a semi-injective replacement of M.*

Proof. Rotation, see (TR2) in E.2, of the given triangle yields a distinguished triangle, $\Sigma^{-1}M''\to M'\to M\to M''$. Since $M'\simeq I'$ and $M''\simeq I''$ hold in $\mathcal{D}(R)$ there is also a distinguished triangle $\Sigma^{-1}I''\xrightarrow{\gamma} I'\longrightarrow M\longrightarrow I''$. The morphism γ is a left fraction β/φ where $\Sigma^{-1}I''\xleftarrow{\varphi} U\xrightarrow{\beta} I$ is a diagram in $\mathcal{K}(R)$ and φ is a quasi-isomorphism. Now 6.3.5 yields a morphism $\alpha\colon\Sigma^{-1}I''\to I$ in $\mathcal{C}(R)$ with $[\alpha]\varphi=\beta$, and hence $\gamma=\beta/\varphi=([\alpha]\varphi)/\varphi=[\alpha]/1$. The diagram $\Sigma^{-1}I''\xrightarrow{\alpha} I'\longrightarrow \operatorname{Cone}\alpha\longrightarrow I''$ in $\mathcal{C}(R)$ is a distinguished triangle when viewed as a diagram in $\mathcal{D}(R)$, see 6.2.3 and 6.5.5. As $\mathcal{D}(R)$ is triangulated, (TR3) in E.2 yields a morphism $\chi\colon M\to\operatorname{Cone}\alpha$ such that the diagram in $\mathcal{D}(R)$,

$$\begin{tikzcd}
\Sigma^{-1}I'' \arrow[r, "\gamma"] \arrow[d, equal] & I' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\chi"] & I'' \arrow[d, equal] \\
\Sigma^{-1}I'' \arrow[r, "\gamma"] & I' \arrow[r] & \operatorname{Cone}\alpha \arrow[r] & I'' ,
\end{tikzcd}$$

is commutative. It follows from 6.5.19 that χ is an isomorphism. Set $I = \operatorname{Cone}\alpha$; by 4.1.5 there is a short exact sequence $0 \to I' \to I \to I'' \to 0$, whence it follows from 5.3.20 that the complex I is semi-injective. □

Minimality and Artinian Rings

By B.26 every complex has a minimal semi-injective resolution.

8.2.15 Proposition. *Let M be an R-complex and n an integer. If $M \xrightarrow{\simeq} I$ is a minimal semi-injective resolution, then the following conditions are equivalent.*

(*i*) $\operatorname{id}_R M \leqslant n$.

(*ii*) $n \geqslant -\inf M$ *and* $I_{-(n+1)} = 0$.

(*iii*) $I_{-v} = 0$ *for all* $v > n$.

In particular, one has $\operatorname{id}_R M = -\inf I^\natural$.

Proof. If $\operatorname{id}_R M \leqslant n$ holds, then there is a semi-injective replacement I' of M with $I'_{-v} = 0$ for all $v > n$. It follows from 6.4.21 that there is a quasi-isomorphism $I \to I'$, and by B.23 it has a left inverse, whence $I_{-v} = 0$ holds for $v > n$. Thus (*i*) implies (*iii*) which, in turn, implies (*ii*). If one has $n \geqslant -\inf M$ and $I_{-(n+1)} = 0$, then the module $\mathrm{Z}_{-n}(I) = I_{-n}$ is injective, whence $\operatorname{id}_R M \leqslant n$ holds by 8.2.8. □

8.2.16 Lemma. *Let I be a semi-injective R-complex and K an R-module. If I is minimal and K is semi-simple, then the complex* $\operatorname{Hom}_R(K, I)$ *has zero differential.*

Proof. By 3.1.27 it suffices to prove the assertion for a simple R-module K. Suppose $\partial^{\operatorname{Hom}_R(K,I)}$ is non-zero. For some $n \in \mathbb{Z}$ there is then a homomorphism $\alpha\colon K \to I_n$ with $\partial^I_n\alpha \neq 0$. In particular, α is non-zero, and since K is simple it follows that α and $\partial^I_n\alpha$ are injective. By 2.5.29 the acyclic complex $\mathrm{D}^n(K)$ is now a subcomplex of I, which by B.24 contradicts the minimality of I. □

Remark. There is a result for minimal semi-projective complexes akin to 8.2.16; see E B.25.

Existence of minimal resolutions and structure theory for injective modules over Artinian rings yields a result akin to 8.1.17.

8.2.17 Theorem. *Assume that R is left Artinian with Jacobson radical $\mathfrak{J}$ and set $\boldsymbol{k} = R/\mathfrak{J}$. Let M be an R-complex and $n \in \mathbb{Z}$. The next conditions are equivalent.*

(*i*) $\operatorname{id}_R M \leqslant n$.

(*ii*) $n \geqslant -\inf M$ *and* $\operatorname{Ext}^{n+1}_R(\boldsymbol{k}, M) \neq 0$.

In particular, there is an equality,

$$\operatorname{id}_R M = -\inf \operatorname{RHom}_R(\boldsymbol{k}, M) .$$

PROOF. Choose by B.26 a semi-injective resolution $M \xrightarrow{\simeq} I$ with I minimal. As $\boldsymbol{k}$ is semi-simple, the complex $\operatorname{Hom}_R(\boldsymbol{k}, I)$ has zero differential by 8.2.16, and hence

$$\mathrm{H}_v(\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)) \;=\; \operatorname{Hom}_R(\boldsymbol{k}, I)_v \;=\; \operatorname{Hom}_R(\boldsymbol{k}, I_v)$$

holds for every $v \in \mathbb{Z}$. For every maximal left ideal $\mathfrak{m}$ in R there is a non-zero homomorphism $\boldsymbol{k} \twoheadrightarrow R/\mathfrak{m}$, so it follows from C.6 that $\operatorname{Hom}_R(\boldsymbol{k}, I_v)$ is non-zero for every $I_v \neq 0$. In particular, one has $\mathrm{H}_v(\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)) = 0$ if and only if $I_v = 0$. Now 8.2.15 yields the equivalence of (*i*) and (*ii*), and from here the displayed equality is immediate. □

THE CASE OF MODULES

8.2.18. Notice from 8.2.8 that a non-zero R-module is injective if and only if it has injective dimension 0 as an R-complex.

8.2.19 Theorem. *Let M be an R-module and $n \geqslant 0$ an integer. The following conditions are equivalent.*

(*i*) $\operatorname{id}_R M \leqslant n$.

(*ii*) *One has* $\operatorname{Ext}_R^m(N, M) = 0$ *for every R-module N and every integer $m > n$.*

(*iii*) *One has* $\operatorname{Ext}_R^{n+1}(R/\mathfrak{a}, M) = 0$ *for every left ideal $\mathfrak{a}$ in R.*

(*iv*) *There is an injective resolution* $0 \to M \to I_0 \to \cdots \to I_{-(n-1)} \to I_{-n} \to 0$.

(*v*) *In every injective resolution* $0 \to M \to I_0 \to \cdots \to I_{-(v-1)} \to I_{-v} \to \cdots$ *the module* $\operatorname{Ker}(I_{-n} \to I_{-(n+1)})$ *is injective.*

In particular, there is an equality,

$$\operatorname{id}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(R/\mathfrak{a}, M) \neq 0 \text{ for some left ideal } \mathfrak{a} \text{ in } R\}\,.$$

PROOF. By 5.3.31 every R-module M has an injective resolution

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{-(v-1)} \longrightarrow I_{-v} \longrightarrow \cdots\,.$$

In every such resolution, the injective homomorphism $M \to I_0$ yields a semi-injective resolution of M, considered as a complex; cf. 5.3.33. Thus the complex $0 \to I_0 \to \cdots \to I_{-(v-1)} \to I_{-v} \to \cdots$ is a semi-injective replacement of M. The equivalence of the five conditions now follows from the equivalence of (*i*)–(*iii*), (*v*), and (*vii*) in 8.2.8. The asserted equality holds by 8.2.8 in view of 7.3.27. □

COPRODUCTS OF INJECTIVE MODULES

8.2.20 Theorem. *The following conditions are equivalent.*

(*i*) *R is left Noetherian.*

(*ii*) *For every U-direct system* $\{\mu^{vu} \colon I^u \to I^v\}_{u \in U}$ *of injective R-modules with U filtered, the colimit* $\operatorname{colim}_{u \in U} I^u$ *is injective.*

(*iii*) *For every countable family* $\{I^u\}_{u \in \mathbb{N}}$ *of injective R-modules the coproduct* $\coprod_{u \in U} I^u$ *is injective.*

Proof. The implication (*ii*) $\Rightarrow$ (*iii*) is evident; cf. 3.3.9.

(*i*) $\Rightarrow$ (*ii*): Let $\mathfrak{a}$ be a left ideal in R. The functor $\mathrm{Ext}^1_R(R/\mathfrak{a},-)$ preserves filtered colimits by 7.3.34, as R is left Noetherian. Thus the implication follows from 8.2.19.

(*iii*) $\Rightarrow$ (*i*): Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be a chain of left ideals in R and set $\mathfrak{a} = \bigcup_{u\in\mathbb{N}} \mathfrak{a}_u$. For every $u \in \mathbb{N}$ choose by 5.3.4 an injective homomorphism $R/\mathfrak{a}_u \longrightarrow I^u$ with I^u injective and identify $R/\mathfrak{a}_u$ with its image in I^u. Set $I = \coprod_{u\in\mathbb{N}} I^u$; the assignment $a \mapsto ([a]_{\mathfrak{a}_u})_{u\in\mathbb{N}}$ defines a homomorphism $\alpha\colon \mathfrak{a} \longrightarrow I$, as one has $a \in \mathfrak{a}_u$ for $u \gg 0$. By (*iii*) the module I is injective, so α lifts to a homomorphism $R \longrightarrow I$; in particular, there is an element $i = (i_u)_{u\in\mathbb{N}}$ in I such that $\alpha(a) = ai$ holds for every $a \in R$. One has $i_u = 0$ for $u \gg 0$ and, therefore, $\mathfrak{a}_u = \mathfrak{a}$ for $u \gg 0$. $\square$

8.2.21 Corollary. *Let $\{M^u\}_{u\in U}$ be a family of R-modules. There is an inequality,*

$$\mathrm{id}_R\Big(\coprod_{u\in U} M^u\Big) \geqslant \sup_{u\in U}\{\mathrm{id}_R\, M^u\}\,,$$

and equality holds if R is left Noetherian.

Proof. For every $u \in U$ the module M^u is a direct summand of $\coprod_{u\in U} M^u$, so the inequality "$\geqslant$" holds by 8.2.12. Now assume that R is left Noetherian. The opposite inequality is trivial if the supremum $\sup_{u\in U}\{\mathrm{id}_R\, M^u\}$ is infinite, so assume that it is not and call it s. By 8.2.19 there is for every u in U an injective resolution $M^u \xrightarrow{\simeq} I^u$ with $I^u_{-v} = 0$ for $v > s$. The complex $\coprod_{u\in U} I^u$ is semi-injective by 8.2.20 and 5.3.12, and there is by 4.2.11 a quasi-isomorphism $\coprod_{u\in U} M^u \longrightarrow \coprod_{u\in U} I^u$, i.e. an injective resolution of $\coprod_{u\in U} M^u$. The asserted inequality now follows from 8.2.2. $\square$

Remark. The proof of 8.2.21 is easily adapted to apply to a family $\{M^u\}_{u\in U}$ of complexes that are uniformly bounded above, see E 8.2.9, and that is the end of its range, since a coproduct of semi-injective complexes need not be semi-injective. In fact, Iacob and Iyengar [140] show that every coproduct of semi-injective R-complexes is semi-injective if and only if R is left Noetherian and regular in the sense that every finitely generated R-module has finite projective dimension. See also 20.2.12.

Exercises

E 8.2.1 Show that an R-module M is injective if $\mathrm{Ext}^1_R(R/\mathfrak{a}, M) = 0$ holds for every left ideal $\mathfrak{a} \subseteq R$ that is essential in R.

E 8.2.2 Let $n > 0$ be an integer and M an R-module with $\mathrm{pd}_R\, M > n$. Show that the functor $\mathrm{Ext}^n_R(M,-)\colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$ is half exact but neither left nor right exact.

E 8.2.3 Let $n > 0$ be an integer and M an R-module with $\mathrm{id}_R\, M > n$. Show that the functor $\mathrm{Ext}^n_R(-,M)\colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$ is half exact but neither left nor right exact.

E 8.2.4 Let R be semi-simple. Show that $\mathrm{id}_R\, M = -\inf M$ holds for every R-complex M.

E 8.2.5 Let M be a complex in $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $u = \inf M$. Show that for every semi-injective replacement I of M one has $\mathrm{id}_R\, M = \mathrm{id}_R\, \mathrm{Z}_u(I) - u$.

E 8.2.6 Let M be an R-complex. Show that $\mathrm{id}_R\, M$ is finite if and only if $\mathrm{H}(\mathrm{RHom}_R(N,M))$ is bounded below for every R-module N.

E 8.2.7 Let M be an R-complex and assume that it is isomorphic in $\mathcal{D}(R)$ to a K-injective complex Y with $Y_v = 0$ for all $v < -n$. Show that $\mathrm{id}_R\, M$ is at most n, and conclude that one could use K-injective replacements in 8.2.2.

E 8.2.8 Let R be left hereditary. Show that $\operatorname{id}_R M \leqslant -\inf M + 1$ holds for every R-complex M.

E 8.2.9 Assume that R is left Noetherian and let $\{M^u\}_{u\in U}$ be a family of R-complexes. Show that if $\sup_{u\in U}\{\sup M^u\} < \infty$, then one has $\operatorname{id}_R(\coprod_{u\in U} M^u) = \sup_{u\in U}\{\operatorname{id}_R M^u\}$.

E 8.2.10 Assume that R is left Noetherian and that every left ideal in R has finite projective dimension. Show that for every family $\{M^u\}_{u\in U}$ of R-complexes the following equality holds, $\operatorname{id}_R(\coprod_{u\in U} M^u) = \sup_{u\in U}\{\operatorname{id}_R M^u\}$.

E 8.2.11 Assume that R is left Noetherian and let $\{\mu^{vu}\colon M^u \to M^v\}_{u\in U}$ be a U-direct system of R-modules. Show that if U is filtered, then $\operatorname{id}_R(\operatorname{colim}_{u\in U} M^u) \leqslant \sup_{u\in U}\{\operatorname{id}_R M^u\}$ holds. *Hint:* 3.3.17.

E 8.2.12 Let $0 \to M \to N \to C \to 0$ and $0 \to M \to N' \to C' \to 0$ be exact sequences of R-modules. Show that if $\operatorname{Ext}^1_R(C, N') = 0 = \operatorname{Ext}^1_R(C', N)$ holds, then there is an isomorphism $N \oplus C' \cong N' \oplus C$.

E 8.2.13 Let M be an R-complex of finite injective dimension n. Show that there exists an injective R-module I with $\operatorname{Ext}^n_R(I, M) \neq 0$.

E 8.2.14 Let M be an R-module and $P \xrightarrow{\simeq} M$ a semi-projective resolution. Show that one has $\operatorname{id}_R \mathrm{C}_n(P) \geqslant n$ for all $n \leqslant \operatorname{pd}_R M$.

E 8.2.15 Let M be an R-module and $M \xrightarrow{\simeq} I$ a semi-injective resolution. Show that one has $\operatorname{pd}_R \mathrm{Z}_{-n}(I) \geqslant n$ for all $n \leqslant \operatorname{id}_R M$.

E 8.2.16 Let R be the ring of 2×2 matrices over a field. (a) Show that R is the direct sum of the two maximal left ideals $\mathfrak{m}$ and $\mathfrak{n}$ made up of matrices with zeroes in the first and second column, respectively. (b) Show that the simple R-modules $R/\mathfrak{m}$ and $R/\mathfrak{n}$ are isomorphic.

E 8.2.17 Show that the full subcategory of R-complexes of finite injective dimension is a triangulated subcategory of $\mathcal{D}_{\sqsupset}(R)$.

E 8.2.18 Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ be an infinite increading chain of ideals in R. For each $u \in \mathbb{N}$ let $R/\mathfrak{a}_u \rightarrowtail E^u$ be the embedding of $R/\mathfrak{a}_u$ into an injective R-module. Show that the module $\coprod_{u\in\mathbb{N}} E^u$ is not injective.

8.3 Flat Dimension

Synopsis. Vanishing of Tor; flat dimension; ~ vs. injective dimension; Schanuel's lemma; flat dimension over Noetherian/perfect ring; flat resolution of module; flat dimension of module; product of flat modules.

The treatments of projective and injective dimension in the previous two sections are not just parallel, they are independent and one could change the order if so inclined. The treatment in this section of the flat dimension, however, does depend on the already established theory for injective dimension. The dependence comes through the flat–injective duality 1.3.48 as it manifests itself in 5.4.9.

8.3.1 Lemma. *Let M be an R-complex, N an R^{o}-complex, and m an integer. There are isomorphisms of $\Bbbk$-modules,*

$$\operatorname{Ext}^m_R(M, \operatorname{Hom}_{\Bbbk}(N, \mathbb{E})) \;\cong\; \operatorname{Hom}_{\Bbbk}(\operatorname{Tor}^R_m(N, M), \mathbb{E}) \;\cong\; \operatorname{Ext}^m_{R^{\mathrm{o}}}(N, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}))\,.$$

Proof. As $\mathbb{E}$ is an injective $\Bbbk$-module one has $\mathsf{RHom}_{\Bbbk}(-, \mathbb{E}) = \operatorname{Hom}_{\Bbbk}(-, \mathbb{E})$. The 1st and 4th isomorphisms below follow from the definitions, 7.3.23 and 7.4.18; the 2nd isomorphism is adjunction 7.5.30, and the 3rd one holds by 2.2.19.

$$\begin{aligned}\operatorname{Ext}_R^m(M, \operatorname{Hom}_{\Bbbk}(N, \mathbb{E})) &\cong \operatorname{H}_{-m}(\mathsf{R}\!\operatorname{Hom}_R(M, \operatorname{Hom}_{\Bbbk}(N, \mathbb{E})))\\ &\cong \operatorname{H}_{-m}(\operatorname{Hom}_{\Bbbk}(N \otimes_R^{\mathsf{L}} M, \mathbb{E}))\\ &\cong \operatorname{Hom}_{\Bbbk}(\operatorname{H}_m(N \otimes_R^{\mathsf{L}} M), \mathbb{E})\\ &\cong \operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_m^R(N, M), \mathbb{E}) \;.\end{aligned}$$

This proves the first of the asserted isomorphisms; the other is proved similarly. □

8.3.2 Lemma. *For an R-module F the following conditions are equivalent.*

(*i*) *F is flat.*
(*ii*) $\sup(N \otimes_R^{\mathsf{L}} F) \leqslant \sup N$ *holds for every R^{o}-complex N.*
(*iii*) *One has* $\operatorname{Tor}_1^R(R/\mathfrak{b}, F) = 0$ *for every finitely generated right ideal $\mathfrak{b}$ in R.*
(*iv*) *There exists a surjective homomorphism $\pi\colon L \to F$ with L flat and such that* $\operatorname{Tor}_1^R(\operatorname{Hom}_{\Bbbk}(\operatorname{Ker}\pi, \mathbb{E}), F) = 0$ *holds.*

Proof. If F is flat, then F is a semi-flat replacement of itself, and by 7.4.17 one has $N \otimes_R^{\mathsf{L}} F = N \otimes_R F$ in $\mathcal{D}(\Bbbk)$. Now 2.5.7(c) yields $\sup(N \otimes_R^{\mathsf{L}} F) \leqslant \sup N$, whence (*i*) implies (*ii*). From (*ii*) it follows that the functor $\operatorname{Tor}_1^R(-, F)$ is zero on modules, and therefore (*ii*) implies (*iii*) and, in view of 1.3.12, (*ii*) implies (*iv*) as well.

(*iii*) ⇒ (*i*): Let $\mathfrak{b}$ be a right ideal in R; by 3.3.5 it is the filtered colimit of its finitely generated right subideals. For each such subideal $\mathfrak{b}'$ the embedding $\mathfrak{b}' \rightarrowtail R$ yields an exact sequence $\operatorname{Tor}_1^R(R/\mathfrak{b}', F) \to \mathfrak{b}' \otimes_R F \to R \otimes_R F$; see 7.4.29 and 7.4.21. From (*iii*) it follows that the morphism $\mathfrak{b}' \otimes_R F \to R \otimes_R F$ is injective, so by exactness of filtered colimits 3.3.10, and the fact that $- \otimes_R F$ preserves colimits 3.2.22, the morphism $\mathfrak{b} \otimes_R F \to R \otimes_R F$ is injective. Thus, F is flat by 1.3.48.

(*iv*) ⇒ (*i*): Set $K = \operatorname{Ker}\pi$; there is an exact sequence of R^{o}-modules,

$$0 \longrightarrow \operatorname{Hom}_{\Bbbk}(F, \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(L, \mathbb{E}) \longrightarrow \operatorname{Hom}_{\Bbbk}(K, \mathbb{E}) \longrightarrow 0\,,$$

where $\operatorname{Hom}_{\Bbbk}(L, \mathbb{E})$ is injective by 1.3.48. By 8.3.1 one has

$$\operatorname{Ext}_{R^{\mathrm{o}}}^1(\operatorname{Hom}_{\Bbbk}(K, \mathbb{E}), \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \cong \operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_1^R(\operatorname{Hom}_{\Bbbk}(K, \mathbb{E}), F), \mathbb{E}) \;.$$

It now follows from (*iv*) and 8.2.1 that the R^{o}-module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is injective, whence F is flat by 1.3.48. □

Semi-Flat Replacements and Flat Dimension

Recall from 7.4.13 that a semi-flat replacement of an R-complex M is a semi-flat complex that is isomorphic to M the derived category $\mathcal{D}(R)$.

8.3.3 Definition. Let M be an R-complex. The *flat dimension* of M, written $\operatorname{fd}_R M$, is defined as

$$\operatorname{fd}_R M = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l}\text{There is a semi-flat replacement}\\ F \text{ of } M \text{ with } F_v = 0 \text{ for all } v > n\end{array}\right\},$$

with the convention $\inf \emptyset = \infty$. One says that $\operatorname{fd}_R M$ is *finite* if $\operatorname{fd}_R M < \infty$ holds.

A comment similar to the one after 8.1.2 justifies the last convention in 8.3.3.

8.3.4. Let M be an R-complex. For every semi-flat replacement F of M one has $\mathrm{H}(F) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\mathrm{fd}_R M \geqslant \sup M \quad \text{and} \quad \mathrm{fd}_R \Sigma^s M = \mathrm{fd}_R M + s \quad \text{for every integer } s\,.$$

Moreover, one has $\mathrm{fd}_R M = -\infty$ if and only if M is acyclic.

Note that the definition of flat dimension could also be written

$$\mathrm{fd}_R M = \inf\{\sup F^\natural \mid F \text{ is a semi-flat replacement of } M\}\,.$$

The projective and injective dimensions could be defined using resolutions—by 6.4.20 and 6.4.21 replacements and resolutions are two sides of a coin—but this is not the case for the flat dimension. The next example illustrates this and shows, in the process, that semi-flat complexes may not allow comparison maps. This, again, contrasts sharply with 6.4.20 and 6.4.21.

8.3.5 Example. As $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, see 1.3.43, one has $\mathrm{fd}_\mathbb{Z} \mathbb{Q} = 0$ by 8.3.4.

Let $L \xrightarrow{\simeq} \mathbb{Q}$ be a free resolution with $L_v = 0$ for $v \neq 0, 1$; cf. 5.4.15. One has $\mathrm{fd}_\mathbb{Z} L = \mathrm{fd}_\mathbb{Z} \mathbb{Q} = 0$, but there does not exist a quasi-isomorphism $F \longrightarrow L$, where F is a (semi-flat) complex with $F_v = 0$ for $v > 0$. Indeed, such a quasi-isomorphism would induce an injective homomorphism $\mathrm{Z}_0(F) \longrightarrow \mathrm{Z}_0(L)$, but that would contradict 1.3.11 as $\mathrm{Z}_0(L) = L_0$ is free while $\mathrm{Z}_0(F) \cong \mathbb{Q}$ is not.

8.3.6 Proposition. *Let M be an R-complex; there is an inequality,*

$$\mathrm{fd}_R M \leqslant \mathrm{pd}_R M\,.$$

Proof. A semi-projective replacement of M is a semi-flat replacement by 5.4.10, so the inequality holds by the definitions of the dimension, 8.1.2 and 8.3.3. □

8.3.7 Example. The inequality in 8.3.6 may be strict. Indeed, $\mathrm{fd}_\mathbb{Z} \mathbb{Q} = 0$ holds by 8.3.5, while 8.1.5 yields $\mathrm{pd}_\mathbb{Z} \mathbb{Q} = 1$. For finitely generated modules over $\mathbb{Z}$ the two dimensions agree, see 8.3.19; it also follows from that result that 8.1.10 provides an example of a module of infinite flat dimension.

8.3.8 Proposition. *Let $R \longrightarrow S$ be a ring homomorphism and M be an R-complex. There is an inequality,*

$$\mathrm{fd}_S(S \otimes^{\mathsf{L}}_R M) \leqslant \mathrm{fd}_R M\,.$$

Proof. For every semi-flat replacement F of the R-complex M, the S-complex $S \otimes_R F$ is a semi-flat replacement of $S \otimes^{\mathsf{L}}_R M$ by 5.4.18(a). As $F_v = 0$ implies $(S \otimes_R F)_v = 0$, the desired inequality follows from 8.3.3. □

The next isomorphism is colloquially referred to as "dimension shifting".

8.3.9 Lemma. *Let M be an R-complex, F a semi-flat replacement of M, and N an R°-module. For all integers $m > 0$ and $n \geqslant \sup M$ one has*

$$\mathrm{Tor}^R_{n+m}(N, M) \cong \mathrm{Tor}^R_m(N, \mathrm{C}_n(F))\,.$$

PROOF. Recall from 5.4.8 that $F_{\geqslant n}$ is a semi-flat R-complex. As one has $n \geqslant \sup M = \sup F$, the canonical morphism $\Sigma^{-n} F_{\geqslant n} \twoheadrightarrow \mathrm{C}_n(F)$ is a quasi-isomorphism; in particular, $\Sigma^{-n} F_{\geqslant n}$ is a semi-flat replacement of $\mathrm{C}_n(F)$. In the next computation, the 1st, 2nd, and 5th identities follow from the definitions of (derived) tensor products; the 4th follows from 2.4.13.

$$\begin{aligned} \mathrm{H}_{n+m}(N \otimes^{\mathsf{L}}_R M) &\cong \mathrm{H}_{n+m}(N \otimes_R F) \\ &= \mathrm{H}_{n+m}(N \otimes_R F_{\geqslant n}) \\ &= \mathrm{H}_m(\Sigma^{-n}(N \otimes_R F_{\geqslant n})) \\ &\cong \mathrm{H}_m(N \otimes_R \Sigma^{-n} F_{\geqslant n}) \\ &\cong \mathrm{H}_m(N \otimes^{\mathsf{L}}_R \mathrm{C}_n(F))\,; \end{aligned}$$

the definition of Tor, 7.4.18, now yields the asserted isomorphism. □

8.3.10 Lemma. *Let F be a semi-flat R-complex and v an integer. The complex $F_{\subseteq v}$ is semi-flat if and only if the module $\mathrm{C}_v(F)$ is flat.*

PROOF. If the complex $F_{\subseteq v}$ is semi-flat, then the module $\mathrm{C}_v(F) = (F_{\subseteq v})_v$ is flat. If the module $\mathrm{C}_v(F)$ is flat, then it is semi-flat as a complex by 5.4.11. There is an exact sequence $0 \to F_{\leqslant v-1} \to F_{\subseteq v} \to \Sigma^v \mathrm{C}_v(F) \to 0$, so by 5.4.12 the complex $F_{\subseteq v}$ is semi-flat if and only if $F_{\leqslant v-1}$ is semi-flat, which it is by 5.4.8 and 5.4.12 applied to the exact sequence $0 \to F_{\leqslant v-1} \to F \to F_{\geqslant v} \to 0$. □

8.3.11 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\mathrm{fd}_R M \leqslant n$.
(*ii*) $\sup(N \otimes^{\mathsf{L}}_R M) \leqslant n + \sup N$ *holds for every R°-complex N.*
(*iii*) $n \geqslant \sup M$ *and* $\mathrm{Tor}^R_{n+1}(R/\mathfrak{b}, M) = 0$ *holds for every finitely generated right ideal $\mathfrak{b}$ in R.*
(*iv*) $n \geqslant \sup M$ *and one has* $\mathrm{Tor}^R_1(\mathrm{Hom}_{\Bbbk}(\mathrm{C}_{n+1}(F), \mathbb{E}), \mathrm{C}_n(F)) = 0$ *for some, equivalently every, semi-flat replacement F of M.*
(*v*) $n \geqslant \sup M$ *and for some, equivalently every, semi-flat replacement F of M the module $\mathrm{C}_n(F)$ is flat.*
(*vi*) $n \geqslant \sup M$ *and for every semi-flat replacement F of M the complex $F_{\subseteq n}$ is a semi-flat replacement of M.*
(*vii*) *There exists a semi-flat replacement F of M with $F_v = 0$ for all $v > n$ and for all $v < \inf M$.*

In particular, there are equalities,

$$\begin{aligned} \mathrm{fd}_R M &= \sup\{\sup(R/\mathfrak{b} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{b} \text{ is a finitely generated right ideal in } R\} \\ &= \sup\{m \in \mathbb{Z} \mid \mathrm{Tor}^R_m(R/\mathfrak{b}, M) \neq 0 \text{ for a finitely generated right ideal } \mathfrak{b} \text{ in } R\}. \end{aligned}$$

PROOF. In addition to the seven conditions in the statement we consider

(*ii′*) $\sup(N \otimes^{\mathsf{L}}_R M) \leqslant n$ holds for every R°-module N, and prove that all eight conditions are equivalent. The implications $(ii) \Rightarrow (ii')$ and $(vii) \Rightarrow (i)$ are trivial.

$(i) \Rightarrow (ii)$: One can assume that N is in $\mathcal{D}_{\sqsubset}(R^{\mathrm{o}})$ and not acyclic; otherwise the inequality is trivial. Set $w = \sup N$; it is an integer and there is an isomorphism $N \otimes^{\mathsf{L}}_R M \simeq N_{\subseteq w} \otimes^{\mathsf{L}}_R M$; see 4.2.4. Choose a semi-flat replacement F of M with $F_v = 0$ for all $v > n$. Now one has $\sup(N \otimes^{\mathsf{L}}_R M) = \sup(N_{\subseteq w} \otimes_R F)$. For $v > n + w$ and $p \in \mathbb{Z}$, one of the inequalities $p > w$ or $v - p \geqslant v - w > n$ holds, so the module

$$(N_{\subseteq w} \otimes_R F)_v = \coprod_{p \in \mathbb{Z}} (N_{\subseteq w})_p \otimes_R F_{v-p}$$

is zero. In particular, $\mathrm{H}_v(N_{\subseteq w} \otimes_R F) = 0$ holds for $v > n+w$, so the desired inequality $\sup(N \otimes^{\mathsf{L}}_R M) \leqslant n + w$ holds.

$(ii') \Rightarrow (iii)$: Apply (ii') with $N = R$ to get $\sup M = \sup(R \otimes_R M) \leqslant n$; the second assertion is immediate from 7.4.19.

$(iii) \Rightarrow (v)$: Let F be a semi-flat replacement of M. It follows from 8.3.9 that $\operatorname{Tor}^R_1(R/\mathfrak{b}, \mathrm{C}_n(F)) = 0$ holds for every finitely generated right ideal $\mathfrak{b}$ in R, whence $\mathrm{C}_n(F)$ is flat by 8.3.2.

$(v) \Rightarrow (iv)$: This implication is immediate from 8.3.2.

$(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (vi)$: Let F be a semi-flat replacement of M. As one has $n \geqslant \sup M = \sup F$, the sequence $0 \to \mathrm{C}_{n+1}(F) \to F_n \to \mathrm{C}_n(F) \to 0$ is exact, and it follows from 8.3.2 that the module $\mathrm{C}_n(F)$ is flat. Further, the canonical morphism $\tau^F_{\subseteq n} \colon F \twoheadrightarrow F_{\subseteq n}$ is a quasi-isomorphism, so M is isomorphic to $F_{\subseteq n}$ in $\mathcal{D}(R)$, and the latter complex is semi-flat by 8.3.10.

This argument shows that the "some" part of (iv) implies (i), and thus (i)–(v) are equivalent. The argument also shows that the "every" part of (iv) implies (vi).

$(vi) \Rightarrow (vii)$: Choose by 5.1.12 a semi-free resolution L of M with $L_v = 0$ for $v < \inf M$. By 5.4.10 the complex L is semi-flat, so $L_{\subseteq n}$ is the desired replacement.

In the last assertion, the first equality follows from the equivalence of (i) and (ii') while the second holds by 7.4.19. □

The next corollary applies, in particular, to a short exact sequence of complexes, see 6.5.24.

8.3.12 Corollary. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $f' = \operatorname{fd}_R M'$, $f = \operatorname{fd}_R M$, and $f'' = \operatorname{fd}_R M''$ there are inequalities,*

$$f' \leqslant \max\{f, f'' - 1\}, \quad f \leqslant \max\{f', f''\}, \quad \textit{and} \quad f'' \leqslant \max\{f' + 1, f\}.$$

In particular, if two of the complexes M', M, and M'' have finite flat dimension, then so has the third.

Proof. For every R^{o}-module N there is a distinguished triangle,

$$N \otimes^{\mathsf{L}}_R M' \longrightarrow N \otimes^{\mathsf{L}}_R M \longrightarrow N \otimes^{\mathsf{L}}_R M'' \longrightarrow \Sigma(N \otimes^{\mathsf{L}}_R M').$$

The inequalities now follow from 8.3.11 and 6.5.20. □

Remark. Corollary 8.3.12 basically shows that the complexes of finite flat dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 8.3.11 and also 10.1.21.

8.3.13 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes; there is an equality,*

$$\operatorname{fd}_R\Big(\coprod_{u\in U} M^u\Big) = \sup_{u\in U}\{\operatorname{fd}_R M^u\}\,.$$

Proof. Let N be an R^{o}-module. The functor $N\otimes^{\mathsf{L}}_R -$ preserves coproducts by 3.1.13 and 7.2.14, so in view of 3.1.11 there are equalities,

$$\sup\big(N\otimes^{\mathsf{L}}_R \coprod_{u\in U} M^u\big) = \sup\big(\coprod_{u\in U}(N\otimes^{\mathsf{L}}_R M^u)\big) = \sup_{u\in U}\{\sup(N\otimes^{\mathsf{L}}_R M^u)\}\,.$$

The desired equality now follows from 8.3.11. □

8.3.14 Proposition. *Assume that every flat R-module has finite projective dimension.*

(a) *There is an integer $n\geqslant 0$ such that $\operatorname{pd}_R F\leqslant n$ holds for every flat R-module.*
(b) *An R-complex has finite flat dimension if and only if it has finite projective dimension.*

Proof. To prove part (a) assume towards a contradiction that for every $n\geqslant 0$ there is a flat R-module F_n with $\operatorname{pd}_R F_n\geqslant n$. By 8.3.13 and 8.1.11 the R-module $\coprod_{n\in\mathbb{N}} F_n$ is a flat of infinite projective dimension, which is a contradiction.

To prove (b) one can assume that M is not acyclic. Set $n=\operatorname{fd}_R M$ and $w=\sup M$; both are integers by assumption and 8.3.4. Let P be a semi-projective replacement of M. Note that $\Sigma^{-w}P_{\geqslant w}$ is a semi-projective replacement of the module $\mathrm{C}_w(P)$ and that $\operatorname{pd}_R M = w+\operatorname{pd}_R \mathrm{C}_w(P)$ holds by 8.1.8. The module $\mathrm{C}_n(P)$ is flat by 8.3.11, so it has finite projective dimension by assumption. It follows that also $\mathrm{C}_w(P)$ has finite projective dimension, whence $\operatorname{pd}_R M = w+\operatorname{pd}_R \mathrm{C}_w(P)$ is finite. □

Notice that 8.1.4, 8.2.4, and 8.3.8 are special cases of the next result. The assumption about non-acyclicity is only there to avoid potentially meaningless expressions like $\infty-\infty$ on the right-hand sides of the displayed inequalities.

8.3.15 Proposition. *Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N an S-complex. If none of the complexes M, X, and N are acyclic, then the following inequalities hold.*

(a) $\operatorname{id}_S \operatorname{RHom}_R(X,M) \leqslant \operatorname{fd}_{S^{\mathrm{o}}} X + \operatorname{id}_R M\,.$
(b) $\operatorname{id}_{S^{\mathrm{o}}} \operatorname{RHom}_R(M,X) \leqslant \operatorname{id}_{S^{\mathrm{o}}} X + \operatorname{pd}_R M\,.$
(c) $\operatorname{fd}_R(X\otimes^{\mathsf{L}}_S N) \leqslant \operatorname{fd}_R X + \operatorname{fd}_S N\,.$
(d) $\operatorname{pd}_R(X\otimes^{\mathsf{L}}_S N) \leqslant \operatorname{pd}_R X + \operatorname{pd}_S N\,.$

Proof. For every left ideal $\mathfrak{a}$ in S, adjunction 7.5.30, 8.2.8, and 8.3.11 yield:

$$\begin{aligned}-\inf \operatorname{RHom}_S(S/\mathfrak{a},\operatorname{RHom}_R(X,M)) &= -\inf \operatorname{RHom}_R(X\otimes^{\mathsf{L}}_S S/\mathfrak{a},M)\\ &\leqslant \operatorname{id}_R M + \sup(X\otimes^{\mathsf{L}}_S S/\mathfrak{a})\\ &\leqslant \operatorname{id}_R M + \operatorname{fd}_{S^{\mathrm{o}}} X\,.\end{aligned}$$

The inequality (a) now follows from another application of 8.2.8.

For every right ideal $\mathfrak{b}$ in S, swap 7.5.24, 8.1.8, and 8.2.8 yield:

$$\begin{aligned}
-\inf \mathsf{RHom}_{S^\mathrm{o}}(S/\mathfrak{b}, \mathsf{RHom}_R(M,X)) &= -\inf \mathsf{RHom}_R(M, \mathsf{RHom}_{S^\mathrm{o}}(S/\mathfrak{b},X))\\
&\leqslant \operatorname{pd}_R M - \inf \mathsf{RHom}_{S^\mathrm{o}}(S/\mathfrak{b},X)\\
&\leqslant \operatorname{pd}_R M + \operatorname{id}_{S^\mathrm{o}} X\,.
\end{aligned}$$

The inequality (b) now follows from another application of 8.2.8.

For every right ideal $\mathfrak{b}$ in R, associativity 7.5.17 and 8.3.11 yield:

$$\begin{aligned}
\sup\,(R/\mathfrak{b} \otimes^{\mathsf{L}}_R (X \otimes^{\mathsf{L}}_S N)) &= \sup\,((R/\mathfrak{b} \otimes^{\mathsf{L}}_R X) \otimes^{\mathsf{L}}_S N)\\
&\leqslant \operatorname{fd}_S N + \sup\,(R/\mathfrak{b} \otimes^{\mathsf{L}}_R X)\\
&\leqslant \operatorname{fd}_S N + \operatorname{fd}_R X\,.
\end{aligned}$$

The inequality (c) now follows from another application of 8.3.11.

For every R-module C, adjunction 7.5.30 and 8.1.8 yield:

$$\begin{aligned}
-\inf \mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S N, C) &= -\inf \mathsf{RHom}_S(N, \mathsf{RHom}_R(X,C))\\
&\leqslant \operatorname{pd}_S N - \inf \mathsf{RHom}_R(X,C)\\
&\leqslant \operatorname{pd}_S N + \operatorname{pd}_R X\,.
\end{aligned}$$

The inequality (d) now follows from another application of 8.1.8. □

The next result, which could be called Schanuel's lemma (for semi-flat complexes), can be seen as a refinement of the statement 8.3.11(v).

8.3.16 Lemma. *Let M be an R-complex with a semi-flat replacement F and a semi-projective replacement P. For every $v \in \mathbb{Z}$ there is an exact sequence of R-modules $0 \to \mathrm{C}_v(P) \to \mathrm{C}_v(F) \oplus L \to G \to 0$ where L is projective and G is flat.*

Proof. By 6.4.20 there is a quasi-isomorphism $\alpha\colon P \to F$. Both complexes P and F are semi-flat, see 5.4.10, so $\operatorname{Cone}\alpha$ is semi-flat and acylic by 4.1.5, 5.4.12, and 4.2.16. By 5.5.22 each module $\mathrm{Z}_n(\operatorname{Cone}\alpha)$ is flat. The morphism $\alpha_{\subseteq v}\colon P_{\subseteq v} \to F_{\subseteq v}$ is a quasi-isomorphism by 4.2.10, and acyclicity of the complex $\operatorname{Cone}(\alpha_{\subseteq v})$:

$$0 \longrightarrow \mathrm{C}_v(P) \longrightarrow \mathrm{C}_v(F) \oplus P_{v-1} \longrightarrow F_{v-1} \oplus P_{v-2} \xrightarrow{\partial^{\operatorname{Cone}\alpha}_{v-1}} F_{v-2} \oplus P_{v-3} \longrightarrow \cdots\,,$$

yields the desired sequence $0 \to \mathrm{C}_v(P) \to \mathrm{C}_v(F) \oplus P_{v-1} \to \mathrm{Z}_{v-1}(\operatorname{Cone}\alpha) \to 0$. □

The next two results exhibit the flat–injective duality.

8.3.17 Proposition. *Let M be an R-complex*; *there is an equality,*

$$\operatorname{id}_{R^\mathrm{o}} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) = \operatorname{fd}_R M\,.$$

Proof. Let K be an R^o-module; by 2.5.7(b), commutativity 7.5.10, adjunction 7.5.30, and 7.2.11 one has

$$\begin{aligned}
\sup\,(K \otimes^{\mathsf{L}}_R M) &= -\inf \mathsf{RHom}_{\Bbbk}(K \otimes^{\mathsf{L}}_R M, \mathbb{E})\\
&= -\inf \mathsf{RHom}_{R^\mathrm{o}}(K, \mathsf{RHom}_{\Bbbk}(M, \mathbb{E}))\\
&= -\inf \mathsf{RHom}_{R^\mathrm{o}}(K, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}))\,;
\end{aligned}$$

the desired equality now follows from 8.2.8 and 8.3.11. □

NOETHERIAN RINGS

8.3.18 Theorem. *Assume that R is left Noetherian and let M be an R-complex. If M belongs to $\mathcal{D}_{\sqsubset}(R)$, then there is an equality,*

$$\operatorname{fd}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E}) = \operatorname{id}_R M .$$

PROOF. The equality is trivial if M is acyclic, so assume that it is not and set $s = \sup M$. Let K be a cyclic R-module and choose by 5.1.19 a free resolution $P \xrightarrow{\simeq} K$ with P degreewise finitely generated. In the following chain of isomorphisms in $\mathcal{D}(\Bbbk)$ the 1st holds by definition, the 2nd and 4th hold by 4.2.4, and the 3rd is homomorphism evaluation 4.5.13(1,a),

$$\begin{aligned} K \otimes^{\mathsf{L}}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E}) &\simeq P \otimes_{R^\circ} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E}) \\ &\simeq P \otimes_{R^\circ} \operatorname{Hom}_{\Bbbk}(M_{\subseteq s},\mathbb{E}) \\ &\cong \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_R(P,M_{\subseteq s}),\mathbb{E}) \\ &\simeq \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_R(P,M),\mathbb{E}) . \end{aligned}$$

Now 2.5.7(b) yields

$$\sup(K \otimes^{\mathsf{L}}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})) = -\inf \operatorname{Hom}_R(P,M) = -\inf \mathsf{R}\!\operatorname{Hom}_R(K,M) ,$$

and the desired inequality follows from 8.2.8 and 8.3.11. ◻

8.3.19 Theorem. *Assume that R is left Noetherian and let M be an R-complex. If M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then there is an equality,*

$$\operatorname{fd}_R M = \operatorname{pd}_R M .$$

PROOF. The inequality $\operatorname{fd}_R M \leqslant \operatorname{pd}_R M$ holds by 8.3.6. To prove the opposite inequality, it is sufficient to consider the case where $n = \operatorname{fd}_R M$ is an integer. Choose by 5.2.16 a semi-projective resolution $P \xrightarrow{\simeq} M$ with P degreewise finitely generated. The complex P is a semi-flat replacement of M by 5.4.10, so the module $\mathrm{C}_n(P)$ is flat by 8.3.11 and hence projective by 1.3.47, as $\mathrm{C}_n(P)$ is finitely presented. Thus $P_{\subseteq n}$ is a semi-projective replacement of M, whence $n \geqslant \operatorname{pd}_R M$ holds. ◻

PERFECT RINGS

Over perfect rings, the equality in 8.3.19 extends to all complexes.

8.3.20 Theorem. *Assume that R is left perfect. Let $\mathfrak{J}$ be the Jacobson radical of R, set $\boldsymbol{k} = R/\mathfrak{J}$, and let M an R-complex. There are equalities,*

$$\operatorname{fd}_R M = \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \operatorname{pd}_R M .$$

PROOF. The inequality $\operatorname{fd}_R M \leqslant \operatorname{pd}_R M$ holds by 8.3.6. From 8.1.17 and 8.3.11 one gets $\operatorname{pd}_R M = \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \leqslant \operatorname{fd}_R M$. ◻

REMARK. A ring over which every flat module has projective dimension at most n is called *left n-perfect*. This quality of a ring is also captured by the splf invariant; see E 8.5.11–E 8.5.14 and see also the Remark after 9.3.30.

THE CASE OF MODULES

8.3.21. Notice from 8.3.11 that a non-zero R-module is flat if and only if it has flat dimension 0 as an R-complex.

Semi-flat replacements subsume the classic notion of flat resolutions of modules in a sense made clear by 8.3.25.

8.3.22 Definition. Let M be an R-module. A *flat resolution* of M consists of a complex $\cdots \to F_v \to F_{v-1} \to \cdots \to F_0 \to 0$ of flat R-modules and a surjective homomorphism $F_0 \to M$, such that the following sequence of R-modules is exact,

$$\cdots \longrightarrow F_v \longrightarrow F_{v-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,.$$

8.3.23 Theorem. *Let M be an R-module and $n \geqslant 0$ an integer. The following conditions are equivalent.*

(i) $\operatorname{fd}_R M \leqslant n$.
(ii) *One has* $\operatorname{Tor}_m^R(N,M) = 0$ *for every R°-module N and every integer $m > n$.*
(iii) *One has* $\operatorname{Tor}_{n+1}^R(R/\mathfrak{b},M) = 0$ *for every finitely generated right ideal $\mathfrak{b}$ in R.*
(iv) *There is a flat resolution* $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$.
(v) *In every flat resolution* $\cdots \to F_v \to F_{v-1} \to \cdots \to F_0 \to M \to 0$ *the module* $\operatorname{Coker}(F_{n+1} \to F_n)$ *is flat.*

In particular, there is an equality,

$$\operatorname{fd}_R M = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(R/\mathfrak{b},M) \neq 0 \text{ for a finitely generated right ideal } \mathfrak{b} \text{ in } R\}.$$

PROOF. By 5.1.16 every R-module M has a flat resolution

$$\cdots \longrightarrow F_v \longrightarrow F_{v-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,.$$

In every such resolution, the surjective homomorphism $F_0 \to M$ is a quasi-isomorphism, so the complex $\cdots \to F_v \to F_{v-1} \to \cdots \to F_0 \to 0$ is a semi-flat replacement of M, cf. 5.1.18. The equivalence of the five conditions now follows from the equivalence of (i)–(iii), (v), and (vii) in 8.3.11. The asserted equality holds by 8.3.11 in view of 7.4.21. ∎

8.3.24 Lemma. *Let F be a semi-flat R-complex. For every integer $v \leqslant \inf F$ the complex $F_{\supseteq v}$ is semi-flat.*

PROOF. If $\inf F = -\infty$ holds, then there is nothing to prove. A bounded below complex of flat modules is semi-flat by 5.4.8, so it suffices to show that $\mathrm{Z}_0(F)$ is flat under the assumption that $\inf F \geqslant 0$ holds. The R°-complex $I = \operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ is semi-injective, by 5.4.9, with $\sup I \leqslant 0$ by 2.5.7(b). The truncated complex $I_{\leqslant -1}$ is semi-injective by 5.3.12, so it follows from 5.3.20 applied to the exact sequence $0 \to I_{\leqslant -1} \to I \to I_{\geqslant 0} \to 0$ that the complex $I_{\geqslant 0}$ is semi-injective. The complex $I' = \cdots \longrightarrow I_1 \longrightarrow I_0 \xrightarrow{\pi} \mathrm{C}_0(I) \longrightarrow 0$ is acyclic, and hence so is $\operatorname{Hom}_R(I', I_{\geqslant 0})$. In particular, the morphism $I' \to I_{\geqslant 0}$ induced by the identity on $I_{\geqslant 0}$ is null-homotopic; see 2.3.3. Thus, there are homomorphisms $\sigma_0 \colon I_0 \to I_1$ and $\sigma_{-1} \colon \mathrm{C}_0(I) \to I_0$ such that $1^{I_0} = \partial_1^I \sigma_0 + \sigma_{-1}\pi$ holds. By 2.1.47 the exact sequence of R-modules,

$$0 \longrightarrow \mathrm{C}_1(I) \xrightarrow{\widetilde{\partial_1^I}} I_0 \xrightarrow{\pi} \mathrm{C}_0(I) \longrightarrow 0\,,$$

where the injective map is induced by ∂_1^I, is split. Thus, $\mathrm{C}_0(I)$ is a direct summand of I_0 and hence injective. By 2.2.19 one has $\mathrm{C}_0(I) \cong \mathrm{Hom}_{\Bbbk}(\mathrm{Z}_0(F), \mathbb{E})$, so the module $\mathrm{Z}_0(F)$ is flat by 5.4.19. □

8.3.25 Proposition. *Let M be an R-module and F a semi-flat replacement of M. The truncated complex $F_{\supseteq 0}$ is a semi-flat replacement of M, in particular, the module $\mathrm{Z}_0(F)$ is flat, and the exact sequence*

$$\cdots \longrightarrow F_v \xrightarrow{\partial_v^F} F_{v-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1^F} \mathrm{Z}_0(F) \longrightarrow M \longrightarrow 0$$

is a flat resolution of M.

Proof. The complex $F_{\supseteq 0}$ is a semi-flat by 8.3.24, and one has $F_{\supseteq 0} \simeq F \simeq M$ by 4.2.4. Finally, the composite of the canonical map $\mathrm{Z}_0(F) \twoheadrightarrow \mathrm{H}_0(F)$ and the isomorphism $\mathrm{H}_0(F) \to M$ yields a surjective homomorphism with kernel $\mathrm{B}_0(F)$. □

Products of Flat Modules

8.3.26 Proposition. *Assume that R is right Noetherian. For every family $\{F^u\}_{u\in U}$ of flat R-modules, the product $\prod_{u\in U} F^u$ is flat.*

Proof. For every right ideal $\mathfrak{b}$ in R one has by 7.4.28 and 8.3.23 equalities,

$$\mathrm{Tor}_1^R\big(R/\mathfrak{b}, \prod_{u\in U} F^u\big) \;=\; \prod_{u\in U} \mathrm{Tor}_1^R\big(R/\mathfrak{b}, F^u\big) \;=\; 0\,.$$

Now another application of 8.3.23 shows that the module $\prod_{u\in U} F^u$ is flat. □

Remark. The proof of 8.3.26 applies under the weaker assumption that R is right coherent. In fact, the property that products of flat modules are flat characterizes right coherent rings; this is a result of Chase [49].

8.3.27 Corollary. *Let $\{M^u\}_{u\in U}$ be a family of R-modules. There is an inequality,*

$$\mathrm{fd}_R\Big(\prod_{u\in U} M^u\Big) \;\geqslant\; \sup_{u\in U}\{\mathrm{fd}_R\, M^u\}\,,$$

and equality holds if R is right Noetherian.

Proof. For every $u \in U$ the module M^u is a direct summand of $\prod_{u\in U} M^u$, so the inequality "$\geqslant$" holds by 8.3.13. Assume now that R is right Noetherian. The opposite inequality is trivial if the supremum $\sup_{u\in U}\{\mathrm{fd}_R\, M^u\}$ is infinite, so assume that it is not and call it s. For every u in U there is by 8.3.23 a flat resolution $F^u \xrightarrow{\simeq} M^u$ with $F_v^u = 0$ for all $v > s$. The complex $\prod_{u\in U} F^u$ is semi-flat by 8.3.26 and 5.4.8, and by 4.2.11 there is a quasi-isomorphism $\prod_{u\in U} F^u \to \prod_{u\in U} M^u$, i.e. a flat resolution of $\prod_{u\in U} M^u$. The asserted inequality now follows from 8.3.3. □

Remark. The proof of 8.3.27 is easily adapted to apply to a family of complexes that are uniformly bounded below, see E 8.3.15, but it reaches no further, as a product of semi-flat complexes need not be semi-flat. In fact, Iacob and Iyengar [140] show that every product of semi-flat R-complexes is semi-flat if and only if R is right coherent and every finitely presented R^{o}-module has finite projective dimension. See also 20.2.14.

8.3.28 Proposition. *Let M be an R-module. There is a pure exact sequence,*

$$0 \longrightarrow M \xrightarrow{\delta_{\mathbb{E}}^{M}} \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(M,\mathbb{E}),\mathbb{E}) \longrightarrow \operatorname{Coker}\delta_{\mathbb{E}}^{M} \longrightarrow 0\,,$$

where $\delta_{\mathbb{E}}^{M}$ is biduality 1.4.2*. If R is right Noetherian and M is flat, then all the R-modules in the exact sequence displayed above are flat.*

Proof. The homomorphism $\delta_{\mathbb{E}}^{M}$ is injective by 4.5.3, and hence the sequence under consideration is exact. To prove that it is pure, it suffices by 5.5.14 and 2.1.47 to argue that $\operatorname{Hom}_{\Bbbk}(\delta_{\mathbb{E}}^{M},\mathbb{E})$ has a right inverse, however, that follows from the zigzag identites associated to the adjunction 4.5.7, which yield:

$$\operatorname{Hom}_{\Bbbk}(\delta_{\mathbb{E}}^{M},\mathbb{E})\,\delta_{\mathbb{E}}^{\operatorname{Hom}_{\Bbbk}(M,\mathbb{E})} \;=\; 1^{\operatorname{Hom}_{\Bbbk}(M,\mathbb{E})}\,.$$

If R is right Noetherian and M is flat, then the R^{o}-module $\operatorname{Hom}_{\Bbbk}(M,\mathbb{E})$ is injective by 5.4.19 and hence the R-module $\operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(M,\mathbb{E}),\mathbb{E})$ is flat by 8.3.18. It now follows from 5.5.18 that $\operatorname{Coker}\delta_{\mathbb{E}}^{M}$ is flat as well. □

Exercises

E 8.3.1 Let $n > 0$ be an integer and M an R-module with $\operatorname{fd}_R M > n$. Show that the functor $\operatorname{Tor}_n^R(-,M)\colon \mathcal{M}(R) \to \mathcal{M}(\Bbbk)$ is half exact but neither left nor right exact.

E 8.3.2 Let $R \to S$ be a ring homomorphism. Show that $\operatorname{fd}_R N \leqslant \operatorname{fd}_S N + \operatorname{fd}_R S$ holds for every S-complex N with $\mathrm{H}(N) \neq 0$.

E 8.3.3 Let $R \to S$ be a ring homomorphism. Show that $\operatorname{id}_R N \leqslant \operatorname{id}_S N + \operatorname{fd}_R S$ holds for every S-complex N with $\mathrm{H}(N) \neq 0$.

E 8.3.4 Assume that R is von Neumann regular. Show that $\operatorname{fd}_R M = \sup M$ holds for every R-complex M. *Hint:* E 3.3.8.

E 8.3.5 Show that a von Neumann regular ring is coherent.

E 8.3.6 Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $w = \sup M$. Show that for every semi-flat replacement F of M one has $\operatorname{fd}_R M = w + \operatorname{fd}_R \mathrm{C}_w(F)$.

E 8.3.7 Let M be an R-complex. Show that $\operatorname{fd}_R M$ is finite if and only if $\mathrm{H}(N \otimes_R^{\mathsf{L}} M)$ is bounded above for every R^{o}-module N.

E 8.3.8 By 1.3.12 there is a surjective homomorphism $\pi\colon L \to \mathbb{Q}$ where L is a free $\mathbb{Z}$-module. Show that it gives rise to a semi-flat $\mathbb{Z}$-complex that is acyclic and not contractible.

E 8.3.9 Let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence of R-modules. Show that the equality $\operatorname{fd}_R M = \max\{\operatorname{fd}_R M', \operatorname{fd}_R M''\}$ holds.

E 8.3.10 Assume that R is left Noetherian and let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence of R-modules. Show that the equality $\operatorname{id}_R M = \max\{\operatorname{id}_R M', \operatorname{id}_R M''\}$ holds.

E 8.3.11 Show that the full subcategory of R-complexes of finite flat dimension is a triangulated subcategory of $\mathcal{D}_{\sqsubset}(R)$.

E 8.3.12 Let M be an R-complex and assume that it is isomorphic in $\mathcal{D}(R)$ to a K-flat complex Z with $Z_v = 0$ for all $v > n$. Show that $\operatorname{fd}_R M$ is at most n, and conclude, that one could use K-flat replacements in 8.3.3.

E 8.3.13 Let $\{\mu^{vu} : M^u \to M^v\}_{u \leqslant v}$ be a U-direct system of R-complexes. Show that if U is filtered, then there is an inequality $\operatorname{fd}_R(\operatorname{colim}_{u\in U} M^u) \leqslant \sup_{u\in U}\{\operatorname{fd}_R M^u\}$, and show that equality need not hold.

E 8.3.14 Let X be a complex of R–S^{o}-bimodules. Show: (a) For every injective R-module E one has $\operatorname{id}_S \operatorname{Hom}_R(X, E) \leqslant \operatorname{fd}_{S^{\mathrm{o}}} X$ and equality holds if E is faithfully injective. (b) For every projective R-module P one has $\operatorname{id}_{S^{\mathrm{o}}} \operatorname{Hom}_R(P, X) \leqslant \operatorname{id}_{S^{\mathrm{o}}} X$ and equality holds if P is faithfully projective. (c) For every flat S-module F one has $\operatorname{fd}_R(X \otimes_R F) \leqslant \operatorname{fd}_R X$ and equality holds if F is faithfully flat. (d) For every projective S-module P one has $\operatorname{pd}_R(X \otimes_R P) \leqslant \operatorname{pd}_R X$ and equality holds if P is faithfully projective.

E 8.3.15 Assume that R is right coherent and let $\{M^u\}_{u\in U}$ be a family of R-complexes. Show that if $\inf_{u\in U}\{\inf M^u\} > -\infty$, then $\operatorname{fd}_R(\prod_{u\in U} M^u) = \sup_{u\in U}\{\operatorname{fd}_R M^u\}$ holds.

E 8.3.16 Assume that R is right coherent and let $\{M^u\}_{u\in U}$ be a family of R-complexes. Show that if every finitely generated right ideal in R has finite projective dimension, then one has $\operatorname{fd}_R(\prod_{u\in U} M^u) = \sup_{u\in U}\{\operatorname{fd}_R M^u\}$. *Hint:* 3.1.30.

E 8.3.17 Let M be an R-complex of finite flat dimension n. Show that there exists an injective R^{o}-module I such that $\operatorname{Tor}_n^R(I, M) \neq 0$.

8.4 Evaluation Morphisms in the Derived Category

Synopsis. Biduality; tensor evaluation; homomorphism evaluation.

The evaluation morphisms in the homotopy category—biduality 7.1.17, tensor evaluation 7.1.18, and homomorphism evaluation 7.1.19—induce morphisms in the derived category. As with the standard isomorphisms, the induced morphisms are by default only $\Bbbk$-linear, but under extra assumptions they are augmented to uphold additional ring actions. Our most general results about augmented evaluation morphisms are 8.4.3 (biduality), 8.4.9 (tensor evaluation), and 8.4.22 (homomorphism evaluation); commonly used special cases are recorded in 8.4.4, 8.4.10, and 8.4.23.

Situations where the evaluation morphisms are invertible are of special interest. For tensor and homomorphism evaluation, the most general results in this direction are 8.4.12 and 8.4.24, with special cases recorded in 8.4.13 and 8.4.25. Invertibility of the biduality morphism is treated in Chap. 10.

Biduality

8.4.1 Construction. Let
$$R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B$$
be a ring homomorphism such that B is flat as an R^{o}-module and as an S-module. Let X be a B-complex; by 5.4.26(b) the complex $I = \mathrm{I}_B(X)$ is semi-injective over R and over S^{o}. Recall from 7.3.6(b′) that there are functors
$$\begin{aligned}
\mathsf{R}\mathrm{Hom}_R(-, X) &: \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} \longrightarrow \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \quad \text{induced by} \quad \operatorname{Hom}_R(-, I)\,,\\
\mathsf{R}\mathrm{Hom}_{S^{\mathrm{o}}}(-, X) &: \mathcal{D}(Q\text{–}S^{\mathrm{o}})^{\mathrm{op}} \longrightarrow \mathcal{D}(R\text{–}Q^{\mathrm{o}}) \quad \text{induced by} \quad \operatorname{Hom}_{S^{\mathrm{o}}}(-, I)\,.
\end{aligned}$$
It follows from 7.2.4, cf. the argument in 7.5.16, that the composite functor

$$\mathsf{RHom}_{S^\mathrm{o}}(\mathsf{RHom}_R(-,X),X) : \mathcal{D}(R\text{–}Q^\mathrm{o}) \longrightarrow \mathcal{D}(R\text{–}Q^\mathrm{o})$$
$$\text{is induced by } \operatorname{Hom}_{S^\mathrm{o}}(\operatorname{Hom}_R(-,I),I)\ .$$

Now, consider the natural transformation

$$\delta_I : \operatorname{Id}_{\mathcal{K}(R\text{–}Q^\mathrm{o})} \longrightarrow \operatorname{Hom}_{S^\mathrm{o}}(\operatorname{Hom}_R(-,I),I)$$

of endofunctors on $\mathcal{K}(R\text{–}Q^\mathrm{o})$ induced by biduality 7.1.17. There is a natural transformation of endofunctors on $\mathcal{D}(R\text{–}Q^\mathrm{o})$ induced by 7.2.5,

$$\boldsymbol{\delta}_X = \tilde{\tilde{\delta}}_X : \operatorname{Id}_{\mathcal{D}(R\text{–}Q^\mathrm{o})} \longrightarrow \mathsf{RHom}_{S^\mathrm{o}}(\mathsf{RHom}_R(-,X),X)\ . \tag{8.4.1.1}$$

8.4.2 Definition. The natural transformation (8.4.1.1) is called *biduality*.

8.4.3 Proposition. *Let $R \otimes_\Bbbk S^\mathrm{o} \to B$ be a ring homomorphism such that B is flat as an R^o-module and as an S-module. Let X be a B-complex; biduality $\boldsymbol{\delta}_X$ is a natural transformation of endofunctors on $\mathcal{D}(R\text{–}Q^\mathrm{o})$. That is, for M in $\mathcal{D}(R\text{–}Q^\mathrm{o})$ there is a morphism in $\mathcal{D}(R\text{–}Q^\mathrm{o})$,*

$$\boldsymbol{\delta}_X^M : M \longrightarrow \mathsf{RHom}_{S^\mathrm{o}}(\mathsf{RHom}_R(M,X),X)\ ,$$

which is natural in M; it is induced by the morphism $\delta^M_{\mathrm{I}(X)}$ in $\mathcal{K}(R\text{–}Q^\mathrm{o})$. As a natural transformation of functors, $\boldsymbol{\delta}_X$ is triangulated.

Proof. The natural transformation $\boldsymbol{\delta}_X = \tilde{\tilde{\delta}}_X$ from 8.4.1 is triangulated by 7.2.5 as δ_X is triangulated by 7.1.17. □

8.4.4 Corollary. *Assume that R and S are flat as $\Bbbk$-modules. For complexes $M \in \mathcal{D}(R\text{–}Q^\mathrm{o})$ and $X \in \mathcal{D}(R\text{–}S^\mathrm{o})$ biduality,*

$$\boldsymbol{\delta}_X^M : M \longrightarrow \mathsf{RHom}_{S^\mathrm{o}}(\mathsf{RHom}_R(M,X),X)\ ,$$

is a morphism in $\mathcal{D}(R\text{–}Q^\mathrm{o})$.

Proof. Apply 8.4.3 with $B = R \otimes_\Bbbk S^\mathrm{o}$ and invoke 7.3.11(b). □

For situations where biduality is an isomorphism, see 10.1.19 and 10.2.1.

Tensor Evaluation

8.4.5 Construction. Recall from 7.3.8 and 7.4.9 that there are functors,

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^\mathrm{o}) \longrightarrow \mathcal{D}(S^\mathrm{o}) \quad \text{induced by} \quad \operatorname{Hom}_R(\mathrm{P}_R(-),-)\ ,$$
$$- \otimes^{\mathsf{L}}_S - : \mathcal{D}(S^\mathrm{o}) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad - \otimes_S \mathrm{P}_S(-)\ .$$

It follows from 7.2.4, cf. the argument in 7.5.16, that the composite functor

$$\begin{aligned}\mathsf{RHom}_R(-,-) \otimes^{\mathsf{L}}_S - : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^\mathrm{o}) \times \mathcal{D}(S) &\longrightarrow \mathcal{D}(\Bbbk)\\ \text{is induced by } \operatorname{Hom}_R(\mathrm{P}_R(-),-) \otimes_S \mathrm{P}_S(-)\ .&\end{aligned} \tag{8.4.5.1}$$

Similarly, the composite

$$\mathsf{RHom}_R(-,-\otimes^{\mathsf{L}}_S -) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(\Bbbk) \quad \text{is induced by } \operatorname{Hom}_R(\mathrm{P}_R(-), -\otimes_S \mathrm{P}_S(-)) . \tag{8.4.5.2}$$

Now, consider the natural transformation,

$$\theta^{\mathrm{P}(-)-\mathrm{P}(-)} : \operatorname{Hom}_R(\mathrm{P}_R(-),-) \otimes_S \mathrm{P}_S(-) \longrightarrow \operatorname{Hom}_R(\mathrm{P}_R(-), -\otimes_S \mathrm{P}_S(-)) ,$$

of functors from $\mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R\text{–}S^{\mathrm{o}}) \times \mathcal{K}(S)$ to $\mathcal{K}(\Bbbk)$ induced by tensor evaluation 7.1.18. There is a natural transformation of functors from $\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S)$ to $\mathcal{D}(\Bbbk)$ induced by 7.2.5,

$$\boldsymbol{\theta} = (\theta^{\mathrm{P}(-)-\mathrm{P}(-)})^{\sim} : \mathsf{RHom}_R(-,-) \otimes^{\mathsf{L}}_S - \longrightarrow \mathsf{RHom}_R(-,-\otimes^{\mathsf{L}}_S -) . \tag{8.4.5.3}$$

8.4.6 Definition. The natural transformation (8.4.5.3) is called *tensor evaluation*.

Tensor evaluation, $\boldsymbol{\theta}$, is by construction a natural transformation of functors from $\mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \times \mathcal{D}(S)$ to $\mathcal{D}(\Bbbk)$. In some cases, $\boldsymbol{\theta}$ can be augmented to a transformation of functors on derived categories of complexes with additional ring actions.

8.4.7 Setup. Consider ring homomorphisms,

$$R \otimes_{\Bbbk} Q^{\mathrm{o}} \longrightarrow A , \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B , \qquad \text{and} \qquad S \otimes_{\Bbbk} T^{\mathrm{o}} \longrightarrow C .$$

Let E_{I} and E_{II} be functors from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\varphi_{\mathrm{I}} : \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes^{\mathsf{L}}_S \operatorname{res}^C_S \longrightarrow \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \quad \text{and}$$
$$\varphi_{\mathrm{II}} : \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S) \longrightarrow \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} ,$$

where the functors on the left are those from (8.4.5.1) and (8.4.5.2) precomposed with $\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_S$.

8.4.8 Definition. Adopt the setup 8.4.7. A natural transformation $\boldsymbol{\theta}_0 : \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is called an *augmentation* of tensor evaluation if the next diagram is commutative,

$$\begin{tikzcd}
\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{I}} \arrow[r, "\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \boldsymbol{\theta}_0"] & \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{E}_{\mathrm{II}} \\
\mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes^{\mathsf{L}}_S \operatorname{res}^C_S \arrow[u, "\varphi_{\mathrm{I}}", "\simeq"'] \arrow[r, "\boldsymbol{\theta}(\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_S)"] & \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S) \arrow[u, "\simeq", "\varphi_{\mathrm{II}}"'] ;
\end{tikzcd}$$

here $\boldsymbol{\theta}$ on the lower horizontal arrow is (8.4.5.3). In this case, $\boldsymbol{\theta}_0 : \mathrm{E}_{\mathrm{I}} \to \mathrm{E}_{\mathrm{II}}$ is written $\boldsymbol{\theta} : \mathsf{RHom}_R(-,-) \otimes^{\mathsf{L}}_S - \to \mathsf{RHom}_R(-,-\otimes^{\mathsf{L}}_S -)$, and tensor evaluation is said to be *augmented* to a natural transformation of functors $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C) \to \mathcal{D}(Q\text{–}T^{\mathrm{o}})$.

8.4.9 Proposition. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $S \otimes_{\Bbbk} T^{\mathrm{o}} \to C$ be ring homomorphisms such that A is projective as an R-module and C is flat as an S-module. Tensor evaluation is augmented to a natural transformation of functors*

$$\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C) \longrightarrow \mathcal{D}(Q\text{–}T^{\mathrm{o}}) \,.$$

That is, for M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$ there is a morphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$,

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N) \,,$$

which is natural in M, X, and N; it is induced by the map $\theta^{\mathrm{P}_A(M) X \mathrm{P}_C(N)}$ in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$. As a natural transformation, this augmented $\boldsymbol{\theta}$ is triangulated in each variable.

Proof. Consider the functors

$$\mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes^{\mathsf{L}}_S \operatorname{res}^C_S \quad \text{and} \quad \mathsf{RHom}_R(\operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes^{\mathsf{L}}_S \operatorname{res}^C_S)$$

from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(\Bbbk)$ that appear in the bottom row of the diagram in 8.4.8. By definition, see 8.4.5, they are per 7.2.2 induced by the functors

$$\begin{aligned} \mathrm{C}_{\mathrm{I}} &= \operatorname{Hom}_R(\mathrm{P}_R \operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes_S \mathrm{P}_S \operatorname{res}^C_S \quad \text{and} \\ \mathrm{C}_{\mathrm{II}} &= \operatorname{Hom}_R(\mathrm{P}_R \operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes_S \mathrm{P}_S \operatorname{res}^C_S) \end{aligned}$$

from $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \times \mathcal{K}(C)$ to $\mathcal{K}(\Bbbk)$. Similarly, see (8.4.5.3), the natural transformation $\boldsymbol{\theta}(\operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_S)$ in the diagram in 8.4.8 is per 7.2.5 induced by $\theta(\mathrm{P}_R \operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \mathrm{P}_S \operatorname{res}^C_S) \colon \mathrm{C}_{\mathrm{I}} \to \mathrm{C}_{\mathrm{II}}$ where θ is tensor evaluation 7.1.18. Now, consider the functors from $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \times \mathcal{K}(C)$ to $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ given by

$$\begin{aligned} \mathrm{D}_{\mathrm{I}} &= \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes_S \operatorname{res}^C_{S\otimes T^{\mathrm{o}}} \mathrm{P}_C \quad \text{and} \\ \mathrm{D}_{\mathrm{II}} &= \operatorname{Hom}_R(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}} \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes_S \operatorname{res}^C_{S\otimes T^{\mathrm{o}}} \mathrm{P}_C) \end{aligned}$$

and let $\vartheta \colon \mathrm{D}_{\mathrm{I}} \to \mathrm{D}_{\mathrm{II}}$ be the natural transformation

$$\vartheta = \theta(\operatorname{res}^A_{R\otimes Q^{\mathrm{o}}} \mathrm{P}_A \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_{S\otimes T^{\mathrm{o}}} \mathrm{P}_C)$$

where θ is tensor evaluation 7.1.18. There is a commutative diagram,

$$\begin{tikzcd}
\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{D}_{\mathrm{I}} \arrow[r, "\operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \vartheta"] \arrow[d, equal] & \operatorname{res}^{Q\otimes T^{\mathrm{o}}}_{\Bbbk} \mathrm{D}_{\mathrm{II}} \arrow[d, equal] \\
\operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes_S \operatorname{res}^C_S \mathrm{P}_C \arrow[r, "\theta(\operatorname{res}^A_R \mathrm{P}_A \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^C_S \mathrm{P}_C)"] & \operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes_S \operatorname{res}^C_S \mathrm{P}_C) \\
\operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes_S \mathrm{P}_S \operatorname{res}^C_S \arrow[r, "\theta(\operatorname{res}^A_R \mathrm{P}_A \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \mathrm{P}_S \operatorname{res}^C_S)"] \arrow[u, "\operatorname{Hom}(1,1)\otimes \varrho^C_S", "\simeq"'] \arrow[d, "\operatorname{Hom}(\varrho^A_R,1)\otimes 1"', "\cong"] & \operatorname{Hom}_R(\operatorname{res}^A_R \mathrm{P}_A, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes_S \mathrm{P}_S \operatorname{res}^C_S) \arrow[u, "\simeq", "\operatorname{Hom}(1,1\otimes \varrho^C_S)"'] \arrow[d, "\cong"', "\operatorname{Hom}(\varrho^A_R,1\otimes 1)"] \\
\operatorname{Hom}_R(\mathrm{P}_R \operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}}) \otimes_S \mathrm{P}_S \operatorname{res}^C_S \arrow[r, "\theta(\mathrm{P}_R \operatorname{res}^A_R \times \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \times \mathrm{P}_S \operatorname{res}^C_S)"] \arrow[d, equal] & \operatorname{Hom}_R(\mathrm{P}_R \operatorname{res}^A_R, \operatorname{res}^B_{R\otimes S^{\mathrm{o}}} \otimes_S \mathrm{P}_S \operatorname{res}^C_S) \arrow[d, equal] \\
\mathrm{C}_{\mathrm{I}} & \mathrm{C}_{\mathrm{II}} \,,
\end{tikzcd}$$

where ϱ^C_S and ϱ^A_R are the natural transformations from 6.3.21. As A is projective as an R-module, 6.3.21 yields that ϱ^A_R is a natural isomorphism; whence so are the natural transformations $\operatorname{Hom}(\varrho_R^A, 1) \otimes 1$ and $\operatorname{Hom}(\varrho_R^A, 1 \otimes 1)$ in the diagram above. For every C-complex N, the S-complex $\mathrm{P}_S \operatorname{res}_S^C(N)$ is semi-flat by 5.4.10, and by 5.4.18(b) so is $\operatorname{res}_S^C \mathrm{P}_C(N)$ since C is flat as an S-module. Thus $(\varrho_S^C)^N$ is a quasi-isomorphism between semi-flat S-complexes. It now follows from 5.4.16 that the natural transformation $\sigma = \operatorname{Hom}(1,1) \otimes \varrho_S^C$ has the property that σ^{MXN} is a quasi-isomorphism for every (M, X, N) in $\mathcal{K}(A)^{\mathrm{op}} \times \mathcal{K}(B) \times \mathcal{K}(C)$. For every A-complex M, the R-complex $\operatorname{res}_R^A \mathrm{P}_A(M)$ is semi-projective by 5.2.23(b), as A is projective as an R-module. From this fact and another application of 5.4.16 it follows that the natural transformation $\tau = \operatorname{Hom}(1, 1 \otimes \varrho_S^C)$ has the property that τ^{MXN} is a quasi-isomorphism for every object (M, X, N).

As the functors $\mathrm{C_I}$ and $\mathrm{C_{II}}$ preserve quasi-isomorphisms, cf. 8.4.5, it follows from the diagram above that $\operatorname{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}} \mathrm{D_I}$ and $\operatorname{res}_{\Bbbk}^{Q\otimes T^{\mathrm{o}}} \mathrm{D_{II}}$ preserve quasi-isomorphisms, and hence so do $\mathrm{D_I}$ and $\mathrm{D_{II}}$. The functors from $\mathcal{D}(A)^{\mathrm{op}} \times \mathcal{D}(B) \times \mathcal{D}(C)$ to $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ induced by $\mathrm{D_I}$ and $\mathrm{D_{II}}$ via 7.2.2 are denoted $\mathrm{E_I}$ and $\mathrm{E_{II}}$. The natural transformation $\mathrm{E_I} \to \mathrm{E_{II}}$ induced by $\vartheta\colon \mathrm{D_I} \to \mathrm{D_{II}}$ via 7.2.5 is denoted $\boldsymbol{\theta}_0$. The commutative diagram above now induces natural isomorphisms φ_{I} and φ_{II} as in 8.4.7, cf. 6.4.18, and a commutative diagram as in 8.4.8. ◻

Notice that the assumptions in the next corollary are satisfied if $\Bbbk$ is a field.

8.4.10 Corollary. *Assume that Q is projective and T is flat as $\Bbbk$-modules. For complexes M in $\mathcal{D}(R\text{–}Q^{\mathrm{o}})$, X in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{D}(S\text{–}T^{\mathrm{o}})$ tensor evaluation,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M, X) \otimes_S^{\mathsf{L}} N \longrightarrow \mathsf{RHom}_R(M, X \otimes_S^{\mathsf{L}} N)\,,$$

is a morphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$.

PROOF. Apply 8.4.9 with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = S \otimes_{\Bbbk} T^{\mathrm{o}}$ and invoke 7.3.11(a,b). ◻

8.4.11 Example. Adopt the setup from 8.4.9. Let F be a semi-flat replacement of the C-complex N. By 6.4.20 there is a quasi-isomorphism $\pi\colon \mathrm{P}_C(N) \to F$ in $\mathcal{K}(C)$, which yields a commutative diagram in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$,

$$\begin{tikzcd}
\operatorname{Hom}_R(\mathrm{P}_A(M), X) \otimes_S \mathrm{P}_C(N) \arrow[r, "\theta^{\mathrm{P}(M)X\mathrm{P}(N)}"] \arrow[d, "\operatorname{Hom}(\mathrm{P}(M),X)\otimes\pi"', "\simeq"] & \operatorname{Hom}_R(\mathrm{P}_A(M), X \otimes_S \mathrm{P}_C(N)) \arrow[d, "\simeq"', "\operatorname{Hom}(\mathrm{P}(M),X\otimes\pi)"] \\
\operatorname{Hom}_R(\mathrm{P}_A(M), X) \otimes_S F \arrow[r, "\theta^{\mathrm{P}(M)XF}"] & \operatorname{Hom}_R(\mathrm{P}_A(M), X \otimes_S F)\,.
\end{tikzcd}$$

As C is flat as an S-module, the complexes $\mathrm{P}_C(N)$ and F are semi-flat over S by 5.4.18(b), and hence it follows from 5.4.16 that the vertical maps in the diagram above are quasi-isomorphisms. Thus $\theta^{\mathrm{P}(M)XF}$ induces a morphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ which is isomorphic to the augmented tensor evaluation $\boldsymbol{\theta}^{MXN}$ from 8.4.9.

8.4.12 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $S \otimes_{\Bbbk} T^{\mathrm{o}} \to C$ be ring homomorphisms such that A is projective as an R-module and C is flat as an S-module. For M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$, consider the morphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$ from* 8.4.9,

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N) .$$

If A is left Noetherian and finitely generated as an R-module, then $\boldsymbol{\theta}^{MXN}$ is an isomorphism if one of the following conditions is satisfied.

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(A)$ and $\operatorname{pd}_A M$ is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(A)$, X is in $\mathcal{D}_{\sqsubset}(B)$, and $\operatorname{fd}_C N$ is finite.*

If C is left Noetherian and finitely presented as an S-module, then $\boldsymbol{\theta}^{MXN}$ is an isomorphism if one of the following conditions is satisfied.

(c) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(C)$ and $\operatorname{pd}_C N$ is finite.*
(d) *$\operatorname{pd}_A M$ is finite, X is in $\mathcal{D}_{\sqsupset}(B)$, and N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(C)$.*

Proof. It must be proved that the morphism $\theta^{\mathrm{P}_A(M) X \mathrm{P}_C(N)}$ in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$ is a quasi-isomorphism under any one of the conditions (a)–(d).

First assume that A is left Noetherian and finitely generated as an R-module.

If (a) holds, then one can assume that $\mathrm{P}_A(M)$ is a bounded complex of finitely generated projective A-modules; see 8.1.14. Since A is a finitely generated and projective R-module, $\mathrm{P}_A(M)$ is a bounded complex of finitely generated projective R-modules, see 1.3.15 and 5.2.26(b). So the desired conclusion follows from 4.5.10(d).

If (b) holds, then one can assume that $\mathrm{P}_A(M)$ is a bounded below complex of finitely generated projective A-modules; see 5.2.16. Thus, as above, $\mathrm{P}_A(M)$ is a bounded below complex of finitely generated projective R-modules. Moreover, by 4.2.4 one can assume that X is bounded above, and by 8.3.11 there exists a bounded above semi-flat replacement F of the C-complex N. It follows from 5.4.24(b) that F is a complex of flat S-modules. By 4.5.10(1,c) the morphism $\theta^{\mathrm{P}_A(M) X F}$ is an isomorphism in $\mathcal{K}(Q\text{–}T^{\mathrm{o}})$, so 8.4.11 yields the desired conclusion.

Next assume that C is left Noetherian and finitely presented as an S-module. Note that since C is also assumed to be flat as an S-module, it follows from 1.3.47 that C is finitely generated and projective as an S-module.

If (c) holds, then one can by 8.1.14 assume that $\mathrm{P}_C(N)$ is a bounded complex of finitely generated projective C-modules; hence it is also a bounded complex of finitely generated projective S-modules. Now 4.5.10(e) yields the desired conclusion.

Finally, if (d) holds, then one can by 8.1.8 assume that $\mathrm{P}_A(M)$ is a bounded above complex of projective A-modules; hence it is also a bounded above complex of projective R-modules. By 4.2.4 one can assume that X is bounded below. Per 5.2.16 one can assume that $\mathrm{P}_C(N)$ is a bounded below complex of finitely generated projective C-modules; hence it is also a bounded below complex of finitely generated projective S-modules. Now apply 4.5.10(2,a) to get the desired conclusion. $\square$

Notice that under the assumptions in 8.4.10 the isomorphism in the next corollary is an isomorphism in $\mathcal{D}(Q\text{–}T^{\mathrm{o}})$.

8.4.13 Corollary. *Let $M \in \mathcal{D}(R)$, $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$, and $N \in \mathcal{D}(S)$. Consider the morphism in $\mathcal{D}(\Bbbk)$, from* 8.4.6*,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N) .$$

If R is left Noetherian, then $\boldsymbol{\theta}^{MXN}$ is an isomorphism if one of the following conditions is satisfied.

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$ and $\operatorname{pd}_R M$ is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, X is in $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$, and $\operatorname{fd}_S N$ is finite.*

If S is left Noetherian, then $\boldsymbol{\theta}^{MXN}$ is an isomorphism if one of the following conditions is satisfied.

(c) *N is in $\mathcal{D}^{\mathrm{f}}_{\Box}(S)$ and $\operatorname{pd}_S N$ is finite.*
(d) *$\operatorname{pd}_R M$ is finite, X is in $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$, and N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$.*

Proof. Apply 8.4.12 with $Q = \Bbbk = T$, $A = R$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = S$. $\square$

Before we move on to treat homomorphism evaluation, we record in 8.4.14 and 8.4.16 two canonical applications of tensor evaluation 8.4.13.

8.4.14 Proposition. *Let $M \in \mathcal{D}(R)$ and $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$ be complexes that are not acyclic. If $\operatorname{pd}_R M$ is finite and condition* (a) *or* (b) *below is satisfied, then the following inequality holds*:

$$\operatorname{fd}_{S^{\mathrm{o}}} \mathsf{RHom}_R(M,X) \leqslant \operatorname{fd}_{S^{\mathrm{o}}} X - \inf M\ .$$

(a) *R is left Noetherian and M belongs to $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$.*
(b) *S is left Noetherian and X belongs to $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$.*

Proof. Let $\mathfrak{a}$ be a left ideal in S. If condition (a) or (b) is satisfied, then tensor evaluation 8.4.13(a) or 8.4.13(d) yields

$$\mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S S/\mathfrak{a} \simeq \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S S/\mathfrak{a})\ .$$

By 7.6.7 and 8.3.11 the right-hand complex has supremum at most $\operatorname{fd}_{S^{\mathrm{o}}} X - \inf M$. The inequality now follows from 8.3.11. $\square$

Remark. An example by Christensen, Ferraro, and Thompson [58] shows that the boundedness condition in 8.4.14(b) is needed.

The next corollary can also be obtained from 1.3.17, 3.1.27, and 8.3.26.

8.4.15 Corollary. *Assume that S is left Noetherian. Let P be an R-module and F an R–S^{o}-bimodule. If P is projective and F is flat as an S^{o}-module, then the S^{o}-module $\operatorname{Hom}_R(P,F)$ is flat.*

Proof. As P is a projective R-module, one has $\operatorname{Hom}_R(P,F) = \mathsf{RHom}_R(P,F)$ by 7.3.21. Now apply 8.4.14. $\square$

Remark. Corollary 8.4.15 holds under the weaker assumption that S is left coherent, cf. E 3.3.4.

It is shown in 17.5.14 that the boundedness condition in part (a) below is needed.

8.4.16 Proposition. *Let $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$ and $N \in \mathcal{D}(S)$ be complexes that are not acyclic. If $\operatorname{fd}_S N$ is finite and condition* (a) *or* (b) *below is satisfied, then one has*

$$\operatorname{id}_R(X \otimes^{\mathsf{L}}_S N) \leqslant \operatorname{id}_R X - \inf N\ .$$

Moreover, if F is a flat S-module and (a) *is satisfied, then one has*

$$\mathrm{id}_R(X \otimes_S F) \leqslant \mathrm{id}_R X\,,$$

and equality holds if F is faithfully flat.

(a) *R is left Noetherian and X belongs to $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$.*
(b) *S is left Noetherian and N belongs to $\mathcal{D}^{\mathrm{f}}_{\Box}(S)$.*

Proof. Let $\mathfrak{a}$ be a left ideal in R. If condition (a) or (b) is satisfied, then tensor evaluation 8.4.13(b) or 8.4.13(c) yields

$$(\diamond) \qquad \mathsf{RHom}_R(R/\mathfrak{a}, X \otimes^{\mathsf{L}}_S N) \simeq \mathsf{RHom}_R(R/\mathfrak{a}, X) \otimes^{\mathsf{L}}_S N\,.$$

By 7.6.8 and 8.2.8 the right-hand complex has minus infimum at most $\mathrm{id}_R X - \inf N$. The inequality now follows from 8.2.8. If $N = F$ is a flat S-module, then by 7.4.17 the asserted inequality is a special case of the one just proved. Finally, if F is faithfully flat, then the complexes $\mathsf{RHom}_R(R/\mathfrak{a}, X \otimes_S F)$ and $\mathsf{RHom}_R(R/\mathfrak{a}, X)$ have the same infima by $(\diamond)$ and 2.5.7(c), so the asserted equality follows from 8.2.8. $\square$

The first part of the next result can also be deduced from 5.5.7, 3.2.23, and 8.2.20.

8.4.17 Corollary. *Assume that R is left Noetherian. Let I be an R–S^{o}-bimodule and F an S-module. If I is injective as an R-module and F is flat, then the R-module $I \otimes_S F$ is injective*; *the converse holds if F is faithfully flat.*

Proof. This follows from 8.4.16 applied with $X = I$. $\square$

Homomorphism Evaluation

8.4.18 Construction. Recall from 7.3.9 and 7.4.8 that there are functors,

$$\begin{aligned} \mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R) &\longrightarrow \mathcal{D}(S) \quad \text{induced by} \quad \mathrm{Hom}_R(-, \mathrm{I}_R(-)) \\ \text{and} \qquad - \otimes^{\mathsf{L}}_S - : \mathcal{D}(S^{\mathrm{o}}) \times \mathcal{D}(S) &\longrightarrow \mathcal{D}(\Bbbk) \quad \text{induced by} \quad \mathrm{P}_{S^{\mathrm{o}}}(-) \otimes_S -\,. \end{aligned}$$

It follows from 7.2.4, cf. the argument in 7.5.16, that the composite functor

$$(8.4.18.1) \qquad \begin{aligned} - \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(-,-) : \mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(S^{\mathrm{o}}) &\longrightarrow \mathcal{D}(\Bbbk) \\ \text{is induced by } \mathrm{P}_{S^{\mathrm{o}}}(-) \otimes_S \mathrm{Hom}_R(-, \mathrm{I}_R(-))\,. \end{aligned}$$

Applied to an object (M, X, N) this functor yields $N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X, M)$. Similarly, the composite

$$(8.4.18.2) \qquad \begin{aligned} \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(-,-),-) : \mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(S^{\mathrm{o}}) &\longrightarrow \mathcal{D}(\Bbbk) \\ \text{is induced by } \mathrm{Hom}_R(\mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),-), \mathrm{I}_R(-))\,. \end{aligned}$$

Applied to an object (M, X, N) this functor yields $\mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N, X), M)$. Now, consider the natural transformation,

$$\eta^{\mathrm{I}(-)-\mathrm{P}(-)} : \mathrm{P}_{S^{\mathrm{o}}}(-) \otimes_S \mathrm{Hom}_R(-, \mathrm{I}_R(-)) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{S^{\mathrm{o}}}(\mathrm{P}_{S^{\mathrm{o}}}(-),-), \mathrm{I}_R(-))\,,$$

of functors from $\mathcal{K}(R) \times \mathcal{K}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{K}(S^{\mathrm{o}})$ to $\mathcal{K}(\Bbbk)$ induced by homomorphism evaluation 7.1.19. There is a natural transformation of functors from the category $\mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(S^{\mathrm{o}})$ to $\mathcal{D}(\Bbbk)$ induced by 7.2.5,

$$(8.4.18.3) \qquad - \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(-,-) \xrightarrow{\boldsymbol{\eta} = (\eta^{\mathrm{I}(-)-\mathrm{P}(-)})^{\sim}} \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(-,-),-)\,.$$

8.4.19 Definition. The natural transformation (8.4.18.3) is called *homomorphism evaluation*.

Homomorphism evaluation, $\boldsymbol{\eta}$, is by construction a natural transformation of functors from $\mathcal{D}(R) \times \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(S^{\mathrm{o}})$ to $\mathcal{D}(\Bbbk)$. In some cases, $\boldsymbol{\eta}$ can be augmented to a transformation of functors on derived categories of complexes with additional ring actions.

8.4.20 Setup. Consider ring homomorphisms,

$$R \otimes_{\Bbbk} Q^{\mathrm{o}} \longrightarrow A\,, \qquad R \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow B\,, \qquad \text{and} \qquad T \otimes_{\Bbbk} S^{\mathrm{o}} \longrightarrow C\,.$$

Let $\mathrm{E_I}$ and $\mathrm{E_{II}}$ be functors from $\mathcal{D}(A) \times \mathcal{D}(B)^{\mathrm{op}} \times \mathcal{D}(C)$ to $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$ and assume that there are natural isomorphisms

$$\varphi_{\mathrm{I}} : \operatorname{res}^{C}_{S^{\mathrm{o}}} \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(\operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}}, \operatorname{res}^{A}_{R}) \longrightarrow \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E_I}$$

and

$$\varphi_{\mathrm{II}} : \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^{C}_{S^{\mathrm{o}}}, \operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}}), \operatorname{res}^{A}_{R}) \longrightarrow \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E_{II}}\,,$$

where the functors on the left are those from (8.4.18.1) and (8.4.18.2) precomposed with $\operatorname{res}^{A}_{R} \times \operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^{C}_{S^{\mathrm{o}}}$.

8.4.21 Definition. Adopt the setup 8.4.20. A natural transformation $\boldsymbol{\eta}_0 : \mathrm{E_I} \to \mathrm{E_{II}}$ is called an *augmentation* of homomorphism evaluation if the diagram

$$\begin{tikzcd}
\operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E_I} \arrow[r, "\operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \boldsymbol{\eta}_0"] & \operatorname{res}^{T\otimes Q^{\mathrm{o}}}_{\Bbbk} \mathrm{E_{II}} \\
\operatorname{res}^{C}_{S^{\mathrm{o}}} \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(\operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}}, \operatorname{res}^{A}_{R}) \arrow[u, "\varphi_{\mathrm{I}}", "\simeq"'] \arrow[r, "\boldsymbol{\eta}(\operatorname{res}^{A}_{R} \times \operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}} \times \operatorname{res}^{C}_{S^{\mathrm{o}}})"] & \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(\operatorname{res}^{C}_{S^{\mathrm{o}}}, \operatorname{res}^{B}_{R\otimes S^{\mathrm{o}}}), \operatorname{res}^{A}_{R}) \arrow[u, "\simeq", "\varphi_{\mathrm{II}}"']
\end{tikzcd}$$

is commutative; here $\boldsymbol{\eta}$ on the lower horizontal arrow is (8.4.18.3). In this case, $\boldsymbol{\eta}_0 : \mathrm{E_I} \to \mathrm{E_{II}}$ is written $\boldsymbol{\eta} : - \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(-,-) \to \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(-,-),-)$, and one says that homomorphism evaluation is *augmented* to a natural transformation of functors from $\mathcal{D}(A) \times \mathcal{D}(B)^{\mathrm{op}} \times \mathcal{D}(C)$ to $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$.

8.4.22 Proposition. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $T \otimes_{\Bbbk} S^{\mathrm{o}} \to C$ be ring homomorphisms where A is flat as an R^{o}-module and C is projective as an S^{o}-module. Homomorphism evaluation is augmented to a natural transformation of functors*

$$\mathcal{D}(A) \times \mathcal{D}(B)^{\mathrm{op}} \times \mathcal{D}(C) \longrightarrow \mathcal{D}(T\text{–}Q^{\mathrm{o}})\,.$$

That is, for M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$ there is a morphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$,

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N, X), M)\,,$$

which is natural in M, X, and N; it is induced by the map $\eta^{\mathrm{I}_A(M) X \mathrm{P}_C(N)}$ in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$. As a natural transformation, this augmented $\boldsymbol{\eta}$ is triangulated in each variable.

Proof. The proof of 8.4.9 provides a template. Let M be an A-complex, X a B-complex, and N a C-complex. The crucial input is that $\mathrm{I}_A(M)$ is a semi-injective R-complex by 5.4.26(b) and $\mathrm{P}_C(M)$ is a semi-projective S^{o}-complex by 5.2.23(b). ◻

Notice that the assumptions in the next corollary are satisfied if $\Bbbk$ is a field.

8.4.23 Corollary. *Assume that Q is flat and T is projective as $\Bbbk$-modules. For complexes M in $\mathcal{D}(R\text{–}Q^{\mathrm{o}})$, X in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and N in $\mathcal{D}(T\text{–}S^{\mathrm{o}})$ homomorphism evaluation,*

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N, X), M)\,,$$

is a morphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$.

Proof. Apply 8.4.22 with $A = R \otimes_{\Bbbk} Q^{\mathrm{o}}$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = T \otimes_{\Bbbk} S^{\mathrm{o}}$ and invoke 7.3.11(a,c). ◻

8.4.24 Theorem. *Let $R \otimes_{\Bbbk} Q^{\mathrm{o}} \to A$, $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and $T \otimes_{\Bbbk} S^{\mathrm{o}} \to C$ be ring homomorphisms such that A is flat as an R^{o}-module and C is projective as an S^{o}-module. For M in $\mathcal{D}(A)$, X in $\mathcal{D}(B)$, and N in $\mathcal{D}(C)$, consider the morphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$ from* 8.4.22,

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N, X), M)\,.$$

If C is left Noetherian and finitely generated as an S^{o}-module, then $\boldsymbol{\eta}^{MXN}$ is an isomorphism if one of the following conditions is satisfied.

(a) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(C)$ and* $\mathrm{pd}_C N$ *is finite.*
(b) *N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(C)$, X is in $\mathcal{D}_{\sqsubset}(B)$, and* $\mathrm{id}_A M$ *is finite.*

Proof. Assume that C is left Noetherian and finitely generated as an S^{o}-module. It must be proved that the morphism $\eta^{\mathrm{I}_A(M) X \mathrm{P}_C(N)}$ in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$ is a quasi-isomorphism if (a) or (b) is satisfied.

(a): One can assume that $\mathrm{P}_C(N)$ is a bounded complex of finitely generated projective C-modules; see 8.1.14. As C is a finitely generated and projective S^{o}-module, $\mathrm{P}_C(N)$ is also a bounded complex of finitely generated projective S^{o}-modules, see 1.3.15 and 5.2.26(b). Thus the desired conclusion follows from 4.5.13(c).

(b): One can assume that $\mathrm{P}_C(N)$ is a bounded below complex of finitely generated projective C-modules; see 5.2.16. As above, $\mathrm{P}_C(N)$ is also a bounded below complex of finitely generated projective S^{o}-modules. Moreover, by 4.2.4 one can assume that X is bounded above, and by 8.2.8 one can assume that $\mathrm{I}_A(M)$ is bounded below. Now it follows from 4.5.13(1,a) that $\eta^{\mathrm{I}_A(M) X \mathrm{P}_C(N)}$ is an isomorphism in $\mathcal{K}(T\text{–}Q^{\mathrm{o}})$. ◻

Notice that under the assumptions in 8.4.23 the isomorphism in the next corollary is an isomorphism in $\mathcal{D}(T\text{–}Q^{\mathrm{o}})$.

8.4.25 Corollary. *Let $M \in \mathcal{D}(R)$, $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$, and $N \in \mathcal{D}(S^{\mathrm{o}})$. The morphism*

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X,M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N,X),M)$$

in $\mathcal{D}(\Bbbk)$, from 8.4.19*, is an isomorphism if S is right Noetherian and one of the following conditions is satisfied.*

(a) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$ and $\mathrm{pd}_{S^{\mathrm{o}}} N$ is finite.*
(b) *N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}})$, X is in $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$, and $\mathrm{id}_R M$ is finite.*

Proof. Apply 8.4.24 with $Q = \Bbbk = T$, $A = R$, $B = R \otimes_{\Bbbk} S^{\mathrm{o}}$, and $C = S^{\mathrm{o}}$. □

We close this section with two canonical applications of 8.4.25.

8.4.26 Proposition. *Assume that S is right Noetherian. Let $N \in \mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$ and $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$ be complexes that are not acyclic. If $\mathrm{pd}_{S^{\mathrm{o}}} N$ is finite, then one has*

$$\mathrm{pd}_R \mathsf{RHom}_{S^{\mathrm{o}}}(N,X) \leqslant \mathrm{pd}_R X - \inf N .$$

Proof. For every R-module M, homomorphism evaluation 8.4.25(a) yields

$$\mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N,X),M) \simeq N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X,M) .$$

By 7.6.8 and 8.1.8 the right-hand complex has minus infimum at most $\mathrm{pd}_R X - \inf N$. The inequality now follows from 8.1.8. □

Example 17.5.15 shows that the boundedness condition in the next result is neecessary; notice that it subsumes 8.3.18.

8.4.27 Proposition. *Assume that S is right Noetherian. Let $M \in \mathcal{D}(R)$ and $X \in \mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$ be complexes that are not acyclic. If $\mathrm{id}_R M$ is finite, then one has*

$$\mathrm{fd}_S \mathsf{RHom}_R(X,M) \leqslant \sup M + \mathrm{id}_{S^{\mathrm{o}}} X .$$

Moreover, if E is an injective R-module, then one has

$$\mathrm{fd}_S \mathrm{Hom}_R(X,E) \leqslant \mathrm{id}_{S^{\mathrm{o}}} X ,$$

and equality holds if E is faithfully injective.

Proof. For every right ideal $\mathfrak{b}$ in S, homomorphism evaluation 8.4.25(b) yields

$$(\dagger) \qquad S/\mathfrak{b} \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X,M) \simeq \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(S/\mathfrak{b},X),M) .$$

By 7.6.7 and 8.2.8 the right-hand complex has supremum at most $\sup M + \mathrm{id}_{S^{\mathrm{o}}} X$, so the desired inequality follows from 8.3.11. If $M = E$ is an injective R-module, then by 7.3.22 the asserted inequality is a special case of the one just proved. Finally, if E is faithfully injective, then one has

$$\sup (S/\mathfrak{b} \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X,E)) = -\inf \mathsf{RHom}_{S^{\mathrm{o}}}(S/\mathfrak{b},X)$$

by (†) and 2.5.7(b), so the asserted equality follows from 8.2.8 and 8.3.11. □

8.4.28 Corollary. *Assume that S is right Noetherian. Let I be an R–S°-bimodule and E an R-module. If I is injective as an S°-module and E is injective, then the S-module* $\operatorname{Hom}_R(I,E)$ *is flat*; *the converse holds if E is faithfully injective.*

Proof. This follows from 8.4.27 applied with $X = I$. □

Remark. Corollary 8.4.28 holds under the weaker assumption that S is right coherent, cf. E 3.3.3.

Exercises

E 8.4.1 Let R be left Noetherian. Show that if $M \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$ has finite projective dimension, then so has the R°-complex $\mathsf{RHom}_R(M,R)$ and one has $M \simeq \mathsf{RHom}_{R^{\circ}}(\mathsf{RHom}_R(M,R),R)$.

E 8.4.2 Let $R \otimes_{\Bbbk} Q^{\circ} \to A$, $R \otimes_{\Bbbk} S^{\circ} \to B$, and $S \otimes_{\Bbbk} T^{\circ} \to C$ be ring homomorphisms such that B is flat as an S°-module and A is finitely generated and projective as an R-module and left Noetherian. For $M \in \mathcal{D}^{\mathrm{f}}_{\square}(A)$, $X \in \mathcal{D}(B)$, and $N \in \mathcal{D}(C)$ show that there is an isomorphism $\mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S N \simeq \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N)$ in $\mathcal{D}(Q\text{–}T^{\circ})$ provided that $\operatorname{pd}_A M$ is finite. *Hint:* E 5.4.10.

E 8.4.3 Let $R \otimes_{\Bbbk} Q^{\circ} \to A$, $R \otimes_{\Bbbk} S^{\circ} \to B$, and $S \otimes_{\Bbbk} T^{\circ} \to C$ be ring homomorphisms such that B is flat as an R°-module and C is finitely generated and projective as an S-module and left Noetherian. For $M \in \mathcal{D}(A)$, $X \in \mathcal{D}(B)$, and $N \in \mathcal{D}^{\mathrm{f}}_{\square}(C)$ show that there is an isomorphism $\mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S N \simeq \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N)$ in $\mathcal{D}(Q\text{–}T^{\circ})$ provided that $\operatorname{pd}_C N$ is finite. *Hint:* E 5.3.16.

E 8.4.4 Let $R \otimes_{\Bbbk} Q^{\circ} \to A$, $R \otimes_{\Bbbk} S^{\circ} \to B$, and $T \otimes_{\Bbbk} S^{\circ} \to C$ be ring homomorphisms such that B is projective as an R-module and C is finitely generated and projective as an S°-module and left Noetherian. For $M \in \mathcal{D}(A)$, $X \in \mathcal{D}(B)$, and $N \in \mathcal{D}^{\mathrm{f}}_{\square}(C)$ show that there is an isomorphism $N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X,M) \simeq \mathsf{RHom}_R(\mathsf{RHom}_{S^{\circ}}(N,X),M)$ in $\mathcal{D}(T\text{–}Q^{\circ})$ provided that $\operatorname{pd}_C N$ is finite. *Hint:* E 5.2.16.

E 8.4.5 Let P be a projective R-module. Show that if S is left Noetherian and X a complex in $\mathcal{D}_{\sqsupset}(R\text{–}S^{\circ})$, then $\operatorname{fd}_{S^{\circ}} \operatorname{Hom}_R(P,X) \leqslant \operatorname{fd}_{S^{\circ}} X$ holds with equality if P is faithfully projective. Apply this to strengthen 8.4.15.

E 8.4.6 Assume that S is right noetherian and let P be a finitely generated projective S°-module. Show that for X in $\mathcal{D}(R\text{–}S^{\circ})$ one has $\operatorname{pd}_R \operatorname{Hom}_{S^{\circ}}(P,X) \leqslant \operatorname{pd}_R X$ with equality if P is faithfully projective.

E 8.4.7 Assume that S is right Noetherian. Let I be an R–S°-bimodule and E an R-module. Show that if I is faithfully injective as an S°-module and E is faithfully injective, then $\operatorname{Hom}_R(I,E)$ is a faithfully flat S-module.

8.5 Global and Finitistic Dimensions

Synopsis. (Weak) global dimension; ∼ of Noetherian ring; finitistic projective/injective/flat dimension; ∼ of perfect ring; ∼ of Noetherian ring; Iwanaga–Gorenstein ring.

The global and finitistic dimensions are invariants of rings, they measure suprema of homological dimensions of all modules, or all modules of finite homological dimension, over a ring.

GLOBAL DIMENSION

8.5.1 Definition. The *global dimension* of R, written $\operatorname{gldim} R$, is defined as

$$\operatorname{gldim} R \;=\; \sup\{\operatorname{pd}_R M \mid M \text{ is an } R\text{-module}\}\,.$$

8.5.2 Example. By 1.3.28 a ring has global dimension 0 if and only if it is semi-simple. By 1.3.11 a principal left ideal domain has global dimension at most 1, so 8.1.5 yields $\operatorname{gldim} \mathbb{Z} = 1$. At the other extreme, 8.1.10 yields $\operatorname{gldim} \mathbb{Z}/4\mathbb{Z} = \infty$.

REMARK. Since semi-simplicity is a left–right symmetric property, one has $\operatorname{gldim} R = 0$ if and only if $\operatorname{gldim} R^{\circ} = 0$. We prove in 8.5.13 that the global dimensions of R and R° agree if R is Noetherian; in general, though, they need not agree. For example, Small [238] shows that the ring

$$R \;=\; \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right) \;=\; \left\{\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \;\middle|\; x \in \mathbb{Z} \text{ and } y, z \in \mathbb{Q}\right\}$$

has $\operatorname{gldim} R = 2$ and $\operatorname{gldim} R^{\circ} = 1$. In fact, the difference $\operatorname{gldim} R - \operatorname{gldim} R^{\circ}$ can be any integer or even infinite: Jategaonkar [145] constructs for $0 < m \leqslant n \leqslant \infty$ a left Noetherian domain R with $\operatorname{gldim} R = m$ and $\operatorname{gldim} R^{\circ} = n$.

8.5.3 Theorem. *There are equalities,*

$$\begin{aligned} \operatorname{gldim} R \;&=\; \sup\{\operatorname{pd}_R M - \sup M \mid M \text{ is a complex in } \mathcal{D}_{\sqsubset}(R) \text{ with } \operatorname{H}(M) \neq 0\} \\ &=\; \sup\{\operatorname{pd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\} \\ &=\; \sup\{\operatorname{id}_R M + \inf M \mid M \text{ is a complex in } \mathcal{D}_{\sqsupset}(R) \text{ with } \operatorname{H}(M) \neq 0\} \\ &=\; \sup\{\operatorname{id}_R M \mid M \text{ is an } R\text{-module}\}\,. \end{aligned}$$

PROOF. Denote the suprema in the display by $s_1, \ldots, s_4$, in the order they appear. The inequalities $s_1 \geqslant \operatorname{gldim} R \geqslant s_2$ and $s_3 \geqslant s_4$ are evident. Let n be an integer. If one has $s_2 \leqslant n$, then 8.1.8 yields $-\inf \mathsf{RHom}_R(R/\mathfrak{a}, M) \leqslant n - \inf M$ for every R-complex M and every left ideal $\mathfrak{a}$, so 8.2.8 yields $\operatorname{id}_R M \leqslant n - \inf M$ for every R-complex M, whence $s_3 \leqslant n$ holds and one has $s_3 \leqslant s_2$. A parallel argument that applies 8.1.8 and 8.2.8 in opposite order yields $s_1 \leqslant s_4$. □

8.5.4 Corollary. *The following conditions are quivalent.*

(*i*) $\operatorname{gldim} R$ *is finite.*
(*ii*) *Every R-module has finite projective dimension.*
(*iii*) *Every complex in $\mathcal{D}_{\sqsubset}(R)$ has finite projective dimension.*
(*iv*) *Every R-module has finite injective dimension.*
(*v*) *Every complex in $\mathcal{D}_{\sqsupset}(R)$ has finite injective dimension.*

PROOF. The conditions are equivalent by 8.5.3, 8.1.11, and 8.2.12. □

WEAK GLOBAL DIMENSION

8.5.5 Definition. The *weak global dimension* of R, written $\operatorname{wgldim} R$, is given by

$$\operatorname{wgldim} R \;=\; \sup\{\operatorname{fd}_R M \mid M \text{ is an } R\text{-module}\}\,.$$

8.5.6. By 8.3.6 there is an inequality

$$\operatorname{wgldim} R \leqslant \operatorname{gldim} R\,,$$

and by 8.3.20 equality holds if R is left perfect.

The equality $\operatorname{wgldim} R = \operatorname{gldim} R$ also holds if R is left Noetherian, see 8.5.13, and left perfect rings need not be left Noetherian; see the Remark after B.51.

8.5.7 Example. In view of 1.3.47 and 8.5.3 one has $\operatorname{gldim} \mathbb{Z} = \operatorname{wgldim} \mathbb{Z} = 1$ and $\operatorname{gldim} \mathbb{Z}/4\mathbb{Z} = \operatorname{wgldim} \mathbb{Z}/4\mathbb{Z} = \infty$; see 8.5.2.

As von Neumann regularity is a left–right symmetric property, it is implicit in the next result that $\operatorname{wgldim} R = 0$ holds if and only if $\operatorname{wgldim} R^{\mathrm{o}} = 0$ holds, but this is just a special case of 8.5.10.

8.5.8 Theorem. *R is von Neumann regular if and only if* $\operatorname{wgldim} R = 0$ *holds.*

Proof. If $\operatorname{wgldim} R = 0$ holds, then every R-module is flat and, therefore, R is von Neumann regular by 1.3.46. Conversely, if R is von Neumann regular, then every finitely generated right ideal $\mathfrak{b}$ in R is generated by an idempotent and hence a direct summand of the R^{o}-module R; thus the R^{o}-module $R/\mathfrak{b}$ is projective. It follows from 8.3.2 that $\operatorname{Tor}_1^R(R/\mathfrak{b}, M) = 0$ holds for every R-module M and for every finitely generated right ideal $\mathfrak{b}$ and, consequently, every R-module is flat. $\square$

8.5.9 Example. The commutative ring $\mathbb{Q}^{\mathbb{N}}$ is von Neumann regular by 1.3.45, so by 8.5.8 one has $\operatorname{wgldim} \mathbb{Q}^{\mathbb{N}} = 0$. As $\mathbb{Q}^{\mathbb{N}}$ is not Noetherian it is, in particular, not semi-simple, so $\operatorname{gldim} \mathbb{Q}^{\mathbb{N}} > 0$ holds by 8.5.2.

Remark. It is a result of Osofsky [200] that $\operatorname{gldim} \mathbb{Q}^{\mathbb{N}} = n+1$ holds if and only if $2^{\aleph_0} = \aleph_n$ where $n \in \mathbb{N}$. Thus, $\operatorname{gldim} \mathbb{Q}^{\mathbb{N}}$ is at least 2, but the exact value can not be decided within ZFC. Moreover if U is a set of cardinality $\geqslant \aleph_\omega$, where ω denotes the first infinite ordinal, then one has $\operatorname{gldim} \mathbb{Q}^U = \infty$. In contrast, $\operatorname{wgldim} \mathbb{Q}^U = 0$ holds by 1.3.45 and 8.5.8.

The weak global dimension is a left–right symmetric invariant.

8.5.10 Theorem. *There is an equality* $\operatorname{wgldim} R = \operatorname{wgldim} R^{\mathrm{o}}$*, and one has*

$$\begin{aligned}\operatorname{wgldim} R &= \sup\{\operatorname{fd}_R M - \sup M \mid M \text{ is a complex in } \mathcal{D}_{\sqsubset}(R) \text{ with } \operatorname{H}(M) \neq 0\}\\ &= \sup\{\operatorname{fd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is a finitely generated left ideal in } R\}\,.\end{aligned}$$

Proof. Let n be an integer. If the inequality $\operatorname{wgldim} R \leqslant n$ holds, then 8.3.11 yields $\operatorname{Tor}_{n+1}^R(N, M) = 0$ for all R-modules M and all R^{o}-modules N. By commutativity 7.5.10 and the definition of Tor, 7.4.18, one has $\operatorname{Tor}_{n+1}^R(N, M) = \operatorname{Tor}_{n+1}^{R^{\mathrm{o}}}(M, N)$, so another application of 8.3.11 yields $\operatorname{fd}_{R^{\mathrm{o}}} N \leqslant n$ for all R^{o}-modules N, whence $\operatorname{wgldim} R^{\mathrm{o}} \leqslant n$ holds. This proves the inequality $\operatorname{wgldim} R^{\mathrm{o}} \leqslant \operatorname{wgldim} R$, and the opposite inequality holds by symmetry.

To prove the equalities in the display, denote the suprema by s_1 and s_2 in the order they appear. The inequalities $s_1 \geqslant \operatorname{wgldim} R \geqslant s_2$ are evident. Let n be an integer. If $\operatorname{wgldim} R \leqslant n$ holds, then $\operatorname{wgldim} R^{\mathrm{o}} \leqslant n$ holds as well, so for every R^{o}-module

N one has $\sup(N \otimes^{\mathsf{L}}_R M) = \sup(M \otimes^{\mathsf{L}}_{R^\mathrm{o}} N) \leqslant n + \sup M$ by 8.3.11. It follows that $\mathrm{fd}_R M \leqslant n + \sup M$ holds, so one has $s_1 \leqslant n$ and hence $s_1 \leqslant \mathrm{wgldim}\, R$. If $s_2 \leqslant n$ holds, then one has $0 = \mathrm{Tor}^R_{n+1}(N, R/\mathfrak{a}) = \mathrm{Tor}^{R^\mathrm{o}}_{n+1}(R/\mathfrak{a}, N)$ for every R^o-module N and every finitely generated left ideal $\mathfrak{a}$ in R. Thus, $n \geqslant \mathrm{wgldim}\, R^\mathrm{o}$ holds, and one has $s_2 \geqslant \mathrm{wgldim}\, R^\mathrm{o} = \mathrm{wgldim}\, R$. □

8.5.11 Corollary. *The following conditions are quivalent.*

(*i*) wgldim R *is finite.*

(*ii*) *Every R-module has finite flat dimension.*

(*iii*) *Every complex in $\mathcal{D}_{\sqsubset}(R)$ has finite flat dimension.*

Proof. The conditions are equivalent by 8.5.10 and 8.3.13. □

Global Dimensions of Noetherian Rings

8.5.12 Proposition. *If R is left Noetherian, then there is an equality,*

$$\mathrm{gldim}\, R \;=\; \sup\{\mathrm{id}_R\, R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\}\ .$$

Proof. In view of 8.5.3 one need only prove the inequality "$\leqslant$". Let n be an integer and assume that $\mathrm{id}_R\, R/\mathfrak{a} \leqslant n$ holds for every left ideal $\mathfrak{a}$ in R. It suffices, again by 8.5.3, to prove that $\mathrm{pd}_R M \leqslant n$ holds for every finitely generated R-module M. By 8.1.14 that follows as $\mathrm{Ext}^{n+1}_R(M, R/\mathfrak{a}) = 0$ holds for every left ideal $\mathfrak{a}$ in R. □

Remark. Osofsky [196] shows that for every $n \geqslant 1$ there is a (valuation) ring R with $\mathrm{gldim}\, R = n$ and $\sup\{\mathrm{id}_R\, R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\} = 1$.

8.5.13 Theorem. *If R is left Noetherian, then* $\mathrm{wgldim}\, R = \mathrm{gldim}\, R$ *holds.*

Proof. By 8.3.19 one has $\mathrm{pd}_R M = \mathrm{fd}_R M$ for every finitely generated R-module M. The equality $\mathrm{gldim}\, R = \mathrm{wgldim}\, R$ now follows from 8.5.10 and 8.5.3. □

8.5.14 Corollary. *If R is left Noetherian and von Neumann regular, then R is semi-simple*; *in particular, R is Artinian.*

Proof. The assertion follows immediately from 8.5.13, 8.5.8, and 8.5.2. □

8.5.15 Corollary. *If R is Noetherian, then* $\mathrm{gldim}\, R = \mathrm{gldim}\, R^\mathrm{o}$ *holds.*

Proof. When R is both left and right Noetherian, 8.5.13 and 8.5.10 yield equalities $\mathrm{gldim}\, R = \mathrm{wgldim}\, R = \mathrm{wgldim}\, R^\mathrm{o} = \mathrm{gldim}\, R^\mathrm{o}$. □

Finitistic Dimensions

8.5.16 Definition. The *finitistic projective dimension* of R, written FPD R, the *finitistic injective dimension* of R, written FID R, and the *finitistic flat dimension* of R, written FFD R, are defined as follows,

$$\begin{aligned} \operatorname{FPD} R &= \sup\{\operatorname{pd}_R M \mid M \text{ is an } R\text{-module with } \operatorname{pd}_R M < \infty\}\,, \\ \operatorname{FID} R &= \sup\{\operatorname{id}_R M \mid M \text{ is an } R\text{-module with } \operatorname{id}_R M < \infty\}\,, \quad \text{and} \\ \operatorname{FFD} R &= \sup\{\operatorname{fd}_R M \mid M \text{ is an } R\text{-module with } \operatorname{fd}_R M < \infty\}\,. \end{aligned}$$

REMARK. The finitistic dimensions are not left–right symmetric invariants. Jensen and Lenzing [149] ascribe the following example to Small. Let $\Bbbk$ be a field and denote by $\mathfrak{n}$ the maximal ideal of the local ring $Q = \Bbbk[x]/(x^2)$; the ring

$$R = \begin{pmatrix} \Bbbk & Q/\mathfrak{n} \\ 0 & Q \end{pmatrix} = \left\{ \left(\begin{smallmatrix} u & v \\ 0 & w \end{smallmatrix}\right) \,\middle|\, u \in \Bbbk,\ v \in Q/\mathfrak{n},\ \text{and } w \in Q \right\}$$

has $\operatorname{FPD} R = 1$ and $\operatorname{FPD} R^{\circ} = 0$. More generally, Green, Kirkman, and Kuzmanovich [108] show that for every $n \in \mathbb{N}$ there exists a ring R with $\operatorname{FPD} R = n$ and $\operatorname{FPD} R^{\circ} = 0$.

8.5.17 Proposition. *There are equalities,*

$$\begin{aligned} \operatorname{FPD} R &= \sup\{\operatorname{pd}_R M - \sup M \mid M \in \mathcal{C}(R) \textit{ with } \operatorname{H}(M) \neq 0 \textit{ and } \operatorname{pd}_R M < \infty\}\,, \\ \operatorname{FFD} R &= \sup\{\operatorname{fd}_R M - \sup M \mid M \in \mathcal{C}(R) \textit{ with } \operatorname{H}(M) \neq 0 \textit{ and } \operatorname{fd}_R M < \infty\}\,, \textit{ and} \\ \operatorname{FID} R &= \sup\{\operatorname{id}_R M + \inf M \mid M \in \mathcal{C}(R) \textit{ with } \operatorname{H}(M) \neq 0 \textit{ and } \operatorname{id}_R M < \infty\}\,. \end{aligned}$$

PROOF. There are three equalities to prove; in each case the inequality "$\leqslant$" is evident. In the following M is an R-complex with $\operatorname{H}(M) \neq 0$.

If M has finite projective dimension, then $w = \sup M$ is an integer by 8.1.3. Let P be a semi-projective replacement of M. The complex $\Sigma^{-w} P_{\geqslant w}$ is a semi-projective replacement of the module $\operatorname{C}_w(P)$, so one has

$$\operatorname{pd}_R M = w + \operatorname{pd}_R \operatorname{C}_w(P) \leqslant w + \operatorname{FPD} R$$

by 8.1.2, 8.1.8, and 8.5.16. This proves the first equality. If M has finite flat dimension, then in view of 5.4.10 one similarly gets $\operatorname{fd}_R M = w + \operatorname{fd}_R \operatorname{C}_w(P) \leqslant w + \operatorname{FFD} R$ by 8.3.3, 8.3.11, and 8.5.16. This proves the third equality.

If M has finite injective dimension, then $\inf M = u$ is an integer by 8.2.3. Let I be a semi-injective replacement of M. The complex $\Sigma^{-u} I_{\leqslant u}$ is a semi-injective replacement of the module $\operatorname{Z}_u(I)$, so one has $\operatorname{id}_R M = \operatorname{id}_R \operatorname{Z}_u(I) - u \leqslant \operatorname{FID} R - u$ by 8.2.2, 8.2.8, and 8.5.16. This establishes the second equality. □

While projective modules are flat, the projective dimension of flat modules is a delicate issue; see the Remark after 17.4.28. It would be too much of a detour to include the proof of the next theorem here, so it has been relegated to an appendix. It is known as Jensen's theorem (on projective dimension of flat modules).

8.5.18 Theorem. *For every flat R-module F one has $\operatorname{pd}_R F \leqslant \operatorname{FPD} R$.*

PROOF. See Appn. D; the proof comes after D.12. □

8.5.19 Corollary. *Let M be an R-complex. If $\operatorname{fd}_R M$ is finite, then one has*

$$\operatorname{pd}_R M \leqslant \operatorname{FPD} R + \sup M\,.$$

PROOF. One can assume that M is not acyclic and $\operatorname{FPD} R$ is finite. Set $n = \operatorname{fd}_R M$ and $w = \sup M$; both are integers by assumption and 8.3.4. Let P be a semi-projective replacement of M. Note that $\Sigma^{-w} P_{\geqslant w}$ is a semi-projective replacement of the module

$C_w(P)$ and that $\mathrm{pd}_R M = w + \mathrm{pd}_R C_w(P)$ holds by 8.1.8. The module $C_n(P)$ is flat by 8.3.11 so it has finite projective dimension by 8.5.18. It follows that $C_w(P)$ has finite projective dimension, so $\mathrm{pd}_R C_w(P) \leqslant \mathrm{FPD}\, R$ holds by definition. □

8.5.20 Corollary. *If* FPD R *is finite, then an* R*-complex has finite flat dimension if and only if it has finite projective dimension.*

Proof. The assertion follows immediately from 8.3.6 and 8.5.19. □

Remark. Without recourse to 8.5.18, we show in 10.3.13 that the flat and projective dimensions are simultaneously finite for complexes over a ring that admits a so-called dualizing complex. This, however, need not imply that such rings have finite finitistic projective dimension. Indeed, every finite dimensional algebra over a field has a dualizing complex, but it remains an open problem if the finitistic projective dimension is finite for every such algebra.

Per the Remark after 8.5.9 there exist rings with flat modules of infinite projective dimension.

8.5.21 Proposition. *The following inequalities hold,*

$$\begin{array}{ccccc} \mathrm{gldim}\, R & \geqslant & \mathrm{wgldim}\, R & \leqslant & \mathrm{gldim}\, R^{\mathrm{o}} \\ \veq & & \veq & & \veq \\ \mathrm{FPD}\, R & \geqslant & \mathrm{FFD}\, R & \leqslant & \mathrm{FID}\, R^{\mathrm{o}} \end{array} .$$

If one of the quantities in the top row is finite, then it equals the quantity below it.

Proof. The inequalities in the first row follow from 8.5.6 and 8.5.10. The vertical inequalities and the last assertion are immediate from the definitions and 8.5.3. The last inequality in the second row follows from 8.3.17. The first one follows from 8.5.19 and 8.3.6. □

By 8.5.21 one has $\mathrm{FPD}\, R = \mathrm{gldim}\, R$ provided that $\mathrm{gldim}\, R$ is finite. As a consequence of 8.5.18, the following stronger result holds.

8.5.22 Proposition. *If* wgldim R *is finite, then* $\mathrm{FPD}\, R = \mathrm{gldim}\, R$ *holds.*

Proof. It is sufficient to prove the inequality $\mathrm{FPD}\, R \geqslant \mathrm{gldim}\, R$. Let M be an R-module; one has $\mathrm{fd}_R M < \infty$ by assumption, so 8.5.18 yields $\mathrm{FPD}\, R \geqslant \mathrm{pd}_R M$. □

Remark. By 8.5.22 and the Remark after 8.5.9 and there exist rings with finitistic flat dimension zero and infinite finitistic projective dimension.

8.5.23 Theorem. *If* R *is left perfect, then there are (in)equalities*

$$\mathrm{FFD}\, R = \mathrm{FPD}\, R \leqslant \mathrm{id}_R R .$$

Proof. The equality is immediate from 8.3.20. To prove the inequality, let M be an R-module with $\mathrm{pd}_R M = n$ finite. Let $\mathfrak{J}$ be the Jacobson radical of R and set $\boldsymbol{k} = R/\mathfrak{J}$. By 8.1.17 one has $\mathrm{Ext}_R^n(M, \boldsymbol{k}) \neq 0$, and by 7.3.35 there is an exact sequence,

$$\mathrm{Ext}_R^n(M,R) \longrightarrow \mathrm{Ext}_R^n(M,\boldsymbol{k}) \longrightarrow \mathrm{Ext}_R^{n+1}(M,\mathfrak{J}) = 0 ,$$

whence one has $\mathrm{Ext}_R^n(M,R) \neq 0$ and, thus, $n \leqslant \mathrm{id}_R R$ by 8.2.8. □

FINITISTIC DIMENSIONS OF NOETHERIAN RINGS

Over Noetherian rings, there are further relations among the quantities in 8.5.21, and they relate to another invariant: the injective dimension of the ring itself.

8.5.24 Theorem. *Assume that R is left Noetherian and let M be an R-module. If M has finite projective dimension, then* $\mathrm{pd}_R\, M \leqslant \mathrm{FFD}\, R + 1$ *holds.*

PROOF. See Appn. D; the proof comes after D.15. □

8.5.25 Corollary. *If R is left Noetherian, then* $\mathrm{FPD}\, R \leqslant \mathrm{FFD}\, R + 1$ *holds.*

PROOF. The inequality follows immediately from 8.5.24. □

8.5.26 Corollary. *If R is left Noetherian, then* $\mathrm{FFD}\, R$ *is finite if and only if* $\mathrm{FPD}\, R$ *is finite.*

PROOF. The assertion follows immediately from 8.5.21 and 8.5.25. □

8.5.27 Theorem. *Assume that R is left Noetherian. There are (in)equalities,*

$$\mathrm{FPD}\, R \;\leqslant\; \mathrm{id}_R\, R \qquad \textit{and} \qquad \mathrm{FID}\, R \;=\; \mathrm{FFD}\, R^{\mathrm{o}}\,.$$

Moreover, if both quantities $\mathrm{id}_R\, R$ *and* $\mathrm{FID}\, R$ *are finite, then they are equal.*

PROOF. To prove the equality, it suffices by 8.5.21 to establish the inequality "$\leqslant$", which follows immediately from 8.3.18.

To prove the inequality, let M be an R-module with $\mathrm{pd}_R\, M = n$ finite. Choose an R-module N with $\mathrm{Ext}_R^n(M,N) \neq 0$ and an exact sequence $0 \to K \to L \to N \to 0$ with L a free R-module; cf. 1.3.12. By 7.3.35 there is an exact sequence

$$\mathrm{Ext}_R^n(M,L) \longrightarrow \mathrm{Ext}_R^n(M,N) \longrightarrow \mathrm{Ext}_R^{n+1}(M,K) \;=\; 0\,,$$

whence one has $\mathrm{Ext}_R^n(M,L) \neq 0$ and, thus, $n \leqslant \mathrm{id}_R\, L = \mathrm{id}_R\, R$ by 8.2.8 and 8.2.21.

Assume that $\mathrm{id}_R\, R$ and $\mathrm{FID}\, R$ are finite; evidently one has $\mathrm{id}_R\, R \leqslant \mathrm{FID}\, R$. To prove the opposite inequality, set $n = \mathrm{FID}\, R$ and note that $\mathrm{Ext}_R^{n+1}(-,K) = 0$ holds for every R-module K with $\mathrm{id}_R\, K$ finite. Furthermore, there exists an R-module M with $\mathrm{id}_R\, M = n$. Consider an exact sequence $0 \to K \to L \to M \to 0$ with L free. As $\mathrm{id}_R\, R$ is finite, it follows from 8.2.21 and 8.2.9 that L and K have finite injective dimension. Let N be an R-module with $\mathrm{Ext}_R^n(N,M) \neq 0$. The exact sequence

$$\mathrm{Ext}_R^n(N,L) \longrightarrow \mathrm{Ext}_R^n(N,M) \longrightarrow \mathrm{Ext}_R^{n+1}(N,K) \;=\; 0\,,$$

shows that $\mathrm{Ext}_R^n(N,L)$ is non-zero, whence $n \leqslant \mathrm{id}_R\, L = \mathrm{id}_R\, R$ holds as desired. □

8.5.28 Corollary. *If R is Noetherian, then there are (in)equalities,*

$$\mathrm{FID}\, R^{\mathrm{o}} \;=\; \mathrm{FFD}\, R \;\leqslant\; \mathrm{FPD}\, R \;\leqslant\; \mathrm{id}_R\, R \;\leqslant\; \mathrm{gldim}\, R\,.$$

PROOF. The equality holds by 8.5.27 as R^{o} is left Noetherian. The first inequality is part of 8.5.21. The second inequality holds by 8.5.27 as R is left Noetherian. The third and final inequality holds by 8.5.3. □

Iwanaga–Gorenstein Rings

The results above call attention to rings of finite self-injective dimension.

8.5.29 Definition. If R is Noetherian with $\operatorname{id}_R R$ and $\operatorname{id}_{R^\circ} R$ finite, then R is called *Iwanaga–Gorenstein*.

We show below that $\operatorname{id}_R R = \operatorname{id}_{R^\circ} R$ holds for an Iwanaga–Gorenstein ring. This equality was first proved by Zaks [262]. See 8.2.10 for examples of commutative Iwanaga–Gorenstein rings.

8.5.30 Corollary. *If R is Iwanaga–Gorenstein, then there are equalities,*

$$\operatorname{id}_R R \;=\; \operatorname{FID} R \;=\; \operatorname{FFD} R \;=\; \operatorname{FPD} R \;=\; \operatorname{id}_{R^\circ} R\,.$$

Proof. By 8.5.28 there are (in)equalities,

$$\operatorname{id}_{R^\circ} R \;\leqslant\; \operatorname{FID} R^\circ \;=\; \operatorname{FFD} R \;\leqslant\; \operatorname{FPD} R \;\leqslant\; \operatorname{id}_R R\,,$$

and since the same inequalities hold with R and R° interchanged, equalities hold. □

8.5.31 Theorem. *Assume that R is Noetherian. The next conditions are equivalent.*

(*i*) *R is Iwanaga–Gorenstein.*
(*ii*) *All flat modules over R and over R° have finite injective dimension.*
(*iii*) *All injective modules over R and over R° have finite flat dimension.*

Proof. Evidently, (*ii*) implies (*i*) as R is flat as a module over R and over R°.

(*i*) ⟹ (*iii*): Let I be an injective R-module, the counitor 4.4.2 and 8.4.27 yield $\operatorname{fd}_R I = \operatorname{fd}_R \operatorname{Hom}_R(R, I) \leqslant \operatorname{id}_{R^\circ} R$. By symmetry an injective R°-module has finite flat dimension.

(*iii*) ⟹ (*ii*): Let F be a flat R-module and recall from 5.4.19 that the R°-module $\operatorname{Hom}_\Bbbk(F, \mathbb{E})$ is injective. By 8.3.18 one has $\operatorname{id}_R F = \operatorname{fd}_{R^\circ} \operatorname{Hom}_\Bbbk(F, \mathbb{E}) < \infty$. By symmetry a flat R°-module has finite injective dimension. □

8.5.32 Corollary. *Let R be Iwanaga–Gorenstein and M a complex in $\mathcal{D}_\square(R)$. If one of the quantities* $\operatorname{pd}_R M$, $\operatorname{fd}_R M$, *or* $\operatorname{id}_R M$ *is finite, then they are all finite.*

Proof. It follows from 8.5.30 and 8.5.20 that $\operatorname{pd}_R M$ is finite if and only if $\operatorname{fd}_R M$ is finite. Assume now that $\operatorname{fd}_R M$ is finite; in the derived category M, is isomorphic to a bounded complex F of flat modules. Assume, without loss of generality, that F is concentrated in non-negative degrees, set $s = \sup F^\natural$, and proceed by induction on s. If $s = 0$, then F is a flat R-module and hence $\operatorname{id}_R F$ is finite by 8.5.31. For $s > 0$ consider the exact seqeuence $0 \to F_{\leqslant s-1} \to F \to \Sigma^s F_s \to 0$ from 2.5.22. By hypothesis, $\operatorname{id}_R F_{\leqslant s-1}$ is finite. The flat module F_s has finite injective dimension by 8.5.31, and per 8.2.3 so has the complex $\Sigma^s F_s$. It now follows from 8.2.9 that $\operatorname{id}_R F$ is finite. If $\operatorname{id}_R M$ is finite, then a similar argument shows that $\operatorname{fd}_R M$ is finite. □

Exercises

E 8.5.1 Show that if R is left Noetherian, then $\operatorname{gldim} R \leqslant \operatorname{gldim} R^{\mathrm{o}}$ holds.

E 8.5.2 Show that the following conditions are equivalent. (*i*) $\operatorname{gldim} R \leqslant 1$ holds. (*ii*) Every submodule of a projective R-module is projective. (*iii*) Every quotient of an injective R-module is injective.

E 8.5.3 Show that $\operatorname{wgldim} R \leqslant 1$ holds if and only if every submodule of a flat R-module is flat. Conclude that over a principal ideal domain submodules of a flat modules are flat.

E 8.5.4 Show that if $\operatorname{gldim} R$ is finite, then every complex of projective R-modules is semi-projective, and every complex of injective R-modules is semi-injective.

E 8.5.5 Let P be a bounded above complex of projective R-modules and assume that $\operatorname{FPD} R$ is finite. (a) Show that if $\operatorname{pd}_R P$ is finite, then P is semi-projective. (b) Show that if P is acyclic, then it is contractible.

E 8.5.6 Let I be a bounded below complex of injective R-modules and assume that $\operatorname{FID} R$ is finite. (a) Show that if $\operatorname{id}_R I$ is finite, then I is semi-injective. (b) Show that if I is acyclic, then it is contractible.

E 8.5.7 Show that if $\operatorname{wgldim} R$ is finite, then every complex of flat R-modules is semi-flat.

E 8.5.8 Assume that R is left Noetherian. Show that the inequality $\operatorname{id}_R M \leqslant \operatorname{id}_R R$ holds for every R-module of finite flat dimension.

E 8.5.9 Assume that R is right Noetherian. Show that the inequality $\operatorname{fd}_R M \leqslant \operatorname{id}_{R^{\mathrm{o}}} R$ holds for every R-module of finite injective dimension.

E 8.5.10 Let F be a bounded above complex of flat R-modules and assume that $\operatorname{FFD} R$ is finite. (a) Show that if $\operatorname{fd}_R F$ is finite, then F is semi-flat. (b) Show that if F is acyclic, then it is pure acyclic.

Exercises E 8.5.11–E 8.5.23 deal with the following invariants of R:

$$\begin{aligned}
\operatorname{splf} R &= \sup\{\operatorname{pd}_R F \mid F \text{ is a flat } R\text{-module}\}\,, \\
\operatorname{sfli} R &= \sup\{\operatorname{fd}_R I \mid I \text{ is an injective } R\text{-module}\}\,, \\
\operatorname{silf} R &= \sup\{\operatorname{id}_R F \mid F \text{ is a flat } R\text{-module}\}\,, \\
\operatorname{silp} R &= \sup\{\operatorname{id}_R P \mid P \text{ is a projective } R\text{-module}\}\,, \text{ and} \\
\operatorname{spli} R &= \sup\{\operatorname{pd}_R I \mid I \text{ is an injective } R\text{-module}\}\,.
\end{aligned}$$

The symbols splf, …, spli are acronyms: splf stands for “supremum of projective lengths of flats” etc. The terminology and notation comes from group representations and first appeared in [105] by Gedrich and Gruenberg. These invariants are studied extensively by Emmanouil and Talelli [84].

E 8.5.11 Show that $\operatorname{splf} R$ is finite if and only if $\operatorname{pd}_R F$ is finite for every flat R-module F. Show also that the analogous statements hold for $\operatorname{sfli} R$, $\operatorname{silf} R$, $\operatorname{silp} R$, and $\operatorname{spli} R$.

E 8.5.12 Show that there are inequalities $\operatorname{spli} R \leqslant \operatorname{sfli} R + \operatorname{splf} R$ and $\operatorname{splf} R \leqslant \operatorname{FPD} R$.

E 8.5.13 Show that there is an inequality $\operatorname{gldim} R \leqslant \operatorname{wgldim} R + \operatorname{splf} R$.

E 8.5.14 Show that if $\operatorname{wgldim} R$ is finite, then $\operatorname{splf} R$ and $\operatorname{FPD} R$ are simultaneously finite.

E 8.5.15 Show that the inequailties $\operatorname{FPD} R \leqslant \operatorname{silp} R$ and $\operatorname{FID} R \leqslant \operatorname{spli} R$ hold.

E 8.5.16 Show that if $\operatorname{silp} R$ and $\operatorname{spli} R$ are finite, then one has $\operatorname{silp} R = \operatorname{FPD} R = \operatorname{FID} R = \operatorname{spli} R$.

E 8.5.17 Let M be an R-module. Show that if $\operatorname{silp} R$ and $\operatorname{spli} R$ are finite, then the quantities $\operatorname{pd}_R M$, $\operatorname{fd}_R M$, and $\operatorname{id}_R M$ are simultaneously finite.

E 8.5.18 Show that there is an inequality $\operatorname{FFD} R \leqslant \operatorname{sfli} R^{\mathrm{o}}$.

E 8.5.19 Show that if $\operatorname{sfli} R$ and $\operatorname{sfli} R^{\mathrm{o}}$ are finite, then $\operatorname{sfli} R = \operatorname{FFD} R = \operatorname{FFD} R^{\mathrm{o}} = \operatorname{sfli} R^{\mathrm{o}}$ holds.

E 8.5.20 Show that the equality $\operatorname{silf} R = \operatorname{silp} R$ holds. *Hint:* 8.5.18.

E 8.5.21 Show that if R is left Noetherian, then $\operatorname{id}_R R = \operatorname{silp} R$ holds.

E 8.5.22 Show that if R is left Noetherian, then one has $\operatorname{silf} R \leqslant \operatorname{id}_R R$ and $\operatorname{splf} R \leqslant \operatorname{id}_R R$.

E 8.5.23 Show that if R is Noetherian, then $\operatorname{sfli} R = \operatorname{id}_{R^\circ} R$ holds.

Chapter 9
Gorenstein Homological Dimensions

The invariants treated in this chapter are refinements of the homological dimensions, sometimes referred to as "classic" or "absolute", that are treated in Chap. 8. The Gorenstein homological dimensions originate in work of Auslander and Bridger [8, 9]. They were introduced to charactarize Iwanaga–Gorenstein rings among commutative Noetherian local rings. Studies of Gorenstein homological dimensions over non-commutative rings started with Enochs and collaborators as consolidated and summarized in [87, Chap. 10]. The approach we take in this chapter builds on this work as further developed by Avramov and Foxby [23] and Christensen, Frankild, and Holm [52, 61, 62, 131, 132]. The overarching theme is that the Gorenstein homological dimensions behave and interact like the absolute dimensions in Chap. 8.

9.1 Gorenstein Projective Dimension

Synopsis. Totally acyclic complex of (finitely generated) projective modules; Gorenstein projective module; Gorenstein projective dimension; ~ vs. projective dimension; ~ over Noetherian ring; ~ of module.

Speaking informally, one determines the projective dimension of a complex M with a semi-projective replacement P by looking for projective cokernels in P above the supremum of M. The Gorenstein projective dimension is defined in much the same way, and the first step is to introduce the kind of cokernels to look for.

Gorenstein Projective Modules

9.1.1 Definition. A complex P of projective R-modules is called *totally acyclic*if it is acyclic and $\operatorname{Hom}_R(P, L)$ is acyclic for every projective R-module L.

An R-module G is called *Gorenstein projective* if one has $G \cong \mathrm{C}_0(P)$ for some totally acyclic complex P of projective R-modules.

Notice that if P is a totally acyclic complex of projective R-modules, then the module $\mathrm{C}_v(P)$ is Gorenstein projective for every $v \in \mathbb{Z}$.

9.1.2 Example. Every projective R-module P is Gorenstein projective as the disk complex $\mathrm{D}^0(P) = 0 \longrightarrow P \xrightarrow{=} P \longrightarrow 0$ is totally acyclic.

A colloquial phrasing of the next lemma could be: To a Gorenstein projective module, every projective module looks injective.

9.1.3 Lemma. *Let G be an R-module; it is Gorenstein projective if and only if it meets the following requirements:*

(1) *For every projective R-module L one has $\operatorname{Ext}_R^m(G,L) = 0$ for all $m > 0$.*
(2) *There exists an exact sequence of R-modules, $0 \to G \to P_0 \to P_{-1} \to \cdots$, where each P_v is projective and the sequence*

$$\cdots \longrightarrow \operatorname{Hom}_R(P_{-1},L) \longrightarrow \operatorname{Hom}_R(P_0,L) \longrightarrow \operatorname{Hom}_R(G,L) \longrightarrow 0$$

is exact for every projective R-module L.

Proof. Assume first that G is Gorenstein projective and let P be a totally acyclic complex of projective R-modules with $G \cong \mathrm{C}_0(P)$; see 9.1.1. The complex $P_{\subseteq 0}$ and the isomorphism $G \cong \mathrm{C}_0(P)$ yield, up to indexing, the exact sequence asserted in (2). The complex $P_{\geqslant 0}$ is a semi-projective replacement of G. For a projective R-module L and $m > 0$ the definition of Ext, 7.3.23, and total acyclicity of P yield

$$\operatorname{Ext}_R^m(G,L) = \mathrm{H}_{-m}(\operatorname{Hom}_R(P_{\geqslant 0},L)) = \mathrm{H}_{-m}(\operatorname{Hom}_R(P,L)) = 0\,.$$

Assuming now that G satisfies the two requirements, let $\pi\colon P' \xrightarrow{\simeq} G$ be a projective resolution, see 5.2.27, and denote by P the complex $0 \to P_0 \to P_{-1} \to \cdots$. The homomorphism $G \to P_0$ induces a quasi-isomorphism $\iota\colon G \to P$, so $\widetilde{P} = \Sigma^{-1}\operatorname{Cone}(\iota\pi)$ is by 4.1.1 and 4.2.16 an acyclic complex of projective R-modules with $G \cong \mathrm{C}_0(\widetilde{P})$. Let L be a projective R-module. By (2) the morphism $\operatorname{Hom}_R(\iota,L)$ is a quasi-isomorphism, and by (1) so is $\operatorname{Hom}_R(\pi,L)$, see 7.3.27. It follows that $\operatorname{Hom}_R(\iota\pi,L)$ is a quasi-isomorphism, whence the complex $\operatorname{Cone}\operatorname{Hom}_R(\iota\pi,L) \cong \operatorname{Hom}_R(\widetilde{P},L)$ is acyclic, see 4.1.17 and 4.2.16. Thus $\widetilde{P}$ is a totally acyclic complex of projective R-modules and G is Gorenstein projective. □

For later reference we prove the following result. It shows that 9.1.3(1) can be strengthened; however, it is strengthened even further in 9.1.9(c).

9.1.4 Lemma. *Let M be an R-module. The following conditions are equivalent.*

(*i*) $\operatorname{Ext}_R^m(M,L) = 0$ *for every projective R-module L and all $m > 0$.*
(*ii*) $\operatorname{Ext}_R^m(M,N) = 0$ *for every R-module N with $\operatorname{pd}_R N$ finite and all $m > 0$.*

Proof. Condition (*ii*) clearly implies (*i*). For the converse, induct on $d = \operatorname{pd}_R N$. The base case $d = 0$ is handled by the assumption (*i*). Now assume that $d > 0$ holds and that one has $\operatorname{Ext}_R^m(M,N') = 0$ for every R-module N' with $\operatorname{pd}_R N' < d$ and all $m > 0$. Let N be an R-module with $\operatorname{pd}_R N = d$ and consider an exact sequence $0 \to N' \to L \to N \to 0$ with L projective. By 8.1.9 one has $\operatorname{pd}_R N' < d$, and hence

$\operatorname{Ext}_R^m(M, N') = 0 = \operatorname{Ext}_R^m(M, L)$ holds for all $m > 0$. Application of the functor $\operatorname{Hom}_R(M, -)$ to $0 \to N' \to L \to N \to 0$ induces by 7.3.35 an exact sequence of Ext modules, which now shows that $\operatorname{Ext}_R^m(M, N) = 0$ holds for all $m > 0$. □

For complexes of finitely generated projective modules, total acyclicity has a couple of alternative characterizations that, if encountered in isolation, would appear to be weaker or stronger than the definition.

9.1.5 Lemma. *Let P be a complex of finitely generated projective R-modules. The following conditions are equivalent.*

(*i*) $\operatorname{Hom}_R(P, R)$ *is acyclic.*

(*ii*) $\operatorname{Hom}_R(P, F)$ *is acyclic for every flat R-module F.*

(*iii*) $E \otimes_R P$ *is acyclic for every injective R°-module E.*

Thus, if P is acyclic and satisfies these conditions, then P is totally acyclic.

Proof. The implication (*ii*) $\Rightarrow$ (*i*) is evident. Considering R as an R–R°-bimodule one has $\operatorname{Hom}_R(P, R) \otimes_R F \cong \operatorname{Hom}_R(P, F)$ for every R-module F by tensor evaluation 1.4.6 and the unitor 1.2.1. This shows that (*i*) implies (*ii*). One also has $E \otimes_R P \cong \operatorname{Hom}_{R^{\circ}}(R, E) \otimes_R P \cong \operatorname{Hom}_{R^{\circ}}(\operatorname{Hom}_R(P, R), E)$ for every R°-module E by the counitor 1.2.2, commutativity 1.2.3, and homomorphism evaluation 1.4.9. From this isomorphism it follows that (*i*) and (*iii*) are equivalent. □

9.1.6 Example. The Dold complex D from 2.1.23 is a totally acyclic complex of projective $\mathbb{Z}/4\mathbb{Z}$-modules; this follows from 9.1.5 as $\operatorname{Hom}_{\mathbb{Z}/4\mathbb{Z}}(D, \mathbb{Z}/4\mathbb{Z}) \cong D$. In particular, the module $\mathrm{C}_0(D) = (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is Gorenstein projective.

A class of modules that is closed under countable (co)powers and under kernels of surjective homomorphisms or cokernels of injective homomorphisms is also closed under summands. The proof uses the technique from Eilenberg's swindle 1.3.20.

9.1.7 Proposition. *Let $\mathcal{X}$ be a class of R-modules and consider these conditions*:

(1) *For every module $X \in \mathcal{X}$, the module $X^{(\mathbb{N})}$ or the module $X^{\mathbb{N}}$ belongs to $\mathcal{X}$.*

(2) *For every exact sequence $0 \to X' \to X \to X'' \to 0$ with $X, X'' \in \mathcal{X}$ also $X' \in \mathcal{X}$.*

(3) *For every exact sequence $0 \to X' \to X \to X'' \to 0$ with $X', X \in \mathcal{X}$ also $X'' \in \mathcal{X}$.*

Assume that $\mathcal{X}$ satisfies (1) *and one of the conditions* (2)–(3). *If $X \cong M \oplus N$ holds in $\mathcal{M}(R)$ and X belongs to $\mathcal{X}$, then M and N belong to $\mathcal{X}$.*

Proof. Let $X \in \mathcal{X}$ and assume that $X \cong M \oplus N$ holds in $\mathcal{M}(R)$. Recall from 1.1.21 that $M \oplus N$ is both a coproduct and a product. By (1) the module $Y = X^{(\mathbb{N})}$ or $Y = X^{\mathbb{N}}$ belongs to $\mathcal{X}$; in either case there is an isomorphism $Y \cong M \oplus Y$. Thus there are short exact sequences $0 \to M \to Y \to Y \to 0$ and $0 \to Y \to Y \to M \to 0$, and it follows from (2) or (3) that M belongs to $\mathcal{X}$. By symmetry N belongs to $\mathcal{X}$. □

The next proposition captures key features of the class of Gorenstein projective modules. The first assertion in part (a) together with 9.1.2 shows that the class is so-called projectively resolving.

9.1.8 Proposition. *The following assertions hold.*

(a) *Let* $0 \to G' \to G \to G'' \to 0$ *be an exact sequence of* R*-modules. If* G'' *is Gorenstein projective, then* G *is Gorenstein projective if and only if* G' *is so. If* G' *and* G *are Gorenstein projective, then* G'' *is Gorenstein projective if and only if* $\operatorname{Ext}^1_R(G'', L) = 0$ *holds for every projective* R*-module* L.

(b) *Let* $\{G^u\}_{u\in U}$ *be a family of* R*-modules. The module* $\coprod_{u\in U} G^u$ *is Gorenstein projective if and only if each* G^u *is Gorenstein projective. In particular, a direct summand of a Gorenstein projective* R*-module is Gorenstein projective.*

Proof. Let $0 \longrightarrow G' \xrightarrow{\alpha'} G \xrightarrow{\alpha} G'' \longrightarrow 0$ be an exact sequence of R-modules where G and G'' are Gorenstein projective. As G and G'' satisfy 9.1.3(1), so does G' by 7.3.35. As G and G'' satisfy 9.1.3(2) there exist complexes

$$P \;=\; 0 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow \cdots \qquad \text{and} \qquad P'' \;=\; 0 \longrightarrow P''_0 \longrightarrow P''_{-1} \longrightarrow \cdots$$

of projective R-modules and quasi-isomorphisms $\varepsilon\colon G \to P$ and $\varepsilon''\colon G'' \to P''$ such that $\operatorname{Hom}_R(\varepsilon, L)$ and $\operatorname{Hom}_R(\varepsilon'', L)$ are quasi-isomorphisms for every projective R-module L. In view of 4.1.17 and 4.2.16 the complex $\operatorname{Hom}_R(\operatorname{Cone}\varepsilon, L)$ is acyclic for every projective R-module L, so A.2 implies that $\operatorname{Hom}_R(\varepsilon, L)$ is a quasi-isomorphism for every complex L of projective R-modules. There is an exact sequence of R-complexes $0 \longrightarrow G \xrightarrow{\varepsilon} P \longrightarrow \operatorname{Coker}\varepsilon \longrightarrow 0$ where $\operatorname{Coker}\varepsilon = 0 \to \operatorname{Coker}\varepsilon_0 \to P_{-1} \to P_{-2} \to \cdots$. The module $\operatorname{Coker}\varepsilon_0$ is Gorenstein projective, so it follows from 7.3.39 that $\operatorname{Hom}_R(\varepsilon, L)$ is surjective for every complex L of projective R-modules. In particular, $\operatorname{Hom}_R(\varepsilon, P'')$ is a surjective quasi-isomorphism, and hence the map $\mathcal{C}(R)(\varepsilon, P'')\colon \mathcal{C}(R)(P, P'') \to \mathcal{C}(R)(G, P'')$ is surjective by 4.2.7 and 2.3.10. Thus there is a morphism of R-complexes $\beta\colon P \to P''$ with $\beta\varepsilon = \varepsilon''\alpha$. Let $\varpi\colon \operatorname{Cone} 1^{P''} \twoheadrightarrow \Sigma P''$ be the canonical morphism from 4.1.5. With $P' = \operatorname{Ker}(\beta\ \ \Sigma^{-1}\varpi)$ there is a commutative diagram in $\mathcal{C}(R)$ with exact rows,

$$(\dagger)\qquad \begin{tikzcd} 0 \arrow[r] & G' \arrow[r, "\alpha'"] \arrow[d, "\varepsilon'"] & G \arrow[r, "\alpha"] \arrow[d, "\underline{\varepsilon} = \left(\begin{smallmatrix}\varepsilon\\ 0\end{smallmatrix}\right)"] & G'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\ 0 \arrow[r] & P' \arrow[r] & \begin{matrix} P \\ \oplus \\ \Sigma^{-1}\operatorname{Cone} 1^{P''} \end{matrix} \arrow[r, "(\beta\ \ \Sigma^{-1}\varpi)"] & P'' \arrow[r] & 0\,, \end{tikzcd}$$

where ε' is the induced morphism. As the complex $\operatorname{Cone} 1^{P''}$ is contractible, see 4.3.31, the morphism $\underline{\varepsilon}$ is a quasi-isomorphism and so is $\operatorname{Hom}_R(\underline{\varepsilon}, L)$ for every projective R-module L. The complex P' consists of projective modules and is concentrated in degrees $\leqslant 0$ as this is the case for P, P'' and $\Sigma^{-1}\operatorname{Cone} 1^{P''}$; see 5.2.3.

As $\underline{\varepsilon}$ and ε'' are quasi-isomorphisms, so is ε' by 4.2.5 applied to the diagram (†). If L is a projective R-module, then application of the functor $\operatorname{Hom}_R(-, L)$ to (†) yields a commutative diagram with exact rows. This is because $\operatorname{Ext}^1_R(G'', L) = 0$ holds and because the bottom row in (†) is degreewise split exact. As $\operatorname{Hom}_R(\underline{\varepsilon}, L)$ and

$\operatorname{Hom}_R(\varepsilon'', L)$ are quasi-isomorphisms, so is $\operatorname{Hom}_R(\varepsilon', L)$ by another application of 4.2.5. This shows that the module G' satisfies 9.1.3(2), and hence we have proved that the class of Gorenstein projective modules is closed under kernels of surjective homomorphisms.

Next we show that the class of Gorenstein projective modules is closed under coproducts. Let $\{G^u\}_{u \in U}$ be a family of Gorenstein projective R-modules. By definition there exists for each $u \in U$ a totally acyclic complex P^u of projective R-modules with $G^u \cong \mathrm{C}_0(P^u)$. The complex $P = \coprod_{u \in U} P^u$ consists of projective modules by 1.3.24 and it is acyclic by 3.1.11. For every R-module L there is by 3.1.27 an isomorphism $\operatorname{Hom}_R(P, L) \cong \prod_{u \in U} \operatorname{Hom}_R(P^u, L)$, and this complex is acyclic if L is projective. Thus, P is a totally acyclic complex of projective R-modules. Since one has $\mathrm{C}_0(P) \cong \coprod_{u \in U} \mathrm{C}_0(P^u) \cong \coprod_{u \in U} G^u$ by 3.1.10(c), it follows that $\coprod_{u \in U} G^u$ is Gorenstein projective.

Having established these properties, it now follows from 9.1.7 that the class of Gorenstein projective modules is closed under direct summands. This proves (b).

We now finish the proof of (a). To show that the class of Gorenstein projective modules is closed under extensions, let $0 \to G' \to G \to G'' \to 0$ be an exact sequence where G' and G'' are Gorenstein projective. By the definition 9.1.1 of Gorenstein projective modules, there exists an exact sequence $0 \to G' \to P' \to G''' \to 0$ with P' projective and G''' Gorenstein projective. By 3.2.28 there is a commutative diagram with exact rows and columns,

$$(\ddagger) \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & G' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P' \arrow[r] \arrow[d] & P' \sqcup_{G'} G \arrow[r] \arrow[d] & G'' \arrow[r] & 0 \\ & G''' \arrow[r, equal] \arrow[d] & G''' \arrow[d] & & \\ & 0 & 0\,. & & \end{tikzcd}$$

As G'' is Gorenstein projective and P' is projective one has $\operatorname{Ext}^1_R(G'', P') = 0$. Thus the second row in ($\ddagger$) is split by 7.3.36, so one has $P' \sqcup_{G'} G \cong P' \oplus G''$. This module is Gorenstein projective by (b). Now the already established part of (a) applied to the second column in ($\ddagger$) shows that G is Gorenstein projective.

It remains to prove the final assertion in (a). The "only if" part is evident. For the converse, consider again the diagram ($\ddagger$). By assumption, P' is projective and G', G, and G''' are Gorenstein projective. The second column and the already established part of (a) show that $P' \sqcup_{G'} G$ is Gorenstein projective. As $\operatorname{Ext}^1_R(G'', P') = 0$ holds, by assumption, it follows from 7.3.36 that the second row is split. Hence part (b) shows that G'' is Gorenstein projective. □

Unlike projective modules, cf. 8.1.1, Gorenstein projective modules are not entirely characterized by Ext-vanishing. Theorem 9.1.31 comes as close as one can to such a characterization—see also the Remark following 9.1.32—and to prove it the next lemma is key.

9.1.9 Lemma. *Let N be an R-module; the following conditions are equivalent.*

(*i*) $\mathrm{Ext}^1_R(G,N) = 0$ *for every Gorenstein projective R-module G.*
(*ii*) $\mathrm{Ext}^m_R(G,N) = 0$ *for every Gorenstein projective R-module G and all $m > 0$.*

The class $\mathcal{N}$ of R-modules N satifying these conditions has the following properties.

(a) *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. If two of the modules N', N, and N'' belong to $\mathcal{N}$, then so does the third.*
(b) *For every family $\{N^u\}_{u\in U}$ of modules in $\mathcal{N}$ one has $\prod_{u\in U} N^u \in \mathcal{N}$.*
(c) *Every R-module N with $\mathrm{pd}_R N$ or $\mathrm{id}_R N$ finite belongs to $\mathcal{N}$.*

Proof. The implication (*ii*) $\Rightarrow$ (*i*) is trivial. For the converse, let G be a Gorenstein projective R-module and P a totally acyclic complex of projective R-modules with $\mathrm{C}_0(P) \cong G$. For $m > 0$ one has $\mathrm{Ext}^m_R(G,N) \cong \mathrm{Ext}^1_R(\mathrm{C}_{m-1}(P),N)$ by 8.1.6, so (*i*) implies (*ii*) as the module $\mathrm{C}_{m-1}(P)$ is Gorenstein projective.

(a): It follows from 7.3.35 that $N', N'' \in \mathcal{N}$ implies $N \in \mathcal{N}$, and that $N', N \in \mathcal{N}$ implies $N'' \in \mathcal{N}$. Now assume $N, N'' \in \mathcal{N}$. Another application of 7.3.35 yields $\mathrm{Ext}^m_R(G,N') = 0$ for every Gorenstein projective R-module G and all $m > 1$. To show that also $\mathrm{Ext}^1_R(G,N') = 0$ holds, note that every Gorenstein projective module G by definition fits into an exact sequence $0 \to G \to P \to G' \to 0$ with P projective and G' Gorenstein projective. Application of 7.3.35 to this sequence yields $\mathrm{Ext}^1_R(G,N') \cong \mathrm{Ext}^2_R(G',N')$, and the right-hand side is zero, as just shown.

(b): For every R-module G the functor $\mathrm{Ext}^1_R(G,-)$ preserves products by 7.3.33. Hence the assertion follows directly from the definition of the class $\mathcal{N}$.

(c): In view of part (a), it suffices to prove that $\mathcal{N}$ contains every projective and every injective R-module, and that is the case by 9.1.3(1) and 8.2.19. □

Gorenstein Projective Dimension

9.1.10 Definition. Let M be an R-complex. The *Gorenstein projective dimension* of M, written $\mathrm{Gpd}_R M$, is defined as

$$\mathrm{Gpd}_R M = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} \text{There exists a semi-projective replacement } P \text{ of } M \text{ with} \\ \mathrm{H}_v(P) = 0 \text{ for all } v > n \text{ and } \mathrm{C}_n(P) \text{ Gorenstein projective} \end{array} \right\}$$

with the convention $\inf \varnothing = \infty$. One says that $\mathrm{Gpd}_R M$ is *finite* if $\mathrm{Gpd}_R M < \infty$ holds.

A comment similar to the one after 8.1.2 justifies the last convention in 9.1.10.

9.1.11. Let M be an R-complex. For every semi-projective replacement P of M one has $\mathrm{H}(P) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\mathrm{Gpd}_R M \geqslant \sup M \quad \text{and} \quad \mathrm{Gpd}_R \Sigma^s M = \mathrm{Gpd}_R M + s \quad \text{for every integer } s\,.$$

Moreover, one has $\operatorname{Gpd}_R M = -\infty$ if and only if M is acyclic.

9.1.12 Lemma. *Let M be an R-complex. For every semi-projective replacement P of M and every integer $v \geqslant \operatorname{Gpd}_R M$ the module $\mathrm{C}_v(P)$ is Gorenstein projective.*

Proof. By 8.1.12 and 9.1.8 it suffices to prove the assertion for some specific semi-projective replacement P of M. One can assume that $\operatorname{Gpd}_R M$ is finite; otherwise the statement is empty. One can also assume that M is not acyclic; otherwise $P = 0$ is a semi-projective replacement of M. Thus $g = \operatorname{Gpd}_R M$ is an integer and by definition, 9.1.10, there is a semi-projective replacement P of M with $\mathrm{H}_v(P) = 0$ for all $v > g$ and $\mathrm{C}_g(P)$ Gorenstein projective. Since there are short exact sequences $0 \to \mathrm{C}_{v+1}(P) \to P_v \to \mathrm{C}_v(P) \to 0$ for all $v \geqslant g$, it follows from 9.1.2 and 9.1.8 that the modules $\mathrm{C}_g(P), \mathrm{C}_{g+1}(P), \dots$ are Gorenstein projective. $\square$

The next result is sometimes expressed by saying that Gpd_R is a refinement of pd_R. It follows, in particular, that a Gorenstein projective module is either projective or has infinite projective dimension. The Gorenstein projective module from 9.1.6 has infinite projective dimension; see 8.1.10.

9.1.13 Theorem. *Let M be an R-complex. There is an inequality,*

$$\operatorname{Gpd}_R M \leqslant \operatorname{pd}_R M ,$$

and equality holds if M has finite projective dimension.

Proof. The inequality is evident from the definitions of the dimensions, see 8.1.2 and 9.1.10, and from the fact that every projective module is Gorenstein projective, see 9.1.2. Now assume that $p = \operatorname{pd}_R M$ is an integer. To prove $\operatorname{Gpd}_R M \geqslant p$ it must be shown that if P is a semi-projective replacement of M with $\mathrm{H}_v(P) = 0$ for all $v > n$ and $\mathrm{C}_n(P)$ Gorenstein projective, then $n \geqslant p$ holds. Suppose one has $n < p$. There are exact sequences $0 \to \mathrm{C}_p(P) \to P_{p-1} \to \cdots \to P_{n+1} \to \mathrm{C}_{n+1}(P) \to 0$ and $0 \to \mathrm{C}_{n+1}(P) \to P_n \to \mathrm{C}_n(P) \to 0$. The first sequence shows that the module $\mathrm{C}_{n+1}(P)$ has finite projective dimension, as $\mathrm{C}_p(P)$ is projective by 8.1.8, and hence 7.3.36 and 9.1.9(c) imply that the latter sequence is split. Thus, $\mathrm{C}_n(P)$ is a direct summand of P_n, in particular, $\mathrm{C}_n(P)$ is projective. Now another application of 8.1.8 yields $p \leqslant n$, which is a contradiction. Thus $n \geqslant p$ holds as desired. $\square$

Equality also holds in 9.1.13 if M has finite injective dimension; see 9.1.20. By 6.5.24 the next result applies, in particular, to a short exact sequence of complexes.

9.1.14 Proposition. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $g' = \operatorname{Gpd}_R M'$, $g = \operatorname{Gpd}_R M$, and $g'' = \operatorname{Gpd}_R M''$ there are inequalities,*

$$g' \leqslant \max\{g, g'' - 1\}\,, \quad g \leqslant \max\{g', g''\}\,, \quad \textit{and} \quad g'' \leqslant \max\{g' + 1, g\}\,.$$

In particular, if two of the complexes M', M, and M'' have finite Gorenstein projective dimension, then so has the third.

Proof. It suffices to prove the second inequality since the first and third inequalities follow by applying the second inequality and 9.1.11 to the distinguished triangles $\Sigma^{-1}M'' \to M' \to M \to M''$ and $M \to M'' \to \Sigma M' \to \Sigma M$; see (TR2) in E.2.

To prove the second inequality, apply 8.1.13 to get an exact sequence of complexes $0 \to P' \to P \to P'' \to 0$ where P', P, and P'' are semi-projective replacements of M', M, and M''. Set $s' = \sup M'$, $s = \sup M$, and $s'' = \sup M''$. One can assume that g' and g'' are finite and that M is not acyclic; otherwise the inequality is trivial. It follows from 6.5.20 that M' and M'' can not both be acyclic, so g' or g'' is an integer, and hence so is $m = \max\{g', g''\}$. Note that 9.1.11 and 6.5.20 yield $m \geqslant s$, so $\mathrm{H}_v(P) = 0$ for $v > m$. As $m + 1 > g'' \geqslant s''$ one has $\mathrm{H}_{m+1}(P'') = 0$, so the sequence $0 \to \mathrm{C}_m(P') \to \mathrm{C}_m(P) \to \mathrm{C}_m(P'') \to 0$ is exact by 2.2.16. As $m \geqslant g'$ and $m \geqslant g''$ the modules $\mathrm{C}_m(P')$ and $\mathrm{C}_m(P'')$ are Gorenstein projective by 9.1.12, and hence so is $\mathrm{C}_m(P)$ by 9.1.8. Thus $g = \operatorname{Gpd}_R M \leqslant m$ holds by 9.1.10. □

Remark. Proposition 9.1.14 essentially shows that the complexes of finite Gorenstein projective dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 9.1.5.

A module of finite Gorenstein projective dimension can be approximated by a Gorenstein projective module and one of finite projective dimension. We derive this as a consequence of the next result about approximations in the derived category.

9.1.15 Proposition. *Let M be an R-complex of finite Gorenstein projective dimension $g = \operatorname{Gpd}_R M$. For every semi-projective replacement P of M and every integer u with $g > u$ there is a distinguished triangle in $\mathcal{D}(R)$,*

$$K \longrightarrow M \longrightarrow N \longrightarrow \Sigma K\,,$$

where the complexes K and N have the following properties:

(a) *There is a degreewise split exact sequence $0 \to P_{\leqslant u} \to K \to \Sigma^u G \to 0$ in $\mathcal{C}(R)$ where G is a Gorenstein projective R-module. Furthermore, one has*

$$\operatorname{Gpd}_R K \leqslant u \qquad \textit{and} \qquad \mathrm{H}_v(K) \cong \begin{cases} 0 & \textit{for } v \geqslant u+1 \\ \mathrm{H}_v(M) & \textit{for } v \leqslant u-1\,. \end{cases}$$

(b) *The complex N satisfies*

$$\operatorname{pd}_R N = g \qquad \textit{and} \qquad \mathrm{H}_v(N) \cong \begin{cases} \mathrm{H}_v(M) & \textit{for } v \geqslant u+2 \\ 0 & \textit{for } v \leqslant u\,. \end{cases}$$

(c) *There is an exact sequence of R-modules,*

$$0 \longrightarrow \mathrm{H}_{u+1}(M) \longrightarrow \mathrm{H}_{u+1}(N) \longrightarrow \mathrm{H}_u(K) \longrightarrow \mathrm{H}_u(M) \longrightarrow 0\,.$$

Proof. If M is acyclic the statement is void as no integer u satisfies $-\infty = g > u$. Now assume that M is not acyclic, in which case g is an integer. By 9.1.12 the module $\mathrm{C}_g(P)$ is Gorenstein projective and by 9.1.1 there exists an acyclic complex $L = 0 \to \mathrm{C}_g(P) \to L_{g-1} \cdots \to L_u \to G \to 0$, concentrated in degrees $g, \dots, u-1$, where the modules $L_{g-1}, \dots, L_u$ are projective and the cokernels are Gorenstein projective. Set $L_g = \mathrm{C}_g(P)$ and $L_{u-1} = G$ and notice that, in particular, the cokernel $\mathrm{C}_u(L) \cong G$ is Gorenstein projective.

Consider the short exact sequence of R-complexes,

$$(\dagger) \qquad 0 \longrightarrow \Sigma^{g-1}\mathrm{C}_g(P) \xrightarrow{\alpha} L_{\leqslant g-1} \longrightarrow L_{\subseteq g-1} \longrightarrow 0\,,$$

where α is a quasi-isomorphism as $L_{\subseteq g-1}$ is acyclic; see 4.2.6. Let F be any complex of projective R-modules. As the complex $L_{\subseteq g-1}$ consists of Gorenstein projective modules, it follows from 7.3.39 and 9.1.9(c) that the functor $\mathrm{Hom}_R(-,F)$ leaves the sequence ($\dagger$) exact. Since $L_{\subseteq g-1}$ is an acyclic complex with $\mathrm{C}_v(L_{\subseteq g-1})$ Gorenstein projective for every v, it follows from 9.1.9(c) and A.1 that $\mathrm{Hom}_R(L_{\subseteq g-1}, F_n)$ is acyclic for every $n \in \mathbb{Z}$, and hence $\mathrm{Hom}_R(L_{\subseteq g-1}, F)$ is acyclic by A.2. It now follows from 4.2.6 that $\mathrm{Hom}_R(\alpha, F)$ is a surjective quasi-isomorphism, whence the morphism $\mathcal{C}(R)(\alpha, F)$ is surjective as well by 4.2.7 and 2.3.10. Surjectivity of $\mathcal{C}(R)(\alpha, P_{\leqslant g-1})$ yields a commutative diagram of R-complexes,

$$\begin{tikzcd}
\Sigma^{g-1}\mathrm{C}_g(P) \arrow[r, "\beta"] \arrow[d, "\alpha"', "\simeq"] & P_{\leqslant g-1} \arrow[d, equal] \\
L_{\leqslant g-1} \arrow[r, dotted, "\gamma"] & P_{\leqslant g-1}\,,
\end{tikzcd}$$

where β is induced by ∂_g^P. This diagram—in conjunction with the definition of distinguished triangles in $\mathcal{D}(R)$, see 6.2.3 and 6.5.5, the axiom (TR3) in E.2, and 6.5.19—shows that the complexes $\operatorname{Cone}\beta$ and $\operatorname{Cone}\gamma$ are isomorphic in $\mathcal{D}(R)$. Evidently, one has $\operatorname{Cone}\beta = P_{\subseteq g}$, and this complex is isomorphic to $P \simeq M$ in $\mathcal{D}(R)$. Consequently, $\operatorname{Cone}\gamma \simeq M$ holds in $\mathcal{D}(R)$.

Set $C = \operatorname{Cone}\gamma$. By 2.5.22 and 6.5.24 there is a distinguished triangle in $\mathcal{D}(R)$,

$$C_{\leqslant u} \longrightarrow C \longrightarrow C_{\geqslant u+1} \longrightarrow \Sigma C_{\leqslant u}\,,$$

which we argue is the desired one. As already noticed, $C \simeq M$ in $\mathcal{D}(R)$. The complex

$$K = C_{\leqslant u} = 0 \longrightarrow P_u \oplus G \longrightarrow P_{u-1} \longrightarrow P_{u-2} \longrightarrow \cdots$$

fits into the degreewise split exact sequence in $\mathcal{C}(R)$,

$$(\ddagger) \qquad 0 \longrightarrow P_{\leqslant u} \longrightarrow K \longrightarrow \Sigma^u G \longrightarrow 0\,.$$

The complex P is semi-projective, and so is $P_{\geqslant u+1}$ by 5.2.8. Hence 5.2.17 applied to the exact sequence $0 \to P_{\leqslant u} \to P \to P_{\geqslant u+1} \to 0$ shows that $P_{\leqslant u}$ is semi-projective. It follows from 8.1.2 and 9.1.13 that $\mathrm{Gpd}_R(P_{\leqslant u}) = \mathrm{pd}_R(P_{\leqslant u}) \leqslant u$ holds. As G is a Gorenstein projective module, one has $\mathrm{Gpd}_R\,\Sigma^u G \leqslant u$, with equality if G is non-zero, so application of 6.5.24 and 9.1.14 to ($\ddagger$) shows that $\mathrm{Gpd}_R\,K \leqslant u$. The assertion about the homology of $K = C_{\leqslant u}$ is immediate as $C \simeq M$ in $\mathcal{D}(R)$. Note that

$$N = C_{\geqslant u+1} = 0 \longrightarrow L_{g-1} \longrightarrow P_{g-1} \oplus L_{g-2} \longrightarrow \cdots \longrightarrow P_{u+1} \oplus L_u \longrightarrow 0$$

is a complex of projective R-modules concentrated in degrees $g, \dots, u+1$. In the extremal case $u = g-1$ one has $N = \Sigma^g L_{g-1}$. In particular, N is semi-projective by 5.2.8 and $\mathrm{pd}_R\,N \leqslant g$ holds. Note that 9.1.14 yields $g \leqslant \max\{\mathrm{Gpd}_R\,K, \mathrm{Gpd}_R\,N\}$.

As $\operatorname{Gpd}_R K \leqslant u < g$ holds one has $\operatorname{Gpd}_R N \geqslant g$, so 9.1.13 yields $\operatorname{pd}_R N = g$. The assertion about the homology of $N = C_{\geqslant u+1}$ is immediate as $C \simeq M$ in $\mathcal{D}(R)$. The exact sequence in part (c) follows by applying 6.5.19 to the constructed distinguished triangle. $\square$

9.1.16 Corollary. *Let M be an R-module of finite Gorenstein projective dimension $g = \operatorname{Gpd}_R M$. The following assertions hold.*

(a) *There is an exact sequence of R-modules $0 \to M \to X \to G \to 0$ where G is Gorenstein projective and $\operatorname{pd}_R X = g$.*

(b) *If $g > 0$, then there is an exact sequence of R-modules $0 \to X \to G \to M \to 0$ where G is Gorenstein projective and $\operatorname{pd}_R X = g - 1$.*

Proof. (a): For $u = -1$ the sequence 9.1.15(c) reads

$$0 \longrightarrow M \longrightarrow \mathrm{H}_0(N) \longrightarrow \mathrm{H}_{-1}(K) \longrightarrow 0\,.$$

If follows from 9.1.15(a) that K is isomorphic to $\Sigma^{-1}\mathrm{H}_{-1}(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gpd}_R \mathrm{H}_{-1}(K) - 1 = \operatorname{Gpd}_R K \leqslant u = -1$. Consequently, the module $\mathrm{H}_{-1}(K)$ is Gorenstein projective. Similarly, one has $N \simeq \mathrm{H}_0(N)$ and $\operatorname{pd}_R \mathrm{H}_0(N) = \operatorname{pd}_R N = g$.

(b): As $g > 0$ one can apply 9.1.15(c) with $u = 0$ to obtain the exact sequence

$$0 \longrightarrow \mathrm{H}_1(N) \longrightarrow \mathrm{H}_0(K) \longrightarrow M \longrightarrow 0\,.$$

If follows from 9.1.15(a) that K is isomorphic to $\mathrm{H}_0(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gpd}_R \mathrm{H}_0(K) = \operatorname{Gpd}_R K \leqslant u = 0$. That is, the module $\mathrm{H}_0(K)$ is Gorenstein projective. Similarly, one has $N \simeq \Sigma\mathrm{H}_1(N)$ and $\operatorname{pd}_R \mathrm{H}_1(N) + 1 = \operatorname{pd}_R N = g$. $\square$

The utility of the following, technical, result becomes clear in 9.1.18.

9.1.17 Proposition. *Let P and L be semi-projective R-complexes. If L has finite projective or finite injective dimension, then the morphism*

$$\operatorname{Hom}_R(\tau^P_{\subseteq n}, L)\colon\ \operatorname{Hom}_R(P_{\subseteq n}, L) \longrightarrow \operatorname{Hom}_R(P, L)$$

is a quasi-isomorphism for every integer $n \geqslant \operatorname{Gpd}_R P$.

Proof. Let $F \xrightarrow{\simeq} L$ be a semi-projective resolution; by 5.2.21 there is a homotopy equivalence $\lambda\colon L \to F$. By 9.1.11 one has $n \geqslant \sup P$, so the map $\tau^P_{\subseteq n}\colon P \to P_{\subseteq n}$ is a quasi-isomorphism by 4.2.4. Choose a semi-injective resolution $\iota\colon F \xrightarrow{\simeq} I$. In the commutative diagram,

$$\begin{tikzcd}
\operatorname{Hom}_R(P_{\subseteq n}, L) \arrow[r, "\operatorname{Hom}(P_{\subseteq n},\lambda)", "\approx"'] \arrow[d, "\operatorname{Hom}(\tau^P_{\subseteq n},L)"] & \operatorname{Hom}_R(P_{\subseteq n}, F) \arrow[r, "\operatorname{Hom}(P_{\subseteq n},\iota)"] \arrow[d, "\operatorname{Hom}(\tau^P_{\subseteq n},F)"] & \operatorname{Hom}_R(P_{\subseteq n}, I) \arrow[d, "\operatorname{Hom}(\tau^P_{\subseteq n},I)", "\simeq"'] \\
\operatorname{Hom}_R(P, L) \arrow[r, "\operatorname{Hom}(P,\lambda)", "\approx"'] & \operatorname{Hom}_R(P, F) \arrow[r, "\operatorname{Hom}(P,\iota)", "\simeq"'] & \operatorname{Hom}_R(P, I)\,,
\end{tikzcd}$$

the left-hand horizontal maps are homotopy equivalences by 4.3.19 while $\operatorname{Hom}_R(P,\iota)$ and $\operatorname{Hom}_R(\tau^P_{\subseteq n}, I)$ are quasi-isomorphisms by semi-projectivity of P and semi-injectivity of I. The diagram shows that it suffices to verify that $\operatorname{Hom}_R(P_{\subseteq n},\iota)$ is a quasi-isomorphism. Set $C = \operatorname{Cone}\iota$ and notice that it is an acyclic complex of modules that are direct sums of projective and injective modules. The goal is to show that the complex $\operatorname{Hom}_R(P_{\subseteq n}, C) \cong \operatorname{Cone}\operatorname{Hom}_R(P_{\subseteq n}, \iota)$ is acyclic; cf. 4.1.16.

By 2.5.24 there is an exact sequence $0 \to P' \to P \to P_{\subseteq n} \to 0$ with P' bounded below. By 9.1.2, 9.1.12, and 9.1.8(a) these are complexes of Gorenstein projective modules, and it follows from 7.3.39 and 9.1.9(c) that the sequence

$$0 \longrightarrow \operatorname{Hom}_R(P_{\subseteq n}, C) \longrightarrow \operatorname{Hom}_R(P, C) \longrightarrow \operatorname{Hom}_R(P', C) \longrightarrow 0$$

is exact. The middle complex is acyclic as P is semi-projective and C is acyclic. To prove acyclicity of $\operatorname{Hom}_R(P_{\subseteq n}, C)$ it suffices to argue that $\operatorname{Hom}_R(P', C)$ is acyclic; see 2.5.6. To that end, it suffices by A.5 to show that $\operatorname{Hom}_R(G, C)$ is acyclic for every Gorenstein projective module G. It is already known from 9.1.9(c) that $\operatorname{Ext}^m_R(G, C_v) = 0$ holds for all $v \in \mathbb{Z}$ and all $m > 0$, so it suffices by A.4 to argue that $\operatorname{Ext}^m_R(G, \mathrm{Z}_v(C)) = 0$ holds for $v \gg 0$ and all $m > 0$.

If L has finite projective dimension, then one can assume that the complexes F and I are bounded above; see 8.1.2 and 5.3.26. It follows that C is bounded above, in particular, $\mathrm{Z}_v(C) = 0$ holds for $v \gg 0$.

If L has finite injective dimension, then one can assume that the complexes I and F are bounded below; see 8.2.2 and 5.2.15. It follows that C is bounded below; in particular, $\mathrm{Z}_v(C) = 0$ holds for for $v \ll 0$. Let G be a Gorenstein projective R-module. For every $v \in \mathbb{Z}$ one has $\operatorname{Ext}^m_R(G, C_v) = 0$ for all $m > 0$, so in view of 9.1.9(a), induction on the exact sequences $0 \to \mathrm{Z}_v(C) \to C_v \to \mathrm{Z}_{v-1}(C) \to 0$ yields $\operatorname{Ext}^m_R(G, \mathrm{Z}_v(C)) = 0$ for all $v \in \mathbb{Z}$ and all $m > 0$. □

The gist of 9.1.3 is that, in homological terms, projective and injective modules interact with Gorenstein projective modules in the same way. This has the following useful consequence:

9.1.18 Corollary. *Let M be an R-complex of finite Gorenstein projective dimension and N an R-complex of finite projective or finite injective dimension. For every semi-projective replacement P of M, every semi-projective replacement L of N, and every integer $n \geqslant \operatorname{Gpd}_R M$ there is an isomorphism in $\mathcal{D}(\Bbbk)$,*

$$\mathsf{R}\!\operatorname{Hom}_R(M, N) \simeq \operatorname{Hom}_R(P_{\subseteq n}, L)\,.$$

Proof. The assertion follows immediately from 9.1.17 and 7.3.7. □

A key difference between the next theorem and the main theorem about projective dimension, 8.1.8, is the a priori assumption that the complex has finite Gorenstein projective dimension. See also the Remark after 9.1.32.

9.1.19 Theorem. *Let M be an R-complex of finite Gorenstein projective dimension and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{Gpd}_R M \leqslant n$.

(*ii*) $-\inf \mathsf{R}\!\operatorname{Hom}_R(M, N) \leqslant n - \inf N$ *holds for every R-complex N with $\operatorname{pd}_R N$ finite or $\operatorname{id}_R N$ finite.*

(*iii*) $-\inf \mathsf{R}\!\operatorname{Hom}_R(M, N) \leqslant n$ *holds for every projective R-module N.*

(*iv*) *$n \geqslant \sup M$ and $\operatorname{Ext}_R^{n+1}(M,N) = 0$ for every R-module N with $\operatorname{pd}_R N$ finite.*
(*v*) *$n \geqslant \sup M$ and for some, equivalently every, semi-projective replacement P of M, the module $\mathrm{C}_v(P)$ is Gorenstein projective for every $v \geqslant n$.*
(*vi*) *There is a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < \inf M$, $\mathrm{H}_v(P) = 0$ for all $v > n$, and $\mathrm{C}_v(P)$ Gorenstein projective for all $v \geqslant n$.*

In particular, there are equalities

$$\begin{aligned}\operatorname{Gpd}_R M &= \sup\{-\inf \mathsf{R}\!\operatorname{Hom}_R(M,N) \mid N \text{ is a projective } R\text{-module}\} \\ &= \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(M,N) \neq 0 \text{ for some projective } R\text{-module } N\}\,.\end{aligned}$$

Proof. We start by establishing the equivalence of (*i*), (*ii*), and (*iii*).

(*i*) $\Rightarrow$ (*ii*): One can assume that N is in $\mathcal{D}_{\sqsupset}(R)$ and not acyclic; otherwise the inequality is trivial. In this case, $u = \inf N$ is an integer. By 5.2.15 there is a semi-projective resolution $L \xrightarrow{\simeq} N$ with $L_v = 0$ for $v < u$. If $\operatorname{pd}_R N$ or $\operatorname{id}_R N$ is finite, then 9.1.18 yields $\mathsf{R}\!\operatorname{Hom}_R(M,N) \simeq \operatorname{Hom}_R(P_{\subseteq n}, L)$, where P is any semi-projective replacement of M. For every $v < u - n$ and $p \in \mathbb{Z}$ one of the inequalities $p > n$ or $p + v \leqslant n + v < u$ holds, so the module

$$\operatorname{Hom}_R(P_{\subseteq n}, L)_v = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R((P_{\subseteq n})_p, L_{p+v})$$

is zero. In particular, $\mathrm{H}_v(\mathsf{R}\!\operatorname{Hom}_R(M,N)) = 0$ holds for $v < u - n = \inf N - n$.

(*ii*) $\Rightarrow$ (*iii*): Trivial.

(*iii*) $\Rightarrow$ (*i*): By assumption $g = \operatorname{Gpd}_R M$ is finite, and it must be shown that (*iii*) implies $g \leqslant n$. One can assume that M is not acyclic as otherwise the inequality is trivial. By definition there exists a semi-projective replacement P of M with $\mathrm{C}_g(P)$ Gorenstein projective. By 9.1.18 there is an isomorphism,

$$(\flat) \qquad \mathsf{R}\!\operatorname{Hom}_R(M,L) \simeq \operatorname{Hom}_R(P_{\subseteq g}, L)\,,$$

for every projective R-module L. Recall from 9.1.11 that $g \geqslant \sup M = \sup P$. We consider two different cases:

First assume that $g = \sup P$ holds; this implies $\mathrm{H}_g(P) \neq 0$ so the homomorphism $\mathrm{C}_g(P) \to P_{g-1}$ is not injective. Since $\mathrm{C}_g(P)$ is Gorenstein projective there exists, in particular, an embedding $\mathrm{C}_g(P) \rightarrowtail L$ where L is a projective module. This map does not admit a factorization $\mathrm{C}_g(P) \to P_{g-1} \to L$, as this would force $\mathrm{C}_g(P) \to P_{g-1}$ to be injective. Thus the map $\operatorname{Hom}_R(P_{g-1}, L) \to \operatorname{Hom}_R(\mathrm{C}_g(P), L)$ is not surjective and hence $\inf \operatorname{Hom}_R(P_{\subseteq g}, L) = -g$. Now ($\flat$) and (*iii*) yield $g = -\inf \mathsf{R}\!\operatorname{Hom}_R(M,L) \leqslant n$.

Next assume that $g > \sup P$. In this case there is a short exact sequence of modules, $0 \to \mathrm{C}_g(P) \to P_{g-1} \to \mathrm{C}_{g-1}(P) \to 0$. As $g > \sup P$ and $g = \operatorname{Gpd}_R M$, the module $\mathrm{C}_{g-1}(P)$ is not Gorenstein projective. Hence 9.1.8 yields $\operatorname{Ext}_R^1(\mathrm{C}_{g-1}(P), L) \neq 0$ for some projective module L, and by 8.1.6 this means that $\mathrm{H}_{-g}(\mathsf{R}\!\operatorname{Hom}_R(M,L)) \neq 0$. Hence $g \leqslant -\inf \mathsf{R}\!\operatorname{Hom}_R(M,L) \leqslant n$, where the last inequality holds by (*iii*).

To finish the proof we show the implications (*ii*) $\Rightarrow$ (*iv*) $\Rightarrow$ (*v*) $\Rightarrow$ (*vi*) $\Rightarrow$ (*i*).

(*ii*) $\Rightarrow$ (*iv*): The second assertion in (*iv*) is immediate from (*ii*). The inequality $n \geqslant \sup M$ follows, in view of 9.1.11, from (*i*), which is equivalent to (*ii*).

(*iv*) $\Rightarrow$ (*v*): First note that by 8.1.12 and 9.1.8 the "some" version and the "every" version of condition (*v*) are equivalent. By assumption, $g = \operatorname{Gpd}_R M$ is finite, so in any semi-projective replacement P of M the module $\mathrm{C}_v(P)$ is Gorenstein projective for every integer $v \geqslant g$; see 9.1.12. Thus, to show (*v*) it is enough to prove $n \geqslant g$. Assume towards a contradiction that $n < g$ holds. Notice that by the assumption $n \geqslant \sup M$, the module $\mathrm{C}_n(P)$ can not be Gorenstein projective. There is an exact sequence $0 \to \mathrm{C}_g(P) \to P_{g-1} \to \cdots \to P_n \to \mathrm{C}_n(P) \to 0$, which shows that $\mathrm{C}_n(P)$ has finite Gorenstein projective dimension, as $P_{\geqslant n}$ is a semi-projective replacement of $\Sigma^n \mathrm{C}_n(P)$. Now 9.1.16 yields an exact sequence $0 \to N \to G \to \mathrm{C}_n(P) \to 0$ where G is Gorenstein projective and N has finite projective dimension. By 8.1.6 and (*iv*) one has $\mathrm{Ext}^1_R(\mathrm{C}_n(P), N) \cong \mathrm{Ext}^{n+1}_R(M, N) = 0$, so the sequence is split by 7.3.36. Now 9.1.8 implies that $\mathrm{C}_n(P)$ is Gorenstein projective, which is a contradiction.

(*v*) $\Rightarrow$ (*vi*): This implication is immediate in view of 5.2.15.

(*vi*) $\Rightarrow$ (*i*): This implication is immediate from the definition, 9.1.10, of Gpd_R.

The equalities in the last assertion follow immediately from the equivalence of conditions (*i*)–(*iii*) and 7.3.24. □

One can take the next result as another manifestation of the inability of Gorenstein projective modules to distinguish between injective and projective modules.

9.1.20 Theorem. *Let M be an R-complex. If M has finite injective dimension, then the equality* $\mathrm{Gpd}_R M = \mathrm{pd}_R M$ *holds.*

Proof. The inequality $\mathrm{Gpd}_R M \leqslant \mathrm{pd}_R M$ holds by 9.1.13. To show the opposite inequality, one can assume that $g = \mathrm{Gpd}_R M$ is an integer. Set $u = \inf M - 1$, which is an integer by 8.2.3, and note that one has $u < \inf M \leqslant \sup M \leqslant g$ by 9.1.11. Thus 9.1.15 yields a distinguished triangle in $\mathcal{D}(R)$,

$$(\dagger) \qquad K \longrightarrow M \longrightarrow N \longrightarrow \Sigma K\,,$$

with $\mathrm{Gpd}_R K \leqslant u$ and $\mathrm{pd}_R N = g$. As $\mathrm{id}_R M$ is finite, 9.1.19 yields

$$-\inf \mathsf{RHom}_R(K, M) \;\leqslant\; \mathrm{Gpd}_R K - \inf M \;\leqslant\; u - \inf M \;=\; -1\,,$$

and hence $\mathrm{H}_0(\mathsf{RHom}_R(K, M)) = 0$. By 7.3.26 this means that $\mathcal{D}(R)(K, M) = 0$, in particular, the morphism $K \longrightarrow M$ in the distinguished triangle above is zero. By E.22 this means the triangle (†) is split, and hence $N \simeq M \oplus \Sigma K$ holds in $\mathcal{D}(R)$. In particular, one has $\mathrm{pd}_R M \leqslant \mathrm{pd}_R N = g$ as claimed. □

9.1.21 Proposition. *Let $\{M^u\}_{u \in U}$ be a family of R-complexes*; *there is an equality,*

$$\mathrm{Gpd}_R\Big(\coprod_{u \in U} M^u\Big) \;=\; \sup_{u \in U}\{\mathrm{Gpd}_R M^u\}\,.$$

Proof. To prove the inequality "$\leqslant$", one can assume that the right-hand side is finite, say, $s \in \mathbb{Z}$. By the definition, 9.1.10, of Gorenstein projective dimension and by 9.1.12 every M^u admits a semi-projective replacement P^u with $\mathrm{H}_v(P^u) = 0$ for all $v > s$ and $\mathrm{C}_v(P^u)$ Gorenstein projective for all $v \geqslant s$. Now $P = \coprod_{u \in U} P^u$ is a semi-projective replacement of $\coprod_{u \in U} M^u$ with $\mathrm{H}_v(P) = 0$ for all $v > s$, see 5.2.18 and 3.1.11, and $\mathrm{C}_v(P) = \coprod_{u \in U} \mathrm{C}_v(P^u)$ Gorenstein projective for all $v \geqslant s$, see 9.1.8.

To prove the opposite inequality "$\geqslant$" it suffices, as each M^u is a direct summand of $\coprod_{u \in U} M^u$, to argue that if M' is a direct summand of an R-complex M, then one has $\operatorname{Gpd}_R M' \leqslant \operatorname{Gpd}_R M$. To this end, one can assume that M is not acyclic and that $g = \operatorname{Gpd}_R M$ is finite. Let M'' be an R-complex with $M = M' \oplus M''$. Let P' and P'' be semi-projective replacements of M' and M''. Now $P = P' \oplus P''$ is a semi-projective replacement of M, see 5.2.18. As $\mathrm{H}_v(P) = 0$ holds for every $v > g$, even for every $v > \sup M$, one has $\mathrm{H}_v(P') = 0$ for every $v > g$ by 3.1.10(d). It follows from 9.1.12 that the module $\mathrm{C}_v(P) = \mathrm{C}_v(P') \oplus \mathrm{C}_v(P'')$ is Gorenstein projective for every $v \geqslant g$, whence $\mathrm{C}_v(P')$ is Gorenstein projective for $v \geqslant g$ by 9.1.8 and $\operatorname{Gpd}_R M' \leqslant g$ holds. □

Noetherian Rings and Homological Finiteness

Like the projective dimension, which it refines, the Gorenstein projective dimension supports stronger statements about finitely generated moules.

9.1.22 Lemma. *Let G be a finitely generated R-module. If G is Gorenstein projective, then there exists an exact sequence $0 \to G \to F \to G' \to 0$ of finitely generated R-modules with F free and G' Gorenstein projective.*

Proof. By assumption there exists a totally acyclic complex P of projective R-modules with $G \cong \mathrm{C}_0(P)$. Without loss of generality, one can assume that the module P_{-1} is free. Indeed, there exists a projective module P' such that $P_{-1} \oplus P'$ is free, see 1.3.17, and the complex $\widetilde{P} = P \oplus \Sigma^{-1}(P')$ is totally acyclic with $\mathrm{C}_0(\widetilde{P}) = \mathrm{C}_0(P)$; cf. 9.1.2. As P is acyclic, there is an injective homomorphism $\iota \colon G \to P_{-1}$, and since G is finitely generated, the image of ι is contained in a finitely generated free submodule F of P_{-1}. It remains to show that $G' = F/\operatorname{Im}\iota$ is Gorenstein projective, and to this end it suffices by 9.1.8 to show that $\operatorname{Ext}^1_R(G', L)$ is zero for every projective R-module L. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & P_{-1} \arrow[r] & \mathrm{C}_{-1}(P) \arrow[r] & 0
\end{tikzcd}$$

yields for every projective R-module L a commutative diagram with exact rows,

$$\begin{tikzcd}
\operatorname{Hom}_R(P_{-1}, L) \arrow[r] \arrow[d] & \operatorname{Hom}_R(G, L) \arrow[r] \arrow[d, equal] & \operatorname{Ext}^1_R(\mathrm{C}_{-1}(P), L) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_R(F, L) \arrow[r] & \operatorname{Hom}_R(G, L) \arrow[r] & \operatorname{Ext}^1_R(G', L) \arrow[r] & 0\,;
\end{tikzcd}$$

see 7.3.35 and 7.3.27. As $\operatorname{Hom}_R(P, L)$ is acyclic, one has $\operatorname{Ext}^1_R(\mathrm{C}_{-1}(P), L) = 0$, and it follows that that $\operatorname{Ext}^1_R(G', L)$ vanishes as well. □

The significance of the next result derives in no small part from 9.1.5. The effects can be seen in 9.1.24 and 9.1.27–9.1.29, which compare to 9.1.9 and 9.1.17–9.1.19.

9.1.23 Proposition. *Assume that R is left Noetherian. A finitely generated R-module G is Gorenstein projective if and only if there exists a totally acyclic complex P of finitely generated free R-modules with $G \cong \mathrm{C}_0(P)$.*

Proof. The "if" statement evident. To prove the converse, assume that G is Gorenstein projective. It follows from 9.1.22 that there is an injectiv quasi-isomorphism $\iota\colon G \to F$, where F is a complex of finitely generated free R-modules with $F_v = 0$ for $v > 0$ and $\mathrm{C}_v(F)$ Gorenstein projective for every $v \in \mathbb{Z}$. By 5.1.19 there is a free resolution $\pi\colon L \xrightarrow{\simeq} G$ where L is degreewise finitely generated. The complex $P = \Sigma^{-1}\mathrm{Cone}(\iota\pi)$ is a complex of finitely generated free R-modules with $\mathrm{C}_0(P) = \mathrm{C}_0(L) \cong G$. Moreover, for $v \ll 0$ the module $\mathrm{C}_v(P) = \mathrm{C}_{v+1}(F)$ is Gorenstein projective, so by 9.1.9(c) and A.1 the complex $\mathrm{Hom}_R(P, L)$ is acyclic for every projective module L. That is, P is totally acyclic. □

Remark. Due to Avramov and Martsinkovsky [27], finitely generated Gorenstein projective modules over Noetherian rings are now commonly called 'totally reflexive' modules; see also 10.4.13. The terminology originally used by Auslander and Bridger [8, 9] was modules 'of G-dimension 0'; see also the Remark after 9.3.34.

9.1.24 Lemma. *Let N be an R-module; the following conditions are equivalent.*

(*i*) $\mathrm{Ext}^1_R(G, N) = 0$ *for all finitely generated Gorenstein projective R-modules G.*
(*ii*) $\mathrm{Ext}^m_R(G, N) = 0$ *for all finitely generated Gorenstein projective R-modules G and all $m > 0$.*

The class $\mathcal{N}$ of R-modules N satifying these conditions has the following properties.

(a) *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. If two of the modules N', N, and N'' belong to $\mathcal{N}$, then so does the third.*
(b) *For every family $\{N^u\}_{u\in U}$ of modules in $\mathcal{N}$ one has $\prod_{u\in U} N^u \in \mathcal{N}$.*
(b′) *Let $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$ be a U-direct system of modules in $\mathcal{N}$. If U is filtered, then $\mathrm{colim}_{u\in U} N^u$ belongs to $\mathcal{N}$.*
(c) *Every R-module N with $\mathrm{fd}_R N$ or $\mathrm{id}_R N$ finite belongs to $\mathcal{N}$.*

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. For the converse, let G be a finitely generated Gorenstein projective R-module and choose by 9.1.23 a totally acyclic complex P of finitely generated free R-modules with $\mathrm{C}_0(P) \cong G$. For $m > 0$ one has $\mathrm{Ext}^m_R(G, N) \cong \mathrm{Ext}^1_R(\mathrm{C}_{m-1}(P), N)$ by 8.1.6, so (i) implies (ii) as the module $\mathrm{C}_{m-1}(P)$ is finitely generated and Gorenstein projective.

(a): It follows from 7.3.35 that $N', N'' \in \mathcal{N}$ implies $N \in \mathcal{N}$, and that $N', N \in \mathcal{N}$ implies $N'' \in \mathcal{N}$. Now assume $N, N'' \in \mathcal{N}$. Another application of 7.3.35 yields $\mathrm{Ext}^m_R(G, N') = 0$ for every finitely generated Gorenstein projective R-module G and all $m > 1$. To show that also $\mathrm{Ext}^1_R(G, N') = 0$ holds, recall from 9.1.22 that every finitely Gorenstein projective module G fits into an exact sequence of finitely generated modules $0 \to G \to F \to G' \to 0$ with F free and G' Gorenstein projective. Application of 7.3.35 to this sequence yields $\mathrm{Ext}^1_R(G, N') \cong \mathrm{Ext}^2_R(G', N')$, and the right-hand side is zero, as just shown.

(b) and (b′): For every finitely generated R-module G the functor $\mathrm{Ext}^1_R(G, -)$ preserves products and filtered colimits by 7.3.33 and 7.3.34. Now the assertions are immediate from the definition of the class $\mathcal{N}$.

(c): In view of part (a), it suffices to prove that $\mathcal{N}$ contains every injective and every flat R-module. The first is true by 8.2.19. Further, by 9.1.3(1) every projective R-module is in $\mathcal{N}$, so part (b′) and 5.5.7 imply that every flat R-module is in $\mathcal{N}$. ◻

The next approximation result applies by 5.2.16 to complexes M in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ of finite Gorenstein projective dimension.

9.1.25 Proposition. *Assume that R is left Noetherian and let M be a complex in $\mathcal{D}^{\mathrm{f}}(R)$ of finite Gorenstein projective dimension $g = \mathrm{Gpd}_R\, M$. For every degreewise finitely generated semi-projective replacement P of M and every integer u with $g > u$ there is a distinguished triangle in $\mathcal{D}^{\mathrm{f}}(R)$,*

$$K \longrightarrow M \longrightarrow N \longrightarrow \Sigma K\,,$$

where the complexes K and N have the properties listed in 9.1.15.

Proof. The proof of 9.1.15 applies with one change: By 9.1.23 one can assume that the acyclic complex L is degreewise finitely generated. ◻

9.1.26 Corollary. *Assume that R is left Noetherian and let M be a finitely generated R-module of finite Gorenstein projective dimension $g = \mathrm{Gpd}_R\, M$.*

(a) *There is an exact sequence $0 \to M \to X \to G \to 0$ of finitely generated R-modules where G is Gorenstein projective and $\mathrm{pd}_R\, X = g$.*
(b) *If $g > 0$, then there is an exact sequence $0 \to X \to G \to M \to 0$ of finitely generated R-modules where G is Gorenstein projective and $\mathrm{pd}_R\, X = g - 1$.*

Proof. The proof of 9.1.16 applies when one replaces references to 9.1.15 with references to 9.1.25. ◻

9.1.27 Proposition. *Assume that R is left Noetherian. Let L be a degreewise finitely generated complex of projective R-modules and F a semi-flat R-complex. If F has finite flat or finite injective dimension, then the morphism*

$$\mathrm{Hom}_R(\tau^L_{\subseteq n}, F) :\ \mathrm{Hom}_R(L_{\subseteq n}, F) \ \longrightarrow\ \mathrm{Hom}_R(L, F)$$

is a quasi-isomorphism for every integer $n \geqslant \mathrm{Gpd}_R\, L$.

Proof. Let $\pi\colon P \xrightarrow{\simeq} F$ be a semi-projective resolution. One has $n \geqslant \sup L$ by 9.1.11, so the map $\tau^L_{\subseteq n}\colon L \to L_{\subseteq n}$ is a quasi-isomorphism by 4.2.4. Choose a semi-injective resolution $\iota\colon P \xrightarrow{\simeq} I$. In the commutative diagram,

$$\begin{tikzcd}
\mathrm{Hom}_R(L_{\subseteq n}, I) \arrow[d, "\mathrm{Hom}(\tau^L_{\subseteq n}, I)", "\simeq"'] & \mathrm{Hom}_R(L_{\subseteq n}, P) \arrow[l, "\mathrm{Hom}(L_{\subseteq n}, \iota)"'] \arrow[r, "\mathrm{Hom}(L_{\subseteq n}, \pi)", "\simeq"'] \arrow[d, "\mathrm{Hom}(\tau^L_{\subseteq n}, P)"] & \mathrm{Hom}_R(L_{\subseteq n}, F) \arrow[d, "\mathrm{Hom}(\tau^L_{\subseteq n}, F)"] \\
\mathrm{Hom}_R(L, I) & \mathrm{Hom}_R(L, P) \arrow[l, "\mathrm{Hom}(L, \iota)", "\simeq"'] \arrow[r, "\mathrm{Hom}(L, \pi)", "\simeq"'] & \mathrm{Hom}_R(L, F)\,,
\end{tikzcd}$$

the right-hand horizontal maps are quasi-isomorphisms by 5.5.23, while $\mathrm{Hom}_R(L, \iota)$ and $\mathrm{Hom}_R(\tau^L_{\subseteq n}, I)$ are quasi-isomorphisms by semi-projectivity of L, see 5.2.8, and semi-injectivity of I. The diagram shows that it suffices to verify that $\mathrm{Hom}_R(L_{\subseteq n}, \iota)$

is a quasi-isomorphism. Set $C = \operatorname{Cone}\iota$ and notice that it is an acyclic complex of modules that are direct sums of projective and injective R-modules. The goal is to show that the complex $\operatorname{Hom}_R(L_{\subseteq n}, C) \cong \operatorname{Cone}\operatorname{Hom}_R(L_{\subseteq n}, \iota)$ is acyclic; cf. 4.1.16.

By 2.5.24 there is an exact sequence $0 \to L' \to L \to L_{\subseteq n} \to 0$ with L' bounded below. By 9.1.2 and 9.1.12 these are complexes of finitely generated Gorenstein projective modules, and it follows from 7.3.39 and 9.1.24(c) that the sequence

$$0 \longrightarrow \operatorname{Hom}_R(L_{\subseteq n}, C) \longrightarrow \operatorname{Hom}_R(L, C) \longrightarrow \operatorname{Hom}_R(L', C) \longrightarrow 0$$

is exact. The middle complex is acyclic as L is semi-projective and C is acyclic. To prove acyclicity of $\operatorname{Hom}_R(L_{\subseteq n}, C)$ it suffices to argue that $\operatorname{Hom}_R(L', C)$ is acyclic; see 2.5.6. To this end, it suffices by A.5 to show that $\operatorname{Hom}_R(G, C)$ is acyclic for every finitely generated Gorenstein projective module G. It is already known from 9.1.24(c) that $\operatorname{Ext}_R^m(G, C_v) = 0$ holds for all $v \in \mathbb{Z}$ and all $m > 0$, so it suffices by A.4 to argue that $\operatorname{Ext}_R^m(G, \mathrm{Z}_v(C)) = 0$ holds for $v \gg 0$ and all $m > 0$.

If F has finite flat dimension, then one can assume that I is bounded above; see 8.3.3 and 5.3.26. Moreover, the modules $\mathrm{C}_v(P)$ are flat for $v \gg 0$ by 5.4.10 and 8.3.11. For $v \gg 0$ one thus has $\mathrm{Z}_v(C) \cong \mathrm{C}_{v+1}(C) = \mathrm{C}_v(P)$, and as these modules are flat, it follows from 9.1.24(c) that $\operatorname{Ext}_R^m(G, \mathrm{Z}_v(C)) = 0$ holds for $v \gg 0$ and all $m > 0$.

If F has finite injective dimension, then one can assume that the complexes I and P are bounded below; see 8.2.2 and 5.2.15. It follows that C is bounded below; in particular, $\mathrm{Z}_v(C) = 0$ holds for for $v \ll 0$. Let G be a Gorenstein projective R-module. For every $v \in \mathbb{Z}$ one has $\operatorname{Ext}_R^m(G, C_v) = 0$ for all $m > 0$, so in view of 9.1.24(a), induction on the exact sequences $0 \to \mathrm{Z}_v(C) \to C_v \to \mathrm{Z}_{v-1}(C) \to 0$ yields $\operatorname{Ext}_R^m(G, \mathrm{Z}_v(C)) = 0$ for all $v \in \mathbb{Z}$ and all $m > 0$. □

9.1.28 Corollary. *Assume that R is left Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}(R)$ of finite Gorenstein projective dimension and N an R-complex of finite flat or finite injective dimension. For every degreewise finitely generated semi-projective replacement L of M, every semi-flat replacement F of N, and every integer $n \geqslant \operatorname{Gpd}_R M$ there is an isomorphism in $\mathcal{D}(\Bbbk)$,*

$$\mathsf{R}\mathrm{Hom}_R(M, N) \simeq \operatorname{Hom}_R(L_{\subseteq n}, F)\,.$$

Proof. The assertion follows immediately from 9.1.27 and 7.3.7. □

9.1.29 Theorem. *Assume that R is left Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite Gorenstein projective dimension and n an integer. The next conditions are equivalent.*

(*i*) $\operatorname{Gpd}_R M \leqslant n$.

(*ii*) $-\inf \mathsf{R}\mathrm{Hom}_R(M, N) \leqslant n - \inf N$ *holds for every R-complex N with* $\operatorname{fd}_R N$ *finite or* $\operatorname{id}_R N$ *finite.*

(*iii*) $-\inf \mathsf{R}\mathrm{Hom}_R(M, R) \leqslant n$ *holds.*

(*iv*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^{n+1}(M, N) = 0$ *holds for every finitely generated R-module N with* $\operatorname{pd}_R N$ *finite.*

(v) *There is a degreewise finitely generated semi-projective resolution* $P \xrightarrow{\simeq} M$ *with* $P_v = 0$ *for all* $v < \inf M$*, and* $\mathrm{H}_v(P) = 0$ *for all* $v > n$*, and* $\mathrm{C}_v(P)$ *Gorenstein projective for all* $v \geqslant n$.

In particular, there are equalities,

$$\operatorname{Gpd}_R M = -\inf \mathsf{RHom}_R(M,R) = \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(M,R) \neq 0\}\,.$$

Proof. The implication (i) ⇒ (iv) follows from 9.1.19 and (v) ⇒ (i) is immediate from the definition, 9.1.10. Moreover, (ii) ⇒ (iii) is trivial.

(i) ⇒ (ii): The complex M has by 5.2.16 a bounded below and degreewise finitely generated semi-projective replacement. The argument in the proof of 9.1.19 now applies when one refers to 9.1.28 in place of 9.1.18.

(iii) ⇒ (i): Let $L \xrightarrow{\simeq} M$ be a semi-projective resolution with L degreewise finitely generated and bounded below; see 5.2.16. The functor $\operatorname{Hom}_R(L,-)$, which by 7.3.21 is $\mathsf{RHom}_R(M,-)$, preserves coproducts of modules by 3.1.33 applied degreewise. It follows that $-\inf \mathsf{RHom}_R(M,P) \leqslant n$ holds for every free, and hence every projective, R-module P; see 1.3.17. Now invoke 9.1.19.

(iv) ⇒ (v): Let $P \xrightarrow{\simeq} M$ be a semi-projective resolution with P degreewise finitely generated and $P_v = 0$ for every $v < \inf M$; see 5.2.16. By assumption, $g = \operatorname{Gpd}_R M$ is finite, so the module $\mathrm{C}_v(P)$ Gorenstein projective for every integer $v \geqslant g$; see 9.1.12. It now suffices to show that $n \geqslant g$. Assume that $n < g$ holds. Notice that by the assumption $n \geqslant \sup M$, the module $\mathrm{C}_n(P)$ can not be Gorenstein projective. There is an exact sequence $0 \to \mathrm{C}_g(P) \to P_{g-1} \to \cdots \to P_n \to \mathrm{C}_n(P) \to 0$, which shows that $\mathrm{C}_n(P)$ has finite Gorenstein projective dimension, as $P_{\geqslant n}$ is a semi-projective replacement of $\Sigma^n \mathrm{C}_n(P)$. Now 9.1.26 yields an exact sequence $0 \to N \to G \to \mathrm{C}_n(P) \to 0$ of finitely generated R-modules where G is Gorenstein projective and N has finite projective dimension. By 8.1.6 and (iv) one has $\operatorname{Ext}_R^1(\mathrm{C}_n(P),N) \cong \operatorname{Ext}_R^{n+1}(M,N) = 0$, so the sequence is split by 7.3.36. Now 9.1.8 implies that $\mathrm{C}_n(P)$ is Gorenstein projective, which is a contradiction.

The equalities in the last assertion follow immediately from the equivalence of (i) and (iii) and 7.3.24. □

The Case of Modules

9.1.30. Notice from 9.1.19 that a non-zero R-module is Gorenstein projective if and only if it has Gorenstein projective dimension 0 as an R-complex.

9.1.31 Theorem. *Let* M *be an* R*-module of finite Gorenstein projective dimension and* $n \geqslant 0$ *an integer. The following conditions are equivalent.*

(i) $\operatorname{Gpd}_R M \leqslant n$.

(ii) $\operatorname{Ext}_R^m(M,N) = 0$ *holds for every* R*-module* N *with* $\operatorname{pd}_R N$ *finite or* $\operatorname{id}_R N$ *finite and all integers* $m > n$.

(iii) $\operatorname{Ext}_R^m(M,N) = 0$ *holds for every projective* R*-module* N *and all integers* $m > n$.

(iv) $\operatorname{Ext}_R^{n+1}(M,N) = 0$ *holds for every* R*-module* N *with* $\operatorname{pd}_R N$ *finite.*

(*v*) *In some/every projective resolution* $\cdots \to P_v \to P_{v-1} \to \cdots \to P_0 \to M \to 0$ *the module* $\operatorname{Coker}(P_{v+1} \to P_v)$ *is Gorenstein projective for every* $v \geqslant n$.

(*vi*) *There is an exact sequence of R-modules* $0 \to G \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *with each* P_i *projective and* G *Gorenstein projective.*

In particular, there is an equality

$$\operatorname{Gpd}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,N) \neq 0 \text{ for some projective } R\text{-module } N\}\,.$$

PROOF. By 5.2.27 every R-module M has a projective resolution

$$\cdots \longrightarrow P_v \longrightarrow P_{v-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0\,.$$

In every such resolution, the surjective homomorphism $P_0 \to M$ yields a semi-projective resolution of M, considered as a complex; cf. 5.2.29. Thus the complex $\cdots \to P_v \to P_{v-1} \to \cdots \to P_0 \to 0$ is a semi-projective replacement of M. The equivalence of conditions (*i*)–(*vi*) is now immediate from 9.1.19, and so is the asserted equality in view of 7.3.27. □

9.1.32 Theorem. *Assume that R is left Noetherian. Let M be a finitely generated R-module of finite Gorenstein projective dimension and* $n \geqslant 0$ *an integer. The following conditions are equivalent.*

(*i*) $\operatorname{Gpd}_R M \leqslant n$.

(*ii*) $\operatorname{Ext}_R^m(M,N) = 0$ *holds for every R-module N with* $\operatorname{fd}_R N$ *finite or* $\operatorname{id}_R N$ *finite and all integers* $m > n$.

(*iii*) $\operatorname{Ext}_R^m(M,R) = 0$ *holds for all integers* $m > n$.

(*iv*) $\operatorname{Ext}_R^{n+1}(M,N) = 0$ *for every finitely generated R-module N with* $\operatorname{pd}_R N$ *finite.*

(*v*) *There is an exact sequence* $0 \to G \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *of finitely generated R-modules with each* P_i *projective and G Gorenstein projective.*

In particular, there is an equality

$$\operatorname{Gpd}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,R) \neq 0\}\,.$$

PROOF. By 5.1.19 every finitely generated R-module M has a projective resolution

$$\cdots \longrightarrow P_v \longrightarrow P_{v-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each module P_v finitely generated. An argument parallel to the proof of 9.1.31 shows that the statement follows from 9.1.29. □

REMARK. Examples by Jorgensen and Şega [151] show that in 9.1.32 one can not dispense with the a priori assumption that $\operatorname{Gpd}_R M$ is finite.

EXERCISES

E 9.1.1 Show that if every R-module has finite Gorenstein projective dimension, then every acyclic complex of projective R-modules is totally acyclic.

E 9.1.2 Show that for an R-module M the next conditions are equivalent. (*i*) For every Gorenstein projective R-module G one has $\operatorname{Ext}_R^m(G, M) = 0$ for all $m \gg 0$. (*ii*) For every Gorenstein projective R-module G one has $\operatorname{Ext}_R^m(G, M) = 0$ for all $m \geqslant 1$. Show that $\operatorname{Gpd}_R M = \operatorname{pd}_R M$ holds for every R-module M that satisfies these conditions.

E 9.1.3 Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\operatorname{H}(M) \neq 0$ and set $w = \sup M$. Show that for every semi-projective replacement P of M one has $\operatorname{Gpd}_R M = w + \operatorname{Gpd}_R \operatorname{C}_w(P)$.

E 9.1.4 Let M be a complex in $\mathcal{D}_{\sqsupset}(R)$. Show that $\operatorname{Gpd}_R M$ is finite if and only if M is isomorphic in $\mathcal{D}(R)$ to a bounded complex of Gorenstein projective R-modules.

E 9.1.5 Show that the full subcategory of R-complexes of finite Gorenstein projective dimension is a triangulated subcategory of $\mathcal{D}_{\sqsubset}(R)$.

E 9.1.6 Let M be a complex in $\mathcal{D}_{\square}(R)$ and G a bounded below complex of Gorenstein projective R-modules with $M \simeq G$ in $\mathcal{D}(R)$. Show that for every R-module N with $\operatorname{pd}_R N$ or $\operatorname{id}_R N$ finite and for all $m > 0$ and $n \geqslant \sup M$ one has $\operatorname{Ext}_R^{n+m}(M, N) \cong \operatorname{Ext}_R^m(\operatorname{C}_n(G), N)$.

E 9.1.7 Let n be an integer. Show that an R-complex M has $\operatorname{Gpd}_R M \leqslant n$ if and only if there is a diagram $P \xrightarrow{\tau} P' \xrightarrow{\pi} M$ in $\mathcal{C}(R)$ where P is a totally acyclic complex of projective R-modules, τ_v is an isomorphism for $v \geqslant n$, and π is a semi-projective resolution.

E 9.1.8 Assume that R is left Noetherian and let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Show that $\operatorname{Gpd}_R M$ is finite if and only if M is isomorphic in $\mathcal{D}(R)$ to a bounded complex of finitely generated Gorenstein projective R-modules.

E 9.1.9 Assume that R is left Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and G a bounded below complex of finitely generated Gorenstein projective R-modules with $M \simeq G$ in $\mathcal{D}(R)$. Show that for every R-module N with $\operatorname{fd}_R N$ or $\operatorname{id}_R N$ finite and for all integers $m > 0$ and $n \geqslant \sup M$ there is an isomorphism $\operatorname{Ext}_R^{n+m}(M, N) \cong \operatorname{Ext}_R^m(\operatorname{C}_n(G), N)$.

E 9.1.10 Assume that R is left Noetherian and let n be an integer. Show that a complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ has $\operatorname{Gpd}_R M \leqslant n$ if and only if there is a diagram $L \xrightarrow{\tau} L' \xrightarrow{\pi} M$ of degreewise finitely generated R-complexes where L is a totally acyclic complex of free R-modules, τ_v is an isomorphism for $v \geqslant n$, and π is a semi-free resolution.

E 9.1.11 Let $\mathcal{N}$ be the class of R°-modules N with $\operatorname{Tor}_m^R(N, G) = 0$ for every finitely generated Gorenstein projective R-module G and all $m > 0$. Show that if in an exact sequence of R°-modules $0 \to N' \to N \to N'' \to 0$ any two of the modules N', N, and N'' are in $\mathcal{N}$, then so is the third. Conclude that every R°-module N with $\operatorname{fd}_{R^{\circ}} N$ is in $\mathcal{N}$. Conclude further that if R is left Noetherian, then every R°-module N with $\operatorname{id}_{R^{\circ}} N$ finite is in $\mathcal{N}$.

E 9.1.12 Assume that R is left Noetherian. Let L be a degreewise finitely generated semi-projective R-complex and E a semi-injective R-complex. Show that if E has finite flat or finite injective dimension, then $E \otimes_R \tau^L_{\subseteq n} \colon E \otimes_R L \to E \otimes_R L_{\subseteq n}$ is a quasi-isomorphism for every integer $n \geqslant \operatorname{Gpd}_R L$.

9.2 Gorenstein Injective Dimension

Synopsis. Totally acyclic complex of injective modules; Gorenstein injective module; Gorenstein injective dimension; ~ vs. injective dimension; ~ of module.

The Gorenstein injective dimension refines the injective dimension like the Gorenstein projective dimension refines the projective dimension. This section thus develops in close parallel with Sect. 9.1, though without a counterpart to the theory surrounding 9.1.29.

GORENSTEIN INJECTIVE MODULES

9.2.1 Definition. A complex I of injective R-modules is called *totally acyclic*if it is acyclic and $\operatorname{Hom}_R(E, I)$ is acyclic for every injective R-module E.

An R-module G is called *Gorenstein injective* if one has $G \cong \mathrm{Z}_0(I)$ for some totally acyclic complex I of injective R-modules.

Notice that if I is a totally acyclic complex of injective R-modules, then the module $\mathrm{Z}_v(I)$ is Gorenstein injective for every $v \in \mathbb{Z}$.

9.2.2 Example. Every injective R-module I is Gorenstein injective as the disk complex $\mathrm{D}^1(I) = 0 \longrightarrow I \xrightarrow{=} I \longrightarrow 0$ is totally acyclic.

9.2.3 Example. The Dold complex D from 2.1.23 is an acyclic complex of injective $\mathbb{Z}/4\mathbb{Z}$-modules; cf. 8.2.10. To see that it is totally acyclic, note that by B.14, and C.23 every injective $\mathbb{Z}/4\mathbb{Z}$-module E is free, i.e. $E \cong (\mathbb{Z}/4\mathbb{Z})^{(U)}$ for some set U. For every such module 3.1.27 and 4.4.2 yield isomorphisms,

$$\operatorname{Hom}_{\mathbb{Z}/4\mathbb{Z}}(E, D) \cong \operatorname{Hom}_{\mathbb{Z}/4\mathbb{Z}}((\mathbb{Z}/4\mathbb{Z})^{(U)}, D) \cong \operatorname{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, D)^U \cong D^U .$$

In particular, the complex $\operatorname{Hom}_{\mathbb{Z}/4\mathbb{Z}}(E, D)$ is acyclic. It follows that $\mathbb{Z}/4\mathbb{Z}$-module $\mathrm{Z}_0(D) = 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ is Gorenstein injective.

A Gorenstein injective module has a “left resolution” by injective precovers and interacts with injective modules as if they were projective.

9.2.4 Lemma. *Let G be an R-module*; *it is Gorenstein injective if and only if it meets the following requirements*:

(1) *For every injective R-module E one has* $\operatorname{Ext}_R^m(E, G) = 0$ *for all* $m > 0$.
(2) *There exists an exact sequence of R-modules,* $\cdots \to I_1 \to I_0 \to M \to 0$, *where each I_v is injective and the sequence*

$$\cdots \longrightarrow \operatorname{Hom}_R(E, I_1) \longrightarrow \operatorname{Hom}_R(E, I_0) \longrightarrow \operatorname{Hom}_R(E, M) \longrightarrow 0$$

is exact for every injective R-module E.

PROOF. Assume first that G is Gorenstein injective and let I be a totally acyclic complex of injective R-modules with $G \cong \mathrm{Z}_0(I)$; see 9.2.1. The complex $I_{\supseteq 0}$ and the isomorphism $G \cong \mathrm{Z}_0(I)$ yield, up to indexing, the exact sequence asserted in (2). The complex $I_{\leqslant 0}$ is a semi-injective replacement of G. For $m > 0$ the definition of Ext, 7.3.23, and total acyclicity of I yield the Ext vanishing asserted in (1):

$$\operatorname{Ext}_R^m(E, G) = \mathrm{H}_{-m}(\operatorname{Hom}_R(E, I_{\leqslant 0})) = \mathrm{H}_{-m}(\operatorname{Hom}_R(E, I)) = 0 .$$

Assuming now that G satisfies the two requirements, let $\iota\colon G \xrightarrow{\simeq} I'$ be an injective resolution, see 5.3.31, and denote by I the complex $\cdots \to I_1 \to I_0 \to 0$. The homomorphism $I_0 \to G$ induces a quasi-isomorphism $\pi\colon I \to G$, so $\widetilde{I} = \operatorname{Cone}(\iota\pi)$ is by 4.1.1 and 4.2.16 an acyclic complex of injective R-modules with $G \cong \mathrm{Z}_0(\widetilde{I})$. Let E be an injective R-module. By (2) the morphism $\operatorname{Hom}_R(E, \pi)$ is a quasi-isomorphism, and by (1) so is $\operatorname{Hom}_R(E, \iota)$, see 7.3.27. It follows that $\operatorname{Hom}_R(E, \iota\pi)$

is a quasi-isomorphism, whence the complex $\operatorname{Cone}\operatorname{Hom}_R(E,\iota\pi) \cong \operatorname{Hom}_R(E,\widetilde{I})$ is acyclic, see 4.1.16 and 4.2.16. Thus $\widetilde{I}$ is a totally acyclic complex of injective R-modules and G is Gorenstein injective. □

For later reference we prove the following result. It shows that 9.2.4(1) can be strengthened; however, it is strengthened even further in 9.2.8(c).

9.2.5 Lemma. *Let M be an R-module. The following conditions are equivalent.*

(*i*) $\operatorname{Ext}_R^m(E,M) = 0$ *for every injective R-module E and all $m > 0$.*
(*ii*) $\operatorname{Ext}_R^m(N,M) = 0$ *for every R-module N with $\operatorname{id}_R N$ finite and all $m > 0$.*

Proof. Condition (*ii*) clearly implies (*i*). For the converse, induct on $d = \operatorname{id}_R N$. The base case $d = 0$ is handled by the assumption (*i*). Now assume that $d > 0$ holds and that one has $\operatorname{Ext}_R^m(N',M) = 0$ for every R-module N' with $\operatorname{id}_R N' < d$ and all $m > 0$. Let N be an R-module with $\operatorname{id}_R N = d$ and consider an exact sequence $0 \to N \to E \to N' \to 0$ with E injective. By 8.2.9 one has $\operatorname{id}_R N' < d$, and hence $\operatorname{Ext}_R^m(N',M) = 0 = \operatorname{Ext}_R^m(E,M)$ holds for all $m > 0$. Application of the functor $\operatorname{Hom}_R(-,M)$ to $0 \to N \to E \to N' \to 0$ induces by 7.3.35 an exact sequence of Ext modules, which now shows that $\operatorname{Ext}_R^m(N,M) = 0$ holds for all $m > 0$. □

The next result picks up essential properties of the class of Gorenstein injective modules. The first assertion in part (a) together with 9.2.2 shows that the class is so-called injectively resolving.

9.2.6 Proposition. *The following assertions hold.*

(a) *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of R-modules. If G' is Gorenstein injective, then G is Gorenstein injective if and only if G'' is so. If G and G'' are Gorenstein injective, then G' is Gorenstein injective if and only if $\operatorname{Ext}_R^1(E,G') = 0$ holds for every injective R-module E.*
(b) *Let $\{G^u\}_{u\in U}$ be a family of R-modules. The module $\prod_{u\in U} G^u$ is Gorenstein injective if and only if each G^u is Gorenstein injective. In particular, a direct summand of a Gorenstein injective R-module is Gorenstein injective.*

Proof. Let $0 \longrightarrow G' \xrightarrow{\alpha'} G \xrightarrow{\alpha} G'' \longrightarrow 0$ be an exact sequence of R-modules where G and G' are Gorenstein injective. As G and G' satisfy condition 9.2.4(1), so does G'' by 7.3.35. As G and G' satisfy 9.2.4(2) there exist complexes

$$I = \cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow 0 \qquad \text{and} \qquad I' = \cdots \longrightarrow I'_1 \longrightarrow I'_0 \longrightarrow 0$$

of injective R-modules and quasi-isomorphisms $\pi\colon I \to G$ and $\pi'\colon I' \to G'$ such that $\operatorname{Hom}_R(E,\pi)$ and $\operatorname{Hom}_R(E,\pi')$ are quasi-isomorphisms for every injective R-module E. By 4.1.16 and 4.2.16 the complex $\operatorname{Hom}_R(E,\operatorname{Cone}\pi)$ is acyclic for every injective R-module E, so A.5 implies that $\operatorname{Hom}_R(E,\pi)$ is a quasi-isomorphism for every complex E of injective R-modules. There is an exact sequence $0 \longrightarrow \operatorname{Ker}\pi \longrightarrow I \xrightarrow{\pi} G \longrightarrow 0$ where $\operatorname{Ker}\pi = \cdots \to I_2 \to I_1 \to \operatorname{Ker}\pi_0 \to 0$. The module $\operatorname{Ker}\pi_0$ is Gorenstein injective, so it follows from 7.3.38 that $\operatorname{Hom}_R(E,\pi)$ is surjective for every complex E of injective R-modules. In particular, $\operatorname{Hom}_R(I',\pi)$ is a surjective quasi-isomorphism, and hence $\mathcal{C}(R)(I',\pi)\colon \mathcal{C}(R)(I',I) \to \mathcal{C}(R)(I',G)$

is surjective by 4.2.7 and 2.3.10. Thus there exists a morphism of R-complexes $\beta\colon I' \to I$ with $\pi\beta = \alpha'\pi'$. Let $\iota\colon I' \rightarrowtail \operatorname{Cone} 1^{I'}$ be the canonical morphism from 4.1.5. With $I'' = \operatorname{Coker}\binom{\beta}{\iota}$ there is a commutative diagram in $\mathcal{C}(R)$ with exact rows,

$$
(\dagger)\qquad
\begin{tikzcd}
0 \arrow[r] & I' \arrow[r, "\binom{\beta}{\iota}"] \arrow[d, "\pi'"] & \begin{matrix} I \\ \oplus \\ \operatorname{Cone} 1^{I'} \end{matrix} \arrow[r] \arrow[d, "\underline{\pi} = (\pi\ 0)"] & I'' \arrow[r] \arrow[d, "\pi''"] & 0 \\
0 \arrow[r] & G' \arrow[r, "\alpha'"] & G \arrow[r, "\alpha"] & G'' \arrow[r] & 0\,,
\end{tikzcd}
$$

where π'' is the induced morphism. As the complex $\operatorname{Cone} 1^{I'}$ is contractible, see 4.3.31, the morphism $\underline{\pi}$ is a quasi-isomorphism and so is $\operatorname{Hom}_R(E, \underline{\pi})$ for every injective R-module E. The complex I'' consists of injective modules and is concentrated in degrees $\geqslant 0$ as this is the case for I, I' and $\operatorname{Cone} 1^{I'}$; see 5.3.7.

As $\underline{\pi}$ and π' are quasi-isomorphisms, so is π'' by 4.2.5 applied to the diagram ($\dagger$). If E is an injective R-module, then application of the functor $\operatorname{Hom}_R(E, -)$ to ($\dagger$) yields a commutative diagram with exact rows. This is because $\operatorname{Ext}^1_R(E, I') = 0$ holds and because the top row in ($\dagger$) is degreewise split exact. As $\operatorname{Hom}_R(E, \underline{\pi})$ and $\operatorname{Hom}_R(E, \pi')$ are quasi-isomorphisms, so is $\operatorname{Hom}_R(E, \pi'')$ by another application of 4.2.5. This shows that the module G'' satisfies condition 9.2.4(2), and it follows that the class of Gorenstein injective modules is closed under cokernels of injective homomorphisms.

Next we show that the class of Gorenstein injective modules is closed under products. Let $\{G^u\}_{u \in U}$ be a family of Gorenstein injective R-modules. By definition there exists for each $u \in U$ a totally acyclic complex I^u of injective R-modules with $G^u \cong \mathrm{Z}_0(I^u)$. The complex $I = \prod_{u \in U} I^u$ consists of injective modules by 1.3.27 and it is acyclic by 3.1.23. For every R-module E there is by 3.1.24 an isomorphism $\operatorname{Hom}_R(E, I) \cong \prod_{u \in U} \operatorname{Hom}_R(E, I^u)$, and this complex is acyclic if E is injective. Thus, I is a totally acyclic complex of injective R-modules. Since one has $\mathrm{Z}_0(I) \cong \prod_{u \in U} \mathrm{Z}_0(I^u) \cong \prod_{u \in U} G^u$, see 3.1.22(a), it follows that $\prod_{u \in U} G^u$ is Gorenstein injective.

Having established these properties, it now follows from 9.1.7 that the class of Gorenstein injective modules is closed under direct summands. This proves (b).

We now finish the proof of (a). To show that the class of Gorenstein injective modules is closed under extensions, let $0 \to G' \to G \to G'' \to 0$ be an exact sequence where G' and G'' are Gorenstein injective. By the definition 9.2.1 of Gorenstein injective modules, there exists an exact sequence $0 \to G''' \to I'' \to G'' \to 0$ with I'' injective and G''' Gorenstein injective. By 3.4.34 there is a commutative diagram with exact rows and columns,

$$
(\ddagger)\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G''' \arrow[r, equal] \arrow[d] & G''' \arrow[d] & \\
0 \arrow[r] & G' \arrow[r] \arrow[d, equal] & G \sqcap_{G''} I'' \arrow[r] \arrow[d] & I'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] & G \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0\,. &
\end{tikzcd}
$$

As G' is Gorenstein injective and I'' is injective one has $\operatorname{Ext}^1_R(I'', G') = 0$. Thus the first row in (‡) is split by 7.3.36, so one has $G \sqcap_{G''} I'' \cong G' \oplus I''$. This module is Gorenstein injective by (b). Now the already established part of (a) applied to the first column in (‡) shows that G is Gorenstein injective.

It remains to prove the final assertion in (a). The "only if" part is evident. For the converse, consider again the diagram (‡). By assumption, I'' is injective and G, G'', and G''' are Gorenstein injective. The first column and the already established part of (a) show that $G \sqcap_{G''} I''$ is Gorenstein injective. As $\operatorname{Ext}^1_R(I'', G') = 0$ holds, by assumption, it follows from 7.3.36 that the first row is split. Hence part (b) shows that G' is Gorenstein injective. □

We learned the next result from an unpublished work [241] of Šťovíček.

9.2.7 Theorem. *Let $\mathcal{Y}$ be a class of R-modules and set*

$$ {}^{\perp}\mathcal{Y} = \{M \in \mathcal{M}(R) \mid \operatorname{Ext}^1_R(M, Y) = 0 \text{ for every } Y \in \mathcal{Y}\}\,. $$

Assume that for every exact sequence $0 \to M' \to M \to M'' \to 0$ of R-modules with $M', M \in {}^{\perp}\mathcal{Y}$ one has $M'' \in {}^{\perp}\mathcal{Y}$. For every U-direct system $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ of modules in ${}^{\perp}\mathcal{Y}$ with U filtered, one has $\operatorname{colim}_{u\in U} M^u \in {}^{\perp}\mathcal{Y}$.

Proof. We start by establishing the following properties of the class ${}^{\perp}\mathcal{Y}$.

(a) For every family $\{M^u\}_{u\in U}$ of modules in ${}^{\perp}\mathcal{Y}$ one has $\coprod_{u\in U} M^u \in {}^{\perp}\mathcal{Y}$.
(b) For every U-direct system $\{\mu^{vu} \colon M^u \rightarrowtail M^v\}_{u\leqslant v}$ of modules in ${}^{\perp}\mathcal{Y}$ with U filtered and each μ^{vu} is injective, one has $\operatorname{colim}_{u\in U} M^u \in {}^{\perp}\mathcal{Y}$.

Property (a) follows from (b) in view of 3.3.9, but it is also an immediate consequence of 7.3.32. To establish property (b) it suffices by 3.3.30 to argue that if an R-module M is the union of a continuous chain $\{M^\alpha\}_{\alpha<\lambda}$ of submodules with $M^\alpha \in {}^{\perp}\mathcal{Y}$ for every $\alpha < \lambda$, then one has $M \in {}^{\perp}\mathcal{Y}$. However, this follows from Eklof's lemma D.2: The assumption on the class ${}^{\perp}\mathcal{Y}$ applied to the exact sequence

$$ 0 \longrightarrow M^\alpha \longrightarrow M^{\alpha+1} \longrightarrow M^{\alpha+1}/M^\alpha \longrightarrow 0 $$

yields $M^{\alpha+1}/M^\alpha \in {}^{\perp}\mathcal{Y}$ for every ordinal α with $\alpha + 1 < \lambda$.

Now, to show the assertion in the theorem, it suffices by another application of 3.3.30 to argue that for every ordinal λ and λ-sequence $\{\mu^{\beta\alpha} \colon M^\alpha \to M^\beta\}_{\alpha\leqslant\beta<\lambda}$ of modules in ${}^{\perp}\mathcal{Y}$ one has $\operatorname{colim}_{\alpha<\lambda} M^{\alpha} \in {}^{\perp}\mathcal{Y}$. If $\lambda = \mu + 1$ is a successor ordinal, then 3.2.8 yields $\operatorname{colim}_{\alpha<\lambda} M^{\alpha} = M^{\mu} \in {}^{\perp}\mathcal{Y}$, so one can assume that λ is a limit ordinal. Consider the homomorphism from 3.2.3,

$$X \;=\; \coprod_{\alpha\leqslant\beta<\lambda} M^{(\alpha,\beta)} \xrightarrow{\;\Delta=\Delta_{\mu}\;} \coprod_{\alpha<\lambda} M^{\alpha} \qquad \text{where} \qquad M^{(\alpha,\beta)} \;=\; M^{\alpha}\,,$$

given by $\varepsilon^{(\alpha,\beta)}(m) \mapsto \varepsilon^{\alpha}(m) - \varepsilon^{\beta}\mu^{\beta\alpha}(m)$ for $m \in M^{(\alpha,\beta)} = M^{\alpha}$. Set $I = \operatorname{Im}\Delta$ and note that by construction there is an exact sequence,

$$0 \longrightarrow I \longrightarrow \coprod_{\alpha<\lambda} M^{\alpha} \longrightarrow \operatorname*{colim}_{\alpha<\lambda} M^{\alpha} \longrightarrow 0\,.$$

By property (a) the module $\coprod_{\alpha<\lambda} M^{\alpha}$ is in ${}^{\perp}\mathcal{Y}$. Hence, to prove that $\operatorname{colim}_{\alpha<\lambda} M^{\alpha}$ is in ${}^{\perp}\mathcal{Y}$, it suffices by the assumption on the class ${}^{\perp}\mathcal{Y}$ to show that one has $I \in {}^{\perp}\mathcal{Y}$. To this end, define for every ordinal $\gamma < \lambda$ submodules,

$$X^{\gamma} \;=\; \coprod_{\alpha<\gamma} M^{(\alpha,\gamma)} \;\subseteq\; X \qquad \text{and} \qquad I^{\gamma} \;=\; \operatorname{Im}(\Delta|_{X^{\gamma}}) \;\subseteq\; I\,.$$

Note that Δ has the following property:

(†) Given $\alpha \leqslant \beta < \gamma < \lambda$ and $m \in M^{(\alpha,\beta)} = M^{\alpha}$ there exists an element $y \in X^{\gamma}$ with $\Delta(y) = \Delta(\varepsilon^{(\alpha,\beta)}(m))$.

Indeed, note that $\mu^{\beta\alpha}(m)$ belongs to $M^{\beta} = M^{(\beta,\gamma)}$, and as $\beta < \gamma$ holds, the element $y = \varepsilon^{(\alpha,\gamma)}(m) - \varepsilon^{(\beta,\gamma)}(\mu^{\beta\alpha}(m))$ is in X^{γ}. A direct computation yields

$$\begin{aligned} \Delta(y) \;&=\; (\varepsilon^{\alpha}(m) - \varepsilon^{\gamma}\mu^{\gamma\alpha}(m)) - (\varepsilon^{\beta}(\mu^{\beta\alpha}(m)) - \varepsilon^{\gamma}\mu^{\gamma\beta}(\mu^{\beta\alpha}(m))) \\ &=\; \varepsilon^{\alpha}(m) - \varepsilon^{\beta}(\mu^{\beta\alpha}(m)) \\ &=\; \Delta(\varepsilon^{(\alpha,\beta)}(m))\,. \end{aligned}$$

Next we argue that the family $\{I^{\gamma}\}_{\gamma<\lambda}$ of submodules of I has the properties:

(1) $I^{\gamma} \subseteq I^{\delta}$ for every $\gamma \leqslant \delta < \lambda$.
(2) $\bigcup_{\gamma<\lambda} I^{\gamma} = I$.
(3) $I^{\gamma} \in {}^{\perp}\mathcal{Y}$ for every $\gamma < \lambda$.

Once this has been proved, properties (1) and (2) together with 3.3.3 show that the family of embeddings $\{I^{\gamma} \rightarrowtail I^{\delta}\}_{\gamma\leqslant\delta<\lambda}$ is a λ-direct system with $\operatorname{colim}_{\gamma<\lambda} I^{\gamma} \cong I$. As $I^{\gamma} \in {}^{\perp}\mathcal{Y}$ holds for every $\gamma < \lambda$ by (3), one concludes from property (b) in the beginning of the proof that I belongs to ${}^{\perp}\mathcal{Y}$, as desired.

The inclusion in (1) is trivial for $\gamma = \delta$, so one can assume that $\gamma < \delta$. It must be argued that for every $x \in X^{\gamma}$ there is a $y \in X^{\delta}$ with $\Delta(y) = \Delta(x)$. By 1.1.20 one can assume that x has the form $x = \varepsilon^{(\alpha,\gamma)}(m)$ for some $\alpha < \gamma$ and $m \in M^{(\alpha,\gamma)} = M^{\alpha}$. The existence of the element y now follows from (†) applied to $\alpha \leqslant \gamma < \delta < \lambda$.

To prove (2) we must argue that for every $x \in X$ there exist $\gamma < \lambda$ and $y \in X^{\gamma}$ with $\Delta(y) = \Delta(x)$. Every $x \in X$ is a finite sum $x = \sum_{i=1}^{n} \varepsilon^{(\alpha_i,\beta_i)}(m_i)$ with $\alpha_i \leqslant \beta_i < \lambda$ and $m_i \in M^{(\alpha_i,\beta_i)} = M^{\alpha_i}$. Set $\gamma = \max\{\beta_1,\ldots,\beta_n\} + 1$ and note that $\gamma < \lambda$ holds as λ is a limit ordinal. For $i \in \{1,\ldots,n\}$ one has $\alpha_i \leqslant \beta_i < \gamma < \lambda$ so (†) yields a y_i in X^{γ} with $\Delta(y_i) = \Delta(\varepsilon^{(\alpha_i,\beta_i)}(m_i))$. Now $y = \sum_{i=1}^{n} y_i \in X^{\gamma}$ satisfies $\Delta(y) = \Delta(x)$.

To prove (3), let $\gamma < \lambda$ be given. Consider the composite,

$$(\ddagger) \qquad X^{\gamma} \xrightarrow{\iota} X = \coprod_{\alpha \leqslant \beta < \lambda} M^{(\alpha,\beta)} \xrightarrow{\varDelta} \coprod_{\alpha<\lambda} M^{\alpha} \xrightarrow{\pi} \coprod_{\alpha<\gamma} M^{\alpha} \xrightarrow[\cong]{\varphi} X^{\gamma}\,,$$

where ι is the embedding, π is the projection, and φ is the isomorphism given by $\varepsilon^{\alpha}(m) \mapsto \varepsilon^{(\alpha,\gamma)}(m)$ for $\alpha < \gamma$ and $m \in M^{\alpha} = M^{(\alpha,\gamma)}$. For such α and m one has

$$\varphi\pi\varDelta\iota(\varepsilon^{(\alpha,\gamma)}(m)) = \varphi\pi(\varepsilon^{\alpha}(m) - \varepsilon^{\gamma}\mu^{\gamma\alpha}(m)) = \varphi(\varepsilon^{\alpha}(m)) = \varepsilon^{(\alpha,\gamma)}(m)\,,$$

so the composite $(\ddagger)$ is the identity. Consequently, the surjective homomorphism $\varDelta\iota\colon X^{\gamma} \twoheadrightarrow I^{\gamma} = \operatorname{Im}(\varDelta\iota)$ has the left inverse $\varphi\pi\jmath$ where $\jmath\colon I^{\gamma} \rightarrowtail \coprod_{\alpha<\lambda} M^{\alpha}$ is the embedding. Hence $\varDelta\iota\colon X^{\gamma} \twoheadrightarrow I^{\gamma}$ is an isomorphism. It remains to note that since each module M^{α} is in ${}^{\perp}\mathcal{Y}$ by assumption, the module $I^{\gamma} \cong X^{\gamma} \cong \coprod_{\alpha<\gamma} M^{\alpha}$ belongs to ${}^{\perp}\mathcal{Y}$ by property (a) in the beginning of the proof. □

Among modules of finite Gorenstein injective dimension, Gorenstein injective modules are characterized by Ext-vanishing. To prove that, the next lemma is key.

9.2.8 Lemma. *Let N be an R-module; the following conditions are equivalent.*

(*i*) $\operatorname{Ext}^1_R(N,G) = 0$ *for every Gorenstein injective R-module G.*

(*ii*) $\operatorname{Ext}^m_R(N,G) = 0$ *for every Gorenstein injective R-module G and all $m > 0$.*

The class $\mathcal{N}$ of R-modules N satifying these conditions has the following properties.

(a) *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. If two of the modules N', N, and N'' belong to $\mathcal{N}$, then so does the third.*

(b) *Let $\{\nu^{vu}\colon N^u \to N^v\}_{u\leqslant v}$ be a U-direct system of modules in $\mathcal{N}$. If U is filtered, then $\operatorname{colim}_{u\in U} N^u$ belongs to $\mathcal{N}$.*

(c) *Every R-module N with $\operatorname{id}_R N$ or $\operatorname{fd}_R N$ finite belongs to $\mathcal{N}$.*

Proof. The implication (*ii*) ⟹ (*i*) is trivial. For the converse, let G be a Gorenstein injective R-module and I a totally acyclic complex of injective R-modules with $G \cong \mathrm{Z}_0(I)$. For $m > 0$ one has $\operatorname{Ext}^m_R(N,G) \cong \operatorname{Ext}^1_R(N,\mathrm{Z}_{-m+1}(I))$ by 8.2.6, so (*i*) implies (*ii*) as the module $\mathrm{Z}_{-m+1}(I)$ is Gorenstein injective.

(a): It follows from 7.3.35 that $N', N'' \in \mathcal{N}$ implies $N \in \mathcal{N}$, and that $N, N'' \in \mathcal{N}$ implies $N' \in \mathcal{N}$. Now assume that $N', N \in \mathcal{N}$. Another application of 7.3.35 yields $\operatorname{Ext}^m_R(N'',G) = 0$ for every Gorenstein injective R-module G and all $m > 1$. To show that also $\operatorname{Ext}^1_R(N'',G) = 0$ holds, note that every Gorenstein injective module G by definition fits into an exact sequence $0 \to G' \to I \to G \to 0$ with I injective and G' Gorenstein injective. Application of 7.3.35 to this sequence gives an isomorphism $\operatorname{Ext}^1_R(N'',G) \cong \operatorname{Ext}^2_R(N'',G')$, and the right-hand side is zero, as just shown.

(b): Let $\mathcal{Y}$ denotes the class of Gorenstein injective R-modules. With the notation from 9.2.7 one has ${}^{\perp}\mathcal{Y} = \mathcal{N}$. By part (a) this class satisfies the assumption in 9.2.7, so the assertion follows from that result.

(c): In view of part (a), it suffices to prove that $\mathcal{N}$ contains every injective and every flat R-module. The first is true by 9.2.4(1). Further, by 8.1.20 every projective R-module is in $\mathcal{N}$, so part (b) and 5.5.7 imply that every flat R-module is in $\mathcal{N}$. □

Remark. Following Xu [256] an R-module M is called *strongly cotorsion* if $\operatorname{Ext}^m_R(N,M) = 0$ holds for every R-module N of finite flat dimension and all $m > 0$. Thus, it follows from 9.2.8(c) that every Gorenstein injective R-module is strongly cotorsion. These two classes of modules are known to coincide over certain rings. Iacob [139], for example, shows that they coincide over Iwanaga–Gorenstein rings, but that is not the final word on the matter, see e.g. Wang and Li [250].

Gorenstein Injective Dimension

9.2.9 Definition. Let M be an R-complex. The *Gorenstein injective dimension* of M, written $\operatorname{Gid}_R M$, is defined as

$$\operatorname{Gid}_R M \;=\; \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{There exists a semi-injective replacement } I \text{ of } M \text{ with} \\ \mathrm{H}_{-v}(I) = 0 \text{ for all } v > n \text{ and } \mathrm{Z}_{-n}(I) \text{ Gorenstein injective} \end{array} \right\}$$

with the convention $\inf \emptyset = \infty$. One says that $\operatorname{Gid}_R M$ is *finite* if $\operatorname{Gid}_R M < \infty$ holds.

A comment similar to the one after 8.1.2 justifies the last convention in 9.2.9.

9.2.10. Let M be an R-complex. For every semi-injective replacement I of M one has $\mathrm{H}(I) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\operatorname{Gid}_R M \;\geqslant\; -\inf M \qquad \text{and} \qquad \operatorname{Gid}_R \Sigma^s M \;=\; \operatorname{Gid}_R M - s \quad \text{for every integer } s\,.$$

Moreover, one has $\operatorname{Gid}_R M = -\infty$ if and only if M is acyclic.

9.2.11 Lemma. *Let M be an R-complex. For every semi-injective replacement I of M and every integer $v \geqslant \operatorname{Gid}_R M$ the module $\mathrm{Z}_{-v}(I)$ is Gorenstein injective.*

Proof. By 8.2.13 and 9.2.6 it suffices to prove the assertion for some specific semi-injective replacement I of M. One can assume that $\operatorname{Gid}_R M$ is finite; otherwise the statement is empty. One can also assume that M is not acyclic; otherwise $I = 0$ is a semi-injective replacement of M. Thus $g = \operatorname{Gid}_R M$ is an integer and by definition, 9.2.9, there is a semi-injective replacement I of M with $\mathrm{H}_{-v}(I) = 0$ for all $v > g$ and $\mathrm{Z}_{-g}(I)$ Gorenstein injective. Since there are short exact sequences $0 \to \mathrm{Z}_{-v}(I) \to I_{-v} \to \mathrm{Z}_{-(v+1)}(I) \to 0$ for all $v \geqslant g$, it follows from 9.2.2 and 9.2.6 that the modules $\mathrm{Z}_{-g}(I), \mathrm{Z}_{-(g+1)}(I), \ldots$ are Gorenstein injective. □

The following result is sometimes expressed by saying that Gid_R is a refinement of id_R. It follows, in particular, that a Gorenstein injective module is either injective or has infinite injective dimension. The Gorenstein injective module from 9.2.3 has infinite injective dimension; see 8.2.11.

9.2.12 Theorem. *Let M be an R-complex. There is an inequality,*

$$\operatorname{Gid}_R M \;\leqslant\; \operatorname{id}_R M\,,$$

and equality holds if M has finite injective dimension.

Proof. The inequality is evident from the definitions of the dimensions, see 8.2.2 and 9.2.9, and from the fact that every injective module is Gorenstein injective, see 9.2.2. Now assume that $\iota = \operatorname{id}_R M$ is an integer. To prove $\operatorname{Gid}_R M \geqslant \iota$ it must be shown that if I is a semi-injective replacement of M with $\mathrm{H}_{-v}(I) = 0$ for all $v > n$

and $\mathrm{Z}_{-n}(I)$ Gorenstein injective, then $n \geqslant \imath$ holds. Suppose one has $n < \imath$. There is an exact sequence $0 \to \mathrm{Z}_{-n}(I) \to I_{-n} \to \mathrm{Z}_{-(n+1)}(I) \to 0$ and an exact sequence $0 \to \mathrm{Z}_{-(n+1)}(I) \to I_{-(n+1)} \to \cdots \to I_{-(\imath-1)} \to \mathrm{Z}_{-\imath}(I) \to 0$. The second sequence shows that the module $\mathrm{Z}_{-(n+1)}(I)$ has finite injective dimension, as $\mathrm{Z}_{-\imath}(I)$ is injective by 8.2.8, and hence 7.3.36 and 9.2.8(c) imply that the first sequence is split. Thus, $\mathrm{Z}_{-n}(I)$ is a direct summand of I_{-n}, in particular, $\mathrm{Z}_{-n}(I)$ is injective. Now another application of 8.2.8 yields $\imath \leqslant n$, which is a contradiction. □

Equality also holds in 9.2.12 if M has finite flat dimension; see 9.2.19. By 6.5.24 the next result applies, in particular, to a short exact sequence of complexes.

9.2.13 Proposition. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $g' = \mathrm{Gid}_R M'$, $g = \mathrm{Gid}_R M$, and $g'' = \mathrm{Gid}_R M''$ there are inequalities,*

$$g' \leqslant \max\{g, g''+1\}\,, \quad g \leqslant \max\{g', g''\}\,, \quad \textit{and} \quad g'' \leqslant \max\{g'-1, g\}\,.$$

In particular, if two of the complexes M', M, and M'' have finite Gorenstein injective dimension, then so has the third.

Proof. It suffices to prove the second inequality since the first and third inequalities follow by applying the second inequality and 9.2.10 to the distinguished triangles $\Sigma^{-1}M'' \to M' \to M \to M''$ and $M \to M'' \to \Sigma M' \to \Sigma M$; see (TR2) in E.2.

To prove the second inequality, apply 8.2.14 to get an exact sequence of complexes $0 \to I' \to I \to I'' \to 0$ where I', I, and I'' are semi-injective replacements of M', M, and M''. Set $u' = \inf M'$, $u = \inf M$, and $u'' = \inf M''$. One can assume that g' and g'' are finite and that M is not acyclic; otherwise the inequality is trivial. It follows from 6.5.20 that M' and M'' can not both be acyclic, so g' or g'' is an integer, and hence so is $m = \max\{g', g''\}$. Note that 9.2.10 and 6.5.20 yield $m \geqslant -u$, so $\mathrm{H}_{-v}(I) = 0$ for $v > m$. As $m + 1 > g' \geqslant -u'$ one has $\mathrm{H}_{-m-1}(I') = 0$, so the sequence $0 \to \mathrm{Z}_{-m}(I') \to \mathrm{Z}_{-m}(I) \to \mathrm{Z}_{-m}(I'') \to 0$ is exact by 2.2.16. As $m \geqslant g'$ and $m \geqslant g''$ the modules $\mathrm{Z}_{-m}(I')$ and $\mathrm{Z}_{-m}(I'')$ are Gorenstein injective by 9.2.11, and hence so is $\mathrm{Z}_{-m}(I)$ by 9.2.6. Thus $g = \mathrm{Gid}_R M \leqslant m$ holds by 9.2.9. □

Remark. Proposition 9.2.13 essentially shows that the complexes of finite Gorenstein injective dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 9.2.6.

A module of finite Gorenstein injective dimension can be approximated by a Gorenstein injective module and one of finite injective dimension. We obtain this as a corollary to the next result about approximations in the derived category.

9.2.14 Proposition. *Let M be an R-complex of finite Gorenstein injective dimension $g = \mathrm{Gid}_R M$. For every semi-injective replacement I of M and every integer w with $g > w$ there is a distinguished triangle in $\mathcal{D}(R)$,*

$$\Sigma^{-1}K \longrightarrow N \longrightarrow M \longrightarrow K\,,$$

where the complexes K and N have the following properties:

(a) *There is a degreewise split exact sequence $0 \to \Sigma^{-w}G \to K \to I_{\geqslant -w} \to 0$ in $\mathcal{C}(R)$ where G is a Gorenstein injective R-module. Furthermore, one has*

$$\operatorname{Gid}_R K \leqslant w \qquad \textit{and} \qquad \mathrm{H}_v(K) \cong \begin{cases} \mathrm{H}_v(M) & \textit{for } v \geqslant -w+1 \\ 0 & \textit{for } v \leqslant -w-1 \,. \end{cases}$$

(b) *The complex N satisfies*

$$\operatorname{id}_R N = g \qquad \textit{and} \qquad \mathrm{H}_v(N) \cong \begin{cases} 0 & \textit{for } v \geqslant -w \\ \mathrm{H}_v(M) & \textit{for } v \leqslant -w-2 \,. \end{cases}$$

(c) *There is an exact sequence of R-modules,*

$$0 \longrightarrow \mathrm{H}_{-w}(M) \longrightarrow \mathrm{H}_{-w}(K) \longrightarrow \mathrm{H}_{-w-1}(N) \longrightarrow \mathrm{H}_{-w-1}(M) \longrightarrow 0 \,.$$

PROOF. If M is acyclic the statement is void as no integer w satisfies $-\infty = g > w$. Now assume that M is not acyclic, in which case g is an integer. By 9.2.11 the module $\mathrm{Z}_{-g}(I)$ is Gorenstein injective and by 9.2.1 there exists an acyclic R-complex $E = 0 \to G \to E_{-w} \cdots \to E_{-g+1} \to \mathrm{Z}_{-g}(I) \to 0$, concentrated in degrees $-w+1, \dots, -g$, where the modules $E_{-w}, \dots, E_{-g+1}$ are injective and the kernels are Gorenstein injective. Set $E_{-w+1} = G$ and $E_{-g} = \mathrm{Z}_{-g}(I)$ and notice that, in particular, the kernel $\mathrm{Z}_{-w}(E) \cong G$ is Gorenstein injective.

Consider the short exact sequence of R-complexes,

$$(\diamond) \qquad 0 \longrightarrow E_{\supseteq -g+1} \longrightarrow E_{\geqslant -g+1} \xrightarrow{\alpha} \Sigma^{-g+1}\mathrm{Z}_{-g}(I) \longrightarrow 0 \,,$$

where α is a quasi-isomorphism as $E_{\supseteq -g+1}$ is acyclic; see 4.2.6. Let J be any complex of injective R-modules. As the complex $E_{\supseteq -g+1}$ consists of Gorenstein injective modules, it follows from 7.3.38 and 9.2.8(c) that the functor $\operatorname{Hom}_R(J, -)$ leaves the sequence $(\diamond)$ exact. As $E_{\supseteq -g+1}$ is an acyclic complex with $\mathrm{Z}_v(E_{\supseteq -g+1})$ Gorenstein injective for every v, it follows from 9.2.8(c) and A.4 that $\operatorname{Hom}_R(J_n, E_{\supseteq -g+1})$ is acyclic for every $n \in \mathbb{Z}$, and hence $\operatorname{Hom}_R(J, E_{\supseteq -g+1})$ is acyclic by A.5. It now follows from 4.2.6 that $\operatorname{Hom}_R(J, \alpha)$ is a surjective quasi-isomorphism, whence the morphism $\mathcal{C}(R)(J, \alpha)$ is surjective as well by 4.2.7 and 2.3.10. Surjectivity of $\mathcal{C}(R)(I_{\geqslant -g+1}, \alpha)$ yields a commutative diagram of R-complexes,

$$\begin{tikzcd} I_{\geqslant -g+1} \arrow[r, dotted, "\gamma"] \arrow[d, equal] & E_{\geqslant -g+1} \arrow[d, "\alpha", "\simeq"'] \\ I_{\geqslant -g+1} \arrow[r, "\beta"] & \Sigma^{-g+1}\mathrm{Z}_{-g}(I) \,, \end{tikzcd}$$

where β is induced by ∂^I_{-g+1}. This diagram—in conjunction with the definition of distinguished triangles in $\mathcal{D}(R)$, see 6.5.5, the axiom (TR3) in E.2, and 6.5.19—shows that the complexes $\operatorname{Cone}\beta$ and $\operatorname{Cone}\gamma$ are isomorphic in $\mathcal{D}(R)$. Evidently, one has $\Sigma^{-1}\operatorname{Cone}\beta = I_{\supseteq -g}$, and this complex is isomorphic to $I \simeq M$ in $\mathcal{D}(R)$. Consequently, $\Sigma^{-1}\operatorname{Cone}\gamma \simeq M$ holds in $\mathcal{D}(R)$.

Set $C = \operatorname{Cone}\gamma$. By 2.5.22 and 6.5.24, and by the axiom (TR2) in E.2, there is a distinguished triangle in $\mathcal{D}(R)$,

$$\Sigma^{-1}(\Sigma^{-1}C_{\geqslant -w+1}) \longrightarrow \Sigma^{-1}C_{\leqslant -w} \longrightarrow \Sigma^{-1}C \longrightarrow \Sigma^{-1}C_{\geqslant -w+1} \,,$$

which we argue is the desired one. As noticed, $\Sigma^{-1}C \simeq M$ in $\mathcal{D}(R)$. The complex

$$K = \Sigma^{-1}C_{\geqslant -w+1} = \cdots \longrightarrow I_{-w+2} \longrightarrow I_{-w+1} \longrightarrow I_{-w} \oplus G \longrightarrow 0$$

is concentrated in degrees $\geqslant -w$ and fits into the degreewise split exact sequence

$$(\star) \qquad 0 \longrightarrow \Sigma^{-w}G \longrightarrow K \longrightarrow I_{\geqslant -w} \longrightarrow 0\,.$$

The complex I is semi-injective, and so is $I_{\leqslant -w-1}$ by 5.3.12. Hence 5.3.20 applied to the exact sequence $0 \to I_{\leqslant -w-1} \to I \to I_{\geqslant -w} \to 0$ shows that $I_{\geqslant -w}$ is semi-injective. It follows from 8.2.2 and 9.2.12 that $\operatorname{Gid}_R(I_{\geqslant -w}) = \operatorname{id}_R(I_{\geqslant -w}) \leqslant w$ holds. As G is a Gorenstein injective module, one has $\operatorname{Gid}_R \Sigma^{-w}G \leqslant w$, with equality if G is non-zero, so application of 6.5.24 and 9.2.13 to $(\star)$ shows that $\operatorname{Gid}_R K \leqslant w$. The assertion about the homology of $K = \Sigma^{-1}C_{\geqslant -w+1}$ follows as $\Sigma^{-1}C \simeq M$ in $\mathcal{D}(R)$. Notice that

$$N = \Sigma^{-1}C_{\leqslant -w} = 0 \longrightarrow E_{-w} \oplus I_{-w-1} \longrightarrow \cdots \longrightarrow E_{-g+2} \oplus I_{-g+1} \longrightarrow E_{-g+1} \longrightarrow 0$$

is a complex of injective R-modules concentrated in degrees $-w-1, \dots, -g$. In the extremal case $w = g-1$ one has $N = \Sigma^{-g}E_{-g+1}$. In particular, N is semi-injective by 5.3.12 and $\operatorname{id}_R N \leqslant g$ holds. Note that 9.2.13 yields $g \leqslant \max\{\operatorname{Gid}_R N, \operatorname{Gid}_R K\}$. As $\operatorname{Gid}_R K \leqslant w < g$ holds one has $\operatorname{Gid}_R N \geqslant g$, so 9.2.12 yields $\operatorname{id}_R N = g$. The assertion about the homology of $N = \Sigma^{-1}C_{\leqslant -w}$ follows as $\Sigma^{-1}C \simeq M$ in $\mathcal{D}(R)$. The exact sequence in part (c) follows by applying 6.5.19 to the constructed distinguished triangle. □

9.2.15 Corollary. *Let M be an R-module of finite Gorenstein injective dimension $g = \operatorname{Gid}_R M$. The following assertions hold.*

(a) *There is an exact sequence of R-modules $0 \to G \to X \to M \to 0$ where G is Gorenstein injective and $\operatorname{id}_R X = g$.*
(b) *If $g > 0$, then there is an exact sequence of R-modules $0 \to M \to G \to X \to 0$ where G is Gorenstein injective and $\operatorname{id}_R X = g-1$.*

Proof. (a): For $w = -1$ the sequence 9.2.14(c) reads

$$0 \longrightarrow \mathrm{H}_1(K) \longrightarrow \mathrm{H}_0(N) \longrightarrow M \longrightarrow 0\,.$$

If follows from 9.2.14(a) that K is isomorphic to $\Sigma\mathrm{H}_1(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gid}_R \mathrm{H}_1(K) - 1 = \operatorname{Gid}_R K \leqslant w = -1$. Consequently, the module $\mathrm{H}_1(K)$ is Gorenstein injective. Similarly, one has $N \simeq \mathrm{H}_0(N)$ and $\operatorname{id}_R \mathrm{H}_0(N) = \operatorname{id}_R N = g$.

(b): As $g > 0$ one can apply 9.2.14(c) with $w = 0$ to obtain the exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{H}_0(K) \longrightarrow \mathrm{H}_{-1}(N) \longrightarrow 0\,.$$

If follows from 9.2.14(a) that K is isomorphic to $\mathrm{H}_0(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gid}_R \mathrm{H}_0(K) = \operatorname{Gid}_R K \leqslant w = 0$. That is, the module $\mathrm{H}_0(K)$ is Gorenstein injective. Similarly, one has $N \simeq \Sigma^{-1}\mathrm{H}_{-1}(N)$ and $\operatorname{id}_R \mathrm{H}_{-1}(N) + 1 = \operatorname{id}_R N = g$. □

The technical result below has a consequence that for our purposes is extremely useful; see 9.2.17.

9.2.16 Proposition. *Let I and E be semi-injective R-complexes. If E has finite flat or finite injective dimension, then the morphism*

$$\operatorname{Hom}_R(E,\tau^I_{\supseteq -n}) :\ \operatorname{Hom}_R(E,I_{\supseteq -n}) \longrightarrow \operatorname{Hom}_R(E,I)$$

is a quasi-isomorphism for every integer $n \geqslant \operatorname{Gid}_R I$.

Proof. Let $E \xrightarrow{\simeq} J$ be a semi-injective resolution; by 5.3.24 there is a homotopy equivalence $\iota\colon J \to E$. By 9.2.10 one has $-n \leqslant \inf I$, so the map $\tau^I_{\supseteq -n}\colon I_{\supseteq -n} \to I$ is a quasi-isomorphism by 4.2.4. Choose a semi-projective resolution $\pi\colon P \xrightarrow{\simeq} J$. In the commutative diagram,

$$\begin{tikzcd}
\operatorname{Hom}_R(E,I_{\supseteq -n}) \arrow[r, "\operatorname{Hom}(\iota,I_{\supseteq -n})", "\approx"'] \arrow[d, "\operatorname{Hom}(E,\tau^I_{\supseteq -n})"] & \operatorname{Hom}_R(J,I_{\supseteq -n}) \arrow[r, "\operatorname{Hom}(\pi,I_{\supseteq -n})"] \arrow[d, "\operatorname{Hom}(J,\tau^I_{\supseteq -n})"] & \operatorname{Hom}_R(P,I_{\supseteq -n}) \arrow[d, "\operatorname{Hom}(P,\tau^I_{\supseteq -n})", "\simeq"'] \\
\operatorname{Hom}_R(E,I) \arrow[r, "\operatorname{Hom}(\iota,I)", "\approx"'] & \operatorname{Hom}_R(J,I) \arrow[r, "\operatorname{Hom}(\pi,I)", "\simeq"'] & \operatorname{Hom}_R(P,I)\ ,
\end{tikzcd}$$

the left-hand horizontal maps are homotopy equivalences by 4.3.19, and $\operatorname{Hom}_R(\pi,I)$ and $\operatorname{Hom}_R(P,\tau^I_{\supseteq -n})$ are quasi-isomorphisms by semi-injectivity of I and semi-projectivity of P. The diagram shows that it suffices to verify that $\operatorname{Hom}_R(\pi,I_{\supseteq -n})$ is a quasi-isomorphism. Set $C = \operatorname{Cone}\pi$ and notice that it is an acyclic complex of modules that are direct sums of projective and injective modules. The goal is to show that the complex $\operatorname{Hom}_R(C,I_{\supseteq -n}) \cong \Sigma^{-1}\operatorname{Cone}\operatorname{Hom}_R(\pi,I_{\supseteq -n})$ is acyclic; cf. 4.1.16.

By 2.5.25 there is an exact sequence $0 \to I_{\supseteq -n} \to I \to I' \to 0$ with I' bounded above. By 9.2.2, 9.2.11, and 9.2.6(a) these are complexes of Gorenstein injective modules, and it follows from 7.3.38, in view of 9.2.8(c), that the sequence

$$0 \longrightarrow \operatorname{Hom}_R(C,I_{\supseteq -n}) \longrightarrow \operatorname{Hom}_R(C,I) \longrightarrow \operatorname{Hom}_R(C,I') \longrightarrow 0$$

is exact. The middle complex is acyclic as I is semi-injective and C is acyclic. To prove acyclicity of $\operatorname{Hom}_R(C,I_{\supseteq -n})$ it now suffices to see that $\operatorname{Hom}_R(C,I')$ is acyclic; see 2.5.6. To that end, it suffices by A.2 to argue that $\operatorname{Hom}_R(C,G)$ is acyclic for every Gorenstein injective module G. It is already known from 9.2.8(c) that $\operatorname{Ext}^m_R(C_v,G) = 0$ holds for all $v \in \mathbb{Z}$ and all $m > 0$, so it suffices by A.1 to argue that $\operatorname{Ext}^m_R(\mathrm{C}_v(C),G) = 0$ holds for $v \ll 0$ and all $m > 0$. With the notation from 9.2.8 we need to show that one has $\mathrm{C}_v(C) \in \mathcal{N}$ for $v \ll 0$.

If E has finite injective dimension, then one can assume that the complexes J and P are bounded below; see 8.2.2 and 5.2.15. It follows that C is bounded below; in particular, $\mathrm{C}_v(C) = 0$ holds for $v \ll 0$.

If E has finite flat dimension, then one can by 5.3.26 assume that the complex J is bounded above, and hence one has $\mathrm{C}_v(C) = \mathrm{C}_{v-1}(P)$ for $v \gg 0$. Further, 8.3.11 shows that the module $\mathrm{C}_{v-1}(P)$ is flat for $v \gg 0$, so 9.2.8(c) implies that one has $\mathrm{C}_v(C) \in \mathcal{N}$ for $v \gg 0$. As already noticed, $C_v \in \mathcal{N}$ holds for every $v \in \mathbb{Z}$. Using 9.2.8(a) and induction on exact sequences $0 \to \mathrm{C}_{v+1}(C) \to C_v \to \mathrm{C}_v(C) \to 0$, it now follows that $\mathrm{C}_v(C) \in \mathcal{N}$ holds for every $v \in \mathbb{Z}$. □

The gist of 9.2.4 is that projective and injective modules have the same homological behavior relative to Gorenstein injective modules. This has the following usseful consequence:

9.2.17 Corollary. *Let M be an R-complex of finite Gorenstein injective dimension and N an R-complex of finite flat or finite injective dimension. For every semi-injective replacement I of M, every semi-injective replacement E of N, and every integer $n \geqslant \operatorname{Gid}_R M$ there is an isomorhism in $\mathcal{D}(\Bbbk)$,*

$$\mathsf{RHom}_R(N,M) \simeq \operatorname{Hom}_R(E, I_{\supseteq -n}) .$$

Proof. The assertion follows immediately from 9.2.16 and 7.3.7. □

A key difference between 9.2.18 below and the main theorem about injective dimension, 8.2.8, is the a priori assumption in 9.2.18 that the complex has finite Gorenstein injective dimension.

9.2.18 Theorem. *Let M be an R-complex of finite Gorenstein injective dimension and n an integer. The following conditions are equivalent.*

(i) $\operatorname{Gid}_R M \leqslant n$.

(ii) $-\inf \mathsf{RHom}_R(N,M) \leqslant n + \sup N$ *holds for every R-complex N with* $\operatorname{fd}_R N$ *finite or* $\operatorname{id}_R N$ *finite.*

(iii) $-\inf \mathsf{RHom}_R(N,M) \leqslant n$ *holds for every injective R-module N.*

(iv) $n \geqslant -\inf M$ *and* $\operatorname{Ext}_R^{n+1}(N,M) = 0$ *for every R-module N with* $\operatorname{id}_R N$ *finite.*

(v) $n \geqslant -\inf M$ *and for some, equivalently every, semi-injective replacement I of M, the module $\mathrm{Z}_{-v}(I)$ is Gorenstein injective for every $v \geqslant n$.*

(vi) *There is a semi-injective resolution $M \xrightarrow{\simeq} I$ with $I_{-v} = 0$ for all $v < -\sup M$, $\mathrm{H}_{-v}(I) = 0$ for all $v > n$, and $\mathrm{Z}_{-v}(I)$ Gorenstein injective for all $v \geqslant n$.*

In particular, there are equalities

$$\begin{aligned}\operatorname{Gid}_R M &= \sup\{-\inf \mathsf{RHom}_R(N,M) \mid N \text{ is an injective } R\text{-module}\}\\ &= \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(N,M) \neq 0 \text{ for some injective } R\text{-module } N\} .\end{aligned}$$

Proof. We start by establishing the equivalence of (i), (ii), and (iii).

(i) $\Rightarrow$ (ii): One can assume that N is in $\mathcal{D}_{\sqsubset}(R)$ and not acyclic; otherwise the inequality is trivial. In this case, $w = \sup N$ is an integer. By 5.3.26 there is a semi-injective resolution $N \xrightarrow{\simeq} J$ with $J_{-v} = 0$ for $v < -w$. If $\operatorname{fd}_R N$ or $\operatorname{id}_R N$ is finite, then 9.2.17 implies that one has $\mathsf{RHom}_R(N,M) \simeq \operatorname{Hom}_R(J, I_{\supseteq -n})$, where I is any semi-injective replacement of M. For every $v > n + w$ and $p \in \mathbb{Z}$ one of the inequalities $p < -w$ or $v + p \geqslant v - w > n$ holds, so the module

$$\operatorname{Hom}_R(J, I_{\supseteq -n})_{-v} = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(J_{-p}, (I_{\supseteq -n})_{-(v+p)})$$

is zero. In particular, $\mathrm{H}_{-v}(\mathsf{RHom}_R(M,N)) = 0$ holds for $v > n + w = n + \sup N$.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): By assumption $g = \operatorname{Gid}_R M$ is finite, and it must be shown that (iii) implies $g \leqslant n$. One can assume that M is not acyclic, as otherwise the inequality is trivial. By definition there exists a semi-injective replacement I of M with $\mathrm{Z}_{-g}(I)$ Gorenstein injective. By 9.2.17 there is an isomorphism,

$$(\star) \qquad \mathsf{RHom}_R(J,M) \simeq \mathrm{Hom}_R(J, I_{\supseteq -g})\,,$$

for every injective R-module J. Recall from 9.2.10 that $g \geqslant -\inf M = -\inf I$. We consider two different cases:

First assume that $g = -\inf I$ holds; this implies $\mathrm{H}_{-g}(I) \neq 0$ so the homomorphism $I_{-g+1} \to \mathrm{Z}_{-g}(I)$ is not surjective. Since $\mathrm{Z}_{-g}(I)$ is Gorenstein injective there exists, in particular, a surjection $J \twoheadrightarrow \mathrm{Z}_{-g}(I)$ where J is an injective module. This map does not admit a factorization $J \to I_{-g+1} \to \mathrm{Z}_{-g}(I)$, as this would force $I_{-g+1} \to \mathrm{Z}_{-g}(I)$ to be surjective. Thus the map $\mathrm{Hom}_R(J, I_{-g+1}) \to \mathrm{Hom}_R(J, \mathrm{Z}_{-g}(I))$ is not surjective, so $\inf \mathrm{Hom}_R(J, I_{\supseteq -g}) = -g$. Now $(\star)$ and (*iii*) yield $g = -\inf \mathsf{RHom}_R(J,M) \leqslant n$.

Next assume that $g > -\inf I$. In this case there is an exact sequence of modules, $0 \to \mathrm{Z}_{-g+1}(I) \to I_{-g+1} \to \mathrm{Z}_{-g}(I) \to 0$. As $g > -\inf I$ and $g = \mathrm{Gid}_R M$, the module $\mathrm{Z}_{-g+1}(I)$ is not Gorenstein injective. Hence 9.2.6 yields $\mathrm{Ext}^1_R(J, \mathrm{Z}_{-g+1}(I)) \neq 0$ for some injective module J, and by 8.2.6 this means that $\mathrm{H}_{-g}(\mathsf{RHom}_R(J,M)) \neq 0$. Hence $g \leqslant -\inf \mathsf{RHom}_R(J,M) \leqslant n$, where the last inequality holds by (*iii*).

To finish the proof we show the implications (*ii*) $\Rightarrow$ (*iv*) $\Rightarrow$ (*v*) $\Rightarrow$ (*vi*) $\Rightarrow$ (*i*).

(*ii*) $\Rightarrow$ (*iv*): The second assertion in (*iv*) is immediate from (*ii*). The inequality $n \geqslant -\inf M$ follows, in view of 9.2.10, from (*i*), which is equivalent to (*ii*).

(*iv*) $\Rightarrow$ (*v*): First note that by 8.2.13 and 9.2.6 the "some" version and the "every" version of condition (*v*) are equivalent. By assumption, $g = \mathrm{Gid}_R M$ is finite, so in any semi-injective replacement I of M the module $\mathrm{Z}_{-v}(I)$ Gorenstein injective for every integer $v \geqslant g$; see 9.2.11. Thus, to show (*v*) it is enough to prove $n \geqslant g$. Assume towards a contradiction that $n < g$ holds. Notice that by the assumption $n \geqslant -\inf M$, the module $\mathrm{Z}_{-n}(I)$ can not be Gorenstein injective. There is an exact sequence $0 \to \mathrm{Z}_{-n}(I) \to I_{-n} \to \cdots \to I_{-g+1} \to \mathrm{Z}_{-g}(I) \to 0$, which shows that $\mathrm{Z}_{-n}(I)$ has finite Gorenstein injective dimension, as $I_{\leqslant -n}$ is a semi-injective replacement of $\Sigma^{-n}\mathrm{Z}_{-n}(I)$. Now 9.2.15 yields an exact sequence $0 \to \mathrm{Z}_{-n}(I) \to G \to N \to 0$ where G is Gorenstein injective and N has finite injective dimension. By 8.2.6 and (*iv*) one has $\mathrm{Ext}^1_R(N, \mathrm{Z}_{-n}(I)) \cong \mathrm{Ext}^{n+1}_R(N,M) = 0$, so the sequence is split by 7.3.36. Now 9.2.6 implies that $\mathrm{Z}_{-n}(I)$ is Gorenstein injective, which is a contradiction.

(*v*) $\Rightarrow$ (*vi*): This implication is immediate in view of 5.3.26.

(*vi*) $\Rightarrow$ (*i*): This implication is immediate from the definition, 9.2.9, of Gid_R.

The equalities in the last assertion follow immediately from the equivalence of (*i*)–(*iii*) and 7.3.24. □

The next result reflects the inability of Gorenstein injective modules to tell injective and projective, even flat, modules apart.

9.2.19 Theorem. *Let M be an R-complex. If M has finite flat dimension, then the equality* $\mathrm{Gid}_R M = \mathrm{id}_R M$ *holds.*

Proof. The inequality $\mathrm{Gid}_R M \leqslant \mathrm{id}_R M$ holds by 9.2.12. To show the opposite inequality, one can assume that $g = \mathrm{Gid}_R M$ is an integer. Set $w = -\sup M - 1$, which is an integer by 8.1.3, and note that one has $w < -\sup M \leqslant -\inf M \leqslant g$ by 9.2.10. Thus 9.2.14 yields a distinguished triangle in $\mathcal{D}(R)$,

$$(\dagger) \qquad \Sigma^{-1}K \longrightarrow N \longrightarrow M \longrightarrow K\,,$$

with $\operatorname{Gid}_R K \leqslant w$ and $\operatorname{id}_R N = g$. As $\operatorname{fd}_R M$ is finite, 9.2.18 yields

$$-\inf \mathsf{R}\!\operatorname{Hom}_R(M,K) \;\leqslant\; \operatorname{Gid}_R K + \sup M \;\leqslant\; w + \sup M \;=\; -1\,,$$

and hence $\operatorname{H}_0(\mathsf{R}\!\operatorname{Hom}_R(M,K)) = 0$. By 7.3.26 this means that $\mathcal{D}(R)(M,K) = 0$, in particular, the morphism $M \to K$ in the distinguished triangle above is zero. By E.22 this means the triangle (†) is split, and hence $N \simeq (\Sigma^{-1}K) \oplus M$ holds in $\mathcal{D}(R)$. In particular, $\operatorname{id}_R M \leqslant \operatorname{id}_R N = g$ holds as claimed. □

9.2.20 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes; there is an equality,*

$$\operatorname{Gid}_R\Big(\prod_{u\in U} M^u\Big) \;=\; \sup_{u\in U}\{\operatorname{Gid}_R M^u\}\,.$$

Proof. To prove the inequality "$\leqslant$", one can assume that the right-hand side is finite, say, $s \in \mathbb{Z}$. By the definition, 9.2.9, of Gorenstein injective dimension and by 9.2.11 every M^u admits a semi-injective replacement I^u with $\operatorname{H}_{-v}(I^u) = 0$ for all $v > s$ and $\operatorname{Z}_{-v}(I^u)$ Gorenstein injective for all $v \geqslant s$. Now $I = \prod_{u\in U} I^u$ is a semi-injective replacement of $\prod_{u\in U} M^u$ with $\operatorname{H}_{-v}(I) = 0$ for all $v > s$, see 5.3.21 and 3.1.22(d), and $\operatorname{Z}_{-v}(I) = \prod_{u\in U}\operatorname{Z}_{-v}(I^u)$ Gorenstein injective for all $v \geqslant s$, see 3.1.22(a) and 9.2.6.

To prove the opposite inequality "$\geqslant$" it suffices, as each M^u is a direct summand of $\prod_{u\in U} M^u$, to argue that if M' is a direct summand of an R-complex M, then one has $\operatorname{Gid}_R M' \leqslant \operatorname{Gid}_R M$. To this end, one may assume that M is not acyclic and that $g = \operatorname{Gid}_R M$ is finite. Let M'' be an R-complex with $M = M' \oplus M''$. Let I' and I'' be semi-injective replacements of M' and M''. Now $I = I' \oplus I''$ is a semi-injective replacement of M, see 5.3.21. As $\operatorname{H}_{-v}(I) = 0$ holds for all $v > g$, even for all $v > -\inf M$, one has $\operatorname{H}_{-v}(I') = 0$ for all $v > g$ by 3.1.23. Per 9.2.11 the module $\operatorname{Z}_{-v}(I) = \operatorname{Z}_{-v}(I') \oplus \operatorname{Z}_{-v}(I'')$ is Gorenstein injective for every $v \geqslant g$, whence $\operatorname{Z}_{-v}(I')$ is Gorenstein injective for $v \geqslant g$, by 9.2.6, and $\operatorname{Gid}_R M' \leqslant g$ holds. □

The Case of Modules

9.2.21. Notice from 9.2.18 that a non-zero R-module is Gorenstein injective if and only if it has Gorenstein injective dimension 0 as an R-complex.

9.2.22 Theorem. *Let M be an R-module of finite Gorenstein injective dimension and $n \geqslant 0$ an integer. The following conditions are equivalent.*

(*i*) $\operatorname{Gid}_R M \leqslant n$.

(*ii*) $\operatorname{Ext}_R^m(N,M) = 0$ *holds for every R-module N with* $\operatorname{fd}_R N$ *finite or* $\operatorname{id}_R N$ *finite and all integers $m > n$.*

(*iii*) $\operatorname{Ext}_R^m(N,M) = 0$ *holds for every injective R-module N and all integers $m > n$.*

(*iv*) $\operatorname{Ext}_R^{n+1}(N,M) = 0$ *holds for every R-module N with* $\operatorname{id}_R N$ *finite.*

(*v*) *In some/every injective resolution* $0 \to M \to I_0 \to \cdots \to I_{-(v-1)} \to I_{-v} \to \cdots$ *the module* $\operatorname{Ker}(I_{-v} \to I_{-(v+1)})$ *is Gorenstein injective for every* $v \geqslant n$.

(*vi*) *There is an exact sequence* $0 \to M \to I_0 \to \cdots \to I_{-(n-1)} \to G \to 0$ *of R-modules with each* I_i *injective and G Gorenstein injective.*

In particular, there is an equality

$$\operatorname{Gid}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}^m_R(N, M) \neq 0 \text{ for some injective } R\text{-module } N\}\,.$$

Proof. By 5.3.31 every R-module M has an injective resolution

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{-(v-1)} \longrightarrow I_{-v} \longrightarrow \cdots\,.$$

In every such resolution, the injective homomorphism $M \to I_0$ yields a semi-injective resolution of M, considered as a complex; cf. 5.3.33. Thus the complex $0 \to I_0 \to \cdots \to I_{-(v-1)} \to I_{-v} \to \cdots$ is a semi-injective replacement of M. The equivalence of conditions (*i*)–(*vi*) now immediate from 9.2.18, and so is the asserted equality in view of 7.3.27. □

Exercises

E 9.2.1 Let $\{M^u\}_{u \in U}$ be a family of R-modules. Show that $\coprod_{u\in U} M^u$ is Gorenstein injective only if M^u is Gorenstein injective for every $u \in U$.

E 9.2.2 Show that if every R-module has finite Gorenstein injective dimension, then every acyclic complex of injective R-modules is totally acyclic.

E 9.2.3 Show that for an R-module M the next conditions are equivalent. (*i*) For every Gorenstein injective R-module G one has $\operatorname{Ext}^m_R(M, G) = 0$ for all $m \gg 0$. (*ii*) For every Gorenstein injective R-module G one has $\operatorname{Ext}^m_R(M, G) = 0$ for all $m \geqslant 1$. Show that $\operatorname{Gid}_R M = \operatorname{id}_R M$ holds for every R-module M that satisfies these conditions.

E 9.2.4 Let M be a complex in $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $u = \inf M$. Show that for every semi-injective replacement I of M one has $\operatorname{Gid}_R M = \operatorname{Gid}_R \mathrm{Z}_u(I) - u$.

E 9.2.5 Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$. Show that $\operatorname{Gid}_R M$ is finite if and only if M is isomorphic in $\mathcal{D}(R)$ to a bounded complex of Gorenstein injective R-modules.

E 9.2.6 Show that the full subcategory of R-complexes of finite Gorenstein injective dimension is a triangulated subcategory of $\mathcal{D}_{\sqsupset}(R)$.

E 9.2.7 Let M be a complex in $\mathcal{D}_{\square}(R)$ and G a bounded above complex of Gorenstein injective R-modules with $M \simeq G$ in $\mathcal{D}(R)$. Show that for every R-module N with $\operatorname{fd}_R N$ or $\operatorname{id}_R N$ finite and for all $m > 0$ and $n \geqslant -\inf M$ one has $\operatorname{Ext}^{n+m}_R(N, M) \cong \operatorname{Ext}^m_R(N, \mathrm{Z}_{-n}(G))$.

E 9.2.8 Show that if the coproduct of every (countable) family of Gorenstein injective R-modules is Gorenstein injective, then R is left Noetherian. *Hint:* 8.2.20.

E 9.2.9 Let n be an integer. Show that an R-complex M has $\operatorname{Gid}_R M \leqslant n$ if and only if there is a diagram $M \xrightarrow{\iota} I' \xrightarrow{\tau} I$ in $\mathcal{C}(R)$ where I is a totally acyclic complex of injective R-modules, τ_{-v} is an isomorphism for $v \geqslant n$, and ι is a semi-injective resolution.

9.3 Gorenstein Flat Dimension

SYNOPSIS. Totally acyclic complex of flat modules; Gorenstein flat module; Gorenstein flat dimension; ~ vs. Gorenstein injective dimension; ~ vs. flat dimension; ~ vs. Gorenstein projective dimension; ~ over Noetherian ring; ~ of module; restricted flat dimension.

The Gorenstein flat dimension refines the flat dimension and, under mild assumptions on the ring, also the Gorenstein projective dimension. The definition of Gorenstein flat modules involves a tensor product, and that sets up a duality with Gorenstein injective modules that is essential for our development of the theory.

GORENSTEIN FLAT MODULES

9.3.1 Definition. A complex F of flat R-modules is called *totally acyclic* if it is acyclic and $E \otimes_R F$ is acyclic for every injective R^{o}-module E.

An R-module G is called *Gorenstein flat* if one has $G \cong \mathrm{C}_0(F)$ for some totally acyclic complex F of flat R-modules.

Notice that if F is a totally acyclic complex of flat R-modules, then the module $\mathrm{C}_v(F)$ is Gorenstein flat for every $v \in \mathbb{Z}$.

9.3.2 Example. A flat R-module F is Gorenstein flat as the disk complex $\mathrm{D}^0(F) = 0 \longrightarrow F \xrightarrow{=} F \longrightarrow 0$ is totally acyclic.

As projective modules are flat, total acyclicity of a complex of projective modules could per 9.1.1 and 9.3.1 mean two potentially different things. We avoid confusion by stating whether we consider such a complex to be totally acyclic as a complex of projective modules or as a complex of flat modules. For a complex of finitely generated projective modules the definitions agree, see 9.3.3.

REMARK. To avoid the potential confusion discussed above, some authors refer to the complexes defined in 9.3.1 as 'F-totally acyclic'. The two notions actually agree under mild assumptions on the ring: Let R be right coherent and P a complex of projective R-modules. If P is totally acyclic in the sense of 9.3.1, then it is totally acyclic in the sense of 9.1.1, and the converse holds if flat R-modules have finite projective dimension. Both assertions are shown by Christensen and Kato [67] under the additional but superfluous assumtion that R is commutative; see also E 9.3.1.

9.3.3 Example. An acyclic complex of finitely presented R-modules is by 1.3.47 and 9.1.5 a totally acyclic complex of flat R-modules if and only if it is totally acyclic complex of projective R-modules; in particular, the cokernels in such a complex are Gorenstein projective and Gorenstein flat.

9.3.4 Example. The Dold complex D from 2.1.23 is by 9.1.6 and 9.3.3 an acyclic complex of flat $\mathbb{Z}/4\mathbb{Z}$-modules. In particular, $\mathrm{C}_0(D) \cong \mathbb{Z}/2\mathbb{Z}$ is a Gorenstein flat $\mathbb{Z}/4\mathbb{Z}$-module.

Relative to a Gorenstein flat module, injective modules act like flat modules.

9.3.5 Lemma. *Let G be an R-module; it is Gorenstein flat if and only if it satisfies the following requirements:*

(1) *For every injective R^{o}-module E one has $\operatorname{Tor}_m^R(E,G) = 0$ for all $m > 0$.*
(2) *There exists an exact sequence of R-modules, $0 \to G \to F_0 \to F_{-1} \to \cdots$, where each F_v is flat and the sequence*

$$0 \longrightarrow E \otimes_R F_0 \longrightarrow E \otimes_R F_{-1} \longrightarrow \cdots$$

is exact for every injective R^{o}-module E.

Proof. Assume first that G is Gorenstein flat and let F be a totally acyclic complex of flat R-modules with $G \cong \mathrm{C}_0(F)$; see 9.3.1. The truncated complex $F_{\subseteq 0}$ and the isomorphism $G \cong \mathrm{C}_0(F)$ yield, up to indexing, the exact sequence asserted in (2). The complex $F_{\geqslant 0}$ is a semi-flat replacement of G; by 7.4.17 it can be used to compute the derived tensor product $- \otimes_R^{\mathsf{L}} G$. For $m > 0$ the definition of Tor, 7.4.18, and total acyclicity of F yield

$$\operatorname{Tor}_m^R(E,G) \;=\; \mathrm{H}_m(E \otimes_R F_{\geqslant 0}) \;=\; \mathrm{H}_m(E \otimes_R F) \;=\; 0\,.$$

Assuming now that G satisfies the two requirements, let $\pi\colon P \xrightarrow{\simeq} G$ be a projective resolution, see 8.3.25, and denote by F the complex $0 \to F_0 \to F_{-1} \to \cdots$. The homomorphism $G \to F_0$ induces a quasi-isomorphism $\iota\colon G \to F$, so $\widetilde{F} = \Sigma^{-1}\mathrm{Cone}(\iota\pi)$ is by 1.3.43, 4.1.1, and 4.2.16 an acyclic complex of flat R-modules with $G \cong \mathrm{C}_0(\widetilde{F})$. Let E be an injective R^{o}-module. By (2) the morphism $E \otimes_R \iota$ is a quasi-isomorphism, and by (1) so is $E \otimes_R \pi$, see 7.4.21. It follows that $E \otimes_R \iota\pi$ is a quasi-isomorphism, whence the complex $\Sigma^{-1}\mathrm{Cone}(E \otimes_R \iota\pi) \cong E \otimes_R \widetilde{F}$ is acyclic, see 4.1.18 and 4.2.16. Thus $\widetilde{F}$ is a totally acyclic complex of flat R-modules and G is Gorenstein flat. □

9.3.6 Proposition. *Let $\{G^u\}_{u \in U}$ be a family of R-modules. If every G^u is Gorenstein flat, then the coproduct $\coprod_{u \in U} G^u$ is Gorenstein flat.*

Proof. For each $u \in U$ there exists a totally acyclic complex F^u of flat R-modules with $G^u \cong \mathrm{C}_0(F^u)$. The complex $F = \coprod_{u \in U} F^u$ consists of flat modules by 5.4.22 and it is acyclic by 3.1.11. For every R^{o}-module E there is by 3.1.13 an isomorphism $E \otimes_R F \cong \coprod_{u \in U} E \otimes_R F^u$, and this complex is acyclic if E is injective. Thus, F is a totally acyclic complex of flat R-modules. As $\mathrm{C}_0(F) \cong \coprod_{u \in U} \mathrm{C}_0(F^u) \cong \coprod_{u \in U} G^u$ holds by 3.1.10(c), it follows that $\coprod_{u \in U} G^u$ is Gorenstein flat. □

The next result smacks of flat–injective duality, but notice that it is not a biconditional statement.

9.3.7 Proposition. *Let G be an R-module. If G is Gorenstein flat, then the R^{o}-module $\operatorname{Hom}_{\Bbbk}(G,\mathbb{E})$ is Gorenstein injective.*

Proof. Let F be a totally acyclic complex of flat R-modules with $G \cong \mathrm{C}_0(F)$. The R^{o}-complex $I = \operatorname{Hom}_{\Bbbk}(F,\mathbb{E})$ is acyclic, and each module I_v is injective; see 5.4.19. For every R^{o}-module E one has $\operatorname{Hom}_{R^{\mathrm{o}}}(E,I) \cong \operatorname{Hom}_{\Bbbk}(E \otimes_R F,\mathbb{E})$ by adjunction 4.4.12, so $\operatorname{Hom}_{R^{\mathrm{o}}}(E,I)$ is acyclic if E is injective. Therefore, I is a totally acyclic complex of injective R°-modules; see 9.2.1. By 2.2.19 one has

$$\mathrm{Z}_0(I) \;\cong\; \mathrm{Hom}_{\Bbbk}(\mathrm{C}_0(F),\mathbb{E}) \;\cong\; \mathrm{Hom}_{\Bbbk}(G,\mathbb{E})\,,$$

so $\mathrm{Hom}_{\Bbbk}(G,\mathbb{E})$ is Gorenstein injective. □

9.3.8 Lemma. *Assume that R is right Noetherian and let G be an R-module. If the R°-module $\mathrm{Hom}_R(G,\mathbb{E})$ is Gorenstein injective, then there exists an exact sequence,*

$$\eta \;=\; 0 \longrightarrow G \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots\,,$$

where each F_v is a flat R-module and $\mathrm{Hom}_R(\eta,P)$ is exact for every flat R-module P. Moreover, the sequence $E\otimes_R \eta$ is exact for every injective R°-module E.

Proof. Let η be a sequence of R-modules and E an R°-module. Adjunction 4.4.12 yields an isomorphism $\mathrm{Hom}_{\Bbbk}(E\otimes_R\eta,\mathbb{E}) \cong \mathrm{Hom}_R(\eta,\mathrm{Hom}_{\Bbbk}(E,\mathbb{E}))$. If E is injective, then $\mathrm{Hom}_{\Bbbk}(E,\mathbb{E})$ is a flat R-module by 8.4.28. Thus if $\mathrm{Hom}_R(\eta,P)$ is exact for every flat R-module P, then $E\otimes_R\eta$ is exact for every injective R°-module E; see 2.5.7(b).

To construct η it is now sufficient to construct a short exact sequence,

$$(\flat)\qquad \eta' \;=\; 0 \longrightarrow G \longrightarrow F \longrightarrow G' \longrightarrow 0\,,$$

where F is flat, G' has the same property as G—that is, $\mathrm{Hom}_{\Bbbk}(G',\mathbb{E})$ is Gorenstein injective—and $\mathrm{Hom}_R(\eta',P)$ is exact for every flat R-module P. To this end, choose by D.22 a flat preenvelope $\varphi\colon G\to F$. As the R°-module $\mathrm{Hom}_{\Bbbk}(G,\mathbb{E})$ is Gorenstein injective there exists, in particular, a surjective homomorphism $I \twoheadrightarrow \mathrm{Hom}_{\Bbbk}(G,\mathbb{E})$ where I is an injective R°-module. There are thus injective homomorphisms,

$$G \rightarrowtail \mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_{\Bbbk}(G,\mathbb{E}),\mathbb{E}) \rightarrowtail \mathrm{Hom}_{\Bbbk}(I,\mathbb{E})\,,$$

where the first map is biduality 4.5.3. As above the R-module $\mathrm{Hom}_{\Bbbk}(I,\mathbb{E})$ is flat, so G maps injectively into a flat R-module. This fact, combined with the defining property D.18 of preenvelopes, implies that φ is injective. Setting $G' = \mathrm{Coker}\,\varphi$ one obtains a short exact sequence ($\flat$). For every flat R-module P the sequence $\mathrm{Hom}_R(\eta',P)$ is exact as the functor $\mathrm{Hom}_R(-,P)$ is left exact and $G\to F$ is a flat preenvelope. It remains to see that $\mathrm{Hom}_{\Bbbk}(G',\mathbb{E})$ is Gorenstein injective. Consider the exact sequence

$$\mathrm{Hom}_{\Bbbk}(\eta',\mathbb{E}) \;=\; 0 \longrightarrow \mathrm{Hom}_{\Bbbk}(G',\mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(F,\mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(G,\mathbb{E}) \longrightarrow 0\,.$$

As the module $\mathrm{Hom}_{\Bbbk}(G,\mathbb{E})$ is Gorenstein injective, by assumption, and $\mathrm{Hom}_{\Bbbk}(F,\mathbb{E})$ is injective by 5.4.19, it suffices by 9.2.6 to show that $\mathrm{Ext}^1_{R^{\circ}}(E,\mathrm{Hom}_{\Bbbk}(G',\mathbb{E}))=0$ holds for all injective R°-modules E. In view of 7.3.35 and 7.3.27, it is by injectivity of $\mathrm{Hom}_{\Bbbk}(F,\mathbb{E})$ enough to show that $\mathrm{Hom}_{R^{\circ}}(E,\mathrm{Hom}_{\Bbbk}(\eta',\mathbb{E}))$ is exact for every injective R°-module E. By adjunction 1.2.6 and commutativity 1.2.3, this sequence is isomorphic to $\mathrm{Hom}_R(\eta',\mathrm{Hom}_{\Bbbk}(E,\mathbb{E}))$, which is exact as $\mathrm{Hom}_{\Bbbk}(E,\mathbb{E})$ is flat. □

The next result is the full analogue of flat–injective duality, 5.4.19.

9.3.9 Theorem. *Assume that R is right Noetherian. An R-module G is Gorenstein flat if and only if the R°-module* $\operatorname{Hom}_{\Bbbk}(G,\mathbb{E})$ *is Gorenstein injective.*

Proof. The "only if" part is 9.3.7. To prove the "if" part assume that $\operatorname{Hom}_{\Bbbk}(G,\mathbb{E})$ is Gorenstein injective. For every injective R°-module E and all $m > 0$ one has $\operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_m^R(E,G),\mathbb{E}) \cong \operatorname{Ext}^m_{R^{\circ}}(E,\operatorname{Hom}_{\Bbbk}(G,\mathbb{E})) = 0$ by 8.3.1 and 9.2.4(1). Thus also $\operatorname{Tor}_m^R(E,G) = 0$ holds; see 2.5.7(b). It follows that every projective resolution,

$$(\diamond) \qquad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0\,,$$

stays exact under application of the functor $E \otimes_R -$ for every injective R°-module E. Splicing together the sequences $(\diamond)$ and η from 9.3.8 one gets a totally acyclic complex of flat R-modules in which G is a cokernel. $\square$

9.3.10. Assume that R is right Noetherian. It follows from 9.3.9 that the cokernels of all the homomorphisms in the sequence η in 9.3.8 are Gorenstein flat.

Remark. Theorem 9.3.9 above is crucial for our development of the theory of Gorenstein flat modules/dimension. Indeed, several of the proofs in this section combine results from Sect. 9.2, which hold over any ring, with 9.3.9, which require the ring to be right Noetherian. It is therefore worth pointing out that 9.3.9 actually holds under the weaker assumption that R is right coherent, see [132]. The reason is that the proof of 9.3.9 relies on the existence of flat preenvelopes D.22, whose proof in turn relies on 8.3.26, and as remarked there, that result holds if the ring is right coherent. This means, for example, that 9.3.13 below holds with the same proof for a right coherent ring.

By 5.5.18 the class of flat modules is closed under pure submodules and pure quotients; over a right Noetherian ring, the class of Gorenstein flat modules exhibits the same behavior.

9.3.11 Corollary. *Assume that R is right Noetherian and consider a pure exact sequence of R-modules,* $0 \to G' \to G \to G'' \to 0$. *If G is Gorenstein flat then so are G' and G''.*

Proof. The sequence $0 \to \operatorname{Hom}_{\Bbbk}(G'',\mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(G,\mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(G',\mathbb{E}) \to 0$ is split exact by 5.5.14, so the assertion follows from 9.3.9 and 9.2.6. $\square$

9.3.12 Lemma. *Let $\{\mu^{vu} \colon M^u \to M^v\}_{u \leqslant v}$ be a U-direct system of R-modules. If U is filtered, then the canonical surjective homomorphism $\coprod_{u\in U} M^u \to \operatorname{colim}_{u\in U} M^u$ from* 3.2.3 *is a pure epimorphism.*

Proof. Write π for the canonical homomorphism. For every finitely presented R-module N the functor $\operatorname{Hom}_R(N,-)$ preserves coproducts and filtered colimits, see 3.1.33 and 3.3.17, so $\operatorname{Hom}_R(N,\pi)$ is isomorphic to the canonical homomorphism $\coprod_{u\in U}\operatorname{Hom}_R(N,M^u) \to \operatorname{colim}_{u\in U}\operatorname{Hom}_R(N,M^u)$ associated with the U-direct system $\{\operatorname{Hom}_R(N,\mu^{vu})\colon \operatorname{Hom}_R(N,M^u) \to \operatorname{Hom}_R(N,M^v)\}_{u\leqslant v}$. As this map is surjective, so is $\operatorname{Hom}_R(N,\pi)$. Hence π is a pure epimorphism; see 5.5.15. $\square$

The next proposition captures key features of the class of Gorenstein flat modules. The first assertion in part (a) together with 9.3.2 shows that the class is so-called projectively resolving.

9.3.13 Proposition. *Assume that R is right Noetherian.*

(a) *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of R-modules. If G'' is Gorenstein flat, then G is Gorenstein flat if and only if G' is so. If G' and G are Gorenstein flat, then G'' is Gorenstein flat if and only if $\operatorname{Tor}_1^R(E, G'') = 0$ holds for every injective R°-module E.*

(b) *Every direct summand of a Gorenstein flat R-module is Gorenstein flat.*

(c) *Let $\{\mu^{vu} : G^u \to G^v\}_{u \leqslant v}$ be a U-direct system of R-modules. If U is filtered and every module G^u is Gorenstein flat, then $\operatorname{colim}_{u \in U} G^u$ is Gorenstein flat.*

Proof. (a): The sequence $0 \to \operatorname{Hom}_{\Bbbk}(G'', \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(G, \mathbb{E}) \to \operatorname{Hom}_{\Bbbk}(G', \mathbb{E}) \to 0$ is exact. Furthermore, one has $\operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_1^R(E, G''), \mathbb{E}) \cong \operatorname{Ext}^1_{R^{\circ}}(E, \operatorname{Hom}_{\Bbbk}(G'', \mathbb{E}))$ for every R°-module E; see 8.3.1. The conclusions now follow from 9.3.9 and 9.2.6.

(b): This is a special case of 9.3.11.

(c): This follows from 9.3.6, 9.3.11, and 9.3.12. $\square$

Among modules of finite Gorenstein flat dimension, Gorenstein flat modules are characterized by Tor vanishing. To prove that, the next lemma is key.

9.3.14 Lemma. *Let N be an R°-module*; *the following conditions are equivalent.*

(i) $\operatorname{Tor}_1^R(N, G) = 0$ *for every Gorenstein flat R-module G.*

(ii) $\operatorname{Tor}_m^R(N, G) = 0$ *for every Gorenstein flat R-module G and all $m > 0$.*

The class $\mathcal{N}$ of R°-modules N satifying these conditions has the following properties.

(a) *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R°-modules. If two of the modules N', N, and N'' beong to $\mathcal{N}$, then so does the third.*

(b) *Let $\{\nu^{vu} : N^u \to N^v\}_{u \leqslant v}$ be a U-direct system of modules in $\mathcal{N}$. If U is filtered, then $\operatorname{colim}_{u \in U} N^u$ belongs to $\mathcal{N}$.*

(c) *Every R°-module N with $\operatorname{id}_R N$ or $\operatorname{fd}_R N$ finite belongs to $\mathcal{N}$.*

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. For the converse, let G be a Gorenstein flat R-module and F a totally acyclic complex of flat R-modules with $\mathrm{C}_0(F) \cong G$. For $m > 0$ one has $\operatorname{Tor}_m^R(N, G) \cong \operatorname{Tor}_1^R(N, \mathrm{C}_{m-1}(F))$ by 8.3.9, so (i) implies (ii) as the module $\mathrm{C}_{m-1}(F)$ is Gorenstein flat.

(a): It follows from 7.4.29 that $N', N'' \in \mathcal{N}$ implies $N \in \mathcal{N}$, and that $N, N'' \in \mathcal{N}$ implies $N' \in \mathcal{N}$. Now assume that $N, N' \in \mathcal{N}$. Another application of 7.4.29 yields $\operatorname{Tor}_m^R(N'', G) = 0$ for every Gorenstein flat R-module G and all $m > 1$. To show that also $\operatorname{Tor}_1^R(N'', G) = 0$ holds, note that every Gorenstein flat module G by definition fits into an exact sequence $0 \to G \to F \to G' \to 0$ with F flat and G' Gorenstein flat. Application of 7.4.29 to this sequence yields $\operatorname{Tor}_1^R(N'', G) \cong \operatorname{Tor}_2^R(N'', G')$, and the right-hand side is zero, as just shown.

(b): For every R-module G the functor $\operatorname{Tor}_1^R(-, G)$ preserves filtered colimits by 7.4.25, so the assertion follows directly from the definition of the class $\mathcal{N}$.

(c): In view of part (a), it suffices to prove that $\mathcal{N}$ contains every injective and every flat R-module, and that is the case by 9.3.5(1) and 8.3.23. $\square$

GORENSTEIN FLAT DIMENSION

9.3.15 Definition. Let M be an R-complex. The *Gorenstein flat dimension* of M, written $\operatorname{Gfd}_R M$, is defined as

$$\operatorname{Gfd}_R M \;=\; \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{There exists a semi-flat replacement } F \text{ of } M \text{ with} \\ \mathrm{H}_v(F) = 0 \text{ for all } v > n \text{ and } \mathrm{C}_n(F) \text{ Gorenstein flat} \end{array} \right\}$$

with the convention $\inf \emptyset = \infty$. One says that $\operatorname{Gfd}_R M$ is *finite* if $\operatorname{Gfd}_R M < \infty$ holds.

A comment similar to the one after 8.1.2 justifies the last convention in 9.3.15.

9.3.16. Let M be an R-complex. For every semi-flat replacement F of M one has $\mathrm{H}(F) \cong \mathrm{H}(M)$; the next (in)equalities are hence immediate from the definition,

$$\operatorname{Gfd}_R M \;\geqslant\; \sup M \qquad \text{and} \qquad \operatorname{Gfd}_R \Sigma^s M \;=\; \operatorname{Gfd}_R M + s \quad \text{for every integer } s\,.$$

Moreover, one has $\operatorname{Gfd}_R M = -\infty$ if and only if M is acyclic.

Under mild assumptions on the ring, one can show that given any semi-flat replacement F of a complex of Gorenstein flat dimension at most n the cokernel $\mathrm{C}_v(F)$ is Gorenstein flat for all $v \geqslant n$; see 9.3.21.

9.3.17 Proposition. *Let M be an R-complex. There is an inequality,*

$$\operatorname{Gid}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \;\leqslant\; \operatorname{Gfd}_R M\,,$$

and equality holds if R is right Noetherian.

PROOF. To prove the inequality "$\leqslant$" one can assume that $g = \operatorname{Gfd}_R M$ is an integer. By definition there is a semi-flat replacement F of M with $\mathrm{H}_v(F) = 0$ for all $v > g$ and $\mathrm{C}_g(F)$ Gorenstein flat. The complex $I = \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is a semi-injective replacement of $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$, see 5.4.9, and for $v \in \mathbb{Z}$ one has $\mathrm{H}_{-v}(I) \cong \operatorname{Hom}_{\Bbbk}(\mathrm{H}_v(F), \mathbb{E})$ and $\mathrm{Z}_{-v}(I) \cong \operatorname{Hom}_{\Bbbk}(\mathrm{C}_v(F), \mathbb{E})$ by 2.2.19. It follows that $\mathrm{H}_{-v}(I) = 0$ holds for all $v > g$ and, in view of 9.3.7, that $\mathrm{Z}_{-g}(I)$ Gorenstein injective. Thus the inequality $\operatorname{Gid}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \leqslant g$ holds by the definition, 9.2.9.

Assume that R is right Noetherian. To prove the inequality "$\geqslant$" one can assume that $d = \operatorname{Gid}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ is an integer. Let F be a semi-flat replacement of M. As the complex $I = \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is a semi-injective replacement of $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$, it follows from 9.2.11 that the module $\mathrm{Z}_{-d}(I) \cong \operatorname{Hom}_{\Bbbk}(\mathrm{C}_d(F), \mathbb{E})$ is Gorenstein injective, and hence $\mathrm{C}_d(F)$ is Gorenstein flat by 9.3.9. As $d \geqslant -\inf \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) = \sup M = \sup F$ holds by 9.2.10 and 2.5.7(b) one has $\mathrm{H}_v(F) = 0$ for all $v > d$. Thus the desired inequality $\operatorname{Gfd}_R M \leqslant d$ follows from the definition 9.3.15. $\square$

The next two results compare the Gorenstein flat dimension to the flat dimension. The first of these is at times expressed by saying that Gfd_R is a refinement of fd_R. It follows, in particular, that a Gorenstein flat module is flat or of infinite flat dimension. The Gorenstein flat module from 9.3.4 has infinite flat dimension; see 8.3.7.

9.3.18 Theorem. *Let M be an R-complex. There is an inequality,*

$$\operatorname{Gfd}_R M \leqslant \operatorname{fd}_R M \,,$$

and equality holds if M has finite flat dimension.

Proof. The inequality is evident from the definitions of the dimensions, see 8.3.3 and 9.3.15, and from the fact that every flat module is Gorenstein flat, see 9.3.2. If M has finite flat dimension, then 8.3.17, 9.2.12, and 9.3.17 yield,

$$\operatorname{fd}_R M = \operatorname{id}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) = \operatorname{Gid}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \leqslant \operatorname{Gfd}_R M \,. \qquad \square$$

Equality also holds in 9.3.18 if M has finite injective dimension. This reflects 9.3.5: To a Gorenstein flat module, injective and flat modules look alike.

9.3.19 Theorem. *Assume that R is left Noetherian and let M be an R-complex. If M has finite injective dimension, then the equality* $\operatorname{Gfd}_R M = \operatorname{fd}_R M$ *holds.*

Proof. One can assume that M belongs to $\mathcal{D}_{\sqsubset}(R)$; otherwise the asserted equality is trivial, see 8.3.4 and 9.3.16. As R is left Noetherian, 8.3.18 now shows that the R°-complex $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ has finite flat dimension, and hence the second equality below holds by 9.2.19. The first equality and the inequality hold by 8.3.17 and 9.3.17.

$$\operatorname{fd}_R M = \operatorname{id}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) = \operatorname{Gid}_{R^\circ} \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \leqslant \operatorname{Gfd}_R M \,.$$

The opposite inequality, $\operatorname{fd}_R M \geqslant \operatorname{Gfd}_R M$, holds by 9.3.18. $\square$

Remark. The assertion in 9.3.19 holds without the assumption that R is left Noetherian; see Christensen, Estrada, and Thompson [56].

Right Noetherian Rings

Just as the treatment of flat dimension in Sect. 8.3 relies crucially on flat–injective duality, see 1.3.48, the development below of the Gorenstein flat dimension hinges on 9.3.9. Hence the assumption in 9.3.20–9.3.30 that the ring is right Noetherian.

Remark. As implied in the Remark after 9.3.10, the theory developed in this section works under the weaker assumption that R is right coherent; in particular, 9.3.17 and 9.3.26 hold under this assumption; see [132].

Šaroch and Šťovíček [248] have proved that the class of Gorenstein flat R-modules has the resolving properties from 9.3.13 without any assumptions on R; they do so by showing that the class is the left half of a complete hereditary cotorsion pair. However, this does not appear to be enough to develop a satisfactory theory of Gorenstein flat dimension. In [55] Christensen, Estrada, Liang, Thompson, Wu, and Yang develop a variation on the Gorenstein flat dimension; one that works well over all rings and agrees with the Gorenstein flat dimension over right coherent rings.

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

9.3.20 Proposition. *Assume that R is right Noetherian and consider a distinguished triangle, $M' \to M \to M'' \to \Sigma M'$, in $\mathcal{D}(R)$. With $g' = \operatorname{Gfd}_R M'$, $g = \operatorname{Gfd}_R M$, and $g'' = \operatorname{Gfd}_R M''$ there are inequalities,*

$$g' \leqslant \max\{g, g''-1\}\,, \quad g \leqslant \max\{g', g''\}\,, \quad \textit{and} \quad g'' \leqslant \max\{g'+1, g\}\,.$$

In particular, if two of the complexes M', M, and M'' have finite Gorenstein flat dimension, then so has the third.

Proof. Application of the triangulated functor $\mathsf{R}\mathrm{Hom}_{\Bbbk}(-,\mathbb{E}) = \mathrm{Hom}_{\Bbbk}(-,\mathbb{E})$ to the given triangle yields a distinguished triangle in $\mathcal{D}(R^{\mathrm{o}})$,

$$\mathrm{Hom}_{\Bbbk}(M'',\mathbb{E}) \to \mathrm{Hom}_{\Bbbk}(M,\mathbb{E}) \to \mathrm{Hom}_{\Bbbk}(M',\mathbb{E}) \to \Sigma\mathrm{Hom}_{\Bbbk}(M'',\mathbb{E})\,.$$

The assertion now follows from 9.2.13 and 9.3.17. □

Remark. Proposition 9.3.20 essentially shows that the complexes of finite Gorenstein flat dimension form a triangulated subcategory of $\mathcal{D}(R)$; see E 9.3.9.

9.3.21 Lemma. *Assume that R is right Noetherian and let M be an R-complex. For every semi-flat replacement F of M and every integer $v \geqslant \operatorname{Gfd}_R M$ the module $\mathrm{C}_v(F)$ is Gorenstein flat.*

Proof. Set $g = \operatorname{Gfd}_R M$. By 9.3.17 one has $\operatorname{Gid}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(M,\mathbb{E}) = g$, and by 5.4.9 the complex $I = \mathrm{Hom}_{\Bbbk}(F,\mathbb{E})$ is a semi-injective replacement of $\mathrm{Hom}_{\Bbbk}(M,\mathbb{E})$. Thus 9.2.11 and 2.2.19 imply that the module $\mathrm{Z}_{-v}(I) \cong \mathrm{Hom}_{\Bbbk}(\mathrm{C}_v(F),\mathbb{E})$ is Gorenstein injective for every $v \geqslant g$. Thus $\mathrm{C}_v(F)$ is Gorenstein flat for $v \geqslant g$ by 9.3.9. □

A module of finite Gorenstein flat dimension can be approximated by a Gorenstein flat module and one of finite flat dimension. We derive this as a corollary to the next result about approximations in the derived category.

9.3.22 Proposition. *Assume that R is right Noetherian and let M be an R-complex of finite Gorenstein flat dimension $g = \operatorname{Gfd}_R M$. For every semi-flat replacement F of M and every integer u with $g > u$ there is a distinguished triangle in $\mathcal{D}(R)$,*

$$K \longrightarrow M \longrightarrow N \longrightarrow \Sigma K\,,$$

where the complexes K and N have the following properties:

(a) *There is a degreewise split exact sequence $0 \to F_{\leqslant u} \to K \to \Sigma^u G \to 0$ in $\mathcal{C}(R)$ where G is a Gorenstein flat R-module. Furthermore, one has*

$$\operatorname{Gfd}_R K \leqslant u \qquad \textit{and} \qquad \mathrm{H}_v(K) \cong \begin{cases} 0 & \textit{for } v \geqslant u+1 \\ \mathrm{H}_v(M) & \textit{for } v \leqslant u-1\,. \end{cases}$$

(b) *The complex N satisfies*

$$\operatorname{fd}_R N = g \qquad \textit{and} \qquad \mathrm{H}_v(N) \cong \begin{cases} \mathrm{H}_v(M) & \textit{for } v \geqslant u+2 \\ 0 & \textit{for } v \leqslant u\,. \end{cases}$$

(c) *There is an exact sequence of R-modules,*

$$0 \longrightarrow \mathrm{H}_{u+1}(M) \longrightarrow \mathrm{H}_{u+1}(N) \longrightarrow \mathrm{H}_u(K) \longrightarrow \mathrm{H}_u(M) \longrightarrow 0\,.$$

PROOF. If M is acyclic the statement is void as no integer u satisfies $-\infty = g > u$. Now assume that M is not acyclic, in which case g is an integer. By 9.3.21 the module $\mathrm{C}_g(F)$ is Gorenstein flat and by 9.3.8 and 9.3.10 there exists an acyclic R-complex $L = 0 \to \mathrm{C}_g(F) \to L_{g-1} \cdots \to L_u \to G \to 0$, concentrated in degrees $g, \ldots, u-1$, where $L_{g-1}, \ldots, L_u$ are flat, the cokernels are Gorenstein flat, and $\operatorname{Hom}_R(L, P')$ is acyclic for every flat R-module P'. Set $L_g = \mathrm{C}_g(F)$ and $L_{u-1} = G$ and notice that, in particular, the cokernel $\mathrm{C}_u(L) \cong G$ is Gorenstein flat.

As L is acyclic so is the truncated complex $L_{\subseteq g-1}$ and the sequence of R-modules $0 \to \mathrm{C}_g(F) \to L_{g-1} \to \mathrm{C}_{g-1}(L) \to 0$ is exact. For every flat R-module P' acyclicity of $\operatorname{Hom}_R(L, P')$ implies acyclicity of $\operatorname{Hom}_R(L_{\subseteq g-1}, P')$ and exactness of the sequence

$$(\dagger) \quad 0 \to \operatorname{Hom}_R(\mathrm{C}_{g-1}(L), P') \to \operatorname{Hom}_R(L_{g-1}, P') \to \operatorname{Hom}_R(\mathrm{C}_g(F), P') \to 0\,.$$

Consider the short exact sequence of R-complexes,

$$(\ddagger) \qquad 0 \longrightarrow \Sigma^{g-1}\mathrm{C}_g(F) \xrightarrow{\alpha} L_{\leqslant g-1} \longrightarrow L_{\subseteq g-1} \longrightarrow 0\,,$$

where α is a quasi-isomorphism as $L_{\subseteq g-1}$ is acyclic; see 4.2.6. Let P be a complex of flat R-modules. It follows from exatness of ($\dagger$) that the functor $\operatorname{Hom}_R(-, P_n)$ leaves the sequence ($\ddagger$) exact for every $n \in \mathbb{Z}$. It now follows from 2.3.19 that the functor $\operatorname{Hom}_R(-, P)$ also leaves the sequence ($\ddagger$) exact. Furthermore, as $\operatorname{Hom}_R(L_{\subseteq g-1}, P_n)$ is acyclic for every $n \in \mathbb{Z}$, it follows from A.2 that $\operatorname{Hom}_R(L_{\subseteq g-1}, P)$ is acyclic. Now 4.2.6 yields that $\operatorname{Hom}_R(\alpha, P)$ is a surjective quasi-isomorphism, whence the morphism $\mathcal{C}(R)(\alpha, P)$ is surjective as well by 4.2.7 and 2.3.10. Surjectivity of $\mathcal{C}(R)(\alpha, F_{\leqslant g-1})$ yields a commutative diagram of R-complexes,

$$\begin{tikzcd}
\Sigma^{g-1}\mathrm{C}_g(F) \arrow[r, "\beta"] \arrow[d, "\alpha"', "\simeq"] & F_{\leqslant g-1} \arrow[d, equal] \\
L_{\leqslant g-1} \arrow[r, dotted, "\gamma"] & F_{\leqslant g-1}\,,
\end{tikzcd}$$

where β is induced by ∂^F_g. This diagram—in conjunction with the definition of distinguished triangles in $\mathcal{D}(R)$, see 6.2.3 and 6.5.5, the axiom (TR3) in E.2, and 6.5.19—shows that the complexes $\operatorname{Cone}\beta$ and $\operatorname{Cone}\gamma$ are isomorphic in $\mathcal{D}(R)$. Evidently, one has $\operatorname{Cone}\beta = F_{\subseteq g}$, and this complex is isomorphic to $F \simeq M$ in $\mathcal{D}(R)$. Consequently, $\operatorname{Cone}\gamma \simeq M$ holds in $\mathcal{D}(R)$.

Set $C = \operatorname{Cone}\gamma$. By 2.5.22 and 6.5.24 there is a distinguished triangle in $\mathcal{D}(R)$,

$$C_{\leqslant u} \longrightarrow C \longrightarrow C_{\geqslant u+1} \longrightarrow \Sigma C_{\leqslant u}\,,$$

which we argue is the desired one. As already noticed, $C \simeq M$ in $\mathcal{D}(R)$. The complex

$$K = C_{\leqslant u} = 0 \longrightarrow F_u \oplus G \longrightarrow F_{u-1} \longrightarrow F_{u-2} \longrightarrow \cdots$$

fits into the degreewise split exact sequence in $\mathcal{C}(R)$,

$$(\flat) \qquad 0 \longrightarrow F_{\leqslant u} \longrightarrow K \longrightarrow \Sigma^u G \longrightarrow 0\,.$$

The complex F is semi-flat, and so is $F_{\geqslant u+1}$ by 5.4.8. Hence 5.4.12 applied to the exact sequence $0 \to F_{\leqslant u} \to F \to F_{\geqslant u+1} \to 0$ shows that $F_{\leqslant u}$ is semi-flat. It follows from 8.3.3 and 9.3.18 that $\operatorname{Gfd}_R(F_{\leqslant u}) = \operatorname{fd}_R(F_{\leqslant u}) \leqslant u$ holds. As G is a Gorenstein flat module, one has $\operatorname{Gfd}_R \Sigma^u G \leqslant u$, with equality if G is non-zero, so application of 6.5.24 and 9.3.20 to (b) shows that $\operatorname{Gfd}_R K \leqslant u$. The assertion about the homology of $K = C_{\leqslant u}$ is immediate as $C \simeq M$ in $\mathcal{D}(R)$. Notice that

$$N = C_{\geqslant u+1} = 0 \longrightarrow L_{g-1} \longrightarrow F_{g-1} \oplus L_{g-2} \longrightarrow \cdots \longrightarrow F_{u+1} \oplus L_u \longrightarrow 0$$

is a complex of flat R-modules concentrated in degrees $g, \dots, u+1$. In the extremal case $u = g-1$ one has $N = \Sigma^g L_{g-1}$. Thus N is semi-flat by 5.4.8 with $\operatorname{fd}_R N \leqslant g$. Note that $g \leqslant \max\{\operatorname{Gfd}_R K, \operatorname{Gfd}_R N\}$ holds by 9.3.20. As $\operatorname{Gfd}_R K \leqslant u < g$ holds one has $\operatorname{Gfd}_R N \geqslant g$, so $\operatorname{fd}_R N = g$ holds by 9.3.18. The assertion about the homology of $N = C_{\geqslant u+1}$ is immediate as $C \simeq M$ in $\mathcal{D}(R)$. The exact sequence in part (c) follows by applying 6.5.19 to the constructed distinguished triangle. □

9.3.23 Corollary. *Assume that R is right Noetherian and let M be an R-module of finite Gorenstein flat dimension $g = \operatorname{Gfd}_R M$. The following assertions hold.*

(a) *There is an exact sequence of R-modules $0 \to M \to X \to G \to 0$ where G is Gorenstein flat and $\operatorname{fd}_R X = g$.*

(b) *If $g > 0$, then there is an exact sequence of R-modules $0 \to X \to G \to M \to 0$ where G is Gorenstein flat and $\operatorname{fd}_R X = g - 1$.*

Proof. (a): For $u = -1$ the sequence 9.3.22(c) reads

$$0 \longrightarrow M \longrightarrow \mathrm{H}_0(N) \longrightarrow \mathrm{H}_{-1}(K) \longrightarrow 0\,.$$

If follows from 9.3.22(a) that K is isomorphic to $\Sigma^{-1}\mathrm{H}_{-1}(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gfd}_R \mathrm{H}_{-1}(K) - 1 = \operatorname{Gfd}_R K \leqslant u = -1$. Consequently, the module $\mathrm{H}_{-1}(K)$ is Gorenstein flat. Similarly, one has $N \simeq \mathrm{H}_0(N)$ and $\operatorname{fd}_R \mathrm{H}_0(N) = \operatorname{fd}_R N = g$.

(b): As $g > 0$ one can apply 9.3.22(c) with $u = 0$ to obtain the exact sequence

$$0 \longrightarrow \mathrm{H}_1(N) \longrightarrow \mathrm{H}_0(K) \longrightarrow M \longrightarrow 0\,.$$

If follows from 9.3.22(a) that K is isomorphic to $\mathrm{H}_0(K)$ in $\mathcal{D}(R)$, whence one has $\operatorname{Gfd}_R \mathrm{H}_0(K) = \operatorname{Gfd}_R K \leqslant u = 0$. That is, the module $\mathrm{H}_0(K)$ is Gorenstein flat. Similarly, one has $N \simeq \Sigma \mathrm{H}_1(N)$ and $\operatorname{fd}_R \mathrm{H}_1(N) + 1 = \operatorname{fd}_R N = g$. □

The utility of the following, technical, result becomes clear in 9.3.25.

9.3.24 Proposition. *Assume that R is right Noetherian. Let F be a semi-flat R-complex and E a semi-injective R°-complex. If E has finite injective or finite flat dimension, then the morphism*

$$E \otimes_R \tau^F_{\subseteq n} : E \otimes_R F \longrightarrow E \otimes_R F_{\subseteq n}$$

is a quasi-isomorphism for every integer $n \geqslant \operatorname{Gfd}_R F$.

Proof. Let $E \xrightarrow{\simeq} J$ be a semi-injective resolution; by 5.3.24 there is a homotopy equivalence $\iota \colon J \to E$. By 9.3.16 one has $n \geqslant \sup F$, so the map $\tau^F_{\subseteq n} \colon F \to F_{\subseteq n}$ is a quasi-isomorphism by 4.2.4. Choose a semi-flat replacement P of E; by 6.4.21 there is a quasi-isomorphism $\pi\colon P \xrightarrow{\simeq} J$. In the commutative diagram,

$$\begin{tikzcd}
P \otimes_R F \arrow[r, "\pi\otimes F", "\simeq"'] \arrow[d, "\simeq"', "P\otimes\tau^F_{\subseteq n}"] & J \otimes_R F \arrow[r, "\iota\otimes F", "\cong"'] \arrow[d, "J\otimes\tau^F_{\subseteq n}"] & E \otimes_R F \arrow[d, "E\otimes\tau^F_{\subseteq n}"] \\
P \otimes_R F_{\subseteq n} \arrow[r, "\pi\otimes F_{\subseteq n}"] & J \otimes_R F_{\subseteq n} \arrow[r, "\iota\otimes F_{\subseteq n}", "\cong"'] & E \otimes_R F_{\subseteq n}\,,
\end{tikzcd}$$

the right-hand horizontal maps are homotopy equivalences by 4.3.20, while $P \otimes_R \tau^F_{\subseteq n}$ and $\pi \otimes_R F$ are quasi-isomorphisms by semi-flatness of P and F. The diagram shows that it is sufficient to verify that $\pi \otimes_R F_{\subseteq n}$ is a quasi-isomorphism. Set $C = \operatorname{Cone}\pi$ and notice that it is an acyclic complex of modules that are direct sums of flat and injective R°-modules. The goal is to prove that the complex $C \otimes_R F_{\subseteq n} \cong \operatorname{Cone}(\pi \otimes_R F_{\subseteq n})$ is acyclic; see 4.1.19.

By 2.5.24 there is an exact sequence $0 \to F' \to F \to F_{\subseteq n} \to 0$ with F' bounded below. By 9.3.2, 9.3.21, and 9.3.13(a) these are all complexes of Gorenstein flat modules, and it follows from 7.4.30 and 9.3.14(c) that the sequence

$$0 \longrightarrow C \otimes_R F' \longrightarrow C \otimes_R F \longrightarrow C \otimes_R F_{\subseteq n} \longrightarrow 0$$

is exact. The complex $C \otimes_R F$ is acyclic as C is acyclic and F is semi-flat, and thus it is enough to show that the left-hand complex $C \otimes_R F'$ is acyclic; see 2.5.6. To that end, it suffices by A.9 to show that $C \otimes_R G$ is acyclic for every Gorenstein flat module G. It is already known from 9.3.14(c) that $\operatorname{Tor}^R_m(C_v, G) = 0$ holds for all $v \in \mathbb{Z}$ and all $m > 0$, so it suffices by A.7 to argue that $\operatorname{Tor}^R_m(\mathrm{C}_v(C), G) = 0$ holds for $v \ll 0$ and all $m > 0$.

If E has finite injective dimension, then one can assume that J and P are bounded below; see 8.2.2, 5.2.15 and 5.4.10. It follows that C is bounded below; in particular, $\mathrm{C}_v(C) = 0$ holds for $v \ll 0$.

If E has finite flat dimension, then one can by 8.3.3 and 5.3.26 assume that the complexes P and J are bounded above. It follows that C is bounded above; in particular, $\mathrm{C}_v(C) = 0$ holds for $v \gg 0$. Let G be a Gorenstein flat module. For every $v \in \mathbb{Z}$ one has $\operatorname{Tor}^R_m(C_v, G) = 0$ for all $m > 0$, so in view of 9.3.14(a), induction on the exact sequences $0 \to \mathrm{C}_{v+1}(C) \to C_v \to \mathrm{C}_v(C) \to 0$ yields $\operatorname{Tor}^R_m(\mathrm{C}_v(C), G) = 0$ for all $v \in \mathbb{Z}$ and all $m > 0$. ☐

The take-away from of 9.3.5 is that flat and injective modules interact with Gorenstein flat modules in the same way. This has the following useful consequence:

9.3.25 Corollary. *Assume that R is right Noetherian. Let M be an R-complex of finite Gorenstein flat dimension and N an R°-complex of finite injective or finite flat dimension. For every semi-flat replacement F of M, every semi-injective replacement I of N, and every integer $n \geqslant \operatorname{Gfd}_R M$ there is in isomorphism $\mathcal{D}(\Bbbk)$,*

$$N \otimes^{\mathsf{L}}_R M \;\simeq\; I \otimes_R F_{\subseteq n}\,.$$

Proof. The assertion follows immediately from 9.3.24 and 7.4.17. ☐

A key difference between the next theorem and the main theorem about flat dimension, 8.3.11, is the a priori assumption that the complex has finite Gorenstein flat dimension. Over a commutative Noetherian ring, 19.1.16 offers another way to compute the Gorenstein flat dimension, in the case it is finite.

9.3.26 Theorem. *Assume that R is right Noetherian. Let M be an R-complex of finite Gorenstein flat dimension and n an integer. The next conditions are equivalent.*

(*i*) $\operatorname{Gfd}_R M \leqslant n$.

(*ii*) $\sup(N \otimes^{\mathsf{L}}_R M) \leqslant n + \sup N$ *holds for every R^{o}-complex N with* $\operatorname{fd}_{R^{\mathrm{o}}} N$ *finite or* $\operatorname{id}_{R^{\mathrm{o}}} N$ *finite.*

(*iii*) $\sup(N \otimes^{\mathsf{L}}_R M) \leqslant n$ *holds for every injective R^{o}-module N.*

(*iv*) $n \geqslant \sup M$ *and* $\operatorname{Tor}^R_{n+1}(N, M) = 0$ *for every R^{o}-module N with* $\operatorname{id}_{R^{\mathrm{o}}} N$ *finite.*

(*v*) $n \geqslant \sup M$ *and for some, equivalently every, semi-flat replacement F of M, the module $\mathrm{C}_v(F)$ is Gorenstein flat for every $v \geqslant n$.*

(*vi*) *There exists a semi-flat replacement F of M with $F_v = 0$ for all $v < \inf M$, $\mathrm{H}_v(F) = 0$ for all $v > n$, and $\mathrm{C}_v(F)$ Gorenstein flat for all $v \geqslant n$.*

In particular, there are equalities,

$$\begin{aligned}\operatorname{Gfd}_R M &= \sup\{\sup(N \otimes^{\mathsf{L}}_R M) \mid N \text{ is an injective } R^{\mathrm{o}}\text{-module}\}\\ &= \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(N, M) \neq 0 \text{ for some injective } R^{\mathrm{o}}\text{-module } N\}\,.\end{aligned}$$

Proof. We start by establishing the equivalence of conditions (*i*)–(*iii*).

(*i*) ⟹ (*ii*): One can assume that N is in $\mathcal{D}_{\sqsubset}(R^{\mathrm{o}})$ and not acyclic; otherwise the inequality is trivial. In this case, $w = \sup N$ is an integer. By 5.3.26 there is a semi-injective resolution $N \xrightarrow{\simeq} I$ with $I_v = 0$ for $v > w$. If $\operatorname{fd}_{R^{\mathrm{o}}} N$ or $\operatorname{id}_{R^{\mathrm{o}}} N$ is finite, then 9.3.25 yields $N \otimes^{\mathsf{L}}_R M \simeq I \otimes_R F_{\subseteq n}$ where F is any semi-flat replacement of M. For $v > n + w$ and $p \in \mathbb{Z}$ either $p > w$ or $v - p \geqslant v - w > n$ holds, so the module

$$(I \otimes_R F_{\subseteq n})_v = \coprod_{p \in \mathbb{Z}} I_p \otimes_R (F_{\subseteq n})_{v-p}$$

is zero. In particular, one has $\mathrm{H}_v(N \otimes^{\mathsf{L}}_R M) = 0$ for $v > n + w$.

(*ii*) ⟹ (*iii*): Trivial.

(*iii*) ⟹ (*i*): By 9.3.17 the R^{o}-complex $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ has finite Gorenstein injective dimension. Furthermore, for every R^{o}-module N one has

$$-\inf \mathsf{RHom}_{R^{\mathrm{o}}}(N, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E})) = -\inf \operatorname{Hom}_{\Bbbk}(N \otimes^{\mathsf{L}}_R M, \mathbb{E}) = \sup(N \otimes^{\mathsf{L}}_R M)\,,$$

where the first equality holds by adjunction 7.5.30 and the second one by 2.5.7(b). Thus condition (*iii*) implies, in view of 9.2.18, that one has $\operatorname{Gid}_R \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \leqslant n$, and hence $\operatorname{Gfd}_R M \leqslant n$ holds by another application of 9.3.17.

(*ii*) ⟹ (*iv*): The vanishing of Tor is immediate from (*ii*). In view of 9.3.16, the inequality $n \geqslant \sup M$ follows from (*i*), which is known to be equivalent to (*ii*).

(*iv*) ⟹ (*i*): As already mentioned, the R^{o}-complex $\operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ has finite Gorenstein injective dimension. Moreover, one has $\sup M = -\inf \operatorname{Hom}_{\Bbbk}(M, \mathbb{E})$ by 2.5.7(b) and $\operatorname{Hom}_{\Bbbk}(\operatorname{Tor}^R_{n+1}(N, M), \mathbb{E}) \cong \operatorname{Ext}^{n+1}_{R^{\mathrm{o}}}(N, \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}))$ by 8.3.1. Thus (*iv*) implies, in view of 9.2.18, that $\operatorname{Gid}_R \operatorname{Hom}_{\Bbbk}(M, \mathbb{E}) \leqslant n$ holds. Therefore, $\operatorname{Gfd}_R M \leqslant n$ holds by 9.3.17.

Before we connect the equivalent conditions (*i*)–(*iv*) to (*v*), note that by 8.3.16 and 9.3.13 the “some” version and the “every” version of condition (*v*) are equivalent.

(*i*) ⇒ (*v*): Follows from 9.3.16 and 9.3.21.

(*v*) ⇒ (*vi*): This implication is immediate in view of 5.2.15 and 5.4.10.

(*vi*) ⇒ (*i*): This implication is immediate from the definition, 9.3.15, of Gfd_R.

The equalities in the last assertion follow immediately from the equivalence of (*i*)–(*iii*) and 7.4.19. □

9.3.27 Proposition. *Assume that R is right Noetherian and let $\{M^u\}_{u\in U}$ be a family of R-complexes. There is an equality,*

$$\operatorname{Gfd}_R\Big(\coprod_{u\in U} M^u\Big) = \sup_{u\in U}\{\operatorname{Gfd}_R M^u\}\,.$$

Proof. By 9.3.17, 3.1.27, and 9.2.20 there are equalities

$$\begin{aligned}\operatorname{Gfd}_R\big(\coprod_{u\in U} M^u\big) &= \operatorname{Gid}_{R^\circ}\operatorname{Hom}_{\Bbbk}\big(\coprod_{u\in U} M^u,\mathbb{E}\big)\\ &= \operatorname{Gid}_{R^\circ}\big(\prod_{u\in U}\operatorname{Hom}_{\Bbbk}(M^u,\mathbb{E})\big)\\ &= \sup_{u\in U}\{\operatorname{Gid}_{R^\circ}\operatorname{Hom}_{\Bbbk}(M^u,\mathbb{E})\}\\ &= \sup_{u\in U}\{\operatorname{Gfd}_R M^u\}\,.\end{aligned}$$ □

Comparison to Gorenstein Projective Dimension

9.3.28 Lemma. *Assume that R is right Noetherian and that every flat R-module has finite projective dimension. Let $\mathcal{X}$ be a class of R-modules with the next properties*:

(1) *Every projective R-module belongs to $\mathcal{X}$.*

(2) *Every module in $\mathcal{X}$ can be embedded into an R-module of finite flat dimension.*

(3) *For every exact sequence $0 \to X' \to X \to X'' \to 0$ of R-modules with X' and X in $\mathcal{X}$, also X'' belongs to $\mathcal{X}$.*

If an R-module M is in $\mathcal{X}$ and $\operatorname{Ext}_R^m(M,L) = 0$ holds for every projective R-module L and all integers $m > 0$, then M is Gorenstein projective.

Proof. To prove that M is Gorenstein projective, it suffices by the assumptions on M to verify that it meets condition 9.1.3(2). To this end, it is enough to construct an exact sequence of R-modules,

$$(\dagger)\qquad 0 \longrightarrow M \longrightarrow P \longrightarrow M' \longrightarrow 0\,,$$

where P is projective, M' has the same properties as M—that is, M' is in $\mathcal{X}$ and $\operatorname{Ext}_R^m(M',L) = 0$ for every projective R-module L and all $m > 0$—and the sequence

$$(\ddagger)\qquad 0 \longrightarrow \operatorname{Hom}_R(M',L) \longrightarrow \operatorname{Hom}_R(P,L) \longrightarrow \operatorname{Hom}_R(M,L) \longrightarrow 0$$

is exact for every projective R-module L. Indeed, having constructed such a sequence, the sequence required in 9.1.3(2) can be constructed recursively.

By property (2) the module M can be embedded into an R-module F of finite flat dimension, say, $n = \operatorname{fd}_R F$. Set $\mathcal{F}_n = \{Y \in \mathcal{M}(R) \mid \operatorname{fd}_R Y \leqslant n\}$. By D.22 there exists an $\mathcal{F}_n$-preenvelope $\alpha\colon M \to Y$, and as F belongs to $\mathcal{F}_n$ it follows from D.20 that α is injective. Let

$$(\flat) \qquad 0 \longrightarrow N \longrightarrow P \xrightarrow{\pi} Y \longrightarrow 0$$

be an exact sequence of R-modules with P projective. As P and Y belong to $\mathcal{F}_n$, it follows from 8.3.12 that N is in $\mathcal{F}_n$, whence $\operatorname{pd}_R N$ is finite by 8.3.14(b). Thus, one has $\operatorname{Ext}^1_R(M,N) = 0$ by 9.1.4, and appliction of the functor $\operatorname{Hom}_R(M,-)$ to the exact sequence ($\flat$) yields by 7.3.35 and 7.3.27 an exact sequence,

$$\operatorname{Hom}_R(M,P) \xrightarrow{\operatorname{Hom}_R(M,\pi)} \operatorname{Hom}_R(M,Y) \longrightarrow 0\,.$$

This gives a homomorphism $\beta\colon M \to P$ with $\pi\beta = \alpha$, and as α is injective, so is β. Thus, with $M' = \operatorname{Coker}\beta$ one obtains an exact sequence (†). If L is a projective R-module, or even a module in $\mathcal{F}_n$, then the sequence (‡) is exact. Indeed, one has

$$\operatorname{Hom}_R(\beta,L)\operatorname{Hom}_R(\pi,L) \;=\; \operatorname{Hom}_R(\alpha,L)\,,$$

and since $\operatorname{Hom}_R(\alpha,L)$ is surjective, see D.20, so is $\operatorname{Hom}_R(\beta,L)$. As $\operatorname{Hom}_R(-,L)$ is left exact, see 2.3.10, exactness of the sequence (‡) follows. Application of the functor $\operatorname{Hom}_R(-,L)$ to (†) yields by 7.3.35 and exactness of (‡) an exact sequence,

$$\operatorname{Hom}_R(M,L) \xrightarrow{0} \operatorname{Ext}^1_R(M',L) \longrightarrow \operatorname{Ext}^1_R(P,L) \longrightarrow \operatorname{Ext}^1_R(M,L) \longrightarrow \cdots\,.$$

As one has $\operatorname{Ext}^m_R(P,L) = 0 = \operatorname{Ext}^m_R(M,L)$ for all $m > 0$ by projectivity of P and the assumption on M, it now follows that $\operatorname{Ext}^m_R(M',L) = 0$ holds for all $m > 0$.

It remains to see that M' is in $\mathcal{X}$. The module P belongs to $\mathcal{X}$ by property (1). The exact sequence (†) and property (3) now imply that M' is in $\mathcal{X}$. □

9.3.29 Lemma. *Assume that R is right Noetherian and that every flat R-module has finite projective dimension. Let M be an R-module and $d \geqslant 0$ an integer. If* $\operatorname{Gfd}_R M$ *is finite and* $\operatorname{Ext}^m_R(M,L) = 0$ *holds for every projective R-module L and all integers $m > d$, then one has* $\operatorname{Gpd}_R M \leqslant d$.

Proof. Per 5.1.16 let $F \xrightarrow{\simeq} M$ be a free resolution. There is an exact sequence,

$$(\dagger) \qquad 0 \longrightarrow \mathrm{C}_d(F) \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,.$$

To prove that $\operatorname{Gpd}_R M \leqslant d$ holds, it suffices by 9.1.10 to show that $\mathrm{C}_d(F)$ is Gorenstein projective. Each module F_v is free, hence flat, and therefore Gorenstein flat by 9.3.2. By assumption, $\operatorname{Gfd}_R M$ is finite, so the exact sequence (†) and 9.3.20 show that $\operatorname{Gfd}_R \mathrm{C}_d(F)$ is finite. For every R-module L and integer $m > 0$ there is by 8.1.6 an isomorphism,

$$\operatorname{Ext}^m_R(\mathrm{C}_d(F),L) \;\cong\; \operatorname{Ext}^{d+m}_R(M,L)\,;$$

whence $\operatorname{Ext}^m_R(\mathrm{C}_d(F),L) = 0$ holds if L is projective. Thus, replacing M with $\mathrm{C}_d(F)$, it suffices to prove the assertion for $d = 0$.

To prove that finiteness of $\operatorname{Gfd}_R M$ and vansishing of $\operatorname{Ext}^m_R(M,L)$ for every projective R-module L and all $m > 0$ imply that M is Gorenstein projective, it suffices by

9.3.28 to argue that the class $\mathcal{X}$ of R-modules of finite Gorenstein flat dimension has the properties (1), (2), and (3) required in that result. Every projective R-module is flat and hence Gorenstein flat by 9.3.2, so (1) holds. Property (2) holds by 9.3.23(a), and (3) follows from 9.3.20. □

The next result is akin to 8.3.6 and 8.5.20. The assumption about finiteness of projective dimension of flat R-modules holds per 8.5.20 if R has has finite finitistic projective dimension; see also the Remark after the proof.

9.3.30 Theorem. *Assume that R is right Noetherian and that every flat R-module has finite projective dimension. Let M be an R-complex; there is an inequality,*

$$\operatorname{Gfd}_R M \leqslant \operatorname{Gpd}_R M .$$

Moreover, $\operatorname{Gfd}_R M$ *is finite if and only if* $\operatorname{Gpd}_R M$ *is finite.*

Proof. First we prove the inequality. By the definitions, 9.1.10 and 9.3.15, of the dimensions, and by the fact that every semi-projective replacement of M is also a semi-flat replacement, see 5.4.10, it suffices to show that every Gorenstein projective R-module is Gorenstein flat. To this end we argue that every totally acyclic complex P of projective R-modules is a totally acyclic per 9.3.1 when considered as a complex of flat R-modules. Certainly P is an acyclic complex of flat modules; see 1.3.43. For every injective R°-module E the R-module $\operatorname{Hom}_{\Bbbk}(E, \mathbb{E})$ is flat by 8.3.18. Hence, by assumption, $\operatorname{Hom}_{\Bbbk}(E, \mathbb{E})$ has finite projective dimension. It now follows from 9.1.9(c) that $\operatorname{Ext}_R^m(G, \operatorname{Hom}_{\Bbbk}(E, \mathbb{E})) = 0$ holds for every Gorenstein projective R-module G and all $m > 0$; consequently the complex $\operatorname{Hom}_R(P, \operatorname{Hom}_{\Bbbk}(E, \mathbb{E}))$ is acyclic by A.1. By adjunction 4.4.12 this complex is isomorphic to $\operatorname{Hom}_{\Bbbk}(E \otimes_R P, \mathbb{E})$ which, therefore, is acyclic too. Finally, 2.5.7(b) yields acyclicity of $E \otimes_R P$, so P is totally acyclic as a complex of flat R-modules.

We now prove the last assertion. From the established inequality it follows that if $\operatorname{Gpd}_R M$ is finite, then so is $\operatorname{Gfd}_R M$. To show the converse, recall from 8.3.14(b) that there is an integer d with $\operatorname{pd}_R F \leqslant d$ for every flat R-module F. Assume that $n = \operatorname{Gfd}_R M$ is finite. To prove finiteness of $\operatorname{Gpd}_R M$ one can assume that M is not acyclic, i.e. n is an integer, see 9.1.11 and 9.3.16. We proceed to verify the inequality,

$$(\dagger) \qquad \operatorname{Gpd}_R M \leqslant n + d .$$

Let P be a semi-projective, and hence also a semi-flat, replacement of M. By 9.3.26 one has $n \geqslant \sup M$ and the module $\mathrm{C}_n(P)$ is Gorenstein flat. We argue below that the inequality $\operatorname{Gpd}_R \mathrm{C}_n(P) \leqslant d$ holds. Having proved this, the exact sequence

$$0 \longrightarrow \mathrm{C}_{n+d}(P) \longrightarrow P_{n+d-1} \longrightarrow \cdots \longrightarrow P_n \longrightarrow \mathrm{C}_n(P) \longrightarrow 0$$

and 9.1.31 show that $\mathrm{C}_{n+d}(P)$ is Gorenstein projective, and thus (†) holds by 9.1.10.

Set $C = \mathrm{C}_n(P)$. To prove that $\operatorname{Gpd}_R C \leqslant d$ holds it suffices by 9.3.29 to argue that one has $\operatorname{Ext}_R^m(C, F) = 0$ for every projective R-module F and every integer $m > d$. Let F be a projective R-module. Define, recursively, R-modules $F_0, F_1, F_2, \ldots$ by setting $F_0 = F$ and $F_{v+1} = \operatorname{Coker} \delta_{\mathbb{E}}^{F_v}$ where $\delta_{\mathbb{E}}^{F_v}$ is biduality 1.4.2. By 8.3.28 there are (pure) exact sequences of R-modules,

$$(\ddagger) \qquad 0 \longrightarrow F_v \longrightarrow \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(F_v, \mathbb{E}), \mathbb{E}) \longrightarrow F_{v+1} \longrightarrow 0 \,,$$

and each module F_v is flat. Thus, each R^{o}-module $\operatorname{Hom}_{\Bbbk}(F_v, \mathbb{E})$ is injective by 5.4.19. Now, let $v \geqslant 0$ and $m > 0$ be integers. Let X be a Gorenstein flat R-module; in the computation below, the first isomorphism holds by 8.3.1 and the second by 9.3.5(1).

$$\operatorname{Ext}_R^m(X, \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(F_v, \mathbb{E}), \mathbb{E})) \;\cong\; \operatorname{Hom}_{\Bbbk}(\operatorname{Tor}_m^R(\operatorname{Hom}_{\Bbbk}(F_v, \mathbb{E}), X), \mathbb{E}) \;=\; 0 \,.$$

In view of this computation the exact sequence of Ext modules, induced per 7.3.35 by the short exact sequence ($\ddagger$), yields for every $v \geqslant 0$ and $m > 0$ an isomorphism,

$$\operatorname{Ext}_R^m(X, F_{v+1}) \;\cong\; \operatorname{Ext}_R^{m+1}(X, F_v) \,.$$

These isomorphisms combine to show that for every integer $m > d$ one has:

$$(\flat) \qquad \operatorname{Ext}_R^m(X, F) \;=\; \operatorname{Ext}_R^m(X, F_0) \;\cong\; \operatorname{Ext}_R^{m-d}(X, F_d) \,.$$

Thus, to prove that $\operatorname{Ext}_R^m(C, F) = 0$ holds for every $m > d$, it suffices by ($\flat$) applied with $X = C$ to argue that one has $\operatorname{Ext}_R^n(C, F_d) = 0$ for every $n > 0$. To that end, we first show that for every flat R-module G and every $n > 0$ one has $\operatorname{Ext}_R^n(G, F_d) = 0$. Applied with $X = G$, see 9.3.2, the isomorphism in ($\flat$) yields

$$\operatorname{Ext}_R^n(G, F_d) \;\cong\; \operatorname{Ext}_R^{n+d}(G, F) \;=\; 0 \,,$$

where the equality follows from 8.1.20, as $\operatorname{pd}_R G \leqslant d$ holds by the definition of d. In particular, one has $\operatorname{Ext}_R^1(F_{d+1}, F_d) = 0$, so the exact sequence

$$0 \longrightarrow F_d \longrightarrow \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(F_d, \mathbb{E}), \mathbb{E}) \longrightarrow F_{d+1} \longrightarrow 0$$

is split by 7.3.36 and hence F_d is a direct summand of $\operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(F_d, \mathbb{E}), \mathbb{E})$. As proved above, one has $\operatorname{Ext}_R^n(C, \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(F_d, \mathbb{E}), \mathbb{E})) = 0$ for every $n > 0$, and it follows that also $\operatorname{Ext}_R^n(C, F_d) = 0$ holds for every $n > 0$, as desired. □

REMARK. The assumption in 9.3.30 about projective dimension of flat modules can be expressed by saying that the invariant splf R is finite; see E 8.5.11. Gruson and Jensen [113, 148] and Simson [236] show that a ring R of cardinality $\aleph_n$ has splf $R \leqslant n + 1$.

9.3.31 Example. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat, see 1.3.43, and hence Gorenstein flat by 9.3.2, so $\operatorname{Gfd}_{\mathbb{Z}} \mathbb{Q} = 0$. However, one has $\operatorname{Gpd}_{\mathbb{Z}} \mathbb{Q} = \operatorname{pd}_{\mathbb{Z}} \mathbb{Q} = 1$ by 8.1.5 and 9.1.13. Thus, the inequality in 9.3.30 can be strict.

NOETHERIAN RINGS AND HOMOLOGICAL FINITENESS

For finitely generated modules—and more generally for complexes with finitely generated homology—over a left Noetherian ring, the projective and flat dimensions agree; that is the content of 8.3.19. To prove the corresponding result for the Gorenstein projective and Gorenstein flat dimensions we have to assume that the ring is not just left Noetherian but Noetherian. This is, of course, because we have developed the theory of Gorenstein flat dimension over right Noetherian rings.

9.3.32 Lemma. *Assume that R is Noetherian and let G be a finitely generated R-module. If G is Gorenstein flat, then there is an exact sequence of finitely generated R-modules, $0 \to G \to L \to G' \to 0$, with L free and G' Gorenstein flat.*

Proof. There exists, by definition, an exact sequence $0 \to G \to F \to G'' \to 0$ with F flat and G'' Gorenstein flat. As R is left Noetherian, it follows from 5.5.7 that the map $G \to F$ admits a factorization $G \xrightarrow{\varphi} L \longrightarrow F$ with L finitely generated free. Notice that φ is injective and that the module $G' = \operatorname{Coker}\varphi$ is finitely generated. There is now a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] & F \arrow[r] & G'' \arrow[r] & 0\,.
\end{tikzcd}$$

For every injective R°-module E, there is per 7.4.29 and 7.4.21 a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^R(E,G') \arrow[r] \arrow[d] & E \otimes_R G \arrow[r] \arrow[d, equal] & E \otimes_R L \arrow[d] \\
0 \arrow[r] & \operatorname{Tor}_1^R(E,G'') \arrow[r] & E \otimes_R G \arrow[r] & E \otimes_R F\,.
\end{tikzcd}$$

Per 9.3.5(1) one has $\operatorname{Tor}_1^R(E,G'') = 0$, and it follows that also $\operatorname{Tor}_1^R(E,G')$ vanishes. As R is right Noetherian it now follows from 9.3.13 that G' is Gorenstein flat. □

9.3.33 Proposition. *Assume that R is left Noetherian and let G be a finitely generated R-module. If G is Gorenstein projective, then it is Gorenstein flat*; *the converse holds if R is Noetherian.*

Proof. If G is Gorenstein projective, then by 9.1.23 it is a cokernel in a totally acyclic complex P of finitely generated free R-modules. It follows from 9.1.5 that P is totally acyclic as a complex of flat modules; see 9.3.1. Thus G is Gorenstein flat.

Conversely, assume that R is Noetherian and that G is Gorenstein flat. It follows from 9.3.32 that there is an injective quasi-isomorphism $\iota\colon G \to L$, where L is a complex of finitely generated free R-modules with $L_v = 0$ for $v > 0$ and $\mathrm{C}_v(L)$ Gorenstein flat for every $v \in \mathbb{Z}$. By 5.1.19 there is a free resolution $\pi\colon F \xrightarrow{\simeq} G$ with F degreewise finitely generated. The complex $P = \Sigma^{-1}\mathrm{Cone}(\iota\pi)$ is a complex of finitely generated free R-modules with $\mathrm{C}_0(P) = \mathrm{C}_0(F) \cong G$. Moreover, for $v \ll 0$ the module $\mathrm{C}_v(P) = \mathrm{C}_{v+1}(L)$ is Gorenstein flat, so by 9.3.5(1) and A.8 the complex $E \otimes_R P$ is acyclic for every injective R°-module E. That is, P is totally acyclic as a complex of flat modules. □

Remark. The similarities in behavior of Gorenstein and absolute homological dimensions are strong but not perfect: It follows from 9.3.33 and 9.3.13(c) that over a Noetherian ring, a filtered colimit of finitely generated Gorenstein projective modules is Gorenstein flat. Over an Iwanaga–Gorenstein ring every Gorenstein flat module is obtained in this way, see Enochs and Jenda [87, 10.3], but in general it is not the case, not even if R is commutative and local. That is, there is no analogue of Govorov and Lazard's theorem 5.5.7; see Holm and Jørgensen [135].

9.3.34 Theorem. *Assume that R is Noetherian and let M be an R-complex. If M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then there is an equality,*

$$\mathrm{Gfd}_R\, M \;=\; \mathrm{Gpd}_R\, M\,.$$

Proof. Choose by 5.2.16 a degreewise finitely generated semi-projective resolution $P \xrightarrow{\simeq} M$. The complex P is, in particular, a semi-projective and a semi-flat replacement of M; see 5.4.10. From the definitions, 9.1.10 and 9.3.15, of the dimensions in question and from 9.1.19 and 9.3.26, it follows that one has

$$\begin{aligned}\mathrm{Gpd}_R\, M \;&=\; \inf\{n \in \mathbb{Z} \mid\; n \geqslant \sup M \text{ and } \mathrm{C}_n(L) \text{ is Gorenstein projective}\} \quad \text{and}\\ \mathrm{Gfd}_R\, M \;&=\; \inf\{n \in \mathbb{Z} \mid\; n \geqslant \sup M \text{ and } \mathrm{C}_n(P) \text{ is Gorenstein flat}\}\end{aligned}$$

As every module $\mathrm{C}_n(P)$ is finitely generated, the assertion follows from 9.3.33. □

Remark. Let R and M be as in 9.3.34. In this context it is standard to write $\text{G-dim}_R\, M$ for the Gorenstein projective/flat dimension of M and refer to it as the 'Gorenstein dimension' or 'G-dimension' of M. This notation and terminology goes back to Auslander and Bridgers original work [8, 9]; in this book we stay with the notation $\mathrm{Gpd}_R\, M$.

The Case of Modules

9.3.35. Notice from 9.3.26 that if R is right Noetherian, then a non-zero R-module is Gorenstein flat if and only if it has Gorenstein flat dimension 0 as an R-complex.

9.3.36 Theorem. *Assume that R is right Noetherian. Let M be an R-module of finite Gorenstein flat dimension and $n \geqslant 0$ an integer. The next conditions are equivalent.*

(*i*) $\mathrm{Gfd}_R\, M \leqslant n$.
(*ii*) $\mathrm{Tor}^R_m(N, M) = 0$ *holds for every R^{o}-module N with* $\mathrm{fd}_{R^{\mathrm{o}}} N$ *finite or* $\mathrm{id}_{R^{\mathrm{o}}} N$ *finite and all integers $m > n$.*
(*iii*) $\mathrm{Tor}^R_m(N, M) = 0$ *holds for every injective R^{o}-module N and all integers $m > n$.*
(*iv*) $\mathrm{Tor}^R_{n+1}(N, M) = 0$ *holds for every R^{o}-module N with* $\mathrm{id}_{R^{\mathrm{o}}} N$ *finite.*
(*v*) *In some/every flat resolution $\cdots \to F_v \to F_{v-1} \to \cdots \to F_0 \to M \to 0$ the module* $\mathrm{Coker}(F_{v+1} \to F_v)$ *is Gorenstein flat for every $v \geqslant n$.*
(*vi*) *There is an exact sequence of R-modules $0 \to G \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ with each F_i flat and G Gorenstein flat.*

In particular, there is an equality

$$\mathrm{Gfd}_R\, M \;=\; \sup\{m \in \mathbb{N}_0 \mid \mathrm{Tor}^R_m(E, M) \neq 0 \text{ for some injective } R^{\mathrm{o}}\text{-module } E\}\,.$$

Proof. By 5.1.16 every R-module M has a flat resolution

$$\cdots \longrightarrow F_v \longrightarrow F_{v-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,.$$

In every such resolution, the surjective homomorphism $F_0 \to M$ is a quasi-isomorphism, so the complex $\cdots \to F_v \to F_{v-1} \to \cdots \to F_0 \to 0$ is a semi-flat replacement of M, cf. 5.1.18. The equivalence of the conditions (i)–(vi) is now immediate from 9.1.19, and so is the asserted equality in view of 7.4.21. □

Restricted Flat Dimension

We end this section by introducing yet another homological dimension. In contrast to the dimensions studied in Sects. 8.1–8.3 and up to this point in this chapter, the homological dimension introduced below is not defined in terms of replacements.

9.3.37 Definition. Let M be an R-complex. The *restricted flat dimension* of M, written $\mathrm{Rfd}_R\, M$, is defined as

$$\mathrm{Rfd}_R\, M \;=\; \sup\{\sup\,(N \otimes^{\mathsf{L}}_R M) \mid N \text{ is an } R^{\mathrm{o}}\text{-module with } \mathrm{fd}_{R^{\mathrm{o}}}\, N < \infty\}$$

with the convention $\sup\varnothing = -\infty$. One says that $\mathrm{Rfd}_R\, M$ is *finite* if $\mathrm{Rfd}_R\, M < \infty$.

The convention that a complex of restricted flat dimension $-\infty$ has finite restricted flat dimension may appear odd; as noticed below it only applies to acyclic complexes.

9.3.38. Let M be an R-complex. The inequality below is immediate from the unitor 7.5.4, and the equality follows from 2.5.5, as the derived tensor product per 7.4.5 is a triangulated functor,

$$\mathrm{Rfd}_R\, M \;\geqslant\; \sup M \qquad \text{and} \qquad \mathrm{Rfd}_R\, \Sigma^s M \;=\; \mathrm{Rfd}_R\, M + s \quad \text{for every integer } s\,.$$

Moreover, one has $\mathrm{Rfd}_R\, M = -\infty$ if and only if M is acyclic.

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

9.3.39 Proposition. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. With $f' = \mathrm{Rfd}_R\, M'$, $f = \mathrm{Rfd}_R\, M$, and $f'' = \mathrm{Rfd}_R\, M''$ there are inequalities,*

$$f' \;\leqslant\; \max\{f, f''-1\}\,, \quad f \;\leqslant\; \max\{f', f''\}\,, \quad \textit{and} \quad f'' \;\leqslant\; \max\{f'+1, f\}\,.$$

In particular, if two of the complexes M', M, and M'' have finite restricted flat dimension, then so has the third.

Proof. For every R^{o}-module N there is a distinguished triangle,

$$N \otimes^{\mathsf{L}}_R M' \longrightarrow N \otimes^{\mathsf{L}}_R M \longrightarrow N \otimes^{\mathsf{L}}_R M'' \longrightarrow \Sigma(N \otimes^{\mathsf{L}}_R M')\,.$$

The inequalities now follow from the definition, 9.3.37, in view of 6.5.20. □

The next result compares to the definition, 8.5.16, of the finitistic flat dimension, see also 9.4.1, but notice that the supremum in 9.3.40 is taken over all modules. Since there exist rings of infinite finitistic flat dimension, see 20.2.21, it follows that the restricted flat dimension of a module need not be finite.

9.3.40 Proposition. *There is an equality,*

$$\operatorname{FFD} R^{\mathrm{o}} = \sup\{\operatorname{Rfd}_R M \mid M \text{ is an } R\text{-module}\}\,.$$

Moreover, there exists an R-module M with $\operatorname{Rfd}_R M = \operatorname{FFD} R^{\mathrm{o}}$.

Proof. Let M be an R-module. For every R^{o}-module N with $\operatorname{fd}_{R^{\mathrm{o}}} N < \infty$ one has

$$\sup(N \otimes^{\mathsf{L}}_R M) \leqslant \operatorname{fd}_{R^{\mathrm{o}}} N \leqslant \operatorname{FFD} R^{\mathrm{o}}$$

by 8.3.11 and 8.5.16. It follows that the inequalities $\operatorname{Rfd}_R M \leqslant \operatorname{FFD} R^{\mathrm{o}}$ and hence $\operatorname{FFD} R^{\mathrm{o}} \geqslant \sup\{\operatorname{Rfd}_R M \mid M \text{ is an } R\text{-module}\}$ hold. It remains to prove the existence of an R-module M with $\operatorname{Rfd}_R M = \operatorname{FFD} R^{\mathrm{o}}$.

First assume that $n = \operatorname{FFD} R^{\mathrm{o}}$ is finite. By 8.5.16 there exists an R^{o}-module N with $\operatorname{fd}_{R^{\mathrm{o}}} N = n$, and by 8.3.11 there exists an R-module M with $\sup(N \otimes^{\mathsf{L}}_R M) = n$, whence one has $\operatorname{Rfd}_R M = n$. If $\operatorname{FFD} R^{\mathrm{o}}$ is infinite, then there exists a sequence $\{N_u\}_{u\in\mathbb{N}}$ of R^{o}-modules with $u \leqslant \operatorname{fd}_{R^{\mathrm{o}}} N_u < \infty$ for every $u \in \mathbb{N}$. By 8.3.11 there exists for each $u \in \mathbb{N}$ an R-module M_u with $\sup(N_u \otimes^{\mathsf{L}}_R M_u) = \operatorname{fd}_R N_u$. Now, set $M = \coprod_{v\in\mathbb{N}} M_v$. For every $u \in \mathbb{N}$ one has by 7.4.5 and 3.1.11:

$$\sup(N_u \otimes^{\mathsf{L}}_R M) = \sup\bigl(\textstyle\coprod_{v\in\mathbb{N}}(N_u \otimes^{\mathsf{L}}_R M_v)\bigr) \geqslant \sup(N_u \otimes^{\mathsf{L}}_R M_u) = \operatorname{fd}_R N_u \geqslant u\,.$$

It follows from 9.3.37 that $\operatorname{Rfd}_R M = \infty$ holds. □

Exercises

E 9.3.1 Assume that R is right coherent and $\operatorname{splf} R$ is finite. Show that a complex of projective R-modules that is totally acyclic in the sense of 9.1.1 is totally acyclic in the sense of 9.3.1. *Hint:* E 3.3.3.

E 9.3.2 Consider an exact sequence $0 \to G' \to F \to G \to 0$ of R-modules. Show that if F is flat and G is Gorenstein flat, then G' is Gorenstein flat. Conclude that a complex of finite Gorenstein flat dimension n has a semi-flat replacement F with $\mathrm{C}_v(F)$ Gorenstein flat for all $v \geqslant n$.

E 9.3.3 Let M be an R-module. Show that if $\operatorname{Tor}^R_m(I, M) = 0$ holds for all injective R^{o}-modules I and all $m > 0$, then one has $\operatorname{Tor}^R_m(N, M) = 0$ for all R^{o}-modules N with $\operatorname{id}_{R^{\mathrm{o}}} N < \infty$ and all $m > 0$.

E 9.3.4 Assume that R is right Noetherian. Show that if every R-module has finite Gorenstein flat dimension, then every acyclic complex of flat R-modules is totally acyclic.

E 9.3.5 Assume that R is right Noetherian and let $\{M^u\}_{u\in U}$ be a family of R-modules. Show that $\prod_{u\in U} M^u$ is Gorenstein flat only if M^u is Gorenstein flat for every $u \in U$.

E 9.3.6 Let R be right Noetherian and $\{\mu^{vu}\colon M^u \to M^v\}_{u\in U}$ a U-direct system of R-modules. Show that $\operatorname{Gfd}_R(\operatorname{colim}_{u\in U} M^u) \leqslant \sup_{u\in U}\{\operatorname{Gfd}_R M^u\}$ holds if U is filtered.

E 9.3.7 Let R be right Noetherian, M a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$, and set $w = \sup M$. Show that for every semi-flat replacement F of M one has $\operatorname{Gfd}_R M = w + \operatorname{Gfd}_R \mathrm{C}_w(F)$.

E 9.3.8 Let R be right Noetherian and M a complex in $\mathcal{D}_{\sqsupset}(R)$. Show that $\operatorname{Gfd}_R M$ is finite if and only if M is isomorphic in $\mathcal{D}(R)$ to a bounded complex of Gorenstein flat R-modules.

E 9.3.9 Assume that R is right Noetherian. Show that the full subcategory of R-complexes of finite Gorenstein flat dimension is a triangulated subcategory of $\mathcal{D}_{\sqsubset}(R)$.

E 9.3.10 Assume that R is right Noetherian. Let M be a complex in $\mathcal{D}_{\square}(R)$ and G a bounded below complex of Gorenstein flat R-modules with $M \simeq G$ in $\mathcal{D}(R)$. Show that for every

R°-module N with $\mathrm{fd}_{R^{\circ}} N$ or $\mathrm{id}_{R^{\circ}} N$ finite and for all integers $m > 0$ and $n \geqslant \sup M$ there is an isomorphism $\mathrm{Tor}^R_{n+m}(N, M) \cong \mathrm{Tor}^R_m(N, \mathrm{C}_n(G))$.

9.4 Gorenstein Global Dimensions

Synopsis. Finitistic projective/injective/flat dimension; Gorenstein global dimension; Gorenstein weak global dimension; Iwanaga–Gorenstein ring.

The finitistic Gorenstein dimensions agree with the ordinary finitistic dimensions, so there is no need to introduce notation for finitistic Gorenstein dimensions.

9.4.1 Proposition. *There are equalities,*

$$\begin{aligned}\operatorname{FPD} R &= \sup\{\operatorname{Gpd}_R M \mid M \text{ is an } R\text{-module with } \operatorname{Gpd}_R M < \infty\}\\ &= \sup\left\{\operatorname{Gpd}_R M - \sup M \;\middle|\; \begin{array}{c} M \text{ is an } R\text{-complex with}\\ \mathrm{H}(M) \neq 0 \text{ and } \operatorname{Gpd}_R M < \infty\end{array}\right\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{FID} R &= \sup\{\operatorname{Gid}_R M \mid M \text{ is an } R\text{-module with } \operatorname{Gid}_R M < \infty\}\\ &= \sup\left\{\operatorname{Gid}_R M + \inf M \;\middle|\; \begin{array}{c} M \text{ is an } R\text{-complex with}\\ \mathrm{H}(M) \neq 0 \text{ and } \operatorname{Gid}_R M < \infty\end{array}\right\}.\end{aligned}$$

Further, if R is right Noetherian, then there are equalities,

$$\begin{aligned}\operatorname{FFD} R &= \sup\{\operatorname{Gfd}_R M \mid M \text{ is an } R\text{-module with } \operatorname{Gfd}_R M < \infty\}\\ &= \sup\left\{\operatorname{Gfd}_R M - \sup M \;\middle|\; \begin{array}{c} M \text{ is an } R\text{-complex with}\\ \mathrm{H}(M) \neq 0 \text{ and } \operatorname{Gfd}_R M < \infty\end{array}\right\}.\end{aligned}$$

Proof. Let s and t denote the suprema in the first display. The inequality $\operatorname{FPD} R \leqslant s$ holds by 9.1.13, and the opposite inequality holds by 9.1.16(a). Evidently, one has $s \leqslant t$. To prove that equality holds, let M be an R-complex with $\operatorname{Gpd}_R M < \infty$ and $\mathrm{H}(M) \neq 0$. Set $w = \sup M$ and note that w is an integer by 9.1.11. Given a semi-projective replacement P of M, the complex $\Sigma^{-w} P_{\geqslant w}$ is a semi-projective replacement of the module $\mathrm{C}_w(P)$, so one has $\operatorname{Gpd}_R M = w + \operatorname{Gpd}_R \mathrm{C}_w(P)$ by 9.1.10 and 9.1.19. It follows that $s = t$ holds.

The first equality in the second display follows from 9.2.12 and 9.2.15(a). For an R-complex M with $\operatorname{Gid}_R M < \infty$ and $\mathrm{H}(M) \neq 0$ set $u = \inf M$. It follows from 9.2.9 and 9.2.18 that for every semi-injective replacement I of M one has $\operatorname{Gid}_R M = \operatorname{Gid}_R \mathrm{Z}_u(I) - u$. This establishes the second equality in the second display.

If R is right Noetherian, then the first equality in the third display follows from 9.3.18 and 9.3.23(a). For an R-complex M with $\operatorname{Gfd}_R M < \infty$ and $\mathrm{H}(M) \neq 0$ set $w = \sup M$. It follows from 9.3.15 and 9.3.26 that for every semi-flat replacement F of M one has $\operatorname{Gfd}_R M = w + \operatorname{Gfd}_R \mathrm{C}_w(F)$. This establishes the second equality in the third display. ∎

GORENSTEIN GLOBAL DIMENSION

9.4.2 Definition. The *Gorenstein global dimension* of R, written $\operatorname{Ggldim} R$, is defined as
$$\operatorname{Ggldim} R \;=\; \sup\{\operatorname{Gpd}_R M \mid M \text{ is an } R\text{-module}\}\,.$$

The Gorenstein global dimension is a refinement of the global dimension.

9.4.3 Proposition. *There is an inequality,*
$$\operatorname{Ggldim} R \;\leqslant\; \operatorname{gldim} R\,,$$
and equality holds if R has finite global dimension.

PROOF. Both assertions follow immediately from 9.1.13 . □

9.4.4 Example. It follows from 8.2.10 and 9.4.15 that one has $\operatorname{Ggldim} \mathbb{Z}/4\mathbb{Z} = 0$ while $\operatorname{gldim} \mathbb{Z}/4\mathbb{Z} = \infty$ holds by 8.5.2.

REMARK. In view of 9.4.3 the Gorenstein global dimension is not a left–right symmetric invariant, see the Remark after 8.5.2, but it follows from 9.4.15 and 8.5.30 that it is left–right symmetric for Noetherian rings. Moreover, Christensen, Estrada, and Thompson [57] prove that $\operatorname{Ggldim} R = 0$ holds if and only if $\operatorname{Ggldim} R^{\mathrm{o}} = 0$ holds, and this condition characterizes quasi-Frobenius rings.

9.4.5 Theorem. *There are equalities,*
$$\begin{aligned}\operatorname{Ggldim} R \;&=\; \sup\{\operatorname{Gpd}_R M - \sup M \mid M \text{ is a complex in } \mathcal{D}_{\sqsubset}(R) \text{ with } \operatorname{H}(M) \neq 0\}\\ &=\; \sup\{\operatorname{Gid}_R M \mid M \text{ is an } R\text{-module}\}\\ &=\; \sup\{\operatorname{Gid}_R M + \inf M \mid M \text{ is a complex in } \mathcal{D}_{\sqsupset}(R) \text{ with } \operatorname{H}(M) \neq 0\}\,.\end{aligned}$$
Moreover, the following inequalities hold,
$$\operatorname{FPD} R \;\leqslant\; \operatorname{Ggldim} R \;\geqslant\; \operatorname{FID} R\,.$$

PROOF. The inequality $\operatorname{FPD} R \leqslant \operatorname{Ggldim} R$ follows from 9.4.1, and so does the other inequality once the equalities in the first display are established.

Denote by s_1, s_2, and s_3 the suprema in the first display in the order they appear. Evidently one has $\operatorname{Ggldim} R \leqslant s_1$. To see that equality holds, let M be a complex with $\operatorname{H}(M) \neq 0$ and $w = \sup M < \infty$. Given a semi-projective replacement P of M, the complex $\Sigma^{-w} P_{\geqslant w}$ is a semi-projective replacement of the module $\operatorname{C}_w(P)$, so by 9.1.10 and 9.1.19 one has $\operatorname{Gpd}_R M = w + \operatorname{Gpd}_R \operatorname{C}_w(P)$. A similarly argument invoking 9.2.9 and 9.2.18 shows that $s_2 = s_3$ holds.

It remains to establish the equality $\operatorname{Ggldim} R = s_2$. To that end let $n \geqslant 0$ be an integer; we proceed to prove that $\operatorname{Ggldim} R \leqslant n$ implies $s_2 \leqslant n$ and vice-versa. Assume first that $\operatorname{Ggldim} R \leqslant n$ holds. From this assumption, 9.1.31, 8.2.19, and 9.1.20 one immediately gets $\operatorname{id}_R P \leqslant n$ for every projective R-module P and $\operatorname{pd}_R E \leqslant n$ for every injective R-module E. Now let M be an R module and I a semi-injective replacement of M; by 9.2.9 it suffices to show that the module $Z = \operatorname{Z}_{-n}(I)$ is Gorenstein injective.

Let $P \xrightarrow{\simeq} M$ be a projective resolution; see 5.2.28. Associated to the short exact sequence $0 \to \mathrm{C}_1(P) \to P_0 \to M \to 0$ there is by 8.2.14 another exact sequence $0 \to I^1 \to J^0 \to I \to 0$ where I^1 and J^0 are semi-injective replacements of $\mathrm{C}_1(P)$ and P_0. It yields by 2.2.16 an exact sequence $0 \to \mathrm{Z}_{-n}(I^1) \to \mathrm{Z}_{-n}(J^0) \to Z \to 0$ where the module $\mathrm{Z}_{-n}(J^0)$ is injective by 8.2.8 since $\mathrm{id}_R P_0 \leqslant n$. Proceeding recursively, the exact sequence $0 \to \mathrm{C}_{v+1}(P) \to P_v \to \mathrm{C}_v(P) \to 0$ yields, as above, an exact sequence $0 \to \mathrm{Z}_{-n}(I^{v+1}) \to \mathrm{Z}_{-n}(J^v) \to \mathrm{Z}_{-n}(I^v) \to 0$, where $\mathrm{Z}_{-n}(J^v)$ is injective. Splicing together the acyclic complexes

$$\cdots \to \mathrm{Z}_{-n}(J^1) \to \mathrm{Z}_{-n}(J^0) \to Z \to 0 \quad \text{and} \quad 0 \to Z \to I_{-n} \to I_{-n-1} \to \cdots$$

one gets an acyclic complex I' of injective R-modules with Z as a cycle module. To see that I' is totally acyclic, let E be an injective R-module. As $\mathrm{pd}_R E \leqslant n$ holds, one has $\mathrm{Ext}^1_R(E, \mathrm{Z}_{-v}(I')) \cong \mathrm{Ext}^{n+1}_R(E, \mathrm{Z}_{n-v}(I')) = 0$ by 8.2.6, since the complex $I'_{\leqslant n-v}$ is a semi-injective replacement of $\mathrm{Z}_{n-v}(I')$. Thus I' is totally acyclic, and Z is Gorenstein injective.

Assuming now that $s_2 \leqslant n$ holds, it follows from 9.2.22, 8.1.20, and 9.2.19 that $\mathrm{pd}_R E \leqslant n$ holds for every injective R-module E and $\mathrm{id}_R P \leqslant n$ holds for every projective R-module P. Let M be an R module and P a semi-projective replacement of M; by 9.1.10 it suffices to show that the module $\mathrm{C}_n(P)$ is Gorenstein projective, and that follows from an argument dual to the one given above. ◻

Remark. In parallel with 8.5.3 one has $\mathrm{Ggldim}\, R = \sup\{\mathrm{Gpd}_R\, R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\}$; this is shown by Bennis, Hu, and Wang [36] under the assumption that R is commutative, but that assumption is irrelevant for the argument. For a Noetherian ring R the equality is proved in 9.4.17.

9.4.6 Corollary. *The following conditions are quivalent.*

(*i*) $\mathrm{Ggldim}\, R$ *is finite.*

(*ii*) *Every R-module has finite Gorenstein projective dimension.*

(*iii*) *Every complex in $\mathcal{D}_{\sqsubset}(R)$ has finite Gorenstein projective dimension.*

(*iv*) *Every R-module has finite Gorenstein injective dimension.*

(*v*) *Every complex in $\mathcal{D}_{\sqsupset}(R)$ has finite Gorenstein injective dimension.*

Moreover, in the case these conditions are satisfied there are equalities,

$$\mathrm{FPD}\, R = \mathrm{Ggldim}\, R = \mathrm{FID}\, R\,.$$

Proof. The equivalence of the conditions follows from 9.4.5, 9.1.21, and 9.2.20. If these conditions are satisfied, then the asserted equalities hold by 9.4.1. ◻

Remark. It transpires from the proof of 9.4.5 that the conditions in 9.4.6 are equivalent to finiteness of the invarianst $\mathrm{spli}\, R$ and $\mathrm{silp}\, R$; see E 8.5.16. The Gorenstein projective and injective cases were originally dealt with separately in works of Gedrich and Gruenberg [105] and Nucinkis [195]; they are brought together by Emmanouil [82].

Gorenstein Weak Global Dimension

9.4.7 Definition. The *Gorenstein weak global dimension* of R, written $\mathrm{Gwgldim}\, R$, is defined as

$$\operatorname{Gwgldim} R \;=\; \sup\{\operatorname{Gfd}_R M \mid M \text{ is an } R\text{-module}\}\,.$$

The Gorenstein weak global dimension refines the weak global dimension.

9.4.8 Proposition. *There is an inequality,*

$$\operatorname{Gwgldim} R \;\leqslant\; \operatorname{wgldim} R\,,$$

and equality holds if wgldim R *is finite.*

Proof. Both assertions follow immediately from 9.3.18 . □

9.4.9 Proposition. *If R is right Noetherian, then there is an inequality,*

$$\operatorname{Gwgldim} R \;\leqslant\; \operatorname{Ggldim} R\,.$$

Proof. Assume that Ggldim R is finite. It follows from 9.4.6 and 8.5.18 that every flat R-module has finite projective dimension, so the inequality holds by 9.3.30. □

For Noetherian rings equality holds in 9.4.9; this is proved in 9.4.16 and compares to 8.5.13. In particular, one has $\operatorname{Gwgldim} \mathbb{Z}/4\mathbb{Z} = 0 < \operatorname{wgldim} \mathbb{Z}/4\mathbb{Z} = \infty$; see 9.4.4 and 8.5.13.

Remark. The inequality in 9.4.9 holds without the Noetherian hypothesis in perfect parallel with the inequality $\operatorname{wgldim} R \leqslant \operatorname{gldim} R$ from 8.5.6; see for example Wang, Yang, Shao, and Zhang [251].

9.4.10 Lemma. *If* Gwgldim R *is finite, then the following assertions hold.*

(a) *Every injective R°-module has finite flat dimension.*

(b) *If R is left Noetherian, then every injective R-module has finite flat dimension.*

Proof. Set $n = \operatorname{Gwgldim} R$ and assume that it is finite. Part (b) is immediate from 9.3.19. To prove part (a), let E be an injective R°-module and M an R-module. It follows from 9.3.15 that M has a semi-flat replacement F with $\mathrm{C}_d(F)$ Gorenstein flat for some $d \leqslant n$. For $m > n$ one now has $\operatorname{Tor}^R_m(E,M) \cong \operatorname{Tor}^R_{m-d}(E,\mathrm{C}_d(F)) = 0$ by 8.3.9 and 9.3.5(1). Thus $\operatorname{fd}_{R^{\circ}} E \leqslant n$ holds by 8.3.23. □

9.4.11 Lemma. *If every injective R-module and every injective R°-module has finite flat dimension, then* Gwgldim R *is finite.*

Proof. Since a product of injective modules by 1.3.27 is injective, it follows from 8.3.27 that $d = \sup\{\operatorname{fd}_R E \mid E \text{ is an injective } R\text{-module}\}$ is a non-negative integer; it suffices to show that every R-module has Gorenstein flat dimension at most d.

Let M be an R-module and P a semi-projective replacement of M; by 9.3.15 it suffices to show that the module $C = \mathrm{C}_d(P)$ is Gorenstein flat. Let $M \xrightarrow{\simeq} I$ be an injective resolution; see 5.3.32. Associated to $0 \to M \to I_0 \to \mathrm{Z}_{-1}(I) \to 0$ there is an exact sequence $0 \to P \to L^0 \to P^{-1} \to 0$ where L^0 and P^{-1} are semi-projective replacements of I_0 and $\mathrm{Z}_{-1}(I)$; see 8.1.13. It yields by 2.2.16 an exact sequence $0 \to C \to \mathrm{C}_d(L^0) \to \mathrm{C}_d(P^{-1}) \to 0$ where the module $\mathrm{C}_d(L^0)$ is flat by 8.3.11. Proceeding recursively, the exact sequence $0 \to \mathrm{Z}_{-v}(I) \to I_{-v} \to \mathrm{Z}_{-v-1}(I) \to 0$ yields, as above, an exact sequence $0 \to \mathrm{C}_d(P^{-v}) \to \mathrm{C}_d(L^{-v}) \to \mathrm{C}_d(P^{-v-1}) \to 0$, where $\mathrm{C}_d(L^{-v})$ is flat. Splicing together the acyclic complexes

$$\cdots \to P_{d+1} \to P_d \to C \to 0 \quad \text{and} \quad 0 \to C \to \mathrm{C}_d(L^0) \to \mathrm{C}_d(L^1) \to \cdots$$

one gets an acyclic complex F of flat R-modules with C as a cokernel module. To see that F is totally acyclic, let E be an injective R^{o}-module. By assumption E has finite flat dimension, say n, so for every $v \in \mathbb{Z}$ one has $\mathrm{Tor}_1^R(E, \mathrm{C}_v(F)) \cong \mathrm{Tor}_{n+1}^R(E, \mathrm{C}_{v-n}(F)) = 0$ by 8.1.6, as $F_{\geqslant v-n}$ is a semi-flat replacement of $\mathrm{C}_{v-n}(F)$. Thus F is totally acyclic and C is Gorenstein flat. □

9.4.12 Proposition. *If R is left Noetherian, then* $\mathrm{Gwgldim}\, R$ *is finite if and only if every injective R-module and every injective R^{o}-module has finite flat dimension.*

Proof. The assertion follows immediately from 9.4.10 and 9.4.11. □

Remark. Proposition 9.4.12 shows that finiteness of $\mathrm{Gwgldim}\, R$ is closely tied to finiteness of the invariants $\mathrm{sfli}\, R$ and $\mathrm{sfli}\, R^{\mathrm{o}}$; see also E 9.4.3.

The Gorenstein weak global dimension is left–right symmetric.

9.4.13 Theorem. *If R is Noetherian, then there is an equality,*

$$\mathrm{Gwgldim}\, R \;=\; \mathrm{Gwgldim}\, R^{\mathrm{o}}\,,$$

and one has

$$\begin{aligned}\mathrm{FFD}\, R \;&\leqslant\; \mathrm{Gwgldim}\, R\\ &=\; \sup\{\mathrm{Gfd}_R\, M - \sup M \mid M \text{ is a complex in } \mathcal{D}_{\sqsubset}(R) \text{ with } \mathrm{H}(M) \neq 0\}\,.\end{aligned}$$

Proof. The inequality $\mathrm{FFD}\, R \leqslant \mathrm{Gwgldim}\, R$ holds by 9.4.1. To verify the equality in the same display, let M be a complex with $\mathrm{H}(M) \neq 0$ and $w = \sup M < \infty$. Given a semi-flat replacement F of M, the complex $\Sigma^{-w} F_{\geqslant w}$ is a semi-flat replacement of the module $\mathrm{C}_w(F)$, so by 9.3.15 and 9.3.26 one has $\mathrm{Gfd}_R\, M = w + \mathrm{Gfd}_R\, \mathrm{C}_w(F)$.

To prove the equality in the first display, notice that $\mathrm{Gwgldim}\, R$ and $\mathrm{Gwgldim}\, R^{\mathrm{o}}$ are simultaneously finite by 9.4.12. Set $n = \mathrm{Gwgldim}\, R$ and assume that it is finite. There exists by 9.3.36 an R-module M and an injective R^{o}-module E with $\mathrm{Tor}_n^R(E, M) \neq 0$. One has $\mathrm{fd}_{R^{\mathrm{o}}}\, E < \infty$ by 9.4.12 and, therefore, $n \leqslant \mathrm{FFD}\, R^{\mathrm{o}}$ by 8.3.23. Thus there are inequalities $\mathrm{FFD}\, R \leqslant \mathrm{Gwgldim}\, R \leqslant \mathrm{FFD}\, R^{\mathrm{o}}$, and equalities hold since the same inequalities hold with R and R^{o} interchanged. □

9.4.14 Corollary. *If R is Noetherian, then the following conditions are equivalent.*

(*i*) $\mathrm{Gwgldim}\, R$ *is finite.*
(*ii*) *Every R-module has finite Gorenstein flat dimension.*
(*iii*) *Every complex in $\mathcal{D}_{\sqsubset}(R)$ has finite Gorenstein flat dimension.*

Moreover, in the case these conditions are satisfied there are equalities,

$$\mathrm{FFD}\, R \;=\; \mathrm{Gwgldim}\, R \;=\; \mathrm{FID}\, R\,.$$

Proof. It follows from 9.4.13 that (*i*) implies (*iii*), which in turn implies (*ii*), and (*ii*) implies (*i*) by 9.3.27. If these conditions are satisfied, then 9.4.1 yields $\mathrm{FFD}\, R = \mathrm{Gwgldim}\, R$. Finally, $\mathrm{Gwgldim}\, R = \mathrm{FFD}\, R^{\mathrm{o}} = \mathrm{FID}\, R$ holds by 9.4.13 and 8.5.27. □

REMARK. In 9.4.13 and 9.4.14 the assumption that R is Noetherian is imposed by the reference to 9.3.19 in the proof of 9.4.10, and just like 9.3.19 the statement above remains valid without that assumption; see Christensen, Estrada, and Thompson [56].

Iwanaga–Gorenstein Rings

A Noetherian ring R is Iwanaga–Gorenstein if $\mathrm{id}_R\, R$ and $\mathrm{id}_{R^{\mathrm{o}}}\, R$ are finite, see 8.5.29, and by 9.2.19 this is equivalent to R having finite Gorenstein injective dimension over R and R^{o}. Theorem 9.4.15 below is a strong converse: All Gorenstein dimensions of all modules over an Iwanaga–Gorenstein ring are finite. By definition, the Iwanaga–Gorenstein property of a Noetherian ring is left–right symmetric, nevertheless it can be detected by finiteness of the Gorenstein global dimension on one side.

REMARK. While a Noetherian ring of finite Gorenstein global dimension on one side is Iwanaga–Gorenstein, it is as discussed by Kirkman, Kuzmanovich, and Small [158] not known if a Noetherian ring of finite self-injective dimension on one side is Iwanaga–Gorenstein. For Artin algebras this is known as the Gorenstein Symmetry Question; it was raised by Auslander and Reiten [13].

9.4.15 Theorem. *If R is Noetherian, then the following conditions are equivalent.*

(*i*) *R is Iwanaga–Gorenstein.*
(*ii*) $\mathrm{Gwgldim}\, R$ *is finite.*
(*iii*) $\mathrm{Ggldim}\, R$ *is finite.*
(*iv*) $\sup\{\mathrm{Gpd}_R\, R/\mathfrak{a} \mid \mathfrak{a}$ is a left ideal in $R\}$ *is finite.*

In the case these conditions are satisfied there are equalities,

$$\begin{aligned}\mathrm{id}_R\, R &= \mathrm{Ggldim}\, R\\ &= \sup\{\mathrm{Gpd}_R\, R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\}\\ &= \mathrm{Gwgldim}\, R\,.\end{aligned}$$

PROOF. Conditions (*i*) and (*ii*) are equivalent by 8.5.31 and 9.4.12.

(*ii*) $\Longrightarrow$ (*iii*): The equivalence of (*i*) and (*ii*) has already been established, so one has $\mathrm{id}_R\, R = n$ for some integer $n \geqslant 0$. Now, it follows from 8.5.27 and 8.5.18 that $\mathrm{pd}_R\, F \leqslant n$ holds for every flat R-module F.

Let M be an R-module and $P' \simeq M$ a semi-projective replacement. By assumption and 9.3.36 the module $G = \mathrm{C}_d(P')$ is Gorenstein flat for some $d \geqslant 0$. The complex $\Sigma^{-d} P'_{\geqslant d}$ is a semi-projective replacement of G, so it suffices by 9.1.19 to show that G has finite Gorenstein projective dimension. Let F be a totally acyclic complex of flat R-modules with $\mathrm{C}_0(F) \cong G$ and hence $G \cong \mathrm{Z}_{-1}(F)$. Let P be a semi-projective replacement of G; it suffices by 9.1.10 to show that the module $C = \mathrm{C}_n(P)$ is Gorenstein projective. Associated to the exact sequence $0 \to G \to F_{-1} \to \mathrm{Z}_{-2}(F) \to 0$ there is an exact sequence $0 \to P \to L^{-1} \to P^{-2} \to 0$ where L^{-1} and P^{-2} are semi-projective replacements of F_{-1} and $\mathrm{Z}_{-2}(F)$; see 8.1.13. It yields by 2.2.16 an exact sequence $0 \to C \to \mathrm{C}_n(L^{-1}) \to \mathrm{C}_n(P^{-2}) \to 0$ where the module $\mathrm{C}_n(L^{-1})$ is projective by 8.1.8. Proceeding recursively, the short exact sequence $0 \to \mathrm{Z}_{-v}(F) \to F_{-v} \to \mathrm{Z}_{-v-1}(F) \to 0$ yields, as above, an exact sequence

$0 \to \mathrm{C}_n(P^{-v}) \to \mathrm{C}_n(L^{-v}) \to \mathrm{C}_n(P^{-v-1}) \to 0$, where $\mathrm{C}_n(L^{-v})$ is projective. Splicing together the acyclic complexes

$$\cdots \to P_{n+1} \to P_n \to C \to 0 \quad \text{and} \quad 0 \to C \to \mathrm{C}_n(L^{-1}) \to \mathrm{C}_n(L^{-2}) \to \cdots$$

one gets an acyclic complex $\widetilde{P}$ of projective R-modules with C as a cokernel module. To see that $\widetilde{P}$ is totally acyclic, let L be a projective R-module; as $\mathrm{id}_R R \leqslant n$ holds, 8.2.21 yields $\mathrm{id}_R L \leqslant n$. For every $v \in \mathbb{Z}$ one now has $\mathrm{Ext}^1_R(\mathrm{C}_v(\widetilde{P}), L) \cong \mathrm{Ext}^{n+1}_R(\mathrm{C}_{v-n}(\widetilde{P}), L) = 0$ by 8.1.6, as $\widetilde{P}_{\geqslant v-n}$ is a semi-projective replacement of $\mathrm{C}_{v-n}(\widetilde{P})$. Thus $\widetilde{P}$ is totally acyclic and C is Gorenstein projective.

(*iii*) $\Rightarrow$ (*iv*): This impilcation is immediate from the definition of $\mathrm{Ggldim}\, R$.

(*iv*) $\Rightarrow$ (*i*): Set $n = \sup\{\mathrm{Gpd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\}$ and assume that it is finite. From 9.1.32 and 8.2.19 one immediately gets $\mathrm{id}_R R \leqslant n$. By 8.3.18 one has $\mathrm{id}_{R^\circ} R = \mathrm{fd}_R \mathrm{Hom}_\Bbbk(R, \mathbb{E})$, and to prove that this quantity is finite it suffices by 9.3.19 to show that the R-module $E = \mathrm{Hom}_\Bbbk(R, \mathbb{E})$ has finite Gorenstein flat dimension. This module is by 3.3.5 a filtered colimit of finitely generated R-modules E^u.

Every finitely generated R-module has Gorenstein projective dimension at most n. Indeed, let M be generated by elements $x_1, \ldots, x_m$ and proceed by induction on the number, m, of generators. For $m = 1$ one has $M \cong R/(0 :_R x_1)$, whence $\mathrm{Gpd}_R M \leqslant n$ holds by assumption. For $m > 1$ set $N = R\langle x_1, \ldots, x_{m-1}\rangle$; the quotient module M/N is then generated by $[x_m]_N$. By the induction hypothesis and the base case one has $\mathrm{Gpd}_R N \leqslant n$ and $\mathrm{Gpd}_R M/N \leqslant n$, so 9.1.14 applied to the exact sequence $0 \to N \to M \to M/N \to 0$ yields $\mathrm{Gpd}_R M \leqslant n$.

The canonical map $\coprod_{u\in U} E^u \to \mathrm{colim}_{u\in U} E^u \cong E$ is by 9.3.12 a pure epimorphism, so $\mathrm{Hom}_\Bbbk(E, \mathbb{E})$ is by 5.5.14 a direct summand of the module

$$(\flat) \qquad \mathrm{Hom}_\Bbbk\Big(\coprod_{u\in U} E^u, \mathbb{E}\Big) \cong \prod_{u\in U} \mathrm{Hom}_\Bbbk(E^u, \mathbb{E})\;;$$

see 3.1.27. As $\mathrm{Gpd}_R E^u \leqslant n$ holds for all $u \in U$ it follows from 9.3.34, 9.3.17, and 9.2.20 that the product module in ($\flat$), and hence the direct summand $\mathrm{Hom}_\Bbbk(E, \mathbb{E})$, has Gorenstein injective dimension at most n. Another application of 9.3.17 now yields $\mathrm{Gfd}_R E \leqslant n$.

Assuming now that conditions (*i*)–(*iv*) are satisfied, we address the equalities. For every left ideal $\mathfrak{a}$ in R the inequality $\mathrm{Gpd}_R R/\mathfrak{a} \leqslant \mathrm{id}_R R$ holds by 9.1.32 and 8.2.19. Moreover, another application of 8.2.19 yields the existence of a left ideal $\mathfrak{a}$ in R with $\mathrm{Gpd}_R R/\mathfrak{a} = \mathrm{id}_R R$. Thus one

$$\sup\{\mathrm{Gpd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\} = \mathrm{id}_R R\,.$$

Further, 8.5.30 yields

$$\mathrm{FID}\, R = \mathrm{FFD}\, R = \mathrm{FPD}\, R = \mathrm{id}_R R\,,$$

and the asserted equalities now hold by 9.4.6 and 9.4.14. □

9.4.16 Corollary. *If R is Noetherian, then the following equalities hold,*

$$\mathrm{Ggldim}\, R = \mathrm{Gwgldim}\, R = \mathrm{Ggldim}\, R^\circ\,.$$

Proof. It follows from 9.4.15 that $\operatorname{Gwgldim} R$ and $\operatorname{Ggldim} R$ are simultaneously finite, and in that case the next equalities hold by 9.4.15 and 9.4.13.

$$\operatorname{Ggldim} R \;=\; \operatorname{Gwgldim} R \;=\; \operatorname{Gwgldim} R^{\mathrm{o}} \;=\; \operatorname{Ggldim} R^{\mathrm{o}}\,. \qquad \square$$

Remark. It is a result of Bouchiba [43] that the equality $\operatorname{Gwgldim} R = \operatorname{Ggldim} R$ in 9.4.16 holds under the weaker assumption that R is left Noetherian, thus providing a perfect parallel to 8.5.13.

9.4.17 Corollary. *If R is Noetherian, then the one has*

$$\operatorname{Ggldim} R \;=\; \sup\{\operatorname{Gpd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is a left ideal in } R\}\,.$$

Proof. It follows from 9.4.15 that the two quantities are simultaneously finite, in which case they agree. $\square$

Exercises

E 9.4.1 Show that if $\operatorname{splf} R$ is finite, then every flat Gorenstein projective R-module is projective.

E 9.4.2 Assume that R is right Noetherian. Show that if $\operatorname{Gwgldim} R$ is finite, then the inequality $\operatorname{Gfd}_R M \leqslant \operatorname{Gpd}_R M$ holds for every R-complex M.

E 9.4.3 Show that the next conditions are equivalent.

(*i*) $\sup\{\operatorname{Gfd}_R R/\mathfrak{a} \mid \mathfrak{a}$ is a finitely generated left ideal in $R\}$ and $\sup\{\operatorname{Gfd}_{R^{\mathrm{o}}}(R/\mathfrak{b}) \mid \mathfrak{b}$ is a finitely generated right ideal in $R\}$ are finite.

(*ii*) $\operatorname{Gwgldim} R$ and $\operatorname{Gwgldim} R^{\mathrm{o}}$ are finite.

(*iii*) $\operatorname{sfli} R$ and $\operatorname{sfli} R^{\mathrm{o}}$ are finite.

Show that the equalities $\operatorname{FFD} R = \operatorname{Gwgldim} R = \operatorname{Gwgldim} R^{\mathrm{o}} = \operatorname{FFD} R^{\mathrm{o}}$ hold in the case these conditions are satsified.

Chapter 10
Dualizing Complexes

In this chapter we treat some fundamental equivalences and dualities of subcategories of derived catgories. The more important ones, in the context of this book, involve a so-called dualizing complex. In a nut shell, a dualizing complex is a computational gadget that in the derived category of a Noetherian ring allows one to mimic the isomorphism of a finite rank vector space over a field to its double dual space. Dualizing complexes came into commutative algebra through Hartshorne's exposition [114] of work of Grothendieck and his student Verdier. Neeman reflects on the history in [193]. The generalization of dualizing complexes to non-commutative rings that we treat in this chapter is essentially due to Yekutieli and Zhang [259].

10.1 Grothendieck Duality

SYNOPSIS. Homothety formation; dualizing complex; Grothendieck Duality.

The goal of this section is to establish the general, non-commutative version, of the Grothendieck Duality Theorem for derived categories. The first step towards that theorem, which is the last result of the section, is to define a dualizing complex. To this end, let X be a complex of R–S°-bimodules and recall from 4.5.5 the homothety formation morphisms

$$\chi^{X}_{S^{\circ}R} : S \longrightarrow \operatorname{Hom}_R(X,X) \qquad \text{and} \qquad \chi^{X}_{RS^{\circ}} : R \longrightarrow \operatorname{Hom}_{S^{\circ}}(X,X)\ .$$

If R is commutative, then an R-complex X is tacitly considered to be a complex of symmetric R–R-bimodules; in this case the two homothety formation morphisms are the same. Hence:

10.1.1 Definition. If R is commutative, then the morphism $\chi^{X}_{R^{\circ}R} = \chi^{X}_{RR^{\circ}}$ in $\mathcal{C}(R)$ is denoted χ^{X}_{R}.

DEFINITION AND EXAMPLES OF DUALIZING COMPLEXES

10.1.2 Definition. Let R be left Noetherian and S right Noetherian. A complex D of R–S^{o}-bimodules is called *dualizing* for (R, S^{o}) if it meets the requirements:

(1) $\mathrm{H}(D)$ is bounded and degreewise finitely generated over R and over S^{o}.
(2) D has finite injective dimension over R and over S^{o}.
(3) There exists a complex J of R–S^{o}-bimodules such that
   - there is an isomorphism $D \simeq J$ in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$,
   - J is semi-injective over R and over S^{o},
   - the next homothety formation maps are quasi-isomorphisms,

$$\chi^{J}_{S^{\mathrm{o}}R} : S \longrightarrow \operatorname{Hom}_R(J, J) \quad \text{and} \quad \chi^{J}_{RS^{\mathrm{o}}} : R \longrightarrow \operatorname{Hom}_{S^{\mathrm{o}}}(J, J) .$$

10.1.3 Proposition. *Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).*

(a) *For every integer s the complex $\Sigma^s D$ is dualizing for (R, S^{o}).*
(b) *D is dualizing for (S^{o}, R).*

PROOF. Part (a) is immediate from 2.3.14, 2.3.16, 8.2.3, and 5.3.10. For part (b) recall that complex of R–S^{o}-bimodules is the same as a complex of S^{o}–R-bimodules, and a ring is left Noetherian if and only if the opposite ring is right Noetherian. $\square$

In the case of a single Noetherian ring it makes sense to talk about a dualizing complex for that ring; see also 10.1.6.

10.1.4 Definition. Let R be Noetherian. A complex of R–R^{o}-bimodules is called *dualizing* for R if it is dualizing for (R, R^{o}) as defined in 10.1.2.

10.1.5 Example. Let $\Bbbk$ be a field and R a finite dimensional $\Bbbk$-algebra; i.e. R has finite rank as a $\Bbbk$-vector space. Consider the R–R^{o}-bimodule $D = \operatorname{Hom}_{\Bbbk}(R, \Bbbk)$. As a $\Bbbk$-vector space D is finitely generated, and hence it is finitely generated as a module over R and over R^{o}. Moreover, D is injective over R and over R^{o} by 5.4.26(a). The isomorphisms in the following commutative diagram in $\mathcal{C}(R\text{–}R^{\mathrm{o}})$ come from 4.5.6, 4.5.4, and 4.4.10; it shows that $\chi^{D}_{RR^{\mathrm{o}}}$ is an isomorphism in $\mathcal{C}(R\text{–}R^{\mathrm{o}})$.

$$\begin{tikzcd}
R \arrow[r, "\chi^D"] \arrow[d, "\chi^R"', "\cong"] & \operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_{\Bbbk}(R,\Bbbk), \operatorname{Hom}_{\Bbbk}(R,\Bbbk)) \\
\operatorname{Hom}_{R^{\mathrm{o}}}(R, R) \arrow[r, "\operatorname{Hom}(R,\delta^R_{\Bbbk})", "\cong"'] & \operatorname{Hom}_{R^{\mathrm{o}}}(R, \operatorname{Hom}_{\Bbbk}(\operatorname{Hom}_{\Bbbk}(R,\Bbbk),\Bbbk)) \arrow[u, "\cong", "\zeta^{RD\Bbbk}"']
\end{tikzcd} .$$

A a similar diagram shows that $\chi^{D}_{R^{\mathrm{o}}R}$ is an isomorphism as well. Thus, with $J = D$ and $S = R$ the conditions in 10.1.2 are met, so D is dualizing for R per 10.1.4.

10.1.6 Definition. Assume that R is commutative and Noetherian. An R-complex D is called *dualizing* for R if it is dualizing as defined in 10.1.4 when considered as a complex of symmetric R–R-bimodules, cf. 10.1.1.

10.1.7 Example. If R is Noetherian and self-injective, then R is a dualizing complex for R, as the conditions in 10.1.2 are met by $D = R = J$; see 4.5.6. Commutative examples include $\mathbb{Z}/n\mathbb{Z}$ and $\Bbbk[x]/(x^n)$ for every $n > 1$ and any field $\Bbbk$; see 8.2.10.

The next example is generalized in 18.2.2.

10.1.8 Example. Assume that R is commutative Artinian and local with unique maximal ideal $\mathfrak{m}$. The chain $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots$ stabilizes, so $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ holds for some n, whence $\mathfrak{m}^n = 0$ holds by Nakayama's lemma B.32. In the notation from C.20 one now has $\mathrm{E}_R(R/\mathfrak{m}) = E^n$, so $\mathrm{E}_R(R/\mathfrak{m})$ is finitely generated by C.22, and C.21(c) yields $\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{m}), \mathrm{E}_R(R/\mathfrak{m})) \cong R$. Thus $\mathrm{E}_R(R/\mathfrak{m})$ is a dualizing complex for R.

Homothety Formation in the Derived Category

Let X be a complex of R–S^{o}-bimodules. Homothety formation from 4.5.5,

$$\chi^X_{S^{\mathrm{o}}R} : S \longrightarrow \mathrm{Hom}_R(X,X) \qquad \text{and} \qquad \chi^X_{RS^{\mathrm{o}}} : R \longrightarrow \mathrm{Hom}_{S^{\mathrm{o}}}(X,X)\,,$$

induce morphisms in $\mathcal{K}(S\text{–}S^{\mathrm{o}})$ and $\mathcal{K}(R\text{–}R^{\mathrm{o}})$ which we denote by the same symbols.

10.1.9 Construction. Assume that S is flat as a $\Bbbk$-module. The functor RHom_R is by 7.3.13 augmented as follows

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}S^{\mathrm{o}})^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^{\mathrm{o}}) \longrightarrow \mathcal{D}(S\text{–}S^{\mathrm{o}})\,,$$

and for $X \in \mathcal{D}(R\text{–}S^{\mathrm{o}})$ and $I = \mathrm{I}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(X)$ one has $\mathsf{RHom}_R(X,X) = \mathrm{Hom}_R(I,I)$. It follows that homothety formation $\chi^I_{S^{\mathrm{o}}R} : S \to \mathrm{Hom}_R(I,I)$ in $\mathcal{K}(S\text{–}S^{\mathrm{o}})$ yields a morphism in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$,

(10.1.9.1) $$\chi^X_{S^{\mathrm{o}}R} : S \longrightarrow \mathsf{RHom}_R(X,X)\,.$$

Similarly, if R is flat as a $\Bbbk$-module, then $\chi^I_{RS^{\mathrm{o}}} : R \to \mathrm{Hom}_{S^{\mathrm{o}}}(I,I)$ in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$ yields a morphism in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$,

(10.1.9.2) $$\chi^X_{RS^{\mathrm{o}}} : R \longrightarrow \mathsf{RHom}_{S^{\mathrm{o}}}(X,X)\,.$$

10.1.10 Definition. The morphisms (10.1.9.1) and (10.1.9.2) are called *homothety formation*. If R is commutative, then the morphism $\chi^X_{R^{\mathrm{o}}R} = \chi^X_{RR^{\mathrm{o}}}$ in $\mathcal{D}(R)$ that comes out of 10.1.9 with $\Bbbk = R$ is denoted χ^X_R; cf. 10.1.1.

10.1.11 Lemma. *Assume that R and S are flat as $\Bbbk$-modules. Let X and J be complexes of R–S^{o}-bimodules such that $X \simeq J$ in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$.*

(a) *If J is semi-injective as an R-complex, then the morphism $\chi^X_{S^{\mathrm{o}}R}$ from* 10.1.9 *is isomorphic in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$ to the morphism induced by $\chi^J_{S^{\mathrm{o}}R}$.*

(b) *If J is semi-injective as an S^{o}-complex, then the morphism $\chi^X_{RS^{\mathrm{o}}}$ from* 10.1.9 *is isomorphic in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ to the morphism induced by $\chi^J_{RS^{\mathrm{o}}}$.*

Proof. We only prove part (a); part (b) follows by interchanging the roles of R and S^{o}. Set $I = \mathrm{I}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(X)$. By 6.4.21 there is a quasi-isomorphism $\varphi\colon J \to I$ in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$. As φ by the assumption on J and 7.3.11(c) is a quasi-isomorphism of semi-injective R-complexes, it follows from 5.3.24 and 4.3.19 that $\mathrm{Hom}_R(M,\varphi)$ is a quasi-isomorphism in $\mathcal{K}(S\text{–}S^{\mathrm{o}})$ for every complex M of R–S^{o}-bimodules. In particular, $\mathrm{Hom}_R(J,\varphi)$ is a quasi-isomorphism. As already established, I is semi-injective as an R-complex, whence $\mathrm{Hom}_R(\varphi,I)$ is a quasi-isomorphism in $\mathcal{K}(S\text{–}S^{\mathrm{o}})$. The assertion now follows from the diagram below, which is commutative by S^{o}-linearity of φ.

$$\begin{tikzcd} S \arrow[r, "\chi^J"] \arrow[d, "\chi^I"'] & \mathrm{Hom}_R(J,J) \arrow[d, "\simeq"', "\mathrm{Hom}(J,\varphi)"] \\ \mathrm{Hom}_R(I,I) \arrow[r, "\mathrm{Hom}(\varphi,I)", "\simeq"'] & \mathrm{Hom}_R(J,I)\,. \end{tikzcd}$$ □

10.1.12 Proposition. *Assume that R and S are flat as $\Bbbk$-modules. Let R be left Noetherian and S right Noetherian. A complex D of R–S^{o}-bimodules is dualizing for (R,S^{o}) if and only if it meets the requirements below.*

(1) *$\mathrm{H}(D)$ is bounded and degreewise finitely generated over R and over S^{o}.*
(2) *D has finite injective dimension over R and over S^{o}.*
(3) *Homothety formation $\chi^D_{S^{\mathrm{o}}R}\colon S \to \mathsf{R}\mathrm{Hom}_R(D,D)$ in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$ and homothety formation $\chi^D_{RS^{\mathrm{o}}}\colon R \to \mathsf{R}\mathrm{Hom}_{S^{\mathrm{o}}}(D,D)$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ are isomorphisms.*

Proof. “Only if” follows from 10.1.11. For “if” let $D \xrightarrow{\simeq} J$ be a semi-injective resolution in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$. One has $D \simeq J$ in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and the complex J is semi-injective over R and over S^{o}; see 7.3.11(c). By assumption, the morphissms $\chi^D_{S^{\mathrm{o}}R}$ and $\chi^D_{RS^{\mathrm{o}}}$ induced by $\chi^J_{S^{\mathrm{o}}R}$ and $\chi^J_{RS^{\mathrm{o}}}$ are isomorphisms in the derived categories $\mathcal{D}(S\text{–}S^{\mathrm{o}})$ and $\mathcal{D}(R\text{–}R^{\mathrm{o}})$, so 6.4.17 yields that $\chi^J_{S^{\mathrm{o}}R}$ and $\chi^J_{RS^{\mathrm{o}}}$ are quasi-isomorphisms. □

Remark. There exists a left Noetherian (even a left Artinian local, and even a commutative Noetherian local) algebra R over a field $\Bbbk$ such that there is no dualizing complex for (R,S^{o}) for any right Noetherian $\Bbbk$-algebra S. See Yekutieli and Zhang [259] and Wu and Zhang [255]. In [255] there is also an example of an Artinian local $\Bbbk$-algebra R that does not have a dualizing complex; however, for some right Artinian $\Bbbk$-algebra S the pair (R,S^{o}) does have a dualizing complex.

We record the next result for use in Sect. 10.4.

10.1.13 Lemma. *Assume that R and S are flat as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S^{o}). There exists a bounded complex of R–S^{o}-bimodules that is isomorphic to D in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ and semi-injective over R and over S^{o}.*

Proof. Let $D \xrightarrow{\simeq} I$ be a semi-injective resolution in $\mathcal{K}(R\text{–}S^{\mathrm{o}})$ with $I_v = 0$ for $v > \sup D$; see 5.3.26. Recall from 7.3.11(c) that I is semi-injective over R and over S^{o}. Thus, for every integer $n \geqslant \max\{\mathrm{id}_R D, \mathrm{id}_{S^{\mathrm{o}}} D\}$ the bounded complex $J = I_{\supseteq -n}$ is semi-injective over R and over S^{o} by 8.2.8, and one has $D \simeq J$ in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$. □

Since projectivity and injectivity are categorically dual notions, it should be no surprise that self-injective rings have excellent duality properties. It is a quality which, in the realm of derived categories, extends to rings of finite self-injective dimension; for that reason Iwanaga–Gorenstein rings, see 8.5.29, come up with frequency in the next chapters.

10.1.14 Example. If R is flat as a $\Bbbk$-module and Iwanaga–Gorenstein, then R is a dualizing complex for R. Indeed, let $\iota\colon R \xrightarrow{\simeq} I$ be an injective resolution in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$. Per 4.5.6 the homothety morphism $\chi^{R}_{R^{\mathrm{o}}R}$ is an isomorphism, so it follows from the commutative diagram,

$$\begin{tikzcd} R \arrow[r, "\chi^R", "\cong"'] \arrow[d, "\chi^I"'] & \operatorname{Hom}_R(R,R) \arrow[d, "\simeq"', "\operatorname{Hom}(R,\iota)"] \\ \operatorname{Hom}_R(I,I) \arrow[r, "\operatorname{Hom}(\iota,I)", "\simeq"'] & \operatorname{Hom}_R(R,I)\ , \end{tikzcd}$$

that $\chi^{I}_{R^{\mathrm{o}}R}$ is a quasi-isomorphism; i.e. the induced morphism $\boldsymbol{\chi}^{R}_{R^{\mathrm{o}}R}$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ is an isomorphism. Similarly, $\boldsymbol{\chi}^{R}_{RR^{\mathrm{o}}}$ is an isomorphism. Now apply 10.1.12 and 10.1.4.

Existence of Dualizing Complexes

To parse the next result, notice that if $\Bbbk$ is Noetherian and R is finitely generated as a $\Bbbk$-module, then R is Noetherian as well. Indeed, every finitely generated R-module and every finitely generated R^{o}-module is finitely generated over $\Bbbk$.

10.1.15 Theorem. *Let $\Bbbk$ be Noetherian and R finitely generated as a $\Bbbk$-module. If D is a dualizing complex for $\Bbbk$, then $\mathsf{R}\mathrm{Hom}_{\Bbbk}(R,D)$ is a dualizing complex for R.*

Proof. As a $\Bbbk$-complex, the homology of $E = \mathsf{R}\mathrm{Hom}_{\Bbbk}(R,D)$ is bounded and degreewise finitely generated by 7.6.16 and 8.2.8; in particular, it is degreewise finitely generated over R and over R^{o}. Moreover, $\mathrm{id}_R\, E$ and $\mathrm{id}_{R^{\mathrm{o}}}\, E$ are finite by 8.2.4.

It remains to show that E satisfies 10.1.2(3). Let $D \xrightarrow{\simeq} I$ be a bounded semi-injective resolution over $\Bbbk$. The complex $J = \operatorname{Hom}_{\Bbbk}(R,I)$ in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$ is by 5.4.26(a) semi-injective over R and over R^{o}, and one has $E \simeq J$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$. There is a commutative diagram in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$,

$$\begin{tikzcd} R \arrow[r, "\chi^J"] & \operatorname{Hom}_R(J,J) \\ R \otimes_{\Bbbk} \Bbbk \arrow[u, "\mu^R\upsilon^{R\Bbbk}", "\cong"'] & \operatorname{Hom}_{\Bbbk}(R \otimes_R J, I) \arrow[u, "\cong", "\rho^{JRI}"'] \\ R \otimes_{\Bbbk} \operatorname{Hom}_{\Bbbk}(I,I) \arrow[u, "R\otimes\chi^I", "\simeq"'] \arrow[r, "\eta^{IIR}", "\cong"'] & \operatorname{Hom}_{\Bbbk}(J,I)\ , \arrow[u, "\cong", "\operatorname{Hom}(\mu^J,I)"'] \end{tikzcd}$$

where adjunction ρ^{JRI} and homomorphism evaluation η^{IIR} are isomorphisms by 4.4.12 and 4.5.13(3,b). The modules in the bounded $\Bbbk$-complex $\operatorname{Hom}_{\Bbbk}(I,I)$ are flat by 8.4.28, so it is semi-flat by 5.4.8. The homothety morphism $\chi^I_{\Bbbk}$ is by 10.1.12 a quasi-isomorphism and by 5.4.16 also $R \otimes_{\Bbbk} \chi^I_{\Bbbk}$ is a quasi-isomorphism. Now it follows from the diagram that $\chi^J_{R^{\mathrm{o}}R}$ is a quasi-isomorphism, whence $\chi^E_{R^{\mathrm{o}}R}$ is an isomorphism in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$. With R and R^{o} interchanged the diagram shows that $\chi^E_{RR^{\mathrm{o}}}$ is an isomorphism. □

Remark. If $\Bbbk$ is Iwanaga–Gorenstein, then by 10.1.15 every $\Bbbk$-algebra that is finitely generated as a $\Bbbk$-module has a dualizing complex. In the commutative realm every ring with a dualizing complex arises in this fashion; se the Remark after 18.2.7.

10.1.16 Definition. Let $\Bbbk$ be Artinian. A $\Bbbk$-algebra that is finitely generated as a $\Bbbk$-module is called an *Artin algebra* or, more elaborately, an *Artin $\Bbbk$-algebra.*

10.1.17 Example. If $\Bbbk$ is Artinian with Jacobson radical $\mathfrak{J}$, then the injective envelope D of $\Bbbk/\mathfrak{J}$ is a dualizing complex for $\Bbbk$; the local case is handled in 10.1.5 and the general case is 18.2.2. By 10.1.15 every Artin $\Bbbk$-algebra R now has a dualizing complex, namely $\mathsf{RHom}_{\Bbbk}(R,D) = \operatorname{Hom}_{\Bbbk}(R,D)$.

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The next example motivates the developments in the rest of this section.

10.1.18 Example. Let R be an Artin algebra, by 10.1.17 it has a dualizing complex D which is, in fact, an R–R^{o}-bimodule that is injective over R and over R^{o}. For every degreewise finitely generated R-complex M there is a commutative diagram in $\mathcal{C}(R)$,

$$\begin{tikzcd} M \otimes_{R^{\mathrm{o}}} R \arrow[r, "M\otimes\chi^D", "\cong"'] \arrow[d, "\mu^M\upsilon^{MR}"', "\cong"] & M \otimes_{R^{\mathrm{o}}} \operatorname{Hom}_{R^{\mathrm{o}}}(D,D) \arrow[d, "\cong"', "\eta^{DDM}"] \\ M \arrow[r, "\delta^M_D"] & \operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_R(M,D),D) \end{tikzcd},$$

where the evaluation morphism η^{DDM} is an isomorphism by 4.5.13(3,b). The diagram shows that the biduality morphism δ^M_D is an isomorphism of R-complexes.

As advertised at the beginning of this chapter, biduality with respect to a dualizing complex is an isomorphism in the derived category.

10.1.19 Theorem. *Assume that R and S are flat as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for* (R,S^{o})*. For every complex M in* $\mathcal{D}^{\mathrm{f}}(R)$ *the biduality morphism from* 8.4.4*,*

$$\delta^M_D : M \longrightarrow \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(M,D),D) ,$$

is an isomorphism in $\mathcal{D}(R)$.

Proof. By A.32(c) the functors $\mathsf{RHom}_R(-,D)$ and $\mathsf{RHom}_{S^\circ}(-,D)$ are bounded, whence the composite functor $\mathsf{RHom}_{S^\circ}(\mathsf{RHom}_R(-,D),D)$ is bounded, and so is the identity functor on $\mathcal{D}(R)$. The transformation $\boldsymbol{\delta}_D$ is triangulated by 8.4.3, so by 7.6.14 and A.28(d) it suffices to show that $\boldsymbol{\delta}_D^M$ is an isomorphism for every finitely generated R-module M. For such a module the evaluation morphism $\boldsymbol{\eta}^{DDM}$ is an isomorphism by 8.4.24(b), whence the commutative diagram,

$$\begin{tikzcd}
M \otimes^{\mathsf{L}}_{R^\circ} R \arrow[r, "M\otimes^{\mathsf{L}}\chi^D", "\simeq"'] \arrow[d, "\boldsymbol{\mu}^M\boldsymbol{\upsilon}^{MR}"', "\simeq"] & M \otimes^{\mathsf{L}}_{R^\circ} \operatorname{Hom}_{S^\circ}(D,D) \arrow[d, "\simeq"', "\boldsymbol{\eta}^{DDM}"] \\
M \arrow[r, "\boldsymbol{\delta}_D^M"] & \mathsf{RHom}_{S^\circ}(\mathsf{RHom}_R(M,D),D)\,,
\end{tikzcd}$$

shows that $\boldsymbol{\delta}_D^M$ is an isomorphism. □

To facilitate the formulation of the Grothendieck Duality Theorem, we introduce notation for a few more subcategories of the derived category.

10.1.20 Definition. The full subcategories $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ of $\mathcal{D}_\square(R)$ are defined by specifying their objects as follows

$$\begin{aligned}
\mathcal{P}(R) &= \{M \in \mathcal{D}_\square(R) \mid \operatorname{pd}_R M < \infty\}\,,\\
\mathcal{I}(R) &= \{M \in \mathcal{D}_\square(R) \mid \operatorname{id}_R M < \infty\}\,, \quad \text{and}\\
\mathcal{F}(R) &= \{M \in \mathcal{D}_\square(R) \mid \operatorname{fd}_R M < \infty\}\,.
\end{aligned}$$

The full subcategory $\mathcal{P}(R) \cap \mathcal{D}^{\mathrm{f}}(R)$ is denoted by $\mathcal{P}^{\mathrm{f}}(R)$. Similarly, one defines the subcategories $\mathcal{I}^{\mathrm{f}}(R)$ and $\mathcal{F}^{\mathrm{f}}(R)$.

10.1.21 Proposition. *The categories $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ are triangulated subcategories of $\mathcal{D}_\square(R)$. Further, if R is left Noetherian, then $\mathcal{P}^{\mathrm{f}}(R)$, $\mathcal{I}^{\mathrm{f}}(R)$, and $\mathcal{F}^{\mathrm{f}}(R)$ are triangulated subcategories of $\mathcal{D}^{\mathrm{f}}_\square(R)$ and one has $\mathcal{P}^{\mathrm{f}}(R) = \mathcal{F}^{\mathrm{f}}(R)$.*

Proof. The full subcategory $\mathcal{P}(R)$ of $\mathcal{D}_\square(R)$ is triangulated by 8.1.3 and 8.1.9. Similarly, $\mathcal{I}(R)$ is triangulated by 8.2.3 and 8.2.9, and $\mathcal{F}(R)$ is triangulated by 8.3.4 and 8.3.12. The remaining assertions follow from 7.6.14 and 8.3.19. □

10.1.22 Proposition. *Assume that R and S are flat as $\Bbbk$-modules and let X be a complex of R–S°-bimodules. There is an adjunction,*

$$\mathcal{D}(S^\circ) \underset{\mathsf{RHom}_R(-,X)}{\overset{\mathsf{RHom}_{S^\circ}(-,X)^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}(R)^{\mathrm{op}}\,.$$

For an S°-complex N the unit of the adjunction is biduality in $\mathcal{D}(S^\circ)$,

$$\boldsymbol{\delta}_X^N : N \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^\circ}(N,X),X)\,,$$

and for R-complex M the counit, viewed as a morphism in $\mathcal{D}(R)$, is biduality

$$\boldsymbol{\delta}_X^M : M \longrightarrow \mathsf{RHom}_{S^\circ}(\mathsf{RHom}_R(M,X),X)\,.$$

Proof. Let M be an R-complex and N and S^{o}-complex. By 7.3.26 and swap 7.5.24 there are natural isomorphisms

$$\begin{aligned}\mathcal{D}(R)^{\mathrm{op}}(\mathsf{RHom}_{S^{\mathrm{o}}}(N,X),M) &\cong \mathrm{H}_0(\mathsf{RHom}_R(M,\mathsf{RHom}_{S^{\mathrm{o}}}(N,X)))\\ &\cong \mathrm{H}_0(\mathsf{RHom}_{S^{\mathrm{o}}}(N,\mathsf{RHom}_R(M,X)))\\ &\cong \mathcal{D}(S^{\mathrm{o}})(N,\mathsf{RHom}_R(M,X))\,.\end{aligned}$$

This establishes the asserted adjunction. The claims about the unit and counit follows from 4.5.7 applied with X replaced by $\mathrm{I}_{R\otimes_{\Bbbk}S^{\mathrm{o}}}(X)$. □

Grothendieck Duality is the phenomenon that for a commutative Noetherian ring $\Bbbk$ with a dualizing complex D the functor $\mathsf{RHom}_{\Bbbk}(-,D)$ yields a duality on $\mathcal{D}^{\mathrm{f}}(\Bbbk)$. This is a special case of the next result; see 18.2.3.

10.1.23 Theorem. *Assume that R and S are flat as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S^{o}). There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}}) \underset{\mathsf{RHom}_R(-,D)}{\overset{\mathsf{RHom}_{S^{\mathrm{o}}}(-,D)^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}}\,.$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)^{\mathrm{op}}\,,\quad \mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)^{\mathrm{op}}\,,\quad \textit{and}\quad \mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\square}(R)^{\mathrm{op}}\,,$$

and further to

$$\mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}}) \rightleftarrows \mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}}\,.$$

Proof. The functors $\mathsf{RHom}_{S^{\mathrm{o}}}(-,D)^{\mathrm{op}}\colon \mathcal{D}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}(R)^{\mathrm{op}} : \mathsf{RHom}_R(-,D)$ are adjoint by 10.1.22, and they are $\Bbbk$-linear and triangulated by 7.3.6. By A.32(c) the functor $\mathsf{RHom}_R(-,D)$ is bounded. To prove that it restricts to a functor $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}}) \leftarrow \mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}}$, it suffices by 7.6.14 and A.34(d) to verify that $\mathsf{RHom}_R(M,D)$ belongs to $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})$ for every finitely generated R-module M, and that was already done in 7.6.16. Similarly, $\mathsf{RHom}_{S^{\mathrm{o}}}(-,D)^{\mathrm{op}}$ is bounded and restricts to a functor $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}}) \to \mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}}$. By 10.1.19 these restrictions yield an adjoint equivalence. The restrictions $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)^{\mathrm{op}}$, $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)^{\mathrm{op}}$, and $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}}) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\square}(R)^{\mathrm{op}}$ now follow from A.31.

To establish the final restriction it suffices to argue that (1) $\mathrm{id}_{S^{\mathrm{o}}}\,\mathsf{RHom}_{S^{\mathrm{o}}}(M,D)$ is finite for every $M \in \mathcal{P}^{\mathrm{f}}(R)$ and that (2) $\mathrm{pd}_R\,\mathsf{RHom}_{S^{\mathrm{o}}}(N,D)$ is finite for every $N \in \mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}})$. Here (1) follows from 8.3.15(b). To prove (2), let $N \in \mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}})$ and $\mathfrak{a}$ be a left ideal in R. Now biduality 10.1.19 and swap 7.5.24 yield,

$$\begin{aligned}&\mathsf{RHom}_R(\,\mathsf{RHom}_{S^{\mathrm{o}}}(N,D),R/\mathfrak{a})\\ &\qquad\simeq \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N,D),\mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(R/\mathfrak{a},D),D))\\ &\qquad\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(R/\mathfrak{a},D),\mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N,D),D))\\ &\qquad\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(R/\mathfrak{a},D),N)\,.\end{aligned}$$

Now 8.2.8 and 7.6.7 yield

$$-\inf \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(R/\mathfrak{a}, D), N) \leqslant \operatorname{id}_{S^{\mathrm{o}}} N + \sup D\,,$$

whence 8.1.14 yields $\operatorname{pd}_R \mathsf{RHom}_{S^{\mathrm{o}}}(N, D) \leqslant \operatorname{id}_{S^{\mathrm{o}}} N + \sup D$. □

REMARK. The equivalence in 10.1.23 specializes to an equivalence $\mathcal{D}(Q\text{–}S^{\mathrm{o}}) \rightleftarrows \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}}$; see E 10.1.3.

10.1.24 Corollary. *Let R be flat as a $\Bbbk$-module and Noetherian. The next conditions are equivalent.*

(*i*) *R is Iwanaga–Gorenstein.*

(*ii*) *R is a dualizing complex for R.*

(*iii*) *R has a dualizing complex D such that* $\operatorname{pd}_R D$ *and* $\operatorname{pd}_{R^{\mathrm{o}}} D$ *are finite.*

PROOF. If R is Iwanaga–Gorenstein, then R is per 10.1.14 a dualizing complex for R; in particular, R has a dualizing complex of finite projective dimension over both R and R^{o}. Conversely, if D is a dualizing complex for R with $\operatorname{pd}_R D < \infty$, then by definition one has $R \simeq \mathsf{RHom}_R(D, D)$ in $\mathcal{D}(R^{\mathrm{o}})$, and 10.1.23 yields $\operatorname{id}_{R^{\mathrm{o}}} \mathsf{RHom}_R(D, D) < \infty$. The same argument applies with the roles of R and R^{o} interchanged. □

EXERCISES

E 10.1.1 Show that every quasi-Frobenius ring has a dualizing complex.

E 10.1.2 Let R be Noetherian. Show that a complex of R–R^{o}-bimodules is dualizing for R if and only if it is dualizing for R^{o}.

E 10.1.3 Assume that R and S are flat as $\Bbbk$-modules and let X be a complex of R–S^{o}-bimodules. Show that there is an adjunction

$$\mathsf{RHom}_{S^{\mathrm{o}}}(-, X)^{\mathrm{op}} : \mathcal{D}(Q\text{–}S^{\mathrm{o}}) \rightleftarrows \mathcal{D}(R\text{–}Q^{\mathrm{o}})^{\mathrm{op}} : \mathsf{RHom}_R(-, X)\,.$$

Hint: Zigzag identities.

E 10.1.4 Assume that R is flat as a $\Bbbk$-module and Noetherian. A complex C of R–R^{o}-bimodules is called *semi-dualizing* for R if the homothety morphisms $\chi^C_{RR^{\mathrm{o}}}$ and $\chi^C_{R^{\mathrm{o}}R}$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ are isomorphisms. Show that R is semi-dualizing for R.

E 10.1.5 Assume that R is flat as a $\Bbbk$-module and Noetherian. Let C be a semi-dualizing complex for R; see E 10.1.4. Show that biduality δ^M_C is an isomorphism in $\mathcal{D}(R)$ for every complex $M \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension and for $M = C$.

E 10.1.6 Assume that R is flat as a $\Bbbk$-module and Noetherian. Let D be a dualizing complex for R and $D \xrightarrow{\simeq} I$ a semi-injective resolution in $\mathcal{K}(R\text{–}R^{\mathrm{o}})$. Show that for a complex P of finitely generated projective R-modules the complex $\operatorname{Hom}_R(P, R)$ is acyclic if and only if $I \otimes_R P$ is acyclic. (For a stronger result see Jørgensen [153].)

E 10.1.7 Assume that R and S are flat as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}). Show that for $M \in \mathcal{D}(R)$ and $N \in \mathcal{D}^{\mathrm{f}}(R)$ one has $\mathsf{RHom}_R(M, N) \simeq \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(N, D), \mathsf{RHom}_R(M, D))$ in $\mathcal{D}(\Bbbk)$.

E 10.1.8 Let R be left Noetherian and S right Noetherian. Let M be an R-complex and X a complex in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$ with $\operatorname{id}_R X$ finite and $\mathrm{H}(X)$ degreewise finitely generated over S^{o}. Show that $\mathsf{RHom}_R(M, X)$ belongs to $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})$ and, further, that it belongs to (a) $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}})$ if X is in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ and (b) $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$ if X is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.

10.2 Morita Equivalence

Synopsis. Derived reflexive complex; invertible complex; Morita Equivalence; inverse complex.

We start by recording another situation in which biduality is an isomorphism in the derived category.

10.2.1 Theorem. *Assume that R is flat as a $\Bbbk$-module and left Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. If $\operatorname{pd}_R M$ is finite, then the biduality morphism*

$$\delta_R^M : M \longrightarrow \mathsf{RHom}_{R^{\mathrm{o}}}(\mathsf{RHom}_R(M,R),R)$$

is an isomorphism in $\mathcal{D}(R)$.

Proof. The vertical isomorphisms in the commutative diagram,

$$\begin{tikzcd}
M \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} R \arrow[r, "M\otimes^{\mathsf{L}}\epsilon^R", "\simeq"'] \arrow[d, "\mu^M v^{MR}"', "\simeq"] & M \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathrm{RHom}_{R^{\mathrm{o}}}(R,R) \arrow[d, "\simeq"', "\eta^{RRM}"] \\
M \arrow[r, "\delta_R^M"] & \mathsf{RHom}_{R^{\mathrm{o}}}(\mathsf{RHom}_R(M,R),R)\,,
\end{tikzcd}$$

come from 7.5.4, 7.5.14, and homomorphism evaluation 8.4.23/8.4.25(a); it shows that δ_R^M is an isomorphism. □

10.2.2 Corollary. *Assume that R is flat as a $\Bbbk$-module and Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. If $\operatorname{pd}_R M$ is finite, then the complex $\mathsf{RHom}_R(M,R)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R^{\mathrm{o}})$ and the next equalities hold,*

$$\operatorname{pd}_R M = -\inf \mathsf{RHom}_R(M,R) \quad \textit{and} \quad \operatorname{pd}_{R^{\mathrm{o}}} \mathsf{RHom}_R(M,R) = -\inf M\,;$$

in particular, the R^{o}-complex $\mathsf{RHom}_R(M,R)$ has finite projective dimension.

Proof. The right-hand equality holds by 8.1.15. This equality, together with 7.6.17 and 8.4.26 show that $\mathsf{RHom}_R(M,R)$ is a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R^{\mathrm{o}})$ of finite projective dimension. Another application of 8.1.15 and biduality 10.2.1 now yield:

$$\operatorname{pd}_{R^{\mathrm{o}}} \mathsf{RHom}_R(M,R) = -\inf \mathsf{RHom}_{R^{\mathrm{o}}}(\mathsf{RHom}_R(M,R),R) = -\inf M\,. \qquad \square$$

10.2.3 Proposition. *Assume that R and S are projective as $\Bbbk$-modules and let M be a complex in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$.*

(a) *If R is left Noetherian, $\mathrm{H}(M)$ is degreewise finitely generated over R, and $\operatorname{pd}_R M$ is finite, then there is a natural isomorphism of functors,*

$$\mathsf{RHom}_R(M,-) \simeq \mathsf{RHom}_R(M,R) \otimes^{\mathsf{L}}_R - \colon \mathcal{D}(R) \longrightarrow \mathcal{D}(S)\,.$$

(b) *If S is right Noetherian, $\mathrm{H}(M)$ is degreewise finitely generated over S^{o}, and $\operatorname{pd}_{S^{\mathrm{o}}} M$ is finite, then there is a natural isomorphism of functors,*

$$M \otimes^{\mathsf{L}}_S - \simeq \mathsf{RHom}_S(\mathsf{RHom}_{S^{\mathrm{o}}}(M,S),-) \colon \mathcal{D}(S) \longrightarrow \mathcal{D}(R)\,.$$

Proof. (a): The first natural isomorphism follows from the unitor 7.5.4 and tensor evaluation 8.4.10/8.4.13(a),

$$\mathsf{RHom}_R(M,-) \simeq \mathsf{RHom}_R(M, R \otimes^{\mathsf{L}}_R -) \simeq \mathsf{RHom}_R(M,R) \otimes^{\mathsf{L}}_R - .$$

(b): The first natural isomorphism follows from the counitor 7.5.8 and homomorphism evaluation 8.4.23/8.4.25(a),

$$M \otimes^{\mathsf{L}}_S - \simeq M \otimes^{\mathsf{L}}_S \mathsf{RHom}_S(S,-) \simeq \mathsf{RHom}_S(\mathsf{RHom}_{S^\circ}(M,S),-) . \qquad \square$$

Derived Reflexive Complexes

10.2.4 Definition. Assume that R is flat as a $\Bbbk$-module. An R-complex M is called *derived reflexive* if it satisfies the conditions:

(1) $M \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$.
(2) $\mathsf{RHom}_R(M,R) \in \mathcal{D}^{\mathrm{f}}_{\square}(R^\circ)$.
(3) Biduality $\boldsymbol{\delta}^M_R \colon M \longrightarrow \mathsf{RHom}_{R^\circ}(\mathsf{RHom}_R(M,R),R)$ is an isomorphism in $\mathcal{D}(R)$.

The full subcategory $\mathcal{R}(R)$ of $\mathcal{D}(R)$ is defined by specifying its objects as follows:

$$\mathcal{R}(R) = \{M \in \mathcal{D}(R) \mid M \text{ is derived reflexive}\} .$$

In 10.4.15 we interpret the derived reflexive complexes in terms of homological dimensions, for now we record:

10.2.5 Example. Assume that R is flat as a $\Bbbk$-module and left Noetherian. It follows from 10.2.1 and 10.2.2 that every complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension is derived reflexive.

10.2.6 Proposition. *Assume that R is flat as a $\Bbbk$-module. If R is left Noetherian, then $\mathcal{R}(R)$ is a triangulated subcategory of $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.*

Proof. The functors $\mathsf{RHom}_R(-,R)$ and $\mathsf{RHom}_{R^\circ}(-,R)$ are triangulated by 7.3.6, and biduality $\boldsymbol{\delta}_R$ is per 8.4.3 a triangulated natural transformation. Further, the subcategories $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and $\mathcal{D}^{\mathrm{f}}_{\square}(R^\circ)$ are triangulated by 7.6.3 and 7.6.14. It now follows from E.19 and E.20 that $\mathcal{R}(R)$ is a triangulated subcategory of $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. $\square$

10.2.7 Theorem. *Assume that R is flat as a $\Bbbk$-module and Noetherian. There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\mathcal{R}(R^\circ) \underset{\mathsf{RHom}_R(-,R)}{\overset{\mathsf{RHom}_{R^\circ}(-,R)^{\mathrm{op}}}{\rightleftarrows}} \mathcal{R}(R)^{\mathrm{op}} ,$$

and it restricts to an adjoint equivalence of triangulated subcategories,

$$\mathcal{P}^{\mathrm{f}}(R^\circ) \rightleftarrows \mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}} .$$

Proof. The functors $\mathsf{RHom}_{R^\circ}(-,R)^{\mathrm{op}} \colon \mathcal{D}(R^\circ) \rightleftarrows \mathcal{D}(R)^{\mathrm{op}} : \mathsf{RHom}_R(-,R)$ are adjoint by 10.1.22 and they are $\Bbbk$-linear and triangulated by 7.3.6. In particular, for every R-complex M and every R°-complex N one has the zigzag identities

$$\mathsf{RHom}_R(\boldsymbol{\delta}_R^M, R)\,\boldsymbol{\delta}_R^{\mathsf{RHom}_R(M,R)} \;=\; 1^{\mathsf{RHom}_R(M,R)}$$

and

$$\mathsf{RHom}_{R^\circ}(\boldsymbol{\delta}_R^N, R)\,\boldsymbol{\delta}_R^{\mathsf{RHom}_{R^\circ}(N,R)} \;=\; 1^{\mathsf{RHom}_{R^\circ}(N,R)}\;.$$

It is now immediate from 10.2.4 that the adjoint functors restrict to an equivalence $\mathcal{R}(R^\circ) \rightleftarrows \mathcal{R}(R)^{\mathrm{op}}$. That they further restrict to an equivalence $\mathcal{P}^{\mathrm{f}}(R^\circ) \rightleftarrows \mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}}$ follows from 10.2.2. □

HOMOTHETY FORMATION IN THE DERIVED CATEGORY

10.2.8. Assume that S is projective as a $\Bbbk$-module. The functor RHom_R is by 7.3.12 augmented as follows

$$\mathsf{RHom}_R(-,-)\colon \mathcal{D}(R\text{–}S^\circ)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S^\circ) \longrightarrow \mathcal{D}(S\text{–}S^\circ)\;,$$

and for $X \in \mathcal{D}(R\text{–}S^\circ)$ an $P = \mathrm{P}_{R\otimes_\Bbbk S^\circ}(X)$ one has $\mathsf{RHom}_R(X,X) = \mathrm{Hom}_R(P,P)$. It follows that $\chi^P_{S^\circ R}\colon S \to \mathrm{Hom}_R(P,P)$ in $\mathcal{K}(S\text{–}S^\circ)$ yields a morphism in $\mathcal{D}(S\text{–}S^\circ)$,

$$S \longrightarrow \mathsf{RHom}_R(X,X)\;.$$

Similarly, if R is projective as a $\Bbbk$-module, then $\chi^P_{RS^\circ}\colon R \to \mathrm{Hom}_{S^\circ}(P,P)$ in $\mathcal{K}(R\text{–}R^\circ)$ yields a morphism in $\mathcal{D}(R\text{–}R^\circ)$,

$$R \longrightarrow \mathsf{RHom}_{S^\circ}(X,X)\;.$$

It is a consequence of the next result that the induced morphisms above agree with homothety formation from 10.1.10.

10.2.9 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let X and L be complexes of R–S°-bimodules such that $X \simeq L$ in $\mathcal{D}(R\text{–}S^\circ)$.*

(a) *If L is semi-projective as an R-complex, then the morphism $\boldsymbol{\chi}^X_{S^\circ R}$ from* 10.1.9 *is isomorphic in $\mathcal{D}(S\text{–}S^\circ)$ to the morphism induced by $\chi^L_{S^\circ R}$.*

(b) *If L is semi-projective as an S°-complex, then the morphism $\boldsymbol{\chi}^X_{RS^\circ}$ from* 10.1.9 *is isomorphic in $\mathcal{D}(R\text{–}R^\circ)$ to the morphism induced by $\chi^L_{RS^\circ}$.*

PROOF. We only prove part (a), as (b) follows by interchanging the roles of R and S°. Set $I = \mathrm{I}_{R\otimes_\Bbbk S^\circ}(X)$; by 6.4.21 there is a quasi-isomorphism $\psi\colon L \to I$ in $\mathcal{K}(R\text{–}S^\circ)$. As L is semi-projective over R, the morphism $\mathrm{Hom}_R(L,\psi)$ is a quasi-isomorphism in $\mathcal{K}(S\text{–}S^\circ)$. It follows from 7.3.11(c) that I is semi-injective over R and hence $\mathrm{Hom}_R(\psi, I)$ is a quasi-isomorphism. The assertion now follows from the diagram below, which is commutative by S°-linearity of φ.

$$\begin{tikzcd} S \arrow[r, "\chi^L"] \arrow[d, "\chi^I"'] & \mathrm{Hom}_R(L,L) \arrow[d, "\mathrm{Hom}(L,\psi)", "\simeq"'] \\ \mathrm{Hom}_R(I,I) \arrow[r, "\mathrm{Hom}(\psi,I)", "\simeq"'] & \mathrm{Hom}_R(L,I)\;. \end{tikzcd}$$

□

Invertible Complexes

Compare the next definition with 10.1.12.

10.2.10 Definition. Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian and S right Noetherian. A complex U of R–S^{o}-bimodules is called *invertible* for (R, S^{o}) if it satisfies the conditions:

(1) $\mathrm{H}(U)$ is bounded and degreewise finitely generated over R and over S^{o}.
(2) U has finite projective dimension over R and over S^{o}.
(3) Homothety formation $\chi^U_{S^{\mathrm{o}}R} \colon S \to \mathsf{RHom}_R(U, U)$ in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$ and homothety formation $\chi^U_{RS^{\mathrm{o}}} \colon R \to \mathsf{RHom}_{S^{\mathrm{o}}}(U, U)$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ are isomorphisms.

Remark. An R-complex that is isomorphic in $\mathcal{D}(R)$ to a bounded complex of finitely generated projective R-modules is called *perfect*. By 8.1.14 an invertible complex for (R, S^{o}) is thus both a perfect R-complex and a perfect S^{o}-complex.

10.2.11 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and U an invertible complex for (R, S^{o}).*

(a) *For every integer s the complex $\Sigma^s U$ is invertible for (R, S^{o}).*
(b) *U is invertible for (S^{o}, R).*

Proof. Part (a) is immediate from 2.3.14, 2.3.16, and 8.1.3. For part (b) recall that complex of R–S^{o}-bimodules is the same as a complex of S^{o}–R-bimodules, and a ring is left Noetherian if and only if the opposite ring is right Noetherian. □

In the case of a single Noetherian ring it makes sense to talk about an invertible complex for that ring.

10.2.12 Definition. Let R be Noetherian. A complex of R–R^{o}-bimodules is called *invertible* for R if it is invertible for (R, R^{o}) as defined in 10.2.10.

Remark. An invertible complex is a special case of a semi-dualizing complex; see E 10.1.4.

10.2.13 Example. If R is projective as a $\Bbbk$-module and Noetherian, then R is invertible for R, cf. 4.5.6 and 10.2.9.

Morita Equivalence

We now aim for a derived version of Morita's [186] original theory of equivalences of module categories. We start with a description of the unit and counit of derived Hom–tensor adjunction.

10.2.14 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. For every complex X of R–S^{o}-bimodules there is an adjunction,*

(10.2.14.1)
$$\mathcal{D}(S) \underset{\mathsf{RHom}_R(X,-)}{\overset{X\otimes^{\mathsf{L}}_S -}{\rightleftarrows}} \mathcal{D}(R)\,.$$

For an S-complex N the unit $\boldsymbol{\alpha}_X^N$ is the unique morphism in $\mathcal{D}(S)$ that makes the following diagram commutative,

$$
\begin{tikzcd}
S \otimes^{\mathsf{L}}_S N \arrow[r, "\chi^X \otimes^{\mathsf{L}} N"] \arrow[d, "\mu^N"', "\simeq"] & \mathsf{RHom}_R(X, X) \otimes^{\mathsf{L}}_S N \arrow[d, "\theta^{XXN}"] \\
N \arrow[r, "\alpha^N_X"] & \mathsf{RHom}_R(X, X \otimes^{\mathsf{L}}_S N) .
\end{tikzcd}
\tag{10.2.14.2}
$$

It is induced by the unit α_P^N from 4.5.14 *with $P = \mathrm{P}_{R\otimes_{\Bbbk}S^{\circ}}(X)$, and if N is a complex in $\mathcal{D}(S\text{–}T^{\circ})$, then $\boldsymbol{\alpha}_X^N$ is a morphism in $\mathcal{D}(S\text{–}T^{\circ})$. Moreover, the natural transformation $\boldsymbol{\alpha}_X$ is triangulated.*

For an R-complex M the counit $\boldsymbol{\beta}_X^M$ is the unique morphism in $\mathcal{D}(R)$ that makes the following diagram commutative,

$$
\begin{tikzcd}
X \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X, M) \arrow[rr, "\beta^M_X"] \arrow[d, "\eta^{MXX}"'] & & M \arrow[d, "\epsilon^M", "\simeq"'] \\
\mathsf{RHom}_R(\mathsf{RHom}_{S^{\circ}}(X, X), M) \arrow[rr, "\mathsf{RHom}(\chi^X, M)"] & & \mathsf{RHom}_R(R, M) .
\end{tikzcd}
\tag{10.2.14.3}
$$

It is induced by the counit β_P^M from 4.5.14 *with $P = \mathrm{P}_{R\otimes_{\Bbbk}S^{\circ}}(X)$, and if M is a complex in $\mathcal{D}(R\text{–}Q^{\circ})$, then $\boldsymbol{\beta}_X^M$ is a morphism in $\mathcal{D}(R\text{–}Q^{\circ})$. Moreover, the natural transformation $\boldsymbol{\beta}_X$ is triangulated.*

PROOF. Set $P = \mathrm{P}_{R\otimes_{\Bbbk}S^{\circ}}(X)$ and consider the adjunction from 4.5.14,

$$
\mathcal{C}(S) \underset{\mathrm{Hom}_R(P,-)}{\overset{P\otimes_S -}{\rightleftarrows}} \mathcal{C}(R) ,
\tag{$\star$}
$$

whose unit α_P and counit β_P fit into the commutative diagrams (4.5.14.1) and (4.5.14.2). The functors in ($\star$) preserve homotopy by 4.3.19 and 4.3.20, so by 6.1.32 there is an induced adjunction,

$$
\mathcal{K}(S) \underset{\mathrm{Hom}_R(P,-)}{\overset{P\otimes_S -}{\rightleftarrows}} \mathcal{K}(R) ,
\tag{$\diamond$}
$$

whose unit and counit we still denote by the symbols α_P and β_P. This is in accordance with the usual convention that the notation $(-)^{\ddot{}}$ gets suppressed. As a natural transformation of endofunctors on $\mathcal{C}(S)$ the unit α_P is a Σ-transformation by 4.5.14. It follows from 6.2.17 that the induced unit of endofunctors on $\mathcal{K}(S)$ is triangulated. Similarly, the counit of endofunctors on $\mathcal{K}(R)$ induced by β_P is triangulated.

The functors in ($\diamond$) preserve quasi-isomorphisms, see 10.2.8, so 6.4.41 yields an adjunction,

$$
\mathcal{D}(S) \underset{\mathrm{Hom}_R(P,-)^{\H{}}}{\overset{(P\otimes_S -)^{\H{}}}{\rightleftarrows}} \mathcal{D}(R) ,
\tag{$\flat$}
$$

which is (10.2.14.1). The unit and counit are $\boldsymbol{\alpha}_X = \tilde{\tilde{\alpha}}_P$ and $\boldsymbol{\beta}_X = \tilde{\tilde{\beta}}_P$. As the natural transformations α_P and β_P are triangulated, so are $\boldsymbol{\alpha}_X$ and $\boldsymbol{\beta}_X$ by 6.5.14. The commutative diagram (4.5.14.1) in $\mathcal{C}(S)$ yields a commutative diagram of endofunctors on $\mathcal{K}(S)$,

$$\begin{tikzcd} S \otimes_S \mathrm{P}_S(-) \arrow[r, "\chi^P \otimes \mathrm{P}(-)"] \arrow[d, "\mu^{\mathrm{P}(-)}"', "\cong"] & \mathrm{Hom}_R(P,P) \otimes_S \mathrm{P}_S(-) \arrow[d, "\theta^{PP\mathrm{P}(-)}"] \\ \mathrm{P}_S(-) \arrow[r, "\alpha_P^{\mathrm{P}(-)}"] & \mathrm{Hom}_R(P, P \otimes_S \mathrm{P}_S(-)) \,. \end{tikzcd}$$

All functors in this diagram preserve quasi-isomorphisms, and by 7.2.11 and 7.2.12 there is a natural isomorphism $\tilde{\tilde{\mathrm{P}}}_S \simeq \mathrm{Id}_{\mathcal{D}(S)}$. Now 6.4.31 and 6.4.33 apply to yield the desired commutative diagram (10.2.14.2). As $\boldsymbol{\mu}^N$ is an isomorphism, $\boldsymbol{\alpha}_X^N$ is the unique morphism in $\mathcal{D}(S)$ that makes the diagram (10.2.14.2) commutative. Evaluated at a complex $N \in \mathcal{D}(S\text{–}T^{\mathrm{o}})$ the morphism $\boldsymbol{\alpha}_X^N = \tilde{\tilde{\alpha}}_P^N$ is a morphism in $\mathcal{D}(S\text{–}T^{\mathrm{o}})$ as α_P^N is a morphism in $\mathcal{C}(S\text{–}T^{\mathrm{o}})$ by 4.5.14. Similar arguments establish the assertions about the counit. □

REMARK. The adjunction (10.2.14.1) exists without the assumption that R and S are projective as $\Bbbk$-modules; see E 7.5.5. The diagrams (10.2.14.2) and (10.2.14.3) that link the unit and counit of the adjunction to the evaluation morphisms do, however, depend on this assumptions on R and S. This is in part due to the conservative approach to the use of the symbols RHom_R and $\otimes_R^{\mathsf{L}}$ imposed in 7.3.5 and 7.4.3. A more ad-hoc approach is taken by Christensen, Frankild, and Holm [61].

10.2.15 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and U an invertible complex for (R, S^{o}). There is an isomorphism in $\mathcal{D}(S\text{–}R^{\mathrm{o}})$,*

$$\mathsf{RHom}_R(U,R) \simeq \mathsf{RHom}_{S^{\mathrm{o}}}(U,S)\,.$$

Denoting this complex U^* there are natural isomorphisms of functors,

$$\mathsf{RHom}_R(U,-) \simeq U^* \otimes_R^{\mathsf{L}} -\colon \mathcal{D}(R) \longrightarrow \mathcal{D}(S) \quad \textit{and}$$
$$U \otimes_S^{\mathsf{L}} - \simeq \mathsf{RHom}_S(U^*,-)\colon \mathcal{D}(S) \longrightarrow \mathcal{D}(R)\,.$$

Finally, $\mathrm{H}(U^*)$ is bounded and U^* has finite projective dimension over S and R^{o}.

PROOF. The isomorphism of complexes follows from the definition of an invertible complex and swap 7.5.28,

$$\begin{aligned} \mathsf{RHom}_R(U,R) &\simeq \mathsf{RHom}_R(U, \mathsf{RHom}_{S^{\mathrm{o}}}(U,U)) \\ &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(U, \mathsf{RHom}_R(U,U)) \\ &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(U,S)\,. \end{aligned}$$

The natural isomorphisms of functors follow from 10.2.3, and the final assertion follows from 10.2.1. □

Rings are said to be *derived Morita equivalent* if their (bounded) derived categories are equivalent as triangulated categories. Thus, the next theorem shows that R and S are derived Morita equivalent if there exists an invertible complex for (R, S^{o}).

10.2.16 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and U an invertible complex for (R, S^{o}). There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\mathcal{D}(S) \underset{\mathsf{RHom}_R(U,-)}{\overset{U\otimes^{\mathsf{L}}_S -}{\rightleftarrows}} \mathcal{D}(R)\,.$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}_{\sqsupset}(S) \rightleftarrows \mathcal{D}_{\sqsupset}(R)\,,\quad \mathcal{D}_{\sqsubset}(S) \rightleftarrows \mathcal{D}_{\sqsubset}(R)\,,\quad \textit{and}\quad \mathcal{D}_{\square}(S) \rightleftarrows \mathcal{D}_{\square}(R)\,,$$

and further to

$$\mathcal{P}(S) \rightleftarrows \mathcal{P}(R)\,,\quad \mathcal{I}(S) \rightleftarrows \mathcal{I}(R)\,,\quad \textit{and}\quad \mathcal{F}(S) \rightleftarrows \mathcal{F}(R)\,.$$

PROOF. The displayed functors are adjoints by 10.2.14 and they are $\Bbbk$-linear and triangulated by 7.3.6 and 7.4.5. By assumption, the homothety morphism $\chi^{U}_{S^{\mathrm{o}}R}$ is an isomorphism in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$, and it follows from 8.4.10/8.4.13(a) that the evaluation morphism θ^{UUN} is an isomorphism in $\mathcal{D}(S)$ for every S-complex N. Thus, the commutative diagram (10.2.14.2) shows that the unit of the adjunction α_U is an isomorphism. Similarly, it follows from 8.4.23/8.4.25(a) and the commutative diagram (10.2.14.3) that the counit β_U is an isomorphism in $\mathcal{D}(R)$.

By A.26(c) the functor $\mathsf{RHom}_R(U,-)$ is bounded, and since $\mathrm{fd}_S\, U$ is finite, see 8.3.6, the functor $U \otimes^{\mathsf{L}}_S -$ is bounded by A.27(c). The restrictions to equivalences $\mathcal{D}_{\sqsupset}(S) \rightleftarrows \mathcal{D}_{\sqsupset}(R)$, $\mathcal{D}_{\sqsubset}(S) \rightleftarrows \mathcal{D}_{\sqsubset}(R)$, and $\mathcal{D}_{\square}(S) \rightleftarrows \mathcal{D}_{\square}(R)$ now follow from A.25. The final equivalences follow in view of 8.3.19 and 10.2.15 from 8.3.15(d,a,c). □

REMARK. An invertible complex for (R, S^{o}) is a special instance of a tilting complex in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and by a theorem of Rickard [212], R and S are derived Morita equivalent if and only if there exists a tilting complex in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$.

The equivalence in 10.2.16 specializes per E 10.2.7 to an equivalence $\mathcal{D}(S\text{–}T^{\mathrm{o}}) \rightleftarrows \mathcal{D}(R\text{–}T^{\mathrm{o}})^{\mathrm{op}}$.

For Noetherian rings the equivalence in 10.2.16 restricts further to the subcategories of complexes with degreewise finitely generated homology; see 10.2.20.

INVERSE COMPLEX

The terminology introduced next is justified by the subsequent results.

10.2.17 Definition. Assume that R and S are projective as $\Bbbk$-modules and Noetherian. For an invertible complex U for (R, S^{o}), the complex $\mathsf{RHom}_R(U,R) \simeq \mathsf{RHom}_{S^{\mathrm{o}}}(U,S)$ in $\mathcal{D}(S\text{–}R^{\mathrm{o}})$, see 10.2.15, is called the *inverse* of U and denoted U^*.

10.2.18 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and U an invertible complex for (R, S^{o}). There are isomorphisms,*

$$U \otimes^{\mathsf{L}}_S U^* \simeq R \qquad \text{and} \qquad U^* \otimes^{\mathsf{L}}_R U \simeq S\,,$$

in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ and $\mathcal{D}(S\text{–}S^{\mathrm{o}})$. Moreover, if $V \in \mathcal{D}(S\text{–}R^{\mathrm{o}})$ satisfies $U \otimes^{\mathsf{L}}_S V \simeq R$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ or $V \otimes^{\mathsf{L}}_R U \simeq S$ in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$, then V is isomorphic to U^*.

Proof. From 10.2.16 one gets an isomorphism $\boldsymbol{\beta}^R_U \colon U \otimes^{\mathsf{L}}_S U^* \to R$ in $\mathcal{D}(R)$ and by 10.2.14 it is an isomorphism in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$. For a complex V with $U \otimes^{\mathsf{L}}_S V \simeq R$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$ the left-hand isomorphism in the next display is $\boldsymbol{\alpha}^V_U$,

$$V \simeq \mathsf{RHom}_R(U, U \otimes^{\mathsf{L}}_S V) \simeq \mathsf{RHom}_R(U, R) \simeq U^*\,.$$

Now apply 10.2.11 to finish the proof. $\square$

10.2.19 Theorem. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. If U is an invertible complex for (R, S^{o}), then the inverse complex U^* is an invertible complex for (S, R^{o}).*

Proof. By 10.2.15 the complex U^* belongs to $\mathcal{D}_{\square}(S\text{–}R^{\mathrm{o}})$ and it has finite projective dimension over S and over R^{o}. Further, it has degreewise finitely generated homology over either ring by 7.6.17. It remains to see that the homothety morphisms are isomorphisms. There is a commutative diagram in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$,

$$\begin{tikzcd} S \arrow[r, "\chi^U", "\simeq"'] \arrow[d, "\chi^{U^*}"'] & \mathsf{RHom}_R(U,U) \arrow[d, "\mathsf{RHom}(U,\delta^U_R)", "\simeq"'] \\ \mathsf{RHom}_{R^{\mathrm{o}}}(U^*,U^*) \arrow[r, "\zeta^{U^*RU}", "\simeq"'] & \mathsf{RHom}_R(U, \mathsf{RHom}_{R^{\mathrm{o}}}(U^*,R))\,, \end{tikzcd}$$

where the isomorphisms come from biduality 8.4.4/10.2.1, swap 7.5.28, and the assumptions on U. It shows that $\boldsymbol{\chi}^{U^*}_{SR^{\mathrm{o}}}$ is an isomorphism in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$; a similar diagram with the rings interchanged shows that $\boldsymbol{\chi}^{U^*}_{R^{\mathrm{o}}S}$ is an isomorphism in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$. $\square$

Remark. For a Noetherian ring R, projective as a $\Bbbk$-module, the invertible complexes for R form a group; see E 10.2.4. It is known as the derived Picard group, see e.g. Rouquier and Zimmerman [221], and it acts on the set of dualizing complexes for R; see 10.3.17.

For Noetherian rings, 10.2.16 restricts to an equivalence of the subcategories of complexes with degreewise finitely generated homology.

10.2.20 Theorem. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. Let U be an invertible complex for (R, S^{o}). There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\begin{tikzcd} \mathcal{D}^{\mathrm{f}}(S) \arrow[r, shift left, "U\otimes^{\mathsf{L}}_S -"] & \mathcal{D}^{\mathrm{f}}(R)\,. \arrow[l, shift left, "\mathsf{RHom}_R(U,-)"] \end{tikzcd}$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)\,,\quad \mathcal{D}^{\mathrm{f}}_{\sqsubset}(S) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)\,,\quad \textit{and}\quad \mathcal{D}^{\mathrm{f}}_{\square}(S) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\square}(R)\,,$$

and further to

$$\mathcal{P}^{\mathrm{f}}(S) \rightleftarrows \mathcal{P}^{\mathrm{f}}(R) \quad \textit{and} \quad \mathcal{I}^{\mathrm{f}}(S) \rightleftarrows \mathcal{I}^{\mathrm{f}}(R)\,.$$

PROOF. It is sufficient to prove that the equivalence $\mathcal{D}(S) \rightleftarrows \mathcal{D}(R)$ established in 10.2.16 restricts to an equivalence of triangulated subcategories $\mathcal{D}^{\mathrm{f}}(S) \rightleftarrows \mathcal{D}^{\mathrm{f}}(R)$. As U has finite flat dimension, see 8.3.6, the functor $U \otimes^{\mathsf{L}}_S -$ is bounded by A.27(c). By 7.6.14 and A.29(d) it is thus enough to show that $U \otimes^{\mathsf{L}}_S N$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$ for every finitely generated S-module N, and that follows from 7.6.18. One has $\mathsf{RHom}_R(U,-) \simeq U^* \otimes^{\mathsf{L}}_R -$ by 10.2.15, and the complex U^* is invertible for (S, R^{o}) by 10.2.19, so it follows from what has already been proved that the functor $\mathsf{RHom}_R(U,-)$ maps $\mathcal{D}^{\mathrm{f}}(R)$ to $\mathcal{D}^{\mathrm{f}}(S)$. □

EXERCISES

E 10.2.1 Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and U an invertible complex for (R, S^{o}). Show that there exists a bounded complex of R–S^{o}-bimodules that is isomorphic to U in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ and semi-projective over R and over S^{o}. *Hint:* See 10.1.13.

E 10.2.2 Let R be Noetherian. Show that a complex of R–R^{o}-bimodules is invertible for R if and only if it is invertible for R^{o}.

E 10.2.3 Let R, S, and T be Noetherian. Show that if U is an invertible complex for (R, S^{o}) and V an invertible complex for (S, T^{o}), then $U \otimes^{\mathsf{L}}_S V$ is invertible for (R, T^{o}).

E 10.2.4 Assume that R is projective as a $\Bbbk$-module and Noetherian. Show that the invertible complexes for R form a group.

E 10.2.5 Assume that R is projective as a $\Bbbk$-module and Noetherian. Show that if D and D' are dualizing complexes for R, then $\mathsf{RHom}_R(D, D')$ is invertible for R.

E 10.2.6 Assume that R is projective as a $\Bbbk$-module and Noetherian. Show that if D is dualizing complex for R and U is an invertible complex for R, then $U \otimes^{\mathsf{L}}_R D$ is dualizing for R.

E 10.2.7 Assume that R and S are projective as $\Bbbk$-modules and let X be a complex of R–S^{o}-bimodules. Show that there is an adjunction

$$X \otimes^{\mathsf{L}}_S - : \mathcal{D}(S\text{–}T^{\mathrm{o}}) \rightleftarrows \mathcal{D}(R\text{–}T^{\mathrm{o}}) : \mathsf{RHom}_R(X,-)\,.$$

Hint: Zigzag identities.

E 10.2.8 Let R be left Noetherian and S right Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ with $\mathrm{pd}_R\, M$ finite and X a complex in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{H}(X)$ degreewise finitely generated over S^{o}. Show that $\mathsf{RHom}_R(M, X)$ is in $\mathcal{D}^{\mathrm{f}}(S^{\mathrm{o}})$ and, further, belongs to (a) $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}})$ if X is in $\mathcal{D}_{\sqsupset}(R\text{–}S^{\mathrm{o}})$; (b) $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$; (c) $\mathcal{P}^{\mathrm{f}}(S^{\mathrm{o}})$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{pd}_{S^{\mathrm{o}}}\, X$ finite; and (d) $\mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}})$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{id}_{S^{\mathrm{o}}}\, X$ finite.

E 10.2.9 Let R and S be left Noetherian. Let N be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(S)$ with $\mathrm{pd}_S\, N$ finite and X a complex in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{H}(X)$ degreewise finitely generated over R. Show that $X \otimes^{\mathsf{L}}_S N$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$ and, further, that it belongs to (a) $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ if X is in $\mathcal{D}_{\sqsubset}(R\text{–}S^{\mathrm{o}})$; (b) $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$; (c) $\mathcal{P}^{\mathrm{f}}(R)$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{pd}_R\, X$ finite; and (d) $\mathcal{I}^{\mathrm{f}}(R)$ if X is in $\mathcal{D}_{\square}(R\text{–}S^{\mathrm{o}})$ with $\mathrm{id}_R\, X$ finite.

10.3 Foxby–Sharp Equivalence

Synopsis. Auslander Category; Bass Category; Foxby–Sharp Equivalence; projective dimension of flat modules; parametrization of dualizing complexes; invertible complex.

Let D be a dualizing complex for (R, S^{o}). In this section, we study the adjunction,

$$\mathcal{D}(S) \underset{\mathsf{RHom}_R(D,-)}{\overset{D\otimes^{\mathsf{L}}_S-}{\rightleftarrows}} \mathcal{D}(R)\,,$$

from 10.2.14.

10.3.1 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).*

(a) *For N in $\mathcal{D}(S)$ with $\operatorname{fd}_S N$ finite the unit $\boldsymbol{\alpha}^N_D \colon N \to \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_S N)$ is an isomorphism in $\mathcal{D}(S)$.*

(b) *For M in $\mathcal{D}(R)$ with $\operatorname{id}_R M$ finite the counit $\boldsymbol{\beta}^M_D \colon D \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(D, M) \to M$ is an isomorphism in $\mathcal{D}(R)$.*

Proof. Part (a) follows from (10.2.14.2) and tensor evaluation 8.4.13(b) and part (b) from (10.2.14.3) and homomorphism evaluation 8.4.25(b). □

Auslander and Bass Categories

10.3.2 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D and D' be dualizing complexes for (R, S^{o}).*

(a) *For an S-complex N the unit $\boldsymbol{\alpha}^N_D$ is an isomorphism in $\mathcal{D}(S)$ if and only if $\boldsymbol{\alpha}^N_{D'}$ is an isomorphism. Furthermore, $D \otimes^{\mathsf{L}}_S N$ has bounded homology if and only if $D' \otimes^{\mathsf{L}}_S N$ has bounded homology.*

(b) *For an R-complex M the counit $\boldsymbol{\beta}^M_D$ is an isomorphism in $\mathcal{D}(R)$ if and only if $\boldsymbol{\beta}^M_{D'}$ is an isomorphism. Furthermore, $\mathsf{RHom}_R(D, M)$ has bounded homology if and only if $\mathsf{RHom}_R(D', M)$ has bounded homology.*

Proof. (a): By symmetry it suffices to prove the "if" statement. Consider the complex $U = \mathsf{RHom}_{S^{\mathrm{o}}}(D, D')$ in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$, cf. 7.3.12. As D is in $\mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}})$ it follows from 10.1.23 that U belongs to $\mathcal{P}^{\mathrm{f}}(R)$. Commutativity $\boldsymbol{\upsilon}^{UD}$ is by 7.5.14 an isomorphism in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$. By 10.1.3 and 10.2.14 the counit $\boldsymbol{\beta}^{D'}_D \colon D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} U \to D'$ is a morphism in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$, and by 10.3.1 it is an isomorphism; set $\boldsymbol{\beta} = \boldsymbol{\beta}^{D'}_D \boldsymbol{\upsilon}^{UD} \colon U \otimes^{\mathsf{L}}_R D \to D'$. The diagram in $\mathcal{D}(S)$,

$$\begin{tikzcd}
N \arrow[rr, "\alpha_{D'}^{N}"] \arrow[d, "\alpha_D^N"'] & & \mathsf{RHom}_R(D', D' \otimes_S^{\mathsf{L}} N) \arrow[d, "\simeq"', "\mathsf{RHom}(\beta, D' \otimes^{\mathsf{L}} N)"] \\
\mathsf{RHom}_R(D, D \otimes_S^{\mathsf{L}} N) \arrow[d, "\mathsf{RHom}(D, \delta_{D'}^{D} \otimes^{\mathsf{L}} N)"', "\simeq"] & & \mathsf{RHom}_R(U \otimes_R^{\mathsf{L}} D, D' \otimes_S^{\mathsf{L}} N) \arrow[d, "\simeq"', "\rho^{DU(D' \otimes^{\mathsf{L}} N)}"] \\
\mathsf{RHom}_R(D, \mathsf{RHom}_R(U, D') \otimes_S^{\mathsf{L}} N) \arrow[rr, "\mathsf{RHom}(D, \theta^{UD'N})", "\simeq"'] & & \mathsf{RHom}_R(D, \mathsf{RHom}_R(U, D' \otimes_S^{\mathsf{L}} N))\,,
\end{tikzcd}$$

is commutative. The vertical isomorphism on the left is induced by biduality 10.1.19, and tensor evaluation 8.4.10/8.4.13(a) induces the lower horizontal isomorphism. Thus $\alpha_{D'}^{N}$ is an isomorphism if α_D^N is an isomorphism.

As in the digram one has $D \otimes_S^{\mathsf{L}} N \simeq \mathsf{RHom}_R(U, D' \otimes_S^{\mathsf{L}} N)$, so if $D' \otimes_S^{\mathsf{L}} N$ has bounded homology, then it follows from 7.6.7 and 8.1.8 that $D \otimes_S^{\mathsf{L}} N$ has bounded homology.

(b): Consider the complex $U' = \mathsf{RHom}_R(D', D)$ in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$, cf. 7.3.12. As D' is in $\mathcal{I}^{\mathrm{f}}(R)$ it follows from 10.1.23 that U' belongs to $\mathcal{P}^{\mathrm{f}}(S^{\mathrm{o}})$. One establishes a commutative diagram in $\mathcal{D}(R)$, similar to the one above, from the following sequences of isomorphisms induced by biduality 10.1.19, homomorphism evaluation 8.4.23/8.4.25(a), and 10.3.1,

$$\begin{aligned}
D' \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D', M) &\simeq D' \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(U', D), M) \\
&\simeq D' \otimes_S^{\mathsf{L}} (U' \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D, M)) \\
&\simeq D \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D, M)\,.
\end{aligned}$$

The diagram shows that the counit $\beta_{D'}^{M}$ is an isomorphism if β_D^M is an isomorphism. Further, as in the display above one has $\mathsf{RHom}_R(D', M) \simeq U' \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D, M)$, so if $\mathsf{RHom}_R(D, M)$ has bounded homology, then it follows from 8.3.6, 8.3.11, and 7.6.8 that $\mathsf{RHom}_R(D', M)$ has bounded homology. $\square$

The lemma above ensures that the next definitions are unambiguous.

10.3.3 Definition. Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).

The *gross Auslander Category* of S, denoted $\hat{\mathcal{A}}(S)$, is the full subcategory of $\mathcal{D}(S)$ defined by specifying its objects as follows,

$$\hat{\mathcal{A}}(S) = \{N \in \mathcal{D}(S) \mid \text{the unit morphism } \alpha_D^N \text{ is an isomorphism}\}\,.$$

The *Auslander Category* of S is the following full subcategory of $\hat{\mathcal{A}}(S)$,

$$\mathcal{A}(S) = \{N \in \hat{\mathcal{A}}(S) \mid N \text{ and } D \otimes_S^{\mathsf{L}} N \text{ have bounded homology}\}\,;$$

we also consider the full subcategory $\mathcal{A}^{\mathrm{f}}(S) = \mathcal{A}(S) \cap \mathcal{D}^{\mathrm{f}}(S)$.

The *gross Bass Category* of R, denoted $\hat{\mathcal{B}}(R)$, is the full subcategory of $\mathcal{D}(R)$ defined by specifying its objects as follows,

$$\hat{\mathcal{B}}(R) = \{M \in \mathcal{D}(R) \mid \text{the counit morphism } \boldsymbol{\beta}_D^M \text{ is an isomorphism}\}\,.$$

The *Bass Category* of R is the following full subcategory of $\hat{\mathcal{B}}(R)$,

$$\mathcal{B}(R) = \{M \in \hat{\mathcal{B}}(R) \mid M \text{ and } \mathsf{RHom}_R(D, M) \text{ have bounded homology}\}\,;$$

we also considered the full subcategory $\mathcal{B}^{\mathrm{f}}(R) = \mathcal{B}(R) \cap \mathcal{D}^{\mathrm{f}}(R)$.

10.3.4 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).*

(a) *The subcategories categories $\hat{\mathcal{A}}(S) \subseteq \mathcal{D}(S)$ and $\mathcal{A}(S) \subseteq \mathcal{D}_{\square}(S)$ are triangulated. If S is Noetherian, then $\mathcal{A}^{\mathrm{f}}(S)$ is a triangulated subcategory of $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.*

(b) *The subcategories $\hat{\mathcal{B}}(R) \subseteq \mathcal{D}(R)$ and $\mathcal{B}(R) \subseteq \mathcal{D}_{\square}(R)$ and $\mathcal{B}^{\mathrm{f}}(R) \subseteq \mathcal{D}^{\mathrm{f}}_{\square}(R)$ are triangulated.*

Proof. The functors $D \otimes^{\mathsf{L}}_S -$ and $\mathsf{RHom}_R(D, -)$ are triangulated by 7.4.5 and 7.3.6, and $\boldsymbol{\alpha}_D$ and $\boldsymbol{\beta}_D$ are triangulated natural transformations by 10.2.14. It follows from E.19 that $\hat{\mathcal{A}}(S)$ and $\hat{\mathcal{B}}(R)$ are triangulated subcategories of $\mathcal{D}(S)$ and $\mathcal{D}(R)$. Further, as $\mathcal{D}_{\square}(S)$ and $\mathcal{D}_{\square}(R)$ by 7.6.3 are triangulated subcategories, it follows from E.20 that $\mathcal{A}(S)$ and $\mathcal{B}(R)$ are triangulated subcategories. Finally, the statements about $\mathcal{A}^{\mathrm{f}}(S)$ and $\mathcal{B}^{\mathrm{f}}(R)$ follow from 7.6.14. $\square$

The (gross) Auslander Category and (gross) Bass Category are by construction equivalent; a precise statement is made in 10.3.7. In 10.4.4 and 10.4.7 we interprete the complexes in these categories in terms of homological dimensions, for now we record that complexes of finite flat/injective dimension are there.

10.3.5 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).*

(a) *Every S-complex of finite flat dimension belongs to $\hat{\mathcal{A}}(S)$; in particular, $\mathcal{F}(S)$ is a subcategory of $\mathcal{A}(S)$ whence $S \in \mathcal{A}(S)$.*

(b) *Every R-complex of finite injective dimension belongs to $\hat{\mathcal{B}}(R)$; in particular, $\mathcal{I}(R)$ is a subcategory of $\mathcal{B}(R)$ whence $D \in \mathcal{B}(R)$.*

Proof. By 10.3.1 every S-complex of finite flat dimension is in $\hat{\mathcal{A}}(S)$, and every R-complex of finite injective dimension is in $\hat{\mathcal{B}}(R)$. It follows from 8.3.11 and 7.6.8 that $\mathcal{F}(S)$ is contained in $\mathcal{A}(S)$, and $\mathcal{I}(R)$ is by 8.2.8 and 7.6.7 contained in $\mathcal{B}(R)$. $\square$

10.3.6 Example. Recall from 10.1.14 that if R is projective over $\Bbbk$ and Iwanaga–Gorenstein, then R is a dualizing complex for R. It follows from 7.5.4 and 7.5.8 that for such a ring there are equalities,

$$\hat{\mathcal{A}}(R) = \mathcal{D}(R) = \hat{\mathcal{B}}(R) \qquad \text{and} \qquad \mathcal{A}(R) = \mathcal{D}_{\square}(R) = \mathcal{B}(R)\,.$$

In particular, every R-module belongs to the Auslander/Bass Category but need not have finite flat/injective dimension; see 8.2.11 and 8.5.32.

The equalities in the example above characterize Iwanaga–Gorenstein rings: If for a Noetherian ring with a dualizing complex the category $\mathcal{A}(R)$ or $\mathcal{B}(R)$ coincides with $\mathcal{D}_{\square}(R)$, then R is Iwanaga–Gorenstein; see 9.4.6, 9.4.15, 10.4.4, and 10.4.7.

Foxby–Sharp Equivalence

10.3.7 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}). There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\hat{\mathcal{A}}(S) \underset{\mathsf{RHom}_R(D,-)}{\overset{D\otimes^{\mathsf{L}}_S-}{\rightleftarrows}} \hat{\mathcal{B}}(R)\,.$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{A}(S) \rightleftarrows \mathcal{B}(R) \quad \text{and} \quad \mathcal{F}(S) \rightleftarrows \mathcal{I}(R)\,.$$

Proof. As the functor $D \otimes^{\mathsf{L}}_S -$ is left adjoint for $\mathsf{RHom}_R(D,-)$, one has for every S-complex N and every R-complex M the zigzag identities:

$$\boldsymbol{\beta}_D^{D\otimes^{\mathsf{L}}_S N} \circ (D \otimes^{\mathsf{L}}_S \boldsymbol{\alpha}_D^N) = 1^{D\otimes^{\mathsf{L}}_S N}$$

and

$$\mathsf{RHom}_R(D, \boldsymbol{\beta}_D^M) \circ \boldsymbol{\alpha}_D^{\mathsf{RHom}_R(D,M)} = 1^{\mathsf{RHom}_R(D,M)}\,.$$

It is now immediate from the definitions in 10.3.3 that the functors $D \otimes^{\mathsf{L}}_S -$ and $\mathsf{RHom}_R(D,-)$ yield an adjoint equivalence $\hat{\mathcal{A}}(S) \rightleftarrows \hat{\mathcal{B}}(R)$. It is evident that this restricts to an equivalence $\mathcal{A}(S) \rightleftarrows \mathcal{B}(R)$. It follows from 10.3.5 that $\mathcal{F}(S)$ and $\mathcal{I}(R)$ are subcategories of $\mathcal{A}(S)$ and $\mathcal{B}(R)$. That the already established equivalence restricts to an equivalence $\mathcal{F}(S) \rightleftarrows \mathcal{I}(R)$ now follows from 8.4.16(a) and 8.4.27. $\square$

For Noetherian rings, 10.3.7 restricts to an equivalence of the subcategories of complexes with degreewise finitely generated homology.

10.3.8 Theorem. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. Let D be a dualizing complex for (R, S^{o}). There is an adjoint equivalence of $\Bbbk$-linear triangulated categories,*

$$\mathcal{A}^{\mathrm{f}}(S) \underset{\mathsf{RHom}_R(D,-)}{\overset{D\otimes^{\mathsf{L}}_S-}{\rightleftarrows}} \mathcal{B}^{\mathrm{f}}(R)\,,$$

and it restricts to an adjoint equivalence of triangulated subcategories,

$$\mathcal{P}^{\mathrm{f}}(S) \rightleftarrows \mathcal{I}^{\mathrm{f}}(R)\,.$$

Proof. For an S-complex N in $\mathcal{A}^{\mathrm{f}}(S)$ it follows from 7.6.18 that the R-complex $D \otimes^{\mathsf{L}}_S N$ has degreewise finitely generated homology, so it is in $\mathcal{B}^{\mathrm{f}}(R)$ by 10.3.7.

For an R-complex M in $\mathcal{B}^{\mathrm{f}}(R)$ one has

$$\begin{aligned}\mathsf{RHom}_R(D,M) &\simeq \mathsf{RHom}_R(D, \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(M,D),D))\\ &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(M,D), \mathsf{RHom}_R(D,D))\\ &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(M,D), S)\end{aligned}$$

by biduality 10.1.19, swap 7.5.28, and 10.1.12. The complex $\mathsf{RHom}_R(M,D)$ belongs by 10.1.23 to $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$, so $\mathsf{RHom}_{S^{\mathrm{o}}}(\mathsf{RHom}_R(M,D),S)$ has degreewise finitely generated homology over S by 7.6.16. Hence, $\mathsf{RHom}_R(D,M)$ belongs to $\mathcal{A}^{\mathrm{f}}(S)$ by 10.3.7.

By 10.3.7 and 10.1.21 the equivalence restricts to $\mathcal{P}^{\mathrm{f}}(S) \rightleftarrows \mathcal{I}^{\mathrm{f}}(R)$. $\square$

Derived reflexive complexes belong to the Auslander Category.

10.3.9 Theorem. *Assume that R is projective as a $\Bbbk$-module and Noetherian. Let D be a dualizing complex for R. A complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ is derived reflexive if and only if it belongs to the Auslander Category. That is, there is an equality,*

$$\mathcal{A}^{\mathrm{f}}(R) = \mathcal{R}(R)\,.$$

Proof. Let M belong to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. We first argue that bidulaity $\boldsymbol{\delta}^M_R$ is an isomorphism if and only if the unit $\boldsymbol{\alpha}^M_D$ is an isomorphism. This follows from the commutaive diagram below, where M^* denotes the R^{o}-complex $\mathsf{RHom}_R(M,R)$ and homomorphism evaluation $\boldsymbol{\eta}^{DRM}$ is an isomorphism by 8.4.25(b).

$$\begin{tikzcd}
M \arrow[r, "\alpha^M_D"] \arrow[d, "\delta^M_R"'] & \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_R M) \arrow[d, "\simeq"', "\mathsf{RHom}(D,(M\otimes^{\mathsf{L}}\epsilon^D)\upsilon^{DM})"] \\
\mathsf{RHom}_{R^{\mathrm{o}}}(M^*, R) \arrow[d, "\mathsf{RHom}(M^*,\chi^D)"', "\simeq"] & \mathsf{RHom}_R(D, M \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathsf{RHom}_R(R,D)) \arrow[d, "\simeq"', "\mathsf{RHom}(D,\eta^{DRM})"] \\
\mathsf{RHom}_{R^{\mathrm{o}}}(M^*, \mathsf{RHom}_R(D,D)) \arrow[r, "\zeta^{M^*DD}", "\simeq"'] & \mathsf{RHom}_R(D, \mathsf{RHom}_{R^{\mathrm{o}}}(M^*,D))\,.
\end{tikzcd}$$

If M is in the Auslander Category, then the complex $D \otimes^{\mathsf{L}}_R M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ by 10.3.8, so the R^{o}-complex

$$\mathsf{RHom}_R(D \otimes^{\mathsf{L}}_R M, D) \simeq \mathsf{RHom}_R(M, \mathsf{RHom}_R(D,D)) \simeq \mathsf{RHom}_R(M,R)$$

belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R^{\mathrm{o}})$ by 10.1.23, whence M is derived reflexive. Conversely, if M is derived reflexive, then $\mathsf{RHom}_R(M,R)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R^{\mathrm{o}})$ and 10.1.23 yields $\mathsf{RHom}_{R^{\mathrm{o}}}(\mathsf{RHom}_R(M,R),D) \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$. As in the diagram above, homomorphism evaluation 8.4.25(b), commutativity 7.5.10, and the counitor 7.5.8 yield

$$\mathsf{RHom}_{R^{\mathrm{o}}}(\mathsf{RHom}_R(M,R),D) \simeq M \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathsf{RHom}_R(R,D) \simeq D \otimes^{\mathsf{L}}_R M\,,$$

so M belongs to the category $\mathcal{A}^{\mathrm{f}}(R)$. $\square$

The next two results are recorded here for later use. To parse the first result, recall from 10.1.3 that a dualizing complex for (R,S^{o}) is dualizing for (S^{o},R).

10.3.10 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S^{o}).*

(a) *Let $M \in \mathcal{D}(R)$; one has $M \in \mathcal{B}(R)$ if and only if* $\operatorname{Hom}_{\Bbbk}(M,\mathbb{E}) \in \mathcal{A}(R^{\mathrm{o}})$.

(b) *Let $N \in \mathcal{D}(S)$; one has $N \in \mathcal{A}(S)$ if and only if* $\operatorname{Hom}_{\Bbbk}(N,\mathbb{E}) \in \mathcal{B}(S^{\mathrm{o}})$.

Proof. (a): Notice from 2.5.7(b) that the complex $M \in \mathcal{D}(R)$ has bounded homology if and only if the R^{o}-complex $\mathrm{Hom}_{\Bbbk}(M,\mathbb{E})$ has bounded homology. Assuming that this is the case, homomorphism evaluation 8.4.23/8.4.25(b) yields an isomorphism

$$\mathrm{Hom}_{\Bbbk}(\mathsf{RHom}_R(D,M),\mathbb{E}) \simeq D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(M,\mathbb{E})\,,$$

and therefore $\mathrm{H}(\mathsf{RHom}_R(D,M))$ is bounded if and only if $\mathrm{H}(D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(M,\mathbb{E}))$ is bounded. This isomorphism induces the right-hand vertical isomorphism in the following commutative diagram in $\mathcal{D}(R^{\mathrm{o}})$,

$$\begin{tikzcd}
\mathrm{Hom}_{\Bbbk}(M,\mathbb{E}) \arrow[r, "\alpha^{\mathrm{Hom}(M,\mathbb{E})}"] \arrow[d, "\mathrm{Hom}(\beta^{M},\mathbb{E})"'] & \mathsf{RHom}_{S^{\mathrm{o}}}(D, D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(M,\mathbb{E})) \arrow[d, "\simeq"', "\mathsf{RHom}(D,\eta^{\mathbb{E}MD})"] \\
\mathrm{Hom}_{\Bbbk}(D \otimes^{\mathsf{L}}_{S} \mathsf{RHom}_R(D,M),\mathbb{E}) \arrow[r, "\simeq"'] & \mathsf{RHom}_{S^{\mathrm{o}}}(D, \mathrm{Hom}_{\Bbbk}(\mathsf{RHom}_R(D,M),\mathbb{E}))\,;
\end{tikzcd}$$

the lower horizontal isomorphism comes from commutativity 7.5.14 and adjunction 7.5.34. The diagram shows that β^{M}_{D} is an isomorphism if and only if $\alpha^{\mathrm{Hom}(M,\mathbb{E})}_{D}$ is an isomorphism.

(b): Notice from 2.5.7(b) that $\mathrm{H}(N)$ is bounded if and only if $\mathrm{H}(\mathrm{Hom}_{\Bbbk}(N,\mathbb{E}))$ is bounded. Commutativity 7.5.14 and adjunction 7.5.34 yield the isomorphism

$$\mathrm{Hom}_{\Bbbk}(D \otimes^{\mathsf{L}}_{S} N,\mathbb{E}) \simeq \mathsf{RHom}_{S^{\mathrm{o}}}(D,\mathrm{Hom}_{\Bbbk}(N,\mathbb{E}))\,,$$

which shows that $\mathrm{H}(D \otimes^{\mathsf{L}}_{S} N)$ is bounded if and only if $\mathrm{H}(\mathsf{RHom}_{S^{\mathrm{o}}}(D,\mathrm{Hom}_{\Bbbk}(N,\mathbb{E})))$ is bounded. Assuming that this is the case, this isomorphism induces the top horizontal isomorphism in the following commutative diagram in $\mathcal{D}(S^{\mathrm{o}})$,

$$\begin{tikzcd}
D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(D \otimes^{\mathsf{L}}_{S} N,\mathbb{E}) \arrow[r, "\simeq"'] \arrow[d, "\eta^{\mathbb{E}(D\otimes^{\mathsf{L}}N)D}"'] & D \otimes^{\mathsf{L}}_{R^{\mathrm{o}}} \mathsf{RHom}_{S^{\mathrm{o}}}(D,\mathrm{Hom}_{\Bbbk}(N,\mathbb{E})) \arrow[d, "\beta^{\mathrm{Hom}(N,\mathbb{E})}"] \\
\mathrm{Hom}_{\Bbbk}(\mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_{S} N),\mathbb{E}) \arrow[r, "\mathrm{Hom}(\alpha^{N},\mathbb{E})"] & \mathrm{Hom}_{\Bbbk}(N,\mathbb{E})\,;
\end{tikzcd}$$

the left-hand vertical isomorphism is homomorphism evaluation 8.4.23/8.4.25(b). The diagram shows that α^{N}_{D} is an isomorphism if and only if $\beta^{\mathrm{Hom}(N,\mathbb{E})}_{D}$ is an isomorphism. □

The next result is recorded for use in Sect. 10.4.

10.3.11 Proposition. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S^{o}).*

(a) *Let $\{N^u\}_{u\in U}$ be a family of S-modules. There is an isomorphism in $\mathcal{D}(R)$,*

$$D \otimes^{\mathsf{L}}_{S} \prod_{u\in U} N^u \simeq \prod_{u\in U} (D \otimes^{\mathsf{L}}_{S} N^u)\,,$$

and $\prod_{u\in U} N^u$ belongs to $\hat{\mathcal{A}}(S)$ if and only if $N^u \in \hat{\mathcal{A}}(S)$ for all $u \in U$.

(b) *Let $\{M^u\}_{u\in U}$ be a family of R-modules. There is an isomorphism in $\mathcal{D}(S)$,*

$$\mathsf{RHom}_R\Big(D, \coprod_{u\in U} M^u\Big) \simeq \coprod_{u\in U} \mathsf{RHom}_R(D, M^u)\,,$$

and $\coprod_{u\in U} M^u$ belongs to $\hat{\mathcal{B}}(R)$ if and only if $M^u \in \hat{\mathcal{B}}(R)$ for all $u \in U$.

PROOF. Set $P = \mathrm{P}_{R\otimes_{\Bbbk} S^{\mathrm{o}}}(D)$ and let $\pi\colon L \to D$ be a semi-free resolution over S^{o} with L bounded below and degreewise finitely generated; see 5.1.12. By 7.3.17 there is a homotopy equivalence $\vartheta\colon P \to L$ of S^{o}-complexes.

(a): There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd} P \otimes_S \prod_{u\in U} N^u \arrow[r, "\varphi"] \arrow[d, "\vartheta\otimes\prod_{u\in U} N^u"', "\cong"] & \prod_{u\in U} (P\otimes_S N^u) \arrow[d, "\cong"', "\prod_{u\in U}(\vartheta\otimes N^u)"] \\ L \otimes_S \prod_{u\in U} N^u \arrow[r, "\varphi'"] & \prod_{u\in U} (L\otimes_S N^u)\,, \end{tikzcd}$$

where the vertical morphisms are homotopy equivalences by 4.3.20 and 4.3.6. The horizontal morphisms are the maps from 3.1.30. As L is degreewise finitely generated, φ' is an isomorphism by 3.1.30 applied degreewise. Thus φ is a homotopy equivalence, in particular a quasi-isomorphism, of $\Bbbk$-complexes; as it is R-linear, it induces the asserted isomorphism in $\mathcal{D}(R)$.

Set $N = \prod_{u\in U} N^u$ and consider the commutative diagram in $\mathcal{C}(S)$,

$$\begin{tikzcd} N \arrow[r, "\alpha_P^N"] \arrow[d, "\prod_{u\in U}\alpha_P^{N^u}"'] & \mathrm{Hom}_R(P, P\otimes_S N) \arrow[d, "\simeq"', "\mathrm{Hom}(P,\varphi)"] \\ \prod_{u\in U} \mathrm{Hom}_R(P, P\otimes_S N^u) \arrow[r, "\cong"'] & \mathrm{Hom}_R\big(P, \prod_{u\in U}(P\otimes_S N^u)\big)\,, \end{tikzcd}$$

where the lower horizontal isomorphism comes from 3.1.24. Now 10.2.14 shows that $\boldsymbol{\alpha}_D^N$ is an isomorphism in $\mathcal{D}(S)$ if and only if $\boldsymbol{\alpha}_D^{N^u}$ is an isomorphism for every $u \in U$.

(b): There is a commutative diagram in $\mathcal{C}(\Bbbk)$,

$$\begin{tikzcd} \coprod_{u\in U} \mathrm{Hom}_R(P, M^u) \arrow[r, "\psi"] \arrow[d, "\coprod_{u\in U}\mathrm{Hom}(\vartheta,M^u)"', "\cong"] & \mathrm{Hom}_R\big(P, \coprod_{u\in U} M^u\big) \arrow[d, "\cong"', "\mathrm{Hom}(\vartheta,\coprod_{u\in U} M^u)"] \\ \coprod_{u\in U} \mathrm{Hom}_R(L, M^u) \arrow[r, "\psi'"] & \mathrm{Hom}_R\big(L, \coprod_{u\in U} M^u\big)\,, \end{tikzcd}$$

where the vertical morphisms are homotopy equivalences by 4.3.19 and 4.3.6. The horizontal morphisms are the maps from 3.1.33. As L is degreewise finitely generated, ψ' is an isomorphism by 3.1.33 applied degreewise. Thus ψ is a homotopy equivalence, in particular a quasi-isomorphism, of $\Bbbk$-complexes; as it is S-linear it induces the asserted isomorphism in $\mathcal{D}(S)$.

By 3.1.27 one has $\operatorname{Hom}_{\Bbbk}(\coprod_{u\in U} M^u, \mathbb{E}) \cong \prod_{u\in U} \operatorname{Hom}_{\Bbbk}(M^u, \mathbb{E})$. As D is a dualizing complex for (S^{o}, R), the final assertion follows from part (a) and 10.3.10. □

Projective Dimension of Flat Modules

The utility and power of dualizing complexes is illustrated by the next results. The theorem compares to Jensen's theorem 8.5.18, and the corollary 10.3.13 provides the same conclusion as Corollary 8.5.20, but under a different assumption.

10.3.12 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, D a dualizing complex for (R, S^{o}), and N an S-complex. If N has finite flat dimension, then one has*

$$\operatorname{pd}_S N \leqslant \max\{\operatorname{id}_R D + \sup(D \otimes^{\mathsf{L}}_S N)\,,\ \sup N\}\ ;$$

in particular, N has finite projective dimension.

Proof. The quantity $n = \max\{\operatorname{id}_R D + \sup(D \otimes^{\mathsf{L}}_S N), \sup N\}$ is finite by 8.3.11. Let $P \xrightarrow{\simeq} N$ be a semi-projective resolution over S. It suffices by 8.1.8 to show that $\operatorname{Ext}^1_S(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P))$ is zero. The S-module $C = \mathrm{C}_{n+1}(P)$ has finite flat dimension; indeed, the sequence $0 \to \mathrm{C}_u(P) \to P_{u-1} \to \cdots \to P_{n+2} \to C \to 0$ is exact for every $u > n+2$, and by 8.3.11 the module $\mathrm{C}_u(P)$ is flat for $u \gg 0$. By 8.1.6, 10.3.5, and adjunction 7.5.34 there are isomorphisms of $\Bbbk$-modules,

$$\begin{aligned}
\operatorname{Ext}^1_S(\mathrm{C}_n(P), C) &\cong \mathrm{H}_{-(n+1)}(\mathsf{R}\mathrm{Hom}_S(N, C))\\
&\cong \mathrm{H}_{-(n+1)}(\mathsf{R}\mathrm{Hom}_S(N, \mathsf{R}\mathrm{Hom}_R(D, D \otimes^{\mathsf{L}}_S C)))\\
&\cong \mathrm{H}_{-(n+1)}(\mathsf{R}\mathrm{Hom}_R(D \otimes^{\mathsf{L}}_S N, D \otimes^{\mathsf{L}}_S C))\ .
\end{aligned}$$

It is now sufficient to show that $-\inf \mathsf{R}\mathrm{Hom}_R(D \otimes^{\mathsf{L}}_S N, D \otimes^{\mathsf{L}}_S C)$ is at most n, and that follows as 8.2.8 and 8.4.16(a) yield,

$$\begin{aligned}
-\inf \mathsf{R}\mathrm{Hom}_R(D \otimes^{\mathsf{L}}_S N, D \otimes^{\mathsf{L}}_S C) &\leqslant \operatorname{id}_R(D \otimes^{\mathsf{L}}_S C) + \sup(D \otimes^{\mathsf{L}}_S N)\\
&\leqslant \operatorname{id}_R D - \inf C + \sup(D \otimes^{\mathsf{L}}_S N)\\
&\leqslant n\ .
\end{aligned}$$

□

10.3.13 Corollary. *Assume that R is projective as a $\Bbbk$-module and Noetherian with a dualizing complex. An R-complex has finite flat dimension if and only if it has finite projective dimension.*

Proof. The assertion is immediate from 8.3.6 and 10.3.12. □

The next lemma is not needed before Sect. 10.4, but it is natural to record it here.

10.3.14 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}). Further, let M be an R-complex and N an S-complex.*

(a) *For every R-complex I of finite injective dimension and with* $\mathrm{H}(I)$ *non-zero and bounded, one has*

$$-\inf \mathsf{RHom}_R(I, M) \leqslant \mathrm{id}_R D + \sup I - \inf \mathsf{RHom}_R(D, M) .$$

(b) *For every S-complex F of finite flat dimension and with* $\mathrm{H}(F)$ *non-zero and bounded, one has*

$$-\inf \mathsf{RHom}_S(N, F) \leqslant \mathrm{id}_R D - \inf F + \sup (D \otimes^{\mathsf{L}}_S N) .$$

(c) *For every S^{o}-complex E of finite injective dimension and with* $\mathrm{H}(E)$ *non-zero and bounded, one has*

$$\sup (E \otimes^{\mathsf{L}}_S N) \leqslant \mathrm{id}_R D + \sup E + \sup (D \otimes^{\mathsf{L}}_S N) .$$

Proof. (a): By 10.3.5 the R-complex I belongs to the Bass Category, so one has

$$(\dagger) \qquad \begin{aligned} \mathsf{RHom}_R(I, M) &\simeq \mathsf{RHom}_R(D \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(D, I), M) \\ &\simeq \mathsf{RHom}_S(\mathsf{RHom}_R(D, I), \mathsf{RHom}_R(D, M)) \end{aligned}$$

where the second isomorphism is adjunction 7.5.34. The S-complex $\mathsf{RHom}_R(D, I)$ has finite flat dimension by 10.3.7, so 10.3.12 yields the first inequality in the chain

$$\begin{aligned} \mathrm{pd}_S \mathsf{RHom}_R(D, I) &\leqslant \max\{\mathrm{id}_R D + \sup (D \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(D, I)), \sup \mathsf{RHom}_R(D, I)\} \\ &\leqslant \max\{\mathrm{id}_R D + \sup I, \sup I - \inf D\} \\ &= \mathrm{id}_R D + \sup I . \end{aligned}$$

The second inequality follows from 10.3.7 and 7.6.7, and finally the equality follows in view of 8.2.3. The desired inequality now follows from (†) and 8.1.8.

(b): By 10.3.5 the complex F belongs to the Auslander Category, so one has

$$\mathsf{RHom}_S(N, F) \simeq \mathsf{RHom}_S(N, \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_S F)) \simeq \mathsf{RHom}_R(D \otimes^{\mathsf{L}}_S N, D \otimes^{\mathsf{L}}_S F)$$

where the second isomorphism is adjunction 7.5.34. Now 8.2.8 and 8.4.16(a) yield

$$\begin{aligned} -\inf \mathsf{RHom}_S(N, F) &\leqslant \mathrm{id}_R(D \otimes^{\mathsf{L}}_S F) + \sup (D \otimes^{\mathsf{L}}_S N) \\ &\leqslant \mathrm{id}_R D - \inf F + \sup (D \otimes^{\mathsf{L}}_S N) . \end{aligned}$$

(c): The S-complex $\mathrm{Hom}_{\Bbbk}(E, \mathbb{E})$ has finite flat dimension by 8.3.18, so by 2.5.7(b), adjunction 7.5.34, and part (b) one has

$$\begin{aligned} \sup (E \otimes^{\mathsf{L}}_S N) &= -\inf \mathsf{RHom}_{\Bbbk}(E \otimes^{\mathsf{L}}_S N, \mathbb{E}) \\ &= -\inf \mathsf{RHom}_S(N, \mathsf{RHom}_{\Bbbk}(E, \mathbb{E})) \\ &\leqslant \mathrm{id}_R D - \inf \mathsf{RHom}_{\Bbbk}(E, \mathbb{E}) + \sup (D \otimes^{\mathsf{L}}_S N) \\ &= \mathrm{id}_R D + \sup E + \sup (D \otimes^{\mathsf{L}}_S N) . \end{aligned}$$ □

Parametrization of Dualizing Complexes

We close this chapter with a theorem which we know from Yekutieli [257]; it says that dualizing complexes are parametrized by invertible complexes.

10.3.15 Lemma. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. If D is a dualizing complex for (R, S^{o}) and U is an invertible complex for S, then the complex $D \otimes_S^{\mathsf{L}} U$ is dualizing for (R, S^{o}).*

Proof. By 10.2.20/10.2.11 and 10.3.8 the complex $D \otimes_S^{\mathsf{L}} U$ in $\mathcal{D}(R\text{–}S^{\mathrm{o}})$ belongs to $\mathcal{I}^{\mathrm{f}}(S^{\mathrm{o}})$ and to $\mathcal{I}^{\mathrm{f}}(R)$. Furthermore, there is a commutative diagram in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$,

$$\begin{tikzcd} S \arrow[r, "\chi^{D\otimes^{\mathsf{L}} U}"] \arrow[d, "\chi^U"', "\simeq"] & \mathsf{RHom}_R(D \otimes_S^{\mathsf{L}} U, D \otimes_S^{\mathsf{L}} U) \arrow[d, "\rho^{(D\otimes^{\mathsf{L}} U)DU}", "\simeq"'] \\ \mathsf{RHom}_S(U, U) \arrow[r, "\mathsf{RHom}(U,\alpha^U)", "\simeq"'] & \mathsf{RHom}_S(U, \mathsf{RHom}_R(D, D \otimes_S^{\mathsf{L}} U)) \,, \end{tikzcd}$$

where the lower horizontal isomorphism comes from 10.3.5 and 10.2.14, while the right-hand vertical isomorphism is adjunction 7.5.34. The diagram shows that $\chi_{S^{\mathrm{o}}R}^{D\otimes^{\mathsf{L}} U}$ is an isomorphism. To see that $\chi_{RS^{\mathrm{o}}}^{D\otimes^{\mathsf{L}} U}$ is an isomorphism in $\mathcal{D}(R\text{–}R^{\mathrm{o}})$, expand the chain of isomorphisms,

$$\begin{aligned} \mathsf{RHom}_{S^{\mathrm{o}}}(D \otimes_S^{\mathsf{L}} U, D \otimes_S^{\mathsf{L}} U) &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(D, \mathsf{RHom}_{S^{\mathrm{o}}}(U, U \otimes_{S^{\mathrm{o}}}^{\mathsf{L}} D)) \\ &\simeq \mathsf{RHom}_{S^{\mathrm{o}}}(D, D) \,, \end{aligned}$$

into a commutative diagram. The first isomorphism is adjunction 7.5.34, and the second is induced by the unit α_U^D, which is an isomorphism by 10.2.16. It now follows from 10.1.12 that $D \otimes_S^{\mathsf{L}} U$ is a dualizing complex for (R, S^{o}). □

10.3.16 Lemma. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. If D and D' are dualizing complexes for (R, S^{o}), then the complex $\mathsf{RHom}_R(D, D')$ is invertible for S with inverse $\mathsf{RHom}_R(D', D)$.*

Proof. It follows from 10.3.8 and from 10.1.23/10.1.3 that the S–S^{o}-bicomplex $U = \mathsf{RHom}_R(D, D')$ belongs to $\mathcal{P}^{\mathrm{f}}(S)$ and $\mathcal{P}^{\mathrm{f}}(S^{\mathrm{o}})$. To prove that it is invertible, it remains to see that the homothety morphisms are isomorphisms; cf. 10.2.10. It is straightforward to verify that there is a commutative diagram in $\mathcal{D}(S\text{–}S^{\mathrm{o}})$,

$$\begin{tikzcd} S \arrow[r, "\chi^U"] \arrow[d, "\chi^{D'}"', "\simeq"] & \mathsf{RHom}_S(\mathsf{RHom}_R(D, D'), \mathsf{RHom}_R(D, D')) \\ \mathsf{RHom}_R(D', D') \arrow[r, "\mathsf{RHom}(\beta^{D'}, D')", "\simeq"'] & \mathsf{RHom}_R(D \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D, D'), D') \,, \arrow[u, "\rho^{D'D\,\mathsf{RHom}(D,D')}"', "\simeq"] \end{tikzcd}$$

where the lower horizontal isomorphism comes from 10.3.5 and 10.2.14, while the right-hand vertical isomorphism is adjunction 7.5.34. It follows that $\chi_{S^{\mathrm{o}}S}^U$ is an isomorphism. In the commutative diagram,

$$\begin{tikzcd}
S \arrow[r, "\chi^U"] \arrow[d, "\chi^D"', "\simeq"] & \mathsf{RHom}_{S^o}(\mathsf{RHom}_R(D,D'),\mathsf{RHom}_R(D,D')) \\
\mathsf{RHom}_R(D,D) \arrow[r, "\mathsf{RHom}(D,\delta^D_{D'})", "\simeq"'] & \mathsf{RHom}_R(D,\mathsf{RHom}_{S^o}(\mathsf{RHom}_R(D,D'),D')) \, , \arrow[u, "\simeq", "\zeta^{DD'\,\mathsf{RHom}(D,D')}"']
\end{tikzcd}$$

the lower horizontal isomorphism is induced by biduality 10.1.19 and the right-hand vertical isomorphism is swap 7.5.28. The diagram shows that also $\chi^U_{SS^o}$ is an isomorphism. Thus U is invertible. Finally, biduality 10.1.19 and swap 7.5.28 yield,

$$\begin{aligned}
\mathsf{RHom}_R(D',D) &\simeq \mathsf{RHom}_R(D',\mathsf{RHom}_{S^o}(\mathsf{RHom}_R(D,D'),D')) \\
&\simeq \mathsf{RHom}_{S^o}(\mathsf{RHom}_R(D,D'),\mathsf{RHom}_R(D',D')) \\
&\simeq \mathsf{RHom}_{S^o}(U,S) \, ,
\end{aligned}$$

whence the inverse of U is $\mathsf{RHom}_R(D',D)$; see 10.2.17. □

10.3.17 Theorem. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. If D is a dualizing complex for (R,S^o), then there is a one-to-one correspondence of isomorphism classes in derived categories,*

$$\{\text{invertible complexes for } S\} \longleftrightarrow \{\text{dualizing complexes for } (R,S^o)\}$$

given by

$$U \longmapsto D \otimes^{\mathsf{L}}_S U \quad \textit{and} \quad \mathsf{RHom}_R(D,D') \longleftarrow\!\shortmid D' \, .$$

Proof. For every invertible complex U for S the complex $D \otimes^{\mathsf{L}}_S U$ is dualizing for (R,S^o) by 10.3.15. By 10.3.5 the complex U belongs to $\mathcal{A}(S)$, and hence there is an isomorphism $U \simeq \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_S U)$ in $\mathcal{D}(S\text{–}S^o)$, see 10.2.14.

For every dualizing complex D' for (R,S^o) the complex $\mathsf{RHom}_R(D,D')$ is invertible for S by 10.3.16. By 10.3.5 the complex D' belongs to $\mathcal{B}(R)$, so there is an isomorphism $D' \simeq D \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(D,D')$ in $\mathcal{D}(R\text{–}S^o)$, see 10.2.14. □

In the case of a commutative Noetherian local ring, a dualizing complex is unique in the derived category, up to a shift; see 18.2.27. It follows that R in this case is the only invertible complex for R.

10.3.18 Corollary. *Assume that R and S are projective as $\Bbbk$-modules and Noetherian. If D is a dualizing complex for (R,S^o), then there is a one-to-one correspondence of isomorphism classes*

$$\{\text{invertible complexes for } S\} \longleftrightarrow \{\text{invertible complexes for } R\}$$

given by

$$U \longmapsto \mathsf{RHom}_{S^o}(D, D \otimes^{\mathsf{L}}_S U) \quad \textit{and} \quad \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_{R^o} V) \longleftarrow\!\shortmid V \, .$$

Proof. Recall from 10.1.3 that a dualizing complex for (R,S^o) is also dualizing for (S^o,R). The assertions are now immediate from 10.3.17. □

Exercises

In exercises E 10.3.1–10.3.6 assume that R and S are projective as $\Bbbk$-modules, let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).

E 10.3.1 Let M and N be S-complexes. Show that if N is in $\hat{\mathcal{A}}(S)$, then there is an isomorphism $\mathsf{RHom}_S(M, N) \simeq \mathsf{RHom}_R(D \otimes^{\mathsf{L}}_S M, D \otimes^{\mathsf{L}}_S N)$ in $\mathcal{D}(\Bbbk)$.

E 10.3.2 Let M and N be R-complexes. Show that if M is in $\hat{\mathcal{B}}(R)$, then there is an isomorphism $\mathsf{RHom}_R(M, N) \simeq \mathsf{RHom}_S(\mathsf{RHom}_R(D, M), \mathsf{RHom}_R(D, N))$ in $\mathcal{D}(\Bbbk)$.

E 10.3.3 Show that the assertions in 10.3.11 hold for families of uniformly bounded complexes. *Hint:* E 3.1.17 and E 3.1.18.

E 10.3.4 Let F be a flat S-module. Show that $\mathrm{pd}_S F \leqslant \mathrm{id}_R D + \sup D$ holds.

E 10.3.5 Show that splf S and splf R^{o} are finite.

E 10.3.6 (a) Let M be a complex of S–R^{o}-bimodules and E an S-complex of finite injective dimension. Show that $M \in \mathcal{A}(R^{\mathrm{o}})$ implies $\mathsf{RHom}_S(M, E) \in \mathcal{B}(R)$, and show that the converse holds if E is a faithfully injective S-module. (b) Let N be a complex of R–S^{o}-bimodules and E an R-complex of finite injective dimension. Show that $N \in \mathcal{B}(S^{\mathrm{o}})$ implies $\mathsf{RHom}_R(N, E) \in \mathcal{A}(S)$, and show that the converse holds if E is a faithfully injective R-module.

E 10.3.7 Assume that R and S are projective as $\Bbbk$-modules and Noetherian. Let D be a dualizing complex for (R, S^{o}) and U an invertible complex for S. Show directly, i.e. without using 10.3.18, that the complex $\mathsf{RHom}_{S^{\mathrm{o}}}(D, D \otimes^{\mathsf{L}}_S U)$ is invertible for R.

10.4 Gorenstein Dimensions vs. Auslander and Bass Categories

Synopsis. Gorenstein projective/flat dimension and the Auslander Category; Gorenstein injective dimension and the Bass Category; derived reflexive complex.

The overarching theme of Chap. 9 is that Gorenstein homological dimensions behave like the absolute homological dimensions treated in Chap. 8, though the parallel is imperfect in the sense that assumptions are needed to get, say, the Gorenstein version of flat–injective duality. Also statements like 8.2.20—a filerted colimit of injective modules is injective over a left Noetherian ring—have their counterparts in the theory of Gorenstein homological dimensions. We arrive at these statements through a categorical characterization of complexes of finite Gorenstein homological dimensions; one that assumes the existence of a dualizing complex.

The Auslander Category

Per 10.3.7 the Auslander Category $\mathcal{A}(R)$ contains the category $\mathcal{F}(R)$ of complexes of finite flat dimension. The first theorem of this section, 10.4.4, shows that all complexes in $\mathcal{A}(R)$ have have finite Gorenstein flat dimension, cf. 9.3.18. We build up to the proof with a result of indepenent interest.

10.4.1 Proposition. *Assume that R is flat as a $\Bbbk$-module. Let R be left Noetherian and S right Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, X a complex in $\mathcal{D}_{\square}(R \otimes_{\Bbbk} S^{\mathrm{o}})$, and N an S-complex. The tensor evaluation morphism* 8.4.6,

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N)\,,$$

is an isomorphism if the invariants $\mathrm{id}_{S^{\mathrm{o}}} X$*,* $\mathrm{fd}_{S^{\mathrm{o}}} \mathsf{RHom}_R(M,X)$*, and* $\mathrm{Gfd}_S N$ *are finite.*

Proof. Choose by 5.3.26 a semi-injective resolution $X \xrightarrow{\simeq} E$ over $R \otimes_{\Bbbk} S^{\mathrm{o}}$ with E bounded above. As R is flat as a $\Bbbk$-module, E is also semi-injective over S^{o}; see 7.3.11(c). Let $P \xrightarrow{\simeq} M$ be a semi-projective resolution with P bounded below and degreewise finitely generated; see 5.2.16. Let $F \simeq N$ be a semi-flat replacement over S and set $n = \mathrm{Gfd}_S N$. By 8.4.11 it suffices to prove that the tensor evaluation morphism θ^{PEF} is a quasi-isomorphism in $\mathcal{D}(\Bbbk)$. There is a commutative diagram,

$$\begin{tikzcd}
\mathrm{Hom}_R(P,E) \otimes_S F \arrow[r, "\theta^{PEF}"] \arrow[d, "\mathrm{Hom}(P,E)\otimes\tau^F_{\subseteq n}"', "\simeq"] & \mathrm{Hom}_R(P, E \otimes_S F) \arrow[d, "\simeq"', "\mathrm{Hom}(P,E\otimes\tau^F_{\subseteq n})"] \\
\mathrm{Hom}_R(P,E) \otimes_S F_{\subseteq n} \arrow[r, "\theta^{PEF_{\subseteq n}}", "\cong"'] & \mathrm{Hom}_R(P, E \otimes_S F_{\subseteq n})\,.
\end{tikzcd}$$

The left-hand vertical morphism is a quasi-isomorphism by 9.3.24. Indeed, it follows from 5.3.25 that $\mathrm{Hom}_R(P,E)$ is a semi-injective S^{o}-complex, and $\mathrm{fd}_{S^{\mathrm{o}}} \mathrm{Hom}_R(P,E)$ is finite by assumption as $\mathrm{Hom}_R(P,E)$ is $\mathsf{RHom}_R(M,X)$. The morphism $E \otimes_S \tau^F_{\subseteq n}$ is a quasi-isomorphism, also by 9.3.24, as $\mathrm{id}_{S^{\mathrm{o}}} E = \mathrm{id}_{S^{\mathrm{o}}} X$ is finite by assumption, and so the right-hand vertical morphism is a quasi-isomorphism by semi-projectivity of P. Finally, $\theta^{PEF_{\subseteq n}}$ is an isomorphism by 4.5.10(1,a). $\square$

10.4.2 Proposition. *Assume that S is projective as a* $\Bbbk$*-module and left Noetherian. Let M be an R-complex, X a complex in* $\mathcal{D}_{\sqsubset}(R \otimes_{\Bbbk} S^{\mathrm{o}})$*, and N a complex in* $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$*. The tensor evaluation morphism* 8.4.6*,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N)\,,$$

is an isomorphism if the invariants $\mathrm{Gpd}_R M$*,* $\mathrm{pd}_R X$*, and* $\mathrm{id}_R(X \otimes^{\mathsf{L}}_S N)$ *are finite.*

Proof. By 5.2.15 and 5.2.16 there exist semi-projective resolutions $P \xrightarrow{\simeq} M$ and $L \xrightarrow{\simeq} N$ with L bounded below and degreewise finitely generated. Further, per 5.2.15 let $F \xrightarrow{\simeq} X$ be a semi-projective resolution over $R \otimes_{\Bbbk} S^{\mathrm{o}}$ with F bounded below; note that since S is projective as a $\Bbbk$-module, F is semi-projective as an R-complex by 7.3.11(a). Now it suffices to show that θ^{PFL} is a quasi-isomorphism. To that end, set $n = \mathrm{Gpd}_R M$ and consider the commutative diagram,

$$\begin{tikzcd}
\mathrm{Hom}_R(P_{\subseteq n},F) \otimes_S L \arrow[r, "\theta^{P_{\subseteq n}FL}", "\cong"'] \arrow[d, "\mathrm{Hom}(\tau^P_{\subseteq n},F)\otimes L"', "\simeq"] & \mathrm{Hom}_R(P_{\subseteq n}, F \otimes_S L) \arrow[d, "\simeq"', "\mathrm{Hom}(\tau^P_{\subseteq n},F\otimes L)"] \\
\mathrm{Hom}_R(P,F) \otimes_S L \arrow[r, "\theta^{PFL}"] & \mathrm{Hom}_R(P, F \otimes_S L)\,.
\end{tikzcd}$$

The map $\mathrm{Hom}_R(\tau^P_{\subseteq n}, F)$ is a quasi-isomorphism by 9.1.17 as F is semi-projective as an R-complex and $\mathrm{pd}_R F = \mathrm{pd}_R X$ is finite. As the S-complex L is semi-flat, see 5.4.10, it follows that the left-hand vertical morphism is a quasi-isomorphism. The R-complex $F \otimes_S L$ is semi-projective by 5.2.22 and $\mathrm{id}_R(F \otimes_S L) = \mathrm{id}_R(X \otimes^{\mathsf{L}}_S N)$ is finite by assumption. Thus, another application of 9.1.17 shows that the right-hand vertical morphism is a quasi-isomorphism. Finally, $\theta^{P \subseteq_n FL}$ is an isomorphism by tensor evaluation 4.5.10(2,a). $\square$

10.4.3 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}). A module G in $\mathcal{A}(S)$ is Gorenstein projective if $\operatorname{Ext}_S^m(G, L) = 0$ holds for all projective S-modules L and all $m > 0$.*

Proof. By 10.3.12 every flat S-module has finite projective dimension. Thus, to prove the assertion, it suffices by 9.3.28 to argue that the class $\mathcal{X} = \mathcal{M}(S) \cap \mathcal{A}(S)$ has the properties (1), (2), and (3) required in that result. By 10.3.5(a) every projective S-module belongs to $\mathcal{A}(S)$, so (1) holds. Property (3) follows from 6.5.24 and 10.3.4(a). To verify (2) let G be a module in the Auslander category $\mathcal{A}(S)$. Choose by 10.1.13 a bounded complex I of $R \otimes_{\Bbbk} S^{\mathrm{o}}$-bimodules that is semi-injective over R and over S^{o} and isomorphic to D in $\mathcal{D}(R \otimes_{\Bbbk} S^{\mathrm{o}})$. By assumption $\mathrm{H}(D \otimes_S^{\mathsf{L}} G)$ is bounded, so by 5.3.26 there is a semi-injective resolution $D \otimes_S^{\mathsf{L}} G \xrightarrow{\simeq} J$ over R with $J_v = 0$ for $v \gg 0$. In $\mathcal{D}(S)$ one now has $G \simeq \mathsf{RHom}_R(D, D \otimes_S^{\mathsf{L}} G) \simeq \operatorname{Hom}_R(I, J)$, as G belongs to $\mathcal{A}(S)$. By 2.5.12 and 8.4.28 the complex $X = \operatorname{Hom}_R(I, J)$ is bounded above and consists of flat S-modules. One has $G \cong \mathrm{H}_0(X)$ and $\mathrm{H}_v(X) = 0$ for $v \neq 0$, so G is per 2.2.12(d) isomorphic to a submodule of $C = \mathrm{C}_0(X)$, and $X_{\geqslant 0} \xrightarrow{\simeq} C$ is a flat resolution. As X is bounded above, C has finite flat dimension; see 8.3.23. $\square$

The equivalence of the first two conditions in the next theorem is already known from 9.3.30 and 10.3.12.

10.4.4 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, D a dualizing complex for (R, S^{o}), and N an S-complex. The following conditions are equivalent.*

(*i*) $\operatorname{Gpd}_S N$ *is finite.*

(*ii*) $\operatorname{Gfd}_S N$ *is finite.*

(*iii*) *N belongs to $\hat{\mathcal{A}}(S)$ and $D \otimes_S^{\mathsf{L}} N$ to $\mathcal{D}_{\sqsubset}(R)$.*

Moreover, if N is in $\mathcal{D}_{\sqsupset}(S)$, then conditions (i)–(iii) are equivalent to

(*iv*) *N belongs to $\mathcal{A}(S)$.*

Proof. An acyclic complex satisfies all four conditions, so one may assume that N is not acyclic. The implication (*i*) $\Rightarrow$ (*ii*) is immediate from 9.3.30 and 10.3.12.

(*ii*) $\Rightarrow$ (*iii*): Let N be an S-complex of finite Gorenstein flat dimension. It follows from 9.3.26 that the homology of the complex $D \otimes_S^{\mathsf{L}} N$ is bounded above. By 10.4.1(a) tensor evaluation $\boldsymbol{\theta}^{DDN}$ is an isomorphism. Now it follows from (10.2.14.2) that the unit $\boldsymbol{\alpha}_D^N$ is an isomorphism, whence N belongs to $\hat{\mathcal{A}}(S)$ per 10.3.3.

(*iii*) $\Rightarrow$ (*i*): Let N be in $\hat{\mathcal{A}}(S)$ and $D \otimes_S^{\mathsf{L}} N$ in $\mathcal{D}_{\sqsubset}(R)$; notice from 7.6.7 that

$$\sup N \;=\; \sup \mathsf{RHom}_R(D, D \otimes_S^{\mathsf{L}} N) \;\leqslant\; \sup\,(D \otimes_S^{\mathsf{L}} N) - \inf D \;<\; \infty$$

holds. Set $n = \max\{\operatorname{id}_R D + \sup\,(D \otimes_S^{\mathsf{L}} N), \sup N\}$; by assumption n is an integer. Choose a semi-projective resolution $P \xrightarrow{\simeq} N$. Per 9.1.10 it suffices to show that the module $\mathrm{C}_n(P)$ is Gorenstein projective. For every projective S-module L one has $-\inf \mathsf{RHom}_S(N, L) \leqslant n$ by 10.3.14(b), it follows from 8.1.6 that $\mathrm{Ext}_S^m(\mathrm{C}_n(P), L) = 0$ holds for all $m > 0$. Moreover, in the exact sequence of S-complexes,

$$0 \longrightarrow P_{\leqslant n-1} \longrightarrow P_{\subseteq n} \longrightarrow \Sigma^n \mathrm{C}_n(P) \longrightarrow 0\,,$$

the middle term $P_{\subseteq n} \simeq N$ belongs to $\hat{\mathcal{A}}(S)$ by assumption, and the left-hand complex is in $\mathcal{A}(S)$ by 10.3.5 as it has finite projective dimension. Thus, $\mathrm{C}_n(P)$ belongs to $\mathcal{A}(S)$, as it is a triangulated subcategory per 10.3.4, and it follows from 10.4.3 that $\mathrm{C}_n(P)$ is Gorenstein projective.

For a complex N in $\mathcal{D}_{\sqsupset}(S)$ conditions (*ii*) and (*iii*) are by 9.3.16 and 7.6.8 equivalent to $N \in \mathcal{A}(S)$. □

The Bass Category

Per 10.3.7 the Bass Category $\mathcal{B}(R)$ contains the category $\mathcal{I}(R)$ of complexes of finite injective dimension. Next we show that all complexes in $\mathcal{B}(R)$ have have finite Gorenstein injective dimenion, cf. 9.2.12.

10.4.5 Proposition. *Assume that S is flat as a $\Bbbk$-module and right Noetherian. Let M be an R-complex, X a complex in $\mathcal{D}_{\square}(R \otimes_{\Bbbk} S^{\mathrm{o}})$, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S^{\mathrm{o}})$. The homomorphism evaluation morphism* 8.4.19*,*

$$\boldsymbol{\eta}^{MXN} : N \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_{S^{\mathrm{o}}}(N, X), M)\,,$$

is an isomorphism if the invariants $\mathrm{Gid}_R\, M$*,* $\mathrm{id}_R\, X$ *and* $\mathrm{fd}_R\, \mathsf{RHom}_{S^{\mathrm{o}}}(N, X)$ *are finite.*

Proof. Choose a semi-injective resolution $X \xrightarrow{\simeq} E$ over $R \otimes_{\Bbbk} S^{\mathrm{o}}$ with $E_v = 0$ for $v \gg 0$. As S is flat as a $\Bbbk$-module, the complex E is also semi-injective over R; see 7.3.11(c). Let $P \xrightarrow{\simeq} N$ be a semi-projective resolution with P bounded below and degreewise finitely generated; see 5.2.16. Let $M \xrightarrow{\simeq} I$ be a semi-injective resolution and set $n = \mathrm{Gid}_R\, M$. By 8.4.19 one has to prove that the homomorphism evaluation morphism η^{IEP} is a quasi-isomorphism. There is a commutative diagram,

$$\begin{CD}
P \otimes_S \mathrm{Hom}_R(E, I_{\supseteq -n}) @>{\eta^{I_{\supseteq -n}EP}}>> \mathrm{Hom}_R(\mathrm{Hom}_{S^{\mathrm{o}}}(P, E), I_{\supseteq -n}) \\
@V{P \otimes \mathrm{Hom}(E, \tau^I_{\supseteq -n})}V{\simeq}V @V{\simeq}V{\mathrm{Hom}(\mathrm{Hom}(P,E), \tau^I_{\supseteq -n})}V \\
P \otimes_S \mathrm{Hom}_R(E, I) @>{\eta^{IEP}}>> \mathrm{Hom}_R(\mathrm{Hom}_{S^{\mathrm{o}}}(P, E), I)\,.
\end{CD}$$

As $\mathrm{id}_R\, E = \mathrm{id}_R\, X$ is finite by assumption, the left-hand vertical morphism is a quasi-isomorphism by 9.2.16 and semi-projectivity of P. The right-hand morphism is also a quasi-isomorphism by 9.2.16. Indeed, it follows from 5.3.25 that $\mathrm{Hom}_{S^{\mathrm{o}}}(P, E)$ is a semi-injective R-complex, and by assumption $\mathrm{fd}_R\, \mathrm{Hom}_{S^{\mathrm{o}}}(P, E) = \mathrm{fd}_R\, \mathsf{RHom}_{S^{\mathrm{o}}}(N, X)$ is finite. Now $\eta^{I_{\supseteq -n}EP}$ is an isomorphism by 4.5.13(1,a). □

10.4.6 Lemma. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}). A module G*

in $\mathcal{B}(R)$ is Gorenstein injective if $\operatorname{Ext}^m_R(E,G) = 0$ *holds for all injective R-modules E and all $m > 0$.*

Proof. By the assumptions on G, it is enough to verify that it meets requirement 9.2.4(2). To that end it suffices to construct an exact sequence of R-modules,

$$(\diamond) \qquad 0 \longrightarrow G' \longrightarrow I \longrightarrow G \longrightarrow 0\,,$$

where I is injective and G' has the same properties as G—that is, G' belongs to $\mathcal{B}(R)$ and $\operatorname{Ext}^m_R(E,G') = 0$ holds for all injective R-modules E and all $m > 0$. Indeed, for every injective R-module E the induced sequence,

$$0 \longrightarrow \operatorname{Hom}_R(E,G') \longrightarrow \operatorname{Hom}_R(E,I) \longrightarrow \operatorname{Hom}_R(E,G) \longrightarrow 0\,,$$

is per 7.3.35 and 7.3.27 exact by vanishing $\operatorname{Ext}^1_R(E,G')$. Therefore, having constructed $(\diamond)$, the sequence required in 9.2.4(2) can be constructed recursively.

First we argue that G is a homomorphic image of a module Z of finite injective dimension. Choose by 10.1.13 a bounded complex J of $R \otimes_{\Bbbk} S^{\mathrm{o}}$-bimodules that is semi-injective over R and over S^{o} and isomorphic to D in $\mathcal{D}(R \otimes_{\Bbbk} S^{\mathrm{o}})$. Choose a semi-projective resolution $P \xrightarrow{\simeq} \mathsf{R}\mathrm{Hom}_R(D,G)$ over S with $P_v = 0$ for $v \ll 0$. In $\mathcal{D}(R)$ one now has $J \otimes_S P \simeq D \otimes^{\mathsf{L}}_S \mathsf{R}\mathrm{Hom}_R(D,G) \simeq G$, as G belongs to $\mathcal{B}(R)$. By 2.5.18 and 8.4.17 the complex $X = J \otimes_S P$ is bounded below and consists of injective R-modules. One has $G \cong \mathrm{H}_0(X)$ and $\mathrm{H}_v(X) = 0$ for $v \neq 0$, so G is per 2.2.12(c) a homomorphic image of $Z = \mathrm{Z}_0(X)$, and $Z \xrightarrow{\simeq} X_{\leqslant 0}$ is an injective resolution. As X is bounded below, Z has finite injective dimension; see 8.2.19.

Next we argue that G is a homomorphic image of an injective module. There is by 5.3.30 and 8.2.9 an exact sequence of R-modules,

$$0 \longrightarrow Z \xrightarrow{\;\varepsilon\;} I' \longrightarrow C \longrightarrow 0\,,$$

where I' is injective and $\mathrm{id}_R\, C$ is finite. By 9.2.5 one has $\operatorname{Ext}^1_R(C,G) = 0$, whence $\operatorname{Hom}_R(\varepsilon,G)\colon \operatorname{Hom}_R(I',G) \to \operatorname{Hom}_R(Z,G)$ is surjective; see 7.3.35 and 7.3.27. Thus the homomorphism $Z \twoheadrightarrow G$ factors through ε; i.e. there is a surjection $I' \twoheadrightarrow G$.

Finally we can construct $(\diamond)$. By C.10 there exists an injective precover $\varphi\colon I \to G$, and since G is a homomorphic image of an injective module, φ is surjective per C.9. Set $G' = \operatorname{Ker}\varphi$; we argue that this yields the desired sequence $(\diamond)$. The injective module I is in $\mathcal{B}(R)$ by 10.3.5, and G is in $\mathcal{B}(R)$ by assumption, so as $\mathcal{B}(R)$ is a triangualted subcategory of $\mathcal{D}(R)$, see 10.3.4, it follows that G' is in $\mathcal{B}(R)$; cf. 6.5.24. Let E be an injective R-module; one has $\operatorname{Ext}^m_R(E,G') \cong \operatorname{Ext}^{m-1}_R(E,G) = 0$ for $m \geqslant 2$ by 7.3.35. To see that $\operatorname{Ext}^1_R(E,G')$ vanishes, consider the exact sequence of $\Bbbk$-modules from 7.3.35 and 7.3.27,

$$(\star) \qquad \operatorname{Hom}_R(E,I) \xrightarrow{\operatorname{Hom}(E,\varphi)} \operatorname{Hom}_R(E,G) \longrightarrow \operatorname{Ext}^1_R(E,G') \longrightarrow 0\,.$$

As $\operatorname{Hom}_R(E,\varphi)$ is surjective, see C.9, exactness of $(\star)$ yields $\operatorname{Ext}^1_R(E,G') = 0$. ◻

10.4.7 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, D a dualizing complex for (R,S^{o}), and M an R-complex. The following conditions are equivalent.*

(i) $\operatorname{Gid}_R M$ *is finite.*

(ii) *M belongs to $\hat{\mathcal{B}}(R)$ and $\mathsf{RHom}_R(D,M)$ to $\mathcal{D}_{\sqsupset}(R)$.*

Moreover, if M is in $\mathcal{D}_{\sqsubset}(R)$, then conditions ($i$) *and* ($ii$) *are equivalent to*

(iii) *M belongs to $\mathcal{B}(R)$.*

Proof. An acyclic complex trivially satisfies all three conditions, so one may assume that M is not acyclic.

(i) $\Rightarrow$ (ii): Let M be an R-complex of finite Gorenstein injective dimension. By 9.2.18 the homology of the complex $\mathsf{RHom}_R(D,M)$ is bounded below. By 10.4.5 homomorphism evaluaion $\boldsymbol{\eta}^{MDD}$ is an isomorphism. Now (10.2.14.3) shows that the counit $\boldsymbol{\beta}_D^M$ is an isomorphism, whence M belongs to $\hat{\mathcal{B}}(R)$ per 10.3.3.

(ii) $\Rightarrow$ (i): Let M be in $\hat{\mathcal{B}}(R)$ and $\mathsf{RHom}_R(D,M)$ in $\mathcal{D}_{\sqsupset}(R)$; note from 7.6.8 that

$$\inf M \;=\; \inf\left(D \otimes_S^{\mathsf{L}} \mathsf{RHom}_R(D,M)\right) \;\geqslant\; \inf D + \inf \mathsf{RHom}_R(D,M) \;>\; -\infty$$

holds. Set $n = \max\{\operatorname{id}_R D - \inf \mathsf{RHom}_R(D,M), -\inf M\}$; by assumption n is an integer. Choose a semi-injective resolution $M \xrightarrow{\simeq} I$. Per 9.2.9 it suffices to show that the module $\mathrm{Z}_{-n}(I)$ is Gorenstein injective. For every injective R-module E one has $-\inf \mathsf{RHom}_R(E,M) \leqslant n$ by 10.3.14(a), so 8.2.6 yields $\operatorname{Ext}_R^m(E,\mathrm{Z}_{-n}(I)) = 0$ for all $m > 0$. Moreover, in the exact sequence,

$$0 \longrightarrow \Sigma^{-n}\mathrm{Z}_{-n}(I) \longrightarrow I_{\sqsupseteq -n} \longrightarrow I_{\geqslant -n+1} \longrightarrow 0\,,$$

of R-complexes the middle term $I_{\sqsupseteq -n} \simeq M$ belongs to $\hat{\mathcal{B}}(R)$ by assumption, and the right-hand complex is in $\mathcal{B}(R)$ by 10.3.5 as it has finite injective dimension. Thus, $\mathrm{Z}_{-n}(I)$ belongs to $\mathcal{B}(R)$, as it is a triangulated subcategory per 10.3.4, and it follows from 10.4.6 that $\mathrm{Z}_{-n}(I)$ is Gorenstein injective.

For a complex M in $\mathcal{D}_{\sqsubset}(R)$ conditions (i) and (ii) are by 9.2.10 and 7.6.7 equivalent to $M \in \mathcal{B}(R)$. $\square$

The next result is a companion to 9.3.17 and parallels 8.3.18.

10.4.8 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S^{o}). If $\operatorname{FFD} R^{\mathrm{o}}$ is finite and M is a complex in $\mathcal{D}_{\sqsubset}(R)$, then the next equality holds*

$$\operatorname{Gfd}_{R^{\mathrm{o}}} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E}) \;=\; \operatorname{Gid}_R M\,.$$

Proof. It follows from 2.5.7(b), 10.3.10, 10.4.4, and 10.4.7 that $\operatorname{Gid}_R M$ is finite if and only if $\operatorname{Gfd}_{R^{\mathrm{o}}} \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})$ is finite.

Let J be a semi-injective replacement of M with $J_v = 0$ for $v \gg 0$; see 5.3.26. It follows from 8.4.28 and 5.4.8 that $\operatorname{Hom}_{\Bbbk}(J,\mathbb{E})$ is a semi-flat replacement of $\operatorname{Hom}_{\Bbbk}(M,\mathbb{E})$. For $n \geqslant -\inf M$ one has $n \geqslant \sup \operatorname{Hom}_{\Bbbk}(M,\mathbb{E})$ and $\operatorname{Hom}_{\Bbbk}(\mathrm{Z}_{-n}(J),\mathbb{E}) = \mathrm{C}_n(\operatorname{Hom}_{\Bbbk}(J,\mathbb{E}))$, see 2.5.7(b) and 2.2.19. To prove the asserted equality, it is by 9.2.18 and 9.3.26 sufficient to show that an R-module Z is Gorenstein injective if and only if the R^{o}-module $\operatorname{Hom}_{\Bbbk}(Z,\mathbb{E})$ is Gorenstein flat.

Set $d = \operatorname{FFD} R^{\mathrm{o}}$ and assume that it is finite. Assume first that Z is Gorenstein injective. Let I be a totally acyclic complex of injective modules with $\mathrm{Z}_{-d}(I) \cong Z$.

The module $\mathrm{Z}_0(I)$ is Gorenstein injective, so by what has already been proved the module $\mathrm{Hom}_{\Bbbk}(\mathrm{Z}_0(I), \mathbb{E})$ has finite Gorenstein flat dimension, and 9.3.23(a) yields $\mathrm{Gfd}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(\mathrm{Z}_0(I), \mathbb{E}) \leqslant d$. The complex $I_{\leqslant 0}$ yields an injective resolution of $\mathrm{Z}_0(I)$, so as above $\mathrm{Hom}_{\Bbbk}(I_{\leqslant 0}, \mathbb{E})$ yields a flat resolution of $\mathrm{Hom}_{\Bbbk}(\mathrm{Z}_0(I), \mathbb{E})$. It now follows from exactness of the sequence

$$0 \to \mathrm{Hom}_{\Bbbk}(Z, \mathbb{E}) \to \mathrm{Hom}_{\Bbbk}(I_{-d+1}, \mathbb{E}) \to \cdots \to \mathrm{Hom}_{\Bbbk}(I_0, \mathbb{E}) \to \mathrm{Hom}_{\Bbbk}(\mathrm{Z}_0(I), \mathbb{E}) \to 0$$

that $\mathrm{Hom}_{\Bbbk}(Z, \mathbb{E})$ is Gorenstein flat; see 9.3.36.

Now assume that $\mathrm{Hom}_{\Bbbk}(Z, \mathbb{E})$ is Gorenstein flat, it follows from what has already been proved that $\mathrm{Gid}_R Z$ is finite. By 9.2.15 there exists an exact sequence

$$0 \longrightarrow G \longrightarrow X \longrightarrow Z \longrightarrow 0\,,$$

where G is Gorenstein injective and $\mathrm{id}_R X = \mathrm{Gid}_R Z$. As already proved, $\mathrm{Hom}_{\Bbbk}(G, \mathbb{E})$ is Gorenstein flat, and by assumption, so is $\mathrm{Hom}_{\Bbbk}(Z, \mathbb{E})$. Therefore, exactness of

$$0 \longrightarrow \mathrm{Hom}_{\Bbbk}(Z, \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(X, \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(G, \mathbb{E}) \longrightarrow 0$$

implies that $\mathrm{Hom}_{\Bbbk}(X, \mathbb{E})$ is Gorenstein flat; see 9.3.13. In particular, one has

$$\mathrm{Gid}_R Z \;=\; \mathrm{id}_R X \;=\; \mathrm{fd}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(X, \mathbb{E}) = \mathrm{Gfd}_{R^{\mathrm{o}}} \mathrm{Hom}_{\Bbbk}(X, \mathbb{E}) \;=\; 0\,.$$

Here the second equality holds by 8.3.18 and the third holds by 9.3.18. □

Recall from 9.3.13 and 9.2.6 that a filtered colimit of Gorenstein flat modules is Gorenstein flat and that a product of Gorenstein injective modules is Gorenstein injective. In the presence of a dualizing complex the roles can be switched.

10.4.9 Theorem. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S^{o}).*

(a) *Let $\{N^u\}_{u\in U}$ be a family of Gorenstein flat S-modules. If* FFD *S is finite, then the module $\prod_{u\in U} N^u$ is Gorenstein flat.*
(b) *Let $\{M^u\}_{u\in U}$ be a family of Gorenstein injective R-modules. If* FID *R is finite, then the module $\coprod_{u\in U} M^u$ is Gorenstein injective.*
(c) *Let $\{\mu^{vu} \colon M^u \to M^v\}_{u\leqslant v}$ be a U-direct system of R-modules. If* FID *R is finite and U is filtered, then the module* $\mathrm{colim}_{u\in U} M^u$ *is Gorenstein injective.*

Proof. (a): Set $d = \mathrm{FFD}\, S$. For every module N^u choose a totally acyclic complex F^u of flat S-modules with $\mathrm{C}_d(F^u) = N^u$ and set $C^u = \mathrm{C}_0(F^u)$. For every $u \in U$ the module C^u is Gorenstein flat. It follows from 9.3.26 that $\sup\,(D \otimes^{\mathsf{L}}_S C^u) \leqslant \sup D$ holds for every u, so 10.3.11 and 3.1.23 yield

$$\sup\big(D \otimes^{\mathsf{L}}_S \prod_{u\in U} C^u\big) \;=\; \sup\big(\prod_{u\in U} (D \otimes^{\mathsf{L}}_S C^u)\big) \;\leqslant\; \sup D\,.$$

It now follows from 10.3.11 and 10.4.4 that the module $C = \prod_{u\in U} C^u$ has finite Gorenstein flat dimension. The complex $F = \prod_{u\in U} F^u_{\geqslant 0}$ yields by 8.3.26 a flat resolution of C. By 9.3.23(a) one has $\mathrm{Gfd}_S C \leqslant d$, so the module $\mathrm{C}_d(F) = \prod_{u\in U} N^u$, see 3.1.22(c), is Gorenstein flat by 9.3.36.

(b): By 3.1.27 there is an isomorphism of R^{o}-modules,

$$(\star) \qquad \operatorname{Hom}_{\Bbbk}\Big(\coprod_{u\in U} M^u, \mathbb{E}\Big) \cong \prod_{u\in U} \operatorname{Hom}_{\Bbbk}(M^u, \mathbb{E})\,.$$

The assumption that $\operatorname{FID} R$ is finite implies by 8.5.27 that $\operatorname{FFD} R^{\circ}$ is finite. Since D by 10.1.3 is dualizing for (S°, R) it follows from 10.4.8 and part (a) that the product $\prod_{u\in U} \operatorname{Hom}_{\Bbbk}(M^u, \mathbb{E})$ is a Gorenstein flat. From $(\star)$ and another application of 10.4.8 it now follows that $\coprod_{u\in U} M^u$ is Gorenstein injective.

(c): By 9.3.12 the canonical homomorphism $\coprod_{u\in U} M^u \twoheadrightarrow \operatorname{colim}_{u\in U} M^u$ is a pure epimorphism, so it follows from 5.5.14 that $\operatorname{Hom}_{\Bbbk}(\operatorname{colim}_{u\in U} M^u, \mathbb{E})$ is a direct summand of the R°-module $\operatorname{Hom}_{\Bbbk}(\coprod_{u\in U} M^u, \mathbb{E})$, which as shown above is Gorenstein flat. From 9.3.13(b) it now follows that $\operatorname{Hom}_{\Bbbk}(\operatorname{colim}_{u\in U} M^u, \mathbb{E})$ is a Gorenstein flat R°-module, so $\operatorname{colim}_{u\in U} M^u$ is Gorenstein injective by 10.4.8. □

10.4.10 Corollary. *Assume that R and S are projective as $\Bbbk$-modules. Let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R, S°).*

(a) *Let $\{N^u\}_{u\in U}$ be a family of S-modules. If* FFD *S is finite, then one has*

$$\operatorname{Gfd}_S\Big(\prod_{u\in U} N^u\Big) = \sup_{u\in U}\{\operatorname{Gfd}_S N^u\}\,.$$

(b) *Let $\{M^u\}_{u\in U}$ be a family of R-modules. If* FID *R is finite, then one has*

$$\operatorname{Gid}_R\Big(\coprod_{u\in U} N^u\Big) = \sup_{u\in U}\{\operatorname{Gid}_R M^u\}\,.$$

Proof. (a): For every $u \in U$ the module N^u is a direct summand of $\prod_{u\in U} N^u$, so the inequality "$\geqslant$" holds by 9.3.27. For the opposite inequality one can assume that the quantity $s = \sup_{u\in U}\{\operatorname{Gfd}_S N^u\}$ is an integer. By 9.3.26 each module N^u has a semi-flat replacement F^u with $F^u_v = 0$ for all $v < 0$ and $\mathrm{C}_s(F^u)$ Gorenstein flat. There is an isomorphism $\prod_{u\in U} F^u \simeq \prod_{u\in U} N^u$ in $\mathcal{D}(S)$, and the complex $\prod_{u\in U} F^u$ is semi-flat by 8.3.26 and 5.4.8. It follows from 3.1.22(c) and 10.4.9(a) that the module $\mathrm{C}_s(\prod_{u\in U} F^u) \cong \prod_{u\in U} \mathrm{C}_s(F^u)$ is Gorenstein flat. Thus "$\leqslant$" holds by 9.3.26.

(b): The assumption that $\operatorname{FID} R$ is finite implies by 8.5.27 that $\operatorname{FFD} R^{\circ}$ is finite. Since D by 10.1.3 is dualizing for (S°, R), the equality (b) follows from 3.1.27, 10.4.8, and part (a). □

Remark. The proof of 10.4.10 applies to families of appropriately bounded complexes; see E 10.4.6 and E 10.4.7, and see also the Remarks after 8.2.21 and 8.3.27.

Our primary interest in the main theorems, 10.4.4 and 10.4.7, of this section is that they allow us to prove statements like 10.4.8–10.4.10; we don't know if these results hold in general, i.e. without the presence of a dualizing complex. It is worth noting that 10.4.4 and 10.4.7 also provide new insight on the Auslander Category and Bass Category; see E 10.4.1 and E 10.4.2.

Derived Reflexive Complexes

Over a Noetherian ring, the category of derived reflexive complexes introduced in 10.2.4 can also be interpreted in terms of Gorenstein homological dimensions; that is the content of 10.4.15, which we move towards with the next result.

10.4.11 Lemma. *Let P be a complex of finitely generated projective R-modules. If P is totally acyclic, then $\operatorname{Hom}_R(P,R)$ is a totally acyclic complex of finitely generated projective R^{o}-modules.*

Proof. Let P be a totally acyclic complex of finitely generated projective R-modules. By 4.5.4 and the assumption, $\operatorname{Hom}_R(P,R)$ is an acyclic complex of finitely generated projective R^{o}-modules. As one has $P \cong \operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_R(P,R),R)$, still by 4.5.4, it follows that $\operatorname{Hom}_R(P,R)$ is totally acyclic; see 9.1.5. □

10.4.12 Proposition. *Assume that R is left Noetherian. For every finitely generated Gorenstein projective R-module G the R^{o}-module $\operatorname{Hom}_R(G,R)$ is finitely generated Gorenstein projective.*

Proof. By 9.1.23 there exists a totally acyclic complex P of finitely generated projective R-modules with $G \cong \mathrm{C}_0(P)$. It now follows from 10.4.11 and the isomorphism $\operatorname{Hom}_R(\mathrm{C}_0(P),R) \cong \mathrm{C}_1(\operatorname{Hom}_R(P,R))$ that the R^{o}-module $\operatorname{Hom}_R(G,R)$ is finitely generated and Gorenstein projective. □

The next characterization of finitely generated Gorenstein projective modules over a Noetherian ring is, in fact, the first published definition of such modules; see Auslander [8]. A module that satisfies the second condition is called *reflexive*, which clarifies the alternate name for Gorenstein projective modules mentioned in the Remark after 9.1.23.

10.4.13 Proposition. *Assume that R is left Noetherian. A finitely generated R-module G is Gorenstein projective if and only if it satisfies the next conditions.*

(1) $\operatorname{Ext}_R^m(G,R) = 0 = \operatorname{Ext}_{R^{\mathrm{o}}}^m(\operatorname{Hom}_R(G,R),R)$ *holds for all $m > 0$.*
(2) *Biduality $\delta_R^G \colon G \longrightarrow \operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_R(G,R),R)$ is an isomorphism.*

Proof. Assume that G is Gorenstein projective. From 9.1.3(1) and 10.4.12 one gets vanishing of $\operatorname{Ext}_R^m(G,R)$ and $\operatorname{Ext}_{R^{\mathrm{o}}}^m(\operatorname{Hom}_R(G,R),R)$ for all $m > 0$. Choose by 9.1.23 a totally acyclic complex P of finitely generated projective R-modules with $\mathrm{C}_0(P) \cong G$. Acyclicity of $P^* = \operatorname{Hom}_R(P,R)$ and $\operatorname{Hom}_{R^{\mathrm{o}}}(P^*,R)$, see 10.4.11, yields

$$\mathrm{C}_0(\operatorname{Hom}_{R^{\mathrm{o}}}(P^*,R)) \;\cong\; \operatorname{Hom}_{R^{\mathrm{o}}}(\mathrm{C}_1(P^*),R) \;\cong\; \operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_R(\mathrm{C}_0(P),R),R)\,,$$

so δ_R^G is a restriction of the biduality isomorphism δ_R^P; see 4.5.4.

Conversely, assume that $\operatorname{Ext}_R^m(G,R) = 0 = \operatorname{Ext}_{R^{\mathrm{o}}}^m(\operatorname{Hom}_R(G,R),R)$ holds for all $m > 0$ and that δ_R^G is an isomorphism. Let $\pi \colon L \xrightarrow{\simeq} G$ and $\pi' \colon L' \xrightarrow{\simeq} \operatorname{Hom}_R(G,R)$ be free resolutions over R and R^{o} with L and L' degreewise finitely generated; see 5.1.19. As $\operatorname{Ext}_{R^{\mathrm{o}}}^m(\operatorname{Hom}_R(G,R),R)$ vanishes for $m > 0$, the homology of $\operatorname{Hom}_{R^{\mathrm{o}}}(L',R)$ is $\operatorname{Hom}_{R^{\mathrm{o}}}(\operatorname{Hom}_R(G,R),R)$, so $\operatorname{Hom}_{R^{\mathrm{o}}}(\pi',R)$ is an injective quasi-isomorphism. As $\operatorname{Hom}_{R^{\mathrm{o}}}(L',R)$ by 4.5.4 is a complex of finitely generated projective R-modules, the complex

$$P \;=\; \Sigma^{-1}\mathrm{Cone}(\operatorname{Hom}_{R^{\mathrm{o}}}(\pi',R)\delta_R^G\pi)$$

is an acyclic complex of finitely generated free R-modules with $\mathrm{C}_0(P) \cong \mathrm{C}_0(L) \cong G$. For $v \leqslant -1$ one has $\mathrm{H}_v(\operatorname{Hom}_R(P,R)) = \operatorname{Ext}_R^{-v}(G,R) = 0$, and for $v > 0$ one has $\mathrm{H}_v(\mathrm{Hom}_R(P,R)) = \mathrm{H}_v(\mathrm{Hom}_R(\mathrm{Hom}_{R^\circ}(L',R),R)) \cong \mathrm{H}_v(L') = 0$ by biduality 4.5.4. Finally, $\mathrm{Z}_0(P) = \mathrm{Ker}(\mathrm{Hom}_{R^\circ}(\pi'_0,R)\delta^G_R\pi_0) = \mathrm{Ker}\,\pi_0$ holds as δ^G_R is an isomorphism and $\mathrm{Hom}_{R^\circ}(\pi'_0,R)$ is injective. Further, one has $\mathrm{Ker}\,\pi_0 = \mathrm{B}_0(L) = \mathrm{B}_0(P)$, and it follows that also $\mathrm{H}_0(\mathrm{Hom}_R(P,R))$ is zero. □

REMARK. Notice from the proof that the "only if" statement in 10.4.13 holds under the assumption that R is left Noetherian, while the proof of "if" uses the full Noetherian hypothesis. This is why the terminology 'totally reflexive' is only used in the Noetherian setting, cf. the Remark after 9.1.23.

10.4.14 Lemma. *Assume that R is flat as a $\Bbbk$-module. Let G be a bounded complex of R-modules. If for every $v \in \mathbb{Z}$ and all $m > 0$ one has*

$$\mathrm{Ext}^m_R(G_v,R) = 0 = \mathrm{Ext}^m_{R^\circ}(\mathrm{Hom}_R(G_v,R),R)\,,$$

then G is derived reflexive if and only if biduality δ^G_R in $\mathcal{C}(R)$ is a quasi-isomorphism.

PROOF. Let $\iota\colon R \xrightarrow{\simeq} I$ be an injective resolution over $R \otimes_\Bbbk R^\circ$; see 5.3.31. Since R is flat as a $\Bbbk$-module, it follows from 7.3.11(c) that I is semi-injective both as an R- and as an R°-complex. Consider the commutative diagram,

$$\begin{tikzcd}
G \arrow[r, "\delta^G_I"] \arrow[d, "\delta^G_R"'] & \mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(G,I),I) \arrow[d, "\simeq"', "\mathrm{Hom}(\mathrm{Hom}(G,\iota),I)"] \\
\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(G,R),R) \arrow[r, "\mathrm{Hom}(\mathrm{Hom}(G,R),\iota)", "\simeq"'] & \mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(G,R),I)\,.
\end{tikzcd}$$

To see that the right-hand map is a quasi-isomorphism, set $C = \mathrm{Cone}\,\iota$; is the complex $0 \to R \to I_0 \to I_{-1} \to \cdots$. By the assumptions on G, it follows from A.4 that $\mathrm{Hom}_R(G,C)$ is acyclic. Thus, $\mathrm{Hom}_R(G,\iota)$ is a quasi-isomorphism, see 4.1.16; in particular, $\mathsf{R}\mathrm{Hom}_R(G,R) \simeq \mathrm{Hom}_R(G,R)$ has bounded homology. By semi-injectivity of I as an R°-complex, also $\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(G,\iota),I)$ is a quasi-isomorphism, and one can repeat the argument above to show that $\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(G,R),\iota)$ is a quasi-isomorphism. By commutativity of the diagram, δ^G_R is a quasi-isomorphism if and only if δ^G_I is a quasi-isomorphism. Per 8.4.1 and 6.4.18 biduality $\boldsymbol{\delta}^G_R$ is an isomorphism in $\mathcal{D}(R)$ if and only if δ^G_I is a quasi-isomorphism. □

The next theorem interprets the category $\mathcal{R}(R)$ of derived reflexive R-complexes. Note that if R has a dualizing complex, 10.4.4 and 10.4.15 combine to give an alternative proof of the equality $\mathcal{A}^{\mathrm{f}}(R) = \mathcal{R}(R)$ from 10.3.9; indeed, both subcategories consist of the complexes in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ with finite Gorenstein projective dimension.

10.4.15 Theorem. *Assume that R is flat as a $\Bbbk$-module and Noetherian. For every complex $M \in \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ the following conditions are equivalent.*

(*i*) $\mathrm{Gpd}_R\, M$ *is finite.*
(*ii*) *M is derived reflexive.*

PROOF. One may assume that M is not acyclic. Let $\pi\colon P \xrightarrow{\simeq} M$ be a semi-projective resolution with P bounded below and degreewise finitely generated; see 5.2.16.

$(i) \Rightarrow (ii)$: If $n = \operatorname{Gpd}_R M$ is finite, then M has bounded homology; see 9.1.11. Now $P_{\subseteq n}$ is by 9.1.19 a bounded complex of finitely generated Gorenstein projective modules. It follows from 10.4.13 that $P_{\subseteq n}$ satisfies the conditions in 10.4.14 and that biduality $\delta_R^{P_{\subseteq n}}$ is an isomorphism, as each component $(\delta_R^{P_{\subseteq n}})_v = \delta_R^{(P_{\subseteq n})_v}$ is so. Since $M \simeq P_{\subseteq n}$ in $\mathcal{D}(R)$ it now follows from 10.4.14 that M is derived reflexive.

$(ii) \Rightarrow (i)$: Set $n = -\inf \mathsf{RHom}_R(M,R)$ and notice that $n \geqslant \sup M$ holds by 7.6.7 and the isomorphism $M \simeq \mathsf{RHom}_{R^\circ}(\mathsf{RHom}_R(M,R),R)$ in $\mathcal{D}(R)$; the goal is to prove that $\mathrm{C}_n(P)$ is Gorenstein projective. To this end, notice that biduality $\boldsymbol{\delta}_R^{P_{\leqslant n}}$ is an isomorphim in $\mathcal{D}(R)$ by 10.2.1, as $P_{\leqslant n}$ is a complex of finite projective dimension. Since $M \simeq P_{\subseteq n}$ holds in $\mathcal{D}(R)$, biduality $\boldsymbol{\delta}_R^{P_{\subseteq n}}$ is an isomorphism by assumption. The canonical exact sequence $0 \to P_{\leqslant n-1} \to P_{\subseteq n} \to \Sigma^n \mathrm{C}_n(P) \to 0$ and biduality 8.4.2 induce by 6.5.24 a commutative diagram in $\mathcal{D}(R)$, which by 6.5.19 shows that also $\boldsymbol{\delta}_R^{\mathrm{C}_n(P)}$ is an isomorphism. By assumption, $\operatorname{Ext}_R^m(\mathrm{C}_n(P),R) = 0$ holds for all $m > 0$, see 8.1.6. It follows that there is an isomorphism $\operatorname{Hom}_R(\mathrm{C}_n(P),R) \simeq \mathsf{RHom}_R(\mathrm{C}_n(P),R)$ in $\mathcal{D}(R)$, and as $\boldsymbol{\delta}_R^{\mathrm{C}_n(P)}$ is an isomorphism one has, for every $v \neq 0$,

$$\begin{aligned}\operatorname{Ext}^v_{R^\circ}(\operatorname{Hom}_R(\mathrm{C}_n(P),R),R) &\cong \mathrm{H}_{-v}(\mathsf{RHom}_{R^\circ}(\mathsf{RHom}_R(\mathrm{C}_n(P),R),R))\\ &\cong \mathrm{H}_{-v}(\mathrm{C}_n(P)) = 0\,.\end{aligned}$$

It now follows from 10.4.14 that biduality $\delta_R^{\mathrm{C}_n(P)}$ is a quasi-isomorphism, and hence an isomorphism as it is a map of modules. Finally it follows from 10.4.13 that $\mathrm{C}_n(P)$ is Gorenstein projective. □

In view of 9.1.13 the next result generalizes 10.2.2.

10.4.16 Corollary. *Assume that R is flat as a $\Bbbk$-module and Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. If $\operatorname{Gpd}_R M$ is finite, then the complex $\mathsf{RHom}_R(M,R)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R^\circ)$ and the next equalities hold,*

$$\operatorname{Gpd}_R M = -\inf \mathsf{RHom}_R(M,R) \quad \textit{and} \quad \operatorname{Gpd}_{R^\circ} \mathsf{RHom}_R(M,R) = -\inf M\,;$$

in particular, $\mathsf{RHom}_R(M,R)$ has finite Gorenstein projective dimension over R°.

Proof. The assertions follow straight from 10.2.7, 9.1.29, and 10.4.15. □

Exercises

In exercises E 10.4.1–10.4.9 assume that R and S are projective as $\Bbbk$-modules, let R be left Noetherian, S right Noetherian, and D a dualizing complex for (R,S°).

E 10.4.1 Assume that $\operatorname{FFD} S$ is finite and let $\{N^u\}_{u\in U}$ be a family of S-modules. Show that $\coprod_{u\in U} N^u$ belongs to $\mathcal{A}(S)$ if and only if each module N^u belongs to $\mathcal{A}(S)$.

E 10.4.2 Assume that $\operatorname{FID} R$ is finite and let $\{M^u\}_{u\in U}$ be a family of R-modules. Show that $\prod_{u\in U} M^u$ belongs to $\mathcal{B}(R)$ if and only if each module M^u belongs to $\mathcal{B}(R)$.

E 10.4.3 Show that there is a complex N in $\hat{\mathcal{A}}(S)$ with $\operatorname{Gfd}_S N = \infty$.

E 10.4.4 Show that there is a complex M in $\hat{\mathcal{B}}(R)$ with $\operatorname{Gid}_R M = \infty$.

E 10.4.5 Assume that FID R is finite and let $0 \to G' \to G \to G'' \to 0$ be a pure exact sequence of R-modules. Show that if G is Gorenstein injective, then so are G' and G''.

E 10.4.6 Assume that FFD S is finite. Let $\{M^u\}_{u \in U}$ be a family of R-complexes. Show that if $\inf_{u \in U}\{\inf M^u\} > -\infty$ holds, then $\operatorname{Gfd}_R(\prod_{u \in U} M^u) = \sup_{u \in U}\{\operatorname{Gfd}_R M^u\}$ holds.

E 10.4.7 Assume that FID R is finite. Let $\{M^u\}_{u \in U}$ be a family of R-complexes. Show that if $\sup_{u \in U}\{\sup M^u\} < \infty$ holds, then $\operatorname{Gid}_R(\coprod_{u \in U} M^u) = \sup_{u \in U}\{\operatorname{Gid}_R M^u\}$ holds.

E 10.4.8 Assume that FID R is finite. Let $\{\mu^{vu} : M^u \to M^v\}_{u \in U}$ be a U-direct system of R-modules. Show that the inequality $\operatorname{Gid}_R(\operatorname{colim}_{u \in U} M^u) \leqslant \sup_{u \in U}\{\operatorname{Gid}_R M^u\}$ holds if U is filtered.

E 10.4.9 Show that 10.4.1(b) is valid without boundedness assumptions on the complex X.

E 10.4.10 Assume that R is flat as a $\Bbbk$-module and S is Noetherian. Let M be an R-complex, X a complex in $\mathcal{D}_{\square}(R \otimes_{\Bbbk} S^{\mathrm{o}})$, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$. Show that the tensor evaluation morphism $\boldsymbol{\theta}^{MXN}$ from 8.4.6 is an isomorphism in $\mathcal{D}(\Bbbk)$ if the dimensions $\operatorname{id}_{S^{\mathrm{o}}} X$, $\operatorname{fd}_{S^{\mathrm{o}}} \mathsf{RHom}_R(M, X)$, and $\operatorname{Gfd}_S N$ are finite.

E 10.4.11 Assume that S is flat as a $\Bbbk$-module and that R is left Noetherian. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, X a complex of R–S^{o}-bimodules, and N an S-complex. Show that the tensor evaluation morphism $\boldsymbol{\theta}^{MXN}$ from 8.4.6 is an isomorphism in $\mathcal{D}(\Bbbk)$ if the dimensions $\operatorname{Gpd}_R M$, $\operatorname{fd}_R X$, $\operatorname{id}_R(X \otimes^{\mathsf{L}}_S N)$ are finite.

E 10.4.12 Assume that R is flat as a $\Bbbk$-module and that S is right Noetherian. Let M be an R-complex, X a complex of R–S^{o}-bimodules, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(S^{\mathrm{o}})$. Show that the homomorphism evaluation morphism $\boldsymbol{\eta}^{MXN}$ from 8.4.19, is an isomorphism if the dimensions $\operatorname{fd}_{S^{\mathrm{o}}} X$, $\operatorname{fd}_S \mathsf{RHom}_R(X, M)$, and $\operatorname{Gpd}_{S^{\mathrm{o}}} N$ are finite. *Hint:* E 9.1.12.

E 10.4.13 Let M be a complex of Gorenstein projective R-modules and $\mathcal{N}$ be as in 9.1.9. (a) Show that $\operatorname{Hom}_R(M, -)$ preserves quasi-isomorphisms of bounded below complexes of modules from $\mathcal{N}$. (b) Show that if M is bounded below, then $\operatorname{Hom}_R(M, -)$ preserves quasi-isomorphisms of bounded above complexes of modules from $\mathcal{N}$.

E 10.4.14 Let M be a complex of Gorenstein injective R-modules and $\mathcal{N}$ be as in 9.2.8. (a) Show that $\operatorname{Hom}_R(-, M)$ preserves quasi-isomorphisms of bounded above complexes of modules from $\mathcal{N}$. (b) Show that if M is bounded above, then $\operatorname{Hom}_R(-, M)$ preserves quasi-isomorphisms of bounded below complexes of modules from $\mathcal{N}$.

E 10.4.15 Let M be a complex of Gorenstein flat R-modules and $\mathcal{N}$ be as in 9.3.14. (a) Show that $- \otimes_R M$ preserves quasi-isomorphisms of bounded above complexes of modules from $\mathcal{N}$. (b) Show that if M is bounded below, then $- \otimes_R M$ preserves quasi-isomorphisms of bounded below complexes of modules from $\mathcal{N}$.

E 10.4.16 Assume that R is projective as a $\Bbbk$-module and Noetherian with a dualizing complex. Show that one has $\mathcal{A}(R) \cap \mathcal{I}(R) = \mathcal{P}(R) \cap \mathcal{I}(R) = \mathcal{P}(R) \cap \mathcal{B}(R)$.

Chapter 11
Torsion and Completion

Torsion, completion, and the associated local (co)homology theories are in their own right the topics of books: Brodman and Sharp's [45] is a classic that focuses on the torsion/local cohomology side as does Lipman's [173]; in [224] Schenzel and Simon treat both theories and their interactions. Compared to these and other treaties, the path we cut here and continue in Chap. 13 is a pretty narrow one. It takes us without many detours to a theorem of Alonso Tarrío, Jeremías López, and Lipman [3] on the equivalence of the categories of derived torsion and derived complete complexes. This theorem—13.4.13 in this text—was inspired by work of Greenlees and May [109] and is known as the Greenlees–May Equivalence. All this is over commutative rings; Vyas and Yekutieli [249] establish a non-commutative version of the equivalence, which they call the MGM Equivalence with the first 'M' referring to Matlis [181] .

Throughout this chapter, **the rings R and S are assumed to be commutative.**

11.1 Completion

Synopsis. The functor $\Lambda^{\mathfrak{a}}$; independence of base; $\mathfrak{a}$-complete complex; $\mathfrak{a}$-completion of ring; $\widehat{R}^{\mathfrak{a}}$-structure on $\mathfrak{a}$-complete complex; $\mathfrak{a}$-dense morphism; idempotence of $\Lambda^{\mathfrak{a}}$.

By convention, a *sequence in R* is understood to be a non-empty finite sequence of elements in R. The standard notation for such a sequence is $\boldsymbol{x}$; it may introduced as $\boldsymbol{x} = x_1, \ldots, x_n$ if the number n of elements is relevant.

Towers from Ideals and Sequences

11.1.1 Definition. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in R. For $u \geqslant 1$ the symbol $\boldsymbol{x}^u$ denotes the sequence $x_1^u, \ldots, x_n^u$ of u^{th} powers.

11.1.2 Definition. Let $\mathfrak{a}$ be an ideal in R; there is a canonical tower of R-modules,

$$\{\vartheta^u_{\mathfrak{a}}\colon R/\mathfrak{a}^u \twoheadrightarrow R/\mathfrak{a}^{u-1}\}_{u>1}\ . \tag{11.1.2.1}$$

For a sequence $\boldsymbol{x}$ in R there is another canonical tower of R-modules,

$$\{\vartheta^u_{\boldsymbol{x}}\colon R/(\boldsymbol{x}^u) \twoheadrightarrow R/(\boldsymbol{x}^{u-1})\}_{u>1}\ . \tag{11.1.2.2}$$

For $\mathfrak{a} = (\boldsymbol{x})$ the inclusions $(\boldsymbol{x}^u) \subseteq (\boldsymbol{x})^u = \mathfrak{a}^u$ yield a canonical comparison morphism of the towers above,

$$\{\xi^u_{\boldsymbol{x}}\colon R/(\boldsymbol{x}^u) \twoheadrightarrow R/(\boldsymbol{x})^u\}_{u\geqslant 1}\ . \tag{11.1.2.3}$$

11.1.3. Let $\mathfrak{a}$ be an ideal in R. Note that there is a commutative diagram,

$$\begin{tikzcd}
R \arrow[dr, two heads, bend left] \arrow[drr, two heads, bend left] & & \\
\cdots \arrow[r, two heads, "\vartheta^3_{\mathfrak{a}}"'] & R/\mathfrak{a}^2 \arrow[r, two heads, "\vartheta^2_{\mathfrak{a}}"'] & R/\mathfrak{a}\ .
\end{tikzcd}$$

Applying a functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ to this diagram, the univeral property of limits 3.4.5 yields a canonical morphism $\mathrm{F}(R) \to \lim_{u\geqslant 1} \mathrm{F}(R/\mathfrak{a}^u)$ in $\mathcal{C}(S)$. More generally, a functor $\mathrm{F}\colon \mathcal{C}(R) \times \mathcal{U} \to \mathcal{C}(S)$ yields a natural transformation,

$$\mathrm{F}(R,-) \longrightarrow \lim_{u\geqslant 1} \mathrm{F}(R/\mathfrak{a}^u,-)\ ,$$

of functors from $\mathcal{U}$ to $\mathcal{C}(S)$. Similarly, a functor $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{U} \to \mathcal{C}(S)$ induces by 3.2.5 a natural transformation of functors from $\mathcal{U}$ to $\mathcal{C}(S)$,

$$\operatorname*{colim}_{u\geqslant 1} \mathrm{G}(R/\mathfrak{a}^u,-) \longrightarrow \mathrm{G}(R,-)\ .$$

Completion Functor

11.1.4 Definition. Let $\mathfrak{a}$ be an ideal in R. The tower (11.1.2.1) yields a functor

$$\Lambda^{\mathfrak{a}} \ =\ \lim_{u\geqslant 1}\,(R/\mathfrak{a}^u \otimes_R -) : \mathcal{C}(R) \ \longrightarrow\ \mathcal{C}(R)\ ;$$

see 3.5.2. It is called the $\mathfrak{a}$-*completion* functor. Denote by

$$\lambda^{\mathfrak{a}} :\ \mathrm{Id}_{\mathcal{C}(R)} \ \longrightarrow\ \Lambda^{\mathfrak{a}}$$

the natural transformation obtained from the unitor 4.4.1 and 11.1.3 with $\mathrm{F} = \otimes_R$.

Remark. The functor $\Lambda^{\mathfrak{a}}$ is in parts of the literature called the '$\mathfrak{a}$-adic completion' functor; we opt for the shorter term '$\mathfrak{a}$-completion'.

Completion with respect to the trivial ideals is a trivial affaire.

11.1.5 Example. Clearly, Λ^R is the zero functor and Λ^0 the identity functor on $\mathcal{C}(R)$. In fact, for every nilpotent ideal $\mathfrak{a}$ one has $\Lambda^{\mathfrak{a}} \cong \mathrm{Id}_{\mathcal{C}(R)}$ 3.5.8(b) and 4.4.1.

11.1.6 Addendum (to 11.1.4). Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. It follows from 11.1.4 and 2.4.10 that the $\mathfrak{a}$-completion functor $\Lambda^{\mathfrak{a}}$ is an endofunctor on $\mathcal{C}(S)$ and $\lambda^{\mathfrak{a}}$ a natural transformation $\mathrm{Id}_{\mathcal{C}(S)} \to \Lambda^{\mathfrak{a}}$.

The next result shows that $\mathfrak{a}$-completion is independent of base.

11.1.7 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. There is a commutative diagram in $\mathcal{C}(S)$ where the horizontal map is an isomorphism,*

$$\begin{tikzcd} & N \arrow[dl, "\lambda_N^{\mathfrak{a}S}"'] \arrow[dr, "\lambda_N^{\mathfrak{a}}"] & \\ \Lambda^{\mathfrak{a}S}(N) \arrow[rr, "\cong"'] & & \Lambda^{\mathfrak{a}}(N)\,. \end{tikzcd}$$

Proof. Recall from 11.1.6 that $\Lambda^{\mathfrak{a}}(N)$ is an S-complex and that $\lambda_N^{\mathfrak{a}} : N \to \Lambda^{\mathfrak{a}}(N)$ is S-linear. In the computation below, the equality is trivial, the first isomorphism follows from 1.1.10, and the second holds by associativity 4.4.7 and the unitor 4.4.1:

$$S/(\mathfrak{a}S)^u \otimes_S N \;=\; S/\mathfrak{a}^u S \otimes_S N \;\cong\; (R/\mathfrak{a}^u \otimes_R S) \otimes_S N \;\cong\; R/\mathfrak{a}^u \otimes_R N\,.$$

As N is an S-complex, so are the tensor product complexes above, see 2.4.10, and the isomorphisms are S-linear. From 11.1.4 one gets an induced isomorphism, $\Lambda^{\mathfrak{a}S}(N) \cong \Lambda^{\mathfrak{a}}(N)$, of S-complexes, which makes the desired diagram commutative. □

11.1.8 Definition. Let $\mathfrak{a}$ be an ideal in R and M an R-complex.

(1) If $\lambda_M^{\mathfrak{a}}$ is injective, then M is called $\mathfrak{a}$-*separated*.
(2) If $\lambda_M^{\mathfrak{a}}$ is a surjective, then M is called $\mathfrak{a}$-*quasi-complete*.
(3) If $\lambda_M^{\mathfrak{a}}$ is an isomorphism, then M is called $\mathfrak{a}$-*complete*.

The next result remains valid, with the same proof, if one replaces "-complete" with "-quasi-complete" or "-separated".

11.1.9 Corollary. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. The complex N is $\mathfrak{a}S$-complete if and only if it is $\mathfrak{a}$-complete as an R-complex.*

Proof. The assertion follows immediately from 11.1.7 and 11.1.8. □

We proceed to develop concrete descriptions of $\Lambda^{\mathfrak{a}}$ and $\lambda^{\mathfrak{a}}$.

11.1.10. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. By 1.1.10 the tensor product $R/\mathfrak{a}^u \otimes_R M$ is isomorphic to $M/\mathfrak{a}^u M$ and via this isomorphism, the composite $M \xrightarrow{\cong} R \otimes_R M \twoheadrightarrow R/\mathfrak{a}^u \otimes_R M$ is identified with the canonical quotient map $\pi_M^u : M \twoheadrightarrow M/\mathfrak{a}^u M$. By the universal property of limits, it follows that $\lambda_M^{\mathfrak{a}}$ is the unique morphism that makes the diagram

$$\begin{tikzcd} M \arrow[dd, "\lambda_M^{\mathfrak{a}}"'] \arrow[dr, two heads, "\pi_M^u"] & \\ & M/\mathfrak{a}^u M \\ \Lambda^{\mathfrak{a}}(M) \arrow[ur, "\nu_M^u"'] & \end{tikzcd}$$

commutative for every $u \geqslant 1$; here ν^u_M is the canonical morphism from (3.4.3.1).

Notice that 3.5.5 yields an isomorphism,

$$\Lambda^{\mathfrak{a}}(M) \;\cong\; \left\{ ([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} \in \prod_{u\geqslant 1} M/\mathfrak{a}^u M \;\middle|\; m^u - m^{u+1} \in \mathfrak{a}^u M \text{ for all } u \geqslant 1 \right\}.$$

Via this isomorphism, the morphism $\lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ is identified with the one given by $m \mapsto ([m]_{\mathfrak{a}^u M})_{u\geqslant 1}$.

11.1.11 Example. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Notice from 11.1.10 that M is $\mathfrak{a}$-separated if and only if $\bigcap_{u\geqslant 1} \mathfrak{a}^u M = 0$ holds. Notice also that if $\mathfrak{a}^u M = 0$ holds for some $u \geqslant 1$, then M is $\mathfrak{a}$-complete.

11.1.12. Let $\mathfrak{a}$ be an ideal in R. The functor $\Lambda^{\mathfrak{a}}$ restricts to a functor $\mathcal{M}(R) \to \mathcal{M}(R)$, and it can be recovered from this restriction by the procedure in 2.1.48. In this sense, $\Lambda^{\mathfrak{a}}$ on $\mathcal{C}(R)$ is extended from $\Lambda^{\mathfrak{a}}$ on $\mathcal{M}(R)$. This follows from 11.1.4 in view of 2.4.1 and 3.4.7. Moreover, for an R-complex M and $v \in \mathbb{Z}$ one has $(\lambda^{\mathfrak{a}}_M)_v = \lambda^{\mathfrak{a}}_{M_v}$.

In addition to the properties listed in the next proposition, the $\mathfrak{a}$-completion functor is idempotent if the ideal $\mathfrak{a}$ is finitely generated; see 11.1.38. The functor does not preserve coproducts; see 11.1.34.

11.1.13 Theorem. *Let $\mathfrak{a}$ be an ideal in R. The $\mathfrak{a}$-completion functor $\Lambda^{\mathfrak{a}}$ is R-linear and bounded. It is also a $\natural$- and Σ-functor, and the natural transformation $\lambda^{\mathfrak{a}}$ is a Σ-transformation. Moreover, if $\mathfrak{a}$ is finitely generated, then $\Lambda^{\mathfrak{a}}$ preserves products.*

Proof. All assertions but the last one follow from properties of tensor products and extended functors: 2.1.48, A.14, 2.1.53, and 4.1.13. If $\mathfrak{a}$ is finitely generated, then each module $R/\mathfrak{a}^u$ is finitely presented, so by 3.1.30 the functors $R/\mathfrak{a}^u \otimes_R -$ preserve products. Finally, limits and products commute by 3.4.8 and 3.4.13. $\square$

Let $\mathfrak{a}$ be an ideal in R. Every R-module M can be equipped with the so-called $\mathfrak{a}$-adic topology, where a basis of open neighbourhoods of 0 are the sets $\{\mathfrak{a}^u M\}_{u\geqslant 1}$. In this topology, a sequence $x^1, x^2, \dots$ of elements in M converges to $x \in M$ if every neighborhood $\mathfrak{a}^v M$ contains the difference $x - x^u$ for $u \gg 0$. The next result shows that for any $m = ([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$ in $\Lambda^{\mathfrak{a}}(M)$ the sequence $\lambda^{\mathfrak{a}}_M(m^1), \lambda^{\mathfrak{a}}_M(m^2), \dots$ converges to m in the $\mathfrak{a}$-adic topology on $\Lambda^{\mathfrak{a}}(M)$, so the image of the morphism $\lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ is dense in $\Lambda^{\mathfrak{a}}(M)$ in this topology. We will not pursue this topological point of view, but in 11.1.29 we give an algebraic analogue of density with respect to $\mathfrak{a}$, and we show in 11.1.37 that the morphism $\lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ is dense in this sense.

11.1.14 Lemma. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-complex. For every element $m = ([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$ in $\Lambda^{\mathfrak{a}}(M)$ and $u \geqslant 1$, the difference $m - \lambda^{\mathfrak{a}}_M(m^u)$ belongs to $\mathfrak{a}^u\,\Lambda^{\mathfrak{a}}(M)$.*

Proof. Fix $u \geqslant 1$ and let $\{x_1, \dots, x_t\}$ be a set of generators for the finitely generated ideal $\mathfrak{a}^u$. For every $i \geqslant 1$ one has $m^{i+1} - m^i \in \mathfrak{a}^i M$, in particular, $m^{i+1} - m^i$ belongs to $\mathfrak{a}^u(\mathfrak{a}^{i-u} M)$ for $i > u$. Thus, for every $i > u$ one can write

$$m^{i+1} - m^i = \sum_{j=1}^{t} x_j n^{ij} \qquad \text{where} \qquad n^{i1}, \dots, n^{it} \in \mathfrak{a}^{i-u} M .$$

It follows that for every $k \geqslant 1$ one has

$$\begin{aligned} m^{u+k} - m^u &= \sum_{i=u}^{u+k-1} (m^{i+1} - m^i) \\ &= \sum_{i=u}^{u+k-1} \sum_{j=1}^{t} x_j n^{ij} \\ &= \sum_{j=1}^{t} x_j \Big(\sum_{i=u}^{u+k-1} n^{ij} \Big) \\ &= \sum_{j=1}^{t} x_j \bar{n}^{kj} , \end{aligned}$$

where the last equality holds with $\bar{n}^{kj} = \sum_{i=u}^{u+k-1} n^{ij} \in M$. For $j \in \{1, \dots, t\}$ set

$$\bar{n}^j = ([\bar{n}^{1j}]_{\mathfrak{a}M}, [\bar{n}^{2j}]_{\mathfrak{a}^2 M}, [\bar{n}^{3j}]_{\mathfrak{a}^3 M}, \dots)$$

and note that $\bar{n}^j$ is in $\Lambda^{\mathfrak{a}}(M)$ since $\bar{n}^{(k+1)j} - \bar{n}^{kj} = n^{(u+k)j} \in \mathfrak{a}^{(u+k)-u} M = \mathfrak{a}^k M$ holds for every $k \geqslant 1$. To finish the proof we show that $m - \lambda_M^{\mathfrak{a}}(m^u) = \sum_{j=1}^{t} x_j \bar{n}^j$ holds; indeed, the right-hand side belongs to $\mathfrak{a}^u \, \Lambda^{\mathfrak{a}}(M)$. For every $l \geqslant 1$ it must be proved that modulo $\mathfrak{a}^l M$ one has the congruence,

$$(\diamond) \qquad m^l - m^u \equiv \sum_{j=1}^{t} x_j \bar{n}^{lj} .$$

For $l \leqslant u$ the left-hand side in $(\diamond)$ is zero modulo $\mathfrak{a}^l M$, and so is the right-hand side as $x_1, \dots, x_t \in \mathfrak{a}^u \subseteq \mathfrak{a}^l$. For $l > u$ write $l = u + k$ where $k \geqslant 1$. By construction, the left-hand side in $(\diamond)$ equals $\sum_{j=1}^{t} x_j \bar{n}^{kj}$, so we must argue that $\sum_{j=1}^{t} x_j \bar{n}^{kj} \equiv \sum_{j=1}^{t} x_j \bar{n}^{lj}$ holds modulo $\mathfrak{a}^l M$. This follows as x_j belongs to $\mathfrak{a}^u$, one has $\bar{n}^{kj} \equiv \bar{n}^{lj}$ modulo $\mathfrak{a}^k M$, and $u + k = l$. □

The powers of an ideal $\mathfrak{a}$ in R form a basis of open neighbourhoods of 0 in the $\mathfrak{a}$-adic topology in R. As said above, we do not pursue this topological point of view in our treatment of $\mathfrak{a}$-completion, but it does shed light on the next definition.

11.1.15 Definition. Ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R are called *topologically equivalent* if there exist natural numbers m and n such that $\mathfrak{a}^m \subseteq \mathfrak{b}$ and $\mathfrak{b}^n \subseteq \mathfrak{a}$ hold.

11.1.16 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. If $\mathfrak{a}$ and $\mathfrak{b}$ are topologically equivalent, then $\Lambda^{\mathfrak{a}}$ and $\Lambda^{\mathfrak{b}}$ are naturally isomorphic endofunctors on $\mathcal{C}(R)$.*

Proof. By assumption, there exist natural numbers m and n with $\mathfrak{a}^m \subseteq \mathfrak{b}$ and $\mathfrak{b}^n \subseteq \mathfrak{a}$. We may assume that $m > 1$. The induced descending chain of ideals,

$$\mathfrak{a} \supseteq \mathfrak{b}^n \supseteq \mathfrak{a}^{mn} \supseteq \mathfrak{b}^{mn^2} \supseteq \mathfrak{a}^{m^2 n^2} \supseteq \mathfrak{b}^{m^2 n^3} \supseteq \cdots$$

yields for every R-complex M a tower,

$$\cdots \twoheadrightarrow R/\mathfrak{b}^{mn^2} \otimes_R M \twoheadrightarrow R/\mathfrak{a}^{mn} \otimes_R M \twoheadrightarrow R/\mathfrak{b}^n \otimes_R M \twoheadrightarrow R/\mathfrak{a} \otimes_R M \,.$$

The limit of this tower agrees with the limits of the towers induced by the morphisms

$$\cdots \twoheadrightarrow R/\mathfrak{a}^{(mn)^2} \twoheadrightarrow R/\mathfrak{a}^{mn} \twoheadrightarrow R/\mathfrak{a}$$

and

$$\cdots \twoheadrightarrow R/\mathfrak{b}^{(mn)^2 n} \twoheadrightarrow R/\mathfrak{b}^{(mn)n} \twoheadrightarrow R/\mathfrak{b}^n \,;$$

this follows from 3.4.15. Thus, with $p = mn > 1$ one has

$$\lim_{u\geqslant 0} (R/\mathfrak{a}^{p^u} \otimes_R M) \;\cong\; \lim_{u\geqslant 0} (R/\mathfrak{b}^{np^u} \otimes_R M) \,.$$

As p is strictly greater that 1, another application of 3.4.15 and the definition, 11.1.4, of the completion functor yield:

$$\Lambda^{\mathfrak{a}}(M) \;\cong\; \lim_{u\geqslant 0} (R/\mathfrak{a}^{p^u} \otimes_R M) \qquad \text{and} \qquad \lim_{u\geqslant 0} (R/\mathfrak{b}^{np^u} \otimes_R M) \;\cong\; \Lambda^{\mathfrak{b}}(M) \,.$$

As the tensor product is a functor, see 2.4.9, all three isomorphisms above are natural in M, cf. 3.4.10, so this proves the desired natural isomorphism of functors. □

11.1.17 Proposition. *Let $\boldsymbol{x}$ be a sequence in R and M an R-complex. The map*

$$\lim_{u\geqslant 1} (\xi^u_{\boldsymbol{x}} \otimes_R M) : \lim_{u\geqslant 1} (R/(\boldsymbol{x}^u) \otimes_R M) \longrightarrow \lim_{u\geqslant 1} (R/(\boldsymbol{x})^u \otimes_R M)$$

induced by (11.1.2.3) *and* 3.5.2 *is an isomorphism. In particular, one has*

$$\Lambda^{(\boldsymbol{x})}(M) \;\cong\; \lim_{u\geqslant 1} (R/(\boldsymbol{x}^u) \otimes_R M) \,.$$

Proof. For every $u \geqslant 1$ the morphism $\xi^u_{\boldsymbol{x}}$ from (11.1.2.3) is surjective, and hence so is $\xi^u_{\boldsymbol{x}} \otimes_R M$ by 2.4.9. It follows that there is a short exact sequence,

$$0 \longrightarrow \operatorname{Ker}(\xi^u_{\boldsymbol{x}} \otimes_R M) \xrightarrow{\iota^u} R/(\boldsymbol{x}^u) \otimes_R M \xrightarrow{\xi^u_{\boldsymbol{x}} \otimes M} R/(\boldsymbol{x})^u \otimes_R M \longrightarrow 0 \,,$$

where ι^u is the embedding. Recall from 11.1.2 that the families $\{R/(\boldsymbol{x}^u) \otimes_R M\}_{u\geqslant 1}$ and $\{R/(\boldsymbol{x})^u \otimes_R M\}_{u\geqslant 1}$ constitute towers, whose morphisms are $\{\vartheta^u_{\boldsymbol{x}} \otimes_R M\}_{u>1}$ and $\{\vartheta^u_{(\boldsymbol{x})} \otimes_R M\}_{u>1}$, and that $\{\xi^u_{\boldsymbol{x}} \otimes_R M\}_{u\geqslant 1}$ is a morphism of these towers. Write χ^u for the morphisms on kernels induced by $\vartheta^u_{\boldsymbol{x}} \otimes_R M$; they form a tower and $\{\iota^u\}_{u\geqslant 1}$ is a morphism of towers.

Let $u \geqslant 1$ be given. Let n be the number of elements in $\boldsymbol{x}$ and set $v = nu$. One has $(\boldsymbol{x})^v \subseteq (\boldsymbol{x}^u)$, so the canonical map $R/(\boldsymbol{x}^v) \twoheadrightarrow R/(\boldsymbol{x}^u)$ admits a factorization $R/(\boldsymbol{x}^v) \twoheadrightarrow R/(\boldsymbol{x})^v \overset{\pi}{\twoheadrightarrow} R/(\boldsymbol{x}^u)$. In the notation above this means $\vartheta^{u+1}_{\boldsymbol{x}} \cdots \vartheta^v_{\boldsymbol{x}} = \pi\xi^v_{\boldsymbol{x}}$. Consequently, one has

$$\iota^u \chi^{u+1} \cdots \chi^v \;=\; (\vartheta^{u+1}_{\boldsymbol{x}} \otimes_R M) \cdots (\vartheta^v_{\boldsymbol{x}} \otimes_R M)\iota^v \;=\; (\pi \otimes_R M)(\xi^v_{\boldsymbol{x}} \otimes_R M)\iota^v \;=\; 0 \,,$$

and hence $\chi^{u+1} \cdots \chi^v = 0$ as ι^u is injective. This shows that the tower of kernels satisfies the trivial Mittag-Leffler Condition; see 3.5.9. Now it follows from 3.5.13 and 3.5.17 that the map $\lim_{u\geqslant 1}(\xi^u_{\boldsymbol{x}} \otimes_R M)$ is an isomorphism. □

COMPLETION OF A RING

Given an ideal $\mathfrak{a}$ in R, the $\mathfrak{a}$-completion of R is itself a ring and it inherits key properties from R.

11.1.18. Let $\mathfrak{a}$ be an ideal in R. A priori, $\Lambda^{\mathfrak{a}}(R)$ is an R-module, however, it is elementary to check that it is a commutative ring with multiplication given by

$$([r^u]_{\mathfrak{a}^u})_{u\geqslant 1}([s^u]_{\mathfrak{a}^u})_{u\geqslant 1} \;=\; ([r^u s^u]_{\mathfrak{a}^u})_{u\geqslant 1}\;,$$

cf. 11.1.10. That is, $\Lambda^{\mathfrak{a}}(R)$ is a subring of the product ring $\prod_{u\geqslant 1} R/\mathfrak{a}^u$. Further, $\Lambda^{\mathfrak{a}}(R)$ is an R-algebra with structure map $\lambda^{\mathfrak{a}}_R$, see 11.1.4.

11.1.19 Definition. Let $\mathfrak{a}$ be an ideal in R. Considered as a ring, the $\mathfrak{a}$-completion $\Lambda^{\mathfrak{a}}(R)$ is denoted $\widehat{R}^{\mathfrak{a}}$.

The next result shows that every $\mathfrak{a}$-complete R-complex is an $\widehat{R}^{\mathfrak{a}}$-complex.

11.1.20 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The complex $\Lambda^{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex with action given by*

$$rm \;=\; ([r^u m^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$$

for elements $r = ([r^u]_{\mathfrak{a}^u})_{u\geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ and $m = ([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} \in \Lambda^{\mathfrak{a}}(M)$. Consequently, $\Lambda^{\mathfrak{a}}$ can be viewed as a functor,

$$\Lambda^{\mathfrak{a}} : \mathcal{C}(R) \longrightarrow \mathcal{C}(\widehat{R}^{\mathfrak{a}})\;.$$

PROOF. We verify that the asserted $\widehat{R}^{\mathfrak{a}}$-action is well-defined; it is then straightforward to see that it makes $\Lambda^{\mathfrak{a}}(M)$ into an $\widehat{R}^{\mathfrak{a}}$-complex. If there are equalities $([r^u]_{\mathfrak{a}^u})_{u\geqslant 1} = ([s^u]_{\mathfrak{a}^u})_{u\geqslant 1}$ in $\widehat{R}^{\mathfrak{a}}$ and $([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} = ([n^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$ in $\Lambda^{\mathfrak{a}}(M)$, then the element $r^u m^u - s^u n^u = (r^u - s^u)m^u + s^u(m^u - n^u)$ belongs to $\mathfrak{a}^u M$ for each $u \geqslant 1$. Hence $([r^u m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} = ([s^u n^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$ holds in $\Lambda^{\mathfrak{a}}(M)$. $\square$

11.1.21. Let $\mathfrak{a}$ be an ideal in R and M an $\widehat{R}^{\mathfrak{a}}$-complex. By 11.1.6 and 11.1.20 the R-complex $\Lambda^{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex in two, potentially different, ways. We argue below that they are the same if $\mathfrak{a}$ is finitely generated; that is, the diagram

$$\begin{tikzcd}
\mathcal{C}(\widehat{R}^{\mathfrak{a}}) \arrow[r, "\Lambda^{\mathfrak{a}}", "\text{(from 11.1.6)}"'] \arrow[d, "\operatorname{res}_R^{\widehat{R}^{\mathfrak{a}}}"'] & \mathcal{C}(\widehat{R}^{\mathfrak{a}}) \arrow[d, equal] \\
\mathcal{C}(R) \arrow[r, "\Lambda^{\mathfrak{a}}", "\text{(from 11.1.20)}"'] & \mathcal{C}(\widehat{R}^{\mathfrak{a}})
\end{tikzcd}$$

is commutative if $\mathfrak{a}$ is finitely generated. Indeed, for elements

$$r \;=\; ([r^u]_{\mathfrak{a}^u})_{u\geqslant 1} \in \widehat{R}^{\mathfrak{a}} \qquad\text{and}\qquad m \;=\; ([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} \in \Lambda^{\mathfrak{a}}(M)$$

the $\widehat{R}^{\mathfrak{a}}$-actions from 11.1.6 and 11.1.20 are given by

$$rm \;=\; ([rm^u]_{\mathfrak{a}^u M})_{u\geqslant 1} \quad\text{and}\quad rm \;=\; ([r^u m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} \;=\; ([\lambda^{\mathfrak{a}}_R(r^u) m^u]_{\mathfrak{a}^u M})_{u\geqslant 1}\;.$$

The last equality holds as M is an R-complex via restriction of scalars along the ring homomorphism $\lambda_R^{\mathfrak{a}} \colon R \to \widehat{R}^{\mathfrak{a}}$. Similarly, with $\mathfrak{b}$ denoting the extension $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$ one has $\mathfrak{a}^u M = \mathfrak{b}^u M$. Thus, to prove that the two $\widehat{R}^{\mathfrak{a}}$-structures on $\Lambda^{\mathfrak{a}}(M)$ agree, it suffices to argue that the element $r - \lambda_R^{\mathfrak{a}}(r^u)$ for every $u \geqslant 1$ belongs to the ideal $\mathfrak{b}^u = \mathfrak{a}^u \widehat{R}^{\mathfrak{a}}$, but this follows from 11.1.14 provided that $\mathfrak{a}$ is finitely generated.

To a commutative algebraist the next statement may appear awkward, as it is common practice in that field to include the Noetherian condition in the definition of a local ring. That is not the convention in this book, but in Part III all rings are Noetherian and this point conveniently becomes moot.

11.1.22 Proposition. *Let $\mathfrak{a}$ be a finitely generated proper ideal in R.*

(a) *If R is Noetherian, then $\widehat{R}^{\mathfrak{a}}$ is Noetherian.*
(b) *If R is local, then $\widehat{R}^{\mathfrak{a}}$ is local.*

Proof. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence that generates $\mathfrak{a}$. Consider the power series algebra $R[\![X_1, \dots, X_n]\!]$. The assignments $X_i \mapsto x_i$ define a family of R-algebra homomorphisms $R[\![X_1, \dots, X_n]\!] \to R/\mathfrak{a}^u$ that are compatible with the homomorphisms $R/\mathfrak{a}^{u+1} \twoheadrightarrow R/\mathfrak{a}^u$ in the tower that defines $\widehat{R}^{\mathfrak{a}}$. By 3.4.5 they determine a unique homomorphism $\alpha \colon R[\![X_1, \dots, X_n]\!] \to \widehat{R}^{\mathfrak{a}}$ of R-modules given by $\alpha(p) = ([p(x_1, \dots, x_n)]_{\mathfrak{a}^u})_{u \geqslant 1}$ for $p \in R[\![X_1, \dots, X_n]\!]$, and it is elementary to verify that it is a homomorphism of R-algebras. To see that it is surjective, let $([r^u]_{\mathfrak{a}^u})_{u \geqslant 1}$ be an element in $\widehat{R}^{\mathfrak{a}}$. For every $u \geqslant 1$ the element $r^{u+1} - r^u$ belongs to $\mathfrak{a}^u$, see 11.1.10, so there is a homogeneous polynomial p^u of degree u in the variables $X_1, \dots, X_n$ with $p^u(x_1, \dots, x_n) = r^u - r^{u+1}$. Set $p^0 = r^1 \in R$ and $p = \sum_{u \geqslant 0} p^u$, which is a well-defined element of $R[\![X_1, \dots, X_n]\!]$. For every $u \geqslant 1$ one now has

$$\begin{aligned}
[p(x_1, \dots, x_n)]_{\mathfrak{a}^u} &= \Big[p^0 + \sum_{i=1}^{u-1} p^i(x_1, \dots, x_n)\Big]_{\mathfrak{a}^u} \\
&= \Big[r^1 + \sum_{i=1}^{u-1} (r^{i+1} - r^i)\Big]_{\mathfrak{a}^u} \\
&= [r^u]_{\mathfrak{a}^u} \,,
\end{aligned}$$

and hence $\alpha(p) = ([r^u]_{\mathfrak{a}^u})_{u \geqslant 1}$. Thus, $\widehat{R}^{\mathfrak{a}}$ is a quotient of $R[\![X_1, \dots, X_n]\!]$.

(a): If R is Noetherian, then so is $R[\![X_1, \dots, X_n]\!]$—this is a version of Hilbert's Basis Theorem—and hence the quotient ring $\widehat{R}^{\mathfrak{a}}$ is Noetherian too.

(b): If R is local with maximal ideal $\mathfrak{m}$, then $R[\![X_1, \dots, X_n]\!]$ is local with maximal ideal generated by $\mathfrak{m}$ and $X_1, \dots, X_n$, whence the quotient ring $\widehat{R}^{\mathfrak{a}}$ is local as well. □

11.1.23 Lemma. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, M an R-complex, and N an S-complex. For $\varphi \in \operatorname{Hom}_R(N, M/\mathfrak{a}M)$, $s \in \mathfrak{a}S$, and $n \in N$ one has $\varphi(sn) = 0$.*

Proof. As s is in $\mathfrak{a}S$ it has the form $s = \sum_{i=1}^k a_i s_i$ where $a_i \in \mathfrak{a}$ and $s_i \in S$ for every $i \in \{1, \dots, k\}$. Thus, one has $\varphi(sn) = \sum_{i=1}^k a_i \varphi(s_i n)$. Each term in this sum is zero as a_i belongs to $\mathfrak{a}$ and $\varphi(s_i n)$ to $M/\mathfrak{a}M$. □

To parse the next result, recall from 11.1.20 that for every ideal $\mathfrak{a}$ in R and every R-complex M there is a canonical $\widehat{R}^{\mathfrak{a}}$-structure on $\Lambda^{\mathfrak{a}}(M)$. Furthermore, recall from 2.3.11 that if N and X are $\widehat{R}^{\mathfrak{a}}$-complexes, then $\operatorname{Hom}_R(N, X)$ is a complex of, not necessarily symmetric, $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules.

11.1.24 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and N an $\widehat{R}^{\mathfrak{a}}$-complex. If $\mathfrak{a}$ is finitely generated, then $\operatorname{Hom}_R(N, \Lambda^{\mathfrak{a}}(M))$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, and there is an equality of $\widehat{R}^{\mathfrak{a}}$-complexes,*

$$\operatorname{Hom}_R(N, \Lambda^{\mathfrak{a}}(M)) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(N, \Lambda^{\mathfrak{a}}(M)) .$$

Proof. It suffices to show that for all elements $r = ([r^u]_{\mathfrak{a}^u})_{u \geqslant 1}$ in $\widehat{R}^{\mathfrak{a}}$, $n \in N$, and $\varphi \in \operatorname{Hom}_R(N, \Lambda^{\mathfrak{a}}(M))$ one has $r\varphi(n) = \varphi(rn)$. By 11.1.10 and 3.4.23 one has

$$\operatorname{Hom}_R(N, \Lambda^{\mathfrak{a}}(M)) = \operatorname{Hom}_R\bigl(N, \lim_{u \geqslant 1} M/\mathfrak{a}^u M\bigr) \cong \lim_{u \geqslant 1} \operatorname{Hom}_R(N, M/\mathfrak{a}^u M) .$$

Consequently, we may identify φ with an element in $\lim_{u \geqslant 1} \operatorname{Hom}_R(N, M/\mathfrak{a}^u M)$, that is, a sequence $(\varphi^u)_{u \geqslant 1}$ with $\varphi^u \in \operatorname{Hom}_R(N, M/\mathfrak{a}^u M)$ such that the composite

$$N \xrightarrow{\varphi^u} M/\mathfrak{a}^u M \twoheadrightarrow M/\mathfrak{a}^{u-1} M$$

is φ^{u-1} for every $u > 1$, see 3.5.5. With this identification, the map $\varphi \colon N \to \Lambda^{\mathfrak{a}}(M)$ is given by $\varphi(n) = (\varphi^u(n))_{u \geqslant 1}$ for $n \in N$. This explains the 1st equality below; the 2nd equality holds by 11.1.20 and the 3rd one holds as each $\varphi^u \colon N \to M/\mathfrak{a}^u M$ is R-linear. The 4th equality holds as N is an R-complex via restriction of scalars along the ring homomorphism $\lambda^{\mathfrak{a}}_R \colon R \to \widehat{R}^{\mathfrak{a}}$.

$$\begin{aligned} r\varphi(n) &= ([r^u]_{\mathfrak{a}^u})_{u \geqslant 1} (\varphi^u(n))_{u \geqslant 1} \\ &= (r^u \varphi^u(n))_{u \geqslant 1} \\ &= (\varphi^u(r^u n))_{u \geqslant 1} \\ &= (\varphi^u(\lambda^{\mathfrak{a}}_R(r^u) n))_{u \geqslant 1} . \end{aligned}$$

As one has $\varphi(rn) = (\varphi^u(rn))_{u \geqslant 1}$, it remains to argue that $\varphi^u(rn) = \varphi^u(\lambda^{\mathfrak{a}}_R(r^u)n)$ holds for each $u \geqslant 1$. To this end notice that $r - \lambda^{\mathfrak{a}}_R(r^u)$ belongs to $\mathfrak{a}^u \widehat{R}^{\mathfrak{a}}$ by 11.1.14. Now 11.1.23 yields

$$\varphi^u(rn) - \varphi^u(\lambda^{\mathfrak{a}}_R(r^u)n) = \varphi^u((r - \lambda^{\mathfrak{a}}_R(r^u))n) = 0 . \qquad \square$$

11.1.25 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If $\mathfrak{a}$ is finitely generated, then $\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M))$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, and there is an isomorphism of $\widehat{R}^{\mathfrak{a}}$-complexes,*

$$\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M)) \xrightarrow{\cong} \Lambda^{\mathfrak{a}}(M) \qquad \text{given by} \qquad \varphi \longmapsto \varphi(1_{\widehat{R}^{\mathfrak{a}}}) ,$$

for $\varphi \in \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \Lambda^{\mathfrak{a}}(M))$.

Proof. The first assertion follows from 11.1.24, applied with $N = \widehat{R}^{\mathfrak{a}}$, and so does the second in view of the counitor 4.4.2. $\square$

11.1.26 Proposition. *Let S be an R-algebra and N an S-module. Homothety formation,*

$$\chi^N_{SR} : S \longrightarrow \operatorname{Hom}_R(N, N)\,,$$

from 4.5.5 *is a morphism of R-algebras.*

Proof. It is known from 4.5.5 that χ^N_{SR} is a map of S–S-bimodules; in particular, it is R-linear. For elements $s, s' \in S$ and $n \in N$ one has

$$\chi^N_{SR}(ss')(n) \;=\; (ss')n \;=\; s(s'n) \;=\; \chi^N_{SR}(s)(s'n) \;=\; (\chi^N_{SR}(s) \circ \chi^N_{SR}(s'))(n)\,,$$

so the map χ^N_{SR} also preserves multiplication. □

11.1.27 Proposition. *Let $\mathfrak{a}$ be an ideal in R. If $\mathfrak{a}$ is finitely generated, then homothety formation* 4.5.5 *is an isomorphism of R-algebras,*

$$\chi^{\widehat{R}^{\mathfrak{a}}}_{\widehat{R}^{\mathfrak{a}}R} : \widehat{R}^{\mathfrak{a}} \xrightarrow{\;\cong\;} \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})\,.$$

Proof. By 11.1.24 one has $\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}}) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})$, so the homothety formation map in question is identical to the map

$$\chi^{\widehat{R}^{\mathfrak{a}}}_{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \longrightarrow \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})$$

from 10.1.1, which is an isomorphism of R-algebras by 11.1.26. □

Density and Failure of Half-Exactness

By the next proposition, the $\mathfrak{a}$-completion functor preserves surjective morphisms. In general, however, it is not right exact; in fact, it is not even half exact, see 11.1.32.

11.1.28 Proposition. *Let $\mathfrak{a}$ be an ideal in R and $\alpha\colon M \to N$ a morphism of R-complexes. The following conditions are equivalent.*

(i) $N = \mathfrak{a}N + \operatorname{Im}\alpha\,.$

(ii) $N = \mathfrak{a}^uN + \operatorname{Im}\alpha$ *and* $\mathfrak{a}^uN = \mathfrak{a}^{u+1}N + \alpha(\mathfrak{a}^uM)$ *hold for every* $u \geqslant 1\,.$

(iii) *The sequence*

$$0 \longrightarrow \lim_{u\geqslant 1} \operatorname{Ker}\alpha^u \longrightarrow \Lambda^{\mathfrak{a}}(M) \xrightarrow{\Lambda^{\mathfrak{a}}(\alpha)} \Lambda^{\mathfrak{a}}(N) \longrightarrow 0$$

is exact; *here* $\alpha^u : M/\mathfrak{a}^uM \to N/\mathfrak{a}^uN$ *is the morphism induced by* $\alpha\,.$

(iv) $\Lambda^{\mathfrak{a}}(\alpha)$ *is surjective.*

In particular, the functor $\Lambda^{\mathfrak{a}}$ preserves surjective morphisms.

Proof. (i) ⟹ (ii): The second equality holds as (i) yields

$$\mathfrak{a}^uN \;=\; \mathfrak{a}^u(\mathfrak{a}N + \alpha(M)) \;=\; \mathfrak{a}^{u+1}N + \mathfrak{a}^u\,\alpha(M) \;=\; \mathfrak{a}^{u+1}N + \alpha(\mathfrak{a}^uM)\,.$$

The first equality is proved by induction on u. Assume that $N = \mathfrak{a}^uN + \operatorname{Im}\alpha$ holds for some $u \geqslant 1$. The already established equality now yields

$$N \;=\; \mathfrak{a}^uN + \operatorname{Im}\alpha \;=\; (\mathfrak{a}^{u+1}N + \alpha(\mathfrak{a}^uM)) + \operatorname{Im}\alpha \;=\; \mathfrak{a}^{u+1}N + \operatorname{Im}\alpha\,.$$

(*ii*) ⟹ (*iii*): The first equality in (*ii*) implies that α^u is surjective, so the sequence

$$0 \longrightarrow \operatorname{Ker}\alpha^u \longrightarrow M/\mathfrak{a}^u M \xrightarrow{\alpha^u} N/\mathfrak{a}^u N \longrightarrow 0$$

is exact. Furthermore, the induced morphism $\operatorname{Ker}\alpha^{u+1} \to \operatorname{Ker}\alpha^u$ is surjective. Indeed, for $[x]_{\mathfrak{a}^u M}$ in $\operatorname{Ker}\alpha^u$ one has $\alpha(x) \in \mathfrak{a}^u N$. By the second equality in (*ii*) one has $\alpha(x) = y + \alpha(z)$ for some $y \in \mathfrak{a}^{u+1}N$ and $z \in \mathfrak{a}^u M$. As $\alpha(x-z) = y \in \mathfrak{a}^{u+1}N$ holds, the element $[x-z]_{\mathfrak{a}^{u+1}M}$ belongs to $\operatorname{Ker}\alpha^{u+1}$, and the map $\operatorname{Ker}\alpha^{u+1} \to \operatorname{Ker}\alpha^u$ sends $[x-z]_{\mathfrak{a}^{u+1}M}$ to $[x-z]_{\mathfrak{a}^u M} = [x]_{\mathfrak{a}^u M}$. Now 3.5.17 and 3.5.10 yield the asserted exact sequence.

(*iii*) ⟹ (*iv*): The exact sequence shows, in particular, that $\Lambda^{\mathfrak{a}}(\alpha)$ is surjective.

(*iv*) ⟹ (*i*): Let n belong to N. Consider the element $y = ([n]_{\mathfrak{a}N}, [n]_{\mathfrak{a}^2 N}, \dots)$ in $\Lambda^{\mathfrak{a}}(N)$; see 11.1.10. As $\Lambda^{\mathfrak{a}}(\alpha)$ is surjective there exists $x = ([m_1]_{\mathfrak{a}M}, [m_2]_{\mathfrak{a}^2 M}, \dots)$ in $\Lambda^{\mathfrak{a}}(M)$ with $\Lambda^{\mathfrak{a}}(\alpha)(x) = y$. In particular, $[\alpha(m_1)]_{\mathfrak{a}N} = [n]_{\mathfrak{a}N}$ holds, so $n - \alpha(m_1)$ is in $\mathfrak{a}N$ and, thus, n belongs to $\mathfrak{a}N + \operatorname{Im}\alpha$.

For the final assertion, note that if α is surjective, then (*i*) holds. □

11.1.29 Definition. Let $\mathfrak{a}$ be an ideal in R. A morphism α in $\mathcal{C}(R)$ that satisfies the equivalent conditions in 11.1.28 is said to be $\mathfrak{a}$-*dense*.

11.1.30 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. One has $\Lambda^{\mathfrak{a}}(M) = 0$ if and only if $R/\mathfrak{a} \otimes_R M = 0$ holds.*

Proof. Consider the zero homomorphism $\zeta\colon 0 \to M$. The condition $\Lambda^{\mathfrak{a}}(M) = 0$ is equivalent to $\Lambda^{\mathfrak{a}}(\zeta)$ being surjective, which by 11.1.28 is equivalent to the identity $M = \mathfrak{a}M + \operatorname{Im}\zeta = \mathfrak{a}M$. The desired conclusion now follows from 1.1.10. □

To demonstrate that the functor $\Lambda^{\mathfrak{a}}$ is not half exact, we need some preparations.

11.1.31. Let $\mathfrak{a}$ be an ideal in R and $0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\alpha} N \longrightarrow 0$ an exact sequence of R-complexes. Consider for every $u \geqslant 1$ the commutative diagram,

$$\begin{tikzcd} & K/\mathfrak{a}^u K \arrow[d, two heads, "\pi^u"'] \arrow[dr, "\iota^u"] & & & \\ 0 \arrow[r] & \operatorname{Ker}\alpha^u \arrow[r] & M/\mathfrak{a}^u M \arrow[r, "\alpha^u"] & N/\mathfrak{a}^u N \arrow[r] & 0\,, \end{tikzcd}$$

where ι^u and α^u are the morphisms induced by ι and α. As $\alpha^u\iota^u = 0$ holds, the morphism ι^u factors uniquely through $\operatorname{Ker}\alpha^u$, and that defines the morphism π^u. As $\operatorname{Ker}\alpha^u = \operatorname{Im}\iota^u$ holds, see 1.1.10 and 2.4.9, the morphism π^u is surjective.

Notice that the complex $\operatorname{Ker}\alpha^u$ may be identified with $K/\iota^{-1}(\mathfrak{a}^u M)$, and with this identification the morphisms $\pi^u\colon K/\mathfrak{a}^u K \to \operatorname{Ker}\alpha^u$ and $\operatorname{Ker}\alpha^u \to M/\mathfrak{a}^u M$ are given by $[x]_{\mathfrak{a}^u K} \mapsto [x]_{\iota^{-1}(\mathfrak{a}^u M)}$ and $[x]_{\iota^{-1}(\mathfrak{a}^u M)} \mapsto [\iota(x)]_{\mathfrak{a}^u M}$ respectively.

Passing to limits one gets, with $\pi = \lim_{u\geqslant 1}\pi^u$, the commutative diagram,

$$\begin{tikzcd} \Lambda^{\mathfrak{a}}(K) \arrow[d, "\pi"'] \arrow[dr, "\Lambda^{\mathfrak{a}}(\iota)"] & & & \\ 0 \arrow[r] & \lim_{u\geqslant 1} \operatorname{Ker}\alpha^u \arrow[r] & \Lambda^{\mathfrak{a}}(M) \arrow[r, "\Lambda^{\mathfrak{a}}(\alpha)"] & \Lambda^{\mathfrak{a}}(N) \arrow[r] & 0\,. \end{tikzcd}$$

The bottom row is exact by 11.1.28. It follows that

$$\Lambda^{\mathfrak{a}}(K) \xrightarrow{\Lambda^{\mathfrak{a}}(\iota)} \Lambda^{\mathfrak{a}}(M) \xrightarrow{\Lambda^{\mathfrak{a}}(\alpha)} \Lambda^{\mathfrak{a}}(N)$$

is an exact sequence if and only if π is surjective.

11.1.32 Example. Let $a > 1$ be an integer and set $\mathfrak{a} = a\mathbb{Z}$. For $n \geqslant 1$ consider the exact sequence

$$0 \longrightarrow \mathbb{Z}/\mathfrak{a} \xrightarrow{a^{n-1}} \mathbb{Z}/\mathfrak{a}^n \longrightarrow \mathbb{Z}/\mathfrak{a}^{n-1} \longrightarrow 0$$

of $\mathbb{Z}$-modules. Forming coproducts yields the exact sequence,

$$0 \longrightarrow \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a} \xrightarrow{\coprod_{n\geqslant 1} a^{n-1}} \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a}^n \longrightarrow \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a}^{n-1} \longrightarrow 0\,.$$

Write $0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\alpha} N \longrightarrow 0$ for this sequence. To show that the functor $\Lambda^{\mathfrak{a}}$ is not half exact we argue that the map $\pi = \lim_{u\geqslant 1} \pi^u$ from 11.1.31 is not surjective.

Fix $u \geqslant 1$. As $\mathfrak{a}K = 0$ it follows that $\mathfrak{a}^u K = 0$ and hence $K/\mathfrak{a}^u K \cong K$. Next note that $\mathfrak{a}^u M$ is the submodule

$$0 \oplus \cdots \oplus 0 \oplus \mathfrak{a}^u/\mathfrak{a}^{u+1} \oplus \mathfrak{a}^u/\mathfrak{a}^{u+2} \oplus \cdots\,,$$

with u leading zeros, of $M = \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a}^n$. It follows that $\iota^{-1}(\mathfrak{a}^u M)$ is the submodule $0 \oplus \cdots \oplus 0 \oplus \mathbb{Z}/\mathfrak{a} \oplus \mathbb{Z}/\mathfrak{a} \oplus \cdots$, with u leading zeros, of $K = \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a}$ and, consequently, one has $K/\iota^{-1}(\mathfrak{a}^u M) \cong (\mathbb{Z}/\mathfrak{a})^u$. Via this isomorphism and the previously established isomorphism $K/\mathfrak{a}^u K \cong K$, the homomorphisms

$$K/\mathfrak{a}^u K \twoheadrightarrow K/\mathfrak{a}^{u-1}K \qquad \text{and} \qquad K/\iota^{-1}(\mathfrak{a}^u M) \twoheadrightarrow K/\iota^{-1}(\mathfrak{a}^{u-1} M)$$

in the towers in 11.1.31 are given by the identity $1^K\colon K \to K$ and the surjection $(\mathbb{Z}/\mathfrak{a})^u \twoheadrightarrow (\mathbb{Z}/\mathfrak{a})^{u-1}$ that maps $(x_1,\dots,x_{u-1},x_u)$ to $(x_1,\dots,x_{u-1})$. Furthermore, the map $\pi^u\colon K/\mathfrak{a}^u K \twoheadrightarrow K/\iota^{-1}(\mathfrak{a}^u M)$ is identified with $K \twoheadrightarrow (\mathbb{Z}/\mathfrak{a})^u$ given by $(x_n)_{n\geqslant 1} \mapsto (x_1,\dots,x_u)$. In view of 3.5.7 it is now elementary to verify that $\pi = \lim_{u\geqslant 1} \pi^u$ is the canonical embedding $K = \coprod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a} \rightarrowtail \prod_{n\geqslant 1} \mathbb{Z}/\mathfrak{a}$, which is not surjective as $a > 1$ holds.

Complete Complexes

Recall from 11.1.8 the notions of $\mathfrak{a}$-seperated/quasi-complete/complete complexes.

11.1.33 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R.

(a) *Let M and N be R-complexes. The direct sum $M \oplus N$ is $\mathfrak{a}$-separated, $\mathfrak{a}$-quasi-complete, or $\mathfrak{a}$-complete if and only if both M and N have said property.*
(b) *Assume that the ideal $\mathfrak{a}$ is finitely generated and let $\{M^u\}_{u\in U}$ be a family of R-complexes. The product $\prod_{u\in U} M^u$ is $\mathfrak{a}$-separated, $\mathfrak{a}$-quasi-complete, or $\mathfrak{a}$-complete if and only if each complex M^u has said property.*

Proof. The functor $\Lambda^{\mathfrak{a}}$ is additive, and if $\mathfrak{a}$ is finitely generated, it preserves products by 11.1.13. Thus, for the direct sum $M \oplus N$ in (a) one has $\lambda^{\mathfrak{a}}_{M\oplus N} \cong \lambda^{\mathfrak{a}}_M \oplus \lambda^{\mathfrak{a}}_N$ and for the product $P = \prod_{u\in U} M^u$ in (b) one has $\lambda^{\mathfrak{a}}_P \cong \prod_{u\in U} \lambda^{\mathfrak{a}}_{M^u}$ by 3.1.20. The conslusions now follow immediately from the definitions in 11.1.8. □

11.1.34 Example. By 11.1.11 the modules K, M, and N in 11.1.32 are coproducts of $\mathfrak{a}$-complete modules. As the sequence $0 \to \Lambda^{\mathfrak{a}}(K) \to \Lambda^{\mathfrak{a}}(M) \to \Lambda^{\mathfrak{a}}(N) \to 0$ is not exact, it follows in particular that $\Lambda^{\mathfrak{a}}$ does not preserve coproducts.

The homology of an $\mathfrak{a}$-complete complex need not be $\mathfrak{a}$-complete; it is, though, $\mathfrak{a}$-quasi-complete. This is proved in 11.1.41; the proof requires a few preliminary results on $\mathfrak{a}$-density and the notions from 11.1.8. Examples of $\mathfrak{a}$-complete complexes with not $\mathfrak{a}$-complete homology are discussed in the Remark after 13.1.34.

11.1.35 Lemma. *Let $\mathfrak{a}$ be an ideal in R and $\alpha\colon M \to N$ and $\beta\colon N \to L$ morphisms of R-complexes. If α is $\mathfrak{a}$-dense, L is $\mathfrak{a}$-separated, and $\beta\alpha = 0$ holds, then $\beta = 0$.*

Proof. As $\beta\alpha = 0$ holds one has $0 = \Lambda^{\mathfrak{a}}(\beta\alpha) = \Lambda^{\mathfrak{a}}(\beta)\,\Lambda^{\mathfrak{a}}(\alpha)$. By assumption, $\Lambda^{\mathfrak{a}}(\alpha)$ is surjective, see 11.1.29, wence $\Lambda^{\mathfrak{a}}(\beta) = 0$ holds. Consequently, one has $\lambda^{\mathfrak{a}}_L\beta = \Lambda^{\mathfrak{a}}(\beta)\lambda^{\mathfrak{a}}_N = 0$, and since $\lambda^{\mathfrak{a}}_L$ is assumed to be injective, $\beta = 0$ holds. □

11.1.36 Lemma. *Let $\mathfrak{a}$ be an ideal in R and $\alpha\colon M \to N$ a morphism of R-complexes.*
(a) *If α is injective and N is $\mathfrak{a}$-separated, then M is $\mathfrak{a}$-separated.*
(b) *If α is $\mathfrak{a}$-dense and M is $\mathfrak{a}$-quasi-complete, then N is $\mathfrak{a}$-quasi-complete.*

Proof. (a): If α and $\lambda^{\mathfrak{a}}_N$ are injective, then so is the composite $\lambda^{\mathfrak{a}}_N\alpha = \Lambda^{\mathfrak{a}}(\alpha)\lambda^{\mathfrak{a}}_M$, whence $\lambda^{\mathfrak{a}}_M$ is injective.

(b): If α is $\mathfrak{a}$-dense, then $\Lambda^{\mathfrak{a}}(\alpha)$ is surjective, see 11.1.29. Thus, if $\lambda^{\mathfrak{a}}_M$ is surjective, then so is the composite $\Lambda^{\mathfrak{a}}(\alpha)\lambda^{\mathfrak{a}}_M = \lambda^{\mathfrak{a}}_N\alpha$, which implies that $\lambda^{\mathfrak{a}}_N$ is surjective. □

11.1.37 Proposition. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-complex.*
(a) *The morphism*
$$\lambda^u : M/\mathfrak{a}^u M \longrightarrow \Lambda^{\mathfrak{a}}(M)/\mathfrak{a}^u\,\Lambda^{\mathfrak{a}}(M)$$
induced by $\lambda^{\mathfrak{a}}_M : M \to \Lambda^{\mathfrak{a}}(M)$ is an isomorphism for every $u \geqslant 1$.
(b) *The morphism $\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M)$ is an isomorphism, in particular, $\lambda^{\mathfrak{a}}_M$ is $\mathfrak{a}$-dense.*

Proof. By definition one has $\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) = \lim_{u\geqslant 1} \lambda^u$, so part (b) follows from (a).

Every element in $\mathfrak{a}^u\,\Lambda^{\mathfrak{a}}(M)$ leads per 11.1.10 with u zeroes, i.e. has the form
$$([0]_{\mathfrak{a}M}, \ldots, [0]_{\mathfrak{a}^u M}, [m^{u+1}]_{\mathfrak{a}^{u+1}M}, [m^{u+2}]_{\mathfrak{a}^{u+2}M}, \ldots)\,.$$
Thus, if $[m]_{\mathfrak{a}^u M}$ is in $\operatorname{Ker}\lambda^u$, then $\lambda^{\mathfrak{a}}_M(m) = ([m]_{\mathfrak{a}M}, [m]_{\mathfrak{a}^2 M}, \ldots)$ is in $\mathfrak{a}^u\,\Lambda^{\mathfrak{a}}(M)$, so in particular $[m]_{\mathfrak{a}^u M} = 0$ holds. Therefore, λ^u is injective.

To see that λ^u is surjective, let $m = ([m^u]_{\mathfrak{a}^u M})_{u \geqslant 1}$ be any element $\Lambda^{\mathfrak{a}}(M)$. We claim that λ^u maps $[m^u]_{\mathfrak{a}^u M}$ to the element $[m]_{\mathfrak{a}^u \Lambda^{\mathfrak{a}}(M)}$. This amounts to showing that the difference $m - \lambda^{\mathfrak{a}}_M(m^u)$ is in $\mathfrak{a}^u \, \Lambda^{\mathfrak{a}}(M)$, which it is by 11.1.14. □

If the ideal $\mathfrak{a}$t is finitely generated, then the completion functor $\Lambda^{\mathfrak{a}}$ is idempotent.

11.1.38 Theorem. *Let $\mathfrak{a}$ be an finitely generated ideal in R and M an R-complex. There is an equality $\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) = \lambda^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}}(M)}$ of morphisms $\Lambda^{\mathfrak{a}}(M) \to \Lambda^{\mathfrak{a}}(\Lambda^{\mathfrak{a}}(M))$, and this map is an isomorphism. In particular, the R-complex $\Lambda^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-complete.*

Proof. Once the equality $\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) = \lambda^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}}(M)}$ has been established, the remaining assertions follow from 11.1.37 and the definition of $\mathfrak{a}$-completeness, see 11.1.8.

To prove the desired equality of morphisms, consider the maps from 11.1.10:

$$\begin{tikzcd} M \arrow[d, "\lambda^{\mathfrak{a}}_M"'] \arrow[dr, two heads, "\pi^u_M"] & \\ \Lambda^{\mathfrak{a}}(M) \arrow[r, "\nu^u_M"'] & M/\mathfrak{a}^u M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Lambda^{\mathfrak{a}}(M) \arrow[d, "\lambda^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}}(M)}"'] \arrow[dr, two heads, "\pi^u_{\Lambda^{\mathfrak{a}}(M)}"] & \\ \Lambda^{\mathfrak{a}}(\Lambda^{\mathfrak{a}}(M)) \arrow[r, "\nu^u_{\Lambda^{\mathfrak{a}}(M)}"'] & \Lambda^{\mathfrak{a}}(M)/\mathfrak{a}^u \, \Lambda^{\mathfrak{a}}(M) \end{tikzcd} .$$

Since $\lambda^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}}(M)}$ is uniquely determined by commutativity of the rightmost diagram for every $u \geqslant 1$, the equality $\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) = \lambda^{\mathfrak{a}}_{\Lambda^{\mathfrak{a}}(M)}$ follows once we establish the identity

$$(\star) \qquad \nu^u_{\Lambda^{\mathfrak{a}}(M)} \Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) = \pi^u_{\Lambda^{\mathfrak{a}}(M)} \, .$$

Every morphism $\gamma \colon M \to L$ induces a morphism $\gamma^u \colon M/\mathfrak{a}^u M \to L/\mathfrak{a}^u L$, and by definition of the functor $\Lambda^{\mathfrak{a}}$ one has $\nu^u_L \, \Lambda^{\mathfrak{a}}(\gamma) = \gamma^u \nu^u_M$. Applied to the morphism $\lambda = \lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ this shows that ($\star$) can be written as:

$$(\diamond) \qquad \lambda^u \nu^u_M = \pi^u_{\Lambda^{\mathfrak{a}}(M)} \, .$$

The morphism $\lambda^u \colon M/\mathfrak{a}^u M \to \Lambda^{\mathfrak{a}}(M)/\mathfrak{a}^u \, \Lambda^{\mathfrak{a}}(M)$ induced by $\lambda = \lambda^{\mathfrak{a}}_M$ satisfies by definition the second equality below, and the first equality holds by commutativity of the leftmost diagram above:

$$(\flat) \qquad \lambda^u \nu^u_M \lambda^{\mathfrak{a}}_M = \lambda^u \pi^u_M = \pi^u_{\Lambda^{\mathfrak{a}}(M)} \lambda^{\mathfrak{a}}_M \, .$$

As the module $\Lambda^{\mathfrak{a}}(M)/\mathfrak{a}^u \, \Lambda^{\mathfrak{a}}(M)$ is annihilated by $\mathfrak{a}^u$, it is $\mathfrak{a}$-complete by 11.1.11; in particular, it is $\mathfrak{a}$-separated. Furthemore, the morphism $\lambda^{\mathfrak{a}}_M$ is $\mathfrak{a}$-dense by 11.1.37; thus by 11.1.35 one may cancel $\lambda^{\mathfrak{a}}_M$ in ($\flat$), which yields ($\diamond$), as desired. □

Remark. In [258] Yekutieli provides an exampel of an ideal $\mathfrak{a}$, of course not finitely generated, and a module M such that the module $\Lambda^{\mathfrak{a}}(M)$ is not $\mathfrak{a}$-complete.

To parse the next statement, recall from 11.1.20 that for an ideal $\mathfrak{a}$ in R and an R-complex M the complex $\Lambda^{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex.

11.1.39 Corollary. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-complex. The $\widehat{R}^{\mathfrak{a}}$-complex $\Lambda^{\mathfrak{a}}(M)$ is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-complete. In particular, the ring $\widehat{R}^{\mathfrak{a}}$ is $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-complete.*

Proof. Considered as an R-complex, $\Lambda^{\mathfrak{a}}(M)$ is $\mathfrak{a}$-complete by 11.1.38 as $\mathfrak{a}$ is finitely generated. Hence $\Lambda^{\mathfrak{a}}(M)$ is also $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-complete by 11.1.9. $\square$

11.1.40 Proposition. *Let $\mathfrak{a}$ be an ideal in R and $\alpha\colon M \to N$ a morphism of $\mathfrak{a}$-complete R-complexes.*

(a) *If $\mathfrak{a}$ is finitely generated, then the complex* $\operatorname{Ker}\alpha$ *is $\mathfrak{a}$-complete.*
(b) *The complex* $\operatorname{Coker}\alpha$ *is $\mathfrak{a}$-quasi-complete, and one has* $\operatorname{Coker}\alpha = 0$ *if and only if $R/\mathfrak{a} \otimes_R \operatorname{Coker}\alpha = 0$ holds.*

Proof. (a): Set $K = \operatorname{Ker}\alpha$ and let $\iota\colon K \to M$ be the embedding. Consider the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, "\lambda^{\mathfrak{a}}_K"] & M \arrow[r, "\alpha"] \arrow[d, "\lambda^{\mathfrak{a}}_M", "\cong"'] & N \arrow[d, "\lambda^{\mathfrak{a}}_N", "\cong"'] \\ & \Lambda^{\mathfrak{a}}(K) \arrow[r, "\Lambda^{\mathfrak{a}}(\iota)"] & \Lambda^{\mathfrak{a}}(M) \arrow[r, "\Lambda^{\mathfrak{a}}(\alpha)"] & \Lambda^{\mathfrak{a}}(N)\,. \end{tikzcd}$$

As $\alpha\iota = 0$ holds, the morphism $(\lambda^{\mathfrak{a}}_M)^{-1}\,\Lambda^{\mathfrak{a}}(\iota)$ satisfies

$$\alpha(\lambda^{\mathfrak{a}}_M)^{-1}\,\Lambda^{\mathfrak{a}}(\iota) \;=\; (\lambda^{\mathfrak{a}}_N)^{-1}\,\Lambda^{\mathfrak{a}}(\alpha)\,\Lambda^{\mathfrak{a}}(\iota) \;=\; (\lambda^{\mathfrak{a}}_N)^{-1}\,\Lambda^{\mathfrak{a}}(\alpha\iota) \;=\; 0\,.$$

Thus $(\lambda^{\mathfrak{a}}_M)^{-1}\,\Lambda^{\mathfrak{a}}(\iota)$ factors uniquely through $\iota\colon K \to M$, that is, there is a unique morphism $\varphi\colon \Lambda^{\mathfrak{a}}(K) \to K$ with $\iota\varphi = (\lambda^{\mathfrak{a}}_M)^{-1}\,\Lambda^{\mathfrak{a}}(\iota)$. Hence $\iota\varphi\lambda^{\mathfrak{a}}_K = \iota = \iota 1^K$, and thus $\varphi\lambda^{\mathfrak{a}}_K = 1^K$ holds as ι is injective. This shows that K is a direct summand of $\Lambda^{\mathfrak{a}}(K)$, so the desired conclusion now follows from 11.1.33 and 11.1.38.

(b): The complex $\operatorname{Coker}\alpha$ is $\mathfrak{a}$-quasi-complete by 11.1.36, and the “only if” part of the last assertion is trivial. To prove the “if” part, set $C = \operatorname{Coker}\alpha = N/\operatorname{Im}\alpha$ and assume that $R/\mathfrak{a} \otimes_R C = 0$ holds, equivalently $\mathfrak{a}C = C$, see 1.1.10. This means that one has $(\mathfrak{a}N + \operatorname{Im}\alpha)/\operatorname{Im}\alpha = N/\operatorname{Im}\alpha$ and hence $\mathfrak{a}N + \operatorname{Im}\alpha = N$. By 11.1.28 this means that $\Lambda^{\mathfrak{a}}(\alpha)$ is surjective. As M and N are $\mathfrak{a}$-complete, it follows from the diagram above that α is surjective, so $C = 0$. $\square$

11.1.41 Proposition. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-complex. If M is $\mathfrak{a}$-complete, then $\mathrm{H}(M)$ is $\mathfrak{a}$-quasi-complete, and for every $v \in \mathbb{Z}$ one has*:

(a) $M_v = 0$ *if and only if* $R/\mathfrak{a} \otimes_R M_v = 0$.
(b) $\mathrm{H}_v(M) = 0$ *if and only if* $R/\mathfrak{a} \otimes_R \mathrm{H}_v(M) = 0$.

Proof. Assuming that M is $\mathfrak{a}$-complete, each module M_v is $\mathfrak{a}$-complete by 11.1.12, so part (a) is an immediate consequence of 11.1.30. By 11.1.40(a) each cycle module $\mathrm{Z}_v(M)$ is $\mathfrak{a}$-complete. As $\mathrm{H}_v(M)$ is the cokernel of the corestricted differential $M_{v+1} \to \mathrm{Z}_v(M)$ it is quasi-complete by 11.1.40(b), which also yields part (b). $\square$

Remark. As the result above suggests, the homology of an $\mathfrak{a}$-complete complex need not be $\mathfrak{a}$-complete; see the Remark after 13.1.34.

EXERCISES

E 11.1.1 (Cf. 11.1.20) Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Verify that $\Lambda^{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex with the action given by $([r^u]_{\mathfrak{a}^u})_{u\geqslant 1}([m^u]_{\mathfrak{a}^u M})_{u\geqslant 1} = ([r^u m^u]_{\mathfrak{a}^u M})_{u\geqslant 1}$.

E 11.1.2 Let $\mathfrak{a}$ be an ideal in R and $u \geqslant 1$ an integer. Show that there is an isomorphism of rings $R/\mathfrak{a}^u \cong R/\mathfrak{a}^u \otimes_R \widehat{R}^{\mathfrak{a}}$.

E 11.1.3 Show that topological equivalence is an equivalence relation on the set of ideals in R.

E 11.1.4 Let $\mathfrak{a}$ and $\mathfrak{b}$ be topologically equivalent ideals in R. Show that there is an isomorphism $\widehat{R}^{\mathfrak{a}} \cong \widehat{R}^{\mathfrak{b}}$ of rings.

E 11.1.5 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that if M is $\mathfrak{a}$-separated and $R/\mathfrak{a} \otimes_R M = 0$, then $M = 0$.

E 11.1.6 Let $\mathfrak{a}$ be a finitely generated ideal in R and M and N be R-complexes. Show that if M is a complex of projective modules and M or N is bounded, then there is an isomorphism $\Lambda^{\mathfrak{a}} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(M, \Lambda^{\mathfrak{a}} N)$ of $\widehat{R}^{\mathfrak{a}}$-complexes.

11.2 Torsion

SYNOPSIS. The functor $\Gamma_{\mathfrak{a}}$; indepence of base; $\mathfrak{a}$-torsion (sub)complex; $\widehat{R}^{\mathfrak{a}}$-structure on $\mathfrak{a}$-torsion complex; idempotence of $\Gamma_{\mathfrak{a}}$; torsion(-free) module; flat module over principal ideal domain.

While the $\mathfrak{a}$-completion functor, treated in the previous section, is defined in terms of limits, the $\mathfrak{a}$-torsion functor is defined in terms of colimits, which are overall simpler to handle, and that alone tempers the technical complexity of this section. In terms of the structure, however, the two sections are closely aligned in order to emphasize parallels, such as independece of base, between $\mathfrak{a}$-completion and $\mathfrak{a}$-torsion.

TORSION FUNCTOR

11.2.1 Definition. Let $\mathfrak{a}$ be an ideal in R. The tower (11.1.2.1) yields a functor

$$\Gamma_{\mathfrak{a}} = \operatorname*{colim}_{u\geqslant 1} \operatorname{Hom}_R(R/\mathfrak{a}^u, -) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R) ;$$

see 3.3.32. It is called the $\mathfrak{a}$-*torsion* functor. Denote by

$$\gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}} \longrightarrow \operatorname{Id}_{\mathcal{C}(R)}$$

the natural transformation obtained from 11.1.3 with G = Hom_R via 4.4.2.

Torsion with respect to the trivial ideals is a trivial affaire.

11.2.2 Example. Clearly, Γ_R is the zero functor and Γ_0 the identity functor on $\mathcal{C}(R)$. In fact, for every nilpotent ideal $\mathfrak{a}$ one has $\Gamma_{\mathfrak{a}} \cong \operatorname{Id}_{\mathcal{C}(R)}$ by 3.3.36(b).

11.2.3 Addendum (to 11.2.1). Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. It follows from 11.2.1 and 2.3.11 that the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ is an endofunctor on $\mathcal{C}(S)$ and $\gamma_{\mathfrak{a}}$ a natural transformation $\Gamma_{\mathfrak{a}} \to \operatorname{Id}_{\mathcal{C}(S)}$.

The next result shows that $\mathfrak{a}$-torsion, like $\mathfrak{a}$-completion, is independent of base.

11.2.4 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. There is a commutative diagram in $\mathcal{C}(S)$ where the horizontal map is an isomorphism,*

$$\begin{tikzcd} \Gamma_{\mathfrak{a}S}(N) \arrow[rr, "\cong"'] \arrow[dr, "\gamma^N_{\mathfrak{a}S}"'] & & \Gamma_{\mathfrak{a}}(N) \arrow[dl, "\gamma^N_{\mathfrak{a}}"] \\ & N\,. & \end{tikzcd}$$

Proof. Recall from 11.2.3 that $\Gamma_{\mathfrak{a}}(N)$ is an S-complex and that $\gamma^N_{\mathfrak{a}} : \Gamma_{\mathfrak{a}}(N) \to N$ is S-linear. In the computation below, the equality is trivial, the first isomorphism follows from 1.1.10, and the second holds by adjunction 4.4.12 and the counitor 4.4.2:

$$\begin{aligned} \operatorname{Hom}_S(S/(\mathfrak{a}S)^u, N) &= \operatorname{Hom}_S(S/\mathfrak{a}^u S, N) \\ &\cong \operatorname{Hom}_S(R/\mathfrak{a}^u \otimes_R S, N) \\ &\cong \operatorname{Hom}_R(R/\mathfrak{a}^u, N)\,. \end{aligned}$$

As N is an S-complex, so are the Hom complexes above, and the isomorphisms are S-linear. From 11.2.1 one gets an induced isomorphism, $\Gamma_{\mathfrak{a}S}(N) \cong \Gamma_{\mathfrak{a}}(N)$, of S-complexes, which makes the desired diagram commutative. $\square$

We start by developing concrete descriptions of $\Gamma_{\mathfrak{a}}$ and $\gamma_{\mathfrak{a}}$.

11.2.5. Let $\mathfrak{a}$ be an ideal in R. By the universal property of colimits, $\gamma^M_{\mathfrak{a}}$ is the unique morphism that for every $u \geqslant 1$ makes the diagram

$$\begin{tikzcd} & \Gamma_{\mathfrak{a}}(M) \arrow[d, "\gamma^M_{\mathfrak{a}}"] \\ \operatorname{Hom}_R(R/\mathfrak{a}^u, M) \arrow[ur, "\mu^u_M"] \arrow[r] & M \end{tikzcd}$$

commutative; here μ^u_M is the canonical morphism from (3.2.3.1) and the unlabeled morphism maps φ to $\varphi([1]_{\mathfrak{a}^u})$.

11.2.6 Definition. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. As the differential ∂^M is R-linear, it maps the graded submodule $(0 :_{M^\natural} \mathfrak{a})$ of $M^\natural$ to itself. Thus the submodule specifies a subcomplex of M, which we henceforth denote $(0 :_M \mathfrak{a})$.

An element m of M is called $\mathfrak{a}$-*torsion* if $\mathfrak{a}^u m = 0$ holds for some $u \geqslant 1$. The $\mathfrak{a}$-torsion elements form a subcomplex $\bigcup_{u \geqslant 1}(0 :_M \mathfrak{a}^u)$ of M, called the $\mathfrak{a}$-*torsion subcomplex*, and M is called $\mathfrak{a}$-*torsion* if this is all of M.

Remark. In parts of the literature, $\mathfrak{a}$-torsion elements and (sub)complexes are called '$\mathfrak{a}$-power torsion' which, albeit more accurate, is longer whence we opt for '$\mathfrak{a}$-torsion'.

11.2.7 Proposition. *Let $\mathfrak{a}$ be an ideal in R and $0 \to M' \to M \to M'' \to 0$ an exact sequence of R-complexes. If M is $\mathfrak{a}$-torsion, then M' and M'' are $\mathfrak{a}$-torsion.*

Proof. The assertion follows immediately from the definition, 11.2.6. □

Remark. If $\mathfrak{a}$ is a finitely generated ideal in R, then the converse of the statement in 11.2.7 also holds, see E 11.2.5, so the class of $\mathfrak{a}$-torsion complexes constitutes a Serre subcategory of $\mathcal{C}(R)$.

The next result justifies the name of the functor $\Gamma_{\mathfrak{a}}$, as it shows that $\Gamma_{\mathfrak{a}}(M)$ can be identified with the $\mathfrak{a}$-torsion subcomplex of M.

11.2.8 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex. One has*

$$\Gamma_{\mathfrak{a}}(M) \;\cong\; \bigcup_{u\geqslant 1} (0 :_M \mathfrak{a}^u)\;,$$

and the morphism $\gamma_{\mathfrak{a}}^M$ *is via this isomorphism the embedding* $\bigcup_{u\geqslant 1}(0 :_M \mathfrak{a}^u) \rightarrowtail M$. *In particular,* M *is* $\mathfrak{a}$*-torsion if and only if* $\gamma_{\mathfrak{a}}^M$ *is an isomorphism.*

Proof. For every $u \geqslant 1$ there is an isomorphism $\operatorname{Hom}_R(R/\mathfrak{a}^u, M) \cong (0 :_M \mathfrak{a}^u)$ of R-complexes; see 1.1.8 and 11.2.6. These subcomplexes form an increasing chain, so one identifies $\Gamma_{\mathfrak{a}}(M) = \operatorname{colim}_{u\geqslant 1} \operatorname{Hom}_R(R/\mathfrak{a}^u, M)$ with $\bigcup_{u\geqslant 1}(0 :_M \mathfrak{a}^u)$; see 3.3.34. Recall from 3.3.2 that every element in $\Gamma_{\mathfrak{a}}(M)$ has the form $\mu_M^u(\varphi)$ for some $\varphi \in \operatorname{Hom}_R(R/\mathfrak{a}^u, M)$; the last assertion now follows from 11.2.5. □

11.2.9 Example. Let R be an integral domain. For an R-module M one has

$$\bigcup_{x\neq 0} \Gamma_{(x)}(M) \;=\; M_{\mathrm{T}}\;.$$

11.2.10 Example. Let $x \in R$ and set $X = \{x^n \mid n \geqslant 0\}$. An R-complex M is (x)-torsion, i.e. $\Gamma_{(x)}(M) = M$, if and only if the localized complex $X^{-1}M$ is zero.

Recall that $\mathrm{H}(X^{-1}M) \cong X^{-1}\,\mathrm{H}(M)$ holds by 2.2.19, as localization is exact by 2.1.50. Consequently, $\mathrm{H}(M)$ is (x)-torsion if and only if $X^{-1}M$ is acyclic.

11.2.11 Proposition. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* S *an* R*-algebra, and* N *an* S*-complex. The complex* N *is* $\mathfrak{a}S$*-torsion if and only if it is* $\mathfrak{a}$*-torsion as an* R*-complex.*

Proof. The statement follows from 11.2.4 and the last assertion in 11.2.8. □

The next result is parallel to 11.1.30.

11.2.12 Lemma. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex. One has* $\Gamma_{\mathfrak{a}}(M) = 0$ *if and only if* $\operatorname{Hom}_R(R/\mathfrak{a}, M) = 0$ *holds.*

Proof. In view of 1.1.8 the "only if" statement is trivial. To prove the converse, one needs by 11.2.8 to argue that the complex $\operatorname{Hom}_R(R/\mathfrak{a}^u, M) \cong (0 :_M \mathfrak{a}^u)$ is zero for every $u \geqslant 1$. We proceed by induction on u. The case $u = 1$ holds by assumption. Now let $u \geqslant 1$ and assume that one has $(0 :_M \mathfrak{a}^u) = 0$. For every x in $(0 :_M \mathfrak{a}^{u+1})$ one has $\mathfrak{a}^u(\mathfrak{a}x) = \mathfrak{a}^{u+1}x = 0$, and hence $\mathfrak{a}x \subseteq (0 :_M \mathfrak{a}^u) = 0$. Therefore, x belongs to $(0 :_M \mathfrak{a})$, and consequently $x = 0$, again by the assumption. □

11.2.13. Let $\mathfrak{a}$ be an ideal in R. The functor $\Gamma_{\mathfrak{a}}$ restricts to a functor $\mathcal{M}(R) \to \mathcal{M}(R)$, and it can be recovered from this restriction by the procedure in 2.1.48. In this sense,

$\Gamma_{\mathfrak{a}}$ on $\mathcal{C}(R)$ is extended from $\Gamma_{\mathfrak{a}}$ on $\mathcal{M}(R)$. This follows from the definition, 11.2.1, in view of 2.3.1 and 3.2.7 but, indeed, it is also evident from 11.2.8. Moreover, for an R-complex M and $v \in \mathbb{Z}$ one has $(\gamma_{\mathfrak{a}}^{M})_v = \gamma_{\mathfrak{a}}^{M_v}$.

11.2.14 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If M is $\mathfrak{a}$-torsion, then $\mathrm{H}(M)$ is $\mathfrak{a}$-torsion and, further, for every $v \in \mathbb{Z}$ one has:*

(a) $M_v = 0$ *if and only if* $\mathrm{Hom}_R(R/\mathfrak{a}, M_v) = 0$.
(b) $\mathrm{H}_v(M) = 0$ *if and only if* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_v(M)) = 0$.

Proof. Assuming that M is $\mathfrak{a}$-torsion, each module M_v is $\mathfrak{a}$-torsion by 11.2.13, and hence so is each cycle module $\mathrm{Z}_v(M)$ and homology module $\mathrm{H}_v(M)$ by 11.2.7. From these facts and 11.2.12 the assertions (a) and (b) follow. □

In addition to the properties listed in the next proposition, the $\mathfrak{a}$-torsion functor is idempotent; see 11.2.18. It does not preserve products; see 11.2.17.

11.2.15 Theorem. *Let $\mathfrak{a}$ be an ideal in R. The $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ is left exact, R-linear, and bounded. It is also a $\natural$- and Σ-functor, and the natural transformation $\gamma_{\mathfrak{a}}$ is a Σ-transformation. Moreover, the functor $\Gamma_{\mathfrak{a}}$ preserves coproducts, and if $\mathfrak{a}$ is finitely generated, then it preserves filtered colimits.*

Proof. All but the last assertion are straightforward consequences of 11.2.8. Alternatively: It follows from 2.3.10 and 3.3.10 that $\Gamma_{\mathfrak{a}}$ is left exact. Since $\Gamma_{\mathfrak{a}}$ and $\gamma_{\mathfrak{a}}$, as noted in 11.2.13, are extended from their restrictions to modules, it follows from properties of extended functors and transformations—see 2.1.48, A.14, 2.1.53, and 4.1.13—that $\Gamma_{\mathfrak{a}}$ is R-linear, bounded, a $\natural$-functor, and a Σ-functor and that $\gamma_{\mathfrak{a}}$ is a Σ-transformation. Further, $\Gamma_{\mathfrak{a}}$ preserves coproducts by 3.1.33, 3.2.9, and 3.2.13. If $\mathfrak{a}$ is finitely generated, then each module $R/\mathfrak{a}^u$ is finitely presented, so by 3.3.17 and another application of 3.2.13 the functor $\Gamma_{\mathfrak{a}}$ preserves filtered colimits. □

11.2.16 Corollary. *Let $\mathfrak{a}$ be a finitely generated ideal in R and $\{M^u \to M^v\}_{u \leqslant v}$ a U-direct system in $\mathcal{C}(R)$. If U is filtered and $\mathrm{H}(M^u)$ is $\mathfrak{a}$-torsion for every $u \in U$, then $\mathrm{H}(\mathrm{colim}_{u \in U} M^u)$ is $\mathfrak{a}$-torsion.*

Proof. The functors H and $\Gamma_{\mathfrak{a}}$ preserve filtered colimits by 3.3.15(d) and 11.2.15. □

11.2.17 Example. The functor $\Gamma_{\mathfrak{a}}$ does not preserve products. Indeed, let $\mathfrak{a}$ be an ideal in R such that the powers $\mathfrak{a}^u$ for $u \geqslant 1$ form a strictly descending chain; while $R/\mathfrak{a}^u$ is $\mathfrak{a}$-torsion for every $u \geqslant 1$ the product $\prod_{u \geqslant 1} R/\mathfrak{a}^u$ is not as the element $([1]_{\mathfrak{a}}, [1]_{\mathfrak{a}^2}, \dots)$ is not annihilated by any power of $\mathfrak{a}$.

The $\mathfrak{a}$-torsion functor is idempotent, even if $\mathfrak{a}$ is not finitely generated, cf. 11.1.38.

11.2.18 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. There is an equality $\Gamma_{\mathfrak{a}}(\gamma_{\mathfrak{a}}^{M}) = \gamma_{\mathfrak{a}}^{\Gamma_{\mathfrak{a}}(M)}$ of morphisms $\Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \to \Gamma_{\mathfrak{a}}(M)$, and this map is an isomorphism. In particular, the R-complex $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion.*

Proof. Set $N = \bigcup_{u \geqslant 1}(0 :_M \mathfrak{a}^u)$. Evidently one has $N = \bigcup_{u \geqslant 1}(0 :_N \mathfrak{a}^u)$, so via the isomorphism in 11.2.8 both $\Gamma_{\mathfrak{a}}(\gamma_{\mathfrak{a}}^{M})$ and $\gamma_{\mathfrak{a}}^{\Gamma_{\mathfrak{a}}(M)}$ are identified with 1^N. □

To parse the next result, recall the notation from 11.1.1.

11.2.19 Proposition. *Let* $\boldsymbol{x} = x_1, \ldots, x_n$ *be a sequence in* R *and* M *an* R*-complex. With the identification from* 11.2.8 *there is an equality of subcomplexes,*

$$\Gamma_{(\boldsymbol{x})}(M) \;=\; \bigcap_{i=1}^{n} \Gamma_{(x_i)}(M)\;.$$

Moreover, one has

$$\Gamma_{(\boldsymbol{x})}(M) \;\cong\; \operatorname*{colim}_{u \geqslant 1} \operatorname{Hom}_R(R/(\boldsymbol{x}^u), M)\;.$$

Proof. For every u one has $(\boldsymbol{x})^{nu} \subseteq (\boldsymbol{x}^u) \subseteq (\boldsymbol{x})^u$ and $\bigcap_{i=1}^{n}(0 :_M x_i^u) = (0 :_M (\boldsymbol{x}^u))$ and, therefore,

$$\bigcup_{u \geqslant 1}(0 :_M (\boldsymbol{x})^u) \;=\; \bigcup_{u \geqslant 1}(0 :_M (\boldsymbol{x}^u)) \;=\; \bigcap_{i=1}^{n} \bigcup_{u \geqslant 1}(0 :_M x_i^u)\;.$$

In view of 11.2.8 these equalities yield the first assertion. The first equality alone yields the second assertion, as one identifies $\operatorname{Hom}_R(R/(\boldsymbol{x}^u), M)$ with $(0 :_M (\boldsymbol{x}^u))$; see 1.1.8 and 3.3.34. $\square$

Recall from 11.1.15 the definition of topological equivalence for ideals.

11.2.20 Proposition. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals in* R*. If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are topologically equivalent, then* $\Gamma_{\mathfrak{a}}$ *and* $\Gamma_{\mathfrak{b}}$ *are naturally isomorphic endofunctors on* $\mathcal{C}(R)$*.*

Proof. Let M be an R-complex. By 11.2.8 one can identify $\Gamma_{\mathfrak{a}}(M)$ with the subcomplex $\bigcup_{u \geqslant 1}(0 :_M \mathfrak{a}^u)$ of M. By assumption, there exist $m, n \in \mathbb{N}$ with $\mathfrak{a}^m \subseteq \mathfrak{b}$ and $\mathfrak{b}^n \subseteq \mathfrak{a}$. Thus, for every $u \geqslant 1$ one has $\mathfrak{a}^{mu} \subseteq \mathfrak{b}^u$ and $\mathfrak{b}^{nu} \subseteq \mathfrak{a}^u$ and, consequently,

$$(0 :_M \mathfrak{b}^u) \subseteq (0 :_M \mathfrak{a}^{mu}) \subseteq \Gamma_{\mathfrak{a}}(M) \quad \text{and} \quad (0 :_M \mathfrak{a}^u) \subseteq (0 :_M \mathfrak{b}^{nu}) \subseteq \Gamma_{\mathfrak{b}}(M)\;.$$

Thus the subcomplexes $\Gamma_{\mathfrak{a}}(M)$ and $\Gamma_{\mathfrak{b}}(M)$ are identical. $\square$

The next result extends 11.1.37(a).

11.2.21 Proposition. *Let* $\mathfrak{a} \subseteq R$ *be an ideal and* M *and* N *be* R*-complexes. If* N *is* $\mathfrak{a}$*-torsion, then there is an isomorphism in* $\mathcal{C}(R)$*,*

$$N \otimes_R \lambda_M^{\mathfrak{a}} : N \otimes_R M \longrightarrow N \otimes_R \Lambda^{\mathfrak{a}}(M)\;.$$

Proof. First note that if $u \geqslant 1$ is an integer and X an $R/\mathfrak{a}^u$-complex, then $X \otimes_R \lambda_M^{\mathfrak{a}}$ is an isomorphism. Indeed, by the unitor 4.4.1 and associativity 4.4.7 one has

$$X \otimes_R \lambda_M^{\mathfrak{a}} \;\cong\; (X \otimes_{R/\mathfrak{a}^u} R/\mathfrak{a}^u) \otimes_R \lambda_M^{\mathfrak{a}} \;\cong\; X \otimes_{R/\mathfrak{a}^u} (R/\mathfrak{a}^u \otimes_R \lambda_M^{\mathfrak{a}})\;,$$

and the map $R/\mathfrak{a}^u \otimes_R \lambda_M^{\mathfrak{a}}$ is an isomorphism by 11.1.37(a) and 1.1.10.

In the computation below, the first isomorphism holds by 11.2.8 as N is $\mathfrak{a}$-torsion, and the second isomorphism follows from 3.2.22.

$$N \otimes_R \lambda_M^{\mathfrak{a}} \;\cong\; \big(\operatorname*{colim}_{u \geqslant 1} \operatorname{Hom}_R(R/\mathfrak{a}^u, N)\big) \otimes_R \lambda_M^{\mathfrak{a}}$$

$$\cong \operatorname*{colim}_{u \geqslant 1} \left(\operatorname{Hom}_R(R/\mathfrak{a}^u, N) \otimes_R \lambda^{\mathfrak{a}}_M\right) .$$

As $\operatorname{Hom}_R(R/\mathfrak{a}^u, N)$ is an $R/\mathfrak{a}^u$-complex, see 2.3.11, the first part of the proof shows that $\operatorname{Hom}_R(R/\mathfrak{a}^u, N) \otimes_R \lambda^{\mathfrak{a}}_M$ is an isomorphism, and hence so is $N \otimes_R \lambda^{\mathfrak{a}}_M$. □

11.2.22 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. If N is $\mathfrak{a}$-torsion, then there is an isomorphism in $\mathcal{C}(R)$,*

$$\operatorname{Hom}_R(N, \gamma^M_{\mathfrak{a}}) : \operatorname{Hom}_R(N, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Hom}_R(N, M) .$$

Proof. Recall the descriptions of $\Gamma_{\mathfrak{a}}$ and $\gamma_{\mathfrak{a}}$ from 11.2.8. As $\gamma^M_{\mathfrak{a}}$ is injective and the functor $\operatorname{Hom}_R(N, –)$ is left exact, see 2.3.10, the morphism $\operatorname{Hom}_R(N, \gamma^M_{\mathfrak{a}})$ is injective. As N is $\mathfrak{a}$-torsion, every homomorphism $\alpha\colon N \to M$ satisfies $\operatorname{Im}\alpha \subseteq \Gamma_{\mathfrak{a}}(M)$, and hence $\operatorname{Hom}_R(N, \gamma^M_{\mathfrak{a}})$ is also surjective. □

$\widehat{R}^{\mathfrak{a}}$-Structure on $\mathfrak{a}$-Torsion Complexes

Recall from 11.1.18 that $\widehat{R}^{\mathfrak{a}}$ is an R-algebra with structure map $\lambda^{\mathfrak{a}}_R\colon R \to \widehat{R}^{\mathfrak{a}}$. The next result shows that every $\mathfrak{a}$-torsion R-complex is an $\widehat{R}^{\mathfrak{a}}$-complex.

11.2.23 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The complex $\Gamma_{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex with action given by*

$$rm = r^u m$$

for elements $r = ([r^v]_{\mathfrak{a}^v})_{v \geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ and $m \in (0 :_M \mathfrak{a}^u) \subseteq \Gamma_{\mathfrak{a}}(M)$. Consequently, $\Gamma_{\mathfrak{a}}$ can be viewed as a functor,

$$\Gamma_{\mathfrak{a}} : \mathcal{C}(R) \longrightarrow \mathcal{C}(\widehat{R}^{\mathfrak{a}}) .$$

Proof. For every $u \geqslant 1$ the R-complex $(0 :_M \mathfrak{a}^u)$ has a canonical $R/\mathfrak{a}^u$-structure given by $[r']_{\mathfrak{a}^u} m = r'm$ for $r' \in R$ and $m \in (0 :_M \mathfrak{a}^u)$. In turn, restriction of scalars along the ring homomorphism $\widehat{R}^{\mathfrak{a}} \to R/\mathfrak{a}^u$ equips $(0 :_M \mathfrak{a}^u)$ with the structure of an $\widehat{R}^{\mathfrak{a}}$-complex given by $rm = r^u m$ for $r = ([r^v]_{\mathfrak{a}^v})_{v \geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ and $m \in (0 :_M \mathfrak{a}^u)$. Note that $(0 :_M \mathfrak{a}^u)$ is an $\widehat{R}^{\mathfrak{a}}$-subcomplex of $(0 :_M \mathfrak{a}^{u+1})$; indeed, as $r^u - r^{u+1} \in \mathfrak{a}^u$ one has $(r^u - r^{u+1})m = 0$, and hence $r^u m = r^{u+1} m$, for every $m \in (0 :_M \mathfrak{a}^u)$. It follows that $\Gamma_{\mathfrak{a}}(M) = \bigcup_{u \geqslant 1}(0 :_M \mathfrak{a}^u)$ is an $\widehat{R}^{\mathfrak{a}}$-complex with the asserted multiplication. □

11.2.24. Let $\mathfrak{a}$ be an ideal in R and M an $\widehat{R}^{\mathfrak{a}}$-complex. By 11.2.3 and 11.2.23 the R-complex $\Gamma_{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex in two, potentially different, ways. We argue below that they are the same if $\mathfrak{a}$ is finitely generated; that is, the diagram

$$\begin{tikzcd}
\mathcal{C}(\widehat{R}^{\mathfrak{a}}) \arrow[r, "\Gamma_{\mathfrak{a}}", "\text{(from 11.2.3)}"'] \arrow[d, "\operatorname{res}^{\widehat{R}^{\mathfrak{a}}}_R"'] & \mathcal{C}(\widehat{R}^{\mathfrak{a}}) \arrow[d, equal] \\
\mathcal{C}(R) \arrow[r, "\Gamma_{\mathfrak{a}}", "\text{(from 11.2.23)}"'] & \mathcal{C}(\widehat{R}^{\mathfrak{a}})
\end{tikzcd}$$

is commutative if $\mathfrak{a}$ is finitely generated. Indeed, for elements

$$r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1} \in \widehat{R}^{\mathfrak{a}} \qquad \text{and} \qquad m \in (0 :_M \mathfrak{a}^u) \subseteq \Gamma_{\mathfrak{a}}(M)$$

the $\widehat{R}^{\mathfrak{a}}$-action from 11.2.23 is given by $rm = r^u m = \lambda^{\mathfrak{a}}_R(r^u)m$, where the last equality holds as M is an R-complex via restriction of scalars along the ring homomorphism $\lambda^{\mathfrak{a}}_R \colon R \to \widehat{R}^{\mathfrak{a}}$. The $\widehat{R}^{\mathfrak{a}}$-structure on $\Gamma_{\mathfrak{a}}(M)$ from 11.2.3 is the restriction to the subcomplex $\Gamma_{\mathfrak{a}}(M)$ of the $\widehat{R}^{\mathfrak{a}}$-structure on M. To prove that the two $\widehat{R}^{\mathfrak{a}}$-structures on $\Gamma_{\mathfrak{a}}(M)$ agree, it suffices to argue that the element $r - \lambda^{\mathfrak{a}}_R(r^u) \in \widehat{R}^{\mathfrak{a}}$ annihilates every $m \in (0 :_M \mathfrak{a}^u)$. To compute the subcomplex $(0 :_M \mathfrak{a}^u)$ one views M as an R-complex; viewing M as an $\widehat{R}^{\mathfrak{a}}$-complex one can also consider the subcomplex $(0 :_M \mathfrak{b}^u)$ where $\mathfrak{b} = \mathfrak{a}\widehat{R}^{\mathfrak{a}}$. Evidently, one has $(0 :_M \mathfrak{a}^u) = (0 :_M \mathfrak{b}^u)$, so it suffices to see that $r - \lambda^{\mathfrak{a}}_R(r^u)$ belongs to the ideal $\mathfrak{b}^u = \mathfrak{a}^u\widehat{R}^{\mathfrak{a}}$. By 11.1.14 this is the case if $\mathfrak{a}$ is finitely generated.

11.2.25 Lemma. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, M an R-complex, and N an S-complex. For elements $m \in (0 :_M \mathfrak{a})$ and $s \in \mathfrak{a}S$ the next assertions hold.*

(a) *For every $\varphi \in \operatorname{Hom}_R(M,N)$ one has $s\varphi(m) = 0$.*
(b) *For every $n \in N$ one has $sn \otimes m = 0$ in $N \otimes_R M$.*

Proof. As s is in $\mathfrak{a}S$ it has the form $s = \sum_{i=1}^k a_i s_i$ with $a_i \in \mathfrak{a}$ and $s_i \in S$ for every $i \in \{1,\dots,k\}$. Thus, for every $\varphi \in \operatorname{Hom}_R(M,N)$ and $n \in N$ one has

$$s\varphi(m) = \textstyle\sum_{i=1}^k s_i\varphi(a_i m) \qquad \text{and} \qquad sn \otimes m = \textstyle\sum_{i=1}^k s_i n \otimes a_i m\,.$$

Each term in these sums is zero because m is in $(0 :_M \mathfrak{a})$ and hence $a_i m = 0$. □

To parse the next result, recall from 11.2.23 that for every ideal $\mathfrak{a}$ in R and every R-complex M there is a canonical $\widehat{R}^{\mathfrak{a}}$-structure on $\Gamma_{\mathfrak{a}}(M)$. Furthermore, recall from 2.3.11 and 2.4.10 that if X and N are $\widehat{R}^{\mathfrak{a}}$-complexes, then $\operatorname{Hom}_R(X,N)$ and $N \otimes_R X$ are complexes of, not necessarily symmetric, $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules.

11.2.26 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and N an $\widehat{R}^{\mathfrak{a}}$-complex. If $\mathfrak{a}$ is finitely generated, then the following assertions hold.*

(a) *$\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(M),N)$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, and there is an equality of $\widehat{R}^{\mathfrak{a}}$-complexes,*

$$\operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(M),N) = \operatorname{Hom}_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(M),N)\,.$$

(b) *$N \otimes_R \Gamma_{\mathfrak{a}}(M)$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, and there is an equality of $\widehat{R}^{\mathfrak{a}}$-complexes,*

$$N \otimes_R \Gamma_{\mathfrak{a}}(M) = N \otimes_{\widehat{R}^{\mathfrak{a}}} \Gamma_{\mathfrak{a}}(M)\,.$$

Proof. (a): It suffices to show that for every $r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1}$ in $\widehat{R}^{\mathfrak{a}}$, $m \in \Gamma_{\mathfrak{a}}(M)$, and $\varphi \in \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(M),N)$ one has $\varphi(rm) = r\varphi(m)$. By 11.2.8 one has $m \in (0 :_M \mathfrak{a}^u)$ for some $u \geqslant 1$, so the first equality below holds by 11.2.23. The middle equality holds as φ is R-linear, and the last one holds as N is an R-complex via restriction of scalars along the ring homomorphism $\lambda^{\mathfrak{a}}_R \colon R \to \widehat{R}^{\mathfrak{a}}$.

$$\varphi(rm) = \varphi(r^u m) = r^u \varphi(m) = \lambda_R^{\mathfrak{a}}(r^u)\varphi(m) .$$

To see that $\lambda_R^{\mathfrak{a}}(r^u)\varphi(m) = r\varphi(m)$ holds, notice that by 11.2.25(a) the difference

$$r\varphi(m) - \lambda_R^{\mathfrak{a}}(r^u)\varphi(m) = (r - \lambda_R^{\mathfrak{a}}(r^u))\varphi(m)$$

is zero, as $r - \lambda_R^{\mathfrak{a}}(r^u)$ belongs to $\mathfrak{a}^u\widehat{R}^{\mathfrak{a}}$ by 11.1.14 and m is in $(0 :_M \mathfrak{a}^u)$.

(b): It suffices to show that for $r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1}$ in $\widehat{R}^{\mathfrak{a}}$, $n \in N$, and $m \in \Gamma_{\mathfrak{a}}(M)$ one has $rn \otimes m = n \otimes rm$ in $N \otimes_R \Gamma_{\mathfrak{a}}(M)$. By 11.2.8 one has $m \in (0 :_M \mathfrak{a}^u)$ for some $u \geqslant 1$, so the first equality below holds by 11.2.23. The middle equality holds as $\otimes$ is middle R-linear, and the last one holds as N is an R-complex via restriction of scalars along the ring homomorphism $\lambda_R^{\mathfrak{a}}$.

$$n \otimes rm = n \otimes r^u m = r^u n \otimes m = \lambda_R^{\mathfrak{a}}(r^u)n \otimes m .$$

To see that $rn \otimes m = \lambda_R^{\mathfrak{a}}(r^u)n \otimes m$ holds, notice that by 11.2.25(b) the difference

$$rn \otimes m - \lambda_R^{\mathfrak{a}}(r^u)n \otimes m = (r - \lambda_R^{\mathfrak{a}}(r^u))n \otimes m$$

is zero, as $r - \lambda_R^{\mathfrak{a}}(r^u)$ belongs to $\mathfrak{a}^u\widehat{R}^{\mathfrak{a}}$ by 11.1.14 and m is in $(0 :_M \mathfrak{a}^u)$. □

11.2.27 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If $\mathfrak{a}$ is finitely generated, then $\widehat{R}^{\mathfrak{a}} \otimes_R \Gamma_{\mathfrak{a}}(M)$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, and there is an isomorphism of $\widehat{R}^{\mathfrak{a}}$-complexes,*

$$\widehat{R}^{\mathfrak{a}} \otimes_R \Gamma_{\mathfrak{a}}(M) \xrightarrow{\cong} \Gamma_{\mathfrak{a}}(M) \qquad \text{given by} \qquad r \otimes m \longmapsto r^u m ,$$

for $r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ and $m \in (0 :_M \mathfrak{a}^u) \subseteq \Gamma_{\mathfrak{a}}(M)$.

Proof. The first assertion follows from 11.2.26(b) applied with $N = \widehat{R}^{\mathfrak{a}}$, and so does the second in view of the unitor 4.4.1. □

Torsion and Flatness over Principal Ideal Domains

First a result that compares to 1.3.31.

11.2.28 Proposition. *Let R be an integral domain with field of fractions Q. Every flat R-module is torsion-free, and for every R-module M there is an exact sequence*

$$0 \longrightarrow M_{\mathrm{T}} \longrightarrow M \longrightarrow Q \otimes_R M .$$

Proof. Application of $- \otimes_R M$ to the embedding $\iota\colon R \rightarrowtail Q$ yields a homomorphism $\iota \otimes_R M$ from $R \otimes_R M \cong M$ to $Q \otimes_R M$. As this can be identified with the canonical map $M \to (R \setminus \{0\})^{-1}M$, it follows that $\operatorname{Ker}(\iota \otimes_R M)$ is the torsion submodule M_{T}. If M is flat, then $\iota \otimes_R M$ is injective, so one has $M_{\mathrm{T}} = 0$. □

Torsion-freeness does not imply flatness.

11.2.29 Example. Let $\Bbbk$ be a field and consider the integral domain $R = \Bbbk[x, y]$. As an R-module, the ideal $\mathfrak{M} = (x, y)$ is evidently torsion-free. However, it is not flat since the homomorphism obtained by applying $- \otimes_R \mathfrak{M}$ to the embedding $\iota\colon \mathfrak{M} \rightarrowtail R$ is not injective. Indeed, in $R \otimes_R \mathfrak{M}$ one has

$$(\iota \otimes_R \mathfrak{M})(x \otimes y - y \otimes x) \;=\; x \otimes y - y \otimes x \;=\; 1 \otimes xy - 1 \otimes yx \;=\; 0\,,$$

but in $\mathfrak{M} \otimes_R \mathfrak{M}$ the element $t = x \otimes y - y \otimes x$ is non-zero. To see why, note that the homomorphism $\mathfrak{M} \otimes_R \mathfrak{M} \to \Bbbk$ given by $f \otimes g \mapsto f'_x(0,0)g'_y(0,0)$ maps t to $1 \neq 0$.

11.2.30 Proposition. *Let R be an integral domain. A finitely generated R-module is torsion-free if and only if it can be embedded into a finitely generated free R-module.*

Proof. A submodule of a free module is evidently torsion-free. Let M be a torsion-free R-module generated by elements $m_1, \dots, m_p$. Consider the injective homomorphism $\alpha\colon M \to Q \otimes_R M$ from 11.2.28. The elements $1 \otimes m_1, \dots, 1 \otimes m_p$ generate the vector space $Q \otimes_R M$, so some subset, say, $1 \otimes m_1, \dots, 1 \otimes m_t$ is a basis. Write $1 \otimes m_v = \sum_{i=1}^{t} q_{vi}(1 \otimes m_i)$ with $q_{vi} \in Q$, and choose $x \neq 0$ in R such that $xq_{vi} \in R$ for all $v \in \{1, \dots, p\}$ and $i \in \{1, \dots, t\}$. The R-submodule $R\langle \frac{1}{x} \otimes m_1, \dots, \frac{1}{x} \otimes m_t \rangle$ of $Q \otimes_R M$ is free and contains $\operatorname{Im}\alpha \cong M$. □

Remark. See E 11.2.10 for an alternative proof of 11.2.30. The number t occuring in the proof of 11.2.30, i.e. the rank of the Q-vector space $Q \otimes_R M$, is known as the rank of the torsion-free module M.

The next theorem compares to 1.3.21 and 1.3.32. Notice that the last assertion also follows from 8.5.2 and 8.5.6.

11.2.31 Theorem. *Let R be a principal ideal domain. An R-module is flat if and only if it is torsion-free. In particular, every submodule of a flat R-module is flat.*

Proof. Every flat R-module is torsion-free by 11.2.28. Let M be a torsion-free R-module. It follows from 3.3.4 that M is the filtered colimit of its finitely generated submodules; so by 5.4.21 it suffices to argue that every finitely generated submodule of M is flat. By 11.2.30 every such module F is a submodule of a free R-module. As R is a principal ideal domain, F is free by 1.3.11, in particular F is flat. □

It transpires from the proof above that every flat module over a principal ideal domain is a colimit of finitely generated free modules. This is true in general and known as Govorov and Lazard's theorem; see 5.5.1.

We close this section with a technical result that, though it is only used in Chap. 19, fits naturally here. It shows, in particular, that to remove the torsion part of a finitely generated module over an integral domain it suffices to invert the powers of a single element, which is far from going all the way to the field of fractions, cf. 11.2.28.

11.2.32 Lemma. *Let R be an integral domain and $M_1, \dots, M_k$ finitely generated R-modules. There exists a non-zero element x in R such that with $X = \{x^n \mid n \geqslant 0\} \subseteq R$ the modules $X^{-1}M_1, \dots, X^{-1}M_k$ are finitely generated free $X^{-1}R$-modules.*

Proof. We start by proving the assertion for $k = 1$. Set $M = M_1$, let $Q = R_{(0)}$ be the field of fractions of R and $\{m_1, \dots, m_s\}$ a set of generators of M. Choose

$e_1, \dots, e_t$ in M such that $\{\frac{e_1}{1}, \dots, \frac{e_t}{1}\}$ is a basis for the Q-vector space $M_{(0)}$. For each $i \in \{1, \dots, s\}$ there exist unique elements $\frac{r_{i1}}{u_{i1}}, \dots, \frac{r_{it}}{u_{it}} \in Q$ with

$$(\star) \qquad \sum_{j=1}^{t} \frac{r_{ij}}{u_{ij}} \frac{e_j}{1} = \frac{m_i}{1} \,.$$

Let x be the product of the denominators u_{ij} for $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$, set $X = \{x^n \mid n \geqslant 0\}$, and consider the homomorphism of $X^{-1}R$-modules,

$$\alpha : (X^{-1}R)^t \longrightarrow X^{-1}M \qquad \text{given by} \qquad (a_1, \dots, a_t) \longmapsto \sum_{i=1}^{t} a_i \frac{e_i}{1} \,.$$

By the definition of the element x, each fraction $\frac{r_{ij}}{u_{ij}}$ is an element of the subring $X^{-1}R \subseteq Q$, so α is surjective by $(\star)$. Further, there is a commutative diagram,

$$\begin{tikzcd} (X^{-1}R)^t \arrow[r, "\alpha"] \arrow[d, tail] & X^{-1}M \arrow[d] \\ Q^t \arrow[r, "\beta", "\cong"'] & M_{(0)} \,, \end{tikzcd}$$

where the vertical maps are the canonical ones and β is the isomorphism given by $(q_1, \dots, q_t) \mapsto \sum_{i=1}^{t} q_i \frac{e_i}{1}$. Consequently, α is injective and thus an isomorphism.

Now, let $M_1, \dots, M_k$ be finitely generated R-modules. As proved above, there exist non-zero elements $x_1, \dots, x_k$ in R such that for each $v \in \{1, \dots, k\}$ the $X_v^{-1}R$-module $X_v^{-1}M_v$ is free where $X_v = \{x_v^n \mid n \geqslant 0\}$. Let x be the non-zero product $x_1 \cdots x_k$ and set $X = \{x^n \mid n \geqslant 0\}$. Fix $v \in \{1, \dots, k\}$ and let Y denote the image of X under the canonical ring homomorphism $R \to X_v^{-1}R$. There are isomorphisms,

$$Y^{-1}(X_v^{-1}R) \cong X^{-1}R \qquad \text{and} \qquad Y^{-1}(X_v^{-1}M_v) \cong X^{-1}M_v \,.$$

Since a localization of a free module is free, see 1.1.11 and 5.1.15(a), it now follows that $X^{-1}M_v$ is a free $X^{-1}R$-module. □

Exercises

E 11.2.1 Let R be local and $x \in R$. Show that if $\Gamma_{(x)}$ is exact, then x is nilpotent or a unit.

E 11.2.2 Show that $\Gamma_{([3]_{6\mathbb{Z}})}$ is exact on the category of $\mathbb{Z}/6\mathbb{Z}$-modules.

E 11.2.3 Let $\mathfrak{a}$ be an ideal in R and M an R-module; show that $\Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ holds.

E 11.2.4 Let $\mathfrak{a}$ be an ideal in R and M a finitely generated R-module. Show that if M is $\mathfrak{a}$-torsion, then it is $\mathfrak{a}$-complete.

E 11.2.5 Let $\mathfrak{a}$ be a finitely generated ideal in R. Show that in an exact sequence of R-complexes, $0 \to M' \to M \to M'' \to 0$, the complex M is $\mathfrak{a}$-torsion if and only if both complexes M' and M'' are $\mathfrak{a}$-torsion.

E 11.2.6 Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. (a) Show that if M is bounded and degreewise finitely presented, then one has $\Gamma_{\mathfrak{a}} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(N))$. (b) Show that if $\mathfrak{a}$ is finitely generated, N is a complex of flat R-modules, and M or N is bounded, then one has $\Gamma_{\mathfrak{a}}(M \otimes_R N) \cong \Gamma_{\mathfrak{a}}(M) \otimes_R N$.

E 11.2.7 Show that for every R-module M one has $M_{\mathrm{T}} = \bigcup \Gamma_{(x)}(M)$ where the union is over all non-zerodivisors $x \in R$.

E 11.2.8 Let M be a finitely generated R-module. Show that the biduality homomorphism δ_R^M is injective if and only if M is a submodule of a free R-module. (A module with this property is called *torsionless*.)

E 11.2.9 Let R be an integral domain, M an R-module, and $m \in M$. Show that $(0 :_R m) = 0$ holds if there is a homomorphism $\varphi\colon M \to R$ with $\varphi(m) \neq 0$. Show that the converse holds if M is finitely generated.

E 11.2.10 Let R be an integral domain and M a finitely generated R-module. Show that the kernel of the biduality homomorphism δ_R^M is the torsion submodule M_{T}. Conclude that a finitely generated torsion-free R-module is a submodule of a free R-module.

E 11.2.11 Show that if every submodule of a free R-module is free, then R is a principal ideal domain.

E 11.2.12 Let R be an integral domain with field of fractions Q. Show that Q is finitely generated as an R-module if and only if R is a field, in which case $R = Q$ holds.

E 11.2.13 Let R be an integral domain, not a field, with field of fractions Q. Show that every free R-submodule of Q is cyclic and that Q is not a submodule of a free R-module.

E 11.2.14 Let R be an integral domain. (a) Show that every submodule of a free R-module is torsion-free. (b) Give an example of a torsion-free R-module that is not a submodule of a free R-module.

E 11.2.15 Let R be an integral domain and M an R-module. Show that M/M_{T} is torsion-free. Conclude that M/M_{T} is injective for every divisible R-module M.

E 11.2.16 Let $\varphi\colon M \to N$ be a homomorphism of R-modules; show that it restricts to a homomorphism $\varphi_{\mathrm{T}}\colon M_{\mathrm{T}} \to N_{\mathrm{T}}$. Conclude that $(-)_{\mathrm{T}}$ is a functor and show that it is left exact but not exact.

E 11.2.17 Let R be an integral domain with field of fractions Q. Show that there is a natural isomorphism $(-)_{\mathrm{T}} \cong \operatorname{Tor}_1^R(-, Q/R)$ of functors on $\mathcal{M}(R)$, cf. E 11.2.16.

E 11.2.18 Let R be an integral domain with field of fractions Q and M be an R-module. Show that one has $M_{\mathrm{T}} \otimes_R Q = 0$ and $M \otimes_R Q \cong (M/M_{\mathrm{T}}) \otimes_R Q$.

E 11.2.19 Let R be an integral domain and M and N be R-modules. Show that $\operatorname{Tor}_m^R(M, N)$ is a torsion module for every $m \in \mathbb{N}$.

E 11.2.20 Let R be an integral domain and M a finitely generated R-module. Show that for every R-module N the equality $\operatorname{Hom}_R(M, N)_{\mathrm{T}} = \operatorname{Hom}_R(M, N_{\mathrm{T}})$ holds.

E 11.2.21 Let R be an integral domain, not a field, and $M \neq 0$ a finitely generated R-module. Show that M is not injective.

E 11.2.22 Let R be an integral domain, not a field, and M a divisible R-module. Show that for every finitely generated R-module N one has $\operatorname{Hom}_R(M, N) = 0$. *Hint:* Start with $N = R$.

E 11.2.23 Let $\mathfrak{a}$ be a finitely generated ideal in R and M and N be R-complexes. Show that if N is a complex of injective modules and M or N is bounded, then there is an isomorphism $\Lambda^{\mathfrak{a}} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(\Gamma_{\mathfrak{a}} M, N)$ of $\widehat{R}^{\mathfrak{a}}$-complexes.

11.3 Local Homology and Cohomology

Synopsis. The functor $\mathsf{L}\Lambda^{\mathfrak{a}}$; local homology $\mathrm{H}^{\mathfrak{a}}$; the functor $\mathsf{R}\Gamma_{\mathfrak{a}}$; local cohomology $\mathrm{H}_{\mathfrak{a}}$.

We start the study of the functors on the derived category obtained from the $\mathfrak{a}$-completion and $\mathfrak{a}$-torsion functors from the previous sections. Here we only develop the standard properties that follow from the general theory of derived functors. In the Noetherian setting there is much more to say, and that is the topic of Chap. 13.

LOCAL HOMOLOGY

The $\mathfrak{a}$-completion functor $\Lambda^{\mathfrak{a}}$ induces by 11.1.12 and 6.1.22 an endofunctor on the homotopy category $\mathcal{K}(R)$, so by 7.2.8 it has a left derived functor $\mathsf{L}\Lambda^{\mathfrak{a}}$ which can be computed by way of semi-projective resolutions.

11.3.1 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R. *The left derived* $\mathfrak{a}$*-completion functor* $\mathsf{L}\Lambda^{\mathfrak{a}}\colon \mathcal{D}(R) \to \mathcal{D}(R)$ *is* R*-linear and triangulated.*

PROOF. It follows from 11.1.13, 6.1.20, and 6.2.16 that the endufunctor on $\mathcal{K}(R)$ induced by $\Lambda^{\mathfrak{a}}$ is R-linear and triangulated. Now invoke 7.2.14. $\square$

We also need notation for the natural transformation induced by the one in 11.1.4.

11.3.2 Definition. Let $\mathfrak{a}$ be an ideal in R. The Σ-transformation $\lambda^{\mathfrak{a}}\colon \mathrm{Id}_{\mathcal{K}(R)} \to \Lambda^{\mathfrak{a}}$ from 11.1.4/11.1.13 yields by 6.2.17 and 6.5.14 a triangulated natural transformation,

$$\boldsymbol{\lambda}^{\mathfrak{a}} = \mathsf{L}\lambda^{\mathfrak{a}} : \mathrm{Id}_{\mathcal{D}(R)} \longrightarrow \mathsf{L}\Lambda^{\mathfrak{a}} ,$$

see 7.2.8 and 7.2.11.

11.3.3 Definition. Let $\mathfrak{a}$ be an ideal in R. An R-complex M is called *derived* $\mathfrak{a}$*-complete* if $\boldsymbol{\lambda}^{\mathfrak{a}}_M\colon M \to \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ from 11.3.2 is an isomorphism in $\mathcal{D}(R)$. The full subcategory $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ of $\mathcal{D}(R)$ is defined by specifying its objects as follows:

$$\mathcal{D}^{\mathfrak{a}\text{-com}}(R) = \{M \in \mathcal{D}(R) \mid M \text{ is derived } \mathfrak{a}\text{-complete}\} .$$

The full subcategory $\mathcal{D}^{\mathfrak{a}\text{-com}}(R) \cap \mathcal{D}_{\sqsupset}(R)$ is denoted by $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsupset}(R)$. Similarly, one defines the subcategories $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsubset}(R)$ and $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\square}(R)$.

Over a Noetherian ring it follows from 11.1.13 that the $\mathfrak{a}$-completion functor preserves products, but it does not follow from the general theory in Chap. 7 that its left derived functor enjoys the same property. Nevertheless, it does preserve products, and that is proved in 13.1.15.

11.3.4 Addendum (to 11.3.1). By definition, $\mathsf{L}\Lambda^{\mathfrak{a}}$ is the endofunctor on $\mathcal{D}(R)$ induced by the endofunctor $\Lambda^{\mathfrak{a}}\,\mathrm{P}_R$ on $\mathcal{K}(R)$, where P_R is the semi-projective resolution functor from 6.3.11. Thus, it follows from 11.1.20 that $\mathsf{L}\Lambda^{\mathfrak{a}}$ can be viewed as a functor

$$\mathsf{L}\Lambda^{\mathfrak{a}} : \mathcal{D}(R) \longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}) .$$

11.3.5. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. By 11.1.6 one can view $\Lambda^{\mathfrak{a}}$ as an endofunctor on $\mathcal{C}(S)$. This functor has per 7.2.8 a left derived functor, $\mathrm{E}\colon \mathcal{D}(S) \to \mathcal{D}(S)$, induced by the endofunctor $\Lambda^{\mathfrak{a}}\,\mathrm{P}_S$ on $\mathcal{K}(S)$, where P_S is the semi-projective resolution functor over S, see 6.3.11. Even though $\mathsf{L}\Lambda^{\mathfrak{a}}$ according to 7.2.8 is the corrrect notation for the derived functor E, we only use it in situations where the following diagram is commutative up to natural isomorphism,

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "\mathrm{E}"] \arrow[d, "\mathrm{res}_R^S"'] & \mathcal{D}(S) \arrow[d, "\mathrm{res}_R^S"] \\ \mathcal{D}(R) \arrow[r, "\mathsf{L}\Lambda^{\mathfrak{a}}"] & \mathcal{D}(R)\,; \end{tikzcd}$$

that is, in situations where E is an augmentation of $\mathsf{L}\Lambda^{\mathfrak{a}}$. It is proved in 13.1.12 that this diagram is commutative up to natural isomorphism if R and S are Noetherian.

11.3.6 Definition. Let $\mathfrak{a}$ be an ideal in R. For $m \in \mathbb{Z}$ denote by $\mathrm{H}_m^{\mathfrak{a}}$ the functor

$$\mathrm{H}_m \mathsf{L}\Lambda^{\mathfrak{a}} : \mathcal{D}(R) \longrightarrow \mathcal{M}(R)\,;$$

it is called the m^{th} *local homology* functor supported at $\mathfrak{a}$.

11.3.7. Let $\mathfrak{a}$ be an ideal in R. It follows from 11.3.1 and 6.5.17 that the local homology functors $\mathrm{H}_m^{\mathfrak{a}}$ are R-linear. Further, for an R-complex M one has $\mathrm{H}_m^{\mathfrak{a}}(M) = 0$ for $m < \inf M$ by 7.2.15.

11.3.8 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. For every pair of integers m, s there is an isomorphism of R-modules,*

$$\mathrm{H}_m^{\mathfrak{a}}(\Sigma^s M) \;\cong\; \mathrm{H}_{m-s}^{\mathfrak{a}}(M)\,,$$

and it is natural in M.

Proof. The functor $\mathsf{L}\Lambda^{\mathfrak{a}}$ is triangulated by 11.3.1, so in $\mathcal{D}(R)$ there is an isomorphism, $\mathsf{L}\Lambda^{\mathfrak{a}}(\Sigma^s M) \simeq \Sigma^s \mathsf{L}\Lambda^{\mathfrak{a}}(M)$, in $\mathcal{D}(R)$ which is natural in M. Apply the functor H_m to this isomorphism and recall from 6.5.17 that one has $\mathrm{H}_m \Sigma^s = \mathrm{H}_{m-s}$. The assertion now follows from the definition, 11.3.6, of local homology. □

Per 6.5.24 the next result applies, in particular, to a commutative diagram in the category of complexes whose rows are short exact sequences.

11.3.9 Theorem. *Let $\mathfrak{a}$ be an ideal in R and consider a morphism of distinguished triangles in $\mathcal{D}(R)$,*

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r, "\alpha''"] \arrow[d, "\varphi''"] & \Sigma M' \arrow[d, "\Sigma\varphi'"] \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \arrow[r, "\beta''"] & \Sigma N'\,. \end{tikzcd}$$

There is a commutative diagram in $\mathcal{M}(R)$ with exact rows,

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}_m^{\mathfrak{a}}(M') \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\alpha')"] \arrow[d, "\mathrm{H}_m^{\mathfrak{a}}(\varphi')"] & \mathrm{H}_m^{\mathfrak{a}}(M) \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\alpha)"] \arrow[d, "\mathrm{H}_m^{\mathfrak{a}}(\varphi)"] & \mathrm{H}_m^{\mathfrak{a}}(M'') \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\alpha'')"] \arrow[d, "\mathrm{H}_m^{\mathfrak{a}}(\varphi'')"] & \mathrm{H}_{m-1}^{\mathfrak{a}}(M') \arrow[r] \arrow[d, "\mathrm{H}_{m-1}^{\mathfrak{a}}(\varphi')"] & \cdots \\ \cdots \arrow[r] & \mathrm{H}_m^{\mathfrak{a}}(N') \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\beta')"'] & \mathrm{H}_m^{\mathfrak{a}}(N) \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\beta)"'] & \mathrm{H}_m^{\mathfrak{a}}(N'') \arrow[r, "\mathrm{H}_m^{\mathfrak{a}}(\beta'')"'] & \mathrm{H}_{m-1}^{\mathfrak{a}}(N') \arrow[r] & \cdots\,. \end{tikzcd}$$

Proof. The functor $\mathsf{L}\Lambda^{\mathfrak{a}}$ is triangulated, see 11.3.1, so the desired conclusion follows from 6.5.21 and the definition, 11.3.6, of local homology modules. $\square$

11.3.10. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Recall from 11.3.4 that $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex. It follows that the isomorphism in 11.3.8 is an isomorphism of $\widehat{R}^{\mathfrak{a}}$-modules, and the second commutative diagram in 11.3.9 is a diagram in $\mathcal{M}(\widehat{R}^{\mathfrak{a}})$.

11.3.11 Proposition. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-complex. For every $m \in \mathbb{Z}$ the local homology module $\mathrm{H}^{\mathfrak{a}}_m(M)$ is $\mathfrak{a}$-quasi-complete, and one has $\mathrm{H}^{\mathfrak{a}}_m(M) = 0$ if and only if $R/\mathfrak{a} \otimes_R \mathrm{H}^{\mathfrak{a}}_m(M) = 0$ holds.*

Proof. Let P be a semi-projective replacement of M; by definition the modules $\mathrm{H}^{\mathfrak{a}}_m(M)$ are the homology modules of the complex $\Lambda^{\mathfrak{a}}(P)$. By 11.1.38 this complex is $\mathfrak{a}$-complete, so the assertion follows from 11.1.41. $\square$

11.3.12. Let $\mathfrak{a}$ be an ideal in R and M an R-module. It follows from 11.3.6 that there is a canonical homomorphism $\mathrm{H}^{\mathfrak{a}}_0(M) \to \Lambda^{\mathfrak{a}}(M)$, which is natural in M.

The functor $\Lambda^{\mathfrak{a}}$ is not right exact, in fact, it is not even half exact as demonstrated in 11.1.32. Thus, for a module M the canonical map $\mathrm{H}^{\mathfrak{a}}_0(M) \to \Lambda^{\mathfrak{a}}(M)$ from 11.3.12 need not be an isomorphism; an example of such behavior follows below. However, as shown in 11.3.14 there is still a close connection between $\mathrm{H}^{\mathfrak{a}}_0(M)$ and $\Lambda^{\mathfrak{a}}(M)$.

11.3.13 Example. Consider the short exact sequence $0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\alpha} N \longrightarrow 0$ from 11.1.32 and the associated commutative diagram,

$$
\begin{tikzcd}
\mathrm{H}^{\mathfrak{a}}_0(K) \arrow[r, "\mathrm{H}^{\mathfrak{a}}_0(\iota)"] \arrow[d, "\varphi^K"] & \mathrm{H}^{\mathfrak{a}}_0(M) \arrow[r, "\mathrm{H}^{\mathfrak{a}}_0(\alpha)"] \arrow[d, "\varphi^M"] & \mathrm{H}^{\mathfrak{a}}_0(N) \arrow[d, "\varphi^N"] \\
\Lambda^{\mathfrak{a}}(K) \arrow[r, "\Lambda^{\mathfrak{a}}(\iota)"] & \Lambda^{\mathfrak{a}}(M) \arrow[r, "\Lambda^{\mathfrak{a}}(\alpha)"] & \Lambda^{\mathfrak{a}}(N)\,,
\end{tikzcd}
\tag{11.3.13.1}
$$

where $\varphi\colon \mathrm{H}^{\mathfrak{a}}_0 \to \Lambda^{\mathfrak{a}}$ is the natural transformation from 11.3.12. By 11.3.7, 11.3.9, and 11.1.28 the maps $\mathrm{H}^{\mathfrak{a}}_0(\alpha)$ and $\Lambda^{\mathfrak{a}}(\alpha)$ are surjective, but this is not important here. By 11.3.9 the upper row in (11.3.13.1) is exact, but as shown in 11.1.32 the lower row is not exact. Thus, at least one of the maps φ^K, φ^M, and φ^N fails to be an isomorphism.

Notice that K is a $\mathbb{Z}/\mathfrak{a}$-module. As the extension of $\mathfrak{a}$ to $\mathbb{Z}/\mathfrak{a}$ is the zero ideal, one has $\Lambda^{\mathfrak{a}}(K) = K$ by 11.1.7 and 11.1.5. It is proved in 13.1.21(a) that also local homology is independent of base, whence one has $\mathrm{H}^{\mathfrak{a}}_0(K) = K$. It follows that φ_K is an isomorphism. Next note that that the modules M and N are identical, though α is not an isomorphism; it follows that neither φ^M nor φ^N is an isomorphism.

11.3.14 Proposition. *Let $\mathfrak{a}$ be a finitely generated ideal in R and M an R-module. One has $\Lambda^{\mathfrak{a}}(M) = 0$ if and only if $\mathrm{H}^{\mathfrak{a}}_0(M) = 0$ holds.*

Proof. Let $L' \to L \to M \to 0$ be a free presentation of M and consider the following commutative diagram,

$$\begin{tikzcd}
L' \arrow[r] \arrow[d, "\lambda^{\mathfrak{a}}_{L'}"] & L \arrow[r] \arrow[d, "\lambda^{\mathfrak{a}}_{L}"] & M \arrow[r] \arrow[d] & 0 \\
\Lambda^{\mathfrak{a}}(L') \arrow[r] & \Lambda^{\mathfrak{a}}(L) \arrow[r] & \mathrm{H}^{\mathfrak{a}}_0(M) \arrow[r] & 0 \,.
\end{tikzcd}$$

In this diagram, the bottom row is exact by definition of $\mathrm{H}^{\mathfrak{a}}_0(M)$, and the homomorphism $M \to \mathrm{H}^{\mathfrak{a}}_0(M)$ is induced by the commutative square to the left. Application of the right exact functor $R/\mathfrak{a} \otimes_R -$ to the diagram yields a commutative diagram with exact rows. By 11.1.37 and 1.1.10 the maps $R/\mathfrak{a} \otimes_R \lambda^{\mathfrak{a}}_{L'}$ and $R/\mathfrak{a} \otimes_R \lambda^{\mathfrak{a}}_{L}$ are isomorphisms, so $R/\mathfrak{a} \otimes_R M \cong R/\mathfrak{a} \otimes_R \mathrm{H}^{\mathfrak{a}}_0(M)$ holds by the Five Lemma 1.1.2. In view of this isomorphism, the assertion follows from 11.1.30 and 11.3.11. $\square$

Local Cohomology

The $\mathfrak{a}$-torsion functor induces by 11.2.13 and 6.1.22 an endofunctor on the homotopy category $\mathcal{K}(R)$, so by 7.2.8 it has a right derived functor $\mathsf{R}\Gamma_{\mathfrak{a}}$ which can be computed by way of semi-injective resolutions.

11.3.15 Proposition. *Let $\mathfrak{a}$ be an ideal in R. The right derived $\mathfrak{a}$-torsion functor $\mathsf{R}\Gamma_{\mathfrak{a}} \colon \mathcal{D}(R) \to \mathcal{D}(R)$ is R-linear and triangulated.*

Proof. It follows from 11.2.15, 6.1.20, and 6.2.16 that the endofunctor on $\mathcal{K}(R)$ induced by $\Gamma_{\mathfrak{a}}$ is R-linear and triangulated. Now invoke 7.2.14. $\square$

We also need notation for the natural transformation induced by the one in 11.2.1.

11.3.16 Definition. Let $\mathfrak{a}$ be an ideal in R. The Σ-transformation $\gamma_{\mathfrak{a}} \colon \Gamma_{\mathfrak{a}} \to \mathrm{Id}_{\mathcal{C}(R)}$ from 11.2.1/11.2.15 yields by 6.2.17 and 6.5.14 a triangulated natural transformation,

$$\boldsymbol{\gamma}_{\mathfrak{a}} \;=\; \mathsf{R}\gamma_{\mathfrak{a}} \colon \mathsf{R}\Gamma_{\mathfrak{a}} \longrightarrow \mathrm{Id}_{\mathcal{D}(R)} \;,$$

see 7.2.8 and 7.2.11.

11.3.17 Definition. Let $\mathfrak{a}$ be an ideal in R. An R-complex M is called *derived $\mathfrak{a}$-torsion* if the map $\boldsymbol{\gamma}^M_{\mathfrak{a}} \colon \mathsf{R}\Gamma_{\mathfrak{a}}(M) \to M$ from 11.3.16 is an isomorphism in $\mathcal{D}(R)$. The full subcategory $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ of $\mathcal{D}(R)$ is defined by specifying its objects as follows:

$$\mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \;=\; \{M \in \mathcal{D}(R) \mid M \text{ is derived } \mathfrak{a}\text{-torsion}\} \,.$$

The full subcategory $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \cap \mathcal{D}_{\sqsubset}(R)$ is denoted by $\mathcal{D}^{\mathfrak{a}\text{-tor}}_{\sqsubset}(R)$. Similarly, one defines the subcategories $\mathcal{D}^{\mathfrak{a}\text{-tor}}_{\sqsupset}(R)$ and $\mathcal{D}^{\mathfrak{a}\text{-tor}}_{\square}(R)$.

By 11.2.15 the $\mathfrak{a}$-torsion functor preserves coproducts, but it does not follow from the general theory in Chap. 7 that its right derived functor enjoys the same property. Over a Noetherian ring it does preserve coproducts, which is proved in 13.3.18.

11.3.18 Addendum (to 11.3.15). By definition, $\mathsf{R}\Gamma_{\mathfrak{a}}$ is the endofunctor on $\mathcal{D}(R)$ induced by the endofunctor $\Gamma_{\mathfrak{a}}\,\mathrm{I}_R$ on $\mathcal{K}(R)$, where I_R is the semi-injective resolution functor from 6.3.17. It follows from 11.2.23 that $\mathsf{R}\Gamma_{\mathfrak{a}}$ can be viewed as a functor

$$\mathsf{R}\Gamma_{\mathfrak{a}} : \mathcal{D}(R) \longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}) .$$

11.3.19. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. By 11.2.3 one can view $\Gamma_{\mathfrak{a}}$ as an endofunctor on $\mathcal{C}(S)$. This functor has per 7.2.8 a right derived functor, $\mathrm{E}\colon \mathcal{D}(S) \to \mathcal{D}(S)$, induced by the endofunctor $\Gamma_{\mathfrak{a}}\,\mathrm{I}_S$ on $\mathcal{K}(S)$, where I_S is the semi-injective resolution functor over S, see 6.3.17. Even though $\mathsf{R}\Gamma_{\mathfrak{a}}$ according to 7.2.8 is the correct notation for the derived functor E, we only use it in situations where the following diagram is commutative up to natural isomorphism,

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "\mathrm{E}"] \arrow[d, "\mathrm{res}^S_R"'] & \mathcal{D}(S) \arrow[d, "\mathrm{res}^S_R"] \\ \mathcal{D}(R) \arrow[r, "\mathsf{R}\Gamma_{\mathfrak{a}}"] & \mathcal{D}(R) \end{tikzcd} ;$$

that is, in situations where E is an augmentation of $\mathsf{R}\Gamma_{\mathfrak{a}}$. It is proved in 13.3.15 that this diagram is commutative up to natural isomorphism if R and S are Noetherian.

11.3.20 Definition. Let $\mathfrak{a}$ be an ideal in R. For $m \in \mathbb{Z}$ denote by $\mathrm{H}^m_{\mathfrak{a}}$ the functor

$$\mathrm{H}_{-m}\,\mathsf{R}\Gamma_{\mathfrak{a}} : \mathcal{D}(R) \longrightarrow \mathcal{M}(R) ;$$

it is called the m^{th} *local cohomology* functor supported at $\mathfrak{a}$.

11.3.21. Let $\mathfrak{a}$ be an ideal in R. It follows from 11.3.15 and 6.5.17 that the local cohomology functors $\mathrm{H}^m_{\mathfrak{a}}$ are R-linear. Further, for an R-complex M one has $\mathrm{H}^m_{\mathfrak{a}}(M) = 0$ for $m < -\sup M$ by 7.2.15.

11.3.22 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. For every pair of integers m, s there is an isomorphism of R-modules,*

$$\mathrm{H}^m_{\mathfrak{a}}(\Sigma^s M) \cong \mathrm{H}^{m+s}_{\mathfrak{a}}(M) ,$$

and it is natural in M.

Proof. The functor $\mathsf{R}\Gamma_{\mathfrak{a}}$ is triangulated by 11.3.15, so there is an isomorphism, $\mathsf{R}\Gamma_{\mathfrak{a}}(\Sigma^s M) \simeq \Sigma^s\,\mathsf{R}\Gamma_{\mathfrak{a}}(M)$, in $\mathcal{D}(R)$ which is natural in M. Apply the functor H_{-m} to this isomorphism and recall from 6.5.17 that one has $\mathrm{H}_{-m}\,\Sigma^s = \mathrm{H}_{-(m+s)}$. The assertion now follows from the definition, 11.3.20, of local cohomology. □

Per 6.5.24 the next result applies, in particular, to a commutative diagram in the category of complexes whose rows are short exact sequences.

11.3.23 Theorem. *Let $\mathfrak{a}$ be an ideal in R and consider a morphism of distinguished triangles in $\mathcal{D}(R)$,*

$$\begin{tikzcd} M' \arrow[r, "\alpha'"] \arrow[d, "\varphi'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M'' \arrow[r, "\alpha''"] \arrow[d, "\varphi''"] & \Sigma M' \arrow[d, "\Sigma\varphi'"] \\ N' \arrow[r, "\beta'"] & N \arrow[r, "\beta"] & N'' \arrow[r, "\beta''"] & \Sigma N' \end{tikzcd} .$$

There is a commutative diagram in $\mathcal{M}(R)$ *with exact rows,*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^m_{\mathfrak{a}}(M') \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\alpha')"] \arrow[d, "\mathrm{H}^m_{\mathfrak{a}}(\varphi')"] & \mathrm{H}^m_{\mathfrak{a}}(M) \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\alpha)"] \arrow[d, "\mathrm{H}^m_{\mathfrak{a}}(\varphi)"] & \mathrm{H}^m_{\mathfrak{a}}(M'') \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\alpha'')"] \arrow[d, "\mathrm{H}^m_{\mathfrak{a}}(\varphi'')"] & \mathrm{H}^{m+1}_{\mathfrak{a}}(M') \arrow[r] \arrow[d, "\mathrm{H}^{m+1}_{\mathfrak{a}}(\varphi')"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^m_{\mathfrak{a}}(N') \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\beta')"'] & \mathrm{H}^m_{\mathfrak{a}}(N) \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\beta)"'] & \mathrm{H}^m_{\mathfrak{a}}(N'') \arrow[r, "\mathrm{H}^m_{\mathfrak{a}}(\beta'')"'] & \mathrm{H}^{m+1}_{\mathfrak{a}}(N') \arrow[r] & \cdots .
\end{tikzcd}$$

Proof. The functor $\mathsf{R}\Gamma_{\mathfrak{a}}$ is triangulated, see 11.3.15, so the desired conclusion follows from 6.5.21 and the definition, 11.3.20, of local cohomology modules. □

11.3.24 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex. For every* $m \in \mathbb{Z}$ *the local cohomology module* $\mathrm{H}^m_{\mathfrak{a}}(M)$ *is* $\mathfrak{a}$*-torsion, and one has* $\mathrm{H}^m_{\mathfrak{a}}(M) = 0$ *if and only if* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^m_{\mathfrak{a}}(M)) = 0$ *holds.*

Proof. Let I be a semi-injective replacement of M; by definition, the modules $\mathrm{H}^m_{\mathfrak{a}}(M)$ are the homology modules of the complex $\Gamma_{\mathfrak{a}}(I)$. By 11.2.18 this complex is $\mathfrak{a}$-torsion, so the assertion follows from 11.2.14. □

11.3.25. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Recall from 11.3.18 that $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex. Thus the isomorphism in 11.3.22 is an isomorphism of $\widehat{R}^{\mathfrak{a}}$-modules, and the second commutative diagram in 11.3.23 is a diagram in $\mathcal{M}(\widehat{R}^{\mathfrak{a}})$.

Exercises

E 11.3.1 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that for every $m \in \mathbb{Z}$ there is a canonical homomorphism $\varphi^M_m \colon \mathrm{H}^{\mathfrak{a}}_m(M) \to \lim_{u\geqslant 1} \mathrm{Tor}^R_m(R/\mathfrak{a}^u, M)$ Show that if M is a module, then φ^M_0 is the canonical homomorphism $\mathrm{H}^{\mathfrak{a}}_0(M) \to \Lambda^{\mathfrak{a}}(M)$ from 11.3.12.

E 11.3.2 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that for every $m \in \mathbb{Z}$ there is a canonical isomorphism $\mathrm{colim}_{u\geqslant 1} \mathrm{Ext}^m_R(R/\mathfrak{a}^u, M) \to \mathrm{H}^m_{\mathfrak{a}}(M)$.

E 11.3.3 Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-module, and E an injective R-module. Show that for every $m \in \mathbb{Z}$ there is an isomorphism $\mathrm{Hom}_R(\mathrm{H}^m_{\mathfrak{a}}(M), E) \cong \mathrm{H}^{\mathfrak{a}}_m(\mathrm{Hom}_R(M, E))$. See also 13.1.4.

11.4 Koszul and Čech Complexes

Synopsis. Koszul complex; Čech complex; base change; semi-free resolution of Čech complex.

In earlier chapters, the Koszul complex mainly served as an example to illustrate basic notions in the theory of complexes. In this section it plays a cental role in the study of a related object: the Čech complex. This study provides the foundation for the treatment in Chap. 13 of the derived $\mathfrak{a}$-torsion and derived $\mathfrak{a}$-completion functors, which culminates in the Greenlees–May Equivalence Theorem.

THE KOSZUL COMPLEX

11.4.1. For every element x in R the complex

$$(11.4.1.1) \qquad 0 \longrightarrow R \xrightarrow{\;x\;} R \longrightarrow 0\,,$$

concentrated in degrees 1 and 0, is isomorphic to the Koszul complex $\mathrm{K}^R(x)$; see (2.2.9.1). We henceforth identify $\mathrm{K}^R(x)$ with the complex (11.4.1.1), and for a sequence $x_1,\dots,x_n$ in R we make per 4.4.8 the identification

$$(11.4.1.2) \qquad \mathrm{K}^R(x_1,\dots,x_n) \;=\; \mathrm{K}^R(x_1)\otimes_R\cdots\otimes_R \mathrm{K}^R(x_n)\,.$$

Where it causes no ambiguity, we suppress the superscript R in the notation for the Koszul complex.

The gist of 11.4.1 is that we ignore the product on the Koszul complex to consider it, simply, as a complex of free R-modules.

11.4.2 Example. Representing the maps between free R-modules by matrices that act by left multiplication on column vectors, the Koszul complexes on sequences of two and three elements look as follows:

$$\mathrm{K}(x_1,x_2) \;=\; 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right)} R \longrightarrow 0$$

and

$$\mathrm{K}(x_1,x_2,x_3) \;=\; 0 \longrightarrow R \xrightarrow{\left(\begin{smallmatrix} x_3 \\ -x_2 \\ x_1 \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} x_1 & x_2 & x_3 \end{smallmatrix}\right)} R \longrightarrow 0\,.$$

We record some frequently used properties of Koszul complexes.

11.4.3 Proposition. *Let $x_1,\dots,x_n$ be a sequence in R and set $K = \mathrm{K}^R(x_1,\dots,x_n)$.*

(a) *One has* $\mathrm{H}_0(K) \cong R/(x_1,\dots,x_n)$.
(b) *K is, up to isomorphism in $\mathcal{C}(R)$, invariant under permutation of $x_1,\dots,x_n$.*
(c) *K is a complex of finitely generated free R-modules and concentrated in degrees $n,\dots,0$; in partiulcar, K is semi-free and $\mathrm{pd}_R\, K \leqslant n$ holds.*
(d) *There is an inclusion $\partial^K(K) \subseteq (x_1,\dots,x_n)K$.*

PROOF. The homology module in part (a) is computed in 2.2.9.

(b): The claim follows from 4.4.8 and commutativity 4.4.4 of the tensor product.

(c): By induction on n, it follows from the definition, 2.4.1, of the tensor product and 1.3.10 that K is a complex of finitely generated free R-modules and concentrated in degrees $n,\dots,0$. Now K is semi-free by 5.1.3 and 8.1.2 yields $\mathrm{pd}_R\, K \leqslant n$.

(d): The inclusion follows from the last display in 2.1.25. Alternatively, it follows from 11.4.1 and the definition, 2.4.1, of the tensor product complex. □

11.4.4 Corollary. *Let $\boldsymbol{x}$ be a sequence in R and M an R-module. One has*:

(a) $$\mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x})\otimes_R M) \;\cong\; R/(\boldsymbol{x})\otimes_R M \;\cong\; M/(\boldsymbol{x})M\,.$$

(b) $\mathrm{H}_0(\mathrm{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), M)) \cong \mathrm{Hom}_R(R/(\boldsymbol{x}), M) \cong (0 :_M (\boldsymbol{x}))$.

Proof. The first isomorphism in (a) follows from 11.4.3 combined with 2.5.18(c), and the last isomorphism holds by 1.1.10. The first isomorphism in (b) follows from 11.4.3 combined with 2.5.12(c), and the last isomorphism holds by 1.1.8. □

A Koszul complex can be viewed as the mapping cone of a homothety, cf. 2.1.9.

11.4.5 Proposition. *Let $x \in R$ and M be an R-complex. There is an isomorphism of R-complexes,* $\mathrm{K}(x) \otimes_R M \cong \mathrm{Cone}\, x^M$.

Proof. There is an isomorphism of complexes $\mathrm{K}(x) \cong \mathrm{Cone}\, x^R$, and the unitor 4.4.1 identifies $x^R \otimes_R M$ with x^M. Now apply 4.1.19. □

11.4.6 Proposition. *Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in R and M an R-complex.*

(a) *One has* $(\boldsymbol{x})\,\mathrm{H}(\mathrm{K}(\boldsymbol{x}) \otimes_R M) = 0$.

(b) *For every $u \geqslant 1$ one has* $(\boldsymbol{x})^{nu}\,\mathrm{H}(\mathrm{K}(\boldsymbol{x}^u) \otimes_R M) = 0$; *in particular, the complex* $\mathrm{H}(\mathrm{K}(\boldsymbol{x}^u) \otimes_R M)$ *is $(\boldsymbol{x})$-torsion and $(\boldsymbol{x})$-complete.*

Proof. (a): For every element x_i one has $\mathrm{K}(\boldsymbol{x}) \otimes_R M \cong \mathrm{K}(x_i) \otimes_R M'$ for some complex M', see 11.4.3(b). Thus it follows from 11.4.5 and 4.1.3 that the complex $\mathrm{H}(\mathrm{K}(\boldsymbol{x}) \otimes_R M)$ is annihilated by each x_i and hence by the ideal $(\boldsymbol{x})$.

(b): The complex $\mathrm{H}(\mathrm{K}(\boldsymbol{x}^u) \otimes_R M)$ is by part (a) annihilated by $(\boldsymbol{x}^u)$, and hence by $(\boldsymbol{x})^{nu} \subseteq (\boldsymbol{x}^u)$, so by 11.2.8 and 11.1.11 it is $(\boldsymbol{x})$-torsion and $(\boldsymbol{x})$-complete. □

The next isomorphism gets referred to as "self-duality" of the Koszul complex.

11.4.7 Lemma. *For every sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in R there are isomorphisms,*

$$\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R) \cong \mathrm{Hom}_R(\mathrm{K}(x_1), R) \otimes_R \cdots \otimes_R \mathrm{Hom}_R(\mathrm{K}(x_n), R) \cong \Sigma^{-n}\,\mathrm{K}(\boldsymbol{x}) .$$

Proof. For an element $x \in R$, the complex $\mathrm{Hom}_R(\mathrm{K}(x), R)$ is concentrated in degrees 0 and -1, and after identification of $\mathrm{Hom}_R(R, R)$ with R the non-trivial differential is multiplication by $-x$; cf. 2.3.1. Thus $\mathrm{Hom}_R(\mathrm{K}(x), R) \cong \Sigma^{-1}\mathrm{K}(x)$ holds. For $n = 1$ this proves the second isomorphism; the first is tautological. The general case now follows by induction: Set $\boldsymbol{x}' = x_1, \ldots, x_{n-1}$; now (11.4.1.2) together with adjunction 4.4.12, the unitor 4.4.1, and tensor evaluation 4.5.10(d) yield

$$\begin{aligned}
\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R) &\cong \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}') \otimes_R \mathrm{K}(x_n), R) \\
&\cong \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}'), \mathrm{Hom}_R(\mathrm{K}(x_n), R)) \\
&\cong \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}'), R) \otimes_R \mathrm{Hom}_R(\mathrm{K}(x_n), R) .
\end{aligned}$$

The first of the asserted isomorphisms is now immediate, and the second follows in view of 2.4.13 and 2.4.14 from another application of (11.4.1.2). □

11.4.8 Construction. For every $x \in R$ and every $u > 1$ there is a morphism in $\mathcal{C}(R)$,

$$
\begin{array}{c}
\mathrm{K}(x^u) \\
\Big\downarrow{\scriptstyle \varkappa_x^u} \\
\mathrm{K}(x^{u-1})
\end{array}
\qquad \text{given by} \qquad
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "x^u"] \arrow[d, "x"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r, "x^{u-1}"] & R \arrow[r] & 0\,.
\end{tikzcd}
$$

The dual morphism of complexes, concentrated indegrees 0 and -1, takes the form

$$
\begin{array}{c}
\mathrm{Hom}_R(\mathrm{K}(x^{u-1}),R) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}(\varkappa_x^u,R)} \\
\mathrm{Hom}_R(\mathrm{K}(x^u),R)
\end{array}
\qquad \text{given by} \qquad
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "-x^{u-1}"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & R \arrow[r, "-x^u"] & R \arrow[r] & 0\,;
\end{tikzcd}
$$

see 11.4.7. For a sequence $\boldsymbol{x} = x_1,\dots,x_n$ in R set

$$\varkappa_{\boldsymbol{x}}^u = \varkappa_{x_1}^u \otimes_R \cdots \otimes_R \varkappa_{x_n}^u \qquad \text{and} \qquad \kappa_{\boldsymbol{x}}^u = \mathrm{Hom}_R(\varkappa_{\boldsymbol{x}}^{u+1},R)\,.$$

The family

$$\{\varkappa_{\boldsymbol{x}}^u : \mathrm{K}(\boldsymbol{x}^u) \longrightarrow \mathrm{K}(\boldsymbol{x}^{u-1})\}_{u>1}$$

is a tower, and the dual family

$$\{\kappa_{\boldsymbol{x}}^u : \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^u),R) \longrightarrow \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^{u+1}),R)\}_{u\geqslant 1}\,. \tag{11.4.8.1}$$

is a telescope. Notice that in view of 11.4.7 one has

$$\kappa_{\boldsymbol{x}}^u \cong \kappa_{x_1}^u \otimes_R \cdots \otimes_R \kappa_{x_n}^u\,. \tag{11.4.8.2}$$

The Čech Complex

11.4.9 Definition. For an element x in R, the complex

$$\check{\mathrm{C}}^R(x) = 0 \longrightarrow R \xrightarrow{\rho_x} \{x^n \mid n \geqslant 0\}^{-1}R \longrightarrow 0\,,$$

concentrated in degrees 0 and -1 and with ρ_x given by $r \mapsto \frac{r}{1}$, is called the *Čech complex* on x. The Čech complex on a sequence $x_1,\dots,x_n$ in R is

$$\check{\mathrm{C}}^R(x_1,\dots,x_n) = \check{\mathrm{C}}^R(x_1) \otimes_R \cdots \otimes_R \check{\mathrm{C}}^R(x_n)\,.$$

It is standard to set $\check{\mathrm{C}}_v^R(x_1,\dots,x_n) = \check{\mathrm{C}}^R(x_1,\dots,x_n)_v$. Where it causes no ambiguity, we often suppress the superscript R in the notation for the Čech complex.

11.4.10 Proposition. *Let $x_1,\dots,x_n$ be a sequence in R and set $C = \check{\mathrm{C}}^R(x_1,\dots,x_n)$.*

(a) *One has $\mathrm{H}_0(C) \cong \Gamma_{(x_1,\dots,x_n)}(R)$.*
(b) *C is, up to isomorphism in $\mathcal{C}(R)$, invariant under permutation of $x_1,\dots,x_n$.*
(c) *C is a complex of flat R-modules and concentrated in degrees $0,\dots,-n$; in particular, C is semi-flat and of flat dimension at most 0.*

Proof. Part (b) follows from 11.4.9 and commutativity 4.4.4 of the tensor product.

(c): The complex C is by 11.4.9 and 2.4.1 concentrated in degrees $0, \dots, -n$. Each module C_v is by 1.3.42 and 5.4.23 flat, whence the complex is semi-flat by 5.4.8 and of flat dimension at most 0 by 8.3.3.

(a): In view of part (c), the module $\mathrm{H}_0(C)$ is the kernel of $\partial_0^C = (\rho_{x_1} \ \dots \ \rho_{x_n})^{\mathrm{T}}$, which maps an element in r in $C_0 \cong R$ to $(\frac{r}{1}, \dots, \frac{r}{1})$ in $C_{-1} = R_{x_1} \oplus \cdots \oplus R_{x_n}$. As the kernel of each map ρ_{x_i} is $\Gamma_{(x_i)}(R)$, one has $\mathrm{H}_0(C) \cong \Gamma_{(x_1,\dots,x_n)}(R)$ by 11.2.19. □

At the end of this section we show that the Čech complex has finite projective dimension. For now, we proceed to show that the Čech complex can be obtained as a colimit of dual Koszul complexes, which amounts to a concrete manifestation of Govorov and Lazard's Theorem 5.5.1.

11.4.11 Construction. For every $x \in R$ and every $u \geqslant 1$ there is a morphism in $\mathcal{C}(R)$,

$$\begin{array}{ccc} \operatorname{Hom}_R(\mathrm{K}(x^u), R) & & 0 \longrightarrow R \xrightarrow{\ -x^u\ } R \longrightarrow 0 \\ \Big\downarrow \varrho_x^u & \text{that is} & \qquad \Big\| \qquad\qquad \Big\downarrow \\ \check{\mathrm{C}}(x) & & 0 \longrightarrow R \xrightarrow{\ \rho_x\ } \{x^n \mid n \geqslant 0\}^{-1}R \longrightarrow 0\,, \end{array}$$

where the map in degree -1 maps r to $-\frac{r}{x^u}$. Evidently, one has $\varrho_x^u = \varrho_x^{u+1}\kappa_x^u$.

Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R. It follows in view of (11.4.8.2) that the morphisms $\varrho_{x_1}^u \otimes \cdots \otimes \varrho_{x_n}^u$ induce a family

$$\{\varrho_{\boldsymbol{x}}^u : \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \longrightarrow \check{\mathrm{C}}(\boldsymbol{x})\}_{u \geqslant 1} \tag{11.4.11.1}$$

of morphisms that are compatible with the morphisms in the telescope (11.4.8.1), that is, $\varrho_{\boldsymbol{x}}^u = \varrho_{\boldsymbol{x}}^{u+1}\kappa_{\boldsymbol{x}}^u$ holds for all $u \geqslant 1$.

11.4.12 Theorem. *Let $\boldsymbol{x}$ be a sequence in R. The unique morphism*

$$\operatorname*{colim}_{u \geqslant 1} \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \longrightarrow \check{\mathrm{C}}(\boldsymbol{x})$$

determined by the family (11.4.11.1) *and* 3.2.5 *is an isomorphism.*

Proof. By 3.3.11 it suffices to consider the case where $\boldsymbol{x} = x$ is a single element. Let $\varrho\colon \operatorname{colim}_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{K}(x^u), R) \to \check{\mathrm{C}}(x)$ be the unique morphism that every ϱ_x^u factors through. The component ϱ_{-1} is an isomorphism by 3.2.7 and 3.3.35, and ϱ_0 is an isomorphism as well as one has $(\kappa_x^u)_0 = 1^R = (\varrho_x^u)_0$ for all $u \geqslant 1$. □

11.4.13 Proposition. *Let $\boldsymbol{x}$ be a sequence in R and M an R-complex. The homology complex* $\mathrm{H}(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R M)$ *is* $(\boldsymbol{x})$*-torsion.*

Proof. By 11.4.7, 11.4.12, and 3.2.22 the complex $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R M$ is isomorphic, up to a shift, to a filtered colimit of the complexes $\mathrm{K}(\boldsymbol{x}^u) \otimes_R M$. The homology of each of these complexes is $(\boldsymbol{x})$-torsion by 11.4.6, so the claim follows from 11.2.16. □

11.4.14 Definition. Let x be an element in R. Denote by ε_x the canonical morphism $\check{\mathrm{C}}(x) \to R$. For a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R set, cf. the unitor 4.4.1,

$$\varepsilon_{\boldsymbol{x}} = \varepsilon_{x_1} \otimes_R \cdots \otimes_R \varepsilon_{x_n} : \check{\mathrm{C}}(\boldsymbol{x}) \longrightarrow R\ .$$

11.4.15 Lemma. *Let $\boldsymbol{x}$ be a sequence in R and M an R-complex. If $\mathrm{H}(M)$ is $(\boldsymbol{x})$-torsion, then $\varepsilon_{\boldsymbol{x}} \otimes_R M : \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R M \to R \otimes_R M$ is a quasi-isomorphism.*

Proof. By the definitions, 11.4.9 and 11.4.14, and semi-flatness of Čech complexes, see 11.4.10(c), it suffices to consider the case where $\boldsymbol{x} = x$ is a single element. The exact sequence $0 \to \Sigma^{-1}(\{x^n \mid n \geqslant 0\}^{-1}R) \to \check{\mathrm{C}}(x) \xrightarrow{\varepsilon_x} R \to 0$ is degreewise split, so per 2.4.12 it remains exact after application of the functor $- \otimes_R M$. The assertion now follows from 2.1.50, 11.2.10, and 4.2.6. □

It follows from 2.2.9 that $\mathrm{H}_1(\mathrm{K}^{\mathbb{Z}}(2))$ is zero while $\mathrm{H}_1(\mathrm{K}^{\mathbb{Z}}(2,4))$ is non-zero as $[4]_{2\mathbb{Z}} = 0$. That is, even though the sequences 2 and 2, 4 generate the same ideal in $\mathbb{Z}$, the Koszul complexes $\mathrm{K}^{\mathbb{Z}}(2)$ and $\mathrm{K}^{\mathbb{Z}}(2,4)$ are not isomorphic in $\mathcal{D}(\mathbb{Z})$. The Čech complex does not exhibit this phenomenon.

11.4.16 Proposition. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences of elements in R. If they generate the same ideal, $(\boldsymbol{x}) = (\boldsymbol{y})$, then there is an isomorphism $\check{\mathrm{C}}(\boldsymbol{x}) \simeq \check{\mathrm{C}}(\boldsymbol{y})$ in $\mathcal{D}(R)$.*

Proof. The homology complex $\mathrm{H}(\check{\mathrm{C}}(\boldsymbol{y}))$ is $(\boldsymbol{x})$-torsion by 11.4.13, so 11.4.15 yields a quasi-isomorphism $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \check{\mathrm{C}}(\boldsymbol{y}) \longrightarrow \check{\mathrm{C}}(\boldsymbol{y})$. The isomorphism in $\mathcal{D}(R)$ now follows by symmetry in $\boldsymbol{x}$ and $\boldsymbol{y}$ and commutativity 4.4.4 of the tensor product. □

11.4.17 Proposition. *Let $\boldsymbol{x}$ be a sequence in R. The next conditions are equivalent.*

(*i*) *The Koszul complex $\mathrm{K}^R(\boldsymbol{x})$ is acyclic.*
(*ii*) *The Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ is acyclic.*
(*iii*) *One has $(\boldsymbol{x}) = R$.*

Proof. By 11.4.3(a) one has $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x})) \cong R/(\boldsymbol{x})$, so (*i*) implies (*iii*).

(*iii*) ⇒ (*ii*): One has $(\boldsymbol{x}) = (1)$, so 11.4.16 yields an isomorphism $\check{\mathrm{C}}(\boldsymbol{x}) \simeq \check{\mathrm{C}}(1)$ in $\mathcal{D}(R)$. The complex $\check{\mathrm{C}}(1)$ is the disk complex $\mathrm{D}^0(R)$, in particular it is acyclic, and hence so is $\check{\mathrm{C}}(\boldsymbol{x})$.

(*ii*) ⇒ (*i*): The homology of $\mathrm{K}(\boldsymbol{x})$ is $(\boldsymbol{x})$-torsion, see 11.4.6(b). Now it follows from 11.4.15 that there is a quasi-isomorphism,

$$\varepsilon_{\boldsymbol{x}} \otimes_R \mathrm{K}(\boldsymbol{x}) : \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \mathrm{K}(\boldsymbol{x}) \xrightarrow{\simeq} R \otimes_R \mathrm{K}(\boldsymbol{x})\ .$$

As $\mathrm{K}(\boldsymbol{x})$ is semi-free, in particular semi-flat, see 11.4.3(c), the complex $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \mathrm{K}(\boldsymbol{x})$ is acyclic and per the unitor 4.4.1 so is the Koszul complex $\mathrm{K}(\boldsymbol{x})$. □

Base Change

If S is an R-algebra, then elements of R can be considered as elements of S via the structure map, and for a sequence $\boldsymbol{x}$ in R one can thus consider the Koszul and Čech complexes $\mathrm{K}^S(\boldsymbol{x})$ and $\check{\mathrm{C}}^S(\boldsymbol{x})$.

11.4.18 Proposition. *Let $\boldsymbol{x}$ be a sequence in R and S be an R-algebra. There are isomorphisms of S-complexes,*

$$S \otimes_R \mathrm{K}^R(\boldsymbol{x}) \;\cong\; \mathrm{K}^S(\boldsymbol{x}) \qquad \textit{and} \qquad S \otimes_R \check{\mathrm{C}}^R(\boldsymbol{x}) \;\cong\; \check{\mathrm{C}}^S(\boldsymbol{x})\,.$$

Proof. For an element $x \in R$ one has per (11.4.1.1), the unitor 4.4.1, and R-linearity of the tensor product, see 2.4.9, an isomorphism of S-complexes,

$$S \otimes_R \mathrm{K}^R(x) \;\cong\; 0 \longrightarrow S \xrightarrow{\;x\;} S \longrightarrow 0 \;=\; \mathrm{K}^S(x)\,.$$

The first of the asserted isomorphisms now follows from 4.4.1 and (11.4.1.2). Similarly, the second isomorphism follows from 4.4.1 and 11.4.9, as there is an isomorphism of S-complexes,

$$S \otimes_R \check{\mathrm{C}}^R(x) \;\cong\; 0 \longrightarrow S \xrightarrow{\;\rho_x\;} \{x^n \mid n \geqslant 0\}^{-1} S \longrightarrow 0 \;=\; \check{\mathrm{C}}^S(x)\,. \qquad \square$$

Semi-Free Resolution of the Čech Complex

11.4.19 Construction. Let x be an element in R. Consider the complex of free R-modules concentrated in degrees 0 and -1,

$$\mathrm{L}(x) \;=\; 0 \longrightarrow R\langle E\rangle \xrightarrow{\;\eth^x\;} R\langle E\rangle \longrightarrow 0\,,$$

where $E = \{e_i \mid i \in \mathbb{N}_0\}$ and the differential $\eth^x$ is given by

$$\eth^x(e_i) \;=\; \begin{cases} e_0 & \text{for } i = 0 \\ e_{i-1} - x e_i & \text{for } i \geqslant 1\,. \end{cases}$$

For every $u \geqslant 1$ the differential $\eth^x$ maps the submodule $R\langle e_0, \dots, e_u\rangle$ of $R\langle E\rangle$ to itself, whence one has the following subcomplex of $\mathrm{L}(x)$,

$$\mathrm{L}^u(x) \;=\; 0 \longrightarrow R\langle e_0, \dots, e_u\rangle \xrightarrow{\;\eth^x\;} R\langle e_0, \dots, e_u\rangle \longrightarrow 0\,.$$

Write $\iota^u_x \colon \mathrm{L}^u(x) \rightarrowtail \mathrm{L}^{u+1}(x)$ and $\bar{\iota}^u_x \colon \mathrm{L}^u(x) \rightarrowtail \mathrm{L}(x)$ for the canonical embeddings.

For a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R and $u \geqslant 1$ set

$$\begin{aligned} \mathrm{L}(\boldsymbol{x}) &= \mathrm{L}(x_1) \otimes_R \cdots \otimes_R \mathrm{L}(x_n)\,, \\ \mathrm{L}^u(\boldsymbol{x}) &= \mathrm{L}^u(x_1) \otimes_R \cdots \otimes_R \mathrm{L}^u(x_n)\,, \\ \iota^u_{\boldsymbol{x}} &= \iota^u_{x_1} \otimes_R \cdots \otimes_R \iota^u_{x_n}\,, \quad \text{and} \\ \bar{\iota}^u_{\boldsymbol{x}} &= \bar{\iota}^u_{x_1} \otimes_R \cdots \otimes_R \bar{\iota}^u_{x_n}\,. \end{aligned}$$

Notice that $\mathrm{L}(\boldsymbol{x})$ and $\mathrm{L}^u(\boldsymbol{x})$ are bounded complexes of free R-modules concentrated in degrees $0, \dots, -n$; in particular, they are semi-free per 5.1.3. Further, $\iota^u_{\boldsymbol{x}}$ and $\bar{\iota}^u_{\boldsymbol{x}}$ are morphisms $\mathrm{L}^u(\boldsymbol{x}) \to \mathrm{L}^{u+1}(\boldsymbol{x})$ and $\mathrm{L}^u(\boldsymbol{x}) \to \mathrm{L}(\boldsymbol{x})$.

The complexes and morphisms constructed in 11.4.19 are denoted by symbols that do refer to the ring R. In situations where this might cause ambiguity, we add a superscript "R"; for example, we write $\mathrm{L}^R(\boldsymbol{x})$ instead of $\mathrm{L}(\boldsymbol{x})$, just as for Koszul and Čech complexes, see 11.4.1 and 11.4.9.

11.4.20 Proposition. *Let $\boldsymbol{x}$ be a sequence in R and S an R-algebra. There is an isomorphisms of S-complexes,*

$$S \otimes_R \mathrm{L}^R(\boldsymbol{x}) \;\cong\; \mathrm{L}^S(\boldsymbol{x})\,.$$

Proof. For an element $x \in R$ there is by 11.4.19, 3.1.13, 4.4.1, and R-linearity of the tensor product, see 2.4.9, an isomorphism of S-complexes,

$$S \otimes_R \mathrm{L}^R(x) \;\cong\; 0 \longrightarrow S\langle E\rangle \xrightarrow{(\eth^x)^S} S\langle E\rangle \longrightarrow 0 \;=\; \mathrm{L}^S(x)\,.$$

The asserted isomorphism now follows from another application of the unitor 4.4.1 and the definitions of $\mathrm{L}^R(\boldsymbol{x})$ and $\mathrm{L}^S(\boldsymbol{x})$, see 11.4.19. □

11.4.21 Lemma. *Let $\boldsymbol{x}$ be a sequence in R. For every $u \geqslant 1$ the morphisms $\iota^u_{\boldsymbol{x}}$ and $\bar{\iota}^u_{\boldsymbol{x}}$ are injective, and the exact sequences*

$$0 \longrightarrow \mathrm{L}^u(\boldsymbol{x}) \xrightarrow{\iota^u_{\boldsymbol{x}}} \mathrm{L}^{u+1}(\boldsymbol{x}) \longrightarrow \mathrm{L}^{u+1}(\boldsymbol{x})/\mathrm{L}^u(\boldsymbol{x}) \longrightarrow 0 \quad \textit{and}$$
$$0 \longrightarrow \mathrm{L}^u(\boldsymbol{x}) \xrightarrow{\bar{\iota}^u_{\boldsymbol{x}}} \mathrm{L}(\boldsymbol{x}) \longrightarrow \mathrm{L}(\boldsymbol{x})/\mathrm{L}^u(\boldsymbol{x}) \longrightarrow 0$$

are degreewise split.

Proof. For a sequence of one element, this is clear from the definitions in 11.4.19. The general case follows in view of (2.4.5.1) and 2.4.11/2.4.12 by induction. □

11.4.22 Lemma. *Let $\boldsymbol{x}$ be a sequence in R. The colimit of the telescope*

$$\{\iota^u_{\boldsymbol{x}} : \mathrm{L}^u(\boldsymbol{x}) \rightarrowtail \mathrm{L}^{u+1}(\boldsymbol{x})\}_{u\geqslant 1}$$

is $\mathrm{L}(\boldsymbol{x})$ *with canonical morphisms* $\bar{\iota}^u_{\boldsymbol{x}} \colon \mathrm{L}^u(\boldsymbol{x}) \rightarrowtail \mathrm{L}(\boldsymbol{x})$.

Proof. The claims are evident for a sequence of one element, see 3.3.34. By 3.3.11 the general case follows by induction on the number of elements in $\boldsymbol{x}$. □

11.4.23 Construction. Let $x \in R$ and $u \geqslant 1$. Consider the morphism of complexes

$$\begin{tikzcd} \mathrm{L}^u(x) \arrow[d, "\pi^u_x"] \\ \mathrm{Hom}_R(\mathrm{K}(x^u), R) \end{tikzcd} \quad \text{that is} \quad \begin{tikzcd} 0 \arrow[r] & R\langle e_0,\dots,e_u\rangle \arrow[r, "\eth^x"] \arrow[d, "(\pi^u_x)_0"] & R\langle e_0,\dots,e_u\rangle \arrow[r] \arrow[d, "(\pi^u_x)_{-1}"] & 0 \\ 0 \arrow[r] & R \arrow[r, "-x^u"] & R \arrow[r] & 0 \end{tikzcd}$$

given by

$$(\pi^u_x)_0(e_i) \;=\; \begin{cases} 1 & \text{for } \; i = 0 \\ 0 & \text{for } \; 1 \leqslant i \leqslant u \end{cases} \qquad \text{and} \qquad (\pi^u_x)_{-1}(e_i) \;=\; -x^{u-i} \quad \text{for } \; 0 \leqslant i \leqslant u\,.$$

Furthermore, let

$$\begin{tikzcd} \mathrm{L}(x) \arrow[d, "\pi_x"] \\ \check{\mathrm{C}}(x) \end{tikzcd} \quad \text{that is} \quad \begin{tikzcd} 0 \arrow[r] & R\langle E\rangle \arrow[r, "\eth^x"] \arrow[d, "(\pi_x)_0"] & R\langle E\rangle \arrow[r] \arrow[d, "(\pi_x)_{-1}"] & 0 \\ 0 \arrow[r] & R \arrow[r, "\rho_x"] & \{x^n \mid n \geqslant 0\}^{-1}R \arrow[r] & 0 \end{tikzcd}$$

be the morphism given by

$$(\pi_x)_0(e_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{for } i \geqslant 1 \end{cases} \qquad \text{and} \qquad (\pi_x)_{-1}(e_i) = \frac{1}{x^i} \quad \text{for } i \geqslant 0\,.$$

Finally, for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R set

$$\pi^u_{\boldsymbol{x}} = \pi^u_{x_1} \otimes_R \cdots \otimes_R \pi^u_{x_n} \qquad \text{and} \qquad \pi_{\boldsymbol{x}} = \pi_{x_1} \otimes_R \cdots \otimes_R \pi_{x_n}\,.$$

Note that $\pi_{\boldsymbol{x}}$ is a morphism $\mathrm{L}(\boldsymbol{x}) \to \check{\mathrm{C}}(\boldsymbol{x})$. In view of 11.4.7 consider $\pi^u_{\boldsymbol{x}}$ as a morphism $\mathrm{L}^u(\boldsymbol{x}) \to \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R)$.

11.4.24 Lemma. *Let $\boldsymbol{x}$ be a sequence in R. The morphisms from* 11.4.23, 11.4.19, *and* (11.4.8.1) *fit into a commutative diagram,*

$$\begin{tikzcd}
\mathrm{L}^u(\boldsymbol{x}) \arrow[r, "\iota^u_{\boldsymbol{x}}"] \arrow[d, "\pi^u_{\boldsymbol{x}}"'] & \mathrm{L}^{u+1}(\boldsymbol{x}) \arrow[d, "\pi^{u+1}_{\boldsymbol{x}}"] \\
\operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \arrow[r, "\kappa^u_{\boldsymbol{x}}"] & \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^{u+1}), R)\,.
\end{tikzcd}$$

Thus, $\{\pi^u_{\boldsymbol{x}}\}_{u \geqslant 0}$ is a morphism between the telescopes from 11.4.22 *and* (11.4.8.1)*.*

Proof. By the definitions of the maps it is sufficient to consider the case where $\boldsymbol{x} = x$ is a single element, and in that case commutativity is straightforward to verify from the definitions. □

11.4.25 Theorem. *Let $\boldsymbol{x}$ be a sequence in R. The following assertions about the morphisms from* 11.4.23 *hold.*

(a) *For every $u \geqslant 1$ the morphism $\pi^u_{\boldsymbol{x}}$ is a homotopy equivalence.*
(b) *There is an identity $\pi_{\boldsymbol{x}} = \operatorname{colim}_{u \geqslant 1} \pi^u_{\boldsymbol{x}}$ of morphisms from $\mathrm{L}(\boldsymbol{x})$ to $\check{\mathrm{C}}(\boldsymbol{x})$.*
(c) *The morphism $\pi_{\boldsymbol{x}} \colon \mathrm{L}(\boldsymbol{x}) \xrightarrow{\simeq} \check{\mathrm{C}}(\boldsymbol{x})$ is a semi-free resolution.*

Proof. (a): By 4.3.20 it suffices in view of (2.4.5.1) to consider the case where $\boldsymbol{x} = x$ is a single element. Since π^u_x is morphism between bounded complexes of free modules, it suffices by 5.2.8 and 5.2.21 to argue that π^u_x is a quasi-isomorphism.

The homomorphism $\mathrm{H}_0(\pi^u_x)$ is the (co)restriction of $(\pi^u_x)_0$ to kernels, i.e. the map

$$R\langle e_0, \dots, e_u \rangle \cap \operatorname{Ker} \eth^x \longrightarrow (0 :_R x^u)$$

that sends $z = r_0 e_0 + \cdots + r_u e_u$ to $(\pi^u_x)_0(z) = r_0$; see 11.4.11 and 11.4.23. From

$$0 = \eth^x(z) = (r_0 + r_1)e_0 + (r_2 - x r_1)e_1 + \cdots + (r_u - x r_{u-1})e_{u-1} - x r_u e_u$$

one gets $r_1 = -r_0$, $r_2 = -x r_0, \dots, r_u = -x^{u-1} r_0$, and $x r_u = 0$. Hence $x^u r_0 = 0$ and z has the form $z = r_0(e_0 - e_1 - x e_2 - \cdots - x^{u-1} e_u)$. Thus $\mathrm{H}_0(\pi^u_x)$ is bijective with inverse given by $r \mapsto r(e_0 - e_1 - x e_2 - \cdots - x^{u-1} e_u)$.

The homomorphism $\mathrm{H}_{-1}(\pi^u_x)$ is the by $(\pi^u_x)_{-1}$ induced map on cokernels,

$$\operatorname{Coker} \eth^x \longrightarrow R/(x^u)\,,$$

where $\eth^x$ is considered as an endomorphism on $R\langle e_0,\dots,e_u\rangle$. Since $(\pi_x^u)_{-1}$ is surjective, so is $\mathrm{H}_{-1}(\pi_x^u)$. To prove injectivity, let $z = r_0e_0+\cdots+r_ue_u$ be an element in $R\langle e_0,\dots,e_u\rangle$ with $(\pi_x^u)_{-1}(z) \in (x^u)$. As one has

$$(\pi_x^u)_{-1}(z) \;=\; -x^ur_0 - x^{u-1}r_1 - \cdots - xr_{u-1} - r_u\,,$$

the equality $-x^ur_0 - x^{u-1}r_1 - \cdots - xr_{u-1} - r_u = x^us$ holds for some $s \in R$. The goal is to construct an element $w = s_0e_0 + \cdots + s_ue_u$ in $R\langle e_0,\dots,e_u\rangle$ with $\eth^x(w) = z$. This amounts to finding elements $s_0,\dots,s_u$ in R with

$$s_0 + s_1 \;=\; r_0\,, \quad s_{i+1} - xs_i \;=\; r_i \;\text{ for }\; i \in \{1,\dots,u-1\}\,, \quad\text{and}\quad -xs_u \;=\; r_u\,.$$

Set $s_0 = -s$. Solving the equations one gets $s_1 = r_0 + s$, $s_2 = r_1 + xs_1 = r_1 + xr_0 + xs$, and generally $s_{i+1} = r_i + \cdots + x^{i-1}r_1 + x^ir_0 + x^is$ for $i \in \{1,\dots,u-1\}$. It remains to see that the last of these elements, that is, $s_u = r_{u-1} + \cdots + x^{u-2}r_1 + x^{u-1}r_0 + x^{u-1}s$, satisfies $-xs_u = r_u$, but this follows from the defining property of s.

(b): By 3.2.10, 11.4.12, and 11.4.22 it is sufficient to argue that the diagram

$$\begin{tikzcd}
\mathrm{L}^u(\boldsymbol{x}) \arrow[r, "\bar{\iota}_{\boldsymbol{x}}^u"] \arrow[d, "\pi_{\boldsymbol{x}}^u"'] & \mathrm{L}(\boldsymbol{x}) \arrow[d, "\pi_{\boldsymbol{x}}"] \\
\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \arrow[r, "\varrho_{\boldsymbol{x}}^u"] & \check{\mathrm{C}}(\boldsymbol{x})
\end{tikzcd}$$

is commutative for every $u \geqslant 1$. By the definitions of the morphisms involved, it suffices to consider the case where $\boldsymbol{x} = x$ is a single element. That is, it must be verified that $(\pi_x)_v(\bar{\iota}_x^u)_v = (\varrho_x^u)_v(\pi_x^u)_v$ holds for $v \in \{-1, 0\}$. For $v = 0$ one has

$$(\pi_x)_0(\bar{\iota}_x^u)_0(e_0) \;=\; (\pi_x)_0(e_0) \;=\; 1 \;=\; (\varrho_x^u)_0(1) \;=\; (\varrho_x^u)_0(\pi_x^u)_0(e_0)$$

and

$$(\pi_x)_0(\bar{\iota}_x^u)_0(e_i) \;=\; (\pi_x)_0(e_i) \;=\; 0 \;=\; (\varrho_x^u)_0(0) \;=\; (\varrho_x^u)_0(\pi_x^u)_0(e_i) \;\text{ for } 1 \leqslant i \leqslant u\,.$$

For $v = -1$ the equality also holds since for every $i \in \{0,\dots,u\}$ one has

$$(\pi_x)_{-1}(\bar{\iota}_x^u)_{-1}(e_i) \;=\; (\pi_x)_{-1}(e_i) \;=\; \frac{1}{x^i} \;=\; (\varrho_x^u)_{-1}(-x^{u-i}) \;=\; (\varrho_x^u)_{-1}(\pi_x^u)_{-1}(e_i)\,.$$

(c): The complex $\mathrm{L}(\boldsymbol{x})$ is bounded and semi-free, see 11.4.19. The morphism $\pi_{\boldsymbol{x}}$ is a quasi-isomorphism by parts (a) and (b), 4.3.4, and 4.2.12. □

11.4.26 Corollary. *Let $\boldsymbol{x}$ be a sequence in R. The Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ has projective dimension at most* 0*.*

Proof. By 11.4.25(c) the complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ has a semi-free resolution concentrated in non-positive degrees, so the assertion follows from 5.2.11 and 8.1.2. □

Exercises

E 11.4.1 Write the dual of the complex $K = \mathrm{K}(x_1, x_2)$ from 11.4.2 explicitly with the differential represented by matrices and determine an isomorphism $\mathrm{Hom}_R(K, R) \to \Sigma^{-2}K$.

E 11.4.2 Give explicit descriptions, as in 11.4.2, of the complexes $\check{\mathrm{C}}(x_1, x_2)$ and $\check{\mathrm{C}}(x_1, x_2, x_3)$.

E 11.4.3 Let K be the Koszul complex on a sequence of elements in R. Show that for every R-complex M there is isomorphism $\operatorname{Hom}_R(M, K) \cong \operatorname{Hom}_R(M, R) \otimes_R K$.

E 11.4.4 Let K be the Koszul complex on a sequence of n elements in R. Show that for every R-complex M there is an isomorphism $\operatorname{Hom}_R(K, M) \cong \Sigma^{-n}(K \otimes_R M)$.

E 11.4.5 Let $x \in R$ and M be an R-module. Show that the homothety x^M is injective if and only if $\mathrm{H}_1(\mathrm{K}(x) \otimes_R M) = 0$ holds and surjective if and only if $\mathrm{H}_0(\mathrm{K}(x) \otimes_R M) = 0$.

E 11.4.6 Let $x \in R$ and show directly from the defintion that $\pi_x\colon \mathrm{L}(x) \to \check{\mathrm{C}}(x)$ is a quasi-isomorphism.

E 11.4.7 Let $\boldsymbol{x}$ be a sequence of n elements in R. Show that $(\boldsymbol{x})^{nu}\,\mathrm{H}(\mathrm{L}^u(\boldsymbol{x}) \otimes_R M) = 0$ holds for every R-complex M.

E 11.4.8 Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in R. Show that if the ideals $(\boldsymbol{x})$ and $(\boldsymbol{y})$ are topologically equivalent, then there is an isomorphism $\check{\mathrm{C}}(\boldsymbol{x}) \simeq \check{\mathrm{C}}(\boldsymbol{y})$ in $\mathcal{D}(R)$.

E 11.4.9 Let $\boldsymbol{x}$ be a sequence in R. Show that the Koszul complex $\mathrm{K}^R(\boldsymbol{x})$ is contractible if and only if the Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ is pure acyclic.

Part III
Applications
in Commutative Algebra

THE MAINSTREAM homological theory of commutative Noetherian rings is the topic of this third part of the book. Results from the previous two parts and from the appendices are used extensively. Because modules over a commutative ring always are considered to be symmetric bimodules, several key results from Parts I and II simplify in the commutative setting. The simplifications are particularly compelling when it comes to statements about the (derived) Hom and tensor product functors, standard isomorphisms, and evaluation morphisms—enough so to warrant restatement here in Part III. The greater part of Chap. 12 is dedicated to these simplified versions of statements from Sects. 4.4–4.5, 7.3–7.6, and 8.4; this is where a commutative algebraist acquainted with derived categories can pick up the track.

Parts I and II and the appendices are essentially self-contained—they rely only on basic facts from ring theory and category theory—but in this third part we appeal to a few "real theorems". They are with one exception classic results, such as the Artin–Rees Lemma, that can be found in any standard textbook on commutative algebra, including [182]. The exception is André's work [5] on the Homological Conjectures, which we invoke through the existence of Big Cohen–Macaulay modules.

The transition to commutative rings happened already in Chap. 11, but the material there is still part of general ring theory rather than commutative algebra. If commutative algebra has a mark of Cain, it must be the reliance on prime ideal spectra. Readers are expected to be familiar with the various sets of prime ideals that get linked to rings and modules in classic commutative algebra, but to fix terminology and notation we include a brief review in Sect. 12.4. Matlis' structure theory for injective modules—in itself a quintessential example of the importance of prime ideal spectra—may also be familiar to many readers, nevertheless it is included in Appn. C. This is not only to honor the promise that "we assume no prerequisites in homological algebra" but also because we need several technical statements from the theory, not just the main structure theorem.

The forerunners of this book, and this third part in particular, are two sets of notes [95, 96] used by Foxby for his lectures at the University of Copenhagen. Both sets ultimately had wider circulation, but only in printed form. Building on the foundations laid down by Dold [73] and by Grothendieck and Hartshorne [114] the lecture notes present systematic extensions of homological invariants from modules to complexes. Further, established results on homological properties of modules are reproved in the broader context of complexes and often improved in the process. Some of these definitions and results made it into Foxby's papers [91, 92, 93, 100, 94] from the late 1970s and others appeared in background sections of his later collaborative works with Avramov [20, 21, 22, 23, 24], Iyengar [98], and his students [59, 97].

THROUGHOUT this part—that is, in Chaps. 12 through 20—**the rings $\Bbbk$, Q, R, S, and T are assumed to be commutative and Noetherian.** When only considering complexes over a single ring, R, one can simply invoke results from Parts I and II with $\Bbbk = R$. In multi-ring situations, that approach assigns priority to R—the rings Q, S, and T are assumed to be algebras over $\Bbbk$—so we keep the ground ring $\Bbbk$ around and only assume relations between Q, R, S, and T where needed.

Chapter 12
A Brief for Commutative Ring Theorists

In this chapter we recapitulate some key results from Parts I and II whose formulations become notably simpler in the setting of commutative Noetherian rings. A convention introduced in Chap. 1, namely that modules over a commutative ring tacitly are considered to be symmetric bimodules, now becomes crucial. The convention is both standard and useful, and not to adopt it might as a matter of fact be outright confusing: As we do not distinguish between a commutative ring and its opposite ring, there would be a clash with the conventions that all modules are left modules. The convention does not surrender any generality: In the case one needs to consider two disinct actions of a commutative ring on the same Abelian group M, one simply uses two different symbols, say R and R', for the ring to distingush the actions. This way, M becomes a symmetric R–R^{o}-bimodule, a symmetric R'–R'^{o}-bimodule, and, provided that the actions are compatible, an R–R'-bimodule.

The convention means that objects in $\mathcal{M}(R)$ are tacitly considered objects in the category $\mathcal{M}(R\text{–}R) = \mathcal{M}(R \otimes_{\Bbbk} R)$. One can, of course, not identify the two categories, but one can avoid considering objects in $\mathcal{M}(R \otimes_{\Bbbk} R)$ that are not symmetric bimodules. This can be achieved quit easily thanks to the orthographic trick discussed above and by way of 12.1.1 and 12.1.3.

12.1 Standard Isomorphisms and Evaluation Morphisms in $\mathcal{C}$

Synopsis. Symmetric bimodule; the functors Hom and $\otimes$; unitor; counitor; commutativity; associativity; swap; adjunction; biduality; tensor evaluation; homomophism evaluation; base change; cobase change; dual numbers.

We start this section by observing that the assumption that R-modules are symmetric bimodules very conveniently implies that the Hom as well as the tensor product of two R-modules is a symmetric bimodule.

Hom Functor

12.1.1. Let M and N be R-modules; by convention they are both symmetric R–R-bimodules, so $\operatorname{Hom}_R(M, N)$ is by 1.1.30 an object in $\mathcal{M}(R\text{–}R) = \mathcal{M}(R \otimes_{\Bbbk} R)$ with one R-action coming from M and the other from N. However, for $\varphi \in \operatorname{Hom}_R(M, N)$, $m \in M$, and $n \in N$ and, one has

$$(r\varphi)(m) = \varphi(mr) = \varphi(rm) = r(\varphi(m)) = (\varphi(m))r = (\varphi r)(m) .$$

Indeed, the 1st and 5th equalities come from 1.1.30, the 2nd and 4th hold because M and N are symmetric bimodules, and the 3rd equality holds by R-linearity of φ. This shows that $\operatorname{Hom}_R(M, N)$ is a symmetric R–R-bimodule. By convention, every object in $\mathcal{M}(R)$ is a symmetric bimodule, so one can without ambiguity talk about the functor

$$\operatorname{Hom}_R(-,-) : \mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R) \longrightarrow \mathcal{M}(R) .$$

For modules $M \in \mathcal{M}(R\text{–}Q)$ and $N \in \mathcal{M}(R\text{–}S)$ the module $\operatorname{Hom}_R(M, N)$, which by 1.1.30 is a Q–S-bimodule, thus has a third compatible ring action, namely the symmetric R-actions through M and N. We do not introduce a notation for such trimodules but point out that in the case Q is an R-algebra and M a Q-module, or S is an R-algebra and N an S-module, the R-action is accounted for as every Q–S-bimodule is an R-module via the structure map.

The Hom functor on R-complexes is established in 2.3.10, but its output is more than a $\Bbbk$-complex now that R is assumed to be commutative.

12.1.2. By 2.3.11 and 12.1.1 there is a functor

$$\operatorname{Hom}_R(-,-) : \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \longrightarrow \mathcal{C}(R) .$$

More generally, 2.3.11 yields a functor

$$\operatorname{Hom}_R(-,-) : \mathcal{C}(R\text{–}Q)^{\mathrm{op}} \times \mathcal{C}(R\text{–}S) \longrightarrow \mathcal{C}(Q\text{–}S) . \tag{12.1.2.1}$$

In particular, if Q and S are R-algebras, then there is a functor

$$\operatorname{Hom}_R(-,-) : \mathcal{C}(Q)^{\mathrm{op}} \times \mathcal{C}(S) \longrightarrow \mathcal{C}(Q\text{–}S) .$$

By 3.1.24 and 3.1.27 these functors preserve products in both variables.

Caveat. To avoid R–R-bimodules that are not symmetric, (12.1.2.1) should for $Q = R$ be interpreted narrowly to mean only that Hom_R is a functor from $\mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R\text{–}S)$ to $\mathcal{C}(R\text{–}S)$. Similarly, for $S = R$ one should interpret it to mean that Hom_R is a functor from $\mathcal{C}(R\text{–}Q)^{\mathrm{op}} \times \mathcal{C}(R)$ to $\mathcal{C}(Q\text{–}R)$. For $Q = S$ there is no guarantee that the Q- and S-actions on Hom_R are symmetric, so the only remedy is use two symbols for the ring.

Tensor Product Functor

12.1.3. Let M and N be R-modules; by convention they are both symmetric R–R-bimodules, so $M \otimes_R N$ is by 1.1.33 an object in $\mathcal{M}(R\text{–}R) = \mathcal{M}(R \otimes_{\Bbbk} R)$ with one R-action coming from M and the other from N. Yet, for $m \in M$ and $n \in N$ one has

$$r(m \otimes n) \;=\; rm \otimes n \;=\; mr \otimes n \;=\; m \otimes rn \;=\; m \otimes nr \;=\; (m \otimes n)r\;.$$

Indeed, the 1st and 5th equalities come from 1.1.33, the 2nd and 4th hold because M and N are symmetric bimodules, and the 3rd equality holds by the definition of the tensor product over R. This shows that $M \otimes_R N$ is a symmetric R–R-bimodule. By convention every object in $\mathcal{M}(R)$ is a symmetric bimodule, so one can without ambiguity talk about the functor

$$- \otimes_R - : \mathcal{M}(R) \times \mathcal{M}(R) \longrightarrow \mathcal{M}(R)\;.$$

For modules $M \in \mathcal{M}(Q\text{–}R)$ and $N \in \mathcal{M}(R\text{–}S)$ the module $M \otimes_R N$, which by 1.1.33 is a Q–S-bimodule, thus has a third compatible ring action, namely the symmetric R-actions through M and N. Notice that in the important special cases where Q is an R-algebra and M a Q-module, or S is an R-algebra and N an S-module, the R-action is accounted for as every Q–S-bimodule is an R-module via the structure map.

The tensor product functor on R-complexes is established in 2.4.9, but its output is more than a $\Bbbk$-complex now that R is assumed to be commutative.

12.1.4. By 2.4.10 and 12.1.3 there is a functor

$$- \otimes_R - : \mathcal{C}(R) \times \mathcal{C}(R) \longrightarrow \mathcal{C}(R)\;.$$

More generally, 2.4.10 yields a functor

$$(12.1.4.1) \qquad - \otimes_R - : \mathcal{C}(Q\text{–}R) \times \mathcal{C}(R\text{–}S) \longrightarrow \mathcal{C}(Q\text{–}S)\;.$$

In particular, if Q and S are R-algebras, then there is a functor

$$- \otimes_R - : \mathcal{C}(Q) \times \mathcal{C}(S) \longrightarrow \mathcal{C}(Q\text{–}S)\;.$$

By 3.1.30 and 3.1.31 these functors preserve coproducts in both variables.

Caveat. To avoid R–R-bimodules that are not symmetric, (12.1.4.1) should for $Q = R$ be interpreted narrowly to mean only that $\otimes_R$ is a functor from $\mathcal{C}(R) \times \mathcal{C}(R\text{–}S)$ to $\mathcal{C}(R\text{–}S)$ and, similarly, for $R = S$ that it is a functor from $\mathcal{C}(Q\text{–}R) \times \mathcal{C}(R)$ to $\mathcal{C}(Q\text{–}R)$. For $Q = S$ there is no guarantee that the Q- and S-actions on the tensor product are symmetric, so one needs two symbols for that ring.

Standard Isomorphisms

Proposition. *For a complex $M \in \mathcal{C}(R\text{–}S)$ the maps*

12.1.5 $$R \otimes_R M \xrightarrow{\;\mu_R^M\;} M \qquad \text{(unitor)}$$

12.1.6 $$M \xrightarrow{\;\epsilon_R^M\;} \operatorname{Hom}_R(R, M) \qquad \text{(counitor)}$$

from 4.4.1 *and* 4.4.2 *are isomorphisms in* $\mathcal{C}(R\text{–}S)$.

Proof. The claims are special cases of 4.4.1 and 4.4.2. ◻

For R-complexes M and N, commutativity υ^{MN} is by default an isomorphism of $\Bbbk$-complexes, see 4.4.4. We now show that υ^{MN} is an isomorphism of R-complexes. One could of course get that by applying 4.4.4 with $\Bbbk = R$, but taking that shortcut would constrain our ability to consider additional ring actions on M and N, as all rings are assumed to be $\Bbbk$-algebras.

12.1.7 Proposition. *For R-complexes M and N the commutativity map,*

$$\upsilon^{MN} : M \otimes_R N \longrightarrow N \otimes_R M \,,$$

is an isomorphism in $\mathcal{C}(R)$. If M belongs to $\mathcal{C}(Q\text{–}R)$, then υ^{MN} is an isomorphism in $\mathcal{C}(Q\text{–}R)$, and if N belongs to $\mathcal{C}(R\text{–}S)$, then υ^{MN} is an isomorphism in $\mathcal{C}(R\text{–}S)$. Moreover, if Q and S are R-algebras, M a Q-complex, and N an S-complex, then υ^{MN} is an isomorphism in $\mathcal{C}(Q\text{–}S)$.

Proof. The assertions about υ^{MN} being an isomorphism in $\mathcal{C}(R)$, $\mathcal{C}(Q\text{–}R)$, or $\mathcal{C}(R\text{–}S)$ are in view of 12.1.4 special cases of 4.4.4. If Q is an R-algebra and M a Q-complex, then M belongs to $\mathcal{C}(Q\text{–}R)$, and similarly an S-complex N is in $\mathcal{C}(R\text{–}S)$ if S is an R-algebra; in this case, υ^{MN} is an isomorphism in $\mathcal{C}(Q\text{–}S)$ by 4.4.4. $\square$

In keeping with the purpose of this chapter, 12.1.7 as well as 12.1.8–12.1.13 below are not stated in their most general form for commutative rings but rather in a generality that serves our needs and does no unduly invite consideration of asymmetric bimodule structures.

Proposition. *For complexes $M \in \mathcal{C}(R)$, $X \in \mathcal{C}(R\text{–}S)$, and $N \in \mathcal{C}(S)$ the maps*

12.1.8 $$(M \otimes_R X) \otimes_S N \xrightarrow{\omega^{MXN}} M \otimes_R (X \otimes_S N) \qquad \text{(associativity)}$$

12.1.9 $$\operatorname{Hom}_R(M, \operatorname{Hom}_S(N,X)) \xrightarrow{\zeta^{MXN}} \operatorname{Hom}_S(N, \operatorname{Hom}_R(M,X)) \qquad \text{(swap)}$$

12.1.10 $$\operatorname{Hom}_R(X \otimes_S N, M) \xrightarrow{\rho^{MXN}} \operatorname{Hom}_S(N, \operatorname{Hom}_R(X,M)) \qquad \text{(adjunction)}$$

from 4.4.6, 4.4.9, *and* 4.4.11 *are isomorphisms in $\mathcal{C}(R\text{–}S)$. Moreover, if Q is an R-algebra and M a Q-complex, and T is an S-algebra and N a T-complex, then associativity ω^{MXN}, swap ζ^{MXN}, and adjunction ρ^{MXN} are isomorphisms in $\mathcal{C}(Q\text{–}T)$.*

Proof. The assertions follow from 4.4.7, 4.4.10, and 4.4.12. Take as an example associativity ω^{MXN}. As M and N are complexes of symmetric bimodules, it follows from 4.4.7 that ω^{MXN} is an isomorphism in $\mathcal{C}(R\text{–}S)$, cf. 12.1.4. Moreover, if Q and T are algebras over R and S, respectively, then a Q-complex belongs to $\mathcal{C}(Q\text{–}R)$ and a T-complex belongs to $\mathcal{C}(S\text{–}T)$, so it follows from 4.4.7 that ω^{MXN} is an isomorphism in $\mathcal{C}(Q\text{–}T)$. $\square$

Remark. For more general versions of 12.1.7–12.1.10 see E 12.1.6–E 12.1.9.

Evaluation Morphisms

12.1.11 Proposition. *For complexes $M \in \mathcal{C}(R)$ and $X \in \mathcal{C}(R\text{–}S)$ the biduality map,*

$$\delta_X^M : M \longrightarrow \operatorname{Hom}_S(\operatorname{Hom}_R(M, X), X) ,$$

from 4.5.1 *is a morphism in* $\mathcal{C}(R)$. *Further, if* Q *is an* R*-algebra and* M *a* Q*-complex, then biduality* δ_X^M *is a morphism in* $\mathcal{C}(Q)$.

Proof. The claims follow from 4.5.2 like in the proof of 12.1.8–12.1.10. □

Proposition. *For complexes* $M \in \mathcal{C}(R)$, $X \in \mathcal{C}(R\text{–}S)$, *and* $N \in \mathcal{C}(S)$ *the maps*

12.1.12 $\operatorname{Hom}_R(M, X) \otimes_S N \xrightarrow{\theta^{MXN}} \operatorname{Hom}_R(M, X \otimes_S N)$ (tensor evaluation)

12.1.13 $N \otimes_S \operatorname{Hom}_R(X, M) \xrightarrow{\eta^{MXN}} \operatorname{Hom}_R(\operatorname{Hom}_S(N, X), M)$ (hom. evaluation)

from 4.5.8 *and* 4.5.11 *are morphisms in* $\mathcal{C}(R\text{–}S)$. *Moreover, if* Q *is an* R*-algebra and* M *a* Q*-complex, and* T *is an* S*-algebra and* N *a* T*-complex, then tensor evaluation* θ^{MXN} *and homomorphism evaluation* η^{MXN} *are morphisms in* $\mathcal{C}(Q\text{–}T)$.

Proof. The claims hold by 4.5.9 and 4.5.12 like in the proof of 12.1.8–12.1.10. □

Remark. For a complex $M \in \mathcal{C}(R\text{–}Q)$ biduality δ_X^M is per 4.5.2 morphism in $\mathcal{C}(R\text{–}Q)$, but we won't need that. See E 12.1.10 and E 12.1.11 for similarly more general statements about tensor evaluation and homomorphism evaluation.

Finally we recall conditions under which the evaluation morphisms are invertible.

12.1.14 Theorem. *For every complex* P *of finitely generated projective* R*-modules,* $\operatorname{Hom}_R(P, R)$ *is a complex of finitely generated projective* R*-modules, and biduality*

$$\delta_R^P : P \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(P, R), R)$$

is an isomorphism.

Proof. This is a special case of 4.5.4. □

12.1.15 Theorem. *Let* $M \in \mathcal{C}(R)$, $X \in \mathcal{C}(R\text{–}S)$, *and* $N \in \mathcal{C}(S)$. *Tensor evaluation,*

$$\theta^{MXN} : \operatorname{Hom}_R(M, X) \otimes_S N \longrightarrow \operatorname{Hom}_R(M, X \otimes_S N) ,$$

is an isomorphism if the complexes meet one of the boundedness conditions (1)–(3) *and one of the conditions* (a)–(c) *on their modules.*

(1) *M is bounded below, and X and N are bounded above.*
(2) *M is bounded above, and X and N are bounded below.*
(3) *Two of the complexes M, X, and N are bounded.*

(a) *M or N is a complex of finitely generated projective modules.*
(b) *M is a complex of projective modules and N is degreewise finitely generated.*
(c) *M is degreewise finitely generated and N is a complex of flat modules.*

Furthermore, θ^{MXN} *is an isomorphism if* M *or* N *is a bounded complex of finitely generated modules and one of the following conditions is satisfied.*

(d) *M is a complex of projective modules.*
(e) *N is a complex of flat modules.*

Proof. This is a specialization of 4.5.10. □

12.1.16 Theorem. *Let $M \in \mathcal{C}(R)$, $X \in \mathcal{C}(R\text{–}S)$, and $N \in \mathcal{C}(S)$. Homorphism evaluation,*

$$\eta^{MXN} : N \otimes_S \operatorname{Hom}_R(X, M) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_S(N, X), M)\,,$$

is an isomorphism if the complexes meet one of the boundedness conditions (1)–(3) *and condition* (a) *or* (b) *on their modules.*

(1) *M and N are bounded below, and X is bounded above.*
(2) *M and N are bounded above, and X is bounded below.*
(3) *Two of the complexes M, X, and N are bounded.*
(a) *N is a complex of finitely generated projective modules.*
(b) *N is degreewise finitely generated and M is a complex of injective modules.*

Furthermore, η^{MXN} is an isomorphism if N is a bounded complex of finitely generated modules and one of the following conditions are satisfied.

(c) *N is a complex of projective modules.*
(d) *M is a complex of injective modules.*

Proof. This is a specialization of 4.5.13. □

Base Change

Assuming that S is an R-algebra, recall from 2.1.49 that for every R-complex M one has the base changed S-complex $S \otimes_R M$.

12.1.17 Proposition. *Let S be an R-algebra and M and N be R-complexes. There is an isomorphism in $\mathcal{C}(S)$,*

$$S \otimes_R (M \otimes_R N) \cong (S \otimes_R M) \otimes_S (S \otimes_R N)\,.$$

Proof. Associativity 12.1.8 and commutativity 12.1.7 of the tensor product together with the unitor 12.1.5 yield isomorphisms in $\mathcal{C}(S)$,

$$\begin{aligned} S \otimes_R (M \otimes_R N) &\cong (S \otimes_R M) \otimes_R N \\ &\cong (S \otimes_S (S \otimes_R M)) \otimes_R N \\ &\cong ((S \otimes_R M) \otimes_S S) \otimes_R N \\ &\cong (S \otimes_R M) \otimes_S (S \otimes_R N)\,. \end{aligned}$$ □

12.1.18 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There are isomorphisms in $\mathcal{C}(S)$,*

$$M \otimes_R N \cong (S \otimes_R M) \otimes_S N \qquad \textit{and} \qquad N \otimes_R M \cong N \otimes_S (S \otimes_R M)\,.$$

Proof. The unitor 12.1.5 in combination with associativity 12.1.8 and commutativity 12.1.7 of the tensor product yields isomorphisms in $\mathcal{C}(S)$,

$$M \otimes_R N \cong M \otimes_R (S \otimes_S N)$$

$$\begin{aligned} &\cong (M \otimes_R S) \otimes_S N \\ &\cong (S \otimes_R M) \otimes_S N \,. \end{aligned}$$

This shows the first of the asserted isomorphisms; the second follows per 12.1.7. □

12.1.19 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There is an isomorphism in $\mathcal{C}(S)$,*

$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_S(S \otimes_R M, N) \,.$$

Proof. The counitor 12.1.6 and adjunction 12.1.10 yield isomorphisms in $\mathcal{C}(S)$,

$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S,N)) \cong \mathrm{Hom}_S(S \otimes_R M, N) \,. \qquad \square$$

Flat Base Change

12.1.20 Proposition. *Let S be an R-algebra and M an R-complex.*

(a) *If M is degreewise finitely generated, then the S-complex $S \otimes_R M$ is degreewise finitely generated. The converse holds if S is faithfully flat as an R-module.*

(b) *If S is flat as an R-module, then there is an isomorphism of S-complexes,*

$$\mathrm{H}(S \otimes_R M) \cong S \otimes_R \mathrm{H}(M) \,.$$

(c) *If S is flat as an R-module and $\mathrm{H}(M)$ is degreewise finitely generated, then the S-complex $\mathrm{H}(S \otimes_R M)$ is degreewise finitely generated. The converse holds if S is faithfully flat as an R-module.*

Proof. (a): One can assume that M is a module. The first statement is a special case of 1.3.14. Conversely, assume that S is faithfully flat as an R-module and that the S-module $S \otimes_R M$ is finitely generated by, say, $x_1, \ldots, x_n$. For each $i \in \{1, \ldots, n\}$ write $x_i = \sum_{j=1}^{k} s_{ij} \otimes m_{ij}$ with $s_{ij} \in S$ and $m_{ij} \in M$. View the free R-module R^{nk} as the module $\mathrm{M}_{n \times k}(R)$ and let $\{e_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k\}$ be its standard basis. We argue that the homomorphism $\alpha\colon R^{nk} \to M$ given by $e_{ij} \mapsto m_{ij}$ is surjective and M hence finitely generated. As S is faithfully flat as an R-module, it suffices to show that $S \otimes_R \alpha\colon S \otimes_R R^{nk} \to S \otimes_R M$ is surjective, but this is evident as $S \otimes_R \alpha$ maps $s_{ij} \otimes e_{ij}$ to $s_{ij} \otimes m_{ij}$.

(b): As S is flat as an R-module, 2.2.19 yields the asserted isomorphism.

(c): In view of the isomorphism in part (b), the assertions follow from (a). □

12.1.21 Proposition. *Let S be an R-algebra, flat as an R-module, and M and N be R-complexes. If M is degreewise finitely generated and condition* (a) *or* (b) *below is satisfied, then there is an isomorphism in $\mathcal{C}(S)$,*

$$S \otimes_R \mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \,.$$

(a) *M or N is bounded.*

(b) *M is bounded below and N is bounded above.*

Proof. Two applications of commutativity 12.1.7 in combination with tensor evaluation 12.1.15(3,c)/(1,c) explain the first isomorphism in $\mathcal{C}(S)$ below; the second isomorphism holds by 12.1.19,

$$S \otimes_R \operatorname{Hom}_R(M, N) \;\cong\; \operatorname{Hom}_R(M, S \otimes_R N) \;\cong\; \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)\,. \qquad \square$$

12.1.22. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra that is flat as an R-module. It follows from 1.3.49 and 12.1.4 that the homomorphism $S \otimes_R \mathfrak{a} \to \mathfrak{a}S$ induced by $S \otimes_R R \xrightarrow{\cong} S$ is an isomorphism of S-modules.

To parse the next result recall from 5.5.11 the definition of a pure exact sequence.

12.1.23 Proposition. *Let S be an R-algebra with structure map $\varphi\colon R \to S$. The following conditions are equivalent.*

(i) *S is faithfully flat as an R-module.*

(ii) *S is flat as an R-module and the sequence $0 \longrightarrow R \xrightarrow{\varphi} S \longrightarrow S/R \longrightarrow 0$ is pure exact.*

(iii) *φ is injective and the R-module S/R is flat.*

Proof. Let η denote the sequence $0 \longrightarrow R \xrightarrow{\varphi} S \longrightarrow S/\operatorname{Im}\varphi \longrightarrow 0$.

(i) $\Rightarrow$ (ii): Let M be an R-module; to prove that $M \otimes_R \eta$ is exact it suffices by faithful flatness of S to show that $S \otimes_R (M \otimes_R \eta)$ is exact. As the tensor product is right exact, this comes down to showing that the canonical map $S \otimes_R (M \otimes_R R) \to S \otimes_R (M \otimes_R S)$ is injective, which follows as the R-linear map given by $s \otimes (m \otimes s') \mapsto ss' \otimes (m \otimes 1)$ is a left inverse. Thus, η is pure exact by 5.5.14; in particular, one has $\operatorname{Im}\varphi \cong R$.

(ii) $\Rightarrow$ (i): By assumption S is flat as an R-module, and for every R-module $M \neq 0$ exactness of $M \otimes_R \eta$ implies that $M \otimes_R S$ is non-zero. Thus S is faithfully flat.

(ii) $\Rightarrow$ (iii): The map φ is given to be injective and S/R is flat by 5.5.18.

(iii) $\Rightarrow$ (ii): By injectivity of φ, the sequence η is exact with $\operatorname{Im}\varphi \cong R$, and flatness of S/R implies by 5.5.18 that η is pure. $\square$

12.1.24 Example. The polynomial and power series algebras $R[x_1, \dots, x_n]$ and $R[\![x_1, \dots, x_n]\!]$ are faithfully flat as R-modules, see 7.3.14.

12.1.25 Proposition. *For an element f in $R[\![x_1, \dots, x_n]\!]$ the next assertions hold.*

(a) *f is a unit in $R[\![x_1, \dots, x_n]\!]$ if and only if $f(0, \dots, 0)$ is a unit in R.*

(b) *f belongs to the Jacobson radical of $R[\![x_1, \dots, x_n]\!]$ if and only if $f(0, \dots, 0)$ belongs to the Jacobson radical of R.*

In particular, if $\mathfrak{J}$ is the Jacobson radical of R, then $\mathfrak{J} + (x_1, \dots, x_n)$ is the Jacobson radical of $R[\![x_1, \dots, x_n]\!]$.

Proof. By recursion, it suffices to prove the assertions for $n = 1$; set $x = x_1$.

(a): If f is a unit in $R[\![x]\!]$, then there exists $g \in R[\![x]\!]$ with $fg = 1$. It follows that one has $f(0)g(0) = 1$, so $f(0)$ is a unit in R. For the converse, write $f = \sum_{i=0}^{\infty} a_i x^i$ and assume that $f(0) = a_0$ is a unit in R. We must prove the existence of an element $g =$

$\sum_{j=0}^{\infty} b_j x^j$ in $R[\![x]\!]$ such that the product $fg = \sum_{k=0}^{\infty}(\sum_{i+j=k} a_i b_j)x^k$ equals 1; i.e. we must prove the existence of elements $b_0, b_1, b_2, \dots$ in R that solve the equations

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ &\vdots \end{aligned}$$

As a_0 is a unit, one can solve the first equation for b_0, the second equation for b_1 etc.

(b): Assume that f does not belong to the Jacobson radical of $R[\![x]\!]$. For some maximal ideal $\mathfrak{M}$ in $R[\![x]\!]$ one then has $f \notin \mathfrak{M}$ and, therefore, $\mathfrak{M} + (f) = R[\![x]\!]$. Thus there exists an element g in $R[\![x]\!]$ with $1 - fg \in \mathfrak{M}$. Hence $1 - fg$ is not a unit, so $1 - f(0)g(0)$ is by part (a) not a unit in R. It follows that there exists a maximal ideal $\mathfrak{m}$ in R with $1 - f(0)g(0) \in \mathfrak{m}$. This shows that $f(0)$ is not in $\mathfrak{m}$ and hence not in the Jacobson radical of R.

Conversely, assume that $f(0)$ does not belong to the Jacobson radical of R. For some maximal ideal $\mathfrak{m}$ in R one then has $f(0) \notin \mathfrak{m}$ and, therefore, $\mathfrak{m} + (f(0)) = R$. Thus there exists an element r in R with $1 - rf(0) \in \mathfrak{m}$. Hence $1 - rf(0)$ is not a unit in R, so $1 - rf$ is by part (a) not a unit in $R[\![x]\!]$. It follows that there exists a maximal ideal $\mathfrak{M}$ in $R[\![x]\!]$ with $1 - rf \in \mathfrak{M}$. This shows that f is not in $\mathfrak{M}$ and hence not in the Jacobson radical of $R[\![x]\!]$. □

Cobase Change

Assuming that S is an R-algebra, recall from 2.1.49 that for every R-complex M one has the cobase changed S-complex $\operatorname{Hom}_R(S, M)$.

12.1.26 Proposition. *Let S be an R-algebra and M an R-complex. If M is degreewise finitely generated and S is finitely generated as an R-module, then* $\operatorname{Hom}_R(S, M)$ *is degreewise finitely generated over R and over S.*

Proof. Considering S and M as R-modules, it follows from 1.3.13 that the complex $\operatorname{Hom}_R(S, M)$ is degreewise finitely generated over R. As the R-action factors through S, it is degreewise finitely generated over S. □

12.1.27 Proposition. *Let S be an R-algebra and M and N be R-complexes. There is an isomorphism in $\mathcal{C}(S)$,*

$$\operatorname{Hom}_R(S, \operatorname{Hom}_R(M, N)) \cong \operatorname{Hom}_S(S \otimes_R M, \operatorname{Hom}_R(S, N)) .$$

Proof. Swap 12.1.9 and 12.1.19 yield isomorphisms in $\mathcal{C}(S)$,

$$\begin{aligned} \operatorname{Hom}_R(S, \operatorname{Hom}_R(M, N)) &\cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(S, N)) \\ &\cong \operatorname{Hom}_S(S \otimes_R M, \operatorname{Hom}_R(S, N)) . \end{aligned}$$ □

12.1.28 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There is an isomorphism in $\mathcal{C}(S)$,*

$$\operatorname{Hom}_R(N,M) \cong \operatorname{Hom}_S(N,\operatorname{Hom}_R(S,M)) .$$

PROOF. The unitor 12.1.5 and adjunction 12.1.10 yield isomorphisms in $\mathcal{C}(S)$,

$$\operatorname{Hom}_R(N,M) \cong \operatorname{Hom}_R(S\otimes_S N,M) \cong \operatorname{Hom}_S(M,\operatorname{Hom}_R(S,M)) . \qquad \square$$

THE DUAL NUMBERS

The ring $R[x]/(x^2)$ is known as the *dual numbers* over R; viewed as an R-module, it is free with basis $\{1,x\}$. There are canonical ring homomorphisms,

$$R \rightarrowtail R[x]/(x^2) \twoheadrightarrow R ,$$

given by $r \mapsto r$ and $r+sx \mapsto r$ for $r,s \in R$. Thus, starting with an R-complex there are a few canonical ways to turn it into an $R[x]/(x^2)$-complex, namely by base change or cobase change along the ring homomorphism $R \rightarrowtail R[x]/(x^2)$ or by restriction of scalars along $R[x]/(x^2) \twoheadrightarrow R$.

12.1.29 Lemma. *There is an isomorphism of $R[x]/(x^2)$-modules,*

$$\xi : R[x]/(x^2) \longrightarrow \operatorname{Hom}_R(R[x]/(x^2),R) ,$$

given by

$$\xi(r+sx)(a+bx) = rb+sa \quad \textit{for} \quad a,b,r,s \in R .$$

The inverse of ξ is given by $\xi^{-1}(\alpha) = \alpha(x)+\alpha(1)x$ for $\alpha \in \operatorname{Hom}_R(R[x]/(x^2),R)$.

PROOF. As the R-algebra $R[x]/(x^2)$ is free as an R-module with basis $\{1,x\}$, the R-module $\operatorname{Hom}_R(R[x]/(x^2),R)$ is free with dual basis $\{1^*,x^*\}$, see 1.4.1. By definition, ξ is given by $1 \mapsto x^*$ and $x \mapsto 1^*$ while ξ^{-1} is given by $1^* \mapsto x$ and $x^* \mapsto 1$, so ξ and ξ^{-1} are mutually inverse homomorphisms of R-modules. Finally, it is straightforward to verify that ξ is $R[x]/(x^2)$-linear. $\square$

12.1.30 Proposition. *Let M be an R-complex. There is an isomorphism,*

$$R[x]/(x^2)\otimes_R M \cong \operatorname{Hom}_R(R[x]/(x^2),M) ,$$

of $R[x]/(x^2)$-complexes which is natural in M; i.e. base change and cobase along the ring homomorphism $R \rightarrowtail R[x]/(x^2)$ are naturally isomorphic functors.

PROOF. Consider the composite of isomorphisms of $R[x]/(x^2)$-complexes that combines 12.1.29, 12.1.4, tensor evaluation 12.1.12/12.1.15(d), and the unitor 12.1.5,

$$\begin{tikzcd}
R[x]/(x^2)\otimes_R M \arrow[r, dotted] \arrow[d, "\xi\otimes M"', "\cong"] & \operatorname{Hom}_R(R[x]/(x^2),M) \\
\operatorname{Hom}_R(R[x]/(x^2),R)\otimes_R M \arrow[r, "\theta^{R[x]/(x^2)RM}", "\cong"'] & \operatorname{Hom}_R(R[x]/(x^2),R\otimes_R M) . \arrow[u, "\cong", "\operatorname{Hom}(R[x]/(x^2),\mu_R^M)"']
\end{tikzcd}$$

Note that all three isomorphisms are natural in M. $\square$

REMARK. As illustrated by 2.1.49 and 12.1.30 it is possible to have functors $\mathsf{F}\colon \mathcal{U} \rightleftarrows \mathcal{V} : \mathsf{G}$ where F is both left and right adjoint to G. Such a situation is called an 'ambidextrous adjunction'. Other names are 'ambiadjunction', 'ambijunction', and 'Frobenius adjunction'.

EXERCISES

E 12.1.1 Show that every commutative ring has IBN.

E 12.1.2 Show that the polynomial ring in countably many variables over a field is coherent but not Noetherian.

E 12.1.3 Show that the ring $\{ f \in \mathbb{Q}[x] \mid f(0) \in \mathbb{Z} \}$ is not Noetherian by constructing an ideal that is not finitely generated.

E 12.1.4 Let S be an R-algebra that is faithfully flat as an R-module. Show: (a) An R-module I is injective if the S-module $S \otimes_R I$ is injective. (b) An R-module P is finitely generated and projective if and only if the S-module $S \otimes_R P$ is finitely generated and projective.

E 12.1.5 Let S be an R-algebra that is is faithfully projective as an R-module. Show that an R-module F is flat if the S-module $\operatorname{Hom}_R(S, F)$ is flat.

In exercises E 12.1.6–12.1.11 let $M \in \mathcal{C}(Q\text{–}R)$, $X \in \mathcal{C}(R\text{–}S)$, and $N \in \mathcal{C}(S\text{–}T)$.

E 12.1.6 Show that the Q-, R-, and S-actions on $M \otimes_R X$ and $X \otimes_R M$ are compatible and that commutativity υ^{MX} is Q-, R-, and S-linear.

E 12.1.7 Show that the Q-, R-, S-, and T-actions on $(M \otimes_R X) \otimes_S N$ and $M \otimes_R (X \otimes_S N)$ are compatible and that associativity ω^{MXN} is Q-, R-, S-, and T-linear.

E 12.1.8 Show that the Q-, R-, S-, and T-actions on the complexes $\operatorname{Hom}_R(M, \operatorname{Hom}_S(N, X))$ and $\operatorname{Hom}_S(N, \operatorname{Hom}_R(M, X))$ are compatible and that swap ζ^{MXN} is Q-, R-, S-, and T-linear.

E 12.1.9 Show that the Q-, R-, S-, and T-actions on the complexes $\operatorname{Hom}_R(X \otimes_S N, M)$ and $\operatorname{Hom}_S(N, \operatorname{Hom}_R(X, M))$ are compatible and that adjunction ρ^{MXN} is Q-, R-, S-, and T-linear.

E 12.1.10 Show that the Q-, R-, S-, and T-actions on the complexes $\operatorname{Hom}_R(M, X) \otimes_S N$ and $\operatorname{Hom}_R(M, X \otimes_S N)$ are compatible and that tensor evaluation θ^{MXN} is Q-, R-, S-, and T-linear.

E 12.1.11 Show that the Q-, R-, S-, and T-actions on the complexes $N \otimes_S \operatorname{Hom}_R(X, M)$ and $\operatorname{Hom}_R(\operatorname{Hom}_S(N, X), M)$ are compatible and that homomorphism evaluation η^{MXN} is Q-, R-, S-, and T-linear.

E 12.1.12 (Cf. 12.1.29) Show that the maps that appear in 12.1.29 are $R[x]/(x^2)$-linear.

12.2 Derived Hom and Tensor Product Functors

SYNOPSIS. The functors RHom and $\otimes^{\mathsf{L}}$; boundedness and finiteness.

In Chap. 7 we handled derived functors on categories of complexes of bimodules by introducing ring homomorphisms, say $R \otimes_{\Bbbk} S^{\mathrm{o}} \to B$, and placing conditions on B as an R- or S-module. This setup also handles the case where S is an R-algebra.

12.2.1. Let S be an R-algebra with structure map $\varphi\colon R \to S$. The composite

$$R \longrightarrow R \otimes_{\Bbbk} S \longrightarrow S \tag{12.2.1.1}$$

of the canonical ring homomorphism $R \to R \otimes_{\Bbbk} S$ and the map given by multiplication, $r \otimes s \mapsto rs$, is a ring homomorphism as R and S are commutative, and it coincides with the structure map φ. In particular, the algebra S is projective/flat as an R-module, i.e. via φ, if and only if it is projective/flat as an R-module via (12.2.1.1).

Notice that for $S = R$ the composite (12.2.1.1) is simply the retract. Notice also that the convention that R-modules are symmetric R–R-bimodules corresponds to restriction of scalars along the second map $R^{\mathrm{e}} \to R$ of the composite; see 7.3.15.

Derived Hom Functor

The derived functor RHom_R is defined in 7.3.1; now that R is assumed to be commutative, RHom_R is as already observed in 7.3.15 augmented to a functor that outputs R-complexes.

12.2.2 Proposition. *The right derived Hom functor is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R\text{–}Q)^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(Q\text{–}R)\,,$$

induced by the functor $\operatorname{Hom}_R(-,\mathrm{I}_R(-))$ *from* $\mathcal{K}(R\text{–}Q)^{\mathrm{op}} \times \mathcal{K}(R)$ *to* $\mathcal{K}(Q\text{–}R)$*, and*

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(R\text{–}S)\,,$$

induced by the functor $\operatorname{Hom}_R(\mathrm{P}_R(-),-)$ *from* $\mathcal{K}(R)^{\mathrm{op}} \times \mathcal{K}(R\text{–}S)$ *to* $\mathcal{K}(R\text{–}S)$.

In particular, if Q and S are R-algebras, then RHom_R *is augmented to functors*

$$\mathcal{D}(Q)^{\mathrm{op}} \times \mathcal{D}(R) \longrightarrow \mathcal{D}(Q) \qquad \textit{and} \qquad \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(S) \longrightarrow \mathcal{D}(S)\,.$$

These functors are $\Bbbk$-bilinear, they preserves products in both variables, and they are triangulated in both variables.

Proof. With $A = R \otimes_{\Bbbk} Q$ and $S = R = B$ it follows from 12.2.1 that condition (b) in 7.3.6 is satisfied, and with $Q = R = A$ and $B = R \otimes_{\Bbbk} S$ condition 7.3.6(a) is satisfied. In either case RHom_R is augmented and induced as asserted. It also follows from 7.3.6 that these functors are $\Bbbk$-bilinear, preserves products in both variables, and that they are triangulated in both variables.

If Q and S are R-algebras, then $\mathcal{D}(Q)^{\mathrm{op}}$, $\mathcal{D}(Q)$, and $\mathcal{D}(S)$ are subcategories of $\mathcal{D}(R\text{–}Q)^{\mathrm{op}}$, $\mathcal{D}(Q\text{–}R)$, and $\mathcal{D}(R\text{–}S)$, so that last assertions follow by (co)restriction of the already established functors. □

Remark. Strong assumptions on $\Bbbk$, see 7.3.14, ensure that RHom_R is always—i.e. without additional assumptions on Q, R, or S—augmented to a functor $\mathcal{D}(R\text{–}Q)^{\mathrm{op}} \times \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(Q\text{–}S)$.

12.2.3 Proposition. *Let Q and S be R-algebras. If Q is projective as an R-module, then* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(Q)^{\mathrm{op}} \times \mathcal{D}(S) \longrightarrow \mathcal{D}(Q\text{–}S)\,;$$

it is induced by $\operatorname{Hom}_R(\mathrm{P}_Q(-),-) : \mathcal{K}(Q)^{\mathrm{op}} \times \mathcal{K}(S) \to \mathcal{K}(Q\text{–}S)$.

Proof. With $A = Q$ and $B = S$ it follows from 12.2.1 that condition (a) in 7.3.6 is satisfied, so RHom_R is augmented and induced as asserted. □

12.2.4 Proposition. *Let Q and S be R-algebras. If S is flat as an R-module, then* RHom_R *is augmented as follows*:

$$\mathsf{RHom}_R(-,-) : \mathcal{D}(Q)^{\mathrm{op}} \times \mathcal{D}(S) \longrightarrow \mathcal{D}(Q\text{–}S) ;$$

it is induced by $\mathrm{Hom}_R(-, \mathrm{I}_S(-)) : \mathcal{K}(Q)^{\mathrm{op}} \times \mathcal{K}(S) \to \mathcal{K}(Q\text{–}S)$.

Proof. With $A = Q$ and $B = S$ it follows from 12.2.1 that condition (b) in 7.3.6 is satisfied, so RHom_R is augmented and induced as as claimed. □

12.2.5. `something about ext`

12.2.6 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S)$, then* $\mathsf{RHom}_R(M,N)$ *belongs to* $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S)$.

Proof. The assertion is a special case of 7.6.16. □

12.2.7 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. If S is finitely generated as an R-module, N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$, and M is in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, then* $\mathsf{RHom}_R(N,M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S)$.

Proof. As S is finitely generated as an R-module, N belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ by 1.3.15, whence $\mathsf{RHom}_R(N,M)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ by 12.2.6 applied with $S = R$. By 12.2.4 the complex $\mathsf{RHom}_R(N,M)$ has an S-structure, so by another application of 1.3.15 the homology $\mathrm{H}(\mathsf{RHom}_R(N,M))$ is also degreewise finitely generated over S. □

Derived Tensor Product Functor

The left derived tensor product functor $\otimes^{\mathsf{L}}_R$ is defined in 7.4.1; now that R is assumed to be commutative, $\otimes^{\mathsf{L}}_R$ is as already observed in 7.4.12 augmented to a functor that outputs R-complexes.

12.2.8 Proposition. *The derived tensor product functor is augmented as follows*:

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(Q\text{–}R) \times \mathcal{D}(R) \longrightarrow \mathcal{D}(Q\text{–}R) ,$$

induced by the functor $- \otimes_R \mathrm{P}_R(-)$ *from* $\mathcal{K}(Q\text{–}R) \times \mathcal{K}(R)$ *to* $\mathcal{K}(Q\text{–}R)$, *and*

$$- \otimes^{\mathsf{L}}_R - : \mathcal{D}(R) \times \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(R\text{–}S) ,$$

induced by the functor $\mathrm{P}_R(-) \otimes_R -$ *from* $\mathcal{K}(R) \times \mathcal{K}(R\text{–}S)$ *to* $\mathcal{K}(R\text{–}S)$.

In particular, if Q and S are R-algebras, then $\otimes^{\mathsf{L}}_R$ is augmented to functors

$$\mathcal{D}(Q) \times \mathcal{D}(R) \longrightarrow \mathcal{D}(Q) \qquad \text{and} \qquad \mathcal{D}(R) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(S) .$$

These functors are $\Bbbk$-bilinear, they preserves coproducts in both variables, and they are triangulated in both variables.

Proof. With $A = Q \otimes_{\Bbbk} R$ and $S = R = B$ it follows from 12.2.1 that condition (b) in 7.4.5 is satisfied, and with $Q = R = A$ and $B = R \otimes_{\Bbbk} S$ condition 7.4.5(a) is satisfied. In either case $\otimes_R^{\mathsf{L}}$ is augmented and induced as asserted. It also follows from 7.4.5 that these functors are $\Bbbk$-bilinear, preserves coproducts in both variables, and that they are triangulated in both variables.

If Q and S are R-algebras, then $\mathcal{D}(Q)$ and $\mathcal{D}(S)$ are subcategories of $\mathcal{D}(Q\text{–}R)$ and $\mathcal{D}(R\text{–}S)$ so that last assertions follow by (co)restriction of the already established functors. □

Remark. Under strong assumptions on $\Bbbk$, the functor $\otimes^{\mathsf{L}}$ is always—i.e. without extra assumptions on Q, R, or S—augmented to a functor $\mathcal{D}(Q\text{–}R) \times \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(Q\text{–}S)$; see the commentary after 7.4.11.

12.2.9 Proposition. *Let Q and S be R-algebras. If Q is flat as an R-module, then $\otimes_R^{\mathsf{L}}$ is augmented as follows*:

$$- \otimes_R^{\mathsf{L}} - : \mathcal{D}(Q) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(Q\text{–}S) \,;$$

it is induced by $\mathrm{P}_Q(-) \otimes_R - : \mathcal{K}(Q) \times \mathcal{K}(S) \to \mathcal{K}(Q\text{–}S)$.

Proof. With $A = Q$ and $B = S$ it follows from 12.2.1 that condition (a) in 7.4.5 is satisfied, so $\otimes_R^{\mathsf{L}}$ is augmented and induced as asserted. □

12.2.10 Proposition. *Let Q and S be R-algebras. If S is flat as an R-module, then $\otimes_R^{\mathsf{L}}$ is augmented as follows*:

$$- \otimes_R^{\mathsf{L}} - : \mathcal{D}(Q) \times \mathcal{D}(S) \longrightarrow \mathcal{D}(Q\text{–}S) \,;$$

it is induced by $- \otimes_R \mathrm{P}_S(-) : \mathcal{K}(Q) \times \mathcal{K}(S) \to \mathcal{K}(Q\text{–}S)$.

Proof. With $A = Q$ and $B = S$ it follows from 12.2.1 that condition (b) in 7.4.5 is satisfied, so $\otimes_R^{\mathsf{L}}$ is augmented and induced as asserted. □

12.2.11. `something about Tor`

12.2.12 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$, then the complex $N \otimes_R^{\mathsf{L}} M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$.*

Proof. The assertion is a special case of 7.6.18. □

12.3 Standard Isomorphisms and Evaluation Morphisms in $\mathcal{D}$

Synopsis. Unitor; counitor; commutativity; associativity; swap; adjunction; biduality; tensor evaluation; homomophism evaluation; derived base change; derived cobase change.

Under mild assumptions the derived Hom and tensor product functors uphold additional ring actions; the morphisms that compare composites of these functors follow suit. We start by recording in 12.3.3–12.3.11 the most basic versions of these morphisms. For the unitor and counitor there is nothing more to say, but more refined versions of commutativity, associativity, swap, adjunction, and the evaluation morphisms are recorded in 12.3.12–12.3.27.

Standard Isomorphisms

12.3.1. By 12.2.8 there is a functor,

$$R \otimes^{\mathsf{L}}_R - : \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(R\text{–}S) \quad \text{induced by} \quad R \otimes_R - .$$

As in 7.5.2 the unitor 7.1.11 induces a natural isomorphism,

$$\boldsymbol{\mu}_R : R \otimes^{\mathsf{L}}_R - \longrightarrow \mathrm{Id}_{\mathcal{D}(R\text{–}S)} ,$$

of endofunctors on $\mathcal{D}(R\text{–}S)$, which is called *unitor*.

12.3.2. By 12.2.2 there is a functor,

$$\mathsf{R}\mathrm{Hom}_R(R,-) : \mathcal{D}(R\text{–}S) \longrightarrow \mathcal{D}(R\text{–}S) \quad \text{induced by} \quad \mathrm{Hom}_R(R,-) .$$

As in 7.5.6 the counitor 7.1.12 induces a natural isomorphism,

$$\boldsymbol{\epsilon}_R : \mathsf{R}\mathrm{Hom}_R(R,-) \longrightarrow \mathrm{Id}_{\mathcal{D}(R\text{–}S)} ,$$

of endofunctors on $\mathcal{D}(R\text{–}S)$, which is called *counitor*.

Proposition. *For a complex $M \in \mathcal{D}(R\text{–}S)$ there are isomorphisms*

12.3.3 $$R \otimes^{\mathsf{L}}_R M \xrightarrow{\mu^M_R} M \qquad \text{(unitor)}$$

12.3.4 $$M \xrightarrow{\epsilon^M_R} \mathsf{R}\mathrm{Hom}_R(R,M) \qquad \text{(counitor)}$$

in $\mathcal{D}(R\text{–}S)$. As natural transformations of functors, $\boldsymbol{\mu}_R$ and $\boldsymbol{\epsilon}_R$ are triangulated.

Proof. The arguments in the proofs of 7.5.4 and 7.5.8 apply verbatim. □

For R-complexes M and N, commutativity $\boldsymbol{\upsilon}^{MN}$ is by default an isomorphism in $\mathcal{D}(\Bbbk)$, see 7.5.10. Just as for commutativity in the category of complexes we now show that $\boldsymbol{\upsilon}^{MN}$ is R-linear and not by applying 7.5.10 with $\Bbbk = R$, as that would constrict our ability to consider additional ring actions on M and N.

12.3.5 Proposition. *For complexes M and N in $\mathcal{D}(R)$ commutativity,*

$$\boldsymbol{\upsilon}^{MN} : M \otimes^{\mathsf{L}}_R N \longrightarrow N \otimes^{\mathsf{L}}_R M ,$$

is an isomorphism in $\mathcal{D}(R)$. If M belongs to $\mathcal{D}(Q\text{–}R)$, then $\boldsymbol{\upsilon}^{MN}$ is an isomorphism in $\mathcal{D}(Q\text{–}R)$, and if N belongs to $\mathcal{D}(R\text{–}S)$, then $\boldsymbol{\upsilon}^{MN}$ is an isomorphism in $\mathcal{D}(R\text{–}S)$. As a natural transformation of functors, $\boldsymbol{\upsilon}$ is triangulated in each variable.

Proof. It follows from 12.2.1 that condition (b) in 7.5.13 is satisfied with $A = Q \otimes_\Bbbk R$ and $S = R = B$; similarly (a) is satisfied with with $Q = R = A$ and $B = R \otimes_\Bbbk S$. □

Proposition. *For complexes $M \in \mathcal{D}(R)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(S)$, the morphisms*

12.3.6
$$(M \otimes^{\mathsf{L}}_R X) \otimes^{\mathsf{L}}_S N \xrightarrow{\omega^{MXN}} M \otimes^{\mathsf{L}}_R (X \otimes^{\mathsf{L}}_S N) \qquad \text{(associativity)}$$

12.3.7
$$\mathsf{RHom}_R(M, \mathsf{RHom}_S(N, X)) \xrightarrow{\zeta^{MXN}} \mathsf{RHom}_S(N, \mathsf{RHom}_R(M, X)) \qquad \text{(swap)}$$

12.3.8
$$\mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S M, N) \xrightarrow{\rho^{MXN}} \mathsf{RHom}_S(M, \mathsf{RHom}_R(X, N)) \qquad \text{(adj.)}$$

defined in 7.5.17, 7.5.24, *and* 7.5.30 *are isomorphisms in* $\mathcal{D}(R\text{–}S)$. *As natural transformations of functors $\boldsymbol{\omega}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\rho}$ are triangulated in each variable.*

Proof. It follows from 12.2.1 that condition (a) in 7.5.20, 7.5.27, and 7.5.33 is satisfied with $Q = R = A$ and $B = R \otimes_{\Bbbk} S$ and $T = S = C$. □

Evaluation Morphisms

12.3.9 Proposition. *For complexes $M \in \mathcal{D}(Q\text{–}R)$ and $X \in \mathcal{D}(R)$ biduality,*

$$\boldsymbol{\delta}^M_X : M \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_R(M, X), X)\,,$$

defined in 8.4.2 *is a morphism in* $\mathcal{D}(Q\text{–}R)$. *As a natural transformation of functors, $\boldsymbol{\delta}_X$ is triangulated.*

Proof. The assertions follow per 12.2.1 from 8.4.3 applied with $S = R = B$. □

Proposition. *For complexes $M \in \mathcal{D}(R)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(S)$ the morphisms*

12.3.10
$$\mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_S N \xrightarrow{\theta^{MXN}} \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N) \qquad \text{(tensor evaluation)}$$

12.3.11
$$N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X, M) \xrightarrow{\eta^{MXN}} \mathsf{RHom}_R(\mathsf{RHom}_S(N, X), M) \qquad \text{(hom. eval.)}$$

defined in 8.4.6 *and* 8.4.19 *are morphisms in* $\mathcal{D}(R\text{–}S)$. *As natural transformations of functors, $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are triangulated in each variable.*

Proof. The assertions follow per 12.2.1 from 8.4.9 and 8.4.22 applied with $Q = R = A$ and $B = R \otimes_{\Bbbk} S$ and $T = S = C$. □

Commutativity

12.3.12 Proposition. *Let Q and S be R-algebras and let $M \in \mathcal{D}(Q)$ and $N \in \mathcal{D}(S)$. If Q or S is flat as an R-module, then commutativity,*

$$\boldsymbol{\upsilon}^{MN} : M \otimes^{\mathsf{L}}_R N \longrightarrow N \otimes^{\mathsf{L}}_R M\,,$$

is an isomorphism in $\mathcal{D}(Q\text{–}S)$.

Proof. Apply 7.5.13 with $A = Q$ and $B = S$, cf. 12.2.1. □

ASSOCIATIVITY

12.3.13 Proposition. *Let Q be an R-algebra and T an S-algebra. For complexes $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$ associativity,*

$$\omega^{MXN} : (M \otimes^{\mathsf{L}}_R X) \otimes^{\mathsf{L}}_S N \longrightarrow M \otimes^{\mathsf{L}}_R (X \otimes^{\mathsf{L}}_S N)\,,$$

is an isomorphism in $\mathcal{D}(Q\text{–}T)$ if one of the following conditions is satisfied.

(a) *Q is flat as an R-module and T is flat as an S-module.*
(b) *Q is flat as an R-module and S is an R-algebra.*
(c) *T is flat as an S-module and R is an S-algebra.*

PROOF. The assertions follow from applications of associativity 7.5.20 with $A = Q$ and $C = T$, cf. 12.2.1. Indeed, if condition (a) is satisfied, then so is (a) in 7.5.20 with $B = R \otimes_{\Bbbk} S$, if condition (b) is satisfied, then so is (b) in 7.5.20 with $B = S$, and if condition (c) is satisfied, then so is (c) in 7.5.20 with $B = R$. □

12.3.14 Corollary. *Let M, X, and N be R-complexes. Associativity,*

$$\omega^{MXN} : (M \otimes^{\mathsf{L}}_R X) \otimes^{\mathsf{L}}_R N \longrightarrow M \otimes^{\mathsf{L}}_R (X \otimes^{\mathsf{L}}_R N)\,,$$

is an isomorphism in $\mathcal{D}(R)$. Moreover, the following assertions hold:

(a) *If Q is an R-algebra and $M \in \mathcal{D}(Q)$, then ω^{MNX} is an isomorphism in $\mathcal{D}(Q)$.*
(b) *If T is an R-algebra and $N \in \mathcal{D}(T)$, then ω^{MNX} is an isomorphism in $\mathcal{D}(T)$.*

PROOF. With $R = S = T$ condition (c) in 12.3.13 is trivially satisfied, and this yields part (a); similarly condition (b) is satisfied with $Q = R = S$, and that yields (b). □

SWAP

12.3.15 Proposition. *Let Q be an R-algebra and T an S-algebra. For complexes $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$ swap,*

$$\zeta^{MXN} : \mathsf{R}\mathrm{Hom}_R(M, \mathsf{R}\mathrm{Hom}_S(N, X)) \longrightarrow \mathsf{R}\mathrm{Hom}_S(N, \mathsf{R}\mathrm{Hom}_R(M, X))\,,$$

is an isomorphism in $\mathcal{D}(Q\text{–}T)$ if one of the following conditions is satisfied.

(a) *Q is projective as an R-module and T is projective as an S-module.*
(b) *Q is projective as an R-module and S is an R-algebra.*
(c) *T is projective as an S-module and R is an S-algebra.*

PROOF. The assertions follow from applications of swap 7.5.27 with $A = Q$ and $C = T$, cf. 12.2.1. Indeed, if condition (a) is satisfied, then so is (a) in 7.5.27 with $B = R \otimes_{\Bbbk} S$, if condition (b) is satisfied, then so is (b) in 7.5.27 with $B = S$, and if condition (c) is satisfied, then so is (c) in 7.5.27 with $B = R$. □

12.3.16 Corollary. *Let M, X, and N be R-complexes. Swap,*

$$\zeta^{MXN} : \mathsf{R}\mathrm{Hom}_R(M, \mathsf{R}\mathrm{Hom}_R(N, X)) \longrightarrow \mathsf{R}\mathrm{Hom}_R(N, \mathsf{R}\mathrm{Hom}_R(M, X))\,,$$

is an isomorphism in $\mathcal{D}(R)$. Moreover, the following assertions hold:

(a) *If Q is an R-algebra and $M \in \mathcal{D}(Q)$, then $\boldsymbol{\zeta}^{MNX}$ is an isomorphism in $\mathcal{D}(Q)$.*
(b) *If T is an R-algebra and $N \in \mathcal{D}(T)$, then $\boldsymbol{\zeta}^{MNX}$ is an isomorphism in $\mathcal{D}(T)$.*

Proof. With $R = S = T$ condition (c) in 12.3.15 is trivially satisfied, and this yields part (a); similarly condition (b) is satisfied with $Q = R = S$, and that yields (b). □

Adjunction

12.3.17 Proposition. *Let Q be an R-algebra and T an S-algebra. For complexes $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$ adjunction,*

$$\boldsymbol{\rho}^{MXN} : \mathsf{RHom}_R(X \otimes^{\mathsf{L}}_S N, M) \longrightarrow \mathsf{RHom}_S(N, \mathsf{RHom}_R(X, M)) ,$$

is an isomorphism in $\mathcal{D}(Q\text{–}T)$ if one of the following conditions is satisfied.

(a) *Q is flat as an R-module and T is projective as an S-module.*
(b) *Q is flat as an R-module and S is an R-algebra.*
(c) *T is projective as an S-module and R is an S-algebra.*

Proof. The assertions follow from applications of adjunction 7.5.33 with $A = Q$ and $C = T$, cf. 12.2.1. Indeed, if condition (a) is satisfied, then so is (a) in 7.5.33 with $B = R \otimes_{\Bbbk} S$, if condition (b) is satisfied, then so is (b) in 7.5.33 with $B = S$, and if condition (c) is satisfied, then so is (c) in 7.5.33 with $B = R$. □

12.3.18 Corollary. *Let M, X, and N be R-complexes. Adjunction,*

$$\boldsymbol{\rho}^{MXN} : \mathsf{RHom}_R(X \otimes^{\mathsf{L}}_R N, M) \longrightarrow \mathsf{RHom}_R(N, \mathsf{RHom}_R(X, M)) ,$$

is an isomorphism in $\mathcal{D}(R)$. Moreover, the following assertions hold:

(a) *If Q is an R-algebra and $M \in \mathcal{D}(Q)$, then $\boldsymbol{\rho}^{MNX}$ is an isomorphism in $\mathcal{D}(Q)$.*
(b) *If T is an R-algebra and $N \in \mathcal{D}(T)$, then $\boldsymbol{\rho}^{MNX}$ is an isomorphism in $\mathcal{D}(T)$.*

Proof. With $R = S = T$ condition (c) in 12.3.17 is trivially satisfied, and this yields part (a); similarly condition (b) is satisfied with $Q = R = S$, and that yields (b). □

Biduality

12.3.19 Proposition. *Let Q be an R-algebra. For complexes $X \in \mathcal{D}(R)$ and $M \in \mathcal{D}(Q)$ biduality,*

$$\boldsymbol{\delta}^M_X : M \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_R(M, X), X) ,$$

is a morphism in $\mathcal{D}(Q)$.

Proof. If Q is an R-algebra, then $\mathcal{D}(Q)$ is a subcategory of $\mathcal{D}(Q\text{–}R)$, so the asserted morphism is a (co)restriction of the morphism from 12.3.9. □

12.3.20 Theorem. *Let M in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ be a complex of finite projective dimension. The biduality morphism $\boldsymbol{\delta}^M_R$ is an isomorphism in $\mathcal{D}(R)$ and the next equalities hold.*

$$\operatorname{pd}_R M \;=\; -\inf \mathsf{RHom}_R(M,R) \qquad \textit{and} \qquad \operatorname{pd}_R \mathsf{RHom}_R(M,R) \;=\; -\inf M\;;$$

in particular, $\mathsf{RHom}_R(M,R)$ is complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension.

Further, there are natural isomorphisms of endofunctors on $\mathcal{D}(R)$,

$$\mathsf{RHom}_R(M,-) \;\simeq\; \mathsf{RHom}_R(M,R) \otimes^{\mathsf{L}}_R -$$

and

$$- \otimes^{\mathsf{L}}_R M \;\simeq\; \mathsf{RHom}_S(\mathsf{RHom}_R(M,R),-)\,.$$

Proof. Biduality $\boldsymbol{\delta}^M_R$ is an isomorphism by 10.2.1 applied with $\Bbbk = R$, and the equalities hold by 10.2.2. The natural isomorphisms of functors come from 10.2.3 and commutativity 12.3.5. □

Tensor Evaluation

12.3.21 Proposition. *Let Q be an R-algebra, projective as an R-module, and T an S-algebra, flat as an S-module. For complexes $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$ tensor evaluation,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_S N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_S N)\,,$$

is a morphism in $\mathcal{D}(Q\text{–}T)$.

Proof. Apply 8.4.9 with $A = Q$ and $B = R \otimes_{\Bbbk} S$ and $C = T$, cf. 12.2.1. □

12.3.22 Theorem. *Let Q be an R-algebra, projective as an R-module, and T an S-algebra, flat as an S-module. Let $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$.*

If Q is finitely generated as an R-module, then the morphism $\boldsymbol{\theta}^{MXN}$ from 12.3.21 *is an isomorphism in $\mathcal{D}(Q\text{–}T)$ provided that one of the next conditions is satisfied.*

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(Q)$ and* $\operatorname{pd}_Q M$ *is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(Q)$, X is in $\mathcal{D}_{\sqsubset}(R\text{–}S)$, and* $\operatorname{fd}_T N$ *is finite.*

If T is finitely generated as an S-module, then the morphism $\boldsymbol{\theta}^{MXN}$ from 12.3.21 *is an isomorphism in $\mathcal{D}(Q\text{–}T)$ provided that one of the next conditions is satisfied.*

(c) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(T)$ and* $\operatorname{pd}_T N$ *is finite.*
(d) $\operatorname{pd}_Q M$ *is finite, X is in $\mathcal{D}_{\sqsupset}(R\text{–}S)$, and N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(T)$.*

Proof. Apply 8.4.12 with $A = Q$ and $B = R \otimes_{\Bbbk} S$ and $C = T$, cf. 12.2.1. □

12.3.23 Corollary. *Let M, X, and N be R-complexes. Tensor evaluation,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M,X) \otimes^{\mathsf{L}}_R N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_R N)\,,$$

is an isomorphism in $\mathcal{D}(R)$ if one of the next conditions is satisfied.

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\operatorname{pd}_R M$ *is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, X is in $\mathcal{D}_{\sqsubset}(R)$, and* $\operatorname{fd}_R N$ *is finite.*

(c) *N is in $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$ and* $\operatorname{pd}_R N$ *is finite.*
(d) $\operatorname{pd}_R M$ *is finite, X is in $\mathcal{D}_{\sqsupset}(R)$, and N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$.*

Proof. This is the special case of 12.3.22 with $Q = R = S = T$. □

12.3.24 Theorem. *Let M, X, and N be R-complexes. Tensor evaluation,*

$$\boldsymbol{\theta}^{MXN} : \mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_R N \longrightarrow \mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_R N)\,,$$

is an isomorphism in $\mathcal{D}(R)$ if one of the next conditions is satisfied.
(a) *$M \in \mathcal{D}_{\Box}(R)$ with $\operatorname{pd}_R M$ finite, $X \in \mathcal{D}_{\Box}(R)$ with $\operatorname{fd}_R X$ finite, and $N \in \mathcal{D}^{\mathrm{f}}(R)$.*
(b) *$M \in \mathcal{D}^{\mathrm{f}}(R)$, $X \in \mathcal{D}_{\Box}(R)$ with $\operatorname{id}_R X$ finite, and $N \in \mathcal{D}_{\Box}(R)$ with $\operatorname{fd}_R N$ finite.*

Proof. (a): By the assumptions on M and X it follows from A.26(c), A.27(c), and commutativity 12.3.5 that the functors $\mathsf{RHom}_R(M, -)$ and $X \otimes^{\mathsf{L}}_R -$ are bounded, and hence so is the composite functor $\mathsf{RHom}_R(M, X \otimes^{\mathsf{L}}_R -)$. Further, the complex $\mathsf{RHom}_R(M, X)$ is in $\mathcal{D}_{\Box}(R)$ and $\operatorname{fd}_R \mathsf{RHom}_R(M, X)$ is finite by 8.4.14(b). Thus, it follows from A.27(c) and 12.3.5 that the functor $\mathsf{RHom}_R(M, X) \otimes^{\mathsf{L}}_R -$ is bounded. To prove that the triangulated natural transformation $\boldsymbol{\theta}^{MXN}$ is an isomorphism for every $N \in \mathcal{D}^{\mathrm{f}}(R)$, one can now by 7.6.14 and A.28(d) assume that N is a finitely generated R-module, and in this case $\boldsymbol{\theta}^{MXN}$ is an isomorphism by 12.3.23(d).

(b): By the assumptions on X and N it follows from A.32(c) and A.27(c) that the functors $\mathsf{RHom}_R(-, X)$ and $- \otimes^{\mathsf{L}}_R N$ are bounded, and hence so is the composite functor $\mathsf{RHom}_R(-, X) \otimes^{\mathsf{L}}_R N$. Moreover, the complex $X \otimes^{\mathsf{L}}_R N$ is in $\mathcal{D}_{\Box}(R)$ and $\operatorname{id}_R(X \otimes^{\mathsf{L}}_R N)$ is finite by 8.4.16(a). Thus, another application of A.32(c) shows that the functor $\mathsf{RHom}_R(-, X \otimes^{\mathsf{L}}_R N)$ is bounded. To prove that the triangulated natural transformation $\boldsymbol{\theta}^{MXN}$ is an isomorphism for every $M \in \mathcal{D}^{\mathrm{f}}(R)$, one can by 7.6.14 and A.33(d) assume that M is a finitely generated R-module, and in that case $\boldsymbol{\theta}^{MXN}$ is an isomorphism by 12.3.23(b). □

Homomorphism Evaluation

12.3.25 Proposition. *Let Q be an R-algebra, flat as an R-module, and T an S-algebra, projective as an S-module. For complexes $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$ homomorphism evaluation,*

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_S \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_S(N, X), M)\,,$$

is a morphism in $\mathcal{D}(Q\text{–}T)$.

Proof. Apply 8.4.22 with $A = Q$ and $B = R \otimes_{\Bbbk} S$ and $C = T$, cf. 12.2.1. □

12.3.26 Theorem. *Let Q be an R-algebra, flat as an R-module, and T an S-algebra, finitely generated and projective as an S-module. Let $M \in \mathcal{D}(Q)$, $X \in \mathcal{D}(R\text{–}S)$, and $N \in \mathcal{D}(T)$. If one of the next conidtions is satisfied, then the morphism $\boldsymbol{\eta}^{MXN}$ from* 12.3.25 *is an isomorphism in $\mathcal{D}(Q\text{–}T)$.*
(a) *N is in $\mathcal{D}^{\mathrm{f}}_{\Box}(T)$ and* $\operatorname{pd}_T N$ *is finite.*

(b) *N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(T)$, X is in $\mathcal{D}_{\sqsubset}(R\text{–}S)$, and* $\mathrm{id}_Q\, M$ *is finite.*

PROOF. Apply 8.4.24 with $A = Q$ and $B = R \otimes_{\Bbbk} S$ and $C = T$, cf. 12.2.1. ◻

12.3.27 Corollary. *Let M, X, and N be R-complexes. Homomorphism evaluation,*

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_R(N, X), M)\,,$$

is an isomorphism in $\mathcal{D}(R)$ if one of the next conditions is satisfied.

(a) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\mathrm{pd}_R\, N$ *is finite.*
(b) *N is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, X is in $\mathcal{D}_{\sqsubset}(R)$, and* $\mathrm{id}_R\, M$ *is finite.*

PROOF. This is the special case of 12.3.26 with $Q = R = S = T$. ◻

12.3.28 Theorem. *Let M, X, and N be R-complexes. Homomorphism evaluation,*

$$\boldsymbol{\eta}^{MXN} : N \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(X, M) \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_R(N, X), M)\,,$$

is an isomorphism in $\mathcal{D}(R)$ if N is in $\mathcal{D}^{\mathrm{f}}(R)$ and X and M are in $\mathcal{D}_{\square}(R)$ with $\mathrm{id}_R\, X$ *and* $\mathrm{id}_R\, M$ *finite.*

PROOF. By the assumptions on X and M it follows from A.32(c) that the functors $\mathsf{RHom}_R(-, X)$ and $\mathsf{RHom}_R(-, M)$ are bounded, and hence so is the composite functor $\mathsf{RHom}_R(\mathsf{RHom}_R(-, X), M)$. Moreover, the complex $\mathsf{RHom}_R(X, M)$ is in $\mathcal{D}_{\square}(R)$ and $\mathrm{fd}_R\, \mathsf{RHom}_R(X, M)$ is finite by 8.4.27. Thus, A.27(c) shows that the functor $- \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(X, M)$ is bounded. To prove that the triangulated natural transformation $\boldsymbol{\eta}^{MXN}$ is an isomorphism for every $N \in \mathcal{D}^{\mathrm{f}}(R)$, one can by 7.6.14 and A.28(d) assume that N is a finitely generated R-module, and in that case $\boldsymbol{\eta}^{MXN}$ is an isomorphism by 12.3.27(b). ◻

DERIVED BASE CHANGE

Assume that S is an R-algebra. For every R-complex M the *derived base changed* complex $S \otimes^{\mathsf{L}}_R M$ is an S-complex; see 12.2.8.

12.3.29 Proposition. *Let S be an R-algebra and M an R-complex. If M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then the complex $S \otimes^{\mathsf{L}}_R M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$. In particular,* $\mathrm{Tor}^R_m(S, M)$ *is a finitely generated S-module for every $m \in \mathbb{Z}$.*

PROOF. The claims follow from 12.2.12 and the definition, 7.4.18, of Tor. ◻

12.3.30 Proposition. *Let S be an R-algebra and M and N be R-complexes. There is an isomorphism in $\mathcal{D}(S)$,*

$$S \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N) \simeq (S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_S (S \otimes^{\mathsf{L}}_R N)\,.$$

PROOF. The asserted isomorphism in $\mathcal{D}(S)$ follows from two applications of associativity 12.3.13 together with commutativity 12.3.5 and the unitor 12.3.3,

$$S \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N) \simeq (S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_R N$$

$$\begin{aligned}
&\simeq\ (S \otimes^{\mathsf{L}}_S (S \otimes^{\mathsf{L}}_R M)) \otimes^{\mathsf{L}}_R N\\
&\simeq\ ((S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_S S) \otimes^{\mathsf{L}}_R N\\
&\simeq\ (S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_S (S \otimes^{\mathsf{L}}_R N)\ . \qquad \square
\end{aligned}$$

12.3.31 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There are isomorphism in $\mathcal{D}(S)$,*

$$M \otimes^{\mathsf{L}}_R N \ \simeq\ (S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_S N \qquad \textit{and} \qquad N \otimes^{\mathsf{L}}_R M \ \simeq\ N \otimes^{\mathsf{L}}_S (S \otimes^{\mathsf{L}}_R M)\ .$$

Proof. The unitor 12.3.3 in combination with associativity 12.3.6 and commutativity 12.3.5 of the tensor product yields isomorphisms in $\mathcal{D}(S)$,

$$\begin{aligned}
M \otimes^{\mathsf{L}}_R N \ &\simeq\ M \otimes^{\mathsf{L}}_R (S \otimes^{\mathsf{L}}_S N)\\
&\simeq\ (M \otimes^{\mathsf{L}}_R S) \otimes^{\mathsf{L}}_S N\\
&\simeq\ (S \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_S N\ .
\end{aligned}$$

This shows the first of the asserted isomorphisms; the second follows per 12.3.5. $\square$

12.3.32 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There is an isomorphism in $\mathcal{D}(S)$,*

$$\mathsf{RHom}_R(M, N) \ \simeq\ \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, N)\ .$$

Proof. The counitor 12.3.4 and adjunction 12.3.8 yield isomorphisms in $\mathcal{D}(S)$,

$$\mathsf{RHom}_R(M, N) \ \simeq\ \mathsf{RHom}_R(M, \mathsf{RHom}_S(S, N)) \ \simeq\ \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, N)\ . \qquad \square$$

12.3.33 Proposition. *Let S be an R-algebra, flat as an R-module, and M and N be R-complexs. There is an isomorphism,*

$$S \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M, N) \ \simeq\ \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, S \otimes^{\mathsf{L}}_R N)\ ,$$

in $\mathcal{D}(S)$ if one of the next conditions is satisfied.

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N is in $\mathcal{D}_{\sqsubset}(R)$.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}(R)$ and N is in $\mathcal{D}_{\square}(R)$ with $\mathrm{id}_R N$ finite.*

Proof. Since S is flat as an R-module, 12.3.21 shows that tensor evaluation,

$$\boldsymbol{\theta}^{MNS} : \mathsf{RHom}_R(M, N) \otimes^{\mathsf{L}}_R S \longrightarrow \mathsf{RHom}_R(M, N \otimes^{\mathsf{L}}_R S)\ ,$$

is a morphism in $\mathcal{D}(S)$. Under the assumptions in part (a) it is an isomorphism by 6.4.37 and 12.3.23(b). Under the assumptions in part (b) it is an isomorphism by 6.4.37 and 12.3.24(b). This, combined with commutativity 12.3.5, explains the first isomorphism in $\mathcal{D}(S)$ below, while the second holds by 12.3.32.

$$S \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M, N) \ \simeq\ \mathsf{RHom}_R(M, S \otimes^{\mathsf{L}}_R N) \ \simeq\ \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, S \otimes^{\mathsf{L}}_R N)\ . \qquad \square$$

Derived Cobase Change

Assume that S is an R-algebra. For every R-complex M the derived *derived cobase changed* complex $\mathsf{RHom}_R(S,M)$ is an S-complex; see 12.2.2.

12.3.34 Proposition. *Let S be an R-algebra and M an R-complex. If S is finitely generated as an R-module and M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, then the complex $\mathsf{RHom}_R(S,M)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S)$. In particular, $\mathrm{Ext}^m_R(S,M)$ is a finitely generated S-module for every $m \in \mathbb{Z}$.*

Proof. The claims follow from 12.2.7 and the definition, 7.3.23, of Ext. □

12.3.35 Proposition. *Let S be an R-algebra and M and N be R-complexes. There is an isomorphism in $\mathcal{D}(S)$,*

$$\mathsf{RHom}_R(S,\mathsf{RHom}_R(M,N)) \simeq \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(S,N)) .$$

Proof. Swap 12.3.16 and 12.3.32 yield isomorphisms in $\mathcal{D}(S)$,

$$\begin{aligned}\mathsf{RHom}_R(S,\mathsf{RHom}_R(M,N)) &\simeq \mathsf{RHom}_R(M,\mathsf{RHom}_R(S,N)) \\ &\simeq \mathsf{RHom}_S(S \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(S,N)) .\end{aligned}$$ □

12.3.36 Proposition. *Let S be an R-algebra, M an R-complex, and N an S-complex. There is an isomorphism in $\mathcal{D}(S)$,*

$$\mathsf{RHom}_R(N,M) \simeq \mathsf{RHom}_S(N,\mathsf{RHom}_R(S,M)) .$$

Proof. The unitor 12.3.3 and adjunction 12.3.8 yield isomorphisms in $\mathcal{D}(S)$,

$$\mathsf{RHom}_R(N,M) \simeq \mathsf{RHom}_R(S \otimes^{\mathsf{L}}_S N, M) \simeq \mathsf{RHom}_S(M,\mathsf{RHom}_R(S,M)) .$$ □

Exercises

E 12.3.1 Let S be an R-algebra, finitely generated as an R-module, and M and N be R-complexes. Show that there is an isomorphism

$$S \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_S(\mathsf{RHom}_R(S,M),\mathsf{RHom}_R(S,N))$$

in $\mathcal{D}(S)$ if S is projective as an R-module.

E 12.3.2 Let S be an R-algebra and M and N be R-complexes. Show that there is an isomorphism $\mathsf{RHom}_R(S, M \otimes^{\mathsf{L}}_R N) \simeq \mathsf{RHom}_R(S,M) \otimes^{\mathsf{L}}_S (S \otimes^{\mathsf{L}}_R N)$ in $\mathcal{D}(S)$ if S is finite finitely generated and projective as an R-module.

E 12.3.3 Let M be an R-complex. Show that there is a distinguished triangle in $\mathcal{D}(R[x]/(x^2))$, $M \to \mathsf{RHom}_R(R[x]/(x^2),M) \to M \to \Sigma M$.

12.4 Prime Ideals

Synopsis. Prime ideal spectrum; Krull dimension; classic support; minimal/associated/maximal prime ideal; vanishing of functor on finitely generated module.

We recall some terminology, notation, and a few facts from commutative algebra which can be found in [182]—or any other textbook on commutative algebra—and are used in this text without further reference.

The set of prime ideals in R is called the *spectrum* of R and denoted $\operatorname{Spec} R$; it is partially ordered under inclusion. The *Krull dimension* of R, denoted $\dim R$, is the supremum of lengths of chains in $\operatorname{Spec} R$. As R is Noetherian, every such chain has finite length, but the Krull dimension of R may well be infinite. The process of adjoining a polynomial variable increases the Krull dimension by one: $\dim R[x] = \dim R + 1$. In particular, for a field $\Bbbk$ the algebra $\Bbbk[x_1, \ldots, x_n]$ of polynomials in n variables with coefficients in $\Bbbk$ has Krull dimension n.

For an R-module M and a prime ideal $\mathfrak{p}$ in R, the localization of M at the multiplicative subset $R \setminus \mathfrak{p}$ is written $M_\mathfrak{p}$. The *support* of M, denoted $\operatorname{Supp}_R M$, is the set of prime ideals $\mathfrak{p}$ with $M_\mathfrak{p} \neq 0$. As a subset of the partially ordered set $\operatorname{Spec} R$, the support of M is *specialization closed*; that is, for a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R M$ every larger prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$ also belongs to $\operatorname{Supp}_R M$. The *Krull dimension* of M is the quantity

$$\dim_R M = \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\} ;$$

it captures the supremum of lengths of chains in the support of M. Adhering to the convention $\sup \varnothing = -\infty$, the Krull dimension of the zero module is $-\infty$. As an R-module, R has full support, i.e. $\operatorname{Supp}_R R = \operatorname{Spec} R$, so $\dim R$ agrees with the Krull dimension of R as an R-module.

A prime ideal $\mathfrak{p}$ in R belongs to the support of an R-module M if and only if it contains the annihilator $(0 :_R m)$ of some element $m \in M$, for in that case the fraction $\frac{m}{1}$ is a non-zero element in $M_\mathfrak{p}$. One says that $\mathfrak{p}$ is *associated* to M if it is the annihilator $(0 :_R m)$ of some element $m \in M$. The associated prime ideals are maximal among annihilator ideals $(0 :_R m)$ for $m \neq 0$ and belong to $\operatorname{Supp}_R M$. In particular, the set $\operatorname{Ass}_R M$ of associated prime ideals of M is empty if and only if $\operatorname{Supp}_R M$ is empty if and only if $M = 0$. The minimal elements in $\operatorname{Supp}_R M$ are associated to M; an associated prime ideal that is not minimal in $\operatorname{Supp}_R M$ is called an *embedded* prime ideal. With the notation $\operatorname{Min}_R M$ for the set of minimal elements in $\operatorname{Supp}_R M$ one has

$$\operatorname{Min}_R M \subseteq \operatorname{Ass}_R M \subseteq \operatorname{Supp}_R M \subseteq \operatorname{Spec} R .$$

One writes $\operatorname{Min} R$ for the set of minimal prime ideals in R and $\operatorname{Max} R$ for the set of maximal ideals in R. In keeping with this, one uses the abridged notation $\operatorname{Ass} R$ for the set of associated prime ideals of the R-module R. The union $\bigcup_{\mathfrak{p} \in \operatorname{Ass} R} \mathfrak{p}$ is the set of zerodivisors in R. Since the support of an R-module M is a specialization closed subset of $\operatorname{Spec} R$, the maximal elements in $\operatorname{Supp}_R M$ are maximal ideals of R. With the notation $\operatorname{Max}_R M$ for the set of maximal elements in $\operatorname{Supp}_R M$ one has

$$\operatorname{Max}_R M = \operatorname{Supp}_R M \cap \operatorname{Max} R .$$

For a family $\{M^u\}_{u\in U}$ of R-modules there are equalities,

$$\operatorname{Ass}_R\Big(\coprod_{u\in U} M^u\Big) = \bigcup_{u\in U}\operatorname{Ass}_R M^u \quad\text{and}\quad \operatorname{Supp}_R\Big(\coprod_{u\in U} M^u\Big) = \bigcup_{u\in U}\operatorname{Supp}_R M^u .$$

Every finitely generated R-module M has a filtration; that is, there is an increasing sequence $0 = M^0 \subset M^1 \subset \cdots \subset M^n = M$ of submodules, such that each quotient M^u/M^{u-1} is isomorphic to $R/\mathfrak{p}_u$ for some prime ideal $\mathfrak{p}_u$. For every such filtration one has $\operatorname{Ass}_R M \subseteq \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\} \subseteq \operatorname{Supp}_R M$. In particular, for a finitely generated R-module M the sets $\operatorname{Ass}_R M$ and, therefore, $\operatorname{Min}_R M$ are finite.

Let U be a multiplicative subset of R. The assignment $\mathfrak{p} \mapsto U^{-1}\mathfrak{p}$ yields an order preserving one-to-one correspondence,

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap U = \varnothing\} \longleftrightarrow \operatorname{Spec} U^{-1}R .$$

For a prime ideal $\mathfrak{P}$ in $U^{-1}R$ the inverse mapping is given by $\mathfrak{P} \mapsto \{x \in R \mid \frac{x}{1} \in \mathfrak{P}\}$. Given a prime ieal $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \varnothing$ there is an isomorphism $(U^{-1}R)_{U^{-1}\mathfrak{p}} \cong R_{\mathfrak{p}}$ of rings and for an R-module M an isomorphism $(U^{-1}M)_{U^{-1}\mathfrak{p}} \cong M_{\mathfrak{p}}$ of modules over these isomorphic rings. The correspondence above restricts to the subsets of spectra affiliated with M and $U^{-1}M$,

$$\begin{aligned}
\{\mathfrak{p} \in \operatorname{Supp}_R M \mid \mathfrak{p} \cap U = \varnothing\} &\longleftrightarrow \operatorname{Supp}_{U^{-1}R} U^{-1}M ,\\
\operatorname{Ass}_R U^{-1}M = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \mathfrak{p} \cap U = \varnothing\} &\longleftrightarrow \operatorname{Ass}_{U^{-1}R} U^{-1}M , \quad\text{and}\\
\operatorname{Min}_R U^{-1}M = \{\mathfrak{p} \in \operatorname{Min}_R M \mid \mathfrak{p} \cap U = \varnothing\} &\longleftrightarrow \operatorname{Min}_{U^{-1}R} U^{-1}M .
\end{aligned}$$

For an ideal $\mathfrak{a}$ in R it is standard to write $\mathrm{V}(\mathfrak{a})$ for the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$. The sets of this form are precisely the closed subsets in the Zariski topology on $\operatorname{Spec} R$. The assignment $\mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{a}$ yields an order preserving one-to-one correspondence,

$$\operatorname{Supp}_R R/\mathfrak{a} = \mathrm{V}(\mathfrak{a}) \longleftrightarrow \operatorname{Spec} R/\mathfrak{a} .$$

For a prime ideal $\mathfrak{P}$ in $R/\mathfrak{a}$ the inverse mapping is given by $\mathfrak{P} \mapsto \{x \in R \mid [x]_{\mathfrak{a}} \in \mathfrak{P}\}$. The *radical* of $\mathfrak{a}$ is the ideal $\sqrt{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$; it is the intersection of the (minimal) prime ideals over $\mathfrak{a}$, that is,

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in \mathrm{V}(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \operatorname{Min}_R R/\mathfrak{a}} \mathfrak{p} .$$

For a primary ideal $\mathfrak{a}$ there is a unique element, say $\mathfrak{p}$, in $\operatorname{Min}_R R/\mathfrak{a} = \operatorname{Ass}_R R/\mathfrak{a}$, so $\sqrt{\mathfrak{a}} = \mathfrak{p}$ holds and $\mathfrak{a}$ is for emphasis called $\mathfrak{p}$-*primary*. If $\mathfrak{p}$ is maximal, then every ideal $\mathfrak{a}$ in R with $\sqrt{\mathfrak{a}} = \mathfrak{p}$ is $\mathfrak{p}$-primary. For an element x in R,

For a sequence $x_1,\dots,x_n$ in R, a result of Krull says that the inequality $\dim R_{\mathfrak{p}} \leqslant n$ holds for every prime ideal $\mathfrak{p}$ in $\operatorname{Min}_R R/(x_1,\dots,x_n)$. The special case $n = 1$ is known as Krull's principal ideal theorem. The result is restated as Corollary 18.4.19, but we recall it here for use in Examples 14.4.22, 17.2.35, and the proof of Lemma 17.4.21.

For an ideal $\mathfrak{a}$ contained in the Jacobson radical of R, Krull's intersection theorem yields $\bigcap_{u\geqslant 1} \mathfrak{a}^u = 0$. This is proved in 15.3.7 but recalled here for use in Example 14.1.8.

Given a multiplicative subset U of R, the set $[U]_{\mathfrak{a}} = \{[u]_{\mathfrak{a}} \mid u \in U\}$ is a multiplicative subset of $R/\mathfrak{a}$, and localization commutes with the quotient construction in the following sense: There are isomorphisms $U^{-1}(R/\mathfrak{a}) \cong U^{-1}R/U^{-1}\mathfrak{a} \cong [U]_{\mathfrak{a}}^{-1}(R/\mathfrak{a})$ of rings and for an R-module M isomorphisms

$$U^{-1}(M/\mathfrak{a}M) \cong U^{-1}M/U^{-1}(\mathfrak{a}M) \cong [U]_{\mathfrak{a}}^{-1}(M/\mathfrak{a}M)$$

of modules over these isomorphic rings. In view of 1.1.10 and 1.1.11 this is a consequence of commutativity 12.1.7 and associativity 12.1.8 of the tensor product. The support of the R-module $M/\mathfrak{a}M$ is evidently contained in $\mathrm{V}(\mathfrak{a})$, and the isomorphisms above yields

$$\operatorname{Supp}_{R/\mathfrak{a}} M/\mathfrak{a}M = \{\mathfrak{p}/\mathfrak{a} \mid \mathfrak{p} \in \operatorname{Supp}_R M/\mathfrak{a}M\}\,.$$

For objects in the derived category there is a more natural notion of support, which is introduced in 15.1.5. For an R-module M it is a subset of the classic support, $\operatorname{Supp}_R M$, discussed above, so it is natural to denote it $\operatorname{supp}_R M$. To distinguish the two notions terminologically, we henceforth refer to the set $\operatorname{Supp}_R M$ as the *classic support* of M and reserve "support" for $\operatorname{supp}_R M$. For finitely generated modules the two sets agree, see 15.1.9.

Vanishing of Half Exact Functors I

The existence of filtrations ensures that vanishing of a functor on finitely generated R-modules can be tested on cyclic modules $R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal.

12.4.1 Lemma. *Let* $\mathcal{U}$ *be an Abelian category,* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{U}$ *a half exact functor, and* M *a finitely generated* R*-module. If* $\mathrm{F}(M) \neq 0$ *holds, then there is a prime ideal* $\mathfrak{p}$ *in* $\operatorname{Supp}_R M$ *with* $\mathrm{F}(R/\mathfrak{p}) \neq 0$*. In particular, the next conditions are equivalent.*

(*i*) $\mathrm{F}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* R.

(*ii*) $\mathrm{F}(M) = 0$ *holds for every finitely generated* R*-module* M.

Proof. Choose a filtration $0 = M^0 \subset M^1 \subset \cdots \subset M^{n-1} \subset M^n = M$ such that for each index u one has $M^u/M^{u-1} \cong R/\mathfrak{p}_u$ for some $\mathfrak{p}_u \in \operatorname{Supp}_R M$. The canonical exact sequences $0 \to M^{u-1} \to M^u \to R/\mathfrak{p}_u \to 0$ yield exact sequences

$$\mathrm{F}(M^{u-1}) \longrightarrow \mathrm{F}(M^u) \longrightarrow \mathrm{F}(R/\mathfrak{p}_u)\,.$$

Recall that F is additive so that one has $\mathrm{F}(0) = 0$. As $\mathrm{F}(M)$ is non-zero, it follows from these sequences that one has $\mathrm{F}(R/\mathfrak{p}_u) \neq 0$ for at least one $u \in \{1, \ldots, n\}$. $\square$

The lemma above is key to a series of results 12.4.2–12.4.11 on vanishing of half exact functors; beware that 12.4.3 deals with a left exact functor.

12.4.2 Proposition. *Let* $\mathcal{U}$ *be an Abelian category,* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{U}$ *a half exact functor, and* $\mathfrak{a}$ *an ideal in* R*. If one has* $\mathrm{F}(R/\mathfrak{a}) \neq 0$*, then there is a prime ideal* $\mathfrak{p} \supseteq \mathfrak{a}$ *in* R *with* $\mathrm{F}(R/\mathfrak{p}) \neq 0$ *such that* $\mathrm{F}(R/\mathfrak{b}) = 0$ *holds for every ideal* $\mathfrak{b} \supset \mathfrak{p}$.

Proof. By assumption the set of ideals $\{\mathfrak{b} \supseteq \mathfrak{a} \mid \mathrm{F}(R/\mathfrak{b}) \neq 0\}$ is non-empty. As R is Noetherian it has a maximal element $\mathfrak{c}$. By 12.4.1 there is a prime ideal $\mathfrak{p}$ in $\mathrm{Supp}_R\, R/\mathfrak{c} = \mathrm{V}(\mathfrak{c})$ with $\mathrm{F}(R/\mathfrak{p}) \neq 0$, and by maximality of $\mathfrak{c}$ one has $\mathfrak{c} = \mathfrak{p}$. □

12.4.3 Proposition. *Let* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ *be a left exact and* R*-linear functor and* M *be a finitely generated* R*-module. If* $\mathrm{F}(M) \neq 0$ *holds, then there exists a prime ideal* $\mathfrak{p}$ *in* $\mathrm{Supp}_R\, M$ *with* $\mathrm{F}(R/\mathfrak{p})_\mathfrak{p} \neq 0$.

Proof. By 12.4.1 there exists a prime ideal $\mathfrak{p}$ in $\mathrm{Supp}_R\, M$ with $\mathrm{F}(R/\mathfrak{p}) \neq 0$. For every x in $R \setminus \mathfrak{p}$ multiplication by x on $R/\mathfrak{p}$ is injective, so it follows from the assumptions on F that the sequence

$$0 \longrightarrow \mathrm{F}(R/\mathfrak{p}) \xrightarrow{\;x\;} \mathrm{F}(R/\mathfrak{p})$$

is exact. Thus the canonical homomorphism $\mathrm{F}(R/\mathfrak{p}) \to \mathrm{F}(R/\mathfrak{p})_\mathfrak{p}$ is injective; in particular, one has $\mathrm{F}(R/\mathfrak{p})_\mathfrak{p} \neq 0$. □

Note that for $\mathfrak{a} = 0$ the next result is a special case of 12.4.1 above.

12.4.4 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ *a half exact and* R*-linear functor with the property that* $R/\mathfrak{a} \otimes_R \mathrm{F}(R/\mathfrak{p}) \neq 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* R *with* $\mathrm{F}(R/\mathfrak{p}) \neq 0$*. Let* M *be a finitely generated* R*-module. If one has* $\mathrm{F}(M) \neq 0$*, then there exists a prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{a}) \cap \mathrm{Supp}_R\, M$ *with* $\mathrm{F}(R/\mathfrak{p}) \neq 0$.

In particular, the following conditions are equivalent.

(*i*) $\mathrm{F}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{a})$.

(*ii*) $\mathrm{F}(M) = 0$ *holds for every finitely generated* R*-module* M.

Proof. By 12.4.1 the set $U = \{\mathfrak{p} \in \mathrm{Supp}_R\, M \mid \mathrm{F}(R/\mathfrak{p}) \neq 0\}$ is not empty; as R is Noetherian, U has a maximal element $\mathfrak{p}$. We argue that $\mathfrak{p}$ is in $\mathrm{V}(\mathfrak{a})$. Assume towards a contradiction that it is not the case and choose an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. Consider the exact sequence,

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;x\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (x)) \longrightarrow 0\,.$$

By half exactness and R-linearity of F there is an induced exact sequence,

$$(\star) \qquad \mathrm{F}(R/\mathfrak{p}) \xrightarrow{\;x\;} \mathrm{F}(R/\mathfrak{p}) \longrightarrow \mathrm{F}(R/(\mathfrak{p} + (x)))\,.$$

If $\mathrm{F}(R/(\mathfrak{p}+(x)))$ were non-zero, then 12.4.1 would yield a prime ideal $\mathfrak{q}$ in $\mathrm{V}(\mathfrak{p}+(x))$ with $\mathrm{F}(R/\mathfrak{q}) \neq 0$, contradicting the maximality of $\mathfrak{p}$ in U. Thus $\mathrm{F}(R/(\mathfrak{p} + (x))) = 0$ holds, so it follows from $(\star)$ that multiplication by x on $\mathrm{F}(R/\mathfrak{p})$ is surjective. As x is in $\mathfrak{a}$, this contradicts per 1.1.10 the assumption $R/\mathfrak{a} \otimes_R \mathrm{F}(R/\mathfrak{p}) \neq 0$. □

12.4.5 Corollary. *Let* $\mathfrak{J}$ *be the Jacobson radical of* R *and* $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ *a half exact and* R*-linear functor such that* $\mathrm{F}(R/\mathfrak{p})$ *is finitely generated for every prime ideal* $\mathfrak{p}$ *in* R*. The following conditions are equivalent.*

(*i*) $\mathrm{F}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{J})$.

(*ii*) $\mathrm{F}(M) = 0$ *holds for every finitely generated* R*-module* M.

Proof. Let $\mathfrak{p}$ be a prime ideal in R. Since $\mathrm{F}(R/\mathfrak{p}) \neq 0$ implies $R/\mathfrak{J} \otimes_R \mathrm{F}(R/\mathfrak{p}) \neq 0$ by Nakayama's lemma B.32, the assertion follows from 12.4.4. □

12.4.6 Corollary. *Let $\mathfrak{a}$ be an ideal in R and $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{F}(R/\mathfrak{p})$ is $\mathfrak{a}$-complete for every prime ideal $\mathfrak{p}$ in R. The following conditions are equivalent.*

(*i*) $\mathrm{F}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{a})$.

(*ii*) $\mathrm{F}(M) = 0$ *holds for every finitely generated R-module M.*

Proof. Let $\mathfrak{p}$ be a prime ideal in R. Since $\mathrm{F}(R/\mathfrak{p}) \neq 0$ implies $R/\mathfrak{a} \otimes_R \mathrm{F}(R/\mathfrak{p}) \neq 0$ by 11.1.30, the assertion follows from 12.4.4. □

Vanishing of Half Exact Functors II

12.4.7 Lemma. *Let $\mathcal{U}$ be an Abelian category, $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{U}$ a half exact functor, and M a finitely generated R-module. If $\mathrm{G}(M) \neq 0$ holds, then there is a prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R M$ with $\mathrm{G}(R/\mathfrak{p}) \neq 0$. In particular, the next conditions are equivalent.*

(*i*) $\mathrm{G}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* R.

(*ii*) $\mathrm{G}(M) = 0$ *holds for every finitely generated R-module M.*

Proof. Apply 12.4.1 to the opposite functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{M}(R) \to \mathcal{U}^{\mathrm{op}}$. □

12.4.8 Proposition. *Let $\mathcal{U}$ be an Abelian category, $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{U}$ a half exact functor, and $\mathfrak{a}$ an ideal in R. If one has $\mathrm{G}(R/\mathfrak{a}) \neq 0$, then there is a prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$ in R with $\mathrm{G}(R/\mathfrak{p}) \neq 0$ such that $\mathrm{G}(R/\mathfrak{b}) = 0$ holds for every ideal $\mathfrak{b} \supset \mathfrak{p}$.*

Proof. Apply 12.4.2 to the opposite functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{M}(R) \to \mathcal{U}^{\mathrm{op}}$. □

12.4.9 Proposition. *Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ be a half exact and R-linear functor and M be a finitely generated R-module. If $\mathrm{G}(M) \neq 0$ holds, then there exists a prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R M$ with $\mathrm{G}(R/\mathfrak{p})_{\mathfrak{p}} \neq 0$.*

Proof. By 12.4.7 the set $U = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \mathrm{G}(R/\mathfrak{p}) \neq 0\}$ is non-empty; as R is Noetherian, the set has a maximal element $\mathfrak{p}$. For every x in $R \setminus \mathfrak{p}$, consider the exact sequence,

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (x)) \longrightarrow 0\,.$$

It follows from the assumptions that G induces an exact sequence,

$$\mathrm{G}(R/(\mathfrak{p} + (x))) \longrightarrow \mathrm{G}(R/\mathfrak{p}) \xrightarrow{x} \mathrm{G}(R/\mathfrak{p})\,.$$

If $\mathrm{G}(R/(\mathfrak{p}+(x)))$ were non-zero, then 12.4.7 would yield a prime ideal $\mathfrak{q}$ in $\mathrm{V}(\mathfrak{p}+(x))$ with $\mathrm{G}(R/\mathfrak{q}) \neq 0$, contradicting the maximality of $\mathfrak{p}$ in U. Thus the sequence $0 \longrightarrow \mathrm{G}(R/\mathfrak{p}) \xrightarrow{x} \mathrm{G}(R/\mathfrak{p})$ is exact, whence one has $\mathrm{G}(R/\mathfrak{p})_{\mathfrak{p}} \neq 0$. □

Note that for $\mathfrak{a} = 0$ the next result is a special case of 12.4.1 above.

12.4.10 Proposition. *Let $\mathfrak{a}$ be an ideal in R and $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ a half exact and R-linear functor with the property that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{G}(R/\mathfrak{p})) \neq 0$ holds for every prime ideal $\mathfrak{p}$ in R with $\mathrm{G}(R/\mathfrak{p}) \neq 0$. Let M be a finitely generated R-module. If one has $\mathrm{G}(M) \neq 0$, then there is a prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a}) \cap \mathrm{Supp}_R M$ with $\mathrm{G}(R/\mathfrak{p}) \neq 0$.*

In particular, the following conditions are equivalent.

(i) $\mathrm{G}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{a})$.

(ii) $\mathrm{G}(M) = 0$ *holds for every finitely generated R-module* M.

Proof. By 12.4.7 the set $U = \{\mathfrak{p} \in \mathrm{Supp}_R M \mid \mathrm{G}(R/\mathfrak{p}) \neq 0\}$ is not empty; as R is Noetherian, U has a maximal element $\mathfrak{p}$. We argue that $\mathfrak{p}$ is in $\mathrm{V}(\mathfrak{a})$. Assume towards a contradiction that it is not the case and choose an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. Consider the exact sequence,

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;x\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (x)) \longrightarrow 0\,.$$

By half exactness and R-linearity of G there is an induced exact sequence,

$$(\star) \qquad \mathrm{G}(R/(\mathfrak{p} + (x))) \longrightarrow \mathrm{G}(R/\mathfrak{p}) \xrightarrow{\;x\;} \mathrm{G}(R/\mathfrak{p})\,.$$

If $\mathrm{G}(R/(\mathfrak{p}+(x)))$ were non-zero, then 12.4.7 would yield a prime ideal $\mathfrak{q}$ in $\mathrm{V}(\mathfrak{p}+(x))$ with $\mathrm{G}(R/\mathfrak{q}) \neq 0$, contradicting the maximality of $\mathfrak{p}$ in U. Thus $\mathrm{G}(R/(\mathfrak{p} + (x))) = 0$ holds, so it follows from $(\star)$ that multiplication by x on $\mathrm{G}(R/\mathfrak{p})$ is injective. As x is in $\mathfrak{a}$, this contradicts per 1.1.8 the assumption $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{G}(R/\mathfrak{p})) \neq 0$. $\square$

An important special case of the next corollary is recorded in 14.2.14; it compares to 12.4.5 the way 12.4.11 compares to 12.4.6..

12.4.11 Corollary. *Let $\mathfrak{a}$ be an ideal in R and $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{G}(R/\mathfrak{p})$ is $\mathfrak{a}$-torsion for every prime ideal $\mathfrak{p}$ in R. The following conditions are equivalent.*

(i) $\mathrm{G}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{a})$.

(ii) $\mathrm{G}(M) = 0$ *holds for every finitely generated R-module* M.

Proof. Let $\mathfrak{p}$ be a prime ideal in R. As $\mathrm{G}(R/\mathfrak{p}) \neq 0$ forces $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{G}(R/\mathfrak{p})) \neq 0$ by 11.2.12, the assertion follows from 12.4.10. $\square$

Exercises

E 12.4.1 Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R, show that they are equal if and only if there is an isomorphims of R-modules $R/\mathfrak{a} \cong R/\mathfrak{b}$. Compare to E 8.2.16.

E 12.4.2 Let M be an R-module; show that one has $\mathrm{Supp}_R M = \bigcup_{\mathfrak{p}\in\mathrm{Min}_R M} \mathrm{V}(\mathfrak{p})$.

E 12.4.3 Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Show that one has $\mathrm{Supp}_R M = \mathrm{Supp}_R M' \cup \mathrm{Supp}_R M''$.

E 12.4.4 Let $\mathfrak{a}$ be an ideal in R and M and $\mathfrak{a}$-torsion R-module. Show that $\mathrm{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$.

E 12.4.5 Let M be an R-module; show that one has $\{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\} = \bigcup_{\mathfrak{p}\in\mathrm{Ass}_R M} \mathfrak{p}$.

E 12.4.6 Let $\mathfrak{p}$ be a maximal ideal in R. Show that an ideal $\mathfrak{a}$ in R is $\mathfrak{p}$-primary if and only if it contains a power of $\mathfrak{p}$.

E 12.4.7 Let F be a flat R-module. Show that one has $\bigcup_{\mathfrak{p} \in \operatorname{Ass}_R F} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R R} \mathfrak{p}$.

E 12.4.8 Let $\Bbbk$ be a field and consider the algebra $R = \Bbbk[\![x, y]\!]$. Let M denote R viewed as an R-module. Show that the set $\{\mathfrak{p} \in \operatorname{Supp}_R M \mid \mathfrak{p} \cap R \setminus (x) = \varnothing\}$, which is in one-to-one correspondence with $\operatorname{Spec} R_{(x)} = \operatorname{Supp}_{R_{(x)}} M_{(x)}$, differs from $\operatorname{Supp}_R M_{(x)}$.

E 12.4.9 Let M be a finitely generated R-module. Show that there is an injective homomorphism $R/(0 :_R M) \to M^n$ for some suitable $n \in \mathbb{N}$.

E 12.4.10 Let M be a finitely generated R-module. Show that if $M \otimes_R R/\mathfrak{p} = 0$ holds for every $\mathfrak{p}$ in $\operatorname{Min}_R M$, then $M = 0$.

E 12.4.11 Let M be a finitely generated R-module and $\{M^u\}_{u \in U}$ a family of R-modules. Show that if every direct sum $\bigoplus_{i=1}^n M^{u_i}$ is a direct summand of M, then $M^u = 0$ holds for all but finitely many $u \in U$. *Hint:* Reduces by E 12.4.10 to the case of an integral domain.

E 12.4.12 Let $\mathcal{U}$ be an Abelian category and $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{U}$ a non-zero half exact functor that preserves filtered colimits. Show that there is a prime ideal $\mathfrak{p}$ in R with $\mathrm{F}(R/\mathfrak{p}) \neq 0$.

E 12.4.13 Show that the next equalities hold for every R-complex M.

$$\inf\{\inf(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \operatorname{Spec} R\} = \inf M$$
$$\inf\{-\sup \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\} = -\sup M\ .$$

Chapter 13
Derived Torsion and Completion

Much of the theory of derived $\mathfrak{a}$-completion and $\mathfrak{a}$-torsion presented in this chapter works beyond the setting of Noetherian rings. The framework and most main results—notably Theorems 13.1.15, 13.3.18, 13.4.1, and 13.4.13—require only a technical condtion on the ideal $\mathfrak{a}$: it has to be generated by a so-called proregular sequence, which ensures that the associated Čech complex has the right homological properties. In this generality the Greenlees–May Equivalence (13.4.13) was first proved by Porta, Shaul, and Yekutieli [204] building on work of Schenzel [223]. Every ideal in a Noetherian ring is generated by a proregular sequence, so Čech complexes automatically have the right properties; this is the contents of 13.1.4.

13.1 Derived Completion

Synopsis. The functor $\mathsf{L}\Lambda^{\mathfrak{a}}$ and local homology $\mathrm{H}^{\mathfrak{a}}$; derived $\mathfrak{a}$-completion via Čech complex; derived $\mathfrak{a}$-completion and change of rings; $\mathfrak{a}$-completion and flatness; derived $\mathfrak{a}$-complete complex.

For an ideal $\mathfrak{a}$ in R the $\mathfrak{a}$-completion functor only depends on the radical of $\mathfrak{a}$.

13.1.1 Lemma. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. The inclusion $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$ holds if and only if $\mathfrak{b}$ contains a power of $\mathfrak{a}$. In particular, $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are topologically equivalent.*

Proof. If $\mathfrak{b}$ contains a power of $\mathfrak{a}$, then the inclusion $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$ holds by the definition of the radical. Now, let $x_1, \dots, x_n$ be a sequence that generates $\mathfrak{a}$. If $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$ holds, then there exists for each generator x_i an $m_i \in \mathbb{N}$ such that $x_i^{m_i}$ belongs to $\mathfrak{b}$. With $m = \sum_{i=1}^{n} m_i$ one has $\mathfrak{a}^m \subseteq \mathfrak{b}$. □

13.1.2 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds, then there is a natural isomorphism $\Lambda^{\mathfrak{a}} \cong \Lambda^{\mathfrak{b}}$ of endofunctors on $\mathcal{C}(R)$.*

Proof. The assertion follows from 11.1.16 in view of 13.1.1. □

Recall that $\mathrm{H}_m^{\mathfrak{a}} = \mathrm{H}_m \, \mathsf{L}\Lambda^{\mathfrak{a}}$ is the m^{th} local homology functor supported at $\mathfrak{a}$.

13.1.3 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds, then there is a natural isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}} \simeq \mathsf{L}\Lambda^{\mathfrak{b}}$ of endofunctors on $\mathcal{D}(R)$ and hence natural isomorphisms of local homology functors $\mathrm{H}_m^{\mathfrak{a}} \cong \mathrm{H}_m^{\mathfrak{b}}$ for all $m \in \mathbb{Z}$.*

Proof. The assertions follow immediately from 13.1.2 and 11.3.6. □

Čech Complexes and Injective Modules

The following technical result is crucial to the next theorem on the docket, 13.1.15, which affords a representation of the derived $\mathfrak{a}$-completion functor.

13.1.4 Proposition. *Let $\boldsymbol{x}$ be a sequence and $\mathfrak{p}$ a prime ideal in R.*

(a) *If $(\boldsymbol{x}) \subseteq \mathfrak{p}$, then one has $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_R(R/\mathfrak{p})$ in $\mathcal{C}(R)$.*
(b) *If $(\boldsymbol{x}) \nsubseteq \mathfrak{p}$, then the complex $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \mathrm{E}_R(R/\mathfrak{p})$ is contractible.*

In particular, $\mathrm{H}_v(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R I) = 0$ holds for every injective R module I and all $v \neq 0$.

Proof. By 11.4.9 and 4.3.20 it is sufficient to consider the case of a single element $\boldsymbol{x} = x$. Set $E = \mathrm{E}_R(R/\mathfrak{p})$; it follows from 11.4.9 and 1.1.11 that the complex $\check{\mathrm{C}}(x) \otimes_R E$ is concentrated in degrees 0 and -1 and isomorphic to the following complex, where ρ maps an element e to $\frac{e}{1}$,

$$(\star) \qquad 0 \longrightarrow E \xrightarrow{\rho} \{x^n \mid n \geqslant 0\}^{-1}E \longrightarrow 0\,.$$

(a): The module E is $\mathfrak{p}$-torsion; see C.14. Thus, if x belongs to $\mathfrak{p}$, then for every $e \in E$ there is a $n > 0$ with $x^n e = 0$; in particular, one has $\{x^n \mid n \geqslant 0\}^{-1}E = 0$.

(b): If x is not in $\mathfrak{p}$, then the homothety $x^E : E \to E$ is an isomorphism by C.17. If $\rho(e) = 0$, then $x^n e = 0$ holds for some $n > 0$, whence $e = 0$ by injectivity of x^E. Thus ρ is injective. To prove surjectivity, let $\frac{e}{x^n} \in \{x^n \mid n \geqslant 0\}^{-1}E$ be given. By surjectivity of x^E there is an element e' with $x^n e' = e$, and hence $\rho(e') = \frac{e}{x^n}$. The complex $(\star)$ is the mapping cone of ρ considered as a morphism of complexes, so it is contractible by 4.3.31.

Tensor products and homology preserve coproducts, see 3.1.13 and 3.1.10(d). For an injective R-module I and $v \neq 0$ it thus follows from parts (a) and (b), in view of Matlis' structure theorem C.23, that the module $\mathrm{H}_v(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R I)$ vanishes. □

13.1.5 Corollary. *Let $\boldsymbol{x}$ be a sequence in R and S an R-algebra. For every injective S-module J one has $\mathrm{H}_v(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R J) = 0$ for all $v \neq 0$.*

Proof. Combining 12.1.18 with 11.4.18 one gets isomorphisms of S-complexes,

$$\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R J \;\cong\; (S \otimes_R \check{\mathrm{C}}^R(\boldsymbol{x})) \otimes_S J \;\cong\; \check{\mathrm{C}}^S(\boldsymbol{x}) \otimes_S J\,.$$

The statement now follows from the last assertion in 13.1.4. □

Derived Completion via Čech Complexes

With the next construction we start the preparations for Theorem 13.1.15, which establishes the representation of the derived $\mathfrak{a}$-completion functor that facilitates the proof of the Greenlees–May Equivalence. The corresponding representation of the derived $\mathfrak{a}$-torsion functor is proved in 13.3.18.

13.1.6 Construction. Let $\boldsymbol{x}$ be a sequence in R. For $u \geqslant 1$ consider the composite,

$$\mathrm{K}(\boldsymbol{x}^u) \xrightarrow[\cong]{\delta_R^{\mathrm{K}(\boldsymbol{x}^u)}} \operatorname{Hom}_R(\operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u),R),R) \xrightarrow[\simeq]{\operatorname{Hom}_R(\pi_{\boldsymbol{x}}^u,R)} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R)\,,$$

where the biduality map is an isomorphism by 12.1.14 and the second morphism is a homotopy equivalence by 11.4.25(a) and 4.3.19. The resulting morphism is a homotopy equivalence denoted $\check{\pi}_{\boldsymbol{x}}^u$. Naturalness of biduality combined with 11.4.24 yields a commutative diagram,

$$(13.1.6.1)\qquad \begin{CD} \mathrm{K}(\boldsymbol{x}^u) @>{\check{\pi}_{\boldsymbol{x}}^u}>{\simeq}> \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R) \\ @V{\varkappa_{\boldsymbol{x}}^u}VV @VV{\operatorname{Hom}_R(\iota_{\boldsymbol{x}}^{u-1},R)}V \\ \mathrm{K}(\boldsymbol{x}^{u-1}) @>{\check{\pi}_{\boldsymbol{x}}^{u-1}}>{\simeq}> \operatorname{Hom}_R(\mathrm{L}^{u-1}(\boldsymbol{x}),R)\,; \end{CD}$$

see 11.4.8. The complex $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R)$ is concentrated in degrees $v \geqslant 0$; this explains the first morphism below and the second isomorphism comes from 11.4.3(a),

$$\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R) \longrightarrow \mathrm{H}_0(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R)) \xrightarrow[\cong]{\mathrm{H}_0(\check{\pi}_{\boldsymbol{x}}^u)^{-1}} \mathrm{H}_0(\mathrm{K}(\boldsymbol{x}^u)) \;\cong\; R/(\boldsymbol{x}^u)\,.$$

Let $\tau_{\boldsymbol{x}}^u$ denote the displayed composite and notice that $\mathrm{H}_0(\tau_{\boldsymbol{x}}^u)$ is an isomorphism.

13.1.7 Lemma. *Let $\boldsymbol{x}$ be a sequence in R. In the tower,*

$$\{\operatorname{Hom}_R(\iota_{\boldsymbol{x}}^{u-1},R)\colon \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R) \to \operatorname{Hom}_R(\mathrm{L}^{u-1}(\boldsymbol{x}),R)\}_{u>1}\,,$$

induced by the telescope in 11.4.22*, every morphism is surjective. Moreover, for every $v > 0$ the induced tower in homology,*

$$\{\mathrm{H}_v(\operatorname{Hom}_R(\iota_{\boldsymbol{x}}^{u-1},R))\colon \mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}),R)) \to \mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^{u-1}(\boldsymbol{x}),R))\}_{u>1}\,,$$

satisfies the trivial Mittag-Leffler Condition.

Proof. The first assertion follows from 11.4.21 and 2.3.13. Fix $v > 0$; the tower induced by H_v is by (13.1.6.1) isomorphic to the tower

$$\{\mathrm{H}_v(\varkappa_{\boldsymbol{x}}^u)\colon \mathrm{H}_v(\mathrm{K}(\boldsymbol{x}^u)) \to \mathrm{H}_v(\mathrm{K}(\boldsymbol{x}^{u-1}))\}_{u>1}\,.$$

For every injective R-module I it induces a telescope $\{\operatorname{Hom}_R(\mathrm{H}_v(\varkappa_{\boldsymbol{x}}^u),I)\}_{u\geqslant 1}$, and we first argue that it suffices to show that every such telescope has colimit zero.

For every u there is by 5.3.30 an injective R-module I and an injective homomorphism ι in $\operatorname{Hom}_R(\mathrm{H}_v(\mathrm{K}(\boldsymbol{x}^u)),I)$. If $\operatorname{colim}_{u\geqslant 1}\operatorname{Hom}_R(\mathrm{H}_v(\mathrm{K}(\boldsymbol{x}^u)),I) = 0$ holds, then 3.3.2(b) yields a $w > u$ such that one has

$$0 = (\operatorname{Hom}_R(\mathrm{H}_v(\varkappa_{\boldsymbol{x}}^w), I) \cdots \operatorname{Hom}_R(\mathrm{H}_v(\varkappa_{\boldsymbol{x}}^{u+1}), I))(\iota) = \iota\, \mathrm{H}_v(\varkappa_{\boldsymbol{x}}^{u+1}) \cdots \mathrm{H}_v(\varkappa_{\boldsymbol{x}}^w)\,.$$

As ι is injective, this implies that the composite $\mathrm{H}_v(\varkappa_{\boldsymbol{x}}^{u+1}) \cdots \mathrm{H}_v(\varkappa_{\boldsymbol{x}}^w)$ is zero. Thus the tower in question satisfies the trivial Mittag-Leffler Condition; see 3.5.9.

Let I be an injective R-module. In the next chain of isomorphisms, the first holds as the exact functor $\operatorname{Hom}_R(-, I)$ commutes with homology and homology preserves filtered colimits; see 2.2.19 and 3.3.15(d). The telescope $\{\operatorname{Hom}_R(\varkappa_{\boldsymbol{x}}^u, I)\}_{u\geqslant 1}$ is by tensor evaluation 12.1.15(d) isomorphic to the telescope $\{\kappa_{\boldsymbol{x}}^u \otimes_R I\}_{u\geqslant 1}$ induced by (11.4.8.1); this explains the second isomorphism. The last two isomorphisms follow from 3.2.22 and 11.4.12.

$$\begin{aligned}
\operatorname*{colim}_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{H}_v(\mathrm{K}(\boldsymbol{x}^u)), I) &\cong \mathrm{H}_{-v}\big(\operatorname*{colim}_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), I)\big)\\
&\cong \mathrm{H}_{-v}\big(\operatorname*{colim}_{u\geqslant 1} (\operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \otimes_R I)\big)\\
&\cong \mathrm{H}_{-v}\big(\big(\operatorname*{colim}_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R)\big) \otimes_R I\big)\\
&\cong \mathrm{H}_{-v}(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R I)\,.
\end{aligned}$$

It now remains to recall that $\mathrm{H}_{-v}(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R I) = 0$ holds by 13.1.4. □

13.1.8 Construction. Let $\boldsymbol{x}$ be a sequence in R and M an R-complex. Denote by $\phi_{\boldsymbol{x}}^M$ the unique morphism that makes the following diagram commutative,

$$\begin{tikzcd}
\operatorname{Hom}_R(\mathrm{L}(\boldsymbol{x}), M) \arrow[rr, "\phi_{\boldsymbol{x}}^M"] \arrow[d, "\cong"] & & \Lambda^{(\boldsymbol{x})}(M) \\
\operatorname{Hom}_R\big(\operatorname*{colim}_{u\geqslant 1} \mathrm{L}^u(\boldsymbol{x}), M\big) \arrow[d, "\cong"] & & \lim_{u\geqslant 1} (R/(\boldsymbol{x}^u) \otimes_R M) \arrow[u, "\lim_{u\geqslant 1}(\xi_{\boldsymbol{x}}^u \otimes M)", "\cong"'] \\
\lim_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), M) & & \lim_{u\geqslant 1} (\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R M)\,. \arrow[ll, "\lim_{u\geqslant 1} \theta^{\mathrm{L}^u(\boldsymbol{x})RM}", "\cong"'] \arrow[u, "\lim_{u\geqslant 1}(\tau_{\boldsymbol{x}}^u \otimes M)"]
\end{tikzcd}$$

The left-hand vertical isomorphisms come from 11.4.22 and 3.4.29, and the morphisms on the right come from 11.1.17 and 13.1.6. The horizontal isomorphism is induced by tensor evaluation 12.1.15(d) and the unitor 12.1.5.

Let S be an R-algebra and N an S-complex. It follows from 12.1.4 and 11.1.6 that $\operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), N)$ and $\Lambda^{(\boldsymbol{x})}(N)$ are S-complexes, and it is straightforward to verify that the morphism,

$$\phi_{\boldsymbol{x}}^N : \operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), N) \longrightarrow \Lambda^{(\boldsymbol{x})}(N)\,,$$

constructed above is S-linear.

13.1.9 Lemma. *Let $\boldsymbol{x}$ be a sequence in R, let S be an R-algebra and N an S-complex. The morphism*

$$\phi_{\boldsymbol{x}}^N : \operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), N) \longrightarrow \Lambda^{(\boldsymbol{x})}(N)$$

constructed in 13.1.8 *is natural in N, and as a natural transformation of functors, $\phi_{\boldsymbol{x}}$ is a Σ-transformation.*

Proof. The claims follow per 4.1.16, 4.1.18, 3.4.21, and 11.1.13 from the construction 13.1.8 of $\phi_{\boldsymbol{x}}^N$. $\square$

13.1.10 Lemma. *Let $\boldsymbol{x}$ be a sequence in R and S an R-algebra. For every complex F of flat S-modules the morphism of S-complexes,*

$$\phi_{\boldsymbol{x}}^F \colon \operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), F) \longrightarrow \Lambda^{(\boldsymbol{x})}(F)$$

from 13.1.9 *is a quasi-isomorphism.*

Proof. First we consider the special case $S = R$. It follows from 2.3.12, 4.1.16, and A.15(c) that $\operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), -)$ is a $\natural$-functor, a Σ-functor, and bounded; by 11.1.13 the functor $\Lambda^{(\boldsymbol{x})}$ has the same properties. Furthermore, $\phi_{\boldsymbol{x}}$ is a Σ-transformation by 13.1.9. Thus, by A.17(d) one can assume that F is a flat R-module.

Proving that $\phi_{\boldsymbol{x}}^F$ is a quasi-isomorphism is by 13.1.8 equivalent to showing that

$$\lim_{u\geqslant 1}(\tau_{\boldsymbol{x}}^u \otimes_R F) \colon \lim_{u\geqslant 1}(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F) \longrightarrow \lim_{u\geqslant 1}(R/(\boldsymbol{x}^u) \otimes_R F)$$

is a quasi-isomorphism. The codomain of this map is a module, so one must prove:

(†) $\mathrm{H}_v(\lim_{u\geqslant 1}(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F)) = 0$ for every $v \neq 0$, and

(‡) $\mathrm{H}_0(\lim_{u\geqslant 1}(\tau_{\boldsymbol{x}}^u \otimes_R F))$ is an isomorphism.

The homology modules in (†) are zero for $v < 0$; cf. 11.4.19. The next step is to verify the assumptions in 3.5.19 in order to conclude that the canonical morphism,

$$\psi_v \colon \mathrm{H}_v\big(\lim_{u\geqslant 1}(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F)\big) \longrightarrow \lim_{u\geqslant 1}\mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F)\,,$$

is an isomorphism for $v \geqslant 0$. The maps in the tower $\{\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F\}_{u\geqslant 1}$ are $\{\operatorname{Hom}_R(\iota_{\boldsymbol{x}}^u, R) \otimes_R F\}_{u\geqslant 1}$, and they are surjective by 13.1.7. In particular, this tower satisfies the Mittag-Leffler Condition; see 3.5.10. As F is flat, 2.2.19 yields

$$\mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F) \;\cong\; \mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R)) \otimes_R F\,,$$

and, therefore, the tower $\{\mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F)\}_{u\geqslant 1}$ is isomorphic to the tower $\{\mathrm{H}_v(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R)) \otimes_R F\}_{u\geqslant 1}$, whose maps are $\{\mathrm{H}_v(\operatorname{Hom}_R(\iota_{\boldsymbol{x}}^u, R)) \otimes_R F\}_{u\geqslant 1}$. For $v > 0$ it follows from 13.1.7 that this tower satisfies the trivial Mittag-Leffler Condition. This establishes that ψ_v is an isomorphism for $v \geqslant 0$, and by 3.5.13 the codomain of ψ_v is zero for $v > 0$, so (†) follows.

The isomorphism ψ_0 fits by 3.4.18 into the commutative diagram,

$$\begin{tikzcd}
\mathrm{H}_0\big(\lim_{u\geqslant 1}(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F)\big) \arrow[r, "\mathrm{H}_0(\lim_{u\geqslant 1}(\tau_{\boldsymbol{x}}^u \otimes F))"] \arrow[d, "\psi_0"', "\cong"] & \mathrm{H}_0\big(\lim_{u\geqslant 1}(R/(\boldsymbol{x}^u) \otimes_R F)\big) \arrow[d, equal] \\
\lim_{u\geqslant 1}\mathrm{H}_0(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R F) \arrow[r, "\lim_{u\geqslant 1}\mathrm{H}_0(\tau_{\boldsymbol{x}}^u \otimes F)"] & \lim_{u\geqslant 1}\mathrm{H}_0(R/(\boldsymbol{x}^u) \otimes_R F)\,,
\end{tikzcd}$$

where the equality on the right holds as $R/(\boldsymbol{x}^u)$ and F are modules. Thus, to establish (‡) it suffices to verify that $\lim_{u\geqslant 1} \mathrm{H}_0(\tau_{\boldsymbol{x}}^u \otimes_R F)$ is an isomorphism. Each map $\mathrm{H}_0(\tau_{\boldsymbol{x}}^u)$ is an isomorphism by 13.1.6 and hence so is $\mathrm{H}_0(\tau_{\boldsymbol{x}}^u) \otimes_R F \cong \mathrm{H}_0(\tau_{\boldsymbol{x}}^u \otimes_R F)$.

Finally, we consider the general case where S is an R-algebra. Let F be an S-complex, 12.1.19 and 11.4.20 yield isomorphisms of S-complexes,

$$\operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), F) \;\cong\; \operatorname{Hom}_S(S \otimes_R \mathrm{L}^R(\boldsymbol{x}), F) \;\cong\; \operatorname{Hom}_S(\mathrm{L}^S(\boldsymbol{x}), F)\ .$$

Moreover, the extension $(\boldsymbol{x})S$ of the ideal $(\boldsymbol{x}) \subseteq R$ to S is the ideal in S generated by $\boldsymbol{x}$ viewed as a sequence in S. Thus, in view of 11.1.7 and the isomorphisms above, it follows that the map $\phi_{\boldsymbol{x}}^F \colon \operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), F) \longrightarrow \Lambda^{(\boldsymbol{x})}(F)$ may be identified with $\phi_{\boldsymbol{x}}^F \colon \operatorname{Hom}_S(\mathrm{L}^S(\boldsymbol{x}), F) \to \Lambda^{(\boldsymbol{x})}(F)$, where $\boldsymbol{x}$ viewed as a sequence in S. By the already established case, the latter map is an isomorphism if F is a complex of flat S-modules. □

We now continue the discussion from 11.3.5.

13.1.11 Construction. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. By 11.1.6 one can view $\Lambda^{\mathfrak{a}}$ as an endofunctor on $\mathcal{C}(S)$; we temporarily denote this functor by the symbol $\mathrm{L}^{\mathfrak{a}}$ to distinguish it from the functor $\Lambda^{\mathfrak{a}} \colon \mathcal{C}(R) \to \mathcal{C}(R)$. There is a diagram, not necssarily commutative, and a natural transformation:

$$\begin{tikzcd} \mathcal{K}(S) \arrow[r, "\mathrm{L}^{\mathfrak{a}}\,\mathrm{P}_S"] \arrow[d, "\operatorname{res}_R^S"'] & \mathcal{K}(S) \arrow[d, "\operatorname{res}_R^S"] \\ \mathcal{K}(R) \arrow[r, "\Lambda^{\mathfrak{a}}\,\mathrm{P}_R"] & \mathcal{K}(R) \end{tikzcd} \quad \text{and} \quad \Lambda^{\mathfrak{a}}\,\mathrm{P}_R \operatorname{res}_R^S \xrightarrow{\Lambda^{\mathfrak{a}}\,\varrho_R^S} \Lambda^{\mathfrak{a}} \operatorname{res}_R^S \mathrm{P}_S \;=\; \operatorname{res}_R^S \mathrm{L}^{\mathfrak{a}}\,\mathrm{P}_S\ ,$$

where ϱ_R^S is the natural transformation of functors $\mathcal{K}(S) \to \mathcal{K}(R)$ from 6.3.21. By 6.4.31, 6.4.40, and the definition, 7.2.8, of left derived functors, one gets an induced diagram and an induced natural transformation:

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "\mathsf{L}\mathrm{L}^{\mathfrak{a}}"] \arrow[d, "\operatorname{res}_R^S"'] & \mathcal{D}(S) \arrow[d, "\operatorname{res}_R^S"] \\ \mathcal{D}(R) \arrow[r, "\mathsf{L}\Lambda^{\mathfrak{a}}"] & \mathcal{D}(R) \end{tikzcd} \quad \text{and} \quad \mathsf{L}\Lambda^{\mathfrak{a}} \operatorname{res}_R^S \longrightarrow \operatorname{res}_R^S \mathsf{L}\mathrm{L}^{\mathfrak{a}}\ .$$

The gist of the next statement is that the functor $\mathsf{L}\mathrm{L}^{\mathfrak{a}}$ on $\mathcal{D}(S)$ is an augmentation of $\mathsf{L}\Lambda^{\mathfrak{a}}$ on $\mathcal{D}(R)$, cf. Chap. 7.

13.1.12 Proposition. *The transformation* $\mathsf{L}\Lambda^{\mathfrak{a}} \operatorname{res}_R^S \longrightarrow \operatorname{res}_R^S \mathsf{L}\mathrm{L}^{\mathfrak{a}}$ *from* 13.1.11 *is a natural isomorphism.*

Proof. We suppress the restriction of scalars functor res_R^S. Set $\varrho = \varrho_R^S$ and write φ for the natural transformation under consideration. Let N be an S-complex. To prove that φ^N is an isomorphism in $\mathcal{D}(R)$ it suffices, by the definition of this morphism, to prove that $\Lambda^{\mathfrak{a}}(\varrho^N)$ is a quasi-isomorphism. Consider the R-complex $P = \mathrm{P}_R(N)$ and the S-complex $L = \mathrm{P}_S(N)$. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$ and $\mathrm{L}^R(\boldsymbol{x})$ be the semi-free replacement of the Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ from 11.4.25(c). By 6.3.21 the map $\varrho^N : P \to L$ is a quasi-isomorphism and hence so is $\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), \varrho^N)$. The natural transformation $\phi_{\boldsymbol{x}}$ from 13.1.9 yields a commutative diagram,

$$\begin{tikzcd}
\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), P) \arrow[r, "\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),\varrho^N)", "\simeq"'] \arrow[d, "\phi_{\boldsymbol{x}}^P"'] & \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), L) \arrow[d, "\phi_{\boldsymbol{x}}^L"] \\
\Lambda^{\mathfrak{a}}(P) \arrow[r, "\Lambda^{\mathfrak{a}}(\varrho^N)"] & \Lambda^{\mathfrak{a}}(L)\,.
\end{tikzcd}$$

Since the vertical maps $\phi_{\boldsymbol{x}}^P$ and $\phi_{\boldsymbol{x}}^L$ are quasi-isomorphisms by 13.1.10, it follows that $\Lambda^{\mathfrak{a}}(\varrho^N)$ is a quasi-isomorphism. ◻

The result above justifies the following extension of 11.3.2.

13.1.13 Definition. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. We write $\mathsf{L}\Lambda^{\mathfrak{a}}$ for the left derived functor of $\Lambda^{\mathfrak{a}}$ viewed as an endofunctor on $\mathcal{C}(S)$. Just as in 11.3.2 the Σ-transformation $\lambda^{\mathfrak{a}} : \mathrm{Id}_{\mathcal{C}(S)} \to \Lambda^{\mathfrak{a}}$ from 11.1.6 induces a triangulated natural transformation of endofunctors on $\mathcal{D}(S)$,

$$\boldsymbol{\lambda}^{\mathfrak{a}} = \mathsf{L}\lambda^{\mathfrak{a}} : \mathrm{Id}_{\mathcal{D}(S)} \longrightarrow \mathsf{L}\Lambda^{\mathfrak{a}}\,.$$

13.1.14. Let $\mathfrak{a}$ be an ideal in R and M an $\widehat{R}^{\mathfrak{a}}$-complex. Let P be a semi-projective replacement of M in $\mathcal{C}(\widehat{R}^{\mathfrak{a}})$. By 11.1.21 the two $\widehat{R}^{\mathfrak{a}}$-structures on $\Lambda^{\mathfrak{a}}(P)$ coming from 11.1.6 and 11.1.20 coincide. It follows that the two $\widehat{R}^{\mathfrak{a}}$-structures on $\mathsf{L}\Lambda^{\mathfrak{a}}(M) = \Lambda^{\mathfrak{a}}(P)$ coming from 13.1.13 and 11.3.4 are the same. In other words, $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules and the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{D}(\widehat{R}^{\mathfrak{a}}) \arrow[r, "\mathsf{L}\Lambda^{\mathfrak{a}}", "\text{(from 13.1.13)}"'] \arrow[d, "\mathrm{res}_R^{\widehat{R}^{\mathfrak{a}}}"'] & \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \arrow[d, equal] \\
\mathcal{D}(R) \arrow[r, "\mathsf{L}\Lambda^{\mathfrak{a}}", "\text{(from 11.3.4)}"'] & \mathcal{D}(\widehat{R}^{\mathfrak{a}})\,.
\end{tikzcd}$$

To parse the next theorem, recall from 11.4.14 and 11.4.25 the natural transformations $\varepsilon_{\boldsymbol{x}}$ and $\pi_{\boldsymbol{x}}$. A consequence of the theorem and 11.4.16 is that the derived $\mathfrak{a}$-completion functor is idempotent. A precise statement is made in 13.4.1.

13.1.15 Theorem. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and S be an R-algebra. The endofunctor $\mathsf{L}\Lambda^{\mathfrak{a}}$ on $\mathcal{D}(S)$ is bounded and preserves products, there are triangulated natural isomorphisms of endofunctors on $\mathcal{D}(S)$,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}), -) \simeq \mathsf{R}\mathrm{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}), -)\,,$$

and isomorphisms of natural transformations,

$$\boldsymbol{\lambda}^{\mathfrak{a}} \simeq \mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}}, -) \simeq \mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}, -)\,.$$

Let N be an S-complex. For every complex F of flat S-modules with $F \simeq M$ in $\mathcal{D}(S)$ there is an isomorphism in $\mathcal{D}(S)$,

$$\mathsf{L}\Lambda^{\mathfrak{a}}(N) \simeq \Lambda^{\mathfrak{a}}(F)\,.$$

PROOF. By 11.4.25(c) the morphism $\pi_{\boldsymbol{x}}\colon \mathrm{L}^R(\boldsymbol{x}) \to \check{\mathrm{C}}^R(\boldsymbol{x})$ is a semi-free resolution. Thus, in the following commutative diagram of natural transformations of endofunctors on $\mathcal{D}(S)$, the horizontal transformation is a natural isomorphism,

$$\begin{tikzcd} & \mathsf{R}\mathrm{Hom}_R(R,-) \arrow[dl, "\mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}},-)"'] \arrow[dr, "\mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-)"] & \\ \mathsf{R}\mathrm{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}),-) \arrow[rr, "\mathsf{R}\mathrm{Hom}_R(\pi_{\boldsymbol{x}},-)", "\simeq"'] & & \mathsf{R}\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-)\,. \end{tikzcd}$$

Hence the functor $\mathsf{R}\mathrm{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}),-)$ and the natural transformation $\mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}},-)$ may be identified with the functor $\mathsf{R}\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-)$ and the natural transformation $\mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-)$. As R and $\mathrm{L}^R(\boldsymbol{x})$ are semi-free R-complexes, one has

$$\mathsf{R}\mathrm{Hom}_R(R,-) = \mathrm{Hom}_R(R,-) \quad\text{and}\quad \mathsf{R}\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-) = \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-)\,,$$

and also $\mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-) = \mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-)$. This establishes the isomorphisms $\mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-) \simeq \mathsf{R}\mathrm{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}),-)$ and $\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-) \simeq \mathsf{R}\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}},-)$.

Now, consider the natural transformations of endofunctors on $\mathcal{K}(S)$,

$$\Lambda^{\mathfrak{a}}(\mathrm{P}_S(-)) \xleftarrow{\phi_{\boldsymbol{x}}^{\mathrm{P}_S(-)}} \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),\mathrm{P}_S(-)) \xrightarrow{\mathrm{Hom}(\mathrm{L}^R(\boldsymbol{x}),\pi_S^-)} \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-)\,,$$

induced by 13.1.9 and 6.3.11. By the same references and 6.2.17, both transformations are triangulated. Evaluated at an S-complex, they are quasi-isomorphisms by 13.1.10 and by 6.3.11 and semi-freeness of $\mathrm{L}^R(\boldsymbol{x})$, see 11.4.25(c). Per 7.2.8, 7.2.11, and 6.5.14 the diagram above yields a triangulated natural isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),-)$. It follows that $\mathsf{L}\Lambda^{\mathfrak{a}}$ is bounded, see 11.4.26 and A.26(c), and it preserves products by 7.3.6 and 7.2.13(b).

Let F be a complex of flat S-modules that is isomorphic to N in $\mathcal{D}(S)$ and let P be a semi-projective replacement of N. By 6.4.20 there is a quasi-isomorphism $P \to F$ of S-complexes, so by semi-freeness of $\mathrm{L}^R(\boldsymbol{x})$ the middle morphism in ($\diamond$) below is a quasi-isomorphism. The left- and right-hand morphisms in ($\diamond$) are quasi-isomorphisms by 13.1.10; this accounts for the isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \Lambda^{\mathfrak{a}}(F)$ in $\mathcal{D}(S)$.

$$(\diamond)\qquad \Lambda^{\mathfrak{a}}(P) \xleftarrow{\phi_{\boldsymbol{x}}^{P}} \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),P) \longrightarrow \mathrm{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),F) \xrightarrow{\phi_{\boldsymbol{x}}^{F}} \Lambda^{\mathfrak{a}}(F)\,.$$

Finally we show that there is an isomorphism between the natural transformations $\lambda^{\mathfrak{a}}$ and $\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},-)$. In $\mathcal{D}(S)$ every S-complex is naturally isomorphic to its semi-projective resolution by 6.3.11. It thus suffices to argue that the morphisms $\lambda_P^{\mathfrak{a}}$ and $\mathrm{Hom}_R(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},P)$ are isomorphic in $\mathcal{D}(S)$ for every semi-projective S-complex P, and that follows from commutativity of the following diagram,

$$\begin{tikzcd} P \arrow[rr, "\lambda^{\mathfrak{a}}_P"] \arrow[d, "\epsilon^P_R"', "\cong"] & & \Lambda^{\mathfrak{a}}(P) \\ \operatorname{Hom}_R(R,P) \arrow[rr, "\operatorname{Hom}(\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}},P)"] & & \operatorname{Hom}_R(\mathrm{L}^R(\boldsymbol{x}),P)\,. \arrow[u, "\simeq", "\phi^P_{\boldsymbol{x}}"'] \end{tikzcd}$$ □

REMARK. The derived $\mathfrak{a}$-completion of an R-complex can also be computed as the $\mathfrak{a}$-completion of an isomorphic K-flat complex; see E 13.1.2.

13.1.16 Corollary. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and F an S-module. If F is flat, then one has*

$$\mathrm{H}^{\mathfrak{a}}_0(F) \;\cong\; \Lambda^{\mathfrak{a}}(F) \qquad \textit{and} \qquad \mathrm{H}^{\mathfrak{a}}_m(F) \;=\; 0 \quad \textit{for all} \quad m > 0\,.$$

PROOF. The assertion is a special case of the last isomorphism in 13.1.15. □

Even the special case $F = S$ of 13.1.16 can be of interest.

13.1.17 Example. Let $\mathfrak{a}$ be an ideal in R. For every ideal $\mathfrak{b}$ in $R[x_1,\dots,x_n]$ the R-algebra $S = R[x_1,\dots,x_n]/\mathfrak{b}$ satisfies

$$\mathrm{H}^{\mathfrak{a}}_0(S) \;\cong\; \Lambda^{\mathfrak{a}}(S) \qquad \text{and} \qquad \mathrm{H}^{\mathfrak{a}}_m(S) \;=\; 0 \quad \text{for all} \quad m > 0\,.$$

REMARK. Let $\mathfrak{a}$ be an ideal in R. Had the $\mathfrak{a}$-completion functor been right exact, which by 11.1.32 it is not, then the modules $\mathrm{H}^{\mathfrak{a}}_0(M)$ and $\Lambda^{\mathfrak{a}}(M)$ would have been isomorphic for every R-module M; see E 7.2.9. By 13.1.16 these modules are isomorphic and the higher local homology modules vanish if M is flat; Simon [233, 234] identifies several other classes of modules with this propterty, among them the $\mathfrak{a}$-complete modules. For a finitely generated R-module M it follows from 13.2.6 that the isomorphism $\mathrm{H}^{\mathfrak{a}}_0(M) \cong \Lambda^{\mathfrak{a}}(M)$ holds, as does $\mathrm{H}^{\mathfrak{a}}_m(M) = 0$ for $m > 0$. It is shown in [234] that this behavior persists for U-fold coproducts of a finitely generated module. In [235] Simon constructs a module M such that $\mathrm{H}^{\mathfrak{a}}_0(M) \cong M$ holds but M is not $\mathfrak{a}$-complete; see also Schenzel and Simon [224, 2.5].

Theorem 13.1.15 provides for simple proofs of several useful properties $\mathsf{L}\Lambda^{\mathfrak{a}}$.

13.1.18 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. There is a commutative diagram in $\mathcal{D}(R)$, where the horizontal morphism is an isomorphism,*

$$\begin{tikzcd} & \mathsf{RHom}_R(M,N) \arrow[dl, "\lambda^{\mathfrak{a}}_{\mathsf{RHom}(M,N)}"'] \arrow[dr, "\mathsf{RHom}(M,\lambda^{\mathfrak{a}}_N)"] & \\ \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{RHom}_R(M,N)) \arrow[rr, "\simeq"'] & & \mathsf{RHom}_R(M,\mathsf{L}\Lambda^{\mathfrak{a}}(N))\,. \end{tikzcd}$$

If S is an R-algebra and M an S-complex, then this is a diagram in $\mathcal{D}(S)$.

PROOF. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. By 12.3.16 there is a commutative diagram in $\mathcal{D}(R)$, where the horizontal morphisms are isomorphisms,

$$\begin{tikzcd} \mathsf{RHom}_R(R,\mathsf{RHom}_R(M,N)) \arrow[r, "\zeta^{RNM}", "\simeq"'] \arrow[d, "\mathsf{RHom}(\varepsilon_{\boldsymbol{x}},\mathsf{RHom}(M,N))"'] & \mathsf{RHom}_R(M,\mathsf{RHom}_R(R,N)) \arrow[d, "\mathsf{RHom}(M,\mathsf{RHom}(\varepsilon_{\boldsymbol{x}},N))"] \\ \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}),\mathsf{RHom}_R(M,N)) \arrow[r, "\zeta^{\check{\mathrm{C}}(\boldsymbol{x})NM}", "\simeq"'] & \mathsf{RHom}_R(M,\mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}),N))\,; \end{tikzcd}$$

if M is an S-complex, then this is a diagram in $\mathcal{D}(S)$. The assertions now follow from 13.1.15 in view of the counitor 12.3.4. $\square$

13.1.19 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. There is a commutative diagram in $\mathcal{D}(R)$,*

$$\begin{tikzcd} & M \otimes^{\mathsf{L}}_R N \arrow[dl, "\lambda^{\mathfrak{a}}_M \otimes^{\mathsf{L}} N"'] \arrow[dr, "\lambda^{\mathfrak{a}}_{M \otimes^{\mathsf{L}} N}"] & \\ \mathsf{L}\Lambda^{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R N \arrow[rr] & & \mathsf{L}\Lambda^{\mathfrak{a}}(M \otimes^{\mathsf{L}}_R N)\,. \end{tikzcd}$$

If S is an R-algebra and N an S-complex, then this is a diagram in $\mathcal{D}(S)$. Furthermore, if one of the conditions (a), (b), *or* (c) *below is satisfied, then the horizontal morphism in the diagram above is an isomorphism.*

(a) *N is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\mathrm{pd}_R\, N$ *is finite.*
(b) *M is in $\mathcal{D}_{\sqsupset}(R)$ and N in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$.*
(c) *M is in $\mathcal{D}_{\square}(R)$,* $\mathrm{fd}_R\, M$ *is finite, and N is in $\mathcal{D}^{\mathrm{f}}(R)$.*

Proof. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. By 12.3.10 one has the commutative diagram below, where the upper horizontal morphism is an isomorphism by tensor evaluation 12.3.23(a). By 13.1.15 and the counitor 12.3.4 this yields the asserted commutative diagram.

$$\begin{tikzcd} \mathsf{RHom}_R(R, M) \otimes^{\mathsf{L}}_R N \arrow[r, "\boldsymbol{\theta}^{RMN}", "\simeq"'] \arrow[d, "\mathsf{RHom}(\varepsilon_{\boldsymbol{x}}, M) \otimes^{\mathsf{L}} N"'] & \mathsf{RHom}_R(R, M \otimes^{\mathsf{L}}_R N) \arrow[d, "\mathsf{RHom}(\varepsilon_{\boldsymbol{x}}, M \otimes^{\mathsf{L}} N)"] \\ \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}), M) \otimes^{\mathsf{L}}_R N \arrow[r, "\boldsymbol{\theta}^{\check{\mathrm{C}}(\boldsymbol{x})MN}"] & \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}), M \otimes^{\mathsf{L}}_R N)\,. \end{tikzcd}$$

Under the assumptions in part (a) the morphism $\boldsymbol{\theta}^{\check{\mathrm{C}}(\boldsymbol{x})MN}$ is an isomorphism by 12.3.23(c). Recall from 11.4.10(c) and 11.4.26 that $\check{\mathrm{C}}(\boldsymbol{x})$ is in $\mathcal{D}_{\square}(R)$ and $\mathrm{pd}_R\, \check{\mathrm{C}}(\boldsymbol{x})$ is finite. Thus, under the assumptions in parts (b) and (c) the morphism $\boldsymbol{\theta}^{\check{\mathrm{C}}(\boldsymbol{x})MN}$ is an isomorphism by 12.3.23(d) and 12.3.24(a). $\square$

13.1.20 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R and M an R-complex*; *there is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{b}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{a}+\mathfrak{b}}(M)\,.$$

Proof. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences that generate $\mathfrak{a}$ and $\mathfrak{b}$, and note that the concatenated sequence $\boldsymbol{x}, \boldsymbol{y}$ generates the ideal $\mathfrak{a} + \mathfrak{b}$. In the computation below, the 1st and 4th isomorphisms follow from 13.1.15. The 2nd isomorphism holds by adjunction 12.3.8, commutativity 12.3.5, and semi-flatness of the Čech complex; see 11.4.10(c). The 3rd isomorphism follows from the definition, 11.4.9, of the Čech complex.

$$\begin{aligned} \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{b}}(M)) &\simeq \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}), \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{y}), M)) \\ &\simeq \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \check{\mathrm{C}}(\boldsymbol{y}), M) \end{aligned}$$

$$
\begin{aligned}
&\simeq \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}), M)\\
&\simeq \mathsf{L}\Lambda^{\mathfrak{a}+\mathfrak{b}}(M)\,. \qquad \square
\end{aligned}
$$

INDEPENDENCE OF BASE

For an ideal $\mathfrak{a}$ in R, an R-algebra S, and an S-complex N the complex $\mathsf{L}\Lambda^{\mathfrak{a}}(N)$ is an S-complex; see 13.1.13. Part (a) of the next result, which is a derived version of 11.1.7, is referred to as "independence of base" for local homology. To parse the statements recall the definition of derived $\mathfrak{a}$-completeness from 11.3.3.

13.1.21 Theorem. *Let $\mathfrak{a}$ be an ideal in R and S an R-algebra.*

(a) *Let N be an S-complex. There is a commutative diagram in $\mathcal{D}(S)$,*

$$
\begin{tikzcd}
 & N \arrow[dl, "\lambda_N^{\mathfrak{a}S}"'] \arrow[dr, "\lambda_N^{\mathfrak{a}}"] & \\
\mathsf{L}\Lambda^{\mathfrak{a}S}(N) \arrow[rr, "\simeq"'] & & \mathsf{L}\Lambda^{\mathfrak{a}}(N)\,,
\end{tikzcd}
$$

where the horizontal morphism is an isomorphism. In particular, N is derived $\mathfrak{a}S$-complete if and only if it is derived $\mathfrak{a}$-complete as an R-complex. Further, for every $m \in \mathbb{Z}$ there is an isomorphism of S-modules,

$$\mathrm{H}_m^{\mathfrak{a}S}(N) \;\cong\; \mathrm{H}_m^{\mathfrak{a}}(N)\,.$$

(b) *Let M be an R-complex. There is a commutative diagram in $\mathcal{D}(S)$,*

$$
\begin{tikzcd}
 & \mathsf{RHom}_R(S,M) \arrow[dl, "\lambda^{\mathfrak{a}S}_{\mathsf{RHom}(S,M)}"'] \arrow[dr, "\mathsf{RHom}(S,\lambda^{\mathfrak{a}}_M)"] & \\
\mathsf{L}\Lambda^{\mathfrak{a}S}(\mathsf{RHom}_R(S,M)) \arrow[rr, "\simeq"'] & & \mathsf{RHom}_R(S,\mathsf{L}\Lambda^{\mathfrak{a}}(M))\,,
\end{tikzcd}
$$

where the horizontal morphism is an isomorphism, If M is derived $\mathfrak{a}$-complete, then $\mathsf{RHom}_R(S,M)$ is derived $\mathfrak{a}S$-complete; the converse holds if S is faithfully projective as an R-module. Moreover, if S is projective as an R-module, then there is for every $m \in \mathbb{Z}$ an isomorphism of S-modules,

$$\mathrm{H}_m^{\mathfrak{a}S}(\mathrm{Hom}_R(S,M)) \;\cong\; \mathrm{Hom}_R(S,\mathrm{H}_m^{\mathfrak{a}}(M))\,.$$

PROOF. Let $\boldsymbol{x}$ be a sequence that generats $\mathfrak{a}$; viewed as a sequence in S it generates $\mathfrak{a}S$.

(a): The asserted commutative diagram in $\mathcal{D}(S)$, with the horizontal morphism an isomorphism, exists by 11.1.7 and the definition, 13.1.13, of the functors $\mathsf{L}\Lambda^{\mathfrak{a}S}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$ and the natural transformations $\boldsymbol{\lambda}^{\mathfrak{a}S}$ and $\boldsymbol{\lambda}^{\mathfrak{a}}$. From this diagram it follows that $\boldsymbol{\lambda}_N^{\mathfrak{a}S}$ is an isomorphism in $\mathcal{D}(S)$ if and only if $\boldsymbol{\lambda}_N^{\mathfrak{a}}$ is an isomorphism in $\mathcal{D}(S)$, which by 6.4.37 is equivalent to $\boldsymbol{\lambda}_N^{\mathfrak{a}}$ being an isomorphism in $\mathcal{D}(R)$. By 11.3.3 this means that N is derived $\mathfrak{a}S$-complete if and only if it is derived $\mathfrak{a}$-complete as an R-complex.

In view of the isomorphism in the commutative diagram, the isomorphisms of local homology modules follow straight from the definition, 11.3.6.

(b): Consider the diagram in $\mathcal{D}(S)$ below. The left-hand triangle is commutative by part (a) applied to $N = \mathsf{RHom}_R(S,M)$. Commutativity of the right-hand triangle follows from 13.1.18.

$$\begin{tikzcd}
 & \mathsf{RHom}_R(S,M) \arrow[dl, "\lambda^{\mathfrak{a}S}_{\mathsf{RHom}(S,M)}"'] \arrow[d, "\lambda^{\mathfrak{a}}_{\mathsf{RHom}(S,M)}"] \arrow[dr, "\mathsf{RHom}(S,\lambda^{\mathfrak{a}}_M)"] & \\
\mathsf{L}\Lambda^{\mathfrak{a}S}(\mathsf{RHom}_R(S,M)) \arrow[r, "\simeq"'] & \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{RHom}_R(S,M)) \arrow[r, "\simeq"'] & \mathsf{RHom}_R(S,\mathsf{L}\Lambda^{\mathfrak{a}}(M))\,.
\end{tikzcd}$$

This establishes the asserted commutative diagram. It follows that if $\lambda^{\mathfrak{a}}_M$ is an isomorphism in $\mathcal{D}(R)$, then $\lambda^{\mathfrak{a}S}_{\mathsf{RHom}_R(S,M)}$ is an isomorphism in $\mathcal{D}(S)$, i.e. if M is derived $\mathfrak{a}$-complete, then $\mathsf{RHom}_R(S,M)$ is derived $\mathfrak{a}S$-complete, see 11.3.3.

Assume now that S is projective as an R-module and let $m \in \mathbb{Z}$. From the isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}S}(\mathsf{RHom}_R(S,M)) \cong \mathsf{RHom}_R(S,\mathsf{L}\Lambda^{\mathfrak{a}}(M))$ it follows, in view of 11.3.6 and 2.2.19, that

$$\mathrm{H}^{\mathfrak{a}S}_m(\mathrm{Hom}_R(S,M)) \;\cong\; \mathrm{Hom}_R(S,\mathrm{H}^{\mathfrak{a}}_m(M))$$

holds. The commutative diagram above and another application of 2.2.19 yield

$$\mathrm{H}(\lambda^{\mathfrak{a}S}_{\mathrm{Hom}(S,M)}) \;\cong\; \mathrm{Hom}_R(S,\mathrm{H}(\lambda^{\mathfrak{a}}_M))\,.$$

Assuming, further, that S is faithfully projective as an R-module, it follows from 6.5.17 that if $\lambda^{\mathfrak{a}S}_{\mathrm{Hom}_R(S,M)}$ is an isomorphism in $\mathcal{D}(S)$, then $\lambda^{\mathfrak{a}}_M$ is an isomorphism in $\mathcal{D}(R)$; i.e. if $\mathrm{Hom}_R(S,M)$ is derived $\mathfrak{a}S$-complete, then M is derived $\mathfrak{a}$-complete. □

In part (b) above, the converse statement—that is, M is derived $\mathfrak{a}$-complete if $\mathsf{RHom}_R(S,M)$ is derived $\mathfrak{a}S$-complete—fails without assumptions on S. Indeed, for every ideal $\mathfrak{a}$ in R and every R-complex M, the $R/\mathfrak{a}$-complex $\mathsf{RHom}_R(R/\mathfrak{a},M)$ is trivially derived 0-complete, see 11.1.5, and the zero ideal is the extension of $\mathfrak{a}$ to $R/\mathfrak{a}$. This last observation combined with part (a) above yields:

13.1.22 Corollary. *Let $\mathfrak{a}$ be an ideal in R. Every $R/\mathfrak{a}$-complex is derived $\mathfrak{a}$-complete as an R-complex.*

Proof. As the extension of $\mathfrak{a}$ to the R-algebra $R/\mathfrak{a}$ is the zero ideal, the assertion follows from 11.1.5 and 13.1.21(a). □

13.1.23 Example. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Considered as R-complexes, $R/\mathfrak{a} \otimes_R M$, $R/\mathfrak{a} \otimes^{\mathsf{L}}_R M$, $\mathrm{Hom}_R(R/\mathfrak{a},M)$, and $\mathsf{RHom}_R(R/\mathfrak{a},M)$ are derived $\mathfrak{a}$-complete, see 12.1.4, 12.2.8, 12.1.2, and 12.2.2.

Recall from 6.4.37 that for every R-algebra S the restriction of scalars functor $\mathcal{D}(S) \to \mathcal{D}(R)$ is conservative. This means that a morphism in $\mathcal{D}(S)$ is an isomorphism if (and only if) it is an isomorphism in $\mathcal{D}(R)$. However, an isomorphism in $\mathcal{D}(R)$ of S-complexes can in general not be lifted to an isomorphism in $\mathcal{D}(S)$, not even if R and S are fields, see 13.1.25. Here is a situation where it can be done.

13.1.24 Proposition. *Let $\mathfrak{a} \subset R$ be an ideal and M and N be $\widehat{R}^{\mathfrak{a}}$-complexes that are derived $\mathfrak{a}$-complete as R-complexes. The complexes M and N are isomorphic in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ if and only if they are isomorphic in $\mathcal{D}(R)$.*

Proof. The "only if" part is evident. Now assume that M and N are isomorphic in $\mathcal{D}(R)$. Recall from 11.3.4 that $\mathsf{L}\Lambda^{\mathfrak{a}}$ is a functor from $\mathcal{D}(R)$ to $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, and as one has $M \simeq N$ in $\mathcal{D}(R)$, it follows that $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(N)$ holds in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To finish the proof it suffices to argue that $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M$ and $\mathsf{L}\Lambda^{\mathfrak{a}}(N) \simeq N$ hold in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Per 13.1.14 the $\widehat{R}^{\mathfrak{a}}$-structure on $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is inherited from the $\widehat{R}^{\mathfrak{a}}$-structure on M; further, $\lambda^{\mathfrak{a}}_M \colon M \to \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is a morphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ by 13.1.13. By assumption, M is derived $\mathfrak{a}$-complete as an R-complex, so $\lambda^{\mathfrak{a}}_M$ is an isomorphism in $\mathcal{D}(R)$ by 11.3.3 and hence also an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ by 6.4.37. This proves that one has $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Similarly, one also has $\mathsf{L}\Lambda^{\mathfrak{a}}(N) \simeq N$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. $\square$

13.1.25 Example. The $\mathbb{R}$-vector spaces $\mathbb{R}$ and $\mathbb{R}^2$ have different ranks and hence they are not isomorphic, neither in $\mathcal{M}(\mathbb{R})$ nor in $\mathcal{D}(\mathbb{R})$, see 6.4.15. However, as $\mathbb{Q}$-vector spaces, $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as they have the same infinite rank, namely $2^{\aleph_0}$.

Completion and Flatness

For an ideal $\mathfrak{a}$ in R and a flat R-module F the next result shows that the R-module $\Lambda^{\mathfrak{a}}(F)$ is flat. Faithful flatness is not necessarily preserved under completion, see 15.3.5, but it is shown in 15.3.6 that $\widehat{R}^{\mathfrak{a}}$ is faithfully flat as an R-module if and only if $\mathfrak{a}$ contained in the Jacobson radical of R.

13.1.26 Theorem. *Let $\mathfrak{a}$ be an ideal in R and F complex of flat R-modules. If F is a bounded below, then there is an inequality*

$$\operatorname{fd}_R \Lambda^{\mathfrak{a}}(F) \leqslant \sup F^{\natural} .$$

In particular, for a flat R-module F the module $\Lambda^{\mathfrak{a}}(F)$ is flat.

Proof. Let $\mathfrak{b}$ be an ideal in R and set $\bar{\mathfrak{a}} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$. In the next sequence of isomorphisms in $\mathcal{D}(R)$, the first follows from commutativity 12.3.5 and 13.1.19(b) and the second from 13.1.21(a); the third holds by semi-flatness of F, see 5.4.8.

$$R/\mathfrak{b} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(F) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(R/\mathfrak{b} \otimes^{\mathsf{L}}_R F) \simeq \mathsf{L}\Lambda^{\bar{\mathfrak{a}}}(R/\mathfrak{b} \otimes^{\mathsf{L}}_R F) \simeq \mathsf{L}\Lambda^{\bar{\mathfrak{a}}}(R/\mathfrak{b} \otimes_R F) .$$

By 13.1.15 one has $\Lambda^{\mathfrak{a}}(F) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(F)$, and since $R/\mathfrak{b} \otimes_R F$ per 5.4.18(a) is a semi-flat $R/\mathfrak{b}$-complex, one also has $\Lambda^{\bar{\mathfrak{a}}}(R/\mathfrak{b} \otimes_R F) \simeq \mathsf{L}\Lambda^{\bar{\mathfrak{a}}}(R/\mathfrak{b} \otimes_R F)$. Therefore,

$$\sup (R/\mathfrak{b} \otimes^{\mathsf{L}}_R \Lambda^{\mathfrak{a}}(F)) = \sup \Lambda^{\bar{\mathfrak{a}}}(R/\mathfrak{b} \otimes_R F) \leqslant \sup F^{\natural} .$$

The inequality $\operatorname{fd}_R \Lambda^{\mathfrak{a}}(F) \leqslant \sup F^{\natural}$ now follows from 8.3.11. The last assertion is now immediate, see 8.3.21. $\square$

13.1.27 Corollary. *Let $\mathfrak{a}$ be an ideal in R. As an R-module, $\widehat{R}^{\mathfrak{a}}$ is flat.*

Proof. The assertion is a special case of 13.1.26. □

13.1.28 Corollary. *Let $\mathfrak{a}$ be an ideal in R and F a semi-flat R-complex. If F is bounded below, then the R-complex $\Lambda^{\mathfrak{a}}(F)$ is semi-flat.*

Proof. For every $v \in \mathbb{Z}$ one has $\Lambda^{\mathfrak{a}}(F)_v = \Lambda^{\mathfrak{a}}(F_v)$, see 11.1.12. Thus $\Lambda^{\mathfrak{a}}(F)$ is by 13.1.26 a bounded below complex of flat R-modules and hence semi-flat by 5.4.8. □

Remark. An example by Christensen, Ferraro, and Thompson [58] shows that the boundedness condition in 13.1.28 is necessary.

13.1.29 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M an R-module. If M is $\mathfrak{a}$-complete, then there is a flat resolution $\cdots \longrightarrow F_v \longrightarrow F_{v-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ where each module F_v is $\mathfrak{a}$-complete.*

Proof. Choose by 1.3.12 a surjective homomorphism $\pi\colon L \longrightarrow M$ where L is free. The induced homomorphism $\Lambda^{\mathfrak{a}}(\pi)\colon\ \Lambda^{\mathfrak{a}}(L) \longrightarrow \Lambda^{\mathfrak{a}}(M)$ is surjective by 11.1.28, and one has $\Lambda^{\mathfrak{a}}(M) \cong M$ as M is $\mathfrak{a}$-complete. By 11.1.38 and 13.1.26 the R-module $\Lambda^{\mathfrak{a}}(L)$ is $\mathfrak{a}$-complete and flat. By 11.1.40 the kernel of $\Lambda^{\mathfrak{a}}(\pi)$ is $\mathfrak{a}$-complete, so the desired resolution is constructed by repeating this procedure. □

Derived Complete Complexes

Recall the definition of derived $\mathfrak{a}$-completeness from 11.3.3.

13.1.30 Proposition. *Let $\mathfrak{a}$ be an ideal in R.*

(a) *The subcategory $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ of $\mathcal{D}(R)$ is triangulated.*
(b) *Every complex in $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsupset}(R)$ has an $\mathfrak{a}$-complete semi-flat replacement.*

Proof. As the functors $\mathrm{Id}_{\mathcal{D}(R)}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$ and the natural transformation $\lambda^{\mathfrak{a}}$ are triangulated, see 11.3.2, it follows from E.19 that the subcategory $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ is triangulated. This proves (a). To prove (b), let M be a complex in $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsupset}(R)$. By 5.2.15 the complex M has a bounded below semi-projective replacement P. The R-complex $\Lambda^{\mathfrak{a}}(P)$ is $\mathfrak{a}$-complete by 11.1.38, and it follows from 5.4.10 and 13.1.28 that it is semi-flat. In $\mathcal{D}(R)$ it is isomorphic to $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M$. □

From 13.1.30(a) and 7.6.3 it follows that $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsubset}(R)$, $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\sqsupset}(R)$, and $\mathcal{D}^{\mathfrak{a}\text{-com}}_{\square}(R)$ are triangulated subcategories of $\mathcal{D}(R)$.

13.1.31 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, M a derived $\mathfrak{a}$-complete R-complex, and N an R-complex.*

(a) *The complex $\mathsf{R}\mathrm{Hom}_R(N, M)$ is derived $\mathfrak{a}$-complete.*
(b) *If M is in $\mathcal{D}_{\sqsupset}(R)$ and N in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then $N \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{a}$-complete.*

Proof. By assumption, $\lambda^{\mathfrak{a}}_M$ is an isomorphism in $\mathcal{D}(R)$. It follows from 13.1.18 that $\lambda^{\mathfrak{a}}_{\mathsf{R}\mathrm{Hom}_R(N,M)}$ is an isomorphism, which proves (a). Under the assumptions in (b), it follows by commutativity 12.3.5 from 13.1.19(b) that $\lambda^{\mathfrak{a}}_{N \otimes^{\mathsf{L}}_R M}$ is an isomorphism. □

Let $\mathfrak{a}$ be an ideal in R and recall from 11.1.8 that an R-complex M is $\mathfrak{a}$-complete if the morphism $\lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ is an isomorphism in $\mathcal{C}(R)$. For a derived $\mathfrak{a}$-complete complex M one has $\mathrm{H}_m(M) \cong \mathrm{H}^{\mathfrak{a}}_m(M)$ for every $m \in \mathbb{Z}$; in particular, $\mathrm{H}(M)$ is $\mathfrak{a}$-quasi-complete, see 11.3.11. The next result comes close to a converse.

13.1.32 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If the homology complex $\mathrm{H}(M)$ is $\mathfrak{a}$-complete, then M is derived $\mathfrak{a}$-complete; in symbols:*

$$\lambda^{\mathfrak{a}}_{\mathrm{H}(M)} \colon \mathrm{H}(M) \xrightarrow{\cong} \Lambda^{\mathfrak{a}}(\mathrm{H}(M)) \quad \Longrightarrow \quad \boldsymbol{\lambda}^{\mathfrak{a}}_M \colon M \xrightarrow{\simeq} \mathsf{L}\Lambda^{\mathfrak{a}}(M)\,.$$

Proof. The functors $\mathrm{Id}_{\mathcal{D}(R)}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$ and the natural transformation $\boldsymbol{\lambda}^{\mathfrak{a}}$ are triangulated, see 11.3.2. The subcategory of $\mathcal{D}(R)$ consisting of complexes with $\mathfrak{a}$-complete homology is evidently closed under shifts and soft truncations. The identity functor is bounded, and by 13.1.15 so is $\mathsf{L}\Lambda^{\mathfrak{a}}$. By A.28(d) it now suffices to show that every $\mathfrak{a}$-complete R-module is derived $\mathfrak{a}$-complete. Let M be an $\mathfrak{a}$-complete R-module. It follows from 13.1.29 that there is a complex F of $\mathfrak{a}$-complete flat R-modules with $F \simeq M$ in $\mathcal{D}(R)$. Now 13.1.15 yields $\mathsf{L}\Lambda^{\mathfrak{a}}(M) = \Lambda^{\mathfrak{a}}(F)$, and $\boldsymbol{\lambda}^{\mathfrak{a}}_M = \lambda^{\mathfrak{a}}_F$ is an isomorphism as F is degreewise $\mathfrak{a}$-complete. □

13.1.33 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-module. If M is $\mathfrak{a}$-complete, then it is derived $\mathfrak{a}$-complete.*

Proof. As $\mathrm{H}(M) = M$ is $\mathfrak{a}$-complete, M is derived $\mathfrak{a}$-complete by 13.1.32. □

13.1.34 Example. Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. It follows from 11.4.6(a), 11.1.11, and 13.1.32 that the complex $\mathrm{K}(\boldsymbol{x}) \otimes_R M$ is derived $\mathfrak{a}$-complete.

Remark. The converse of the implication in 13.1.32 fails—for $\mathfrak{a}$-torsion the situation is better, see 13.3.29. The example by Simon discussed in the Remark after 13.1.17 serves as an example of a module which is derived $\mathfrak{a}$-complete but not $\mathfrak{a}$-complete, see Schenzel and Simon [224, 2.5]. Porta, Shaul, and Yekutieli [204] give an example of a complex of amplitude 1 which is derived $\mathfrak{a}$-complete though its homology is not $\mathfrak{a}$-complete.

Notice that given a derived $\mathfrak{a}$-complete complex M such that $\mathrm{H}(M)$ is not complete and a semi-projective replacement P of M, the complex $\Lambda^{\mathfrak{a}}(P)$ is by 11.1.38 $\mathfrak{a}$-complete but its homology $\mathrm{H}(\Lambda^{\mathfrak{a}}(P)) \cong \mathrm{H}(P) \cong \mathrm{H}(M)$ is not $\mathfrak{a}$-complete. Recall, though, from 11.1.41 that the homology of an $\mathfrak{a}$-complete complex is $\mathfrak{a}$-quasi-complete.

13.1.35 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M a derived $\mathfrak{a}$-complete R-complex. For every $v \in \mathbb{Z}$ one has $\mathrm{H}_v(M) = 0$ if and only if $R/\mathfrak{a} \otimes_R \mathrm{H}_v(M) = 0$.*

Proof. As M is derived $\mathfrak{a}$-complete one has $\mathrm{H}_v(M) \cong \mathrm{H}^{\mathfrak{a}}_v(M)$ for all $v \in \mathbb{Z}$. The assertion now follows from 11.3.11. □

Exercises

In exercises E 13.1.1–13.1.4 let $\mathfrak{a}$ be an ideal in R.

E 13.1.1 Let Z be a K-flat R-complex. Show that if Z is acyclic, then $\Lambda^{\mathfrak{a}}(Z)$ is acyclic. *Hint:* 3.5.16.

E 13.1.2 Let M be an R-complex and Z a K-flat R-complex isomorphic to M in $\mathcal{D}(R)$. Show that there is an isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \Lambda^{\mathfrak{a}}(Z)$ in $\mathcal{D}(R)$.

E 13.1.3 Let M be a complex in $\mathcal{D}_{\square}(R)$. (a): Show that if $\mathrm{fd}_R\, M$ is finite, then $\mathrm{fd}_R\, \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is finite. (b): Show that if $\mathrm{id}_R\, M$ is finite, then $\mathrm{id}_R\, \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is finite.

E 13.1.4 Let M be an R-complex and E an injective R-module. Show that for every $m \in \mathbb{Z}$ there is an isomorphism $\mathrm{Hom}_R(\mathrm{H}^m_{\mathfrak{a}}(M), E) \cong \mathrm{H}^{\mathfrak{a}}_m(\mathrm{Hom}_R(M, E))$.

E 13.1.5 With the convention that the Čech complex on the empty sequence is R, show that 13.1.15 remains valid for the empty sequence generating the zero ideal.

E 13.1.6 Calculate the $\mathbb{Z}$-modules $\mathrm{H}^{p\mathbb{Z}}_m(\mathbb{Z})$, $\mathrm{H}^{p\mathbb{Z}}_m(\mathbb{Q})$, and $\mathrm{H}^{p\mathbb{Z}}_m(\mathbb{Q}/\mathbb{Z})$ for $p \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. *Hint:* 3.5.3.

E 13.1.7 Let $\Bbbk$ be a field and consider in $R = \Bbbk[\![x, y]\!]/(x^2)$ the ideals $\mathfrak{p} = (x)$ and $\mathfrak{m} = (x, y)$. (a) Show that there is an isomorphism $R_{\mathfrak{p}} \cong \{y^n \mid n \geqslant 0\}^{-1}R$ and conclude that there is an exact sequence $0 \to \Sigma^{-1}R_{\mathfrak{p}} \to \check{\mathrm{C}}^R(y) \to R \to 0$. (b) Calculate the radical $\sqrt{(y)}$. (c) Show that $\mathsf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M) \to M \to \mathsf{L}\Lambda^{\mathfrak{m}}(M) \to \Sigma\mathsf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M)$ is a distinguished triangle in $\mathcal{D}(R)$ for every R-complex M.

13.2 Derived Completion and Homological Finiteness

Synopsis. Exactness of $\mathfrak{a}$-completion functor; $\mathfrak{a}$-completion of degreewise finitely generated complex; derived $\mathfrak{a}$-completion of complex with degreewise finitely generated homology.

The theory of $\mathfrak{a}$-completion simplifies significantly when one restricts attention to degreewise finitely generated complexes: As discussed in 11.1.20 the $\mathfrak{a}$-completion of an R-complex M is an $\widehat{R}^{\mathfrak{a}}$-complex, and in this case it is simply the base changed complex $\widehat{R}^{\mathfrak{a}} \otimes_R M$. Further, $\mathfrak{a}$-completion is an exact functor on $\mathcal{C}^{\mathrm{f}}(R)$, so it becomes a functor on $\mathcal{D}^{\mathrm{f}}(R)$.

To get started we need the following classic result; it is proved in [182, §8].

The Artin–Rees Lemma. *Let $\mathfrak{a}$ be an ideal in R. Let M be a finitely generated R-module and K a submodule of M. There is an integer $c > 0$ such that*

$$K \cap \mathfrak{a}^i M = \mathfrak{a}^{i-c}(K \cap \mathfrak{a}^c M)$$

holds for all $i \geqslant c$.

Completion and Finiteness

Recall from 11.1.32 that the completion functor, in general, is not even half exact.

13.2.1 Proposition. *Let $\mathfrak{a}$ be an ideal in R. The restricted $\mathfrak{a}$-completion functor $\Lambda^{\mathfrak{a}}(-) : \mathcal{C}^{\mathrm{f}}(R) \longrightarrow \mathcal{C}(R)$ is exact.*

Proof. Let $0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\alpha} N \longrightarrow 0$ be an exact sequence of degreewise finitely generated R-complexes; in view of 11.1.12, one can assume that K, M, and N are finitely generated R-modules. Adopt the notation from 11.1.31; it suffices to show that π is an isomorphism. Without loss of generality, assume that K is a submodule of M and that ι is the embedding. In this case, one has $\iota^{-1}(\mathfrak{a}^u M) = K \cap \mathfrak{a}^u M$, so there is an exact sequence,

$$0 \longrightarrow (K \cap \mathfrak{a}^u M)/\mathfrak{a}^u K \longrightarrow K/\mathfrak{a}^u K \xrightarrow{\pi^u} K/\iota^{-1}(\mathfrak{a}^u M) \longrightarrow 0\,.$$

To prove that $\pi = \lim_{u \geqslant 1} \pi^u$ is an isomorphism, it suffices by 3.5.13 and 3.5.17 to show that the tower $\{(K \cap \mathfrak{a}^u M)/\mathfrak{a}^u K \to (K \cap \mathfrak{a}^{u-1} M)/\mathfrak{a}^{u-1} K\}_{u>1}$ satisfies the trivial Mittag-Leffler Condition 3.5.9. Let $c > 0$ be as in the Artin–Rees Lemma. For every $u \geqslant 1$ one now has $K \cap \mathfrak{a}^{u+c} M = \mathfrak{a}^u (K \cap \mathfrak{a}^c M) \subseteq \mathfrak{a}^u K$, so the composite

$$(K \cap \mathfrak{a}^{u+c} M)/\mathfrak{a}^{u+c} K \longrightarrow \cdots \longrightarrow (K \cap \mathfrak{a}^u M)/\mathfrak{a}^u K$$

is zero; i.e the tower in question satisfies the trivial Mittag-Leffler Condition. ◻

13.2.2. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Recall from (3.4.16.1) that there is a canonical morphism of $\widehat{R}^{\mathfrak{a}}$-complexes, which is easily seen to be natural in M,

$$\Lambda^{\mathfrak{a}}(R) \otimes_R M \;=\; \big(\lim_{u \geqslant 1} R/\mathfrak{a}^u\big) \otimes_R M \xrightarrow{\alpha^M} \lim_{u \geqslant 1} (R/\mathfrak{a}^u \otimes_R M) \;=\; \Lambda^{\mathfrak{a}}(M)\,.$$

As a transformation of functors, α is by 4.1.13 a Σ-transformation.

13.2.3 Theorem. *Let* $\mathfrak{a}$ *be an ideal in* R. *The transformation from* 13.2.2 *yields a natural isomorphism of functors,*

$$\Lambda^{\mathfrak{a}}(-) \;\cong\; \Lambda^{\mathfrak{a}}(R) \otimes_R - : \mathcal{C}^{\mathrm{f}}(R) \;\longrightarrow\; \mathcal{C}(R)\,.$$

Proof. To prove that α^M is an isomorphism for every complex M in $\mathcal{C}^{\mathrm{f}}(R)$, it suffices in view of 11.1.12 to consider the case of a finitely generated R-module. By 2.4.9 the functor $\Lambda^{\mathfrak{a}}(R) \otimes_R -$ is right exact, and by 13.2.1 the functor $\Lambda^{\mathfrak{a}}$ is exact on the category of finitely generated R-modules. Both functors are additive, and the map α^R is evidently an isomorphism, so α^L is an isomorphism for every finitely generated free R-module L. A free presentation $L' \to L \to M \to 0$ with L and L' finitely generated now yields a commutative diagram with exact rows,

$$\begin{tikzcd}
\Lambda^{\mathfrak{a}}(R) \otimes_R L' \arrow[r] \arrow[d, "\alpha^{L'}", "\cong"'] & \Lambda^{\mathfrak{a}}(R) \otimes_R L \arrow[r] \arrow[d, "\alpha^{L}", "\cong"'] & \Lambda^{\mathfrak{a}}(R) \otimes_R M \arrow[r] \arrow[d, "\alpha^{M}"] & 0 \\
\Lambda^{\mathfrak{a}}(L') \arrow[r] & \Lambda^{\mathfrak{a}}(L) \arrow[r] & \Lambda^{\mathfrak{a}}(M) \arrow[r] & 0\,,
\end{tikzcd}$$

and it follows from the Five Lemma 1.1.2 that α^M is an isomorphism. ◻

Recall from 11.1.19 that $\widehat{R}^{\mathfrak{a}}$ is notation for $\Lambda^{\mathfrak{a}}(R)$ considered as an R-algebra; by 11.1.22 it is a Noetherian ring. It follows from 13.2.1 and 13.2.3 that $\widehat{R}^{\mathfrak{a}}$ as an R-module is flat, which is already known from 13.1.27. In 15.3.6 it is proved that $\widehat{R}^{\mathfrak{a}}$ is faithfully flat if and only if $\mathfrak{a}$ contained in the Jacobson radical of R.

13.2.4 Corollary. *Let* $\mathfrak{a}$ *be an ideal in* R; *there is a natural isomorphism of functors,*

$$\Lambda^{\mathfrak{a}}(-) \;\cong\; \widehat{R}^{\mathfrak{a}} \otimes_R - : \mathcal{C}^{\mathrm{f}}(R) \;\longrightarrow\; \mathcal{C}^{\mathrm{f}}(\widehat{R}^{\mathfrak{a}})\,.$$

Proof. For an R-complex M the complexes $\Lambda^{\mathfrak{a}}(M)$ and $\widehat{R}^{\mathfrak{a}} \otimes_R M$ are $\widehat{R}^{\mathfrak{a}}$-complexes by 11.1.20 and 2.1.49. For M in $\mathcal{C}^{\mathrm{f}}(R)$ the complex $\widehat{R}^{\mathfrak{a}} \otimes_R M$ belongs to $\mathcal{C}^{\mathrm{f}}(\widehat{R}^{\mathfrak{a}})$ by 2.5.19. Per 13.2.2 the assertion now follows from 13.2.3. □

Derived Completion and Homological Finiteness

13.2.5 Theorem. *Let* $\mathfrak{a}$ *be an ideal in* R; *there is a triangulated natural isomorphism of functors,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(-) \;\simeq\; \Lambda^{\mathfrak{a}}(R) \otimes_R - : \mathcal{D}^{\mathrm{f}}(R) \longrightarrow \mathcal{D}(R)\,.$$

Proof. Consider the natural transformations of endofunctors on $\mathcal{K}(R)$,

$$\Lambda^{\mathfrak{a}}(R) \otimes_R - \xleftarrow{\Lambda^{\mathfrak{a}}(R)\otimes\pi^{-}} \Lambda^{\mathfrak{a}}(R) \otimes_R \mathrm{P}(-) \xrightarrow{\alpha^{\mathrm{P}(-)}} \Lambda^{\mathfrak{a}}(\mathrm{P}(-))\,,$$

induced by 13.2.2 and 6.3.11. By the same references, 6.2.17, and 7.1.8 both transformations are triangulated, and for every R-complex M the morphism $\Lambda^{\mathfrak{a}}(R) \otimes_R \pi^M$ is a quasi-isomorphism by flatness of $\Lambda^{\mathfrak{a}}(R)$, see 13.1.27. The diagram above establishes a triangulated natural transformation $\tau\colon \Lambda^{\mathfrak{a}}(R) \otimes_R - \longrightarrow \mathsf{L}\Lambda^{\mathfrak{a}}(-)$ of endofunctors on $\mathcal{D}(R)$. The functors are bounded by 13.1.15 and A.27(c). Thus, to see that τ^M is an isomorphism for M in $\mathcal{D}^{\mathrm{f}}(R)$, it suffices per 7.6.14 and A.28(d) to consider the case where M is a finitely generated R-module. By 5.2.16 one can take $\mathrm{P}(M)$ to be bounded below and degreewise finitely generated. It now follows from 13.2.3 that $\alpha^{\mathrm{P}(M)}$ is an isomorphism in $\mathcal{C}(R)$, whence τ^M is an isomorphism in $\mathcal{D}(R)$. □

13.2.6 Corollary. *Let* $\mathfrak{a}$ *be an ideal in* R. *For every complex* M *in* $\mathcal{C}^{\mathrm{f}}(R)$ *there is an isomorphism* $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \Lambda^{\mathfrak{a}}(M)$ *in* $\mathcal{D}(R)$.

In particular, for every finitely generated R*-module* M *one has*

$$\mathrm{H}_0^{\mathfrak{a}}(M) \;\cong\; \Lambda^{\mathfrak{a}}(M) \qquad \text{and} \qquad \mathrm{H}_m^{\mathfrak{a}}(M) \;=\; 0 \quad \text{for all} \quad m > 0\,.$$

Proof. A complex in $\mathcal{C}^{\mathrm{f}}(R)$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$, so the isomorphism follows from 13.2.5 and 13.2.3. The last assertion is now immediate from the definition, 11.3.6, of local homology. □

Recall from 11.1.19 that $\widehat{R}^{\mathfrak{a}}$ is notation for $\Lambda^{\mathfrak{a}}(R)$ considered as an R-algebra; by 11.1.22 it is a Noetherian ring.

13.2.7 Corollary. *Let* $\mathfrak{a}$ *be an ideal in* R; *there is a triangulated natural isomorphism of functors,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(-) \;\simeq\; \widehat{R}^{\mathfrak{a}} \otimes_R - : \mathcal{D}^{\mathrm{f}}(R) \longrightarrow \mathcal{D}^{\mathrm{f}}(\widehat{R}^{\mathfrak{a}})\,.$$

Proof. For an R-complex M it follows from 11.3.10 and 2.1.49 that $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ and $\widehat{R}^{\mathfrak{a}} \otimes_R M$ are $\widehat{R}^{\mathfrak{a}}$-complexes. For M in $\mathcal{D}^{\mathrm{f}}(R)$ it follows by flatness of $\widehat{R}^{\mathfrak{a}}$, see 13.1.27, from 12.1.20(c) that the complex $\widehat{R}^{\mathfrak{a}} \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}(\widehat{R}^{\mathfrak{a}})$. The assertion now follows from 13.2.5. □

Exercises

In the following exercises let $\mathfrak{a}$ be an ideal in R.

E 13.2.1 Show by example that the isomorphism $\Lambda^{\mathfrak{a}}(M) \cong \widehat{R}^{\mathfrak{a}} \otimes_R M$ may fail if M is an R-module that is not finitely generated.

E 13.2.2 Let S be an R-algebra and N a complex in $\mathcal{C}^{\mathrm{f}}(S)$. Show that there is an isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(N) \simeq \Lambda^{\mathfrak{a}}(N)$ in $\mathcal{D}(R)$.

13.3 Derived Torsion

Synopsis. The functor $\mathsf{R}\Gamma_{\mathfrak{a}}$ and local cohomology $\mathrm{H}_{\mathfrak{a}}$; $\mathfrak{a}$-torsion and semi-injectivity; derived $\mathfrak{a}$-torsion via Čech complex; derived $\mathfrak{a}$-torsion and change of rings; derived $\mathfrak{a}$-torsion complex.

For an ideal $\mathfrak{a}$ in R the $\mathfrak{a}$-torsion functor only depends on the radical of $\mathfrak{a}$.

13.3.1 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds, then there is a natural isomorphism $\Gamma_{\mathfrak{a}} \cong \Gamma_{\mathfrak{b}}$ of endofunctors on $\mathcal{C}(R)$.*

Proof. The assertion follows from 11.2.20 in view of 13.1.1. □

Recall that $\mathrm{H}^m_{\mathfrak{a}} = \mathrm{H}_{-m}\,\mathsf{R}\Gamma_{\mathfrak{a}}$ is the m^{th} local cohomology functor supported at $\mathfrak{a}$.

13.3.2 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds, then there is a natural isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}} \simeq \mathsf{R}\Gamma_{\mathfrak{b}}$ of endofunctors on $\mathcal{D}(R)$ and hence natural isomorphisms of local cohomology functors $\mathrm{H}^m_{\mathfrak{a}} \cong \mathrm{H}^m_{\mathfrak{b}}$ for all $m \in \mathbb{Z}$.*

Proof. The assertions follow immediately from 13.3.1 and 11.3.20. □

Torsion and Injectivity

For an ideal $\mathfrak{a}$ in R and a faithfully injective R-module E the R-module $\Gamma_{\mathfrak{a}}(E)$ is injective but not necessarily faithfully so, see 15.3.3.

13.3.3 Lemma. *Let $\mathfrak{a}$ be an ideal in R. For every prime ideal $\mathfrak{p}$ in R one has*

$$\Gamma_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{p})) = \begin{cases} \mathrm{E}_R(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p}\,. \end{cases}$$

Proof. By C.14 the module $\mathrm{E}_R(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion, so $\Gamma_{\mathfrak{a}}(\mathrm{E}_R(R/\mathfrak{p})) = \mathrm{E}_R(R/\mathfrak{p})$ holds if $\mathfrak{a}$ is contained in $\mathfrak{p}$. On the other hand, if there exists an element $x \in \mathfrak{a} \setminus \mathfrak{p}$, then multiplication by any power of this x on $\mathrm{E}_R(R/\mathfrak{p})$ is an automorphism by C.17, so no non-zero element in $\mathrm{E}_R(R/\mathfrak{p})$ is $\mathfrak{a}$-torsion. □

13.3.4 Proposition. *Let I be an injective R-module decomposed per* C.23 *as*

$$I \cong \coprod_{\mathfrak{p} \in \operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{p})^{(U(\mathfrak{p}))}\,.$$

There is an isomorphism,

$$\Gamma_{\mathfrak{a}}(I) \;\cong\; \coprod_{\mathfrak{p}\supseteq\mathfrak{a}} \mathrm{E}_R(R/\mathfrak{p})^{(U(\mathfrak{p}))}\;;$$

in particular, $\Gamma_{\mathfrak{a}}(I)$ *is an injective* R*-module.*

Proof. The isomorphism follows immediately from 13.3.3 as the $\mathfrak{a}$-torsion functor preserves coproduct by 11.2.15. □

13.3.5 Corollary. *Let* $\mathfrak{p}$ *be a prime ideal in* R *and* I *an injective* R*-module decomposed per* C.23 *as*

$$I \;\cong\; \coprod_{\mathfrak{q}\in\operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{q})^{(U(\mathfrak{q}))}\;.$$

There are isomorphisms of $R_{\mathfrak{p}}$*-modules,*

$$\Gamma_{\mathfrak{p}_{\mathfrak{p}}}(I_{\mathfrak{p}}) \;\cong\; \mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}})^{(U(\mathfrak{p}))} \;\cong\; \Gamma_{\mathfrak{p}}(I)_{\mathfrak{p}}\;.$$

Proof. The isomorphisms follow immediately from 13.3.4 and C.24. □

13.3.6 Example. Let $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ a prime ideal in R. If $\mathfrak{a} \subseteq \mathfrak{p}$ holds, then 11.2.23 and 13.3.3 show that $\mathrm{E}_R(R/\mathfrak{p})$ is an $\widehat{R}^{\mathfrak{a}}$-module with the action given by

$$rm \;=\; r^u m$$

for $r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ and $m \in (0 :_{\mathrm{E}_R(R/\mathfrak{p})} \mathfrak{p}^u) \subseteq \mathrm{E}_R(R/\mathfrak{p})$.

13.3.7 Proposition. *Let* $\mathfrak{a}$ *be an ideal and* $\mathfrak{p}$ *a prime ideal in* R*. If* $\mathfrak{a} \subseteq \mathfrak{p}$ *holds, then there is an isomorphism of* $\widehat{R}^{\mathfrak{a}}$*-modules,*

$$\widehat{R}^{\mathfrak{a}} \otimes_R \mathrm{E}_R(R/\mathfrak{p}) \xrightarrow{\;\cong\;} \mathrm{E}_R(R/\mathfrak{p}) \qquad \text{given by} \qquad r\otimes m \longmapsto r^u m\;,$$

for $r = ([r^v]_{\mathfrak{a}^v})_{v\geqslant 1} \in \widehat{R}^{\mathfrak{a}}$ *and* $m \in (0 :_{\mathrm{E}_R(R/\mathfrak{p})} \mathfrak{p}^u) \subseteq \mathrm{E}_R(R/\mathfrak{p})$.

Proof. The assertion follows from 13.3.3 and 11.2.27 with $M = \mathrm{E}_R(R/\mathfrak{p})$. □

The boundedness condition in the next result is necessary, see 17.5.16.

13.3.8 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* I *a semi-injective* R*-complex. If* I *is bounded above, then the* R*-complex* $\Gamma_{\mathfrak{a}}(I)$ *is semi-injective.*

Proof. For every $v \in \mathbb{Z}$ one has $\Gamma_{\mathfrak{a}}(I)_v = \Gamma_{\mathfrak{a}}(I_v)$, see 11.2.13. It now follows from 13.3.4 that $\Gamma_{\mathfrak{a}}(I)$ is a bounded above complex of injective R-modules and hence semi-injective by 5.3.12. □

13.3.9 Lemma. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* M *a graded* R*-module, and* $N \subseteq M$ *a graded submodule. If* N *is essential in* M*, then* $\Gamma_{\mathfrak{a}}(N)$ *is an essential submodule of* $\Gamma_{\mathfrak{a}}(M)$.

Proof. By 11.2.8 one identifies $\Gamma_{\mathfrak{a}}(M)$ with a graded submodule of M. Let $M' \neq 0$ be a graded submodule of $\Gamma_{\mathfrak{a}}(M)$ and thus of M. By assumption the module $M' \cap N$ is non-zero and, clearly, one has $M' \cap N = M' \cap \Gamma_{\mathfrak{a}}(M) \cap N = M' \cap \Gamma_{\mathfrak{a}}(N)$. □

13.3.10 Proposition. *Let $\mathfrak{a}$ be an ideal in R and I a complex of injective R-modules. If I is minimal, then $\Gamma_{\mathfrak{a}}(I)$ is a minimal complex of injective R-modules.*

Proof. For every $v \in \mathbb{Z}$ one has $\Gamma_{\mathfrak{a}}(I)_v = \Gamma_{\mathfrak{a}}(I_v)$, see 11.2.13, so it follows from 13.3.4 that $\Gamma_{\mathfrak{a}}(I)$ is a complex of injective R-modules. By 11.2.8 one identifies $\Gamma_{\mathfrak{a}}(I)$ with a subcomplex of I. It follows from B.21 that $\mathrm{Z}(I)$ is an essential submodule of $I^{\natural}$, so by 13.3.9 the submodule $\Gamma_{\mathfrak{a}}(\mathrm{Z}(I))$ is essential in $\Gamma_{\mathfrak{a}}(I^{\natural}) = \Gamma_{\mathfrak{a}}(I)^{\natural}$. The cycle functor is left exact, see 2.2.16, so one has $\mathrm{Z}(\Gamma_{\mathfrak{a}}(I)) = \Gamma_{\mathfrak{a}}(\mathrm{Z}(I))$, see 1.1.41. Thus, $\mathrm{Z}(\Gamma_{\mathfrak{a}}(I))$ is an essential submodule of $\Gamma_{\mathfrak{a}}(I)^{\natural}$, whence $\Gamma_{\mathfrak{a}}(I)$ is minimal by B.21. $\square$

Derived Torsion via Čech Complexes

13.3.11 Construction. Let $\boldsymbol{x}$ be a sequence in R. One has $\mathrm{H}_0(\mathrm{K}(\boldsymbol{x})) \cong R/(\boldsymbol{x})$ by 11.4.3(a), so 2.5.10 yields a canonical morphism $\pi_{\boldsymbol{x}} \colon \mathrm{K}(\boldsymbol{x}) \twoheadrightarrow R/(\boldsymbol{x})$. For every R-complex M and every $u \geqslant 1$ this morphism conspires with those from (11.1.2.2) and 11.4.8 to yield a commutative diagram,

$$\begin{tikzcd}
\mathrm{Hom}_R(R/(\boldsymbol{x}^u), M) \arrow[r, "(\theta^{\mathrm{K}(\boldsymbol{x}^u)RM})^{-1}\,\mathrm{Hom}(\pi_{\boldsymbol{x}^u}, M)"] \arrow[d, "\mathrm{Hom}(\vartheta_{\boldsymbol{x}}^{u+1}, M)"'] & \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \otimes_R M \arrow[d, "\kappa_{\boldsymbol{x}}^u \otimes M"] \\
\mathrm{Hom}_R(R/(\boldsymbol{x}^{u+1}), M) \arrow[r, "(\theta^{\mathrm{K}(\boldsymbol{x}^{u+1})RM})^{-1}\,\mathrm{Hom}(\pi_{\boldsymbol{x}^{u+1}}, M)"] & \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^{u+1}), R) \otimes_R M\,.
\end{tikzcd}$$

Thus, the composites $\varpi_{\boldsymbol{x}^u}^M = (\theta^{\mathrm{K}(\boldsymbol{x}^u)RM})^{-1}\,\mathrm{Hom}_R(\pi_{\boldsymbol{x}^u}, M)$ form a morphism of telescopes. Let $\varpi_{\boldsymbol{x}}^M$ be the morphism defined by commutativity of the diagram,

$$\begin{tikzcd}
\operatorname*{colim}_{u\geqslant 1} \mathrm{Hom}_R(R/(\boldsymbol{x}^u), M) \arrow[r, "\operatorname{colim}_{u\geqslant 1} \varpi_{\boldsymbol{x}^u}^M"] \arrow[d, "\cong"] & \operatorname*{colim}_{u\geqslant 1} (\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}^u), R) \otimes_R M) \arrow[d, "\cong"'] \\
\Gamma_{(\boldsymbol{x})}(M) \arrow[r, "\varpi_{\boldsymbol{x}}^M"] & \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R M\,,
\end{tikzcd}$$

where the vertical isomorphisms come from 11.2.19, 3.2.22, and 11.4.12.

Let S be an R-algebra and N an S-complex. It follows from 11.1.6 and 12.1.4 that $\Gamma_{(\boldsymbol{x})}(N)$ and $\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R N$ are S-complexes, and it is straightforward to verify that the morphism,

$$\varpi_{\boldsymbol{x}}^N \colon \Gamma_{(\boldsymbol{x})}(N) \longrightarrow \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R N\,,$$

from above is S-linear.

The isomorphism in the next lemma specializes to 11.4.10(a).

13.3.12 Lemma. *Let $\boldsymbol{x}$ be a sequence in R, let S be an R-algebra and N an S-complex. The morphism*

$$\varpi_{\boldsymbol{x}}^M \colon \Gamma_{(\boldsymbol{x})}(N) \longrightarrow \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R N$$

constructed in 13.3.11 *is natural in N, and as a natural transformation of functors, $\varpi_{\boldsymbol{x}}$ is a Σ-transformation. Moreover, if N is an S-module, then the induced map*

$$\mathrm{H}_0(\varpi_{\boldsymbol{x}}^N) : \Gamma_{(\boldsymbol{x})}(N) \longrightarrow \mathrm{H}_0(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R N)$$

is an isomorphism.

Proof. The functors $\Gamma_{(\boldsymbol{x})}$ and $\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R -$ are Σ-functors, see 11.2.15 and 4.1.18, so in view of 3.2.20 it is straightforward to verify that $\varpi_{\boldsymbol{x}}$ is a Σ-transformation.

Let N be an S-module. It suffices by 3.3.15(d) to show that each map $\mathrm{H}_0(\varpi_{\boldsymbol{x}^u}^N)$ is an isomorphism. To this end it is by the definition of $\varpi_{\boldsymbol{x}^u}^N$ enough to prove that

$$\mathrm{H}_0(\mathrm{Hom}_R(\pi_{\boldsymbol{x}^u}, N)) : \mathrm{H}_0(\mathrm{Hom}_R(R/(\boldsymbol{x}^u), N)) \longrightarrow \mathrm{H}_0(\mathrm{Hom}_R(\mathrm{K}^R(\boldsymbol{x}^u), N))$$

is an isomorphism, and that follows from 2.5.10 as $\mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x}^u)) \cong R/(\boldsymbol{x}^u)$. □

13.3.13 Lemma. *Let $\boldsymbol{x}$ be a sequence in R and S an R-algebra. For every complex J of injective S-modules the morphism of S-complexes,*

$$\varpi_{\boldsymbol{x}}^J : \Gamma_{(\boldsymbol{x})}(J) \longrightarrow \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R J\,,$$

from 13.3.12 *is a quasi-isomorphism.*

Proof. By 13.3.11 the map $\varpi_{\boldsymbol{x}}^J$ is S-linear for every S-complex J. Thus, proving that $\varpi_{\boldsymbol{x}}^J$ is a quasi-isomorphism in $\mathcal{C}(S)$ is equivalent to showing that it is a quasi-isomorphism in $\mathcal{C}(R)$, see 6.1.24. Viewing $\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R -$ and $\Gamma_{(\boldsymbol{x})}$ as endofunctors on $\mathcal{C}(R)$, it follows from 2.4.11, 4.1.18, and A.16 that $\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R -$ is a $\natural$-functor, a Σ-functor, and bounded; by 11.2.15 the functor $\Gamma_{(\boldsymbol{x})}$ has the same properties. Moreover, $\varpi_{\boldsymbol{x}}$ is a Σ-transformation by 13.3.12. Thus, by A.17 one can assume that J is an injective S-module, and the assertion now follows from 13.1.5 and 13.3.12. □

We now continue the discussion from 11.3.19.

13.3.14 Construction. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. By 11.2.3 one can view $\Gamma_{\mathfrak{a}}$ as an endofunctor on $\mathcal{C}(S)$; we temporarily denote this functor by the symbol $\mathrm{G}_{\mathfrak{a}}$ to distinguish it from the functor $\Gamma_{\mathfrak{a}} : \mathcal{C}(R) \to \mathcal{C}(R)$. There is a diagram, not necssarily commutative, and a natural transformation:

$$\begin{tikzcd} \mathcal{K}(S) \arrow[r, "\mathrm{G}_{\mathfrak{a}}\mathrm{I}_S"] \arrow[d, "\mathrm{res}_R^S"'] & \mathcal{K}(S) \arrow[d, "\mathrm{res}_R^S"] \\ \mathcal{K}(R) \arrow[r, "\Gamma_{\mathfrak{a}}\mathrm{I}_R"] & \mathcal{K}(R) \end{tikzcd} \quad \text{and} \quad \mathrm{res}_R^S\,\mathrm{G}_{\mathfrak{a}}\mathrm{I}_S = \Gamma_{\mathfrak{a}}\,\mathrm{res}_R^S\,\mathrm{I}_S \xrightarrow{\Gamma_{\mathfrak{a}}\,\varepsilon_R^S} \Gamma_{\mathfrak{a}}\,\mathrm{I}_R\,\mathrm{res}_R^S\,,$$

where ε_R^S is the natural transformation of functors $\mathcal{K}(S) \to \mathcal{K}(R)$ from 6.3.22. By 6.4.31, 6.4.40, and the definition, 7.2.8, of right derived functors, one gets an induced diagram and a natural transformation:

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "\mathsf{R}\mathrm{G}_{\mathfrak{a}}"] \arrow[d, "\mathrm{res}_R^S"'] & \mathcal{D}(S) \arrow[d, "\mathrm{res}_R^S"] \\ \mathcal{D}(R) \arrow[r, "\mathsf{R}\Gamma_{\mathfrak{a}}"] & \mathcal{D}(R) \end{tikzcd} \quad \text{and} \quad \mathrm{res}_R^S\,\mathsf{R}\mathrm{G}_{\mathfrak{a}} \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}\,\mathrm{res}_R^S\ .$$

The gist of the next statement is that $\mathsf{RG}_{\mathfrak{a}}$ on $\mathcal{D}(S)$ is an augmentation of $\mathsf{R}\Gamma_{\mathfrak{a}}$ on $\mathcal{D}(R)$, cf. Chap. 7.

13.3.15 Proposition. *The transformation* $\operatorname{res}_R^S \mathsf{RG}_{\mathfrak{a}} \to \mathsf{R}\Gamma_{\mathfrak{a}} \operatorname{res}_R^S$ *from* 13.3.14 *is a natural isomorphism.*

Proof. We suppress the restriction of scalars functor res_R^S, set $\varepsilon = \varepsilon_R^S$, and write φ for the natural transformation under consideration. Let N be an S-complex. To prove that φ^N is an isomorphism in $\mathcal{D}(R)$ it suffices, by the definition of this morphism, to prove that $\Gamma_{\mathfrak{a}}(\varepsilon^N)$ is a quasi-isomorphism. Consider the S-complex $J = \mathrm{I}_S(N)$ and the R-complex $I = \mathrm{I}_R(N)$. By 6.3.22 the map $\varepsilon^N : J \to I$ is a quasi-isomorphism and hence so is $\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R \varepsilon^N$ by semi-flatness of the Čech complex, see 11.4.10(c). Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. The natural transformation $\varpi_{\boldsymbol{x}}$ from 13.3.12 yields a commutative diagram,

$$\begin{tikzcd}
\Gamma_{\mathfrak{a}}(J) \arrow[r, "\Gamma_{\mathfrak{a}}(\varepsilon^N)"] \arrow[d, "\varpi_{\boldsymbol{x}}^J"'] & \Gamma_{\mathfrak{a}}(I) \arrow[d, "\varpi_{\boldsymbol{x}}^I"] \\
\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R J \arrow[r, "\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes \varepsilon^N", "\simeq"'] & \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R I\,.
\end{tikzcd}$$

As the maps $\varpi_{\boldsymbol{x}}^J$ and $\varpi_{\boldsymbol{x}}^I$ are quasi-isomorphisms by 13.3.13, it follows that $\Gamma_{\mathfrak{a}}(\varepsilon^N)$ is a quasi-isomorphism. $\square$

The result above justifies the following extension of 11.3.16.

13.3.16 Definition. Let $\mathfrak{a}$ be an ideal in R and S an R-algebra. We write $\mathsf{R}\Gamma_{\mathfrak{a}}$ for the right derived functor of $\Gamma_{\mathfrak{a}}$ viewed as an endofunctor on $\mathcal{C}(S)$. Just as in 11.3.16 the Σ-transformation $\gamma_{\mathfrak{a}} : \Gamma_{\mathfrak{a}} \to \mathrm{Id}_{\mathcal{C}(S)}$ from 11.2.3 induces a triangulated natural transformation on endofunctors on $\mathcal{D}(S)$,

$$\boldsymbol{\gamma}_{\mathfrak{a}} \;=\; \mathsf{R}\gamma_{\mathfrak{a}} : \; \mathsf{R}\Gamma_{\mathfrak{a}} \;\longrightarrow\; \mathrm{Id}_{\mathcal{D}(S)}\,.$$

13.3.17. Let $\mathfrak{a}$ be an ideal in R and M an $\widehat{R}^{\mathfrak{a}}$-complex. Let I be a semi-injective replacement of M in $\mathcal{C}(\widehat{R}^{\mathfrak{a}})$. By 11.2.24 the two $\widehat{R}^{\mathfrak{a}}$-structures on $\Gamma_{\mathfrak{a}}(I)$ coming from 11.2.3 and 11.2.23 coincide. It follows that the two $\widehat{R}^{\mathfrak{a}}$-structures on $\mathsf{R}\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(I)$ coming from 13.3.16 and 11.3.18 are the same. In other words, $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules and the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{D}(\widehat{R}^{\mathfrak{a}}) \arrow[r, "\mathsf{R}\Gamma_{\mathfrak{a}}", "\text{(from 13.3.16)}"'] \arrow[d, "\operatorname{res}_R^{\widehat{R}^{\mathfrak{a}}}"'] & \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \arrow[d, equal] \\
\mathcal{D}(R) \arrow[r, "\mathsf{R}\Gamma_{\mathfrak{a}}", "\text{(from 11.3.18)}"'] & \mathcal{D}(\widehat{R}^{\mathfrak{a}})\,.
\end{tikzcd}$$

To parse the next theorem, recall from 11.4.14 the natural transformation $\varepsilon_{\boldsymbol{x}}$. An immediate consequence of the theorem and 11.4.16 is that the derived $\mathfrak{a}$-torsion functor is idempotent. A more precise statement is made in 13.4.1.

13.3.18 Theorem. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and S be an R-algebra. The endofunctor $\mathsf{R}\Gamma_{\mathfrak{a}}$ on $\mathcal{D}(S)$ is bounded and preserves coproducts, there is a triangulated natural isomorphism of endofunctors on $\mathcal{D}(S)$,*

$$\mathsf{R}\Gamma_{\mathfrak{a}}(-) \simeq \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R - = \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R - ,$$

and an isomorphism of natural transformations,

$$\boldsymbol{\gamma}_{\mathfrak{a}} \simeq \varepsilon_{\boldsymbol{x}} \otimes_R - = \varepsilon_{\boldsymbol{x}} \otimes^{\mathsf{L}}_R - .$$

Let N be an S-complex. For every complex J of injective S-modules with $N \simeq J$ in $\mathcal{D}(S)$ there is an isomorphism,

$$\mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq \Gamma_{\mathfrak{a}}(J) ,$$

in $\mathcal{D}(S)$, and if N is a module, then there is an isomorphism $\mathrm{H}^0_{\mathfrak{a}}(N) \cong \Gamma_{\mathfrak{a}}(N)$.

Proof. Consider the natural transformations of endofunctors on $\mathcal{K}(S)$,

$$\Gamma_{\mathfrak{a}}(\mathrm{I}_S(-)) \xrightarrow{\varpi_{\boldsymbol{x}}^{\mathrm{I}_S(-)}} \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R \mathrm{I}_S(-) \xleftarrow{\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes \iota_S^-} \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R - ,$$

induced by 13.3.12 and 6.3.17. By the same references and 6.2.17, both transformations are triangulated. Evaluated at an S-complex they are quasi-isomorphisms by 13.3.13, 6.3.17, and semi-flatness of the Čech complex, see 11.4.10(c). Per 7.2.8 and 6.5.14 the diagram above establishes the triangulated natural isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}}(-) \simeq \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R -$. Further, one has $\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R - = \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R -$ by semi-flatness of the Čech complex; see 7.4.16. The final assertion about modules is now immediate from 13.3.12. Furthermore, it follows from 11.4.10(c) and A.27(c) that $\mathsf{R}\Gamma_{\mathfrak{a}}$ is bounded and from 7.1.8 and 6.4.31 that $\mathsf{R}\Gamma_{\mathfrak{a}}$ preserves coproducts.

Let J be a complex of injective S-modules that is isomorphic to N in $\mathcal{D}(S)$ and let I be a semi-injective replacement of N. By 6.4.21 there is a quasi-isomorphism $J \to I$ of S-complexes, so by semi-flatness of $\check{\mathrm{C}}^R(\boldsymbol{x})$ the middle morphism in (†) below is a quasi-isomorphism. The left- and right-hand morphisms in (†) are quasi-isomorphisms by 13.3.13; this accounts for the isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq \Gamma_{\mathfrak{a}}(J)$ in $\mathcal{D}(S)$.

$$(\dagger) \qquad \Gamma_{\mathfrak{a}}(I) \xrightarrow{\varpi_{\boldsymbol{x}}^I} \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R I \longleftarrow \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R J \xleftarrow{\varpi_{\boldsymbol{x}}^J} \Gamma_{\mathfrak{a}}(J) .$$

Next we show that there is an isomorphism between the natural transformations $\boldsymbol{\gamma}_{\mathfrak{a}}$ and $\varepsilon_{\boldsymbol{x}} \otimes_R -$. In $\mathcal{D}(S)$ every S-complex is naturally isomorphic to its semi-injective resolution, so it suffices to argue that the morphisms $\gamma_{\mathfrak{a}}^I$ and $\varepsilon_{\boldsymbol{x}} \otimes_R I$ are naturally isomorphic in $\mathcal{D}(S)$ for every semi-injective S-complex I. The desired conclusion now follows from commutativity of the diagram:

$$\begin{tikzcd} \Gamma_{\mathfrak{a}}(I) \arrow[r, "\gamma_{\mathfrak{a}}^I"] \arrow[d, "\varpi_{\boldsymbol{x}}^I"', "\simeq"] & I \\ \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R I \arrow[r, "\varepsilon_{\boldsymbol{x}} \otimes I"] & R \otimes_R I . \arrow[u, "\cong", "\mu_R^I"'] \end{tikzcd}$$

Finally, one has $\varepsilon_{\boldsymbol{x}} \otimes_R - = \varepsilon_{\boldsymbol{x}} \otimes^{\mathsf{L}}_R -$ since $\varepsilon_{\boldsymbol{x}} \colon \check{\mathrm{C}}^R(\boldsymbol{x}) \longrightarrow R$ is a morphism of semi-flat R-complexes. □

Theorem 13.3.18 provides for simple proofs of several useful properties $\mathsf{R}\Gamma_{\mathfrak{a}}$.

13.3.19 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. There is a commutative diagram in $\mathcal{D}(R)$, where the horizontal morphism is an isomorphism,*

$$\begin{tikzcd}
\mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R N \arrow[rr, "\simeq"'] \arrow[dr, "\gamma_{\mathfrak{a}}^M \otimes^{\mathsf{L}} N"'] & & \mathsf{R}\Gamma_{\mathfrak{a}}(M \otimes^{\mathsf{L}}_R N) \arrow[dl, "\gamma_{\mathfrak{a}}^{M \otimes^{\mathsf{L}} N}"] \\
& M \otimes^{\mathsf{L}}_R N \,. &
\end{tikzcd}$$

If S is an R-algebra and N an S-complex, then this is a diagram in $\mathcal{D}(S)$.

Proof. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. By 12.3.14 there is a commutative diagram where the horizontal morphisms are isomorphisms,

$$\begin{tikzcd}
(\check{\mathrm{C}}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_R N \arrow[r, "\omega^{\check{\mathrm{C}}(\boldsymbol{x})MN}", "\simeq"'] \arrow[d, "(\varepsilon_{\boldsymbol{x}} \otimes^{\mathsf{L}} N) \otimes^{\mathsf{L}} N"'] & \check{\mathrm{C}}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N) \arrow[d, "\varepsilon_{\boldsymbol{x}} \otimes^{\mathsf{L}} (M \otimes^{\mathsf{L}} N)"] \\
(R \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_R N \arrow[r, "\omega^{RMN}", "\simeq"'] & R \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N) \,;
\end{tikzcd}$$

if M is an S-complex, then this is a diagram in $\mathcal{D}(S)$. The assertions now follow from 13.3.18 in view of the unitor 12.3.3. □

13.3.20 Theorem. *Let $\mathfrak{a}$ an ideal in R and M and N be R-complexes. There is a commutative diagram in $\mathcal{D}(R)$,*

$$\begin{tikzcd}
\mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{R}\mathrm{Hom}_R(M,N)) \arrow[rr] \arrow[dr, "\gamma_{\mathfrak{a}}^{\mathsf{R}\mathrm{Hom}(M,N)}"'] & & \mathsf{R}\mathrm{Hom}_R(M, \mathsf{R}\Gamma_{\mathfrak{a}}(N)) \arrow[dl, "\mathsf{R}\mathrm{Hom}(M,\gamma_{\mathfrak{a}}^N)"] \\
& \mathsf{R}\mathrm{Hom}_R(M,N) \,. &
\end{tikzcd}$$

If S is an R-algebra and N an S-complex, then this is a diagram in $\mathcal{D}(S)$. Furthermore, if one of the conditions (a), (b), *or* (c) *below is satisfied, then the horizontal morphism in the diagram above is an isomorphism.*

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\mathrm{pd}_R\, M$ *is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}_{\sqsubset}(R)$.*
(c) *M is in $\mathcal{D}^{\mathrm{f}}(R)$, N is in $\mathcal{D}_{\square}(R)$, and* $\mathrm{id}_R\, N$ *is finite.*

Proof. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. By 12.3.10 one has the commutative diagram below, where the lower horizontal morphism is an isomorphism by 12.3.23(c). In view of 13.3.18, commutativity 12.3.5, and the unitor 12.3.3 this yields the asserted commutative diagram.

$$
\begin{tikzcd}
\mathsf{RHom}_R(M,N) \otimes^{\mathsf{L}}_R \check{\mathrm{C}}(\boldsymbol{x}) \arrow[r, "\boldsymbol{\theta}^{MN\check{\mathrm{C}}(\boldsymbol{x})}"] \arrow[d, "\mathsf{RHom}(M,N)\otimes^{\mathsf{L}}\varepsilon_{\boldsymbol{x}}"'] & \mathsf{RHom}_R(M, N\otimes^{\mathsf{L}}_R \check{\mathrm{C}}(\boldsymbol{x})) \arrow[d, "\mathsf{RHom}(M,N\otimes^{\mathsf{L}}\varepsilon_{\boldsymbol{x}})"] \\
\mathsf{RHom}_R(M,N) \otimes^{\mathsf{L}}_R R \arrow[r, "\boldsymbol{\theta}^{MNR}", "\simeq"'] & \mathsf{RHom}_R(M, N\otimes^{\mathsf{L}}_R R)\,.
\end{tikzcd}
$$

Under the assumptions in part (a) the morphism $\boldsymbol{\theta}^{MN\check{\mathrm{C}}(\boldsymbol{x})}$ is an isomorphism by 12.3.23(a). Recall from 11.4.10(c) that $\check{\mathrm{C}}(\boldsymbol{x})$ is in $\mathcal{D}_{\square}(R)$ and $\operatorname{fd}_R \check{\mathrm{C}}(\boldsymbol{x})$ is finite. Thus, under the assumptions in parts (b) and (c) the morphism $\boldsymbol{\theta}^{MN\check{\mathrm{C}}(\boldsymbol{x})}$ is an isomorphism by 12.3.23(b) and 12.3.24(b). □

13.3.21 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R and M an R-complex; there is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{b}}(M)) \;\simeq\; \mathsf{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(M)\,.$$

PROOF. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences that generate $\mathfrak{a}$ and $\mathfrak{b}$, and note that the concatenated sequence $\boldsymbol{x},\boldsymbol{y}$ generates the ideal $\mathfrak{a}+\mathfrak{b}$. In the computation below, the 1st and 4th isomorphisms follow from 13.3.18. The 2nd isomorphism holds by associativity 12.3.6 and semi-flatness of the Čech complex; see 11.4.10(c). The 3rd isomorphism follows from the definition, 11.4.9, of the Čech complex.

$$
\begin{aligned}
\mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{b}}(M)) &\simeq \check{\mathrm{C}}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R (\check{\mathrm{C}}(\boldsymbol{y}) \otimes^{\mathsf{L}}_R M)\\
&\simeq (\check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \check{\mathrm{C}}(\boldsymbol{y})) \otimes^{\mathsf{L}}_R M\\
&\simeq \check{\mathrm{C}}(\boldsymbol{x},\boldsymbol{y}) \otimes^{\mathsf{L}}_R M\\
&\simeq \mathsf{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(M)\,.
\end{aligned}
$$
□

13.3.22 Proposition. *Let $\mathfrak{a}$ be an ideal in R and $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ a U-direct system in $\mathcal{C}(R)$. If U is filtered, then there is an isomorphism of R-modules,*

$$\mathrm{H}^{\mathfrak{a}}_m\big(\operatorname*{colim}_{u\in U} M^u\big) \;\cong\; \operatorname*{colim}_{u\in U} \mathrm{H}^m_{\mathfrak{a}}(M^u)\,,$$

for every $m \in \mathbb{Z}$.

PROOF. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. By 3.2.23 and 3.3.15(d) there are isomorphisms,

$$
\begin{aligned}
\mathrm{H}\big(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R \operatorname*{colim}_{u\in U} M^u\big) &\cong \mathrm{H}\big(\operatorname*{colim}_{u\in U} (\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R M^u)\big)\\
&\cong \operatorname*{colim}_{u\in U} \mathrm{H}(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes_R M^u)\,.
\end{aligned}
$$

The asserted isomorphisms now follow from 13.3.18 and the definition, 11.3.20, of local cohomology. □

INDEPENDENCE OF BASE

For an ideal $\mathfrak{a}$ in R, an R-algebra S, and an S-complex N the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(N)$ is an S-complex; see 13.3.16. Part (a) of the next result, which is a derived version of

11.2.4, is traditionally referred to as “independence of base” for local cohomology. To parse the statements recall the definition of derived $\mathfrak{a}$-torsionness from 11.3.17.

13.3.23 Theorem. *Let $\mathfrak{a}$ be an ideal in R and S an R-algebra.*

(a) *Let N be an S-complex. There is a commutative diagram in $\mathcal{D}(S)$,*

$$\begin{tikzcd} \mathsf{R}\Gamma_{\mathfrak{a}S}(N) \arrow[rr, "\simeq"'] \arrow[dr, "\gamma_{\mathfrak{a}S}^N"'] & & \mathsf{R}\Gamma_{\mathfrak{a}}(N) \arrow[dl, "\gamma_{\mathfrak{a}}^N"] \\ & N\,, & \end{tikzcd}$$

where the horizontal morphism is an isomorphism. In particular, N is derived $\mathfrak{a}S$-torsion if and only if it is derived $\mathfrak{a}$-torsion as an R-complex. Moreover, for every $m \in \mathbb{Z}$ there is an isomorphism of S-modules,

$$\mathrm{H}_{\mathfrak{a}S}^m(N) \;\cong\; \mathrm{H}_{\mathfrak{a}}^m(N)\,.$$

(b) *Let M be an R-complex. There is a commutative diagram in $\mathcal{D}(S)$,*

$$\begin{tikzcd} \mathsf{R}\Gamma_{\mathfrak{a}S}(S \otimes_R^{\mathsf{L}} M) \arrow[rr, "\simeq"'] \arrow[dr, "\gamma_{\mathfrak{a}S}^{S\otimes^{\mathsf{L}} M}"'] & & S \otimes_R^{\mathsf{L}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) \arrow[dl, "S\otimes^{\mathsf{L}}\gamma_{\mathfrak{a}}^M"] \\ & S \otimes_R^{\mathsf{L}} M & \end{tikzcd}$$

where the horizontal morphism is an isomorphism. If M is derived $\mathfrak{a}$-torsion, then $S \otimes_R^{\mathsf{L}} M$ is derived $\mathfrak{a}S$-torsion; the converse holds if S is faithfully flat as an R-module. Moreover, if S is flat as an R-module, then there is for every $m \in \mathbb{Z}$ an isomorphism of S-modules,

$$\mathrm{H}_{\mathfrak{a}S}^m(S \otimes_R M) \;\cong\; S \otimes_R \mathrm{H}_{\mathfrak{a}}^m(M)\,.$$

Proof. Let $\boldsymbol{x}$ be a sequence that generats $\mathfrak{a}$; viewed as a sequence in S it generates $\mathfrak{a}S$.

(a): The asserted commutative diagram in $\mathcal{D}(S)$, with the horizontal morphism an isomorphism, exists by 11.2.4, the definition, 13.3.16, of the functors $\mathsf{R}\Gamma_{\mathfrak{a}S}$ and $\mathsf{R}\Gamma_{\mathfrak{a}}$ and the natural transformations $\boldsymbol{\gamma}_{\mathfrak{a}S}$ and $\boldsymbol{\gamma}_{\mathfrak{a}}$. From this diagram it follows that $\boldsymbol{\gamma}_{\mathfrak{a}S}^N$ is an isomorphism in $\mathcal{D}(S)$ if and only if $\boldsymbol{\gamma}_{\mathfrak{a}}^N$ is an isomorphism in $\mathcal{D}(S)$, which by 6.4.37 is equivalent to saying that $\boldsymbol{\gamma}_{\mathfrak{a}}^N$ is an isomorphism in $\mathcal{D}(R)$. By 11.3.17 this means that N is derived $\mathfrak{a}S$-torsion if and only if it is derived $\mathfrak{a}$-torsion as an R-complex. In view of the isomorphism in the commutative diagram, the isomorphisms of local cohomology modules follow straight from the definition, 11.3.20.

(b): Consider the diagram in $\mathcal{D}(S)$ below. The left-hand triangle is commutative by part (a) applied to $N = S \otimes_R^{\mathsf{L}} M$. Commutativity of the right-hand triangle follows by 12.3.5 from 13.3.19.

$$\begin{tikzcd}
\mathsf{R}\Gamma_{\mathfrak{a}S}(S \otimes^{\mathsf{L}}_R M) \arrow[r, "\simeq"'] \arrow[dr, "\gamma_{\mathfrak{a}S}^{S\otimes^{\mathsf{L}} M}"'] & \mathsf{R}\Gamma_{\mathfrak{a}}(S \otimes^{\mathsf{L}}_R M) \arrow[r, "\simeq"'] \arrow[d, "\gamma_{\mathfrak{a}}^{S\otimes^{\mathsf{L}} M}"] & S \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \arrow[dl, "S\otimes^{\mathsf{L}} \gamma_{\mathfrak{a}}^{M}"] \\
& S \otimes^{\mathsf{L}}_R M &
\end{tikzcd}$$

This establishes the asserted commutative diagram. It follows that if $\gamma_{\mathfrak{a}}^M$ is an isomorphism in $\mathcal{D}(R)$, then $\gamma_{\mathfrak{a}S}^{S\otimes^{\mathsf{L}}_R M}$ is an isomorphism in $\mathcal{D}(S)$, i.e. if M is derived $\mathfrak{a}$-torsion, then $S \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{a}S$-torsion, see 11.3.17.

Assume now that S is flat as an R-module and let $m \in \mathbb{Z}$. From the isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}S}(S \otimes^{\mathsf{L}}_R M) \cong S \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ it follows, in view of 11.3.20 and 12.1.20(b), that

$$\mathrm{H}^m_{\mathfrak{a}S}(S \otimes_R M) \;\cong\; S \otimes_R \mathrm{H}^m_{\mathfrak{a}}(M)$$

holds. The commutative diagram above and another application of 12.1.20(b) yield

$$\mathrm{H}(\gamma_{\mathfrak{a}S}^{S\otimes_R M}) \;\cong\; S \otimes_R \mathrm{H}(\gamma_{\mathfrak{a}}^M)\,.$$

Assuming, further, that S is faithfully flat as an R-module, it follows from 6.5.17 that if $\gamma_{\mathfrak{a}S}^{S\otimes_R M}$ is an isomorphism in $\mathcal{D}(S)$, then $\gamma_{\mathfrak{a}}^M$ is an isomorphism in $\mathcal{D}(R)$. That is, if $S \otimes_R M$ is derived $\mathfrak{a}S$-torsion, then M is derived $\mathfrak{a}$-torsion. □

In part (b) above, the converse statement—that is, M is derived $\mathfrak{a}$-torsion if $S \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{a}S$-torsion—fails without assumptions on S. Indeed, for every ideal $\mathfrak{a}$ in R and every R-complex M, the $R/\mathfrak{a}$-complex $R/\mathfrak{a} \otimes^{\mathsf{L}}_R M$ is trivially derived 0-torsion, see 11.2.2, and the zero ideal is the extension of $\mathfrak{a}$ to $R/\mathfrak{a}$. This last observation combined with part (a) above yields:

13.3.24 Corollary. *Let $\mathfrak{a}$ be an ideal in R. Every $R/\mathfrak{a}$-complex is derived $\mathfrak{a}$-torsion as an R-complex.*

Proof. As the extension of $\mathfrak{a}$ to the R-algebra $R/\mathfrak{a}$ is the zero ideal, the assertion follows from 11.2.2 and 13.3.23(a). □

13.3.25 Example. Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Considered as R-complexes, $R/\mathfrak{a} \otimes_R M$, $R/\mathfrak{a} \otimes^{\mathsf{L}}_R M$, $\mathrm{Hom}_R(R/\mathfrak{a}, M)$, and $\mathsf{RHom}_R(R/\mathfrak{a}, M)$ are derived $\mathfrak{a}$-torsion, see 12.1.4, 12.2.8, 12.1.2, and 12.2.2.

Example 13.1.25 and the text before 13.1.24 contextualize the next result.

13.3.26 Proposition. *Let $\mathfrak{a} \subset R$ be an ideal and M and N be $\widehat{R}^{\mathfrak{a}}$-complexes that are derived $\mathfrak{a}$-torsion as R-complexes. The complexes M and N are isomorphic in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ if and only if they are isomorphic in $\mathcal{D}(R)$.*

Proof. The "only if" part is evident. Now assume that M and N are isomorphic in $\mathcal{D}(R)$. Recall from 11.3.18 that $\mathsf{R}\Gamma_{\mathfrak{a}}$ is a functor from $\mathcal{D}(R)$ to $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, and as one has $M \simeq N$ in $\mathcal{D}(R)$, it follows that $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(N)$ holds in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To finish the proof it suffices to argue that $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq M$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq N$ hold in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Per 13.3.17 the $\widehat{R}^{\mathfrak{a}}$-structure on $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is inherited from the $\widehat{R}^{\mathfrak{a}}$-structure on M; further, $\gamma_{\mathfrak{a}}^M \colon \mathsf{R}\Gamma_{\mathfrak{a}}(M) \to M$ is a morphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ by 13.3.16. By assumption,

M is derived $\mathfrak{a}$-torsion as an R-complex, so $\boldsymbol{\gamma}_{\mathfrak{a}}^{M}$ is an isomorphism in $\mathcal{D}(R)$ by 11.3.17 and hence also an isomorphism in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ by 6.4.37. This proves that one has $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq M$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. Similarly, one also has $\mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq N$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. □

Derived Torsion Complexes

Recall the definition of derived $\mathfrak{a}$-torsionness from 11.3.17.

13.3.27 Proposition. *Let $\mathfrak{a}$ be an ideal in R.*

(a) *The subcategory $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ of $\mathcal{D}(R)$ is triangulated.*
(b) *Every complex in $\mathcal{D}_{\sqsubset}^{\mathfrak{a}\text{-tor}}(R)$ has an $\mathfrak{a}$-torsion semi-injective replacement.*

Proof. As the functors $\mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{Id}_{\mathcal{D}(R)}$ and the natural transformation $\boldsymbol{\gamma}_{\mathfrak{a}}$ are triangulated, see 11.3.16, the subcategory $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ is triangulated by E.19. This proves (a). To prove (b), let M be a complex in $\mathcal{D}_{\sqsubset}^{\mathfrak{a}\text{-tor}}(R)$. It follows from 5.3.26 that M has a bounded above semi-injective replacement I. By 13.3.8 the complex $\Gamma_{\mathfrak{a}}(I)$ is semi-injective, and it is isomorphic in $\mathcal{D}(R)$ to $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq M$. □

From 13.3.27(a) and 7.6.3 it follows that $\mathcal{D}_{\sqsubset}^{\mathfrak{a}\text{-tor}}(R)$, $\mathcal{D}_{\sqsupset}^{\mathfrak{a}\text{-tor}}(R)$, and $\mathcal{D}_{\square}^{\mathfrak{a}\text{-tor}}(R)$ are triangulated subcategories of $\mathcal{D}(R)$.

13.3.28 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, M a derived $\mathfrak{a}$-torsion R-complex, and N an R-complex.*

(a) *The complex $M \otimes_R^{\mathsf{L}} N$ is derived $\mathfrak{a}$-torsion.*
(b) *If M is in $\mathcal{D}_{\sqsubset}(R)$ and N in $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$, then $\mathsf{R}\mathrm{Hom}_R(N, M)$ is derived $\mathfrak{a}$-torsion.*

Proof. By assumption, $\boldsymbol{\gamma}_{\mathfrak{a}}^{M}$ is an isomorphism in $\mathcal{D}(R)$. It follows from 13.3.19 that $\boldsymbol{\gamma}_{\mathfrak{a}}^{M \otimes_R^{\mathsf{L}} N}$ is an isomorphism, which proves (a). Under the assumptions in (b), it follows from 13.3.20(b) that $\boldsymbol{\gamma}_{\mathfrak{a}}^{\mathsf{R}\mathrm{Hom}_R(N,M)}$ is an isomorphism. □

Let $\mathfrak{a}$ be an ideal in R and recall from 11.2.8 that an R-complex M is $\mathfrak{a}$-torsion if the morphism $\gamma_{\mathfrak{a}}^{M} \colon \Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism in $\mathcal{C}(R)$. For a derived $\mathfrak{a}$-torsion complex M one has $\mathrm{H}_{-m}(M) \cong \mathrm{H}_{\mathfrak{a}}^{m}(M)$ for every $m \in \mathbb{Z}$; in particular $\mathrm{H}(M)$ is $\mathfrak{a}$-torsion, see for example 11.3.24. Next we prove that the converse holds.

13.3.29 Theorem. *Let $\mathfrak{a}$ be an ideal in R. An R-complex M is derived $\mathfrak{a}$-torsion if and only if the homology complex $\mathrm{H}(M)$ is $\mathfrak{a}$-torsion; in symbols:*

$$\gamma_{\mathfrak{a}}^{\mathrm{H}(M)} \colon \Gamma_{\mathfrak{a}}(\mathrm{H}(M)) \xrightarrow{\cong} \mathrm{H}(M) \quad \Longleftrightarrow \quad \boldsymbol{\gamma}_{\mathfrak{a}}^{M} \colon \mathsf{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\simeq} M\,.$$

Proof. The "only if" is immediate from 13.3.18 and 11.4.13. Conversely, if $\mathrm{H}(M)$ is $\mathfrak{a}$-torsion, then the morphism $\varepsilon_{\boldsymbol{x}} \otimes_R M \colon \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R M \to R \otimes_R M$, where $\boldsymbol{x}$ is a sequence that generates $\mathfrak{a}$, is an isomorphism in $\mathcal{D}(R)$ by 11.4.15, and hence so is $\boldsymbol{\gamma}_{\mathfrak{a}}^{M}$ by 13.3.18. □

13.3.30 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If M is $\mathfrak{a}$-torsion, then it is derived $\mathfrak{a}$-torsion.*

Proof. If M is $\mathfrak{a}$-torsion, then the homology complex $\mathrm{H}(M)$ is $\mathfrak{a}$-torsion by 11.2.14, so it follows from 13.3.29 that M is derived $\mathfrak{a}$-torsion. □

13.3.31 Example. Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. Per 11.4.6 and 13.3.29 the complex $\mathrm{K}(\boldsymbol{x}) \otimes_R M$ is derived $\mathfrak{a}$-torsion.

13.3.32 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M a derived $\mathfrak{a}$-torsion R-complex. For every $v \in \mathbb{Z}$ one has $\mathrm{H}_v(M) = 0$ if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_v(M)) = 0$.*

Proof. As M is derived $\mathfrak{a}$-torsion one has $\mathrm{H}_v(M) \cong \mathrm{H}_{\mathfrak{a}}^{-v}(M)$ for all $v \in \mathbb{Z}$. The assertion now follows from 11.3.24. □

Exercises

E 13.3.1 Let $\boldsymbol{x}$ be a sequence in R. Prove without recourse to 13.3.12 that $\mathrm{Z}_0(\check{\mathrm{C}}(\boldsymbol{x}))$ is $\Gamma_{(\boldsymbol{x})}(R)$.

E 13.3.2 Let $\mathfrak{a}$ be an ideal in R and M a complex in $\mathcal{D}_{\square}(R)$. (a): Show that if $\operatorname{pd}_R M$ is finite, then $\operatorname{pd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is finite. (b): Show that if $\operatorname{fd}_R M$ is finite, then $\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is finite. (c): Show that if $\operatorname{id}_R M$ is finite, then $\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is finite.

E 13.3.3 With the convention that the Čech complex on the empty sequence is R, show that 13.3.18 remains valid for the empty sequence generating the zero ideal.

E 13.3.4 Let $\mathfrak{a}$ be an ideal in R. Show that the category $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ is closed under soft truncations.

E 13.3.5 Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideal in R and M an R-complex. Show that if M is derived $\mathfrak{a}$-torsion, then it is derived $\mathfrak{b}$-torsion.

E 13.3.6 Calculate the $\mathbb{Z}$-modules $\mathrm{H}^m_{p\mathbb{Z}}(\mathbb{Z})$, $\mathrm{H}^m_{p\mathbb{Z}}(\mathbb{Q})$, and $\mathrm{H}^m_{p\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$ for $p \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. *Hint:* C.13.

13.4 Greenlees–May Equivalence

Synopsis. Idempotence of the functors $\mathsf{L}\Lambda^{\mathfrak{a}}$ and $\mathsf{R}\Gamma_{\mathfrak{a}}$; derived $\mathfrak{a}$-complete complex; derived $\mathfrak{a}$-torsion complex; adjointness of $\mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$; Greenlees–May Equivalence.

The isomorphisms (a) and (b) in the next theorem show that the functors $\mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$ are idempotent. The isomorphisms (c) and (d) say, loosely speaking, that when the two functors act successively, only the last action survives.

13.4.1 Theorem. *Let $\mathfrak{a}$ be an ideal in R. The following morphisms in $\mathcal{D}(R)$ are isomorphisms.*

(a) $\mathsf{L}\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M)\colon\ \mathsf{L}\Lambda^{\mathfrak{a}}(M) \longrightarrow \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M))$.

(b) $\mathsf{R}\Gamma_{\mathfrak{a}}(\gamma^M_{\mathfrak{a}})\colon\ \mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}(M)$.

(c) $\mathsf{L}\Lambda^{\mathfrak{a}}(\gamma^M_{\mathfrak{a}})\colon\ \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathsf{L}\Lambda^{\mathfrak{a}}(M)$.

(d) $\mathsf{R}\Gamma_{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M)\colon\ \mathsf{R}\Gamma_{\mathfrak{a}}(M) \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M))$.

Furthermore, there are equalities of morphisms in $\mathcal{D}(R)$,

$$\lambda^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)} = \mathsf{L}\Lambda^{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M) \qquad \textit{and} \qquad \gamma^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}} = \mathsf{R}\Gamma_{\mathfrak{a}}(\gamma^M_{\mathfrak{a}})\,.$$

Proof. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$ and denote by $\omega_{\boldsymbol{x}}$ the composite $\varepsilon_{\boldsymbol{x}}\pi_{\boldsymbol{x}}\colon \mathrm{L}(\boldsymbol{x}) \to R$; see 11.4.14 and 11.4.23. The functors and natural transformations from 11.3.2 and 11.3.16 can be realized as follows:

$$\begin{aligned} \mathsf{L}\Lambda^{\mathfrak{a}} &\simeq \mathrm{Hom}_R(\mathrm{L}(\boldsymbol{x}), -)\,, & \qquad & \boldsymbol{\lambda}^{\mathfrak{a}} \simeq \mathrm{Hom}_R(\omega_{\boldsymbol{x}}, -)\,, \\ \mathsf{R}\Gamma_{\mathfrak{a}} &\simeq \mathrm{L}(\boldsymbol{x}) \otimes_R -\,, & \text{and} \qquad & \boldsymbol{\gamma}_{\mathfrak{a}} \simeq \omega_{\boldsymbol{x}} \otimes_R -\,; \end{aligned}$$

see 13.1.15, 11.4.25(c), and 13.3.18.

Swap 12.1.9 yields

$$(\flat)\qquad \begin{aligned} \mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M) &\simeq \mathrm{Hom}_R(\mathrm{L}(\boldsymbol{x}), \mathrm{Hom}_R(\omega_{\boldsymbol{x}}, M)) \\ &\cong \mathrm{Hom}_R(\omega_{\boldsymbol{x}}, \mathrm{Hom}_R(\mathrm{L}(\boldsymbol{x}), M)) \\ &\simeq \boldsymbol{\lambda}^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)}\,, \end{aligned}$$

and associativity 12.1.8 and commutativity 12.1.7 yield

$$(\diamond)\qquad \mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}}) \simeq \mathrm{L}(\boldsymbol{x}) \otimes_R (\omega_{\boldsymbol{x}} \otimes_R M) \cong \omega_{\boldsymbol{x}} \otimes_R (\mathrm{L}(\boldsymbol{x}) \otimes_R M) \simeq \boldsymbol{\gamma}^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}}\,.$$

Below we show that the morphisms $\mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M)$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}})$ from parts (a) and (b) are isomorphisms. Once this has been proved it follows from ($\flat$) and ($\diamond$) that $\boldsymbol{\lambda}^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)}$ and $\boldsymbol{\gamma}^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}}$ are isomorphisms too, and hence 6.3.20 yields the asserted equalities,

$$\boldsymbol{\lambda}^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)} = \mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M) \qquad \text{and} \qquad \boldsymbol{\gamma}^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}} = \mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}})\,.$$

(a): By ($\flat$) combined with adjunction 12.1.10 one has

$$\mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M) \simeq \mathrm{Hom}_R(\mathrm{L}(\boldsymbol{x}) \otimes_R \omega_{\boldsymbol{x}}, M)\,.$$

Thus, it follows from 4.3.19 and the fact that $\mathrm{L}(\boldsymbol{x}) \otimes_R \omega_{\boldsymbol{x}}$ is a homotopy equivalence that $\mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M)$ is an isomorphism in $\mathcal{D}(R)$.

(b): The tensor product preserves homotopy, see 4.3.20, so to prove that the morphism $\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}})$ is an isomorphism in $\mathcal{D}(R)$, it suffices per ($\diamond$) to show that $\mathrm{L}(\boldsymbol{x}) \otimes_R \omega_{\boldsymbol{x}}$ is a homotopy equivalence. In the commutative diagram,

$$\begin{tikzcd} \mathrm{L}(\boldsymbol{x}) \otimes_R \mathrm{L}(\boldsymbol{x}) \arrow[r, "\mathrm{L}(\boldsymbol{x}) \otimes \omega_{\boldsymbol{x}}"] \arrow[d, "\pi_{\boldsymbol{x}} \otimes \pi_{\boldsymbol{x}}"', "\simeq"] & \mathrm{L}(\boldsymbol{x}) \otimes_R R \arrow[d, "\simeq"', "\pi_{\boldsymbol{x}} \otimes R"] \\ \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R \check{\mathrm{C}}(\boldsymbol{x}) \arrow[r, "\check{\mathrm{C}}(\boldsymbol{x}) \otimes \varepsilon_{\boldsymbol{x}}", "\simeq"'] & \check{\mathrm{C}}(\boldsymbol{x}) \otimes_R R\,, \end{tikzcd}$$

the vertical maps are quasi-isomorphisms by 11.4.25(c) and semi-flatness of the complexes $\mathrm{L}(\boldsymbol{x})$ and $\check{\mathrm{C}}(\boldsymbol{x})$; see 5.4.10 and 11.4.10(c). The lower horizontal morphism is a quasi-isomorphism by 11.4.13 and 11.4.15. The diagram shows that $\mathrm{L}(\boldsymbol{x}) \otimes_R \omega_{\boldsymbol{x}}$ is a quasi-isomorphism and hence a homotopy equivalence as both domain and codomain are semi-free complexes; see 5.1.10, 5.2.11, and 5.2.21.

(d): To show that $\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M) \simeq \mathrm{L}(\boldsymbol{x}) \otimes_R \mathrm{Hom}_R(\omega_{\boldsymbol{x}}, M)$ is an isomorphism in $\mathcal{D}(R)$ is per 6.4.18 equivalent to proving that $\mathrm{L}(\boldsymbol{x}) \otimes_R \mathrm{Hom}_R(\omega_{\boldsymbol{x}}, M)$ is a quasi-isomorphism in $\mathcal{C}(R)$. The first two isomorphisms in the next computation follow from 11.4.22 and 3.2.22,

$$
\begin{aligned}
\mathrm{L}(\boldsymbol{x}) \otimes_R \operatorname{Hom}_R(\omega_{\boldsymbol{x}}, M) &\cong \big(\operatorname*{colim}_{u\geqslant 1} \mathrm{L}^u(\boldsymbol{x})\big) \otimes_R \operatorname{Hom}_R(\omega_{\boldsymbol{x}}, M)\\
&\cong \operatorname*{colim}_{u\geqslant 1} (\mathrm{L}^u(\boldsymbol{x}) \otimes_R \operatorname{Hom}_R(\omega_{\boldsymbol{x}}, M))\\
&\cong \operatorname*{colim}_{u\geqslant 1} \operatorname{Hom}_R(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}}), M)\,,
\end{aligned}
$$

the last isomorphism holds by homomorphism evaluation 12.1.16(c), as $\mathrm{L}^u(\boldsymbol{x})$ is a bounded complex of finitely generated free R-modules, see 11.4.19. To show that the morphism $\operatorname{colim}_{u\geqslant 1} \operatorname{Hom}_R(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}}), M)$ is a quasi-isomorphism, it suffices by 4.2.12, 4.3.4(b), and 4.3.19 to argue that each morphism $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}})$ is a homotopy equivalence. As the domain and codomain of $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}})$, that is, $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \mathrm{L}(\boldsymbol{x}))$ and $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R)$, are bounded complexes of free R-modules, and hence semi-projective by 5.2.8, it suffices by 5.2.21 to prove that $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}})$ is a quasi-isomorphism. By definition one has

$$
\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}}) = \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \varepsilon_{\boldsymbol{x}}) \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \pi_{\boldsymbol{x}})\,.
$$

The morphism $\pi_{\boldsymbol{x}}$ is a quasi-isomorphism by 11.4.25(c), thus so is $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \pi_{\boldsymbol{x}})$. It remains to show that $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \varepsilon_{\boldsymbol{x}})$ is a quasi-isomorphism. Biduality 12.1.14, homomorphism evaluation 12.1.16(c), and the counitor 12.1.6 yield isomorphisms,

$$
\begin{aligned}
\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \varepsilon_{\boldsymbol{x}}) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R), R), \varepsilon_{\boldsymbol{x}})\\
&\cong \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R \operatorname{Hom}_R(R, \varepsilon_{\boldsymbol{x}})\\
&\cong \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R \varepsilon_{\boldsymbol{x}}\,.
\end{aligned}
$$

By 13.1.6 one has $\mathrm{H}(\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R)) \cong \mathrm{H}(\mathrm{K}(\boldsymbol{x}^u))$, and this complex is $(\boldsymbol{x})$-torsion by 11.4.6. Now 11.4.15 yields that $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R) \otimes_R \varepsilon_{\boldsymbol{x}}$ is a quasi-isomorphism.

(c): Showing that $\mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}}) \simeq \operatorname{Hom}_R(\mathrm{L}(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M)$ in an isomorphism in $\mathcal{D}(R)$ is per 6.4.18 equivalent to proving that $\operatorname{Hom}_R(\mathrm{L}(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M)$ is a quasi-isomorphism. From 11.4.22 and 3.4.29 one gets,

$$
\begin{aligned}
\operatorname{Hom}_R(\mathrm{L}(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M) &\cong \operatorname{Hom}_R\big(\operatorname*{colim}_{u\geqslant 1} \mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M\big)\\
&\cong \lim_{u\geqslant 1} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M)\,.
\end{aligned}
$$

Set $\omega^{u,M}_{\boldsymbol{x}} = \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M)$; it must be proved that $\lim_{u\geqslant 1} \omega^{u,M}_{\boldsymbol{x}}$ is a quasi-isomorphism. To this end, set $K_{\boldsymbol{x}} = \operatorname{Ker} \omega_{\boldsymbol{x}}$ and consider the exact sequence,

$$
0 \longrightarrow K_{\boldsymbol{x}} \xrightarrow{\alpha_{\boldsymbol{x}}} \mathrm{L}(\boldsymbol{x}) \xrightarrow{\omega_{\boldsymbol{x}}} R \longrightarrow 0\,,
$$

which is degreewise split, see 5.2.2. It induces by 2.4.12 exact sequences,

$$
\begin{aligned}
0 \longrightarrow \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), K_{\boldsymbol{x}} \otimes_R M) &\xrightarrow{\alpha^{u,M}_{\boldsymbol{x}}} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \mathrm{L}(\boldsymbol{x}) \otimes_R M)\\
&\xrightarrow{\omega^{u,M}_{\boldsymbol{x}}} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), R \otimes_R M) \longrightarrow 0\,.
\end{aligned}
$$

The families $\{\alpha^{u,M}_{\boldsymbol{x}}\}_{u\geqslant 1}$ and $\{\omega^{u,M}_{\boldsymbol{x}}\}_{u\geqslant 1}$ are morphisms of towers. The morphisms in all three towers are surjective as they are induced by the degreewise split embeddings $\mathrm{L}^u(\boldsymbol{x}) \rightarrowtail \mathrm{L}^{u+1}(\boldsymbol{x})$ from 11.4.21. By 3.5.10 all three towers satisfy the

Mittag-Leffler Condition, so 3.5.17 implies that $\lim_{u \geqslant 1} \omega_{\boldsymbol{x}}^{u,M}$ is surjective with kernel isomorphic to $\lim_{u \geqslant 1} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), K_{\boldsymbol{x}} \otimes_R M)$. Thus, showing that $\lim_{u \geqslant 1} \omega_{\boldsymbol{x}}^{u,M}$ is a quasi-isomorphism is by 4.2.6 equivalent to showing that the complex

$$\lim_{u \geqslant 1} \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), K_{\boldsymbol{x}} \otimes_R M)$$

is acyclic. As already mentioned, the tower $\{\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), K_{\boldsymbol{x}} \otimes_R M)\}_{u \geqslant 1}$ satisfies the Mittag-Leffler Condition, so by 3.5.16 it it enough to argue that each complex $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), K_{\boldsymbol{x}} \otimes_R M)$ is acyclic. By another application of 4.2.6 this is equivalent to showing that $\omega_{\boldsymbol{x}}^{u,M}$ is a quasi-isomorphism. One has

$$\omega_{\boldsymbol{x}}^{u,M} = \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}} \otimes_R M) \cong \operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}}) \otimes_R M$$

by tensor evaluation 12.1.15(d). In the proof of (c) we showed that $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}})$ is a homotopy equivalence, and hence so is $\operatorname{Hom}_R(\mathrm{L}^u(\boldsymbol{x}), \omega_{\boldsymbol{x}}) \otimes_R M$ by 4.3.20. □

Derived Complete Complexes

13.4.2 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is derived $\mathfrak{a}$-complete.*

Proof. It follows from last assertion in 13.4.1 combined with 13.4.1(a) that the morphism $\boldsymbol{\lambda}^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)}$ is an isomorphism. □

13.4.3 Proposition. *Let $\mathfrak{a}$ be an ideal in R. There is an adjunction,*

$$\mathcal{D}(R) \underset{\longleftarrow}{\xrightarrow{\mathsf{L}\Lambda^{\mathfrak{a}}}} \mathcal{D}^{\mathfrak{a}\text{-com}}(R)\,,$$

where the right adjoint is the inclusion functor. For a complex M in $\mathcal{D}(R)$ the unit of the adjunction is the morphism $\boldsymbol{\lambda}^{\mathfrak{a}}_M \colon M \to \mathsf{L}\Lambda^{\mathfrak{a}}(M)$. For a complex N in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ the counit is the isomorphism $(\boldsymbol{\lambda}^{\mathfrak{a}}_N)^{-1} \colon \mathsf{L}\Lambda^{\mathfrak{a}}(N) \to N$.

Proof. It follows from 13.4.2 that the image of $\mathsf{L}\Lambda^{\mathfrak{a}}$ is contained in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$. To see that the functors are adjoint with the asserted unit and counit, it suffices to verify the zigzag identities. That is, for M in $\mathcal{D}(R)$ and N in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ the composites

$$\mathsf{L}\Lambda^{\mathfrak{a}}(M) \xrightarrow{\mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M)} \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \xrightarrow{(\boldsymbol{\lambda}^{\mathfrak{a}}_{\mathsf{L}\Lambda^{\mathfrak{a}}(M)})^{-1}} \mathsf{L}\Lambda^{\mathfrak{a}}(M)$$

and

$$N \xrightarrow{\boldsymbol{\lambda}^{\mathfrak{a}}_N} \mathsf{L}\Lambda^{\mathfrak{a}}(N) \xrightarrow{(\boldsymbol{\lambda}^{\mathfrak{a}}_N)^{-1}} N$$

must be identities. The last assertion in 13.4.1 shows that the first composite is the identity on $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$. Evidently, the second composite is the identity on N. □

Remark. It is not surprising that the counit of adjunction in 13.4.3 is an isomorphism; indeed this is well-known to be equivalent to the right adjont being fully faithful, which is evident, as it is the inclusion functor $\mathcal{D}^{\mathfrak{a}\text{-com}}(R) \to \mathcal{D}(R)$. The fact that this inclusion functor has a left adjoint expresses that $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ is a reflective subcategory of $\mathcal{D}(R)$.

We collect characterizations of what it means to be derived $\mathfrak{a}$-complete.

13.4.4 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The following conditions are equivalent.*

(i) *M is derived $\mathfrak{a}$-complete.*
(ii) *There is an isomorphism $M \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ in $\mathcal{D}(R)$.*
(iii) *For some R-complex X there is an isomorphism $M \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(X)$ in $\mathcal{D}(R)$.*

Moreover, the next condition implies (i)–(iii).

(iv) *$\mathrm{H}(M)$ is $\mathfrak{a}$-complete.*

Proof. The three first conditions are equivalent by 13.4.2. Further, if $\mathrm{H}(M)$ is $\mathfrak{a}$-complete, then M is derived $\mathfrak{a}$-complete by 13.1.32. □

13.4.5 Corollary. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals in R and M an R-complex. If M is derived $\mathfrak{a}$-complete, then it is derived $\mathfrak{b}$-complete.*

Proof. By the assumption on M and 13.1.20 there are isomorphisms,

$$\mathsf{L}\Lambda^{\mathfrak{b}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{b}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{b}+\mathfrak{a}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M,$$

in $\mathcal{D}(R)$, whence M is derived $\mathfrak{b}$-complete by 13.4.4. □

For a prime ideal $\mathfrak{p}$ in R and an R-complex M, the complex $R/\mathfrak{p} \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{p}$-complete by 13.1.22 and, therefore, derived $\mathfrak{a}$-complete for every ideal $\mathfrak{a}$ contained in $\mathfrak{p}$, see 13.4.5. The assumption on M in the next lemma is thus trivially satisfied for prime ideals in $\mathrm{V}(\mathfrak{a})$. Notice from 13.1.31(b) that the assumption is also satisfied if M is derived $\mathfrak{a}$-complete and belongs to $\mathcal{D}_{\sqsupset}(R)$.

13.4.6 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the complex $R/\mathfrak{p} \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{a}$-complete for every prime ideal $\mathfrak{p}$ in R. For every integer m, the following conditions are equivalent.*

(i) $\operatorname{Tor}^R_m(R/\mathfrak{p}, M) = 0$ *holds for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$.*
(ii) $\operatorname{Tor}^R_m(K, M) = 0$ *holds for every R-module K.*

In particular, if $\mathrm{H}_m(M) \neq 0$ *then* $\mathrm{H}_m(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \neq 0$ *for some* $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$*, whence one has the inequality,*

$$\sup\{\sup(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \geqslant \sup M .$$

Proof. By 3.3.5 every R-module is a filtered colimit of finitely generated modules and the functor $\operatorname{Tor}^R_m(-, M)$ preserves filtered colimits by 7.4.25. Thus, condition (ii) is equivalent to the following condition:

(ii') $\operatorname{Tor}^R_m(K, M) = 0$ holds for every finitely generated R-module K.

The functor $\mathrm{F} = \operatorname{Tor}^R_m(-, M)$ is R-linear, see 7.4.18, and half exact by 7.4.29. For every prime ideal $\mathfrak{p}$ in R, the complex $R/\mathfrak{p} \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{a}$-complete by assumption, so 13.1.35 shows that $\mathrm{F}(R/\mathfrak{p}) \neq 0$ implies $R/\mathfrak{a} \otimes_R \mathrm{F}(R/\mathfrak{p}) \neq 0$. Now the equivalence of conditions (i) and (ii') follows from 12.4.4.

Finally, for $m \in \mathbb{Z}$ with $\mathrm{H}_m(M) \neq 0$ one has $\mathrm{H}_m(R \otimes^{\mathsf{L}}_R M) = \mathrm{Tor}^R_m(R, M) \neq 0$, so $\mathrm{H}_m(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) = \mathrm{Tor}^R_m(R/\mathfrak{p}, M) \neq 0$ holds for some $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. From this observation the asserted inequality follows. □

Derived Torsion Complexes

13.4.7 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is derived $\mathfrak{a}$-torsion.*

Proof. It follows from last assertion in 13.4.1 combined with 13.4.1(b) that the morphism $\boldsymbol{\gamma}_{\mathfrak{a}}^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}$ is an isomorphism. □

13.4.8 Proposition. *Let $\mathfrak{a}$ be an ideal in R. There is an adjunction,*

$$\mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \underset{\mathsf{R}\Gamma_{\mathfrak{a}}}{\overset{}{\rightleftarrows}} \mathcal{D}(R)\;,$$

where the left adjoint is the inclusion functor. For a complex M in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ the unit of the adjunction is the isomorphism $(\boldsymbol{\gamma}_{\mathfrak{a}}^M)^{-1} \colon M \to \mathsf{R}\Gamma_{\mathfrak{a}}(M)$. For a complex N in $\mathcal{D}(R)$ the counit is the morphism $\boldsymbol{\gamma}_{\mathfrak{a}}^N \colon \mathsf{R}\Gamma_{\mathfrak{a}}(N) \to N$.

Proof. It follows from 13.4.7 that the image of $\mathsf{R}\Gamma_{\mathfrak{a}}$ is contained in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$. To see that the functors are adjoint with the asserted unit and counit, it suffices to verify the zigzag identities. That is, for M in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ and N in $\mathcal{D}(R)$ the composites

$$M \xrightarrow{(\gamma_{\mathfrak{a}}^M)^{-1}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\gamma_{\mathfrak{a}}^M} M$$

and

$$\mathsf{R}\Gamma_{\mathfrak{a}}(N) \xrightarrow{(\gamma_{\mathfrak{a}}^{\mathsf{R}\Gamma_{\mathfrak{a}}(N)})^{-1}} \mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{a}}(N)) \xrightarrow{\mathsf{R}\Gamma_{\mathfrak{a}}(\gamma_{\mathfrak{a}}^N)} \mathsf{R}\Gamma_{\mathfrak{a}}(N)$$

must be identities. Evidently, the first composite is the identity on M. The last assertion in 13.4.1 shows that the second composite is the identity on $\mathsf{R}\Gamma_{\mathfrak{a}}(N)$. □

Remark. It is not surprising that the unit of adjunction in 13.4.8 is an isomorphism; indeed, by adjoint functor theory this is equivalent to the left adjont being fully faithful, which is evident in this case, as it is the inclusion functor $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \to \mathcal{D}(R)$. The fact that this inclusion functor has a right adjoint expresses that $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ is a coreflective subcategory of $\mathcal{D}(R)$.

We collect characterizations of what it means to be derived $\mathfrak{a}$-torsion.

13.4.9 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The next conditions are equivalent.*

(*i*) *$\mathrm{H}(M)$ is $\mathfrak{a}$-torsion.*
(*ii*) *M is derived $\mathfrak{a}$-torsion.*
(*iii*) *There is an isomorphism $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ in $\mathcal{D}(R)$.*
(*iv*) *For some R-complex X there is an isomorphism $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(X)$ in $\mathcal{D}(R)$.*

Proof. The conditions are equivalent by 13.3.29 and 13.4.7. □

13.4.10 Corollary. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals in R and M an R-complex. If M is derived $\mathfrak{a}$-torsion, then it is derived $\mathfrak{b}$-torsion.*

Proof. By the assumption on M and 13.3.21 there are isomorphisms,

$$\mathsf{R}\Gamma_{\mathfrak{b}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{b}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathsf{R}\Gamma_{\mathfrak{b}+\mathfrak{a}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq M,$$

in $\mathcal{D}(R)$, whence M is derived $\mathfrak{b}$-torsion by 13.4.9. □

For a prime ideal $\mathfrak{p}$ in R and an R-complex M, the complex $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)$ is derived $\mathfrak{p}$-torsion by 13.3.24 and, therefore, derived $\mathfrak{a}$-torsion for every ideal $\mathfrak{a}$ contained in $\mathfrak{p}$, see 13.4.10. The assumption on M in the next lemma is thus trivially satisfied for prime ideals in $\mathrm{V}(\mathfrak{a})$. Notice from 13.3.28(b) that the assumption is also satisfied if M is derived $\mathfrak{a}$-torsion and belongs to $\mathcal{D}_{\sqsubset}(R)$.

13.4.11 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the complex $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)$ is derived $\mathfrak{a}$-torsion for every prime ideal $\mathfrak{p}$ in R. For every integer m, the following conditions are equivalent.*

(*i*) $\mathrm{Ext}^m_R(R/\mathfrak{p}, M) = 0$ *holds for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$.*

(*ii*) $\mathrm{Ext}^m_R(K, M) = 0$ *holds for every finitely generated R-module K.*

In particular, if $\mathrm{H}_m(M) \neq 0$ then $\mathrm{H}_m(\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)) \neq 0$ for some $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, whence one has the inequality,

$$\sup\{-\inf \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \geqslant -\inf M .$$

Proof. The functor $\mathrm{G} = \mathrm{Ext}^m_R(-, M)$ is R-linear, see 7.3.23, and half exact by 7.3.35. For every prime ideal $\mathfrak{p}$ in R, the complex $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)$ is derived $\mathfrak{a}$-torsion by assumption, so 13.3.32 shows that $\mathrm{G}(R/\mathfrak{p}) \neq 0$ implies $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{G}(R/\mathfrak{p})) \neq 0$. Now the equivalence of conditions (*i*) and (*ii*) follows from 12.4.10.

Finally, for every integer m with $\mathrm{H}_m(M) \neq 0$ the module $\mathrm{H}_m(\mathsf{R}\mathrm{Hom}_R(R, M)) = \mathrm{Ext}^{-m}_R(R, M)$ is non-zero, so $\mathrm{H}_m(\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)) = \mathrm{Ext}^{-m}_R(R/\mathfrak{p}, M) \neq 0$ holds for some $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. From this observation the asserted inequality follows. □

Greenlees–May Equivalence

To parse the formulas for the unit and counit in the next theorem see 13.4.1(c,d).

13.4.12 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N) \simeq \mathsf{R}\mathrm{Hom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(N)) ,$$

which is natural in M and N. In particular, $\mathsf{R}\Gamma_{\mathfrak{a}}$ is left adjoint for $\mathsf{L}\Lambda^{\mathfrak{a}}$. The unit and counit of this adjunction are given by the composites

$$M \xrightarrow{\lambda^{\mathfrak{a}}_M} \mathsf{L}\Lambda^{\mathfrak{a}}(M) \xrightarrow{\mathsf{L}\Lambda^{\mathfrak{a}}(\gamma^M_{\mathfrak{a}})^{-1}} \mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M))$$

and

$$\mathsf{R}\Gamma_{\mathfrak{a}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \xrightarrow{\mathsf{R}\Gamma_{\mathfrak{a}}(\lambda^{\mathfrak{a}}_M)^{-1}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\gamma^M_{\mathfrak{a}}} M .$$

Proof. The isomorphism follows from 13.3.18, adjunction 12.3.18, and 13.1.15. Indeed, given a sequence $\boldsymbol{x}$ that generates $\mathfrak{a}$ one has

$$\begin{aligned} \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N) &\simeq \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R M, N) \\ &\simeq \mathsf{RHom}_R(M, \mathsf{RHom}_R(\check{\mathrm{C}}(\boldsymbol{x}), N)) \\ &\simeq \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(N)) \,. \end{aligned}$$

In view of 7.3.26 this shows, in particular, that $\mathsf{R}\Gamma_{\mathfrak{a}}$ is left adjoint for $\mathsf{L}\Lambda^{\mathfrak{a}}$. To see that the unit and counit of the adjunction are given by the asserted formulas, it suffices to verify the zigzag identities. That is, with

$$\mathrm{F} = \mathsf{R}\Gamma_{\mathfrak{a}}\,, \quad \mathrm{G} = \mathsf{L}\Lambda^{\mathfrak{a}}\,, \quad \alpha = (\mathrm{G}\boldsymbol{\gamma}_{\mathfrak{a}})^{-1} \circ \boldsymbol{\lambda}^{\mathfrak{a}}\,, \quad \text{and} \quad \beta = \boldsymbol{\gamma}_{\mathfrak{a}} \circ (\mathrm{F}\boldsymbol{\lambda}^{\mathfrak{a}})^{-1}$$

it must be shown that the composites $\mathrm{F} \xrightarrow{\mathrm{F}\alpha} \mathrm{FGF} \xrightarrow{\beta\mathrm{F}} \mathrm{F}$ and $\mathrm{G} \xrightarrow{\alpha\mathrm{G}} \mathrm{GFG} \xrightarrow{\mathrm{G}\beta} \mathrm{G}$ are the identities on F and G. As $\boldsymbol{\lambda}^{\mathfrak{a}} \colon \mathrm{Id}_{\mathcal{D}(R)} \to \mathrm{G}$ is a natural transformation, one has for every morphism $\varphi \colon N \to M$ in $\mathcal{D}(R)$ the left-hand commutative diagram below.

$$\begin{tikzcd} N \arrow[r, "\lambda^{\mathfrak{a}}_N"] \arrow[d, "\varphi"'] & \mathrm{G}(N) \arrow[d, "\mathrm{G}(\varphi)"] \\ M \arrow[r, "\lambda^{\mathfrak{a}}_M"] & \mathrm{G}(M) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{FF}(M) \arrow[r, "\mathrm{F}(\lambda^{\mathfrak{a}}_{\mathrm{F}(M)})", "\simeq"'] \arrow[d, "\mathrm{F}(\gamma^M_{\mathfrak{a}})"', "\simeq"] & \mathrm{FGF}(M) \arrow[d, "\simeq"', "\mathrm{FG}(\gamma^M_{\mathfrak{a}})"] \\ \mathrm{F}(M) \arrow[r, "\mathrm{F}(\lambda^{\mathfrak{a}}_M)", "\simeq"'] & \mathrm{FG}(M) \end{tikzcd}$$

The right-hand commutative diagram arises by applying the functor F to the diagram on the left in the case where φ is the morphism $\boldsymbol{\gamma}^M_{\mathfrak{a}} \colon \mathrm{F}(M) \to M$; all morphisms in this diagram are isomorphisms by 13.4.1. By definition, one has

$$\mathrm{F}(\alpha^M) = \mathrm{FG}(\boldsymbol{\gamma}^M_{\mathfrak{a}})^{-1} \circ \mathrm{F}(\boldsymbol{\lambda}^{\mathfrak{a}}_M) \qquad \text{and} \qquad \beta^{\mathrm{F}(M)} = \boldsymbol{\gamma}^{\mathrm{F}(M)}_{\mathfrak{a}} \circ \mathrm{F}(\boldsymbol{\lambda}^{\mathfrak{a}}_{\mathrm{F}(M)})^{-1} \,.$$

From these formulas and the right-hand commutative diagram above, one gets

$$\beta^{\mathrm{F}(M)} \circ \mathrm{F}(\alpha^M) = \boldsymbol{\gamma}^{\mathrm{F}(M)}_{\mathfrak{a}} \circ \mathrm{F}(\boldsymbol{\gamma}^M_{\mathfrak{a}})^{-1} \,.$$

By 13.4.1 the morphisms $\boldsymbol{\gamma}^{\mathrm{F}(M)}_{\mathfrak{a}} = \boldsymbol{\gamma}^{\mathsf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}}$ and $\mathrm{F}(\boldsymbol{\gamma}^M_{\mathfrak{a}}) = \mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}})$ are equal, so the composite $\beta^{\mathrm{F}(M)} \circ \mathrm{F}(\alpha^M)$ is the identity on $\mathrm{F}(M) = \mathsf{R}\Gamma_{\mathfrak{a}}(M)$. This proves the first of the two zigzag identities; the other is proved similarly. □

13.4.13 Theorem. *Let $\mathfrak{a}$ be an ideal in R. There is an adjoint equivalence of R-linear triangulated categories,*

$$\mathcal{D}^{\mathfrak{a}\text{-com}}(R) \underset{\mathsf{L}\Lambda^{\mathfrak{a}}}{\overset{\mathsf{R}\Gamma_{\mathfrak{a}}}{\rightleftarrows}} \mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \,.$$

Proof. By 13.4.7 the image of $\mathsf{R}\Gamma_{\mathfrak{a}}$ is contained in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ and the image of $\mathsf{L}\Lambda^{\mathfrak{a}}$ is by 13.4.2 contained in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$. It is immediate from 13.4.12 that the unit of the adjuction is an isomorphism for every complex in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ and that the counit is an isomorphism for every complex in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$. The functors $\mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$ are R-linear and triangulated, see 11.3.15 and 11.3.1. □

13.4.14 Example. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. The Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x}) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(R)$, see 13.3.18, belongs by 13.4.7 to $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$. Under the equivalence in 13.4.13, it corresponds to $\mathsf{L}\Lambda^{\mathfrak{a}}(\mathsf{R}\Gamma_{\mathfrak{a}}(R)) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(R) \simeq \Lambda^{\mathfrak{a}}(R)$ in $\mathcal{D}^{\mathfrak{a}\text{-com}}(R)$, see 13.4.1(c) and 13.2.5. Every complex in $\mathcal{D}^{\mathfrak{a}\text{-tor}}(R) \cap \mathcal{D}^{\mathfrak{a}\text{-com}}(R)$ corresponds to itself under the equivalence 13.4.13; in particular, every $R/\mathfrak{a}$-complex corresponds to itself, see 13.1.23 and 13.3.25.

Derived Completion and Torsion of Derived Hom and Tensor Products

13.4.15. Let $\mathfrak{a}$ be an ideal in R. As an R-module, $\widehat{R}^{\mathfrak{a}}$ is flat by 13.1.27, so it follows from 12.2.4 and 12.2.9 that $\mathsf{R}\mathrm{Hom}_R$ and $\otimes^{\mathsf{L}}_R$ are augmented to functors,

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(-,-) : \mathcal{D}(\widehat{R}^{\mathfrak{a}})^{\mathrm{op}} \times \mathcal{D}(\widehat{R}^{\mathfrak{a}}) &\longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}\text{–}\widehat{R}^{\mathfrak{a}}) \quad \text{and} \\ -\otimes^{\mathsf{L}}_R - : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \times \mathcal{D}(\widehat{R}^{\mathfrak{a}}) &\longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}\text{–}\widehat{R}^{\mathfrak{a}})\,, \end{aligned}$$

induced by $\mathrm{Hom}_R(-, \mathrm{I}_{\widehat{R}^{\mathfrak{a}}}(-))$ and $\mathrm{P}_{\widehat{R}^{\mathfrak{a}}}(-) \otimes_R -$. On the level of objects, the output of these functors is complexes of $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules, which are not necessarily symmetric. As $\mathsf{L}\Lambda^{\mathfrak{a}}$ and $\mathsf{R}\Gamma_{\mathfrak{a}}$ are functors from $\mathcal{D}(R)$ to $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, see 11.3.4 and 11.3.18, there are composite functors,

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_R(-, \mathsf{L}\Lambda^{\mathfrak{a}}(-)) : \mathcal{D}(\widehat{R}^{\mathfrak{a}})^{\mathrm{op}} \times \mathcal{D}(R) &\longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}\text{–}\widehat{R}^{\mathfrak{a}})\,, \\ \mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(-), -) : \mathcal{D}(R)^{\mathrm{op}} \times \mathcal{D}(\widehat{R}^{\mathfrak{a}}) &\longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}\text{–}\widehat{R}^{\mathfrak{a}})\,, \quad \text{and} \\ -\otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(-) : \mathcal{D}(\widehat{R}^{\mathfrak{a}}) \times \mathcal{D}(R) &\longrightarrow \mathcal{D}(\widehat{R}^{\mathfrak{a}}\text{–}\widehat{R}^{\mathfrak{a}})\,. \end{aligned}$$

The functors recalled above are studied in the next result, which yields derived versions of 11.1.24 and 11.2.26. The proofs of parts (b) and (c) use 11.2.26; however, the proof of part (a) is different in nature and does not make use of 11.1.24.

13.4.16 Theorem. *Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and N an $\widehat{R}^{\mathfrak{a}}$-complex. The following objects are complexes of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules,*

$$\mathsf{R}\mathrm{Hom}_R(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M))\,, \quad \mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N)\,, \quad \textit{and} \quad N \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,,$$

and there are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$,

(a) $$\mathsf{R}\mathrm{Hom}_R(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M))\,.$$

(b) $$\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N) \simeq \mathsf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N)\,.$$

(c) $$N \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq N \otimes^{\mathsf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,.$$

Proof. (b): Let I be a semi-injective replacement of the R-complex M and J a semi-injective replacement of the $\widehat{R}^{\mathfrak{a}}$-complex N. The complex $\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N)$ of $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules is $\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), J)$. By 11.2.26(a) this complex consists of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules and one has $\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), J) = \mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\Gamma_{\mathfrak{a}}(I), J)$. The right-hand complex is $\mathsf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N)$.

(c): Let P be a semi-projective replacement of the $\widehat{R}^{\mathfrak{a}}$-complex N and I a semi-injective replacement of the R-complex M. The complex $N \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ of

$\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules is $P \otimes_R \Gamma_{\mathfrak{a}}(I)$. By 11.2.26(b) this complex consists of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules and there is an equality $P \otimes_R \Gamma_{\mathfrak{a}}(I) = P \otimes_{\widehat{R}^{\mathfrak{a}}} \Gamma_{\mathfrak{a}}(I)$. The right-hand side is $N \otimes^{\mathsf{L}}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M)$.

(a): The morpism $\boldsymbol{\gamma}^N_{\mathfrak{a}} \colon \mathsf{R}\Gamma_{\mathfrak{a}}(N) \to N$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ from 13.3.16 induces a morphism,

$$\mathsf{RHom}_R(\boldsymbol{\gamma}^N_{\mathfrak{a}}, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \colon \mathsf{RHom}_R(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \longrightarrow \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(N), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) ,$$

in $\mathcal{D}(\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}})$. To show that this is an isomorphism, it suffices by 6.4.37 to argue that it is an isomorphism in $\mathcal{D}(R)$. By the adjunction 13.4.12 one has

$$\mathsf{RHom}_R(\boldsymbol{\gamma}^N_{\mathfrak{a}}, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^N_{\mathfrak{a}}), M)$$

in $\mathcal{D}(R)$, and the right-hand side is an isomorphism by 13.4.1(b). This shows that $\mathsf{RHom}_R(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M))$ and $\mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(N), \mathsf{L}\Lambda^{\mathfrak{a}}(M))$ are isomorphic in $\mathcal{D}(\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}})$. Part (b) shows that the latter, and hence also the former, is a complex of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules. In the following chain of isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$, the 1st one has just been established and the 2nd holds by part (b). The 3rd isomorphism holds by 13.3.23(a) and the 4th follows from the adjunction 13.4.12. The $\widehat{R}^{\mathfrak{a}}$-complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is derived $\mathfrak{a}\widehat{R}^{\mathfrak{a}}$-complete; indeed, this follows from 13.1.21(a) and the fact that $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is derived $\mathfrak{a}$-complete as an R-complex, see 13.4.2. This explains the 5th and last isomorphism.

$$\begin{aligned} \mathsf{RHom}_R(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(N), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \\ &\simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(\mathsf{R}\Gamma_{\mathfrak{a}}(N), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \\ &\simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(\mathsf{R}\Gamma_{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(N), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \\ &\simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(N, \mathsf{L}\Lambda^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M))) \\ &\simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(N, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) . \end{aligned}$$ □

As a consequence of 13.4.16 one gets the following formulas for derived cobase change of the left derived completion functor and base change of the right derived torsion functor.

13.4.17 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. There are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$,*

$$\mathsf{RHom}_R(\widehat{R}^{\mathfrak{a}}, \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(M) \quad \textit{and} \quad \widehat{R}^{\mathfrak{a}} \otimes_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M) .$$

In particular, if M is a derived $\mathfrak{a}$-complete module, then one has $\operatorname{Ext}^m_R(\widehat{R}^{\mathfrak{a}}, M) = 0$ for every $m > 0$.

Proof. The first isomorphism is the special case $N = \widehat{R}^{\mathfrak{a}}$ of 13.4.16(a) combined with the counitor 12.3.4. As $\widehat{R}^{\mathfrak{a}}$ is flat as an R-module, see 13.1.27, the second isomorphism the special case $N = \widehat{R}^{\mathfrak{a}}$ of 13.4.16(c) combined with the unitor 12.3.3. If M is a derived $\mathfrak{a}$-complete R-module, then one has $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M$ in $\mathcal{D}(R)$, and it follows from the first isomorphism that the homology of the complex $\mathsf{RHom}_R(\widehat{R}^{\mathfrak{a}}, M)$ is concentrated in degree 0. Thus, the last assertion follows from the definition, 7.3.23, of Ext. □

REMARK. If the ideal $\mathfrak{a}$ is contained in the Jacobson radical of R and M is a finitely generated R-module, then vanishing of $\operatorname{Ext}_R^m(\widehat{R}^{\mathfrak{a}}, M)$ for all $m > 0$ means that M is $\mathfrak{a}$-complete. This is a partial converse to the last assertion in 13.4.17; it was proved by Frankild and Sather-Wagstaff [101].

13.4.18 Corollary. *Let $\mathfrak{a} \subseteq R$ be an ideal, M an $R/\mathfrak{a}$-complex, and N an $\widehat{R}^{\mathfrak{a}}$-complex. There are isomorphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$:*

(a) $$\mathsf{RHom}_R(N, M) \simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(N, M)\,.$$

(b) $$\mathsf{RHom}_R(M, N) \simeq \mathsf{RHom}_{\widehat{R}^{\mathfrak{a}}}(M, N)\,.$$

(c) $$N \otimes^{\mathsf{L}}_R M \simeq N \otimes^{\mathsf{L}}_{\widehat{R}^{\mathfrak{a}}} M\,.$$

In particular, these isomorphisms hold for $M = R/\mathfrak{a}$.

PROOF. Every $R/\mathfrak{a}$-complex is both derived $\mathfrak{a}$-complete and derived $\mathfrak{a}$-torsion, see 13.1.22 and 13.3.24. Thus the isomorphisms follow directly from 13.4.16. □

The final results in this section complement 13.1.18 and 13.3.19. The adjunctions in 13.4.3 and 13.4.8 are per 7.3.26 special cases of the first isomorphisms in parts (a) and (b) below. In each of the three parts, the first and last objects are by 13.4.16 complexes of symmetric $\widehat{R}^{\mathfrak{a}}$–$\widehat{R}^{\mathfrak{a}}$-bimodules. The middle complex in each part has only a single $\widehat{R}^{\mathfrak{a}}$-structure.

13.4.19 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes.*

(a) *The morphisms $\mathsf{RHom}_R(\boldsymbol{\lambda}^{\mathfrak{a}}_M, \mathsf{L}\Lambda^{\mathfrak{a}}(N))$ and $\mathsf{RHom}_R(\boldsymbol{\gamma}^M_{\mathfrak{a}}, \mathsf{L}\Lambda^{\mathfrak{a}}(N))$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ are isomorphisms*:
$$\begin{aligned}\mathsf{RHom}_R(\mathsf{L}\Lambda^{\mathfrak{a}}(M), \mathsf{L}\Lambda^{\mathfrak{a}}(N)) &\simeq \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(N))\\ &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \mathsf{L}\Lambda^{\mathfrak{a}}(N))\,.\end{aligned}$$

(b) *The morphisms $\mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \boldsymbol{\gamma}^N_{\mathfrak{a}})$ and $\mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \boldsymbol{\lambda}^{\mathfrak{a}}_N)$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ are isomorphisms*:
$$\begin{aligned}\mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \mathsf{R}\Gamma_{\mathfrak{a}}(N)) &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N)\\ &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \mathsf{L}\Lambda^{\mathfrak{a}}(N))\,.\end{aligned}$$

(c) *The morphisms $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R \boldsymbol{\gamma}^N_{\mathfrak{a}}$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R \boldsymbol{\lambda}^{\mathfrak{a}}_N$ in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$ are isomorphisms*:
$$\mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R N \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(N)\,.$$

PROOF. (a): Set $\alpha = \mathsf{RHom}_R(\boldsymbol{\lambda}^{\mathfrak{a}}_M, \mathsf{L}\Lambda^{\mathfrak{a}}(N))$ and $\beta = \mathsf{RHom}_R(\boldsymbol{\gamma}^M_{\mathfrak{a}}, \mathsf{L}\Lambda^{\mathfrak{a}}(N))$; they are morphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To see that they are isomorphisms, it suffices by 6.4.37 to argue that they are isomorphisms in $\mathcal{D}(R)$. In this category one has
$$\alpha \simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\lambda}^{\mathfrak{a}}_M), N) \quad\text{and}\quad \beta \simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}^M_{\mathfrak{a}}), N)$$
by 13.4.12, so the conclusion follows from 13.4.1(b,d).

(b): Set $\alpha = \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \boldsymbol{\gamma}_{\mathfrak{a}}^N)$ and $\beta = \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \boldsymbol{\lambda}_N^{\mathfrak{a}})$; they are morphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To see that they are isomorphisms, it suffices by 6.4.37 to argue that they are isomorphisms in $\mathcal{D}(R)$. In this category one has

$$\alpha \simeq \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\gamma}_{\mathfrak{a}}^N)) \quad \text{and} \quad \beta \simeq \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(\boldsymbol{\lambda}_N^{\mathfrak{a}}))$$

by 13.4.12, so the conclusion follows from 13.4.1(a,c).

(c): Set $\alpha = \mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes_R^{\mathsf{L}} \boldsymbol{\gamma}_{\mathfrak{a}}^N$ and $\beta = \mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes_R^{\mathsf{L}} \boldsymbol{\lambda}_N^{\mathfrak{a}}$; they are morphisms in $\mathcal{D}(\widehat{R}^{\mathfrak{a}})$. To see that they are isomorphisms, it suffices by 6.4.37 to argue that they are isomorphisms in $\mathcal{D}(R)$. In this category, 13.3.19 and commutativity 12.3.5 yield

$$\alpha \simeq M \otimes_R^{\mathsf{L}} \mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\gamma}_{\mathfrak{a}}^N) \quad \text{and} \quad \beta \simeq M \otimes_R^{\mathsf{L}} \mathsf{R}\Gamma_{\mathfrak{a}}(\boldsymbol{\lambda}_N^{\mathfrak{a}}) \,,$$

so the conclusion follows from 13.4.1(b,d). □

13.4.20 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes.*

(a) *If N is derived $\mathfrak{a}$-complete, then $\mathsf{RHom}_R(M,N)$ is derived $\mathfrak{a}$-complete and the morphisms $\mathsf{RHom}_R(\boldsymbol{\lambda}_M^{\mathfrak{a}}, N)$ and $\mathsf{RHom}_R(\boldsymbol{\gamma}_{\mathfrak{a}}^M, N)$ in $\mathcal{D}(R)$ are isomorphisms*:

$$\mathsf{RHom}_R(\mathsf{L}\Lambda^{\mathfrak{a}}(M), N) \simeq \mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), N) \,.$$

(b) *If M is derived $\mathfrak{a}$-torsion, then $\mathsf{RHom}_R(M,N)$ is derived $\mathfrak{a}$-complete and the morphisms $\mathsf{RHom}_R(M, \boldsymbol{\gamma}_{\mathfrak{a}}^N)$ and $\mathsf{RHom}_R(M, \boldsymbol{\lambda}_N^{\mathfrak{a}})$ in $\mathcal{D}(R)$ are isomorphisms*:

$$\mathsf{RHom}_R(M, \mathsf{R}\Gamma_{\mathfrak{a}}(N)) \simeq \mathsf{RHom}_R(M,N) \simeq \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(N)) \,.$$

(c) *If M is derived $\mathfrak{a}$-torsion, then the complex $M \otimes_R^{\mathsf{L}} N$ is derived $\mathfrak{a}$-torsion and the morphisms $M \otimes_R^{\mathsf{L}} \boldsymbol{\gamma}_{\mathfrak{a}}^N$ and $M \otimes_R^{\mathsf{L}} \boldsymbol{\lambda}_N^{\mathfrak{a}}$ in $\mathcal{D}(R)$ are isomorphisms,*

$$M \otimes_R^{\mathsf{L}} \mathsf{R}\Gamma_{\mathfrak{a}}(N) \simeq M \otimes_R^{\mathsf{L}} N \simeq M \otimes_R^{\mathsf{L}} \mathsf{L}\Lambda^{\mathfrak{a}}(N) \,.$$

Proof. (a): Assume that N is derived $\mathfrak{a}$-complete. The complex $\mathsf{RHom}_R(M,N)$ is derived $\mathfrak{a}$-complete by 13.1.31(a). By 11.3.3 there is an isomorphism $N \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(N)$ in $\mathcal{D}(R)$, so the asserted isomorphisms follow directly from 13.4.19(a).

(b): Assume that M is derived $\mathfrak{a}$-torsion. By 11.3.17 there is an isomorphism $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ in $\mathcal{D}(R)$, so the asserted isomorphisms follow from 13.4.19(b). The complex $\mathsf{L}\Lambda^{\mathfrak{a}}(N)$ is derived $\mathfrak{a}$-complete by 13.4.2, and hence so is $\mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{a}}(N))$ by 13.1.31(a). As already argued, the latter complex is isomorphic to $\mathsf{RHom}_R(M,N)$, which is therefore also derived $\mathfrak{a}$-complete.

(c): Assume that M is derived $\mathfrak{a}$-torsion. The complex $M \otimes_R^{\mathsf{L}} N$ is derived $\mathfrak{a}$-torsion by commutativity 12.3.5 and 13.3.28(a). By 11.3.17 there is an isomorphism $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ in $\mathcal{D}(R)$, so the asserted isomorphisms follow directly from 13.4.19(c). □

13.4.21 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. The following conditions are equivalent.*

(*i*) *There is an isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(N)$ in $\mathcal{D}(R)$.*

(*ii*) *There is an isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(N)$ in $\mathcal{D}(R)$.*

If M and N satisfy these conditions, then there are natural isomorphisms:

(a) $\mathsf{RHom}_R(M,Y) \simeq \mathsf{RHom}_R(N,Y)$ *for every derived $\mathfrak{a}$-complete complex Y.*

(b) $\mathsf{RHom}_R(X,M) \simeq \mathsf{RHom}_R(X,N)$ *for every derived* $\mathfrak{a}$*-torsion complex* X.
(c) $X \otimes^{\mathsf{L}}_R M \simeq X \otimes^{\mathsf{L}}_R N$ *for every derived* $\mathfrak{a}$*-torsion complex* X.

Proof. The natural isomorphisms $\mathsf{R}\Gamma_{\mathfrak{a}}\mathsf{L}\Lambda^{\mathfrak{a}} \simeq \mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}\mathsf{R}\Gamma_{\mathfrak{a}} \simeq \mathsf{L}\Lambda^{\mathfrak{a}}$ from 13.4.1(c,d) yield the equivalence of (*i*) and (*ii*). For a derived $\mathfrak{a}$-complete complex Y and a derived $\mathfrak{a}$-torsion complex X there are by 13.4.20 natural isomorphisms:

$$\begin{aligned} \mathsf{RHom}_R(M,Y) &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M),Y)\ , \\ \mathsf{RHom}_R(X,M) &\simeq \mathsf{RHom}_R(X,\mathsf{R}\Gamma_{\mathfrak{a}}(M))\ , \quad \text{and} \\ X \otimes^{\mathsf{L}}_R M &\simeq X \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)\ . \end{aligned}$$

Thus, if one has $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(N)$, then the asserted isomorphisms follow. □

Exercises

In following exercises let $\mathfrak{a}$ be an ideal in R.

E 13.4.1 Let $M \in \mathcal{D}_{\sqsubset}(R)$ be a derived $\mathfrak{a}$-torsion complex. Show that there is an equality $\inf\{-\sup \mathsf{RHom}_R(R/\mathfrak{p},M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} = -\sup M$.

E 13.4.2 Let $M \in \mathcal{D}_{\sqsupset}(R)$ be a derived $\mathfrak{a}$-complete complex. Show that there is an equality $\inf\{\inf (R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} = \inf M$.

E 13.4.3 Let $\{M^u \to M^v\}_{u \leqslant v}$ be a U-direct system in $\mathcal{C}(R)$. Show that if U is filtered and each complex M^u is derived $\mathfrak{a}$-torsion, then the complex $\operatorname{colim}_{u\in U} M^u$ is derived $\mathfrak{a}$-torsion.

E 13.4.4 With $\mathcal{F}^{\mathfrak{a}\text{-tor}}(R) = \mathcal{F}(R) \cap \mathcal{D}^{\mathfrak{a}\text{-tor}}(R)$ etc. show that the equivalence in 13.4.13 restricts to $\mathcal{F}^{\mathfrak{a}\text{-tor}}(R) \rightleftarrows \mathcal{F}^{\mathfrak{a}\text{-com}}(R)$ and $\mathcal{I}^{\mathfrak{a}\text{-tor}}(R) \rightleftarrows \mathcal{I}^{\mathfrak{a}\text{-com}}(R)$.

E 13.4.5 Let M be an R-complex. Show that $\mathsf{RHom}_R(M, R/\mathfrak{a})$ is derived $\mathfrak{a}$-complete and derived $\mathfrak{a}$-torsion and that there are isomorphisms in $\mathcal{D}(R)$,

$$\mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), R/\mathfrak{a}) \simeq \mathsf{RHom}_R(M, R/\mathfrak{a}) \simeq \mathsf{RHom}_R(\mathsf{L}\Lambda^{\mathfrak{a}}(M), R/\mathfrak{a})\ .$$

Chapter 14
Krull Dimension, Depth, and Width

A fundamental idea in commutative algebra is to link a module with particular subsets of the prime ideal spectrum of the ring. The study of these sets and the information they capture goes by the name of support theories. The classic support of a module, with its subsets of minimal and associated prime ideals, was already recalled in Sect. 12.4. The next chapter covers more modern notions of support that conform better to derived category methods, but here we are concerned with the classic notion. In the first sections it is extended to complexes along with the related numerical invariant: the Krull dimension. Later sections extend the classic notion of depth, and the dual notion of width, to complexes.

14.1 Classic Support for Complexes

SYNOPSIS. Classic support for complexes; localization; classic support of (derived) tensor product; classic suppoort of derived Hom; colocalization.

In Sect. 12.4 we recalled basic properties of the prime ideal spectrum of R and the classic support of an R-module. For ease of reference, we open this chapter with a standard result that can certainly be found in any textbook on commutative algebra.

14.1.1 Proposition. *Let M be an R-module. If M is finitely generated, then one has*

$$\operatorname{Supp}_R M = \mathrm{V}(0 :_R M) \qquad \textit{and} \qquad \dim_R M = \dim R/(0 :_R M) .$$

PROOF. Let $m_1, \ldots, m_n$ be a set of generators for M. If a prime ideal $\mathfrak{p}$ in R belongs to $\operatorname{Supp}_R M$, then it contains the annihilator $(0 :_R m)$ for some $m \in M$, in particular, one has $(0 :_R M) \subseteq \mathfrak{p}$. Conversely, if a prime ideal $\mathfrak{p}$ contains $(0 :_R M) = \bigcap_{i=1}^{n}(0 :_R m_i)$, then it contains $(0 :_R m_i)$ for some $i \in \{1, \ldots, n\}$ and therefore $\mathfrak{p}$ is in $\operatorname{Supp}_R M$. This proves the first equality. The equality of Krull dimensions holds as $\mathrm{V}(0 :_R M)$ is in an order preserving one-to-one correspondence with $\operatorname{Spec} R/(0 :_R M)$. $\square$

14.1.2 Example. For $n \in \mathbb{N}$ one has $\operatorname{Supp}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{p\mathbb{Z} \mid p \text{ is prime and } p|n\}$.

14.1.3 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-module. There is an inclusion* $\operatorname{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$ *if and only if M is $\mathfrak{a}$-torsion.*

Proof. A prime ideal belongs to $\operatorname{Supp}_R M$ if and only if it contains the annihilator $(0:_R m)$ of some element $m \in M$. If M is $\mathfrak{a}$-torsion, then every annihilator $(0:_R m)$ contains a power of $\mathfrak{a}$, and a prime ideal contains a power of $\mathfrak{a}$ if and only if it contains $\mathfrak{a}$. For the converse, assume that $\operatorname{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$ holds and let m be an element of M. The assumption implies that every prime ideal that contains $(0:_R m)$ contains $\mathfrak{a}$, so one has $\mathfrak{a} \subseteq \sqrt{(0:_R m)}$. Now it follows from 13.1.1 that $(0:_R m)$ contains a power of $\mathfrak{a}$, whence m is $\mathfrak{a}$-torsion. □

An elementary way to extend invariants and other notions from the realm of modules to complexes is to apply them to the homology modules of a complex.

14.1.4 Definition. Let M be an R-complex. The *classic support* of M is the set

$$\operatorname{Supp}_R M = \bigcup_{v\in\mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_v(M)\,.$$

Since an R-module has empty classic support if and only if it is the zero module, an R-complex has empty classic support if and only if it is acyclic; see also 14.2.4. Notice that the classic support of a complex is a union of specialization closed subsets of $\operatorname{Spec} R$ and hence itself a specialization closed subset of $\operatorname{Spec} R$.

14.1.5 Example. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. From 11.4.6(a) and 14.1.1 one gets $\operatorname{Supp}_R \mathrm{K}^R(\boldsymbol{x}) \subseteq \mathrm{V}(\mathfrak{a})$, and it follows from 11.4.3(a) that equality holds. Similarly, one gets $\operatorname{Supp}_R \check{\mathrm{C}}^R(\boldsymbol{x}) \subseteq \mathrm{V}(\mathfrak{a})$ from 11.4.13 and 14.1.3; in fact, equality holds, this follows from 15.1.27 in view of 15.1.9.

14.1.6 Example. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. For all but finitely many $v \in \mathbb{Z}$ one has $\mathrm{H}_v(M) = 0$ and thus $(0:_R \mathrm{H}_v(M)) = R$, so 14.1.1 yields

$$\operatorname{Supp}_R M = \mathrm{V}\Big(\bigcap_{v\in\mathbb{Z}} (0:_R \mathrm{H}_v(M))\Big)\,.$$

14.1.7 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes*; *one has*

$$\operatorname{Supp}_R\Big(\coprod_{u\in U} M^u\Big) = \bigcup_{u\in U} \operatorname{Supp}_R M^u\,.$$

Proof. Homology commutes with coproducts, see 3.1.10(d), and for every $v \in \mathbb{Z}$ one has $\operatorname{Supp}_R \coprod_{u\in U} \mathrm{H}_v(M^u) = \bigcup_{u\in U} \operatorname{Supp}_R \mathrm{H}_v(M^u)$. □

The classic support of a product $\prod_{u\in U} M^u$ of R-modules contains by 14.1.7 the classic support of each module M^u, but it may also contain ideals not present in any of the sets $\operatorname{Supp}_R M^u$. The next example shows how big the discrepancy can be.

14.1.8 Example. Let R be local with unique maximal ideal $\mathfrak{m}$. By C.15(b) one has

$$\operatorname{Supp}_R \mathrm{E}_R(R/\mathfrak{m}) = \{\mathfrak{m}\}\,.$$

As in C.20 set $E = \mathrm{E}_R(R/\mathfrak{m})$ and $E^u = (0 :_E \mathfrak{m}^u)$ for $u \in \mathbb{N}$; by C.22 each of these submodules has finite length. For each u let $e_{u,1}, \ldots, e_{u,n_u}$ be a minimal set of generators for E^u and consider the element $e = (e_{1,1}, \ldots, e_{1,n_1}, e_{2,1}, \ldots, e_{2,n_2}, \ldots)$ in the countable product $P = \prod_{u \in \mathbb{N}} \prod_{i=1}^{n_u} E$. The homomorphism $R \to P$ defined by the assignment $1 \mapsto e$ is injective. Indeed, one has $(0 :_R E^u) = \mathfrak{m}^u$ by C.21(c), so an element in the kernel belongs to $\bigcap_{u \in \mathbb{N}} \mathfrak{m}^u$, which is 0 by Krull's intersection theorem, see also 15.3.7. As localization is exact, one now has $P_\mathfrak{p} \neq 0$ for every $\mathfrak{p} \in \operatorname{Spec} R$ and, therefore,

$$\operatorname{Supp}_R \left(\mathrm{E}_R(R/\mathfrak{m})^{\mathbb{N}} \right) = \operatorname{Spec} R \, .$$

Thus, for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$ one has $\mathrm{E}_R(R/\mathfrak{m})_\mathfrak{p} = 0$ but $(\mathrm{E}_R(R/\mathfrak{m})^{\mathbb{N}})_\mathfrak{p} \neq 0$.

Localization

Recall from 2.1.50 and 6.4.32 that localization at a multiplicative subset U of R is a functor $\mathcal{C}(R) \to \mathcal{C}(U^{-1}R)$ and $\mathcal{D}(R) \to \mathcal{D}(U^{-1}R)$.

We extend a standard notation for modules, recalled in Sect. 12.4, to complexes.

14.1.9 Definition. Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. The localization of M at the multiplicative subset $R \setminus \mathfrak{p}$ is denoted $M_\mathfrak{p}$.

14.1.10. Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. By 2.1.50, flatness of $R_\mathfrak{p}$ as an R-module, see 1.3.42, and 7.4.16 there are isomorphisms in $\mathcal{C}(R_\mathfrak{p})$ and $\mathcal{D}(R_\mathfrak{p})$:

$$M_\mathfrak{p} \cong R_\mathfrak{p} \otimes_R M \qquad \text{and} \qquad M_\mathfrak{p} \simeq R_\mathfrak{p} \otimes^{\mathsf{L}}_R M \, .$$

Viewed as a functor $\mathcal{C}(R) \to \mathcal{C}(R_\mathfrak{p})$, localization $(-)_\mathfrak{p}$ is exact, see 2.1.50, and as a functor $\mathcal{D}(R) \to \mathcal{D}(R_\mathfrak{p})$ it is triangulated, see 12.2.8.

14.1.11 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. One has*:

(a) $\quad \mathrm{H}(M_\mathfrak{p}) \cong \mathrm{H}(M)_\mathfrak{p}$.

(b) $\quad \operatorname{Supp}_{R_\mathfrak{p}} M_\mathfrak{p} = \{\mathfrak{q}_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Supp}_R M \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$.

(c) $\quad \inf M_\mathfrak{p} \geqslant \inf M \, , \quad \sup M_\mathfrak{p} \leqslant \sup M \, , \quad$ *and* $\quad \operatorname{amp} M_\mathfrak{p} \leqslant \operatorname{amp} M$.

Moreover, if M belongs to $\mathcal{C}^{\mathrm{f}}(R)$, then $M_\mathfrak{p}$ belongs to $\mathcal{C}^{\mathrm{f}}(R_\mathfrak{p})$, and if M belongs to $\mathcal{D}^{\mathrm{f}}(R)$, then $M_\mathfrak{p}$ belongs to $\mathcal{D}^{\mathrm{f}}(R_\mathfrak{p})$.

Proof. The isomorphism (a) follows from 12.1.20(b) and implies the (in)equalities in (b) and (c). The last assertions follow from 12.1.20(a,c). □

14.1.12 Corollary. *Let M be an R-complex*; *one has*

$$\operatorname{Supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(M_\mathfrak{p}) \neq 0\} \, .$$

Proof. The equality is immediate from 14.1.4 and 14.1.11(a). □

The second equality below is improved in 17.6.4; the first compares to 17.6.11.

14.1.13 Corollary. *Let M be an R-complex; there are equalities,*

$$\inf\{\inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \inf M \quad \textit{and} \quad \sup\{\sup M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup M \,.$$

Proof. It follows from 14.1.11(c) that the inequalities $\inf\{\inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \geqslant \inf M$ and $\sup\{\sup M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \leqslant \sup M$ hold. On the other hand, for every integer v with $\mathrm{H}_v(M) \neq 0$ one can choose a prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R \mathrm{H}_v(M)$, and then 14.1.11(a) yields $\inf M_{\mathfrak{p}} \leqslant v$ and $\sup M_{\mathfrak{p}} \geqslant v$. □

Let $\mathfrak{p}$ be a prime ideal in R. Recall from 12.1.4 and 12.2.9 that there are functors,

$$\begin{aligned} -\otimes_R - :\ & \mathcal{C}(R_{\mathfrak{p}}) \times \mathcal{C}(R_{\mathfrak{p}}) \longrightarrow \mathcal{C}(R_{\mathfrak{p}}\text{–}R_{\mathfrak{p}}) \quad \text{and} \\ -\otimes^{\mathsf{L}}_R - :\ & \mathcal{D}(R_{\mathfrak{p}}) \times \mathcal{D}(R_{\mathfrak{p}}) \longrightarrow \mathcal{D}(R_{\mathfrak{p}}\text{–}R_{\mathfrak{p}}) \,, \end{aligned}$$

where the latter per 14.1.10 is induced by $\mathrm{P}_{R_{\mathfrak{p}}}(-) \otimes_R -$. We now show that the output of these functors, in fact, is complexes of symmetric $R_{\mathfrak{p}}$–$R_{\mathfrak{p}}$-bimodules. Note that the result below has the same flavor as 11.2.26(b) and 13.4.16(c).

14.1.14 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M and N be $R_{\mathfrak{p}}$-complexes.*

(a) *The object $M \otimes_R N$ is a complex of symmetric $R_{\mathfrak{p}}$–$R_{\mathfrak{p}}$-bimodules, and there is an equality of $R_{\mathfrak{p}}$-complexes,*

$$M \otimes_R N = M \otimes_{R_{\mathfrak{p}}} N \,.$$

In particular, one has $R_{\mathfrak{p}} \otimes_R N \cong N$.

(b) *The object $M \otimes^{\mathsf{L}}_R N$ is a complex of symmetric $R_{\mathfrak{p}}$–$R_{\mathfrak{p}}$-bimodules and there is an isomorphism in $\mathcal{D}(R_{\mathfrak{p}})$,*

$$M \otimes^{\mathsf{L}}_R N \simeq M \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} N \,.$$

In particular, one has $R_{\mathfrak{p}} \otimes^{\mathsf{L}}_R N \simeq N$.

Proof. (a): It suffices to show that for all elements $\frac{r}{u} \in R_{\mathfrak{p}}$, $m \in M$, and $n \in N$ there is an equality $\frac{r}{u}m \otimes n = m \otimes \frac{r}{u}n$ in $M \otimes_R N$; and that follows as one has:

$$\tfrac{r}{u}m \otimes n = \tfrac{1}{u}m \otimes rn = \tfrac{1}{u}m \otimes \tfrac{r}{u}un = \tfrac{u}{u}m \otimes \tfrac{r}{u}n = m \otimes \tfrac{r}{u}n \,.$$

The isomorphism $R_{\mathfrak{p}} \otimes_R N \cong N$ now follows from the unitor 12.1.5.

(b): Let $P \xrightarrow{\simeq} M$ be a semi-projective resolution in $\mathcal{C}(R_{\mathfrak{p}})$. The complex $M \otimes^{\mathsf{L}}_R N$ of $R_{\mathfrak{p}}$–$R_{\mathfrak{p}}$-bimodules is $P \otimes_R N$, see 12.2.9. By part (a) this is a complex of symmetric $R_{\mathfrak{p}}$–$R_{\mathfrak{p}}$-bimodules and there is an equality $P \otimes_R N = P \otimes_{R_{\mathfrak{p}}} N$. Since the $R_{\mathfrak{p}}$-complex $P \otimes_{R_{\mathfrak{p}}} N$ is $M \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} N$, the assertion follows. The isomorphism $R_{\mathfrak{p}} \otimes^{\mathsf{L}}_R N \simeq N$ now follows from the unitor 12.3.3. □

14.1.15 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M and N be R-complexes. There are isomorphisms in $\mathcal{C}(R_{\mathfrak{p}})$ and $\mathcal{D}(R_{\mathfrak{p}})$,*

$$(M \otimes_R N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \qquad \textit{and} \qquad (M \otimes^{\mathsf{L}}_R N)_{\mathfrak{p}} \simeq M_{\mathfrak{p}} \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \,.$$

In particular, for every $m \in \mathbb{Z}$ there is an isomorphism of $R_{\mathfrak{p}}$-modules,

$$\operatorname{Tor}_m^R(M,N)_{\mathfrak{p}} \;\cong\; \operatorname{Tor}_m^{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) \,.$$

Proof. The isomorphisms of complexes are special cases of 12.1.17 and 12.3.30. The last isomorphism is immediate from 14.1.11(a), the isomorphism in $\mathcal{D}(R_{\mathfrak{p}})$, and the definition, 7.4.18, of Tor. □

14.1.16 Corollary. *Let $\mathfrak{p}$ be a prime ideal in R, let M be an R-complex and X an $R_{\mathfrak{p}}$-complex.*

(a) *There are isomorphisms in $\mathcal{C}(R_{\mathfrak{p}})$,*

$$(X \otimes_R M)_{\mathfrak{p}} \;\cong\; X \otimes_R M \;\cong\; X \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \,.$$

(b) *There are isomorphisms in $\mathcal{D}(R_{\mathfrak{p}})$,*

$$(X \otimes_R^{\mathsf{L}} M)_{\mathfrak{p}} \;\simeq\; X \otimes_R^{\mathsf{L}} M \;\simeq\; X \otimes_{R_{\mathfrak{p}}}^{\mathsf{L}} M_{\mathfrak{p}} \,.$$

Proof. Since $X \otimes_R M$ and $X \otimes_R^{\mathsf{L}} M$ are $R_{\mathfrak{p}}$-complexes by 12.1.4 and 12.2.8, the first isomorphisms in (a) and (b) follow from idempotence of localization, see the "in particular" statements in 14.1.14. In view of 14.1.10 the remaining isomorphisms are special cases of 12.1.18 and 12.3.31. □

14.1.17 Corollary. *Let M and N be R-complexes*; *one has*

$$\operatorname{Supp}_R(M \otimes_R^{\mathsf{L}} N) \;\subseteq\; \operatorname{Supp}_R M \cap \operatorname{Supp}_R N \,.$$

Proof. The inclusion follows from 14.1.12 and 14.1.15. □

14.1.18 Theorem. *Let M and N be R-modules. There are inclusions,*

$$\operatorname{Supp}_R(M \otimes_R N) \;\subseteq\; \operatorname{Supp}_R(M \otimes_R^{\mathsf{L}} N) \;\subseteq\; \operatorname{Supp}_R M \;\cap\; \operatorname{Supp}_R N \,.$$

If M and N are finitely generated, then both inclusions are equalities and the following conditions are equivalent.

(*i*) $M \otimes_R N = 0$.
(*ii*) $\mathrm{H}(M \otimes_R^{\mathsf{L}} N) = 0$.
(*iii*) $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \emptyset$.

Proof. The second inclusion was proved in 14.1.17. The first inclusion follows from the definition, 14.1.4, as one has $\mathrm{H}_0(M \otimes_R^{\mathsf{L}} N) \cong M \otimes_R N$ by 7.6.8. Assume now that M and N are finitely generated, it must be shown that $(M \otimes_R N)_{\mathfrak{p}} \neq 0$ holds for $\mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Supp}_R N$. By the isomorphism $(M \otimes_R N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ from 14.1.15, it is sufficient to show that for finitely generated modules $K \neq 0$ and $L \neq 0$ over a local ring S one has $K \otimes_S L \neq 0$. Denote by $\mathfrak{n}$ the maximal ideal of S; it is the Jacobson radical of S, so $S/\mathfrak{n} \otimes_S K$ and $S/\mathfrak{n} \otimes_S L$ are by 1.1.10 and Nakayama's lemma B.32 non-zero vector spaces over the field $S/\mathfrak{n}$. By 12.1.17 one has

$$S/\mathfrak{n} \otimes_S (K \otimes_S L) \;\cong\; (S/\mathfrak{n} \otimes_S K) \otimes_{S/\mathfrak{n}} (S/\mathfrak{n} \otimes_S L) \,,$$

so $S/\mathfrak{n} \otimes_S (K \otimes_S L)$ is non-zero by 1.3.10, whence $K \otimes_S L$ is non-zero. The equivalence of conditions (*i*)–(*iii*) is now immediate. □

14.1.19 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M a finitely generated R-module. There is an equality,*

$$\operatorname{Supp}_R M/\mathfrak{a}M \;=\; \mathrm{V}(\mathfrak{a}) \cap \operatorname{Supp}_R M\,.$$

PROOF. As $\operatorname{Supp}_R R/\mathfrak{a} = \mathrm{V}(\mathfrak{a})$ holds, this is per 1.1.10 a special case of 14.1.18. □

14.1.20 Example. Let $p \in \mathbb{Z}$ be a prime; per 14.1.2 one has $\operatorname{Supp}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \{p\mathbb{Z}\}$. Evidently, $\operatorname{Supp}_{\mathbb{Z}} \mathbb{Q} = \operatorname{Spec} \mathbb{Z}$, and as $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, see 1.3.43, one has

$$\operatorname{Supp}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathsf{L}}_{\mathbb{Z}} \mathbb{Q}) \;=\; \operatorname{Supp}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \;=\; \operatorname{Supp}_{\mathbb{Z}} 0 \;=\; \varnothing\,.$$

Thus, the inclusion in 14.1.17 may be strict.

Let $\mathfrak{p}$ be a prime ideal in R. Recall from 12.1.2 and 12.2.4 that there are functors,

$$\begin{aligned} \operatorname{Hom}_R(-,-) : \mathcal{C}(R_\mathfrak{p})^{\mathrm{op}} \times \mathcal{C}(R_\mathfrak{p}) \;&\longrightarrow\; \mathcal{C}(R_\mathfrak{p}\text{–}R_\mathfrak{p}) \quad \text{and} \\ \mathsf{RHom}_R(-,-) : \mathcal{D}(R_\mathfrak{p})^{\mathrm{op}} \times \mathcal{D}(R_\mathfrak{p}) \;&\longrightarrow\; \mathcal{D}(R_\mathfrak{p}\text{–}R_\mathfrak{p})\,, \end{aligned}$$

where the latter per 14.1.10 is induced by $\operatorname{Hom}_R(-,\mathrm{I}_{R_\mathfrak{p}}(-))$. We now show that the output of these functors, in fact, is complexes of symmetric $R_\mathfrak{p}$–$R_\mathfrak{p}$-bimodules. Note that the result below has the same flavor as 11.1.24/11.2.26(a) and 13.4.16(a,b).

14.1.21 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M and N be $R_\mathfrak{p}$-complexes.*

(a) *The object $\operatorname{Hom}_R(M,N)$ is a complex of symmetric $R_\mathfrak{p}$–$R_\mathfrak{p}$-bimodules, and there is an equality of $R_\mathfrak{p}$-complexes,*

$$\operatorname{Hom}_R(M,N) \;=\; \operatorname{Hom}_{R_\mathfrak{p}}(M,N)\,.$$

In particular, one has $\operatorname{Hom}_R(R_\mathfrak{p},N) \cong N$.

(b) *The object $\mathsf{RHom}_R(M,N)$ is a complex of symmetric $R_\mathfrak{p}$–$R_\mathfrak{p}$-bimodules and there is an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,*

$$\mathsf{RHom}_R(M,N) \;\simeq\; \mathsf{RHom}_{R_\mathfrak{p}}(M,N)\,.$$

In particular, one has $\mathsf{RHom}_R(R_\mathfrak{p},N) \simeq N$.

PROOF. (a): It suffices to show that for $\frac{r}{u} \in R_\mathfrak{p}$, $m \in M$, and $\alpha \in \operatorname{Hom}_R(M,N)$ there is an equality $\alpha(\frac{r}{u}m) = \frac{r}{u}\alpha(m)$, and that follows as one has:

$$\alpha(\tfrac{r}{u}m) \;=\; r\alpha(\tfrac{1}{u}m) \;=\; \tfrac{r}{u}u\alpha(\tfrac{1}{u}m) \;=\; \tfrac{r}{u}\alpha(\tfrac{u}{u}m) \;=\; \tfrac{r}{u}\alpha(m)\,.$$

The isomorphism $\operatorname{Hom}_R(R_\mathfrak{p},N) \cong N$ now follows from the counitor 12.1.6.

(b): Let $N \xrightarrow{\simeq} I$ be a semi-injective resolution in $\mathcal{C}(R_\mathfrak{p})$. The complex $\mathsf{RHom}_R(M,N)$ of $R_\mathfrak{p}$–$R_\mathfrak{p}$-bimodules is $\operatorname{Hom}_R(M,I)$, see 12.2.4. By part (a) this is a complex of symmetric $R_\mathfrak{p}$–$R_\mathfrak{p}$-bimodules and there is an equality $\operatorname{Hom}_R(M,I) = \operatorname{Hom}_{R_\mathfrak{p}}(M,I)$. Since the $R_\mathfrak{p}$-complex $\operatorname{Hom}_{R_\mathfrak{p}}(M,I)$ is $\mathsf{RHom}_{R_\mathfrak{p}}(M,N)$, the assertion follows. The isomorphism $\mathsf{RHom}_R(R_\mathfrak{p},N) \simeq N$ now follows from the counitor 12.3.4. □

14.1.22 Lemma. *Let $\mathfrak{p}$ be a prime ideal in R and M and N be R-complexes. If M is degreewise finitely generated and condition* (a) *or* (b) *below is satisfied, then there is an isomorphism in $\mathcal{C}(R_\mathfrak{p})$,*

$$\operatorname{Hom}_R(M,N)_\mathfrak{p} \;\cong\; \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})\;.$$

(a) *M is bounded or N is bounded.*
(b) *M is bounded below and N is bounded above.*

Proof. This is per 14.1.10 a special case of 12.1.21. □

For an R-module N and a finitely generated R-module M one can fom 14.1.22 conclude that there is an inclusion $\operatorname{Supp}_R \operatorname{Hom}_R(M,N) \subseteq \operatorname{Supp}_R M \cap \operatorname{Supp}_R N$, cf. 14.1.18. However, a stronger result is proved in 17.1.1.

14.1.23 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M and N be R-complexes. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}_{\sqsubset}(R)$, then there is an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,*

$$\mathsf{R}\!\operatorname{Hom}_R(M,N)_\mathfrak{p} \;\simeq\; \mathsf{R}\!\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})\;.$$

In particular, for every $m \in \mathbb{Z}$ there is an isomorphism of $R_\mathfrak{p}$-modules,

$$\operatorname{Ext}^m_R(M,N)_\mathfrak{p} \;\cong\; \operatorname{Ext}^m_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})\;.$$

Proof. The first isomorphism is per 14.1.10 a special case of 12.3.33(a). Now the second isomorphism follows from 14.1.11(a) and the definition, 7.3.23, of Ext. □

14.1.24 Corollary. *Let M and N be R-complexes. If M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N to $\mathcal{D}_{\sqsubset}(R)$, then there is an inclusion,*

$$\operatorname{Supp}_R \mathsf{R}\!\operatorname{Hom}_R(M,N) \;\subseteq\; \operatorname{Supp}_R M \cap \operatorname{Supp}_R N\;.$$

Proof. The inclusion follows from 14.1.12 and 14.1.23. □

14.1.25 Proposition. *Let $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ a prime ideal in R; let M be an R-complex. There is an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,*

$$\mathsf{R}\Gamma_\mathfrak{a}(M)_\mathfrak{p} \;\simeq\; \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})\;.$$

In particular, for every $m \in \mathbb{Z}$ there is an isomorphism of $R_\mathfrak{p}$-modules,

$$\mathrm{H}^m_\mathfrak{a}(M)_\mathfrak{p} \;\cong\; \mathrm{H}^m_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})\;.$$

Proof. This is per 14.1.10 a special case of 13.3.23(b). □

14.1.26 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and L an R-complex. If L is semi-free, then the $R_\mathfrak{p}$-complex $L_\mathfrak{p}$ is semi-free.*

Proof. Per 14.1.10 the assertion follows from 5.1.11(a). □

14.1.27 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and P an R-complex. If P is semi-projective, then the $R_\mathfrak{p}$-complex $P_\mathfrak{p}$ is semi-projective.*

Proof. Per 14.1.10 the assertion follows from 5.2.23(a). □

14.1.28 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and F an R-complex.*

(a) *If F is semi-flat, then the $R_\mathfrak{p}$-complex $F_\mathfrak{p}$ is semi-flat.*
(b) *Every semi-flat $R_\mathfrak{p}$-complex is semi-flat over R.*

Proof. Per 14.1.10 the statement is a special case of 5.4.18. □

For semi-injectivity the situation is not quite so simple. Example 17.5.14 shows that the boundedness condition is necessary in the next result.

14.1.29 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and I an R-complex.*

(a) *If I is semi-injective and bounded above, then the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ is semi-injective and bounded above.*
(b) *Every semi-injective $R_\mathfrak{p}$-complex is semi-injective over R.*

Proof. It follows from C.24 that $I_\mathfrak{p}$ is a complex of injective $R_\mathfrak{p}$-modules. If I is bounded above, then so is $I_\mathfrak{p}$ whence it is semi-injective by 5.3.12. This proves part (a), while (b) per 1.3.42 is a special case of 5.4.26(b). □

14.1.30 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and I a complex of injective R-modules. If I is minimal, then $I_\mathfrak{p}$ is a minimal complex of injective $R_\mathfrak{p}$-modules.*

Proof. It follows from C.24 that $I_\mathfrak{p}$ is a complex of injective $R_\mathfrak{p}$-modules. Thus, to show that $I_\mathfrak{p}$ is minimal, one must per B.21(b) argue that the $R_\mathfrak{p}$-submodule $\mathrm{Z}_v(I_\mathfrak{p}) \subseteq (I_\mathfrak{p})_v$ is essential for every $v \in \mathbb{Z}$. As the localization functor $(-)_\mathfrak{p}$ is exact, one has $\mathrm{Z}_v(I_\mathfrak{p}) = \mathrm{Z}_v(I)_\mathfrak{p}$, see 2.2.19. By assumption and B.21(b) the R-submodule $\mathrm{Z}_v(I) \subseteq I_v$ is essential for every $v \in \mathbb{Z}$, so it follows from B.12 that $\mathrm{Z}_v(I)_\mathfrak{p}$ is essential in $(I_v)_\mathfrak{p} = (I_\mathfrak{p})_v$. □

14.1.31 Corollary. *Let $\mathfrak{p}$ be a prime ideal in R and M a complex in $\mathcal{D}_\sqsubset(R)$. If $M \xrightarrow{\simeq} I$ is a minimal semi-injective resolution, then the induced morphism $M_\mathfrak{p} \longrightarrow I_\mathfrak{p}$ is a minimal semi-injective resolution in $\mathcal{C}(R_\mathfrak{p})$.*

Proof. Localization preserves quasi-isomorphisms, see 6.4.32, so $M \xrightarrow{\simeq} I$ induces a quasi-isomorphism $M_\mathfrak{p} \longrightarrow I_\mathfrak{p}$. The complex I is bounded above, see B.26, so the complex $I_\mathfrak{p}$ is semi-injective and minimal by 14.1.29(a) and 14.1.30. □

Colocalization

Let U be a multiplicative subset of R; (derived) cobase change along the canonical homomorphism $R \longrightarrow U^{-1}R$ is in many places referred to as '(derived) colocalization' at U. Base change along the same map is simply localization at U, and here there is only one variety as $U^{-1}R$ is a flat R-module. Since we rarely consider underived cobase change along $R \longrightarrow U^{-1}R$ we make the following definition.

14.1.32 Definition. Let M be an R-complex and U a multiplicative subset of R. The $U^{-1}R$-complex $\mathsf{RHom}_R(U^{-1}R, M)$ is called the *colocalization* of M at U.

14.1.33 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R, let M be an R-complex and X an $R_\mathfrak{p}$-complex. There are isomorphisms in $\mathcal{C}(R_\mathfrak{p})$:*

(a) $$\operatorname{Hom}_R(M,X) \cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},X)\,.$$

(b) $$\operatorname{Hom}_R(X,M) \cong \operatorname{Hom}_{R_\mathfrak{p}}(X,\operatorname{Hom}_R(R_\mathfrak{p},M))\,.$$

There are isomorphisms in $\mathcal{D}(R_\mathfrak{p})$:

(c) $$\mathsf{R}\!\operatorname{Hom}_R(M,X) \simeq \mathsf{R}\!\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},X)\,.$$

(d) $$\mathsf{R}\!\operatorname{Hom}_R(X,M) \simeq \mathsf{R}\!\operatorname{Hom}_{R_\mathfrak{p}}(X,\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},M))\,.$$

PROOF. Parts (b) and (d) are special cases of 12.1.28 and 12.3.36. Further, in view of 14.1.10 parts (a) and (c) are special cases of 12.1.19 and 12.3.32. □

14.1.34 Proposition. *Let $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ a prime ideal in R; let M be an R-complex. There is an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,*

$$\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{a}_\mathfrak{p}}(\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},M))\,.$$

PROOF. The assertion is immediate from 13.1.21(b). □

REMARK. Let M be an R-complex. In analogy with 14.1.12 the set

$$\{\mathfrak{p}\in\operatorname{Spec}R \mid \mathrm{H}(\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},M))\neq 0\}$$

is in some places, see for example Sather-Wagstaff and Wicklein [222], denoted $\operatorname{Cosupp}_R M$. See also 15.2.1 and E 15.2.1.

14.1.35 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and I an R-complex. If I is semi-injective, then the $R_\mathfrak{p}$-complex $\operatorname{Hom}_R(R_\mathfrak{p},I)$ is semi-injective.*

PROOF. The assertion is a special case of 5.4.26(a). □

EXERCISES

E 14.1.1 Determine $\operatorname{Spec}(\mathbb{Z}/4\mathbb{Z})$ and $\operatorname{Spec}(\mathbb{Z}/6\mathbb{Z})$ and compute $\dim(\mathbb{Z}/4\mathbb{Z})$ and $\dim(\mathbb{Z}/6\mathbb{Z})$.

E 14.1.2 Let $\mathfrak{p}$ be a prime ideal in R and M and N be R-complexes. Show that there is an isomorphism $\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},\mathsf{R}\!\operatorname{Hom}_R(M,N)) \simeq \mathsf{R}\!\operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},N))$.

E 14.1.3 Let M be an R-complex. Show that one has $\inf\{\inf M_\mathfrak{m} \mid \mathfrak{m}\in\operatorname{Max}R\} = \inf M$ and $\sup\{\sup M_\mathfrak{m} \mid \mathfrak{m}\in\operatorname{Max}R\} = \sup M$.

E 14.1.4 Let $\mathfrak{p}\subseteq R$ be a prime ideal, M a complex in $\mathcal{D}_{\sqsupset}(R)$, and $N\in\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Show that if $\operatorname{splf}R$ is finite, then one has $\mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},M\otimes^{\mathsf{L}}_R N) \simeq \mathsf{R}\!\operatorname{Hom}_R(R_\mathfrak{p},M)\otimes^{\mathsf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}$.

E 14.1.5 Let $\mathfrak{p}$ be a prime ideal in R and M an $R_\mathfrak{p}$-complex. Show that there are isomorphisms $\mathsf{R}\Gamma_\mathfrak{p}(M)\simeq\mathsf{R}\Gamma_{\mathfrak{p}_\mathfrak{p}}(M)$ and $\mathsf{L}\Lambda^\mathfrak{p}(M)\simeq\mathsf{L}\Lambda^{\mathfrak{p}_\mathfrak{p}}(M)$ in $\mathcal{D}(R_\mathfrak{p})$.

E 14.1.6 Let $\mathrm{F}\colon\mathcal{M}(R)\to\mathcal{M}(R)$ be a half exact functor with the property that there is an exact sequence $\mathrm{F}(R/(\mathfrak{p}+(x)))\longrightarrow\mathrm{F}(R/\mathfrak{p})\xrightarrow{x}\mathrm{F}(R/\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ in R and every $x\in R\setminus\mathfrak{p}$. Show that if the set $\{\mathfrak{p}\in\operatorname{Spec}R \mid \mathrm{F}(R/\mathfrak{p})\neq 0\}$ is non-empty and $\mathfrak{p}$ is a maximal element of that set, then one has $\mathrm{F}(R/\mathfrak{p})_\mathfrak{p}\neq 0$.

E 14.1.7 Show that $\operatorname{fd}_R M = \sup\{m\in\mathbb{Z} \mid \exists\,\mathfrak{p}\in\operatorname{Supp}_R M\colon \operatorname{Tor}^{R_\mathfrak{p}}_m(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p},M_\mathfrak{p})\neq 0\}$ holds for every R-complex M. *Hint:* Apply E 14.1.6 with F a coproduct of Tor functors.

14.2 Krull Dimension for Complexes

SYNOPSIS. Krull dimension; ~ vs. localization; ~ of derived tensor product; module of finite length; Artinian module; Artinian ring.

The classic support for complexes from 14.1.4 and the Krull dimension, defined right below, are homological invariants in the strong sense that there are equalities,

$$\operatorname{Supp}_R M = \operatorname{Supp}_R \mathrm{H}(M) \qquad \text{and} \qquad \dim_R M = \dim_R \mathrm{H}(M) \,.$$

In particular, complexes that are isomorphic in $\mathcal{D}(R)$ have the same classic support and the same Krull dimension.

14.2.1 Definition. Let M be an R-complex. The *Krull dimension* of M, written $\dim_R M$, is defined as

$$\dim_R M = \sup\{\dim_R \mathrm{H}_v(M) - v \mid v \in \mathbb{Z}\} \,.$$

One says that $\dim_R M$ is *finite* if $\dim_R M < \infty$ holds.

The convention that a complex of Krull dimension $-\infty$ has finite Krull dimension may appear odd, but it only happens for acyclic complexes; see 14.2.4 below.

14.2.2. Let M be an R-complex. The next equality is immediate from 14.2.1,

$$\dim_R \Sigma^s M = \dim_R M - s \quad \text{for every integer } s \,.$$

REMARK. The notion of Krull dimension for complexes introduced above appeared in [93], and so did the notion of depth for complexes that is covered in the next two sections. Around the same time, Iversen [141, 142] worked with definitions of these invariants that were superficially different, but only in the sense that the invariants defined here depend on the homological position of the complex, while Iversen's versions are invariant under shift.

14.2.3 Example. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. By 14.1.5 and 11.4.3(a) one has $\operatorname{Supp}_R \mathrm{K}^R(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a}) = \operatorname{Supp}_R \mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x}))$. As $\mathrm{H}_v(\mathrm{K}^R(\boldsymbol{x}))$ is zero for $v < 0$ one has $\dim_R \mathrm{K}^R(\boldsymbol{x}) = \dim R/\mathfrak{a}$.

14.2.4 Proposition. *Let M be an R-complex; the next conditions are equivalent.*

(*i*) $\mathrm{H}(M) = 0$.
(*ii*) $\operatorname{Supp}_R M = \emptyset$.
(*iii*) $\dim_R M = -\infty$.

Moreover, if M is not acyclic, then the following inequalities hold,

$$-\inf M \leqslant \dim_R M \leqslant \dim R - \inf M \,.$$

PROOF. For every non-zero R-module H one has $0 \leqslant \dim_R H \leqslant \dim R$, and the zero module has Krull dimension $-\infty$. Together with the definition, 14.2.1, of the Krull dimension of a complex, this explains the inequalities. The equivalence of the three conditions follows in view of 14.1.4 and 14.2.1 as the only module with empty classic support and/or Krull dimension $-\infty$ is the zero module. □

14.2.5 Proposition. *Let $\mathfrak{a}$ be a proper ideal in R and M an $R/\mathfrak{a}$-complex*; *one has*

$$\dim_R M \;=\; \dim_{R/\mathfrak{a}} M\;.$$

PROOF. For every $v \in \mathbb{Z}$ the set $\operatorname{Supp}_R \mathrm{H}_v(M)$ is contained in $\mathrm{V}(\mathfrak{a})$, which is order isomorphic to $\operatorname{Spec} R/\mathfrak{a}$, so one has $\dim_R \mathrm{H}_v(M) = \dim_{R/\mathfrak{a}} \mathrm{H}_v(M)$; the desired equality now follows from 14.2.1. ◻

LOCALIZATION

For a module M the first equality in the next theorem simply recovers the definition of the Krull dimension.

14.2.6 Theorem. *Let M be an R-complex. One has*

$$\dim_R M \;=\; \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}\;.$$

Moreover, if $\dim R$ *is finite, then one has*

$$\dim_R M \;=\; \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\;.$$

PROOF. Per 14.2.4 one can assume that $\mathrm{H}(M)$ is bounded below and not zero.

"$\geqslant$": Let $\mathfrak{p} \in \operatorname{Supp}_R M$ and set $n = \inf M_\mathfrak{p}$. Since $\mathfrak{p}$ is in $\operatorname{Supp}_R \mathrm{H}_n(M)$, there is an inequality $\dim_R \mathrm{H}_n(M) - n \geqslant \dim R/\mathfrak{p} - \inf M_\mathfrak{p}$. Now invoke 14.2.1.

"$\leqslant$": Let $n \in \mathbb{Z}$. If $\mathrm{H}_n(M) = 0$, then the inequality

$$(\dagger)\qquad \dim_R \mathrm{H}_n(M) - n \;\leqslant\; \dim R/\mathfrak{p} - \inf M_\mathfrak{p}$$

holds for every prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R M$. If $\mathrm{H}_n(M) \neq 0$, choose a prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R \mathrm{H}_n(M)$ with $\dim_R \mathrm{H}_n(M) = \dim R/\mathfrak{p}$; for such a prime ideal one has $n \geqslant \inf M_\mathfrak{p}$ whence (†) holds. Finally recall from 14.1.4 that $\mathfrak{p}$ is in $\operatorname{Supp}_R M$.

Finally, if $\dim R$ is finite, then $\dim R/\mathfrak{p} - \inf M_\mathfrak{p} = -\infty$ holds for $\mathfrak{p} \notin \operatorname{Supp}_R M$; this proves the last assertion. ◻

14.2.7 Proposition. *Let M be an R-complex. There is an equality,*

$$\dim_R M \;=\; \sup\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\;,$$

and for every $\mathfrak{p} \in \operatorname{Supp}_R M$ *the next inequality holds,*

$$\dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} \;\leqslant\; \dim_R M\;.$$

Moreover, if $\dim R$ *is finite, then this inequality holds for every prime ideal* $\mathfrak{p}$ *in* R.

PROOF. Let $\mathfrak{p}$ be a prime ideal in R. The elements of $\operatorname{Supp}_{R_\mathfrak{p}} M_\mathfrak{p}$ are the prime ideals in $R_\mathfrak{p}$ of the form $\mathfrak{q}_\mathfrak{p}$ where $\mathfrak{q} \in \operatorname{Supp}_R M$ and $\mathfrak{q} \subseteq \mathfrak{p}$, so 14.2.6 yields:

$$(\dagger)\qquad \dim_{R_\mathfrak{p}} M_\mathfrak{p} \;=\; \sup\{\dim R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p} - \inf M_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Supp}_R M \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}\;.$$

This explains the first equality below. The penultimate equality follows from the definition of Krull dimension for commutative Noetherian rings, and the last equality holds by another application of 14.2.6; the remaining equalities are trivial.

$$\begin{aligned}
&\sup\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\
&\qquad = \sup\big\{\sup\{\dim R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p} - \inf M_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Supp}_R M \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\} \bigm| \mathfrak{p} \in \operatorname{Spec} R\big\} \\
&\qquad = \sup\big\{\sup\{\dim(R/\mathfrak{q})_\mathfrak{p} - \inf M_\mathfrak{q} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\} \bigm| \mathfrak{q} \in \operatorname{Supp}_R M\big\} \\
&\qquad = \sup\big\{\sup\{\dim(R/\mathfrak{q})_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\} - \inf M_\mathfrak{q} \bigm| \mathfrak{q} \in \operatorname{Supp}_R M\big\} \\
&\qquad = \sup\{\dim R/\mathfrak{q} - \inf M_\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Supp}_R M\} \\
&\qquad = \dim_R M\,.
\end{aligned}$$

This establishes the asserted equality. To prove the inequality, let $\mathfrak{p}$ be a prime ideal $\operatorname{Supp}_R M$. For every $\mathfrak{q}$ in $\operatorname{Supp}_R M$ with $\mathfrak{q} \subseteq \mathfrak{p}$ the inequalities below follow from the definition of Krull dimension for commutative Noetherian rings and from 14.2.6.

$$\begin{aligned}
\dim R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p} - \inf M_\mathfrak{q} + \dim R/\mathfrak{p} &= \dim(R/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}} + \dim(R/\mathfrak{q})/(\mathfrak{p}/\mathfrak{q}) - \inf M_\mathfrak{q} \\
&\leqslant \dim R/\mathfrak{q} - \inf M_\mathfrak{q} \;\leqslant\; \dim_R M\,.
\end{aligned}$$

Combining this with (†) one gets $\dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} \leqslant \dim_R M$. Further, if $\dim R$ is finite, then $\dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} = -\infty$ holds for $\mathfrak{p} \notin \operatorname{Supp}_R M$, see 14.1.12 and 14.2.4; this proves the last assertion. □

Krull Dimension of Derived Tensor Product

Under extra assumptions, the next inequality can be strengthened, 18.3.25, or the Krull dimension of the derived tensor product even computed exactly, 17.6.19.

14.2.8 Proposition. *Let M and N be R-complexes that are not acyclic*; *one has*

$$\dim_R(M \otimes^{\mathsf{L}}_R N) \;\leqslant\; \dim_R M - \inf N\,.$$

Proof. The inequality holds trivially if M or N does not have bounded below homology, see 14.2.4. Assuming now that both complexes belong to $\mathcal{D}_{\sqsupset}(R)$, the assertion follows from 14.2.6, 14.1.17, 7.6.8, 14.1.15, and 14.1.11(c):

$$\begin{aligned}
\dim_R M &= \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\} \\
&\geqslant \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M \otimes^{\mathsf{L}}_R N)\} \\
&\geqslant \sup\{\dim R/\mathfrak{p} - \inf\,(M_\mathfrak{p} \otimes^{\mathsf{L}}_{R_\mathfrak{p}} N_\mathfrak{p}) + \inf N_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M \otimes^{\mathsf{L}}_R N)\} \\
&= \sup\{\dim R/\mathfrak{p} - \inf\,(M \otimes^{\mathsf{L}}_R N)_\mathfrak{p} + \inf N_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M \otimes^{\mathsf{L}}_R N)\} \\
&\geqslant \sup\{\dim R/\mathfrak{p} - \inf\,(M \otimes^{\mathsf{L}}_R N)_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M \otimes^{\mathsf{L}}_R N)\} + \inf N \\
&= \dim_R(M \otimes^{\mathsf{L}}_R N) + \inf N\,.
\end{aligned}$$
□

Artinian Modules

Recall that an R-module has finite length if it admits a filtration with simple quotients, in which case all such filtrations, called composition series, have the same length. Modules of finite length are evidently finitely generated.

14.2.9 Proposition. *Let M be a finitely generated R-module. The following conditions are equivalent.*

(i) *M has finite length.*

(ii) *Every ideal in $\operatorname{Ass}_R M$ is maximal.*

(iii) *Every ideal in $\operatorname{Supp}_R M$ is maximal.*

If these conditions are satisfied, then $\operatorname{Ass}_R M = \operatorname{Supp}_R M$ holds; in particular, the set $\operatorname{Supp}_R M$ is finite.

Proof. (i) $\Rightarrow$ (ii): Let $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^n = M$ be a composition series. For every $\mathfrak{p} \in \operatorname{Ass}_R M$ there is a quotient M^i/M^{i-1} isomorphic to $R/\mathfrak{p}$, and since all these quotients are simple, $\mathfrak{p}$ is maximal.

(ii) $\Rightarrow$ (iii): As every prime ideal in $\operatorname{Supp}_R M$ contains an associated prime ideal of M, one has $\operatorname{Ass}_R M = \operatorname{Supp}_R M$.

(iii) $\Rightarrow$ (i): Let $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^n = M$ be a filtration with quotients $M^i/M^{i-1} \cong R/\mathfrak{p}_i$ for ideals $\mathfrak{p}_i \in \operatorname{Supp}_R M$. As $\operatorname{Supp}_R M$ consists of maximal ideals, all these quotients are simple, whence M has finite length. □

Recall that an R-module is Artinian if it satisfies the Descending Chain Condition on submodules.

14.2.10 Proposition. *Let M be an Artinian R-module. The classic support of M is a finite set of maximal ideals and there are equalities,*

$$\operatorname{Min}_R M \;=\; \operatorname{Ass}_R M \;=\; \operatorname{Supp}_R M \;.$$

In particular, if M is non-zero then one has $\dim_R M = 0$.

Proof. One can assume that M is non-zero. Let $\mathfrak{p} \in \operatorname{Supp}_R M$ and choose an element $m \in M$ with $(0 :_R m) \subseteq \mathfrak{p}$. The assignment $rm \mapsto [r]_{\mathfrak{p}}$ defines a surjective homomorphism $R\langle m\rangle \to R/\mathfrak{p}$; in particular $R/\mathfrak{p}$ is an Artinian R-module and hence an Artinian ring. For $x \notin \mathfrak{p}$ the descending chain $([x]_{\mathfrak{p}}) \supseteq ([x^2]_{\mathfrak{p}}) \supseteq \cdots$ stabilizes. As $R/\mathfrak{p}$ is an integral domain, this implies that $[x]_{\mathfrak{p}}$ is a unit. I.e. $R/\mathfrak{p}$ is a field, whence $\mathfrak{p}$ is maximal. In particular, every minimal prime ideal in $\operatorname{Supp}_R M$ is maximal, whence the asserted equalities follow. To see that the set is finite, choose for every $\mathfrak{m} \in \operatorname{Ass}_R M$ an element $x^{\mathfrak{m}} \in M$ with $(0 :_R x^{\mathfrak{m}}) = \mathfrak{m}$. Notice that each module $R\langle x^{\mathfrak{m}}\rangle$ is simple, as it is isomorphic to $R/\mathfrak{m}$. Let V be a finite subset of $\operatorname{Ass}_R M$; one has

$$\Big(0 : \sum_{\mathfrak{m}\in V} R\langle x^{\mathfrak{m}}\rangle\Big) \;=\; \bigcap_{\mathfrak{m}\in V} \mathfrak{m} \;.$$

Now let $\mathfrak{n} \in (\operatorname{Ass}_R M) \setminus V$. If the intersection $(\sum_{\mathfrak{m}\in V} R\langle x^{\mathfrak{m}}\rangle) \cap R\langle x^{\mathfrak{n}}\rangle$ is not zero, then it is all of $R\langle x^{\mathfrak{n}}\rangle$, which means that the maximal ideal $\mathfrak{n} = (0 : x^{\mathfrak{n}})$ contains $\bigcap_{\mathfrak{m}\in V} \mathfrak{m}$ and hence it contains one of the maximal ideals $\mathfrak{m} \in V$, a contradiction. Thus the family $\{R\langle x^{\mathfrak{m}}\rangle\}_{\mathfrak{m}\in\operatorname{Ass}_R M}$ of submodules of M is independent by 1.1.24(a). Since M is Artinian it follows from 1.1.24(b) that the set $\operatorname{Ass}_R M$ is finite. □

The description above of the classic support of Artinian modules does not characterize such modules.

14.2.11 Example. Let M be an Artinian R-module. By 14.1.7 one has $\operatorname{Supp}_R M^{(\mathbb{N})} = \operatorname{Supp}_R M$, yet the coproduct $M^{(\mathbb{N})}$ has a countable descending chain of submodules.

14.2.12 Corollary. *An R-module has finite length if and only if it is Artinian and finitely generated.*

PROOF. An R-module M of finite length is finitely generated. To see that it is Artinian notice that a strictly descending chain of submodules $M^0 \supset M^1 \supset \cdots$ in M yields a descending sequence of numbers $\operatorname{length}_R M^0 > \operatorname{length}_R M^1 > \cdots$, which is impossible as $\operatorname{length}_R M^0 \leqslant \operatorname{length}_R M$ is finite.

Conversely, let M be an Artinian R-module. By 14.2.10 all ideals in $\operatorname{Supp}_R M$ are maximal, so if M it finitely generated, then it has finite length by 14.2.9. □

14.2.13 Example. For every maximal ideal $\mathfrak{m}$ in R, the injective module $\mathrm{E}_R(R/\mathfrak{m})$ is Artinian; see C.18 and 16.1.26. If R is local, then $\mathrm{E}_R(R/\mathfrak{m})$ is finitely generated if and only if R is Artinian, see 18.1.4, while $\mathrm{E}_R(R/\mathfrak{p})$ for $\mathfrak{p} \neq \mathfrak{m}$ is neither Artinian nor finitely generated, see 14.2.10, C.15(a), 16.2.27, and 16.2.29.

We record a special case of 12.4.11.

14.2.14 Proposition. *Let $\mathfrak{J}$ be the Jacobson radical of R and $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{G}(R/\mathfrak{p})$ is Artinian for every prime ideal $\mathfrak{p}$ in R. The following conditions are equivalent.*

(*i*) $\mathrm{G}(R/\mathfrak{p}) = 0$ *holds for every prime ideal* $\mathfrak{p}$ *in* $\mathrm{V}(\mathfrak{J})$.

(*ii*) $\mathrm{G}(M) = 0$ *holds for every finitely generated R-module M.*

PROOF. It follows from 14.2.10 and 14.1.3 that an Artinian R-module is $\mathfrak{J}$-torsion, so the statement is a special case of 12.4.11. □

14.2.15 Definition. An R-complex M is called *degreewise Artinian* if the R-module M_v is Artinian for every $v \in \mathbb{Z}$. Similarly, M is called *degreewise of finite length* if the R-module M_v is of finite length for every $v \in \mathbb{Z}$.

14.2.16 Definition. The full subcategories $\mathcal{D}^{\mathrm{art}}(R)$ and $\mathcal{D}^{\ell}(R)$ of $\mathcal{D}(R)$ are defined by specifying their objects as follows,

$$\mathcal{D}^{\mathrm{art}}(R) = \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is degreewise Artinian}\}$$

and

$$\mathcal{D}^{\ell}(R) = \{M \in \mathcal{D}(R) \mid \mathrm{H}(M) \text{ is degreewise of finite length}\}\,.$$

Notice that per 14.2.12 one has

$$\mathcal{D}^{\ell}(R) = \mathcal{D}^{\mathrm{art}}(R) \cap \mathcal{D}^{\mathrm{f}}(R)\,.$$

The full subcategory $\mathcal{D}^{\mathrm{art}}(R) \cap \mathcal{D}_{\sqsubset}(R)$ is denoted by $\mathcal{D}^{\mathrm{art}}_{\sqsubset}(R)$. Similarly, one defines the subcategories $\mathcal{D}^{\mathrm{art}}_{\sqsupset}(R)$, $\mathcal{D}^{\mathrm{art}}_{\square}(R)$, $\mathcal{D}^{\ell}_{\sqsubset}(R)$, $\mathcal{D}^{\ell}_{\sqsupset}(R)$, and $\mathcal{D}^{\ell}_{\square}(R)$.

14.2.17 Proposition. *The categories $\mathcal{D}^{\mathrm{f}}(R)$, $\mathcal{D}^{\mathrm{art}}(R)$, and $\mathcal{D}^{\ell}(R)$ are triangulated subcategories of $\mathcal{D}(R)$ and closed under soft truncations.*

Proof. The assertions about $\mathcal{D}^{\mathrm{f}}(R)$ are proved in 7.6.14, and the same argument applies for $\mathcal{D}^{\mathrm{art}}(R)$ and $\mathcal{D}^{\ell}(R)$. $\square$

It follows from 14.2.17 and 7.6.3 the intersection of any of the categories $\mathcal{D}^{\mathrm{f}}(R)$, $\mathcal{D}^{\mathrm{art}}(R)$, and $\mathcal{D}^{\ell}(R)$ with $\mathcal{D}_{\sqsubset}(R)$, $\mathcal{D}_{\sqsupset}(R)$, or $\mathcal{D}_{\square}(R)$ yields a triangulated subcategory of $\mathcal{D}(R)$ which is closed under soft truncations.

14.2.18 Proposition. *Let U be a multiplicative subset of R and M an R-complex. If M is degreewise Artinian, then the $U^{-1}R$-complex $U^{-1}M$ is degreewise Artinian.*

Proof. As one has $(U^{-1}M)_v = U^{-1}M_v$ for every $v \in \mathbb{Z}$ one can assume that M is an R-module M. For every $U^{-1}R$-submodule N of $U^{-1}M$, set $N' = \{m \in M \mid \frac{m}{1} \in N\}$. Evidently, N' is an R-submodule of M and one has $U^{-1}N' = N$. If $N_1 \supseteq N_2 \supseteq \cdots$ is a descending chain of $U^{-1}R$-submodules of $U^{-1}M$, then $N_1' \supseteq N_2' \supseteq \cdots$ is a descending chain of R-submodules of M. If M is Artinian, then this chain becomes stationary, and hence $U^{-1}N_1' \supseteq U^{-1}N_2' \supseteq \cdots$ becomes stationary too. This chain coincides with the given chain $N_1 \supseteq N_2 \supseteq \cdots$. $\square$

Artinian Rings

14.2.19 Theorem. *The following conditions are equivalent.*

(*i*) *R is Artinian.*
(*ii*) Spec *R is a finite set of maximal ideals.*
(*iii*) *R has Krull dimension* 0.
(*iv*) *R has finite length as an R-module.*
(*v*) *Every finitely generated R-module has finite length.*

Moreover, if R is Artinian with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and Jacobson radical $\mathfrak{J}$, then the following assertions hold.

(a) *The Jacobson radical is the product $\mathfrak{J} = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ and it is nilpotent.*
(b) *For $u \in \{1, \ldots, n\}$ and $p \in \mathbb{N}$ with $\mathfrak{J}^p = 0$ there is an isomorphism of rings, $R_{\mathfrak{m}_u} \cong R/\mathfrak{m}_u^p$; in particular $R_{\mathfrak{m}_u}$ is an Artinian local ring.*
(c) *There is an isomorphism of rings, $R \cong R_{\mathfrak{m}_1} \times \cdots \times R_{\mathfrak{m}_n}$.*
(d) *For every ideal $\mathfrak{a} \subseteq \mathfrak{J}$ the ring R is $\mathfrak{a}$-complete, i.e. $R \cong \widehat{R}^{\mathfrak{a}}$.*

Proof. The implication (*i*) $\Rightarrow$ (*ii*) follows from 14.2.10. By the definition of Krull dimension, condition (*ii*) implies (*iii*), which means that all prime ideals in R are maximal. Thus (*iii*) implies (*v*) by 14.2.9, and (*iv*) is a special case of (*v*) and implies (*i*) by 14.2.12.

Now, let R be Artinian with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and Jacobson radical $\mathfrak{J}$. As the ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are coprime, the intersection $\mathfrak{J} = \bigcap_{u=1}^{n} \mathfrak{m}_u$ equals the product $\mathfrak{m}_1 \cdots \mathfrak{m}_n$. The descending chain $\mathfrak{J} \supseteq \mathfrak{J}^2 \supseteq \cdots$ stabilizes, so one has $\mathfrak{J}^p = \mathfrak{J}^{p+1}$ for some p; Nakayama's lemma B.32 now yields $\mathfrak{J}^p = 0$. This proves part (a), and part (d) follows in view of 11.1.11. As powers of coprime ideals are coprime, the Chinese Remainder Theorem yields $R/\mathfrak{J}^p \cong R/\mathfrak{m}_1^p \times \cdots \times R/\mathfrak{m}_n^p$ for every $p \in \mathbb{N}$.

In particular, one has $R \cong R/\mathfrak{m}_1^p \times \cdots \times R/\mathfrak{m}_n^p$ for p with $\mathfrak{J}^p = 0$. Fix such a p and a maximal ideal $\mathfrak{m}_u$; to finish the proof it suffices to establish the isomorphism in (b). For $w \neq u$ one has $(R/\mathfrak{m}_w^p)_{\mathfrak{m}_u} = 0$, whence $R_{\mathfrak{m}_u} \cong (R/\mathfrak{m}_u^p)_{\mathfrak{m}_u}$ holds. The ring $R/\mathfrak{m}_u^p$ is already local with maximal ideal $\mathfrak{m}_u/\mathfrak{m}_u^p$, whence $R_{\mathfrak{m}_u} \cong R/\mathfrak{m}_u^p$. □

REMARK. It follows from 14.2.19 that a commutative Artinian ring is semi-local; see the Remark after B.43.

14.2.20 Lemma. *Let R be an integral domain with field of fractions Q. If Q is finitely generated as an R-module, then R is a field, i.e. $R = Q$.*

PROOF. Let $x_1, \ldots, x_n$ be elements in R such that the fractions $\frac{1}{x_1}, \ldots, \frac{1}{x_n}$ generate Q as an R-module. In particular, one has $x_1^2 x_2^2 \cdots x_n^2 (\sum_{i=1}^n \frac{r_i}{x_i}) = 1$ for elements $r_1, \ldots, r_n$ in R. It follows that each fraction $\frac{1}{x_i}$ belongs to R. Thus, one has $R = Q$. □

14.2.21 Theorem. *The following conditions are equivalent.*

(*i*) *R is Artinian.*
(*ii*) *Every indecomposable injective R-module has finite length.*
(*iii*) *Every indecomposable injective R-module is finitely generated.*

PROOF. The proof if cyclic. It is evident that (*ii*) implies (*iii*).

(*i*) $\Rightarrow$ (*ii*): Let E be an indecomposable injective R-module, by C.6 it has the form $E \cong \mathrm{E}_R(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ in R. As R is Artinian, there is an $n \geqslant 1$ such that $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. With the notation from C.20 one now has $E = E^n$, and this module has finite length by C.22.

(*iii*) $\Rightarrow$ (*i*): To prove that R is Artinian, it is by 14.2.19 sufficient to prove that every prime ideal $\mathfrak{p}$ in R is maximal. Set $E = \mathrm{E}_R(R/\mathfrak{p})$, it is an indecomposable injective R-module by C.12, and hence it is finitely generated by assumption. By C.18 there is an isomorphism of R-modules $E \cong \mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p})$, so in particular the submodule $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ of E is finitely generated over R. That is, the field of fractions of the integral domain $R/\mathfrak{p}$ is finitely generated as an R-module and hence as an $R/\mathfrak{p}$-module. Now it follows from 14.2.20 that $R/\mathfrak{p}$ is a field, whence the ideal $\mathfrak{p}$ is maximal. □

14.2.22 Corollary. *Let R be Artinian. For every finitely generated R-module M the injective envelope $\mathrm{E}_R(M)$ has finite length.*

PROOF. By C.5 the module $\mathrm{E}_R(M)$ is a direct sum of indecomposable injective R-modules, and each of these has finite length by 14.2.21. □

EXERCISES

E 14.2.1 Let M be an R-complex. Show that $\dim_R M = \sup\{\dim_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max} R\}$ holds.

E 14.2.2 Let M be a complex in $\mathcal{D}_\sqsupset(R)$ with $\mathrm{H}(M) \neq 0$ and $w = \inf M$. (a) Show that if one has $\dim_R M = -w$, then $\dim_R \mathrm{H}_w(M) = 0$ holds. (b) Show that if $\dim R$ is finite, then one has $\dim_R M = \dim R - w$ if and only if $\dim_R \mathrm{H}_w(M) = \dim R$ holds.

E 14.2.3 Let M be an R-complex. Show that if $\dim R$ is finite, then $\dim_R M = \dim_R M_{\subseteq n}$ holds for $n \gg 0$.

E 14.2.4 Let M and N be R-modules. Show that one has $\dim_R(M \otimes^{\mathsf{L}}_R N) \geqslant \dim_R(M \otimes_R N)$ and $\dim_R \mathsf{RHom}_R(M, N) \geqslant \dim_R \operatorname{Hom}_R(M, N)$.

E 14.2.5 Let R be of finite Krull dimension. Show that for complexes M in $\mathcal{D}(R)$ and N in $\mathcal{D}_{\square}(R)$ one has $\dim_R(M \otimes^{\mathsf{L}}_R N) \leqslant \sup\{\dim_R(M \otimes^{\mathsf{L}}_R \mathrm{H}_v(N)) - v \mid v \in \mathbb{Z}\}$. *Hint:* 7.6.10.

E 14.2.6 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and N an R-complex. Show that if $\operatorname{pd}_R M$ is finite, then the inequality $\dim_R \mathsf{RHom}_R(M, N) \leqslant \dim_R N + \operatorname{pd}_R M$ holds.

E 14.2.7 Let $\mathfrak{J}$ be the Jacobson radical of R. Show that one has $\mathcal{D}^{\mathrm{art}}(R) \subseteq \mathcal{D}^{\mathfrak{J}\text{-tor}}(R)$.

14.3 Koszul Homology

Synopsis. Koszul complex; Koszul homology; $\mathfrak{a}$-depth; $\mathfrak{a}$-width.

For a sequence $\boldsymbol{x}$ in R and an R-module M it is standard to refer to the homology of the complex $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M$ as the 'Koszul homology' of M with respect to $\boldsymbol{x}$. While we do not formally employ this terminology, the homology of such complexes plays a central role in our treatment of the invariants depth and width, hence the title of this section.

14.3.1 Lemma. *Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R and M an R-complex. For subsequences $\boldsymbol{x}' = x'_1, \dots, x'_m$ and $\boldsymbol{x}'' = x''_1, \dots x''_{m-n}$ that partition $\boldsymbol{x}$ one has*

$$\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \;\cong\; \mathrm{K}^R(\boldsymbol{x}') \otimes_R (\mathrm{K}^R(\boldsymbol{x}'') \otimes_R M)\,.$$

Proof. The isomorphism follows in view of (11.4.1.2) and 11.4.3(b) from associativity 12.1.8 of the tensor product. $\square$

14.3.2 Proposition. *Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R and M an R-complex. There is an isomorphism,*

$$\operatorname{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), M) \;\cong\; \Sigma^{-n}(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\,.$$

Proof. A straightforward computation based on the unitor 12.1.5, tensor evaluation 12.1.15(d), 11.4.7, and 2.4.14 yields:

$$\operatorname{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), M) \;\cong\; \operatorname{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), R) \otimes_R M \;\cong\; \Sigma^{-n}(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\,. \qquad \square$$

The conditions in parts (b) and (d) of the next lemma are interpreted in 14.3.16 and 14.3.28.

14.3.3 Lemma. *Let x be an element in R and M an R-complex. There is an exact sequence of R-modules,*

$$\cdots \longrightarrow \mathrm{H}_v(M) \xrightarrow{\;x\;} \mathrm{H}_v(M) \longrightarrow \mathrm{H}_v(\mathrm{K}^R(x) \otimes_R M) \longrightarrow \mathrm{H}_{v-1}(M) \xrightarrow{\;x\;} \cdots\,,$$

and the following assertions hold.

(a) *There are inequalities,*

$$\sup (R/(x) \otimes_R \mathrm{H}(M)) \leqslant \sup (\mathrm{K}^R(x) \otimes_R M) ,$$
$$\sup \operatorname{Hom}_R(R/(x), \mathrm{H}(M)) + 1 \leqslant \sup (\mathrm{K}^R(x) \otimes_R M) \leqslant \sup M + 1 .$$

In particular, if $x\,\mathrm{H}(M) = 0$ *holds, then* $\sup (\mathrm{K}^R(x) \otimes_R M) = \sup M + 1$ *holds.*

(b) *Assume that* M *is not acyclic and set* $s = \sup M$. *If* $s < \infty$, *then one has*:
- $\sup (\mathrm{K}^R(x) \otimes_R M) \geqslant s$ *holds if one has* $R/(x) \otimes_R \mathrm{H}_s(M) \neq 0$.
- $\sup (\mathrm{K}^R(x) \otimes_R M) = s + 1$ *if and only if* $\operatorname{Hom}_R(R/(x), \mathrm{H}_s(M)) \neq 0$.

(c) *There are inequalities,*

$$\inf (\mathrm{K}^R(x) \otimes_R M) \leqslant \inf \operatorname{Hom}_R(R/(x), \mathrm{H}(M)) + 1 ,$$
$$\inf M \leqslant \inf (\mathrm{K}^R(x) \otimes_R M) \leqslant \inf (R/(x) \otimes_R \mathrm{H}(M)) .$$

In particular, if $x\,\mathrm{H}(M) = 0$ *holds, then one has* $\inf (\mathrm{K}^R(x) \otimes_R M) = \inf M$.

(d) *Assume that* M *is not acyclic and set* $w = \inf M$. *If* $w > -\infty$, *then one has*
- $\inf (\mathrm{K}^R(x) \otimes_R M) \leqslant w + 1$ *holds if one has* $\operatorname{Hom}_R(R/(x), \mathrm{H}_w(M)) \neq 0$.
- $\inf (\mathrm{K}^R(x) \otimes_R M) = w$ *holds if and only if one has* $R/(x) \otimes_R \mathrm{H}_w(M) \neq 0$.

Proof. The complex $\mathrm{K}^R(x) \otimes_R M$ is by 11.4.5 isomorphic to the mapping cone of the homothety x^M, so the asserted exact sequence comes from 4.2.15. The inequalities in (a) and (c) follow by inspection of this sequence. Indeed, non-vanishing of

$$\operatorname{Hom}_R(R/(x), \mathrm{H}_v(M)) \cong \operatorname{Ker}(x^{\mathrm{H}_v(M)})$$

implies $\mathrm{H}_{v+1}(\mathrm{K}^R(x) \otimes_R M) \neq 0$, and non-vanishing of

$$R/(x) \otimes_R \mathrm{H}_v(M) \cong \operatorname{Coker}(x^{\mathrm{H}_v(M)})$$

implies $\mathrm{H}_v(\mathrm{K}^R(x) \otimes_R M) \neq 0$.

(b): The first assertion is immediate from the first inequality in (a), and the second assertion follows by inspection of the exact sequence established above.

(d): The first assertion is immediate from the first inequality in (c), and the second assertion follows by inspection of the exact sequence established above. □

14.3.4 Proposition. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* $\boldsymbol{y}$ *a sequence in* R, *and* M *an* R-*complex.*

(a) *If* M *belongs to* $\mathcal{D}^{\mathrm{f}}(R)$, *then also the complex* $\mathrm{K}^R(\boldsymbol{y}) \otimes_R M$ *is in* $\mathcal{D}^{\mathrm{f}}(R)$.

(b) *If* M *belongs to* $\mathcal{D}^{\mathrm{art}}(R)$, *then also the complex* $\mathrm{K}^R(\boldsymbol{y}) \otimes_R M$ *is in* $\mathcal{D}^{\mathrm{art}}(R)$.

(c) *If* M *is derived* $\mathfrak{a}$-*complete, then also* $\mathrm{K}^R(\boldsymbol{y}) \otimes_R M$ *is derived* $\mathfrak{a}$-*complete.*

(d) *If* M *is derived* $\mathfrak{a}$-*torsion, then also* $\mathrm{K}^R(\boldsymbol{y}) \otimes_R M$ *is derived* $\mathfrak{a}$-*torsion.*

Proof. For a sequence $\boldsymbol{y} = y$ of length one, the assertions (a) and (b) are immediate from the exact sequence in 14.3.3, and per (11.4.1.2) the general case of either assertion now follows by induction. Part (c) follows via 14.3.2 from 13.1.31(a), and part (d) is by commutativity 12.1.7 immediate from 13.3.28(a). □

14.3.5 Lemma. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* $\boldsymbol{x} = x_1,\dots,x_n$ *a sequence in* $\mathfrak{a}$, *and* M *an* R-*complex. There are* (*in*)*equalities,*

$$\sup M + n \geqslant \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \geqslant \sup M \quad \textit{and} \quad \inf(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = \inf M\,,$$

provided that one of the following conditions is satisfied.

(a) *M is derived $\mathfrak{a}$-complete.*
(b) *M belongs to $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathfrak{a}$ is contained in the Jacobson radical of R.*

PROOF. By (11.4.1.2), associativity 12.1.8, and 14.3.4(c,a) it is sufficient to establish the (in)equalities for a sequence $\boldsymbol{x} = x$ in $\mathfrak{a}$ of length one. If M is derived $\mathfrak{a}$-complete, then 13.1.35 yields

$$\sup M = \sup(R/(x) \otimes_R \mathrm{H}(M)) \quad \text{and} \quad \inf M = \inf(R/(x) \otimes_R \mathrm{H}(M))\,,$$

and under the assumptions in (b) the same equalities hold by Nakayama's lemma B.32. The asserted (in)equalities now follow from 14.3.3(a,c). □

14.3.6 Lemma. *Let $\mathfrak{a} \subseteq R$ be an ideal, $\boldsymbol{x} = x_1, \dots, x_n$ a sequence in $\mathfrak{a}$, and M an R-complex. If M is derived $\mathfrak{a}$-torsion, then there are (in)equalities,*

$$\sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = \sup M + n \quad \textit{and} \quad \inf M + n \geqslant \inf(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \geqslant \inf M\,.$$

PROOF. By (11.4.1.2), associativity 12.1.8, and 14.3.4(d) it is sufficient to establish the (in)equalities for a sequence $\boldsymbol{x} = x$ in $\mathfrak{a}$ of length one. By 13.3.32 one has $\sup M = \sup \mathrm{Hom}_R(R/(x), \mathrm{H}(M))$ and $\inf M = \inf \mathrm{Hom}_R(R/(x), \mathrm{H}(M))$. The asserted (in)equalities now follow from 14.3.3(a,c). □

The next result shows that for a finitely generated R-module M and a sequence $\boldsymbol{x}$ in the Jacobson radical of R the Koszul homology $\mathrm{H}(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$ has "no holes."

14.3.7 Theorem. *Let $\boldsymbol{x}$ be a sequence in the Jacobson radical of R and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ that is not acyclic. Set*

$$s = \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \qquad \textit{and} \qquad u = \inf(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\,.$$

If $\mathrm{H}_v(M)$ is non-zero for every integer $\sup M \geqslant v \geqslant \inf M$, then $\mathrm{H}_v(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$ is non-zero for every integer $s \geqslant v \geqslant u$.

PROOF. The equality $\inf M = u$ holds by 14.3.5(b). Assume that $\mathrm{H}_v(M)$ is non-zero for all $\sup M \geqslant v \geqslant u$. Let $\boldsymbol{x} = x_1, \dots, x_n$ and proceed by induction on n.

For $n = 1$ consider the exact sequence from 14.3.3,

$$(\flat) \quad \cdots \longrightarrow \mathrm{H}_v(M) \xrightarrow{x_1} \mathrm{H}_v(M) \longrightarrow \mathrm{H}_v(\mathrm{K}^R(x_1) \otimes_R M) \longrightarrow \mathrm{H}_{v-1}(M) \xrightarrow{x_1} \cdots\,.$$

To prove that $\mathrm{H}_v(\mathrm{K}(x_1) \otimes_R M)$ is non-zero for every integer $s \geqslant v \geqslant u$, it suffices to argue that if $\mathrm{H}_v(\mathrm{K}(x_1) \otimes_R M) \neq 0$ holds for some integer $v > u$, then one has $\mathrm{H}_{v-1}(\mathrm{K}(x_1) \otimes_R M) \neq 0$. Thus, assume that $\mathrm{H}_v(\mathrm{K}(x_1) \otimes_R M) \neq 0$ holds. First we notice that the module $\mathrm{H}_{v-1}(M)$ is non-zero. Indeed, in the case $\mathrm{H}_v(M) = 0$ this follows directly from the exact sequence $(\flat)$, and if $\mathrm{H}_v(M) \neq 0$, then $\mathrm{H}_{v-1}(M)$ is non-zero by the assumption on M as $v > u = \inf M$ holds. Now $\mathrm{H}_{v-1}(M)$ is a non-zero finitely generated R-module. As x_1 is in the Jacobson radical of R, Nakayama's lemma B.32 shows that the homothety $\mathrm{H}_{v-1}(M) \xrightarrow{x_1} \mathrm{H}_{v-1}(M)$ is not surjective, whence the exact sequence $(\flat)$ shows that $\mathrm{H}_{v-1}(\mathrm{K}(x_1) \otimes_R M) \neq 0$ holds.

Next let $n > 1$ and set $M' = \mathrm{K}(x_2,\ldots,x_n) \otimes_R M$. As noticed above one has $\inf M' = u$; in particular, M' is not acyclic. Further, M' is in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ by 14.3.4(a) and 14.3.5(b). Thus, by the induction hypothesis, the module $\mathrm{H}_v(M')$ is non-zero for every integer $\sup M' \geqslant v \geqslant u$. Now the base case yields $\mathrm{H}_v(\mathrm{K}(x_1) \otimes_R M') \neq 0$ for every integer $\sup(\mathrm{K}(x_1) \otimes_R M') \geqslant v \geqslant u$. It remains to recall the isomorphism $\mathrm{K}(x_1) \otimes_R M' \cong \mathrm{K}(\boldsymbol{x}) \otimes_R M$ from 14.3.1. $\square$

14.3.8 Corollary. *Let $\boldsymbol{x}$ be a sequence in the Jacobson radical of R. The homology module $\mathrm{H}_v(\mathrm{K}^R(\boldsymbol{x}))$ is non-zero for every integer* $\sup \mathrm{K}^R(\boldsymbol{x}) \geqslant v \geqslant 0$.

Proof. In view of the unitor 12.1.5, the assertion is the special case $M = R$ of 14.3.7; indeed, the infimum of the complex $\mathrm{K}^R(\boldsymbol{x})$ is 0 by 11.4.3. $\square$

14.3.9 Lemma. *Let $\mathfrak{a}$ be an ideal in R generated by sequences $\boldsymbol{x} = x_1,\ldots,x_n$ and $\boldsymbol{y} = y_1,\ldots,y_m$ and let M be an R-complex. There are equalities,*

$$\sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) - n = \sup(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) - m \quad \textit{and}$$
$$\inf(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = \inf(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M)\,.$$

Proof. Let $\boldsymbol{a}$ denote the concatenated sequence $x_1,\ldots,x_n,y_1,\ldots,y_m$. By symmetry, it suffices to prove that one has $\inf(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) = \inf(\mathrm{K}^R(\boldsymbol{a}) \otimes_R M)$ and $\sup(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) = \sup(\mathrm{K}^R(\boldsymbol{a}) \otimes_R M) - n$. By 14.3.1 it is enough to prove the equalities for $n = 1$, i.e. for $\boldsymbol{x}$ consisting of a single element x, in which case one has

$$\mathrm{K}^R(\boldsymbol{a}) \otimes_R M \cong \mathrm{K}^R(x) \otimes_R (\mathrm{K}^R(\boldsymbol{y}) \otimes_R M)\,.$$

By 11.4.6(a) one has $x\,\mathrm{H}(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) = 0$, so the desired equalities hold by the second assertions in 14.3.3(a,c). $\square$

Depth

In view of 14.3.9 one can make the following definition.

14.3.10 Definition. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x} = x_1,\ldots,x_n$. For an R-complex M the $\mathfrak{a}$-*depth* is defined as

$$\mathfrak{a}\text{-depth}_R M = n - \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\,.$$

One says that $\mathfrak{a}\text{-depth}_R M$ is *finite* if $\mathfrak{a}\text{-depth}_R M < \infty$ holds.

For the $\mathfrak{a}$-depth of the R-module R one uses the simplified notation $\mathfrak{a}\text{-depth}\, R$.

The convention that a complex with $\mathfrak{a}$-depth equal to $-\infty$ has finite $\mathfrak{a}$-depth aligns with the convetion for homological dimensions.

14.3.11. Let $\mathfrak{a}$ and M be as in 14.3.10. Notice that one has

$$\mathfrak{a}\text{-depth}_R \Sigma^s M = \mathfrak{a}\text{-depth}_R M - s \quad \text{for every integer } s\,.$$

Moreover, one has $\mathfrak{a}\text{-depth}_R M = \infty$ if M is acyclic.

14.3.12. For an R-complex M one has $0\text{-depth}_R\, M = -\sup M$ and $R\text{-depth}_R\, M = \infty$.

14.3.13 Example. Let $\mathfrak{p}$ be a prime ideal in R. For every ideal $\mathfrak{a}$ not contained in $\mathfrak{p}$ one has $\mathfrak{a}\text{-depth}_R\, \mathrm{E}_R(R/\mathfrak{p}) = \infty$. Indeed, for $x \in \mathfrak{a} \setminus \mathfrak{p}$ it follows from C.17, 4.3.31, and 11.4.5 that the complex $\mathrm{K}^R(x) \otimes_R \mathrm{E}_R(R/\mathfrak{p})$ is contractible, and hence so is $\mathrm{K}^R(\boldsymbol{x}) \otimes_R \mathrm{E}_R(R/\mathfrak{p})$ for any sequence $\boldsymbol{x}$ that generates $\mathfrak{a}$. In particular, one has $\mathfrak{a}\text{-depth}_{\mathbb{Z}}\, \mathbb{Q} = \infty$ for every non-zero ideal $\mathfrak{a}$ in $\mathbb{Z}$, see B.15.

14.3.14 Proposition. *Let $\{M^u\}_{u \in U}$ be a family of R-complexes; one has*

$$\mathfrak{a}\text{-depth}_R \Big(\coprod_{u \in U} M^u \Big) = \inf_{u \in U} \{ \mathfrak{a}\text{-depth}_R\, M^u \} = \mathfrak{a}\text{-depth}_R \Big(\prod_{u \in U} M^u \Big) .$$

Proof. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$. The functor $K \otimes_R -$ commutes by 3.1.13 with coproducts; this explains the first equality in the next display, and the second equality holds by 3.1.11. It remains to invoke the definition of $\mathfrak{a}$-depth to obtain the first of the asserted equalities.

$$\begin{aligned} -\sup \big(K \otimes_R \coprod_{u \in U} M^u \big) &= -\sup \big(\coprod_{u \in U} (K \otimes_R M^u) \big) \\ &= -\sup_{u \in U} \{ \sup (K \otimes_R M^u) \} \\ &= \inf_{u \in U} \{ -\sup (K \otimes_R M^u) \} . \end{aligned}$$

Similarly one gets the second equality: By 3.1.30 the functor $K \otimes_R -$ also commutes with products; this explains the first equality below and the second holds by 3.1.23.

$$\begin{aligned} -\sup \big(K \otimes_R \prod_{u \in U} M^u \big) &= -\sup \big(\prod_{u \in U} (K \otimes_R M^u) \big) \\ &= -\sup_{u \in U} \{ \sup (K \otimes_R M^u) \} \\ &= \inf_{u \in U} \{ -\sup (K \otimes_R M^u) \} . \end{aligned}$$ □

14.3.15 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and F a faithfully flat R-module. There is an equality,*

$$\mathfrak{a}\text{-depth}_R (F \otimes_R M) = \mathfrak{a}\text{-depth}_R\, M .$$

In particular, one has $\mathfrak{a}\text{-depth}_R\, F = \mathfrak{a}\text{-depth}\, R$.

Proof. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence that generates $\mathfrak{a}$. The 1st and 4th equalities below hold by 14.3.10, the 2nd follows from associativity 12.1.8 and commutativity 12.1.7, and the 3rd holds by 2.5.7(c).

$$\begin{aligned} \mathfrak{a}\text{-depth}_R (F \otimes_R M) &= n - \sup (\mathrm{K}^R(\boldsymbol{x}) \otimes_R (F \otimes_R M)) \\ &= n - \sup (F \otimes_R (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)) \\ &= n - \sup (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \\ &= \mathfrak{a}\text{-depth}_R\, M . \end{aligned}$$ □

14.3.16 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. There is an inequality*

$$\mathfrak{a}\text{-depth}_R M \geqslant -\sup M\,,$$

and the following assertions hold.

(a) *If M is derived $\mathfrak{a}$-torsion, then one has $\mathfrak{a}\text{-depth}_R M = -\sup M$.*

(b) *If M is not acyclic and belongs to $\mathcal{D}_{\sqsubset}(R)$ with $s = \sup M$, then the following conditions are equivalent.*

  (i) $\mathfrak{a}\text{-depth}_R M = -\sup M$.

  (ii) $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s(M)) \neq 0$ *i.e.* $(0 :_{\mathrm{H}_s(M)} \mathfrak{a}) \neq 0$.

  (iii) $\Gamma_{\mathfrak{a}}(\mathrm{H}_s(M)) \neq 0$.

  (iv) $\mathfrak{a}$ *is contained in a prime ideal* $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(M)$.

Proof. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence that generates $\mathfrak{a}$. As $\mathrm{K}^R(\boldsymbol{x})$ is semi-free with $\inf \mathrm{K}^R(\boldsymbol{x}) \geqslant 0$, see 11.4.3(c), one gets from 14.3.2 and 7.6.7:

$$(\diamond)\qquad \begin{aligned} \mathfrak{a}\text{-depth}_R M &= n - \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \\ &= -\sup \mathsf{RHom}_R(\mathrm{K}^R(\boldsymbol{x}), M) \\ &\geqslant -\sup M\,. \end{aligned}$$

(a): If M is derived $\mathfrak{a}$-torsion, then 14.3.6 yields

$$\mathfrak{a}\text{-depth}_R M = n - \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = n - (n + \sup M) = -\sup M\,.$$

(b): Assume now that M belongs to $\mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $s = \sup M$. By 11.4.3(a) one has $\mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x})) \cong R/\mathfrak{a}$, so by 7.6.7 equality holds in ($\diamond$) if and only if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_s(M))$ is non-zero. That is, conditions (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is known from 11.2.12. If a non-zero homomorphism $R/\mathfrak{a} \to \mathrm{H}_s(M)$ exists, then there is by C.1 an element $m \neq 0$ in $\mathrm{H}_s(M)$ with $\mathfrak{a} \subseteq (0 :_R m)$. The annihilator $(0 :_R m)$ is contained in a prime ideal in $\operatorname{Ass}_R \mathrm{H}_s(M)$; thus, ($ii$) implies ($iv$). For the converse notice that if $\mathfrak{a}$ is contained in an associated prime ideal $\mathfrak{p} = (0 :_R m)$ of $\mathrm{H}_s(M)$, then the assignment $[1]_{\mathfrak{a}} \mapsto m$ defines a non-zero homomorphism $R/\mathfrak{a} \to \mathrm{H}_s(M)$. □

For a complex M as in part (b) above, the equality $\mathfrak{a}\text{-depth}_R M = -\sup M$ holds precisely if the critical module, $\mathrm{H}_{\sup M}(M)$, has $\mathfrak{a}$-depth zero:

14.3.17 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-module. The next conditions are equivalent.*

(i) $\mathfrak{a}\text{-depth}_R M = 0$.

(ii) $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$ *i.e.* $(0 :_M \mathfrak{a}) \neq 0$.

(iii) $\Gamma_{\mathfrak{a}}(M) \neq 0$.

(iv) $\mathfrak{a}$ *is contained in a prime ideal* $\mathfrak{p} \in \operatorname{Ass}_R M$.

Proof. The zero module has infinite $\mathfrak{a}$-depth and no associated prime ideals. The equivalence of the four conditions now follows from 14.3.16(b). □

REMARK. The $\mathfrak{a}$-depth of a complex is not determined by the $\mathfrak{a}$-depth of its homology modules, not even for a complex with finitely generated homology modules over a local ring with maximal ideal $\mathfrak{a}$; see Iyengar [143]. From this perspective the depth invariant as defined in 14.3.10 is qualitatively different from the Krull dimension 14.2.1. However, 14.3.10 is the definition that allows for extension to complexes of classic equalities for modules that involve the depth invariant, such as the Auslander–Buchsbaum Formula 16.4.2.

The next result is supplemented by 17.6.8 which expresses the $\mathfrak{a}\mathfrak{b}$- and $(\mathfrak{a} \cap \mathfrak{b})$-depth of a complex in terms of its $\mathfrak{a}$- and $\mathfrak{b}$-depth.

14.3.18 Proposition. *Let* $\mathfrak{b} \subseteq \mathfrak{a}$ *be ideals in* R *and* M *an* R*-complex*; *one has*

$$\mathfrak{b}\text{-depth}_R M \;\leqslant\; \mathfrak{a}\text{-depth}_R M \,.$$

PROOF. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence that generates $\mathfrak{a}$ such that the subsequence $\boldsymbol{x}' = x_1, \dots, x_m$ generates $\mathfrak{b}$. In the next computation, the equalities hold by definition and the inequality follows, by induction on the quantity $n-m$, from the last inequality in 14.3.3(a).

$$\begin{aligned}
\mathfrak{b}\text{-depth}_R M &= m - \sup\left(\mathrm{K}^R(\boldsymbol{x}') \otimes_R M\right) \\
&\leqslant m - \left(m - n + \sup\left(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M\right)\right) \\
&= \mathfrak{a}\text{-depth}_R M \,.
\end{aligned}$$ □

14.3.19 Proposition. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* S *an* R*-algebra, and* M *an* S*-complex. There is an equality,*

$$\mathfrak{a}\text{-depth}_R M \;=\; \mathfrak{a}S\text{-depth}_S M \,.$$

PROOF. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence that generates $\mathfrak{a}$. The image of $\boldsymbol{x}$ in S generates the ideal $\mathfrak{a}S$. Now 12.1.18 and 11.4.18 yield

$$\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \;\cong\; (S \otimes_R \mathrm{K}^R(\boldsymbol{x})) \otimes_S M \;\cong\; \mathrm{K}^S(\boldsymbol{x}) \otimes_S M \,,$$

and the desired equality now follows from 14.3.10. □

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

14.3.20 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* $M' \to M \to M'' \to \Sigma M'$ *a distinguished triangle in* $\mathcal{D}(R)$*. With the notation* $d' = \mathfrak{a}\text{-depth}_R M'$, $d = \mathfrak{a}\text{-depth}_R M$, *and* $d'' = \mathfrak{a}\text{-depth}_R M''$ *there are inequalities,*

$$d' \;\geqslant\; \min\{d, d''+1\}\,, \quad d \;\geqslant\; \min\{d', d''\}\,, \quad \textit{and} \quad d'' \;\geqslant\; \min\{d'-1, d\}\,.$$

PROOF. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$; it is a semi-free R-complex, see 11.4.3(c), and hence one has $K \otimes_R - = K \otimes^{\mathsf{L}}_R -$. Application of this functor to $M' \to M \to M'' \to \Sigma M'$ yields a distinguished triangle,

$$K \otimes_R M' \longrightarrow K \otimes_R M \longrightarrow K \otimes_R M'' \longrightarrow \Sigma(K \otimes_R M') \,.$$

The inequalities now follow from the definition, 14.3.10, in view of 6.5.20. □

WIDTH

In view of 14.3.9 one can make the following definition.

14.3.21 Definition. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x} = x_1, \ldots, x_n$. For an R-complex M the $\mathfrak{a}$-*width* is defined as

$$\mathfrak{a}\text{-width}_R\, M \;=\; \inf\,(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\;.$$

One says that $\mathfrak{a}\text{-width}_R\, M$ is *finite* if $\mathfrak{a}\text{-width}_R\, M < \infty$ holds.

The convention that a complex of $\mathfrak{a}$-width equal to $-\infty$ has finite $\mathfrak{a}$-width aligns with the convetion for homological dimensions.

14.3.22. Let $\mathfrak{a}$ and M be as in 14.3.21. Notice that one has

$$\mathfrak{a}\text{-width}_R\, \Sigma^s M \;=\; \mathfrak{a}\text{-width}_R\, M + s \quad \text{for every integer } s\;.$$

Moreover, one has $\mathfrak{a}\text{-width}_R\, M = \infty$ if M is acyclic.

14.3.23. For an R-complex M one has $0\text{-width}_R\, M = \inf M$ and $R\text{-width}_R\, M = \infty$.

14.3.24 Example. Let $\mathfrak{p}$ be a prime ideal in R. For every ideal $\mathfrak{a}$ not contained in $\mathfrak{p}$ one has $\mathfrak{a}\text{-width}_R\, \mathrm{E}_R(R/\mathfrak{p}) = \infty$; see 14.3.13. In particular, one has $\mathfrak{a}\text{-width}_{\mathbb{Z}}\, \mathbb{Q} = \infty$ for every non-zero ideal $\mathfrak{a}$ in $\mathbb{Z}$, see B.15.

14.3.25 Proposition. *Let* $\{M^u\}_{u\in U}$ *be a family of* R*-complexes*; *one has*

$$\mathfrak{a}\text{-width}_R \Big(\coprod_{u\in U} M^u\Big) \;=\; \inf_{u\in U}\{\mathfrak{a}\text{-width}_R\, M^u\} \;=\; \mathfrak{a}\text{-width}_R \Big(\prod_{u\in U} M^u\Big)\;.$$

PROOF. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$. The functor $K \otimes_R -$ commutes by 3.1.13 with coproducts; this explains the first equality in the next display, and the second holds by 3.1.11. It remains to invoke the definition of $\mathfrak{a}$-width to obtain the first of the asserted equalities.

$$\inf\big(K \otimes_R \coprod_{u\in U} M^u\big) \;=\; \inf\big(\coprod_{u\in U}(K \otimes_R M^u)\big) \;=\; \inf_{u\in U}\{\inf\,(K \otimes_R M^u)\}\;.$$

Similarly one gets the second equality: By 3.1.30 the functor $K \otimes_R -$ also commutes with products; this explains the first equality below and the second holds by 3.1.23.

$$\inf\big(K \otimes_R \prod_{u\in U} M^u\big) \;=\; \inf\big(\prod_{u\in U}(K \otimes_R M^u)\big) \;=\; \inf_{u\in U}\{\inf\,(K \otimes_R M^u)\}\;. \qquad \square$$

14.3.26 Proposition. *Let* $\mathfrak{a} \subseteq R$ *be an ideal,* M *an* R*-complex, and* F *a faithfully flat* R*-module. There is an equality,*

$$\mathfrak{a}\text{-width}_R(F \otimes_R M) \;=\; \mathfrak{a}\text{-width}_R\, M\;.$$

In particular, if $\mathfrak{a}$ *is a proper ideal, then one has* $\mathfrak{a}\text{-width}_R\, F \;=\; 0$.

PROOF. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. The 1st and 4th equalities below hold by 14.3.21, the 2nd follows from associativity 12.1.8 and commutativity 12.1.7, and the 3rd holds by 2.5.7(c).

$$\begin{aligned}\mathfrak{a}\text{-depth}_R(F \otimes_R M) &= \inf\left(\mathrm{K}^R(\boldsymbol{x}) \otimes_R (F \otimes_R M)\right)\\ &= \inf\left(F \otimes_R (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\right)\\ &= \inf\left(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M\right)\\ &= \mathfrak{a}\text{-width}_R M\,.\end{aligned}$$

If $\mathfrak{a}$ is a proper ideal, then $\mathfrak{a}\text{-width}_R F = \inf \mathrm{K}^R(\boldsymbol{x}) = 0$ holds by 11.4.3(a,c). □

14.3.27 Proposition. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. The following conditions are equivalent.*

(*i*) $\mathfrak{a}\text{-depth}_R M$ *is finite.*
(*ii*) $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M$ *is not acyclic.*
(*iii*) $\mathfrak{a}\text{-width}_R M$ *is finite.*

Further, if M is a finitely generated module, then conditions (*i*)–(*iii*) *are equivalent to*

(*iv*) $R/\mathfrak{a} \otimes_R M \neq 0$ *i.e.* $\mathfrak{a}M \neq M$.

PROOF. The equivalence of conditions (*i*)–(*iii*) is immediate from the definitions, 14.3.10 and 14.3.21. Now let M be a finitely generated R-module. It follows from 11.4.4(a) that (*iv*) implies (*ii*). Conversely, the Koszul complex is by 11.4.3(c) and 14.2.3 semi-free with $\mathrm{Supp}_R \mathrm{K}^R(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a})$, so (*ii*) implies in view of 14.2.4 and 14.1.17 that $\mathrm{V}(\mathfrak{a}) \cap \mathrm{Supp}_R M$ is non-empty, which per 14.1.19 implies that the quotient module $M/\mathfrak{a}M$ is non-zero. □

14.3.28 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. There is an inequality*

$$\mathfrak{a}\text{-width}_R M \geqslant \inf M\,,$$

and the following assertions hold.

(a) *If M is derived $\mathfrak{a}$-complete, or if M is in $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathfrak{a}$ is contained in the Jacobson radical of R, then one has $\mathfrak{a}\text{-width}_R M = \inf M$.*

(b) *If M is not acyclic and belongs to $\mathcal{D}_{\sqsupset}(R)$ with $w = \inf M$, then the following conditions are equivalent.*

  (*i*) $\mathfrak{a}\text{-width}_R M = \inf M$.
  (*ii*) $R/\mathfrak{a} \otimes_R \mathrm{H}_w(M) \neq 0$ *i.e.* $\mathfrak{a}\,\mathrm{H}_w(M) \neq \mathrm{H}_w(M)$.
  (*iii*) $\Lambda^{\mathfrak{a}}(\mathrm{H}_w(M)) \neq 0$.
  (*iv*) $\mathrm{H}_0^{\mathfrak{a}}(\mathrm{H}_w(M)) \neq 0$.

  Moreover, if M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then conditions (*i*)–(*iv*) *are equivalent to*

  (*v*) $\mathfrak{a}$ *is contained in a prime ideal* $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{H}_w(M)$.

PROOF. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. As $\mathrm{K}^R(\boldsymbol{x})$ by 11.4.3(c) is semi-free with $\inf \mathrm{K}^R(\boldsymbol{x}) \geqslant 0$, one gets from 7.6.8:

$$(\star)\qquad \mathfrak{a}\text{-width}_R M = \inf\left(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M\right) = \inf\left(\mathrm{K}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R M\right) \geqslant \inf M\,.$$

(a): If M is derived $\mathfrak{a}$-complete, or if M is in $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathfrak{a}$ is contained in the Jacobson radical of R, then equality holds in ($\star$) by 14.3.5.

(b): Assume now that M belongs to $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $w = \inf M$. By 11.4.3(a) one has $\mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x})) \cong R/\mathfrak{a}$, so by 7.6.8 equality holds in ($\star$) if and only if $R/\mathfrak{a} \otimes_R \mathrm{H}_w(M)$ is non-zero. That is, conditions (*i*) and (*ii*) are equivalent; the equivalence of conditions (*ii*)–(*iv*) is already known from 11.1.30 and 11.3.14.

Under the additional assumption that M has degreewise finitely generated homology, conditions (*ii*) and (*v*) are equivalent by 14.1.19. □

For a complex M as in part (b) above, the equality $\mathfrak{a}\text{-width}_R M = \inf M$ holds precisely if the critical module, $\mathrm{H}_{\inf M}(M)$, has $\mathfrak{a}$-width zero:

14.3.29 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-module. The following conditions are equivalent.*

(*i*) $\mathfrak{a}\text{-width}_R M = 0$.
(*ii*) $R/\mathfrak{a} \otimes_R M \neq 0$ *i.e.* $\mathfrak{a}M \neq M$.
(*iii*) $\Lambda^{\mathfrak{a}}(M) \neq 0$.
(*iv*) $\mathrm{H}_0^{\mathfrak{a}}(M) \neq 0$.

Moreover, if M is finitely generated, then conditions (*i*)–(*iv*) *are equivalent to*

(*v*) $\mathfrak{a}$ *is contained in a prime ideal* $\mathfrak{p} \in \mathrm{Supp}_R M$.

Proof. The zero module has infinite $\mathfrak{a}$-width and empty classic support. The equivalence of the conditions now follows from 14.3.28(b). □

The next result is supplemented by 17.6.8 which expresses the $\mathfrak{a}\mathfrak{b}$- and $(\mathfrak{a} \cap \mathfrak{b})$-width of a complex in terms of its $\mathfrak{a}$- and $\mathfrak{b}$-width.

14.3.30 Proposition. *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals in R and M an R-complex*; *one has*

$$\mathfrak{b}\text{-width}_R M \leqslant \mathfrak{a}\text{-width}_R M .$$

Proof. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence that generates $\mathfrak{a}$ such that the subsequence $\boldsymbol{x}' = x_1, \ldots, x_m$ generates $\mathfrak{b}$. In the next computation, the equalities hold by definition and the inequality follows, by induction on the quantity $n - m$, from 14.3.3(c).

$$\begin{aligned} \mathfrak{b}\text{-width}_R M &= \inf (\mathrm{K}^R(\boldsymbol{x}') \otimes_R M) \\ &\leqslant \inf (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \\ &= \mathfrak{a}\text{-width}_R M . \end{aligned}$$ □

14.3.31 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and M an S-complex. There is an equality,*

$$\mathfrak{a}\text{-width}_R M = \mathfrak{a}S\text{-width}_S M .$$

Proof. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. The image of $\boldsymbol{x}$ in S generates the ideal $\mathfrak{a}S$. Now 12.1.18 and 11.4.18 yield

$$\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \cong (S \otimes_R \mathrm{K}^R(\boldsymbol{x})) \otimes_S M \cong \mathrm{K}^S(\boldsymbol{x}) \otimes_S M ,$$

and the desired equality now follows from 14.3.21. □

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

14.3.32 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* $M' \to M \to M'' \to \Sigma M'$ *a distinguished triangle in* $\mathcal{D}(R)$. *With the notation* $w' = \mathfrak{a}\text{-width}_R M'$, $w = \mathfrak{a}\text{-width}_R M$, *and* $w'' = \mathfrak{a}\text{-width}_R M''$ *there are inequalities,*

$$w' \geqslant \min\{w, w'' - 1\}\,, \quad w \geqslant \min\{w', w''\}\,, \quad \textit{and} \quad w'' \geqslant \min\{w' + 1, w\}\,.$$

Proof. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$; it is a semi-free R-complex, see 11.4.3(c), and hence one has $K \otimes_R - = K \otimes^{\mathsf{L}}_R -$. Application of this functor to $M' \to M \to M'' \to \Sigma M'$ yields a distinguished triangle,

$$K \otimes_R M' \longrightarrow K \otimes_R M \longrightarrow K \otimes_R M'' \longrightarrow \Sigma(K \otimes_R M')\,.$$

The inequalities now follow from the definition, 14.3.21, in view of 6.5.20. □

Exercises

E 14.3.1 Let $\boldsymbol{y}$ be a sequence in R and $\mathcal{T}$ a triangulated subcategory of $\mathcal{D}(R)$. Show that for every object $M \in \mathcal{T}$ one has $\mathrm{K}^R(\boldsymbol{y}) \otimes_R M \in \mathcal{T}$.

E 14.3.2 Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. Show that one has $\mathfrak{a}\text{-depth}_R \mathrm{K}^R(\boldsymbol{x}) = -\sup \mathrm{K}^R(\boldsymbol{x})$.

E 14.3.3 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that if M has finite $\mathfrak{a}$-depth, then one has $\mathfrak{a}\text{-depth}_R M = \mathfrak{a}\text{-depth}_R M_{\supseteq n}$ for $n \ll 0$.

E 14.3.4 Let R be an integral domain and $\mathfrak{a} \neq 0$ an ideal in R. Show that $\mathfrak{a}\text{-depth}\, R \geqslant 1$ holds.

E 14.3.5 Let $\mathfrak{a}$ be an ideal in R and M an R-complex of finite $\mathfrak{a}$-depth. Let $\boldsymbol{x}$ and $\boldsymbol{x}'$ be sequences that generate $\mathfrak{a}$ and set $s = \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$ and $s' = \sup(\mathrm{K}^R(\boldsymbol{x}') \otimes_R M)$. Show that the modules $\mathrm{H}_s(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$ and $\mathrm{H}_{s'}(\mathrm{K}^R(\boldsymbol{x}') \otimes_R M)$ are isomorphic.

E 14.3.6 Let $\mathfrak{a}$ be an ideal in R. Show that the R-complexes with $\mathfrak{a}$-depth greater than $-\infty$ form a triangulated subcategory of $\mathcal{D}(R)$.

E 14.3.7 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that $\mathfrak{a}\text{-depth}_R M = -\sup M$ holds under each of the following conditions: (a) One has $\sup M \in \mathbb{Z}$ and $\mathrm{H}_{\sup M}(M)$ is $\mathfrak{a}$-torsion. (b) One has $\sup M = \infty$ and $\mathrm{H}_v(M)$ is $\mathfrak{a}$-torsion for $v \gg 0$.

E 14.3.8 Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x} = x_1, \dots, x_n$, and M an R-complex. Show that if M has finite $\mathfrak{a}$-depth, then one has $\mathfrak{a}\text{-depth}_R M + \mathfrak{a}\text{-width}_R M \leqslant n$.

E 14.3.9 Let $\mathfrak{a}$ be an ideal in R and M an R-complex with $\sup M = \infty$. Show that if M is derived $\mathfrak{a}$-complete, or if M is in $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathfrak{a}$ is contained in the Jacobson radical of R, then $\mathfrak{a}\text{-depth}_R M = -\infty$ holds.

E 14.3.10 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that if M has finite $\mathfrak{a}$-width, then one has $\mathfrak{a}\text{-width}_R M = \mathfrak{a}\text{-width}_R M_{\subseteq n}$ for $n \gg 0$.

E 14.3.11 Let $\mathfrak{a}$ be an ideal in R and M an R-complex with $\inf M = -\infty$. Show that if M is derived $\mathfrak{a}$-torsion, then $\mathfrak{a}\text{-width}_R M = -\infty$ holds.

E 14.3.12 Let $\mathfrak{a}$ be an ideal in R. Show that the R-complexes with $\mathfrak{a}$-width greater than $-\infty$ form a triangulated subcategory of $\mathcal{D}(R)$.

E 14.3.13 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that $\mathfrak{a}\text{-width}_R M = \inf M$ holds under each of the following conditions: (a) One has $\inf M \in \mathbb{Z}$ and $\mathrm{H}_{\inf M}(M)$ is $\mathfrak{a}$-complete. (b) One has $\inf M = -\infty$ and $\mathrm{H}_v(M)$ is $\mathfrak{a}$-complete for $v \ll 0$.

E 14.3.14 By convention, the empty sequence of elements in R generates the zero ideal, and the Koszul complex on the empty sequence is R, cf. 2.1.25. Show that 14.3.10 and 14.3.21 if extended to include the empty sequence would correctly define the 0-depth and 0-width of complexes.

14.4 Depth and Width via Local Cohomology and Homology

Synopsis. Local cohomology $\mathrm{H}_{\mathfrak{a}}$ vs. $\mathfrak{a}$-depth; local homology $\mathrm{H}^{\mathfrak{a}}$ vs. $\mathfrak{a}$-width; finiteness of $\mathfrak{a}$-depth and $\mathfrak{a}$-width; regular sequence; $\mathfrak{a}$-depth vs. maximal length of regular sequence in $\mathfrak{a}$.

An R-complex has the same $\mathfrak{a}$-depth and $\mathfrak{a}$-width as its derived $\mathfrak{a}$-completion and derived $\mathfrak{a}$-torsion complexes.

14.4.1 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. One has*

$$\begin{aligned}\mathfrak{a}\text{-depth}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;&=\; \mathfrak{a}\text{-depth}_R M \;=\; \mathfrak{a}\text{-depth}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \quad \textit{and}\\ \mathfrak{a}\text{-width}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;&=\; \mathfrak{a}\text{-width}_R M \;=\; \mathfrak{a}\text{-width}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,.\end{aligned}$$

Proof. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$. As K is derived $\mathfrak{a}$-torsion, see 13.3.31, there are by 13.4.20(c) isomorphisms,

$$K \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;\simeq\; K \otimes^{\mathsf{L}}_R M \;\simeq\; K \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,.$$

The equalities now follow from the definitions of $\mathfrak{a}$-depth and $\mathfrak{a}$-width. □

The next result is a “true” lemma: It is crucial for the proof of 14.4.3, from which one can derive that the inequality in the lemma actually holds under the weaker assumption, cf. 13.4.9, that the complex K is derived $\mathfrak{a}$-torsion; see 14.4.6.

14.4.2 Lemma. *Let $\mathfrak{a}$ be an ideal in R and K and M be R-complexes. If K is not acyclic, belongs to $\mathcal{D}_{\square}(R)$, and $\mathfrak{a}\,\mathrm{H}(K) = 0$ holds, then there is an inequality,*

$$-\sup \mathsf{R}\mathrm{Hom}_R(K, M) \;\geqslant\; \inf K - \sup \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)\,.$$

Proof. By 7.6.9 one has

$$-\sup \mathsf{R}\mathrm{Hom}_R(K, M) \;\geqslant\; \inf\{-\sup \mathsf{R}\mathrm{Hom}_R(\mathrm{H}_v(K), M) + v \mid v \in \mathbb{Z}\}\,,$$

so it suffices to prove that $-\sup \mathsf{R}\mathrm{Hom}_R(H, M) \geqslant -\sup \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)$ holds for every R-module H with $\mathfrak{a}H = 0$. Such a module is an $R/\mathfrak{a}$-module, so 12.3.36 yields

$$\mathsf{R}\mathrm{Hom}_R(H, M) \;\simeq\; \mathsf{R}\mathrm{Hom}_{R/\mathfrak{a}}(H, \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M))\,.$$

Now 7.6.7 yields the asserted inequality. □

14.4.3 Theorem. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. There are equalities,*

$$\begin{aligned}\mathfrak{a}\text{-depth}_R M \;&=\; -\sup \mathrm{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), M)\\ &=\; -\sup \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)\end{aligned}$$

$$= -\sup \mathsf{R}\Gamma_{\mathfrak{a}}(M) \,.$$

In particular, one has

$$\mathfrak{a}\text{-depth}_R M \;=\; \inf\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(R/\mathfrak{a}, M) \neq 0\} \;=\; \inf\{m \in \mathbb{Z} \mid \mathrm{H}_{\mathfrak{a}}^m(M) \neq 0\} \,.$$

PROOF. Recall from 11.4.3(c) and 11.4.6(a) that $\mathrm{K}^R(\boldsymbol{x})$ is a bounded complex with $\inf \mathrm{K}^R(\boldsymbol{x}) \geqslant 0$ and $\mathfrak{a}\,\mathrm{H}(\mathrm{K}^R(\boldsymbol{x})) = 0$. In the next computation, the first equality is immediate from 14.3.2 and 2.5.5; the second equality holds by 13.3.24 and 13.4.20(b). The first inequality holds by 14.4.2, and the second inequality holds by 7.6.7.

$$\begin{aligned}
\mathfrak{a}\text{-depth}_R M &= -\sup \operatorname{Hom}_R(\mathrm{K}^R(\boldsymbol{x}), M) \\
&\geqslant -\sup \mathsf{R}\!\operatorname{Hom}_R(R/\mathfrak{a}, M) \\
&= -\sup \mathsf{R}\!\operatorname{Hom}_R(R/\mathfrak{a}, \mathsf{R}\Gamma_{\mathfrak{a}}(M)) \\
&\geqslant -\sup \mathsf{R}\Gamma_{\mathfrak{a}}(M) \,.
\end{aligned}$$

Finally, it suffices to recall from 13.4.7, 14.3.16(a) and 14.4.1 that one has

$$-\sup \mathsf{R}\Gamma_{\mathfrak{a}}(M) \;=\; \mathfrak{a}\text{-depth}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \;=\; \mathfrak{a}\text{-depth}_R M \,. \qquad \square$$

14.4.4 Corollary. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R and M an R-complex. If $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ holds, then one has $\mathfrak{a}\text{-depth}_R M = \mathfrak{b}\text{-depth}_R M$.*

PROOF. In view of 13.3.2 the equality follows immediately from 14.4.3. $\square$

14.4.5 Proposition. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. If $d = \mathfrak{a}\text{-depth}_R M$ is an integer, then there are isomorphisms,*

$$\operatorname{Ext}_R^d(R/\mathfrak{a}, M) \;\cong\; \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^d(M)) \;\cong\; \operatorname{Ext}_R^d(\mathrm{K}^R(\boldsymbol{x}), M) \,.$$

PROOF. Let K denote the complex $R/\mathfrak{a}$ or $\mathrm{K}^R(\boldsymbol{x})$. In the computation below, the first isomorphism follows from the definition, 7.3.23, of Ext. As K is derived $\mathfrak{a}$-torsion, see 13.3.24 and 13.3.31, the second isomorphism below follows from 13.4.20(b). Note that one has $\inf K = 0$ and $\mathrm{H}_0(K) \cong R/\mathfrak{a}$ by 11.4.3, and further $\sup \mathsf{R}\Gamma_{\mathfrak{a}}(M) = -d$ and $\mathrm{H}_{-d}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) \;\cong\; \mathrm{H}_{\mathfrak{a}}^d(M)$ by 14.4.3 and 11.3.20. Hence, the third isomorphism below follows from 7.6.7.

$$\begin{aligned}
\operatorname{Ext}_R^d(K, M) &\cong \mathrm{H}_{-d}(\mathsf{R}\!\operatorname{Hom}_R(K, M)) \\
&\cong \mathrm{H}_{-d}(\mathsf{R}\!\operatorname{Hom}_R(K, \mathsf{R}\Gamma_{\mathfrak{a}}(M))) \\
&\cong \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^d(M)) \,. \qquad \square
\end{aligned}$$

14.4.6 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes of finite $\mathfrak{a}$-depth. If M is derived $\mathfrak{a}$-torsion or N is derived $\mathfrak{a}$-complete, then one has*

$$-\sup \mathsf{R}\!\operatorname{Hom}_R(M, N) \;\geqslant\; \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-depth}_R N \,.$$

PROOF. Assume first that M is derived $\mathfrak{a}$-torsion so that $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds. One can assume that M belongs to $\mathcal{D}_{\sqsupset}(R)$, otherwise the inequality is trivial. Similarly, the inequality holds trivially if $\mathfrak{a}\text{-depth}_R N = -\infty$, so one can per 14.4.3 assume that $\mathsf{R}\Gamma_{\mathfrak{a}}(N)$ belongs to $\mathcal{D}_{\sqsubset}(R)$. From 13.4.20(b), 7.6.7, and 14.4.3 one now gets

$$\begin{aligned}-\sup \mathsf{RHom}_R(M,N) &= -\sup \mathsf{RHom}_R(M,\mathsf{R}\Gamma_{\mathfrak{a}}(N))\\ &\geqslant \inf M - \sup \mathsf{R}\Gamma_{\mathfrak{a}}(N)\\ &= \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-depth}_R N\,.\end{aligned}$$

Assume now that the complex N is derived $\mathfrak{a}$-complete. By 13.4.20(a) there is an equality $\sup \mathsf{RHom}_R(M,N) = \sup \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M),N)$, and the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is by 13.4.7 and 14.4.1 derived $\mathfrak{a}$-torsion of finite $\mathfrak{a}$-depth. As $\mathsf{R}\Gamma_{\mathfrak{a}}$ is idempotent, see 13.4.1(b), the asserted equality follows from the one proved above. □

Remark. Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. Under certain assumptions there are inequalities $\mathfrak{a}\text{-depth}_R \mathsf{RHom}_R(M,N) \geqslant -\sup \mathsf{RHom}_R(M,N) \geqslant \inf M - \sup N$, see 14.3.16 and 7.6.7, and $\mathfrak{a}\text{-depth}_R \mathsf{RHom}_R(M,N) \geqslant \mathfrak{a}\text{-width}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-depth}_R N \geqslant \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) - \sup N$, see E 14.4.1, 14.4.1, 14.3.28, and 14.3.16. The inequalitites in 14.4.6 and E 14.4.3 can be seen as hybrids of these inequalities.

Width and Vanishing of Local Homology

The next lemma has a fate similar to 14.4.2: It is crucial for the proof of 14.4.8, from which one can derive that the inequality in the lemma actually holds under the weaker assumption, cf. 13.4.9, that the complex K is derived $\mathfrak{a}$-torsion; see 14.4.11.

14.4.7 Lemma. *Let $\mathfrak{a}$ be an ideal in R and K and M be R-complexes. If K is not acyclic, belongs to $\mathcal{D}_{\square}(R)$, and $\mathfrak{a}\,\mathrm{H}(K) = 0$ holds, then there is an inequality,*

$$\inf (K \otimes^{\mathsf{L}}_R M) \geqslant \inf K + \inf (R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)\,.$$

Proof. By 7.6.10 one has

$$\inf (K \otimes^{\mathsf{L}}_R M) \geqslant \inf\{\inf (\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M) + v \mid v \in \mathbb{Z}\}\,,$$

so it is sufficient to prove that $\inf (H \otimes^{\mathsf{L}}_R M) \geqslant \inf (R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)$ holds for every R-module H with $\mathfrak{a}H = 0$. Such a module is an $R/\mathfrak{a}$-module, so 12.3.31 yields

$$H \otimes^{\mathsf{L}}_R M \simeq H \otimes^{\mathsf{L}}_{R/\mathfrak{a}} (R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)\,.$$

Now 7.6.8 yields the asserted inequality. □

14.4.8 Theorem. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x} = x_1,\dots,x_n$, and M be an R-complex. There are equalities,*

$$\begin{aligned}\mathfrak{a}\text{-width}_R M &= n + \inf \mathrm{Hom}_R(\mathrm{K}^R(\boldsymbol{x}),M)\\ &= \inf (R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)\\ &= \inf \mathsf{L}\Lambda^{\mathfrak{a}}(M)\,.\end{aligned}$$

In particular, one has

$$\mathfrak{a}\text{-width}_R M = \inf\{m \in \mathbb{Z} \mid \mathrm{Tor}^R_m(R/\mathfrak{a},M) \neq 0\} = \inf\{m \in \mathbb{Z} \mid \mathrm{H}^{\mathfrak{a}}_m(M) \neq 0\}\,.$$

PROOF. The first of the asserted equalities is immediate from 14.3.2 and 2.5.5. Recall from 11.4.3(c) and 11.4.6(a) that $\mathrm{K}^R(\boldsymbol{x})$ is a bounded complex with $\inf \mathrm{K}^R(\boldsymbol{x}) \geqslant 0$ and $\mathfrak{a}\,\mathrm{H}(\mathrm{K}^R(\boldsymbol{x})) = 0$. Thus, the first inequality below follows from 14.4.7; the equality holds by 13.3.24 and 13.4.20(c), and the last inequality holds by 7.6.8.

$$\mathfrak{a}\text{-width}_R\, M \;\geqslant\; \inf\,(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) \;=\; \inf\,(R/\mathfrak{a} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \;\geqslant\; \inf \mathsf{L}\Lambda^{\mathfrak{a}}(M)\,.$$

Finally, it suffices to recall from 13.4.2, 14.3.28(a), and 14.4.1 that one has

$$\inf \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;=\; \mathfrak{a}\text{-width}_R\, \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;=\; \mathfrak{a}\text{-width}_R\, M\,. \qquad \square$$

14.4.9 Corollary. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals in* R *and* M *be an* R*-complex. If* $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ *holds, then one has* $\mathfrak{a}\text{-width}_R\, M = \mathfrak{b}\text{-width}_R\, M$.

PROOF. In view of 13.1.3 the equality follows immediately from 14.4.8. $\square$

14.4.10 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R*, generated by a sequence* $\boldsymbol{x}$*, and* M *be an* R*-complex. If* $w = \mathfrak{a}\text{-width}_R\, M$ *is an integer, then there are isomorphisms,*

$$\mathrm{Tor}^R_w(R/\mathfrak{a}, M) \;\cong\; R/\mathfrak{a} \otimes_R \mathrm{H}^{\mathfrak{a}}_w(M) \;\cong\; \mathrm{Tor}^R_w(\mathrm{K}^R(\boldsymbol{x}), M)\,.$$

PROOF. Let K denote the complex $R/\mathfrak{a}$ or $\mathrm{K}^R(\boldsymbol{x})$. In the computation below, the first isomorphism follows from the definition, 7.4.18, of Tor. As K is derived $\mathfrak{a}$-torsion, see 13.3.24 and 13.3.31, the second isomorphism below follows from 13.4.20(c). Note that one has $\inf K = 0$ and $\mathrm{H}_0(K) \cong R/\mathfrak{a}$ by 11.4.3, and further $\inf \mathsf{L}\Lambda^{\mathfrak{a}}(M) = w$ and $\mathrm{H}_w(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \;\cong\; \mathrm{H}^{\mathfrak{a}}_w(M)$ by 14.4.8 and 11.3.6. Hence, the third isomorphism below follows from 7.6.8.

$$\mathrm{Tor}^R_w(K, M) \;\cong\; \mathrm{H}_w(K \otimes^{\mathsf{L}}_R M) \;\cong\; \mathrm{H}_w(K \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \;\cong\; R/\mathfrak{a} \otimes_R \mathrm{H}^{\mathfrak{a}}_w(M)\,. \qquad \square$$

14.4.11 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *and* N *be* R*-complexes of finite* $\mathfrak{a}$*-width. If* M *or* N *is derived* $\mathfrak{a}$*-torsion, then one has*

$$\inf\,(M \otimes^{\mathsf{L}}_R N) \;\geqslant\; \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-width}_R\, N\,.$$

PROOF. Assume first that M is derived $\mathfrak{a}$-torsion, so that $M \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds. One can assume that M belongs to $\mathcal{D}_{\sqsupset}(R)$, otherwise the inequality is trivial. Similarly, the inequality holds trivially if $\mathfrak{a}\text{-width}_R\, N = -\infty$, so one can per 14.4.8 assume that $\mathsf{L}\Lambda^{\mathfrak{a}}(N)$ belongs to $\mathcal{D}_{\sqsupset}(R)$. From 13.4.20(c), 7.6.8, and 14.4.8 one now gets

$$\begin{aligned} \inf\,(M \otimes^{\mathsf{L}}_R N) \;&=\; \inf\,(M \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}}(N)) \\ &\geqslant\; \inf M + \inf \mathsf{L}\Lambda^{\mathfrak{a}}(N) \\ &=\; \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-width}_R\, N\,. \end{aligned}$$

If N is derived $\mathfrak{a}$-torsion, then 13.4.20(c) yields $\inf\,(M \otimes^{\mathsf{L}}_R N) = \inf\,(\mathsf{R}\Gamma_{\mathfrak{a}}(M) \otimes^{\mathsf{L}}_R N)$, and the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is per 13.4.7 and 14.4.1 derived $\mathfrak{a}$-torsion of finite $\mathfrak{a}$-width. As $\mathsf{R}\Gamma_{\mathfrak{a}}$ is idempotent, see 13.4.1(b), the asserted equality follows from the one proved above. $\square$

REMARK. Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. Under certain assumptions there are inequalities $\mathfrak{a}\text{-width}_R(M \otimes^{\mathsf{L}}_R N) \geqslant \inf(M \otimes^{\mathsf{L}}_R N) \geqslant \inf M + \inf N$, see 14.3.28 and 7.6.8, and $\mathfrak{a}\text{-width}_R(M \otimes^{\mathsf{L}}_R N) \geqslant \mathfrak{a}\text{-width}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) + \mathfrak{a}\text{-width}_R N \geqslant \inf M + \inf N$, see E 14.4.2, 14.4.1, and 14.3.28. The inequality in 14.4.11 can be seen as a hybrid of these inequalities.

DEPTH VS. WIDTH

14.4.12 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The following conditions are equivalent.*

(*i*) $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)$ *is not acyclic.*
(*ii*) $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ *is not acyclic.*
(*iii*) $\mathfrak{a}\text{-depth}_R M$ *is finite.*
(*iv*) $R/\mathfrak{a} \otimes^{\mathsf{L}}_R M$ *is not acyclic.*
(*v*) $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ *is not acyclic.*
(*vi*) $\mathfrak{a}\text{-width}_R M$ *is finite.*

PROOF. The equivalence of these conditions is immediate from 14.4.3 and 14.4.8 in view of 14.3.27. □

The next result compares to 11.1.30 and 11.2.12.

14.4.13 Corollary. *Let $\mathfrak{a}$ be an ideal in R.*

(a) *Let F be a semi-flat R-complex. The complex $R/\mathfrak{a} \otimes_R F$ is acyclic if and only if $\Lambda^{\mathfrak{a}}(F)$ is acyclic.*
(b) *Let I be a semi-injective R-complex. The complex $\mathrm{Hom}_R(R/\mathfrak{a}, I)$ is acyclic if and only if $\Gamma_{\mathfrak{a}}(I)$ is acyclic.*

PROOF. Part (a) follows per 13.1.15 from the equivalence of conditions (*iv*) and (*v*) in 14.4.12; part (b) follows from the equivalence of conditions (*i*) and (*ii*). □

14.4.14 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, E a faithfully injective R-module, and M an R-complex. The following equalities hold.*

$$\begin{aligned} \mathfrak{a}\text{-depth}_R M &= \mathfrak{a}\text{-width}_R \mathrm{Hom}_R(M, E) \quad \textit{and} \\ \mathfrak{a}\text{-width}_R M &= \mathfrak{a}\text{-depth}_R \mathrm{Hom}_R(M, E) \,. \end{aligned}$$

PROOF. Let K be the Koszul complex on a sequence that generates $\mathfrak{a}$. By 2.5.7(b), and homomorphism evaluation 12.1.16(c) one has

$$-\sup \mathrm{Hom}_R(K, M) = \inf \mathrm{Hom}_R(\mathrm{Hom}_R(K, M), E) = \inf (K \otimes_R \mathrm{Hom}_R(M, E)) \,.$$

Now the first equality follows from 14.4.3 and the definition, 14.3.21, of $\mathfrak{a}$-width. The second equality follows similarly from 2.5.7(b) and adjunction 12.1.10. □

REGULAR SEQUENCES

14.4.15 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex. For every sequence* $\boldsymbol{x} = x_1, \dots, x_n$ *in* $\mathfrak{a}$ *one has*

$$\begin{aligned} \mathfrak{a}\text{-depth}_R(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) &= \mathfrak{a}\text{-depth}_R M - n \quad \textit{and} \\ \mathfrak{a}\text{-width}_R(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) &= \mathfrak{a}\text{-width}_R M \,. \end{aligned}$$

PROOF. Let $\boldsymbol{y} = y_1, \dots, y_m$ be a sequence that generates $\mathfrak{a}$ and denote by $\boldsymbol{a}$ the concatenated sequence $y_1, \dots, y_m, x_1, \dots, x_n$. Since this sequence also generates $\mathfrak{a}$, the next equalities hold by 14.3.9,

$$(\diamond) \qquad \begin{aligned} \sup(\mathrm{K}^R(\boldsymbol{a}) \otimes_R M) &= \sup(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) + n \quad \text{and} \\ \inf(\mathrm{K}^R(\boldsymbol{a}) \otimes_R M) &= \inf(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) \,. \end{aligned}$$

By 14.3.1 one has $\mathrm{K}^R(\boldsymbol{a}) \otimes_R M \cong \mathrm{K}^R(\boldsymbol{y}) \otimes_R (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$, so the second equality in $(\diamond)$ immediately yields the second of the asserted equalities. The first equality in $(\diamond)$ explains the third equality in the computation below. The remaining equalities hold by the definition of $\mathfrak{a}$-depth and 14.3.1.

$$\begin{aligned} \mathfrak{a}\text{-depth}_R(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) &= m - \sup(\mathrm{K}^R(\boldsymbol{y}) \otimes_R (\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)) \\ &= m - \sup(\mathrm{K}^R(\boldsymbol{a}) \otimes_R M) \\ &= m - \sup(\mathrm{K}^R(\boldsymbol{y}) \otimes_R M) - n \\ &= \mathfrak{a}\text{-depth}_R M - n \,. \end{aligned}$$ □

14.4.16 Definition. Let M be an R-module. An element $x \in R$ is called *M-regular*, or *regular for M*, if the homothety x^M is injective but not surjective. A sequence $\boldsymbol{x} = x_1, \dots, x_n$ in R is called *M-regular*, or *regular for M*, if the element x_1 is M-regular and the element x_i is $M/(x_1, \dots, x_{i-1})M$-regular for every $i \in \{2, \dots, n\}$.

Note that no element is regular for the zero module and 0 is regular for no module.

REMARK. The notion of regular elements for modules extends to complexes [53, 54].

14.4.17 Example. An element in R is R-regular if and only if it is neither a unit nor a zerodivisor. Thus, if R is an integral domain, then every non-unit $x \neq 0$ is R-regular.

14.4.18 Proposition. *Let* $\boldsymbol{x} = x_1, \dots, x_n$ *be a sequence in* R *and* M *an* R*-module. The following conditions are equivalent.*

(*i*) $\boldsymbol{x}$ *is* M*-regular.*

(*ii*) $(\boldsymbol{x})M \neq M$ *holds and for every* $i \in \{1, \dots, n\}$ *there is an isomorphism in* $\mathcal{D}(R)$,

$$\mathrm{K}^R(x_1, \dots, x_i) \otimes_R M \simeq M/(x_1, \dots, x_i)M \,.$$

(*iii*) *One has* $\mathrm{H}_0(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) \neq 0$ *and for every* $i \in \{1, \dots, n\}$ *the* R*-complex* $\mathrm{H}(\mathrm{K}^R(x_1, \dots, x_i) \otimes_R M)$ *is concentrated in degree* 0.

Furthermore, if these conditions are satisfied and $\mathfrak{a}$ *is an ideal in* R *that contains* $\boldsymbol{x}$*, then for every* $i \in \{1, \dots, n\}$ *one has*

$$\mathfrak{a}\text{-depth}_R\, M/(x_1, \dots, x_i)M \;=\; \mathfrak{a}\text{-depth}_R\, M - i\,.$$

Proof. Recall from 11.4.4(a) that for every $i \in \{1, \dots, n\}$ there is an isomorphism,

$$(\star) \qquad \mathrm{H}_0(\mathrm{K}^R(x_1, \dots, x_i) \otimes_R M) \;\cong\; M/(x_1, \dots, x_i)M\,.$$

The equivalence of conditions (*ii*) and (*iii*) follows from these isomorphisms combined with 7.3.29. To prove the equivalence of (*i*) and (*ii*) we proceed by induction on the length, n, of the sequence $\boldsymbol{x}$. If $n = 1$, i.e. $\boldsymbol{x} = x$, then the complex

$$\mathrm{K}^R(x) \otimes_R M \;\cong\; 0 \longrightarrow M \xrightarrow{\;x\;} M \longrightarrow 0$$

is concentrated in degrees 1 and 0 with $\mathrm{H}_0(\mathrm{K}^R(x) \otimes_R M) \cong M/(x)M$, see $(\star)$. Thus, x is M-regular if and only if one has $\mathrm{H}_1(\mathrm{K}^R(x) \otimes_R M) = 0$ and $M/(x)M \neq 0$, which by 7.3.29 is tantamount to $\mathrm{K}^R(x) \otimes_R M \simeq \mathrm{H}_0(\mathrm{K}^R(x) \otimes_R M)$ in $\mathcal{D}(R)$.

Now let $n > 1$ and denote by $\boldsymbol{x}'$ the sequence $x_1, \dots, x_{n-1}$; by 14.3.1 one has

$$(\diamond) \qquad \mathrm{K}^R(\boldsymbol{x}) \otimes_R M \;\cong\; \mathrm{K}^R(x_n) \otimes_R (\mathrm{K}^R(\boldsymbol{x}') \otimes_R M)\,.$$

If $\boldsymbol{x}$ is M-regular, then $\boldsymbol{x}'$ is M-regular, so by the induction hypothesis it suffices to show that one has $(\boldsymbol{x})M \neq M$ and $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \simeq M/(\boldsymbol{x})M$. By $(\diamond)$ and the induction hypothesis there is an isomorphism,

$$\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \;\simeq\; \mathrm{K}^R(x_n) \otimes_R M/(\boldsymbol{x}')M\,,$$

in $\mathcal{D}(R)$, so the desired conclusion follows from the base case as x_n is regular for the module $M/(\boldsymbol{x}')M$.

Conversely, if one has $(\boldsymbol{x})M \neq M$ and $\mathrm{K}^R(x_1, \dots, x_i) \otimes_R M \simeq M/(x_1, \dots, x_i)M$ for all $i \in \{1 \dots, n\}$, then $\boldsymbol{x}'$ is M-regular by the induction hypothesis and x_n is per $(\diamond)$ regular for $M/(\boldsymbol{x}')M$ by the base case. Thus $\boldsymbol{x}$ is M-regular.

The last assertion follows immediately from 14.4.15. □

14.4.19 Corollary. *Let* $\boldsymbol{x}$ *be a sequence in* R*. If* $\boldsymbol{x}$ *is* R*-regular, then the Koszul complex* $\mathrm{K}^R(\boldsymbol{x})$ *is a semi-free replacement of* $R/(\boldsymbol{x})$*; in particular,* $\mathrm{pd}_R\, R/(\boldsymbol{x})$ *is finite.*

Proof. The assertions follow immediately from 11.4.3(c) and 14.4.18. □

Per 14.4.18 factoring out a regular element lowers the depth, but that property does not characterize regular elements.

14.4.20 Example. Let $\Bbbk$ be a field and set $Q = \Bbbk[x_1, x_2]$ and $R = Q/(x_1x_2)$. Consider the complex $K = \mathrm{K}^Q(x_1, x_2) \otimes_Q R = \mathrm{K}^R(x_1, x_2)$, see 11.4.18, where x_1 and x_2 by a standard abuse of notation also denote the cosets in R of the indeterminates. As Q is a unique factorization domain, it is simple to verify that $\mathrm{H}_2(K) = 0$ and $\mathrm{H}_1(K) \neq 0$ hold, see 11.4.2. With $\mathfrak{a} = (x_1, x_2) \subseteq Q$ one thus has $\mathfrak{a}\text{-depth}_Q\, R = 2 - \sup K = 1$. As $x_1^2 \in Q$ annihilates the element x_2 of the Q-module R it is not R-regular. Nevertheless, the $\mathfrak{a}$-depth of $M = R/(x_1^2) = Q/(x_1^2, x_1x_2)$ is zero by 14.3.17 as $x_1 \in (0 :_M \mathfrak{a})$.

Part (b) of the next theorem, as well as the two results that follow, deal with the $\mathfrak{a}$-depth of a finitely generated R-module M. Recall from 14.3.27 that $\mathfrak{a}\text{-depth}_R M$ is finite if and only if $\mathfrak{a}M \neq M$ holds.

14.4.21 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-module.*

(a) *If $x_1, \ldots, x_n$ is an M-regular sequence in $\mathfrak{a}$, then one has $n \leqslant \mathfrak{a}\text{-depth}_R M$.*

(b) *If M is finitely generated and $d = \mathfrak{a}\text{-depth}_R M$ is finite, then there exists an M-regular sequence $x_1, \ldots, x_d$ in $\mathfrak{a}$.*

Proof. Part (a) follows from 14.4.18 and 14.3.16 which yield

$$n \;=\; \mathfrak{a}\text{-depth}_R M - \mathfrak{a}\text{-depth}_R M/(\boldsymbol{x})M \;\leqslant\; \mathfrak{a}\text{-depth}_R M \;.$$

To prove part (b) let M be a finitely generated R-module with $d = \mathfrak{a}\text{-depth}_R M$ finite. It suffices in view of the last assertion in 14.4.18 to show that if d is positive, then there exists an M-regular element in $\mathfrak{a}$. The assumption $d > 0$ implies by 14.3.17 that $\mathfrak{a}$ is not contained in any of the prime ideals in $\operatorname{Ass}_R M$. By Prime Avoidance, one can thus choose an element $x \in \mathfrak{a}$ that does not belong to any of the associated prime ideals. As the annihilator $(0 :_R m)$ of every element $m \neq 0$ in M is contained in an associated prime ideal of M, it follows that x^M is injective. The assumption that d is finite ensures by 14.3.27 that one has $\mathfrak{a}M \neq M$, in particular, x^M is not surjective. □

The next example shows, simultaneously, that the finite generation assumption in part (b) of 14.4.21 is necessary and that a ring R of Krull dimension 2 or more has infinitely many prime ideals $\mathfrak{p}$ with $\dim R_{\mathfrak{p}} = 1$. Prime Avoidance yields a simple direct proof of this last fact; that argument is embedded in the proof of 17.4.21.

14.4.22 Example. Let R be of Krull dimension at least 2 and be $\mathfrak{m}$ a prime ideal in R with $\dim R_{\mathfrak{m}} \geqslant 2$. Set $U = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R_{\mathfrak{p}} \leqslant 1\}$. The R-module $M = \coprod_{\mathfrak{p} \in U} R/\mathfrak{p}$ has $\operatorname{Ass}_R M = U$ and, therefore, $\mathfrak{m}\text{-depth}_R M > 0$, see 14.3.17. For $\mathfrak{p}$ in U with $\mathfrak{p} \subseteq \mathfrak{m}$ one has $\mathfrak{m}(R/\mathfrak{p}) = \mathfrak{m}/\mathfrak{p} \neq R/\mathfrak{p}$, so $R/\mathfrak{p}$ has finite $\mathfrak{m}$-depth by 14.3.27. Thus the $\mathfrak{m}$-depth of M is finite by 14.3.14. Nevertheless, every element $x \in \mathfrak{m}$ belongs by Krull's principal ideal theorem, see also 18.4.19, to a prime ideal $\mathfrak{p} \in U$, so x^M is not injective and x is hence not M-regular.

14.4.23 Definition. Let $\mathfrak{a}$ be an ideal in R and M an R-module. An M-regular sequence in $\mathfrak{a}$ is called *maximal* if it is not part of a longer M-regular sequence in $\mathfrak{a}$.

Note from the proof of the next theorem that, without the assumption that M is finitely generated, conditions (*ii*), (*iii*), and (*iv*) remain equivalent and imply (*i*). The example above shows that the assumption is necessary for the equivalence of (*i*)–(*iv*).

14.4.24 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M a finitely generated R-module. If $\mathfrak{a}\text{-depth}_R M$ is finite, then the following conditions are equivalent for an M-regular sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{a}$.*

(*i*) *$\boldsymbol{x}$ is a maximal M-regular sequence in $\mathfrak{a}$.*

(*ii*) *$\mathfrak{a}$ is contained in a prime ideal $\mathfrak{p} \in \operatorname{Ass}_R(M/(\boldsymbol{x})M)$.*

(*iii*) $\mathfrak{a}\text{-depth}_R(M/(\boldsymbol{x})M) = 0$.

(*iv*) $\mathfrak{a}\text{-depth}_R M = n$.

Proof. The equivalence of conditions (*ii*) and (*iii*) is part of 14.3.17. Since $\boldsymbol{x}$ is M-regular, the last assertion in 14.4.18 yields the equivalence of (*iii*) and (*iv*). If $\boldsymbol{x}$ is not maximal, then it can be extended to a regular sequence of length $n + 1$, so per 14.4.21(a) condition (*iv*) implies (*i*) by contraposition. Similarly (*i*) implies (*iv*): One has $\mathfrak{a}\text{-depth}_R M \geqslant n$ by 14.4.21(a); if equality does not hold, then the last assertion in 14.4.18 yields $\operatorname{depth}_R M/(\boldsymbol{x})M > 0$, so there exists an $M/(\boldsymbol{x})M$-regular element y in $\mathfrak{a}$ by 14.4.21(b). Now $x_1, \dots, x_n, y$ is M-regular, so $\boldsymbol{x}$ is not maximal in $\mathfrak{a}$. □

14.4.25 Corollary. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *a finitely generated* R*-module with* $d = \mathfrak{a}\text{-depth}_R M$ *finite.*

(a) *The maximal length of an* M*-regular sequence in* $\mathfrak{a}$ *is* d.
(b) *There exists an* M*-regular sequence in* $\mathfrak{a}$ *of length* d.
(c) *Every maximal* M*-regular sequence in* $\mathfrak{a}$ *has length* d.

Proof. Parts (a) and (b) constitute a restatement of 14.4.21. Part (c) is an immediate consequence of 14.4.24. □

Remark. Let $\mathfrak{a}$ be an ideal in R. For a finitely generated R-module M the classic definition of the $\mathfrak{a}$-depth of M is the maximal length of an M-regular sequence in $\mathfrak{a}$, so 14.4.25 reconciles the more general definition 14.3.10 with the classical one. Bruns and Herzog [46, 1.2] call the $\mathfrak{a}$-depth of M the 'grade of $\mathfrak{a}$' on M; the Remark after 18.5.17 provides context to that terminology.

14.4.26 Proposition. *Let* $\boldsymbol{x}$ *be a sequence in* R*. Let* S *be an* R*-algebra, flat as an* R*-module, and* M *an* R*-module.*

(a) *If* $\boldsymbol{x}$ *is regular for the* R*-module* M *and* $S \otimes_R M/(\boldsymbol{x})M \neq 0$ *holds, then* $\boldsymbol{x}$ *is regular for the* S*-module* $S \otimes_R M$.
(b) *If* $\boldsymbol{x}$ *is regular for the* S*-module* $S \otimes_R M$ *and* S *is faithfully flat as an* R*-module, then* $\boldsymbol{x}$ *is regular for the* R*-module* M.

Proof. Set $\boldsymbol{x} = x_1, \dots, x_n$. For every $i \in \{1, \dots, n\}$ there is by 11.4.18, 12.1.17, and 12.1.20(b) an isomorphism,

$$\mathrm{H}(\mathrm{K}^S(x_1, \dots, x_i) \otimes_S (S \otimes_R M)) \;\cong\; S \otimes_R \mathrm{H}(\mathrm{K}^R(x_1, \dots, x_i) \otimes_R M)\,.$$

In view of 11.4.4(a) the assertions now follow from 14.4.18. □

14.4.27 Corollary. *Let* M *be a finitely generated* R*-module,* $\boldsymbol{x} = x_1, \dots, x_n$ *an* M*-regular sequence, and* $\mathfrak{p}$ *a prime ideal in* $\operatorname{Supp}_R M$*. If* $\boldsymbol{x}$ *is contained in* $\mathfrak{p}$*, then the sequence* $\boldsymbol{x} = \frac{x_1}{1}, \dots, \frac{x_n}{1}$ *in* $R_\mathfrak{p}$ *is* $M_\mathfrak{p}$*-regular.*

Proof. As $\mathfrak{p} \in \mathrm{V}(\boldsymbol{x})$ it follows from 14.1.18 that $\mathfrak{p}$ is in $\operatorname{Supp}_R(R/(\boldsymbol{x}) \otimes_R M)$, so the module $(R/(\boldsymbol{x}) \otimes_R M)_\mathfrak{p}$ is non-zero. The assertion now follows from 14.4.26(a). □

Finally we record a result that, compared to 14.4.18, simplifies the identification of regular sequences for, among others, finitely generated modules.

14.4.28 Proposition. *Let $\mathfrak{a} \subseteq R$ be an ideal, $\boldsymbol{x} = x_1,\dots,x_n$ a sequence in $\mathfrak{a}$, and $M \neq 0$ an R-module. If M is derived $\mathfrak{a}$-complete or if M is finitely generated and $\mathfrak{a}$ is contained in the Jacobson radical of R, then the next conditions are equivalent.*

(*i*) *$\boldsymbol{x}$ is M-regular.*
(*ii*) $\sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = 0$.
(*iii*) *There is an isomorphism $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \simeq M/(\boldsymbol{x})M$.*

Proof. It is immediate from 14.4.18 that (*i*) implies (*iii*). If M is finitely generated and $\mathfrak{a}$ is contained in the Jacobson radical, then one has $(\boldsymbol{x})M \neq M$ by Nakayama's lemma B.32 and $\mathrm{K}^R(x_1,\dots,x_i) \otimes_R M$ has degreewise finitely generated homology by 14.3.4(a); if M is derived $\mathfrak{a}$-complete, then one has $(\boldsymbol{x})M \neq M$ by 13.1.35 and the complex $\mathrm{K}^R(x_1,\dots,x_i) \otimes_R M$ is derived $\mathfrak{a}$-complete by 14.3.4(c). Thus, (*iii*) evidently implies (*ii*) and it remains to show that (*ii*) implies (*i*). Let $i \in \{1,\dots,n-1\}$. Now 14.3.1 and 14.3.5 yield,

$$\begin{aligned} \sup(\mathrm{K}^R(x_1,\dots,x_{i+1}) \otimes_R M) &= \sup(\mathrm{K}^R(x_{i+1}) \otimes_R (\mathrm{K}^R(x_1,\dots,x_i) \otimes_R M)) \\ &\geqslant \sup(\mathrm{K}^R(x_1,\dots,x_i) \otimes_R M)\ . \end{aligned}$$

Thus, the assumption $\sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = 0$ forces that $\mathrm{H}(\mathrm{K}^R(x_1,\dots,x_i) \otimes_R M)$ is concentrated in degree 0, which by 14.4.18 means that $\boldsymbol{x}$ is M regular. ∎

Exercises

E 14.4.1 Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. Show that if M has finite $\mathfrak{a}$-width and N has finite $\mathfrak{a}$-depth, then the inequality $\mathfrak{a}\text{-depth}_R \mathsf{RHom}_R(M,N) \geqslant \mathfrak{a}\text{-width}_R M + \mathfrak{a}\text{-depth}_R N$ holds.

E 14.4.2 Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. Show that if M and N have finite $\mathfrak{a}$-width, then one has $\mathfrak{a}\text{-width}_R(M \otimes^{\mathsf{L}}_R N) \geqslant \mathfrak{a}\text{-width}_R M + \mathfrak{a}\text{-width}_R N$.

E 14.4.3 Let $\mathfrak{a}$ be an ideal and M and N be R-complexes of finite $\mathfrak{a}$-width. Show that if M is derived $\mathfrak{a}$-torsion or N is derived $\mathfrak{a}$-complete, then one has $-\sup \mathsf{RHom}_R(M,N) \geqslant \mathfrak{a}\text{-width}_R M - \sup \mathsf{L}\Lambda^{\mathfrak{a}}(N)$.

E 14.4.4 Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals in R and M an R-complex. Show that there are equalities $\mathfrak{b}\text{-depth}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \mathfrak{b}\text{-depth}_R M = \mathfrak{b}\text{-depth}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ and $\mathfrak{b}\text{-width}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \mathfrak{b}\text{-width}_R M = \mathfrak{b}\text{-width}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M)$.

E 14.4.5 Let $\mathfrak{a}$ be an ideal in R and M a finitely generated R-module with $d = \mathfrak{a}\text{-depth}_R M$ finite. Show that if $x_1,\dots,x_n$ is an M-regular sequence in $\mathfrak{a}$ and $n < d$, then there exist elements $x_{n+1},\dots,x_d$ such that the concatenated sequence $x_1,\dots,x_d$ is a maximal M-regular sequence in $\mathfrak{a}$.

E 14.4.6 Let $x \in R$ and M be an R-module. Show that if x is R-regular and M-regular, then one has $\mathrm{pd}_{R/(x)} M/(x)M \leqslant \mathrm{pd}_R M$.

E 14.4.7 Let x be an element in R and M an R-module. Show that x is M-regular if and only if $(x)\text{-depth}_R M = 1$ holds.

E 14.4.8 Let $\mathfrak{a} \subseteq R$ be an ideal, $\boldsymbol{x}$ a sequence in $\mathfrak{a}$, and M an R-module. Assume that M is derived $\mathfrak{a}$-complete or that M is finitely generated and $\mathfrak{a}$ is contained in the Jacobson radical of R. Show that if $\boldsymbol{x}$ is M-regular, then every permutation of $\boldsymbol{x}$ is M-regular.

E 14.4.9 Adding to the conventions in E 14.3.14 that the empty sequence is regular for every non-zero module, show that 14.4.18, 14.4.21(a), and 14.4.24 remain valid for the empty sequence.

Chapter 15
Support Theories

The notion of support studied in this chapter agrees with the classic support from Chap. 14 for finitely generated modules but conforms better to derived category methods: Compare, for example, 15.1.15 to 14.1.17. In commutative algebra this notion of support was introduced in the paper [93] from the late 1970s. In the late 1990s the dual notion of cosupport was introduced by Hovey and Strickland [137] in the context of spectra. Its importance in commutative algebra was only recognized later, initially through work of Benson, Iyengar, and Krause [38].

15.1 Support

Synopsis. Residue field; support; ~ vs. classic support; Support Formula; faithfully flat module; localization.

For a local ring R with unique maximal ideal $\mathfrak{m}$, the field $R/\mathfrak{m}$ is called as the residue field. For a prime ideal $\mathfrak{p}$ in R the localization $R_\mathfrak{p}$ of R is a local ring with unique maximal ideal $\mathfrak{p}_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, and it is convenient to have notation for its residue field $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$; notice that this is the field of fractions of the integral domain $R/\mathfrak{p}$.

15.1.1 Definition. Let $\mathfrak{p} \in \operatorname{Spec} R$; the residue field of the local ring $R_\mathfrak{p}$ is denoted $\kappa(\mathfrak{p})$, i.e. $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p}$, and called the *residue field of R at* $\mathfrak{p}$. If R is local with unique maximal ideal $\mathfrak{m}$, then $\kappa(\mathfrak{m}) = R/\mathfrak{m}$ is simply called the *residue field of R*.

15.1.2. Let $\mathfrak{a}$ be an ideal in R. The ideals in $\operatorname{Spec} R/\mathfrak{a}$ have the form $\mathfrak{p}/\mathfrak{a}$ for prime ideals $\mathfrak{p}$ in R with $\mathfrak{a} \subseteq \mathfrak{p}$. For such an ideal, one has $(R/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} \cong R_\mathfrak{p}/\mathfrak{a}_\mathfrak{p}$ and, therefore,

$$\kappa(\mathfrak{p}/\mathfrak{a}) \cong \kappa(\mathfrak{p}) .$$

15.1.3. Let U be a multiplicative subset of R. The ideals in $\operatorname{Spec} U^{-1}R$ have the form $U^{-1}\mathfrak{p}$ for prime ideals $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \emptyset$. For such an ideal, one has $(U^{-1}R)_{U^{-1}\mathfrak{p}} \cong R_\mathfrak{p}$ and, therefore,

$$\kappa(U^{-1}\mathfrak{p}) \cong \kappa(\mathfrak{p}) .$$

15.1.4 Lemma. *Let* $\mathfrak{a}$ *be an ideal and* $\mathfrak{p}$ *a prime ideal in* R. *There are isomorphisms,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(\kappa(\mathfrak{p})) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(\kappa(\mathfrak{p})) \simeq \begin{cases} \kappa(\mathfrak{p}) & \textit{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \textit{if } \mathfrak{a} \not\subseteq \mathfrak{p} . \end{cases}$$

PROOF. Consider the composite ring homomorphism $R \to R/\mathfrak{p} \to \kappa(\mathfrak{p})$ and the extension $\mathfrak{b} = \mathfrak{a}\kappa(\mathfrak{p})$ of the ideal $\mathfrak{a}$ to the field $\kappa(\mathfrak{p})$. By 13.1.21(a), 13.3.23(a), and the fact that $\kappa(\mathfrak{p})$ is both projective and injective as a module over itself, one has

$$\mathsf{L}\Lambda^{\mathfrak{a}}(\kappa(\mathfrak{p})) \simeq \mathsf{L}\Lambda^{\mathfrak{b}}(\kappa(\mathfrak{p})) \simeq \Lambda^{\mathfrak{b}}(\kappa(\mathfrak{p})) \quad \text{and} \quad \mathsf{R}\Gamma_{\mathfrak{a}}(\kappa(\mathfrak{p})) \simeq \mathsf{R}\Gamma_{\mathfrak{b}}(\kappa(\mathfrak{p})) \simeq \Gamma_{\mathfrak{b}}(\kappa(\mathfrak{p})) .$$

As one has $\mathfrak{b} = 0$ if $\mathfrak{a} \subseteq \mathfrak{p}$ and $\mathfrak{b} = \kappa(\mathfrak{p})$ if $\mathfrak{a} \not\subseteq \mathfrak{p}$, the assertion follows from 11.1.5 and 11.2.2. □

SUPPORT

15.1.5 Definition. Let M be an R-complex. The *support* of M is the set

$$\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \neq 0\} .$$

REMARK. In the literature, one can find the set defined in 15.1.5 referred to as the 'small support' since that was the term used in [93]. This terminology is inspired by 15.1.9.

15.1.6 Example. Let F be a faithfully flat R-module. For every prime ideal $\mathfrak{p}$ one has $\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R F \simeq \kappa(\mathfrak{p}) \otimes_R F \neq 0$, so $\operatorname{supp}_R F = \operatorname{Spec} R$. See 15.1.18 for a converse.

As is the case with the classic support, see 14.1.7, the support of a coproduct of complexes is the union of their supports.

15.1.7 Proposition. *Let* $\{M^u\}_{u\in U}$ *be a family of* R*-complexes*; *one has*

$$\operatorname{supp}_R \Big(\coprod_{u\in U} M^u \Big) = \bigcup_{u\in U} \operatorname{supp}_R M^u .$$

PROOF. Homology and the functor $\otimes^{\mathsf{L}}_R$ preserve coproducts by 3.1.10(d) and 7.4.5. The asserted equality now follows straight from the definition of support. □

The support of a product $\prod_{u\in U} M^u$ of R-modules contains by 15.1.7 the support of each module M^u, but it may also include prime ideals not contained in any of the sets $\operatorname{supp}_R M^u$; see 15.1.11 and E 15.1.6.

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

15.1.8 Proposition. *Let* $M' \to M \to M'' \to \Sigma M'$ *be a distinguished triangle in* $\mathcal{D}(R)$. *Any one of the sets* $\operatorname{supp}_R M$, $\operatorname{supp}_R M'$, *and* $\operatorname{supp}_R M''$ *is contained in the union of the two other sets.*

Proof. For every prime ideal $\mathfrak{p}$ in R there is a distinguished triangle,

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M' \longrightarrow \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M \longrightarrow \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M'' \longrightarrow \Sigma(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M')\,,$$

in $\mathcal{D}(R)$. It now follows from 6.5.20 and the definition, 15.1.5, of support that $\mathfrak{p}$ is either in at least two of the three supports or not in any of them. □

Support vs. Classic Support

Next we prove that the support is a subset of the classic support. Among the prime ideals in the classic support of a module, only the associated prime ideals are guaranteed to show up in the support, see 17.1.9. For injective modules they are, indeed, the only prime ideals in the support, see 15.1.14.

15.1.9 Theorem. *Let M be an R-complex. There is an inclusion,*

$$\operatorname{supp}_R M \subseteq \operatorname{Supp}_R M\,,$$

and equality holds if M belongs to $\mathcal{D}^{\mathrm{f}}(R)$.

Proof. The inclusion is immediate as 14.1.16(b) yields

$$(\dagger) \qquad \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M \simeq \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}\,.$$

Assume that M is in $\mathcal{D}^{\mathrm{f}}(R)$ and let $\mathfrak{p} \in \operatorname{Supp}_R M$; the localized complex $M_\mathfrak{p}$ belongs by 12.1.20 to $\mathcal{D}^{\mathrm{f}}(R_\mathfrak{p})$ and is not acyclic. As the ring $R_\mathfrak{p}$ is local with unique maximal ideal $\mathfrak{p}_\mathfrak{p}$, it follows from 14.3.28(a) and the assumption on $\mathfrak{p}$ that $\mathfrak{p}_\mathfrak{p}\text{-width}_{R_\mathfrak{p}} M_\mathfrak{p} = \inf M_\mathfrak{p} < \infty$ holds. Now it follows from 14.4.8 that the complex in $(\dagger)$ is not acyclic, so $\mathfrak{p}$ belongs to $\operatorname{supp}_R M$. □

15.1.10 Example. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. One has

$$\operatorname{supp}_R R/\mathfrak{a} = \operatorname{Supp}_R R/\mathfrak{a} = \mathrm{V}(\mathfrak{a}) = \operatorname{Supp}_R \mathrm{K}^R(\boldsymbol{x}) = \operatorname{supp}_R \mathrm{K}^R(\boldsymbol{x})$$

by 15.1.9 in view of 14.1.5.

15.1.11 Example. Let R be local with unique maximal ideal $\mathfrak{m}$. As the maps in the tower (11.1.2.1) are surjective, 3.5.14 and the definition, 11.1.4, of $\Lambda^{\mathfrak{m}}(R)$ yields an exact sequence,

$$0 \longrightarrow \Lambda^{\mathfrak{m}}(R) \longrightarrow \prod_{u\in\mathbb{N}} R/\mathfrak{m}^u \longrightarrow \prod_{u\in\mathbb{N}} R/\mathfrak{m}^u \longrightarrow 0\,.$$

Now it follows from 15.1.8 that the support of the product $\prod_{u\in\mathbb{N}} R/\mathfrak{m}^u$ contains the support of $\Lambda^{\mathfrak{m}}(R) = \widehat{R}$, which by 15.3.6 and 15.1.6 is all of $\operatorname{Spec} R$. Thus one has $\operatorname{supp}_R(\prod_{u\in\mathbb{N}} R/\mathfrak{m}^u) = \operatorname{Spec} R$. On the other hand, $\operatorname{supp}_R R/\mathfrak{m}^u = \mathrm{V}(\mathfrak{m}^u) = \{\mathfrak{m}\}$ holds for every $u \in \mathbb{N}$ by 15.1.10.

For injective modules the support and classic support can differ widely. The support is, in particular, not specialization closed.

15.1.12 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R; one has*

$$\operatorname{supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \{\mathfrak{p}\} \quad \textit{and} \quad \operatorname{Supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})\,.$$

Proof. The equality $\operatorname{Supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ holds by C.15(b). Injectivity of the module $\mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ and C.21(a) yield an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,

$$\mathsf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))) \cong \kappa(\mathfrak{p})\,.$$

Thus it follows from 14.4.12, applied to the ideal $\mathfrak{p}_\mathfrak{p}$ in the ring $R_\mathfrak{p}$, that the complex $\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ is not acyclic. Now C.18 and 14.1.16(b) yield

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p})) \simeq \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R} \mathrm{E}_{R}(R/\mathfrak{p})$$

whence $\mathfrak{p} \in \operatorname{supp}_R \mathrm{E}_R(R/\mathfrak{p})$. Now, let $\mathfrak{q} \in \operatorname{supp}_R \mathrm{E}_R(\mathfrak{p})$, by 15.1.9 one has $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$, i.e. $\mathfrak{p} \subseteq \mathfrak{q}$. Associativity 12.3.14 yields $\kappa(\mathfrak{q}) \otimes^{\mathsf{L}}_R \mathrm{E}_R(R/\mathfrak{p}) \simeq (R/\mathfrak{q} \otimes^{\mathsf{L}}_R \mathrm{E}_R(R/\mathfrak{p}))_\mathfrak{q}$, so one has $\mathrm{H}(R/\mathfrak{q} \otimes^{\mathsf{L}}_R \mathrm{E}_R(R/\mathfrak{p})) \neq 0$ and hence $\mathrm{H}(\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{q}, \mathrm{E}_R(R/\mathfrak{p}))) \neq 0$ by 14.4.12. As $\mathrm{E}_R(R/\mathfrak{p})$ is injective this means that the module $\mathrm{Hom}_R(R/\mathfrak{q}, \mathrm{E}_R(R/\mathfrak{p}))$ is non-zero, so the inclusion $\mathfrak{q} \subseteq \mathfrak{p}$ holds by C.15(c). Thus one has $\mathfrak{q} = \mathfrak{p}$. □

15.1.13 Example. Per B.15 one has $\mathrm{E}_\mathbb{Z}(\mathbb{Z}) = \mathbb{Q}$, so 15.1.12 yields

$$\operatorname{supp}_\mathbb{Z} \mathbb{Q} = \{0\} \quad \text{and} \quad \operatorname{Supp}_\mathbb{Z} \mathbb{Q} = \operatorname{Spec} \mathbb{Z}\,.$$

Further, for every prime p one has $\mathrm{E}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$ and hence

$$\operatorname{supp}_\mathbb{Z} \mathbb{Z}(p^\infty) = \{p\mathbb{Z}\} = \operatorname{Supp}_\mathbb{Z} \mathbb{Z}(p^\infty)\,.$$

To parse the next result recall Matlis' structure theorem C.23.

15.1.14 Proposition. *Let I be an injective R-module; one has*

$$\operatorname{supp}_R I = \operatorname{Ass}_R I = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{E}_R(R/\mathfrak{p}) \text{ is a direct summand of } I\}\,.$$

Proof. For $\mathfrak{p} \in \operatorname{Spec} R$ one has $\operatorname{supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \{\mathfrak{p}\} = \operatorname{Ass}_R \mathrm{E}_R(R/\mathfrak{p})$ by 15.1.12 and C.15(a). By Matlis' structure theorem C.23 an injective R-module is a coproduct of such indecomposable modules, so the claim follows from 15.1.7 and the corresponding result for associated prime ideals. □

Support Formula

15.1.15 Theorem. *An R-complex M is acyclic if and only if* $\operatorname{supp}_R M = \emptyset$ *holds.*

Proof. If M is acyclic, then it follows from the definition that $\operatorname{supp}_R M$ is empty. For the converse, assume that M is not acyclic. Considered as a functor on the module category, $\mathrm{F} = \coprod_{m\in\mathbb{Z}} \operatorname{Tor}^R_m(-, M)$ is half exact by 7.4.29 and 3.1.6. The unitor 12.3.3 yields $\mathrm{F}(R) = \coprod_{m\in\mathbb{Z}} \mathrm{H}_m(M)$, which is non-zero, so it follows from 12.4.2 that there is a prime ideal $\mathfrak{p}$ in R with $\mathrm{F}(R/\mathfrak{p}) \neq 0$ such that $\mathrm{F}(R/\mathfrak{b}) = 0$ holds for every ideal $\mathfrak{b} \supset \mathfrak{p}$. Let $r \in R \setminus \mathfrak{p}$; the ideal $\mathfrak{p} + (r)$ is strictly larger that $\mathfrak{p}$, so the complex

$R/(\mathfrak{p}+(r)) \otimes^{\mathsf{L}}_R M$ is acyclic. Application of $- \otimes^{\mathsf{L}}_R M$ to the triangle induced per 6.5.24 by the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;r\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}+(r)) \longrightarrow 0$$

shows in view of 6.5.21 that multiplication by r on $\mathrm{H}(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)$ is an isomorphism. The residue field $\kappa(\mathfrak{p})$ is the field of fractions of the integral domain $R/\mathfrak{p}$; in particular, it is a flat $R/\mathfrak{p}$-module, see 1.3.42. Now 12.3.31 and 7.6.11(b) yield

$$\mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \;\cong\; \mathrm{H}(\kappa(\mathfrak{p}) \otimes_{R/\mathfrak{p}} (R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)) \;\cong\; \mathrm{H}(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)\,.$$

It follows that $\mathfrak{p}$ belongs to $\operatorname{supp}_R M$; in particular $\operatorname{supp}_R M \neq \emptyset$. □

The next result is known as the Support Formula.

15.1.16 Theorem. *Let M and N be R-complexes*; *there is an equality,*

$$\operatorname{supp}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{supp}_R M \cap \operatorname{supp}_R N\,.$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$; by 12.3.30 and 7.6.12 there are isomorphisms,

$$\begin{aligned}\mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N)) \;&\cong\; \mathrm{H}((\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R N))\\ &\cong\; \mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \otimes_{\kappa(\mathfrak{p})} \mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R N)\,.\end{aligned}$$

The last complex is a tensor product of $\kappa(\mathfrak{p})$-vector spaces and hence non-zero if and only if both spaces are non-zero. Now invoke the definition of support. □

15.1.17 Corollary. *Let M and N be complexes in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$*; *there is an equality,*

$$\operatorname{Supp}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{Supp}_R M \cap \operatorname{Supp}_R N\,.$$

Proof. In view of 12.2.12, the equality follows from 15.1.9 and 15.1.16. □

15.1.18 Proposition. *Let F be a flat R-module*; *the next conditions are equivalent.*

(*i*) *F is faithfully flat.*
(*ii*) $\operatorname{supp}_R F = \operatorname{Spec} R$.
(*iii*) $\operatorname{Max} R \subseteq \operatorname{supp}_R F$.

Proof. As noted in 15.1.6, condition (*i*) implies (*ii*). Conversely, if (*ii*) holds, then the Support Formula 15.1.16 yields $\operatorname{supp}_R(F \otimes_R M) = \operatorname{supp}_R M$ for every R-module M. Hence $F \otimes_R M = 0$ implies $M = 0$ by 15.1.15, so F is faithfully flat. The implication (*ii*) $\Rightarrow$ (*iii*) is trivial. We prove that (*iii*) implies (*ii*) by contraposition. Let $\mathfrak{p}$ be a prime ideal in R that does not belong to $\operatorname{supp}_R F$, that is, the module $F \otimes_R \kappa(\mathfrak{p}) \simeq F \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p})$ is zero. The embedding $R/\mathfrak{p} \rightarrowtail \kappa(\mathfrak{p})$ induces an exact sequence $0 \to F \otimes_R R/\mathfrak{p} \to F \otimes_R \kappa(\mathfrak{p})$, which yields $F \otimes_R R/\mathfrak{p} = 0$. Let $\mathfrak{m}$ be a maximal ideal in R that contains $\mathfrak{p}$. The canonical homomorphism $R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m}$ yields an exact sequence $F \otimes_R R/\mathfrak{p} \to F \otimes_R R/\mathfrak{m} \to 0$, which shows that the module $F \otimes_R R/\mathfrak{m} \simeq F \otimes^{\mathsf{L}}_R \kappa(\mathfrak{m})$ is zero. Thus, $\mathfrak{m}$ does not belong to $\operatorname{supp}_R F$. □

LOCALIZATION

15.1.19 Lemma. *Let U be a multiplicative subset of R; one has*

$$\operatorname{supp}_R U^{-1}R \;=\; \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap U = \emptyset\}\,.$$

PROOF. Let $\mathfrak{p} \in \operatorname{Spec} R$; one has $U^{-1}R \otimes_R R/\mathfrak{p} \neq 0$ if and only if $\mathfrak{p} \cap U = \emptyset$, and in that case one has $R_\mathfrak{p} \otimes_R U^{-1}R \cong R_\mathfrak{p}$. Flatness of $R_\mathfrak{p}$ and $U^{-1}R$, see 1.3.42, together with commutativity 12.1.7 and associativity 12.1.8 yields,

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R U^{-1}R \;\simeq\; R_\mathfrak{p} \otimes_R (U^{-1}R \otimes_R R/\mathfrak{p}) \;\cong\; \begin{cases} \kappa(\mathfrak{p}) & \text{if } \;\mathfrak{p} \cap U = \emptyset \\ \;\;0 & \text{if } \;\mathfrak{p} \cap U \neq \emptyset\,. \end{cases} \qquad \square$$

Here is another simple example of the support not being specialization closed.

15.1.20 Example. For every $\mathfrak{p} \in \operatorname{Spec} R$ one has $\operatorname{supp}_R R_\mathfrak{p} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.

The fact that the support is not specialization closed may make it harder to picture than the classic support. On the upside it allows for the equality in 15.1.19 which fails for the classic support, see E 12.4.8.

15.1.21 Proposition. *Let U be a multiplicative subset of R and M an R-complex. There is an equality,*

$$\operatorname{supp}_R U^{-1}M \;=\; \{\mathfrak{p} \in \operatorname{supp}_R M \mid \mathfrak{p} \cap U = \emptyset\}\,.$$

For every prime ideal $\mathfrak{p}$ with $\mathfrak{p} \cap U = \emptyset$ there is an isomorphism in $\mathcal{D}(U^{-1}R)$,

$$\kappa(U^{-1}\mathfrak{p}) \otimes^{\mathsf{L}}_{U^{-1}R} U^{-1}M \;\simeq\; \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M\,.$$

The assignment $\mathfrak{p} \leftrightarrow U^{-1}\mathfrak{p}$ yields an order preserving one-to-one correspondence,

$$\operatorname{supp}_R U^{-1}M \;\longleftrightarrow\; \operatorname{supp}_{U^{-1}R} U^{-1}M\,.$$

In particular, a prime ideal $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \emptyset$ belongs to $\operatorname{supp}_R M$ if and only if $U^{-1}\mathfrak{p}$ belongs to $\operatorname{supp}_{U^{-1}R} U^{-1}M$.

PROOF. By flatness of $U^{-1}R$ one has $U^{-1}M \simeq U^{-1}R \otimes^{\mathsf{L}}_R M$, so 15.1.19 and the Support Formula 15.1.16 yield $\operatorname{supp}_R U^{-1}M = \{\mathfrak{p} \in \operatorname{supp}_R M \mid \mathfrak{p} \cap U = \emptyset\}$. It follows from 12.3.31 and 15.1.3 that for every prime ideal $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \emptyset$ one has,

$$\kappa(U^{-1}\mathfrak{p}) \otimes^{\mathsf{L}}_{U^{-1}R} U^{-1}M \;\simeq\; \kappa(U^{-1}\mathfrak{p}) \otimes^{\mathsf{L}}_{U^{-1}R} (U^{-1}R \otimes^{\mathsf{L}}_R M) \;\simeq\; \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M\,.$$

The asserted one-to-one correspondence is thus a restriction of the order preserving one-to-one correspondence between $\operatorname{Spec} U^{-1}R$ and $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap U = \emptyset\}$. $\square$

15.1.22 Corollary. *Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. One has*

$$\operatorname{supp}_R M_\mathfrak{p} \;=\; \{\mathfrak{q} \in \operatorname{supp}_R M \mid \mathfrak{q} \subseteq \mathfrak{p}\}\,.$$

For every prime ideal $\mathfrak{q}$ contained in $\mathfrak{p}$ there is an isomorphism in $\mathcal{D}(R_\mathfrak{p})$,

$$\kappa(\mathfrak{q}_\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} M_\mathfrak{p} \;\simeq\; \kappa(\mathfrak{q}) \otimes^{\mathsf{L}}_R M\,.$$

The assignment $\mathfrak{q} \leftrightarrow \mathfrak{q}_\mathfrak{p}$ *yields an order preserving one-to-one correspondence,*

$$\operatorname{supp}_R M_\mathfrak{p} \longleftrightarrow \operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p} .$$

In particular, a prime ideal $\mathfrak{q}$ *contained in* $\mathfrak{p}$ *belongs to* $\operatorname{supp}_R M$ *if and only if* $\mathfrak{q}_\mathfrak{p}$ *belongs to* $\operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p}$.

Proof. This is the special case $U = R \setminus \mathfrak{p}$ of 15.1.21. □

15.1.23 Example. For every $\mathfrak{p} \in \operatorname{Spec} R$ one has $\operatorname{supp}_R R/\mathfrak{p} = \mathrm{V}(\mathfrak{p})$ by 15.1.10 and hence $\operatorname{supp}_R \kappa(\mathfrak{p}) = \{\mathfrak{p}\}$ by 15.1.1 and 15.1.22.

15.1.24 Example. Let $\mathfrak{p}$ be a prime ideal in R and F a faithfully flat $R_\mathfrak{p}$-module. Idempotence of localization, see 14.1.14(a), together with 15.1.18, and 15.1.22 yields

$$\operatorname{supp}_R F = \operatorname{supp}_R F_\mathfrak{p} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\} .$$

Notice that this subsumes 15.1.20.

Recall from 15.1.9 that the support of a complex is a subset of its classic support. The next result gives more details, see also 17.1.8.

15.1.25 Proposition. *Let M be an R-complex. Every prime ideal in* $\operatorname{Supp}_R M$ *contains one from* $\operatorname{supp}_R M$. *In particular, the two sets have the same minimal elements.*

Proof. Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Supp}_R M$. As the R-complex $M_\mathfrak{p}$ is not acyclic, see 14.1.12, there is by 15.1.15 and 15.1.22 a prime ideal $\mathfrak{q}$ in $\operatorname{supp}_R M$ with $\mathfrak{q} \subseteq \mathfrak{p}$. The last assertion is now immediate in view of 15.1.9. □

Derived Torsion Complexes

15.1.26 Lemma. *Let $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ a prime ideal in R; let M be an R-complex. There is an isomorphism,*

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_\mathfrak{a}(M) \simeq \begin{cases} \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p} . \end{cases}$$

Proof. The next isomorphism follows from two applications of 13.3.19 combined with commutativity 12.3.5,

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_\mathfrak{a}(M) \simeq \mathsf{R}\Gamma_\mathfrak{a}(\kappa(\mathfrak{p})) \otimes^{\mathsf{L}}_R M .$$

Now the assertions follow from 15.1.4. □

Given an ideal $\mathfrak{a}$ in R and an R-complex M, the next result shows that the support of the complex $\mathsf{R}\Gamma_\mathfrak{a}(M)$ is contained in $\mathrm{V}(\mathfrak{a})$; in fact, this characterizes derived $\mathfrak{a}$-torsion complexes, see 15.3.23.

15.1.27 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. One has*

$$\operatorname{supp}_R \mathsf{R}\Gamma_\mathfrak{a}(M) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M = \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R \mathsf{L}\Lambda^\mathfrak{a}(M) .$$

In particular, $\operatorname{supp}_R \check{\mathrm{C}}^R(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a})$ *holds for every sequence* $\boldsymbol{x}$ *that generates* $\mathfrak{a}$.

Proof. Per the definition of support, 15.1.5, the first equality is an immediate consequence of 15.1.26. The second equality follows by 13.4.1(d) from the first equality applied to the complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$. In view of 13.3.18, the final assertion follows from the first equality applied to $M = R$. □

For a faithfully flat R-module F, it follows from 15.1.27, in view of 15.1.18, that $\operatorname{supp}_R \mathsf{R}\Gamma_{\mathfrak{a}}(F) = \mathrm{V}(\mathfrak{a})$ holds. The support of $\mathsf{L}\Lambda^{\mathfrak{a}}(F) \simeq \Lambda^{\mathfrak{a}}(F)$ is more elusive, but some information about this set is provided by 15.3.5.

Exercises

E 15.1.1 Let $\mathfrak{p}$ be a prime ideal in R. Show that $\operatorname{Supp}_R \kappa(\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$ holds.

E 15.1.2 For $M \in \mathcal{D}(R)$ show that $\operatorname{supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathsf{R}\mathrm{Hom}_R(M, \kappa(\mathfrak{p}))) \neq 0\}$.

E 15.1.3 Let M be an R-complex, $\mathfrak{a}$ an ideal, and $\mathfrak{p}$ a prime ideal in R. Show that one has

$$\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(M), \kappa(\mathfrak{p})) \simeq \begin{cases} \mathsf{R}\mathrm{Hom}_R(M, \kappa(\mathfrak{p})) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p}\,. \end{cases}$$

E 15.1.4 Let M be an R-complex. Show that for every prime ideal $\mathfrak{p}$ in $\operatorname{supp}_R M$ there is an integer n such that $\mathfrak{p} \in \operatorname{supp}_R M_{\subseteq n}$.

E 15.1.5 Let M be an R-complex. Show that for every prime ideal $\mathfrak{p}$ in $\operatorname{supp}_R M$ there is an integer n such that $\mathfrak{p} \in \operatorname{supp}_R M_{\supseteq n}$. *Hint:* E 3.3.11.

E 15.1.6 Let R be local with unique maximal ideal $\mathfrak{m}$ and set $E = \mathrm{E}_R(R/\mathfrak{m})$. Assuming that R is not Artinian, show that $E^{\mathbb{N}}$ contains an element that is not $\mathfrak{m}$-torsion and conclude that $\mathrm{E}_R(R/\mathfrak{p})$ is a direct summand of $E^{\mathbb{N}}$ for some $\mathfrak{p} \neq \mathfrak{m}$ and, therefore, $\mathfrak{p} \in \operatorname{supp}_R E^{\mathbb{N}}$.

E 15.1.7 Let M, X, and N be R-complexes with $\operatorname{supp}_R N \subseteq \operatorname{supp}_R M$. (a) Show that if $M \otimes^{\mathsf{L}}_R X$ is acyclic, then $N \otimes^{\mathsf{L}}_R X$ is acyclic. (b) Show that if $\mathsf{R}\mathrm{Hom}_R(M, X)$ is acyclic, then $\mathsf{R}\mathrm{Hom}_R(N, X)$ is acyclic.

E 15.1.8 Let R be local. Show that $\operatorname{supp}_R M = \operatorname{Supp}_R M$ holds for $M \in \mathcal{D}^{\mathrm{art}}(R)$.

E 15.1.9 Let R be an integral domain with field of fractions Q. Show that one has $\operatorname{supp}_R Q = \{0\}$ and $\operatorname{Supp}_R Q = \operatorname{Spec} R$.

15.2 Cosupport

Synopsis. Cosupport; Cosupport Formula; faithfully injective module; colocalization; maximal elements of (co)support.

The support is defined by way of the functors $\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R -$ but could per E 15.1.2 as well be defined in terms of $\mathsf{R}\mathrm{Hom}_R(-, \kappa(\mathfrak{p}))$. Here we consider the dual notion.

15.2.1 Definition. Let M be an R-complex. The *cosupport* of M is the set

$$\operatorname{cosupp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathsf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), M)) \neq 0\}\,.$$

15.2.2 Example. Let E be a faithfully injective R-module. For every prime ideal $\mathfrak{p}$ in R one has $\mathsf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), E) \simeq \mathrm{Hom}_R(\kappa(\mathfrak{p}), E) \neq 0$, so $\operatorname{cosupp}_R E = \operatorname{Spec} R$ holds; see 15.2.11 for a converse. In particular, 1.3.35 yields $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = \operatorname{Spec} \mathbb{Z}$.

15.2.3 Proposition. *Let $\{M^u\}_{u\in U}$ be a family of R-complexes*; *one has*

$$\operatorname{cosupp}_R\Big(\prod_{u\in U} M^u\Big) = \bigcup_{u\in U}\operatorname{cosupp}_R M^u\,.$$

PROOF. Homology and the functor RHom_R preserve products by 3.1.22(d) and 7.3.6. The asserted equality now follows straight from the definition of cosupport. □

The cosupport of a coproduct $\coprod_{u\in U} M^u$ of R-modules contains by 15.2.3 the cosupport of each module M^u, but it may also contain prime ideals not included in any of the sets $\operatorname{cosupp}_R M^u$; see 15.2.12.

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

15.2.4 Proposition. *Let $M' \to M \to M'' \to \Sigma M'$ be a distinguished triangle in $\mathcal{D}(R)$. Any one of the sets $\operatorname{cosupp}_R M$, $\operatorname{cosupp}_R M'$, and $\operatorname{cosupp}_R M''$ is contained in the union of the two other sets.*

PROOF. For every prime ideal $\mathfrak{p}$ in R there is a distinguished triangle,

$$\begin{aligned}\mathsf{RHom}_R(\kappa(\mathfrak{p}),M') &\longrightarrow \mathsf{RHom}_R(\kappa(\mathfrak{p}),M)\\ &\longrightarrow \mathsf{RHom}_R(\kappa(\mathfrak{p}),M'') \longrightarrow \Sigma\mathsf{RHom}_R(\kappa(\mathfrak{p}),M')\,,\end{aligned}$$

in $\mathcal{D}(R)$. It now follows from 6.5.20 and the definition, 15.2.1, of cosupport that $\mathfrak{p}$ is either in at least two of the three cosupports or not in any of them. □

15.2.5 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R*; *one has*

$$\operatorname{cosupp}_R \mathrm{E}_R(R/\mathfrak{p}) = \{\mathfrak{q}\in\operatorname{Spec} R \mid \mathfrak{q}\subseteq\mathfrak{p}\}\,.$$

PROOF. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals. For every non-zero element $m \in \kappa(\mathfrak{q})$ one has $(0 :_R m) = \mathfrak{q}$ and for every non-zero element $n \in \mathrm{E}_R(R/\mathfrak{p})$ one has $(0 :_R n) \subseteq \mathfrak{p}$ by C.15(a). As $\mathrm{E}_R(R/\mathfrak{p})$ is an injective R-module, it now follows from C.1 that $\operatorname{Hom}_R(\kappa(\mathfrak{q}),\mathrm{E}_R(R/\mathfrak{p}))$ is non-zero if and only if $\mathfrak{q}$ is contained in $\mathfrak{p}$, and this proves the asserted equality. □

15.2.6 Example. From 15.2.5 it follows that for every prime ideal $p\mathbb{Z}$ in $\mathbb{Z}$ one has $\operatorname{cosupp}_{\mathbb{Z}} \mathrm{E}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) = \{0, p\mathbb{Z}\}$. In particular, $\operatorname{cosupp}_{\mathbb{Z}}\mathbb{Q} = \{0\}$, see B.15.

15.2.7 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R*; *one has*

$$\operatorname{cosupp}_R \kappa(\mathfrak{p}) = \{\mathfrak{p}\} = \operatorname{supp}_R \kappa(\mathfrak{p})\,.$$

PROOF. The equality $\operatorname{supp}_R \kappa(\mathfrak{p}) = \{\mathfrak{p}\}$ was alredy noted in 15.1.23. To establish the first equality let $\mathfrak{q}\in\operatorname{Spec} R$ and notice that $H = \mathrm{H}(\mathsf{RHom}_R(\kappa(\mathfrak{q}),\kappa(\mathfrak{p})))$ is a $\kappa(\mathfrak{q})$-vector space and a $\kappa(\mathfrak{p})$-vector space; see 12.2.2. Thus, if $H \neq 0$ then one has $\mathfrak{q} = (0 :_R H) = \mathfrak{p}$, so one has $\operatorname{cosupp}_R \kappa(\mathfrak{p}) \subseteq \{\mathfrak{p}\}$. On the other hand, 7.3.27 yields $\mathrm{H}_0(\mathsf{RHom}_R(\kappa(\mathfrak{p}),\kappa(\mathfrak{p}))) \cong \operatorname{Hom}_R(\kappa(\mathfrak{p}),\kappa(\mathfrak{p})) \neq 0$, so $\mathfrak{p}$ is in $\operatorname{cosupp}_R \kappa(\mathfrak{p})$. □

COSUPPORT FORMULA

15.2.8 Theorem. *An R-complex M is acyclic if and only if* $\operatorname{cosupp}_R M = \emptyset$ *holds.*

PROOF. If M is acyclic, then it follows from the definition that $\operatorname{cosupp}_R M$ is empty. For the converse, assume that M is not acyclic. Considered as a functor on the module category, $\mathrm{G} = \prod_{m\in\mathbb{Z}} \operatorname{Ext}^m_R(-, M)$ is half exact by 7.3.35 and 3.1.18. The counitor 12.3.4 yields $\mathrm{G}(R) = \prod_{m\in\mathbb{Z}} \mathrm{H}_m(M)$, which is non-zero, so it follows from 12.4.8 that there is a prime ideal $\mathfrak{p}$ in R with $\mathrm{G}(R/\mathfrak{p}) \neq 0$ such that $\mathrm{G}(R/\mathfrak{b}) = 0$ holds for every ideal $\mathfrak{b} \supset \mathfrak{p}$. Consider an element $r \in R \setminus \mathfrak{p}$. The ideal $\mathfrak{p} + (r)$ is strictly larger that $\mathfrak{p}$, so the complex $\mathsf{RHom}_R(R/(\mathfrak{p}+(r)), M)$ is acyclic. Application of $\mathsf{RHom}_R(-, M)$ to the triangle induced per 6.5.24 by the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;r\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (r)) \longrightarrow 0$$

shows in view of 6.5.21 that multiplication by r on $\mathrm{H}(\mathsf{RHom}_R(R/\mathfrak{p}, M))$ is an isomorphism. The residue field $\kappa(\mathfrak{p})$ is the field of fractions of the integral domain $R/\mathfrak{p}$, so in the next chain, the last two isomorphism follow from 7.6.11(a,c). The first follows from 12.3.36.

$$\begin{aligned}
\mathrm{H}(\mathsf{RHom}_R(\kappa(\mathfrak{p}), M)) &\cong \mathrm{H}(\mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), \mathsf{RHom}_R(R/\mathfrak{p}, M)))\\
&\cong \mathrm{H}(\mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{H}(\mathsf{RHom}_R(R/\mathfrak{p}, M))))\\
&\cong \operatorname{Hom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{H}(\mathsf{RHom}_R(R/\mathfrak{p}, M)))\;.
\end{aligned}$$

As $\mathrm{H}(\mathsf{RHom}_R(R/\mathfrak{p}, M))$ is non-zero and by 7.6.11(b) a complex of $\kappa(\mathfrak{p})$-vector spaces, it follows that $\mathfrak{p}$ is in $\operatorname{cosupp}_R M$, in particular $\operatorname{cosupp}_R M$ is non-empty. □

The next result is known as the Cosupport Formula.

15.2.9 Theorem. *Let M and N be R-complexes*; *there is an equality,*

$$\operatorname{cosupp}_R \mathsf{RHom}_R(M, N) \;=\; \operatorname{supp}_R M \cap \operatorname{cosupp}_R N\;.$$

PROOF. Let $\mathfrak{p} \in \operatorname{Spec} R$; by 12.3.35 and 7.6.12 there are isomorphisms,

$$\begin{aligned}
&\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{RHom}_R(M, N))\\
&\qquad\simeq \mathsf{RHom}_{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(\kappa(\mathfrak{p}), N))\\
&\qquad\simeq \operatorname{Hom}_{\kappa(\mathfrak{p})}(\mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M), \mathrm{H}(\mathsf{RHom}_R(\kappa(\mathfrak{p}), N)))\;.
\end{aligned}$$

The last complex is a Hom of $\kappa(\mathfrak{p})$-vector spaces and hence non-zero if and only if both spaces are non-zero. Now invoke the definitions of support and cosupport. □

For an R-complex M, a faithfully flat R-module F, and a faithfully injective R-module E, the complexes M and $F \otimes^{\mathsf{L}}_R M$ and $\mathsf{RHom}_R(M, E)$ are simultaneously acyclic, see 2.5.7, 7.3.22, and 7.4.16. The next corollary runs deeper.

15.2.10 Corollary. *Let F be a faithfully flat R-module. An R-complex M is acyclic if and only if* $\mathsf{RHom}_R(F, M)$ *is acyclic.*

PROOF. By 15.1.18 one has $\operatorname{supp}_R F = \operatorname{Spec} R$, so the Cosupport Formula 15.2.9 gives $\operatorname{cosupp}_R \mathsf{R}\mathrm{Hom}_R(F, M) = \operatorname{cosupp}_R M$, and 15.2.8 yields the assertion. □

The next result compares to the characterization of faithfully flat modules in 15.1.18. In fact, one can derive 15.1.18 from 15.2.11 via 1.3.48 and 15.2.9, but we lead the exposition with support, as it is a more established notion than cosupport.

15.2.11 Proposition. *Let E be an injective R-module. The following conditions are equivalent.*

(*i*) *E is faithfully injective.*
(*ii*) $\operatorname{cosupp}_R E = \operatorname{Spec} R$.
(*iii*) $\operatorname{Max} R \subseteq \operatorname{cosupp}_R E$.

PROOF. As noted in 15.2.2, condition (*i*) implies (*ii*). Conversely, if (*ii*) holds, then the Cosupport Formula 15.2.9 yields $\operatorname{cosupp}_R \operatorname{Hom}_R(M, E) = \operatorname{cosupp}_R M$ for every R-module M. Hence $\operatorname{Hom}_R(M, E) = 0$ implies $M = 0$ by 15.2.8, so E is faithfully injective. The implication (*ii*) $\Rightarrow$ (*iii*) is trivial. We prove that (*iii*) implies (*ii*) by contraposition. Let $\mathfrak{p}$ be a prime ideal in R that does not belong to $\operatorname{cosupp}_R E$, i.e. the module $\operatorname{Hom}_R(\kappa(\mathfrak{p}), E) \simeq \mathsf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), E)$ is zero. The embedding $R/\mathfrak{p} \rightarrowtail \kappa(\mathfrak{p})$ induces an exact sequence $\operatorname{Hom}_R(\kappa(\mathfrak{p}), E) \to \operatorname{Hom}_R(R/\mathfrak{p}, E) \to 0$, which yields $\operatorname{Hom}_R(R/\mathfrak{p}, E) = 0$. Let $\mathfrak{m}$ be a maximal ideal in R that contains $\mathfrak{p}$. There is an exact sequence $0 \to \operatorname{Hom}_R(R/\mathfrak{m}, E) \to \operatorname{Hom}_R(R/\mathfrak{p}, E)$, induced by the canonical map $R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{m}$. It shows that the module $\operatorname{Hom}_R(R/\mathfrak{m}, E) \simeq \mathsf{R}\mathrm{Hom}_R(\kappa(\mathfrak{m}), E)$ is zero. Thus, $\mathfrak{m}$ does not belong to $\operatorname{cosupp}_R E$. □

15.2.12 Example. Let R be local with unique maximal ideal $\mathfrak{m}$. As in C.20 set $E = \mathrm{E}_R(R/\mathfrak{m})$ and $E^u = (0 :_E \mathfrak{m}^u)$ for $u \in \mathbb{N}$. By 3.3.34 there is an isomorphism $E \cong \operatorname{colim}_{u\in\mathbb{N}} E^u$, so by 3.3.37 there is an exact sequence,

$$0 \longrightarrow \coprod_{u\in\mathbb{N}} E^u \longrightarrow \coprod_{u\in\mathbb{N}} E^u \longrightarrow E \longrightarrow 0\,.$$

It now follows from 15.2.4 that the cosupport of the coproduct $\coprod_{u\in\mathbb{N}} E^u$ contains $\operatorname{cosupp}_R E$, which by 15.2.11 is all of $\operatorname{Spec} R$. Thus $\operatorname{cosupp}_R(\coprod_{u\in\mathbb{N}} E^u) = \operatorname{Spec} R$. For every $u \in \mathbb{N}$ there is by 1.1.8 an isomorphism $E^u \cong \operatorname{Hom}_R(R/\mathfrak{m}^u, E)$, and this module is isomorphic to $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{m}^u, E)$ in the derived category. Hence the Cosupport Formula 15.2.9 yields $\operatorname{cosupp}_R E^u = \operatorname{supp}_R R/\mathfrak{m}^u$. Finally, $\operatorname{supp}_R R/\mathfrak{m}^u = \mathrm{V}(\mathfrak{m}^u) = \{\mathfrak{m}\}$ holds by 15.1.10.

COLOCALIZATION

15.2.13 Proposition. *Let U be a multiplicative subset of R and M an R-complex. There is an equality,*

$$\operatorname{cosupp}_R \mathsf{R}\mathrm{Hom}_R(U^{-1}R, M) \;=\; \{\mathfrak{p} \in \operatorname{cosupp}_R M \mid \mathfrak{p} \cap U = \varnothing\}\,.$$

For every prime ideal $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \varnothing$ there is an isomorphism in $\mathcal{D}(U^{-1}R)$,

$$\mathsf{RHom}_{U^{-1}R}(\kappa(U^{-1}\mathfrak{p}), \mathsf{RHom}_R(U^{-1}R, M)) \;\cong\; \mathsf{RHom}_R(\kappa(\mathfrak{p}), M)\;.$$

The assignment $\mathfrak{p} \leftrightarrow U^{-1}\mathfrak{p}$ *yields an order preserving one-to-one correspondence,*

$$\operatorname{cosupp}_R \mathsf{RHom}_R(U^{-1}R, M) \;\longleftrightarrow\; \operatorname{cosupp}_{U^{-1}R} \mathsf{RHom}_R(U^{-1}R, M)\;.$$

In particular, a prime ideal $\mathfrak{p}$ *in* R *with* $\mathfrak{p} \cap U = \varnothing$ *belongs to* $\operatorname{cosupp}_R M$ *if and only if* $U^{-1}\mathfrak{p}$ *belongs to* $\operatorname{cosupp}_{U^{-1}R} \mathsf{RHom}_R(U^{-1}R, M)$.

Proof. The equality $\operatorname{cosupp}_R \mathsf{RHom}_R(U^{-1}R, M) = \{\mathfrak{p} \in \operatorname{cosupp}_R M \mid \mathfrak{p} \cap U = \varnothing\}$ follows from the Cosupport Formula 15.2.9 and 15.1.19. For every prime ideal $\mathfrak{p}$ in R with $\mathfrak{p} \cap U = \varnothing$ one gets from 12.3.36 and 15.1.3 the isomorphism,

$$\mathsf{RHom}_{U^{-1}R}(\kappa(U^{-1}\mathfrak{p}), \mathsf{RHom}_R(U^{-1}R, M)) \;\simeq\; \mathsf{RHom}_R(\kappa(\mathfrak{p}), M)\;.$$

The asserted one-to-one correspondence is thus a restriction of the order preserving one-to-one correspondence between $\operatorname{Spec} U^{-1}R$ and $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap U = \varnothing\}$. □

15.2.14 Corollary. *Let* $\mathfrak{p}$ *be a prime ideal in* R *and* M *an* R*-complex. One has*

$$\operatorname{cosupp}_R \mathsf{RHom}_R(R_\mathfrak{p}, M) \;=\; \{\mathfrak{q} \in \operatorname{cosupp}_R M \mid \mathfrak{q} \subseteq \mathfrak{p}\}\;.$$

For every prime ideal $\mathfrak{q}$ *contained in* $\mathfrak{p}$ *there is an isomorphism in* $\mathcal{D}(R_\mathfrak{p})$,

$$\mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{q}_\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M)) \;\simeq\; \mathsf{RHom}_R(\kappa(\mathfrak{q}), M)\;.$$

The assignment $\mathfrak{q} \leftrightarrow \mathfrak{q}_\mathfrak{p}$ *yields an order preserving one-to-one correspondence,*

$$\operatorname{cosupp}_R \mathsf{RHom}_R(R_\mathfrak{p}, M) \;\longleftrightarrow\; \operatorname{cosupp}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)\;.$$

In particular, a prime ideal $\mathfrak{q}$ *contained in* $\mathfrak{p}$ *belongs to* $\operatorname{cosupp}_R M$ *if and only if* $\mathfrak{q}_\mathfrak{p}$ *belongs to* $\operatorname{cosupp}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$.

Proof. This is the special case $U = R \setminus \mathfrak{p}$ of 15.2.13. □

15.2.15 Example. Let $\mathfrak{p}$ be a prime ideal in R and E a faithfully injective $R_\mathfrak{p}$-module. From 14.1.21(b), 15.2.11, and 15.2.14 one gets

$$\operatorname{cosupp}_R E \;=\; \operatorname{cosupp}_R \mathsf{RHom}_R(R_\mathfrak{p}, E) \;=\; \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}\;.$$

Derived Complete Complexes

15.2.16 Lemma. *Let* $\mathfrak{a}$ *be an ideal and* $\mathfrak{p}$ *a prime ideal in* R; *let* M *be an* R*-complex. There is an isomorphism,*

$$\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \;\simeq\; \begin{cases} \mathsf{RHom}_R(\kappa(\mathfrak{p}), M) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p}\;. \end{cases}$$

Proof. By 13.4.12 there is an isomorphism,

$$\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{L}\Lambda^{\mathfrak{a}}(M)) \;\simeq\; \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{a}}(\kappa(\mathfrak{p})), M)\;,$$

in $\mathcal{D}(R)$, so the assertion follows from 15.1.4. □

Given an ideal $\mathfrak{a}$ in R and an R-complex M, the next result shows that the cosupport of the complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is contained in $\mathrm{V}(\mathfrak{a})$; in fact, this characterizes derived $\mathfrak{a}$-complete complexes, see 15.3.19.

15.2.17 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. One has*

$$\operatorname{cosupp}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \;=\; \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R M \;=\; \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,.$$

In particular, one has $\operatorname{cosupp}_R \Lambda^{\mathfrak{a}}(R) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R R$*, so if R is $\mathfrak{a}$-complete, then* $\operatorname{cosupp}_R R$ *is contained in* $\mathrm{V}(\mathfrak{a})$.

Proof. Per the definition of cosupport, 15.2.1, the first equality in the first display is an immediate consequence of 15.2.16. The second equality in that display follows by 13.4.1(c) from the first equality applied to the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$. Finally, one has $\mathsf{L}\Lambda^{\mathfrak{a}}(R) = \Lambda^{\mathfrak{a}}(R)$ by 13.1.15, and hence the last assertion follows. □

For a faithfully injective R-module E, it follows from 15.2.17, in view of 15.2.11, that $\operatorname{cosupp}_R \mathsf{L}\Lambda^{\mathfrak{a}}(E) = \mathrm{V}(\mathfrak{a})$ holds. The cosupport of $\mathsf{R}\Gamma_{\mathfrak{a}}(E) \simeq \Gamma_{\mathfrak{a}}(E)$ is more elusive. Some information about this set can be obtained from Matlis' structure theorem C.23 and 15.2.5, but beware that the cosupport is not well-behaved on infinite coproducts; cf. 15.2.3. Additional information is provided in 15.3.3.

Support Compared to Cosupport

15.2.18 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The following conditions are equivalent.*

(*i*) $\mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M \neq \emptyset$.
(*ii*) $\mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R M \neq \emptyset$.
(*iii*) $\mathfrak{a}\text{-}\operatorname{width}_R M$ *is finite.*
(*iv*) $\mathfrak{a}\text{-}\operatorname{depth}_R M$ *is finite.*

Proof. One has (*iii*) ⇔ (*iv*) by 14.4.12. By 15.1.10 one has $\operatorname{supp}_R R/\mathfrak{a} = \mathrm{V}(\mathfrak{a})$. The Support Formula 15.1.16 yields $\operatorname{supp}_R(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M$, and it follows from 15.1.15 and 14.4.12 that conditions (*i*) and (*iii*) are equivalent. Similarly, $\operatorname{cosupp}_R \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R M$ holds by the Cosupport Formula 15.2.9, so conditions (*ii*) and (*iv*) are equivalent by 15.2.8 and 14.4.12. □

Computing the support or cosupport of a complex can be a delicate task, and the two sets may differ as much as allowed by the next theorem; compare 15.1.12 to 15.2.5 and see also 16.1.19. For a large class of rings, the cosupport of a complex with degreewise finitely generated homology is a subset of the support; see 17.1.19.

15.2.19 Corollary. *Let M be an R-complex. The sets* $\operatorname{supp}_R M$ *and* $\operatorname{cosupp}_R M$ *have the same maximal elements.*

Proof. The equivalence of conditions (*i*) and (*ii*) in 15.2.18 shows that every element of $\operatorname{supp}_R M$ is contained in an element of $\operatorname{cosupp}_R M$ and, conversely, every element of $\operatorname{cosupp}_R M$ is contained in an element of $\operatorname{supp}_R M$. The assertion now follows. □

15.2.20 Example. For every prime p the maximal ideal $p\mathbb{Z}$ belongs to $\operatorname{supp}_{\mathbb{Z}} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$ and hence, by 15.2.19, to $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z}$. Further, one has $\operatorname{Ext}^1_R(\mathbb{Q}, \mathbb{Z}) \neq 0$ by 7.3.28, so also the zero ideal is in the cosupport of $\mathbb{Z}$, i.e. $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$. Notice that 15.2.19, in view of 15.1.13, also yields $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z} = \{0\}$, which was already computed in 15.2.6.

EXERCISES

E 15.2.1 Let M be an R-complex. Show that the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathsf{RHom}_R(R_{\mathfrak{p}}, M)) \neq 0\}$ contains $\operatorname{cosupp}_R M$ and that every prime ideal in the set contains one from $\operatorname{cosupp}_R M$. Conclude that the two sets have the same minimal elements.

E 15.2.2 Let M be an R-complex. Show that for every prime ideal $\mathfrak{p}$ in $\operatorname{cosupp}_R M$ there is an integer n such that $\mathfrak{p} \in \operatorname{cosupp}_R M_{\supseteq n}$.

E 15.2.3 Let M be an R-complex. Show that for every prime ideal $\mathfrak{p}$ in $\operatorname{cosupp}_R M$ there is an integer n such that $\mathfrak{p} \in \operatorname{cosupp}_R M_{\subseteq n}$. *Hint:* E 3.5.2.

E 15.2.4 Let M, X, and N be R-complexes with $\operatorname{cosupp}_R N \subseteq \operatorname{cosupp}_R M$. Show that if $\mathsf{RHom}_R(X, M)$ is acyclic, then $\mathsf{RHom}_R(X, N)$ is acyclic.

E 15.2.5 Let M be an R-complex. Show that if $\operatorname{supp}_R M$ or $\operatorname{cosupp}_R M$ is a subset of $\operatorname{Min} R$, then $\operatorname{supp}_R M = \operatorname{cosupp}_R M$ holds.

E 15.2.6 Let $n \in \mathbb{N}$. Show that one has $\mathsf{RHom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/n\mathbb{Z}) \simeq 0 \simeq \mathbb{Q} \otimes^{\mathsf{L}}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ and conclude that $\operatorname{supp}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ holds.

E 15.2.7 Let M be a finitely generated $\mathbb{Z}$-module; show that one has $\operatorname{supp}_{\mathbb{Z}} M = \operatorname{cosupp}_{\mathbb{Z}} M$. Conclude via 6.4.23 that $\operatorname{supp}_{\mathbb{Z}} M = \operatorname{cosupp}_{\mathbb{Z}} M$ holds for every complex M in $\mathcal{D}^{\mathrm{f}}(\mathbb{Z})$.

15.3 Applications of Support and Cosupport

SYNOPSIS. Krull's intersection theorem; detecting isomorphisms; cosupport of derived $\mathfrak{a}$-complete complex; support of derived $\mathfrak{a}$-torsion complex; derived annihilator.

A homomorphism of modules that induces an isomorphism upon localization outside any prime ideal is necessarily an isomorphism, and of course the same holds for morphisms of complexes; see 15.3.8. One can similarly recognize isomorphisms in the derived category, and that is one of three themes explored in this section. The other themes are: Describing the support and cosupport of derived complete and derived torsion complexes, and extending the concept of annihilators to the derived category setting.

15.3.1 Lemma. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R and M an R-complex. If $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, then one has*

$$\mathrm{V}(\mathfrak{b}) \cap \operatorname{supp}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) = \emptyset \qquad \textit{and} \qquad \mathrm{V}(\mathfrak{b}) \cap \operatorname{cosupp}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \emptyset\,.$$

PROOF. By 13.1.20 one has $\mathsf{L}\Lambda^{\mathfrak{b}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{a}+\mathfrak{b}}(M) = \mathsf{L}\Lambda^{R}(M)$, and in view of 7.2.11 this complex is acyclic by 11.1.5. Similarly one has $\mathsf{R}\Gamma_{\mathfrak{b}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathsf{R}\Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \mathsf{R}\Gamma_R(M)$ by 13.3.21 and this complex is acyclic by 11.2.2. The asserted equalities now follow from 14.4.12 and 15.2.18. $\square$

15.3.2 Lemma. *Let* $\mathfrak{a}$ *be an ideal in* R *and* E *a faithfully injective* R*-module*; *one has*

$$\mathrm{V}(\mathfrak{a}) \subseteq \operatorname{cosupp}_R \Gamma_{\mathfrak{a}}(E) \quad \textit{and} \quad \mathrm{V}(\mathfrak{a}) \cap \operatorname{Max} R = \operatorname{cosupp}_R \Gamma_{\mathfrak{a}}(E) \cap \operatorname{Max} R\,.$$

PROOF. One has $\Gamma_{\mathfrak{a}}(E) \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(E)$ in $\mathcal{D}(R)$, see 13.3.18. As $\operatorname{cosupp}_R E = \operatorname{Spec} R$ holds by 15.2.11, the inclusion $\mathrm{V}(\mathfrak{a}) \subseteq \operatorname{cosupp}_R \Gamma_{\mathfrak{a}}(E)$ follows from 15.2.17. If $\mathfrak{m} \in \operatorname{cosupp}_R \Gamma_{\mathfrak{a}}(E)$ is a maximal ideal in R, then $\mathrm{V}(\mathfrak{m}) \cap \operatorname{cosupp}_R \mathsf{R}\Gamma_{\mathfrak{a}}(E) = \{\mathfrak{m}\}$ holds, so it follows from 15.3.1 that $\mathfrak{a}$ and $\mathfrak{m}$ are not comaximal. As $\mathfrak{m}$ is maximal, this means that $\mathfrak{m}$ contains $\mathfrak{a}$; that is, one has $\mathfrak{m} \in \mathrm{V}(\mathfrak{a})$. ◻

15.3.3 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* E *a faithfully injective* R*-module. The injective* R*-module* $\Gamma_{\mathfrak{a}}(E)$ *is faithfully injective if and only if* $\mathfrak{a}$ *is contained in the Jacobson radical of* R.

PROOF. Recall from 13.3.4 that the R-module $\Gamma_{\mathfrak{a}}(E)$ is injective. Thus 15.2.11 shows that $\Gamma_{\mathfrak{a}}(E)$ is faithfully injective if and only if $\operatorname{Max} R \subseteq \operatorname{cosupp}_R \Gamma_{\mathfrak{a}}(E)$ holds. By the equality in 15.3.2 this is tantamount to the inclusion $\operatorname{Max} R \subseteq \mathrm{V}(\mathfrak{a})$, which again is equivalent to $\mathfrak{a}$ being contained in the Jacobson radical of R. ◻

15.3.4 Lemma. *Let* $\mathfrak{a}$ *be an ideal in* R *and* F *a faithfully flat* R*-module*; *one has*

$$\mathrm{V}(\mathfrak{a}) \subseteq \operatorname{supp}_R \Lambda^{\mathfrak{a}}(F) \quad \textit{and} \quad \mathrm{V}(\mathfrak{a}) \cap \operatorname{Max} R = \operatorname{supp}_R \Lambda^{\mathfrak{a}}(F) \cap \operatorname{Max} R\,.$$

PROOF. One has $\Lambda^{\mathfrak{a}}(F) \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(F)$ in $\mathcal{D}(R)$ by 13.1.15. As $\operatorname{supp}_R F = \operatorname{Spec} R$ holds, see 15.1.18, the inclusion $\mathrm{V}(\mathfrak{a}) \subseteq \operatorname{supp}_R \Lambda^{\mathfrak{a}}(F)$ follows from 15.1.27. If $\mathfrak{m} \in \operatorname{supp}_R \Lambda^{\mathfrak{a}}(F)$ is a maximal ideal in R, then one has $\mathrm{V}(\mathfrak{m}) \cap \operatorname{supp}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) = \{\mathfrak{m}\}$, so it follows from 15.3.1 that $\mathfrak{a}$ and $\mathfrak{m}$ are not comaximal. As $\mathfrak{m}$ is maximal, this means that $\mathfrak{m}$ contains $\mathfrak{a}$; that is, one has $\mathfrak{m} \in \mathrm{V}(\mathfrak{a})$. ◻

15.3.5 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* F *a faithfully flat* R*-module. The flat* R*-module* $\Lambda^{\mathfrak{a}}(F)$ *is faithfully flat if and only if* $\mathfrak{a}$ *is contained in the Jacobson radical of* R.

PROOF. Recall from 13.1.26 that the R-module $\Lambda^{\mathfrak{a}}(F)$ is flat. Thus 15.1.18 shows that $\Lambda^{\mathfrak{a}}(F)$ is faithfully flat if and only if $\operatorname{Max} R \subseteq \operatorname{supp}_R \Lambda^{\mathfrak{a}}(F)$ holds. By the equality in 15.3.4 this is tantamount to the inclusion $\operatorname{Max} R \subseteq \mathrm{V}(\mathfrak{a})$, which again is equivalent to $\mathfrak{a}$ being contained in the Jacobson radical of R. ◻

KRULL'S INTERSECTION THEOREM

15.3.6 Corollary. *Let* $\mathfrak{a}$ *be an ideal in* R*. The* R*-algebra* $\widehat{R}^{\mathfrak{a}}$ *is faithfully flat as an* R*-module if and only if* $\mathfrak{a}$ *is contained in the Jacobson radical of* R.

PROOF. Per 11.1.19 the assertion follows by applying 15.3.5 to $F = R$. ◻

The next result is known as Krull's intersection theorem.

15.3.7 Theorem. *Let* $\mathfrak{a}$ *be an ideal in* R *contained in the Jacobson radical and* M *an* R*-module. If* M *is finitely generated or* $\mathfrak{a}$*-complete, then one has* $\bigcap_{u \geqslant 1} \mathfrak{a}^u M = 0$.

PROOF. As noted in 11.1.11, the equality $\bigcap_{u\geqslant 1} \mathfrak{a}^u M = 0$ holds if and only if M is $\mathfrak{a}$-separated. Evidently, every $\mathfrak{a}$-complete R-module is $\mathfrak{a}$-separated, see 11.1.8. As an R-module, $\widehat{R}^{\mathfrak{a}}$ is faithfully flat by 15.3.6, and hence the structure map $R \to \widehat{R}^{\mathfrak{a}}$ is a pure monomorphism by 12.1.23; see also 5.5.15. Thus, for every R-module M the map $M \cong R \otimes_R M \to \widehat{R}^{\mathfrak{a}} \otimes_R M$ is injective by 5.5.14. If M is finitely generated, this map is by 13.2.4 identified with the canonical one $\lambda^{\mathfrak{a}}_M \colon M \to \Lambda^{\mathfrak{a}}(M)$ from 11.1.4, and hence M is $\mathfrak{a}$-separated. ◻

DETECTING ISOMORPHISMS

Our main focus is applications of support and cosupport to detect isomorphisms in the derived category. However, we open with the complex version of a standard test for isomorphisms of modules.

15.3.8 Proposition. *Let $\alpha \colon M \to N$ be a morphism of R-complexes.*

(a) *α is injective if and only if $\alpha_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in \operatorname{Spec} R$.*
(b) *α is surjective if and only if $\alpha_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in \operatorname{Spec} R$.*
(c) *α is an isomorphism if and only if $\alpha_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in \operatorname{Spec} R$.*

PROOF. (a): For each integer v and prime ideal $\mathfrak{p}$ in R one has $(\operatorname{Ker} \alpha_v)_{\mathfrak{p}} = \operatorname{Ker}((\alpha_v)_{\mathfrak{p}})$, as the localization functor $(-)_{\mathfrak{p}}$ is exact. This equality, combined with the fact that a module is zero if and only if it has empty classic support, yields the assertion. The same argument applies to the modules $\operatorname{Coker} \alpha_v$ and yields (b). Part (c) follows from parts (a) and (b). ◻

15.3.9 Proposition. *Let $\alpha \colon M \to N$ be a morphism in $\mathcal{D}(R)$ and X an R-complex with $\operatorname{supp}_R M \cup \operatorname{supp}_R N \subseteq \operatorname{supp}_R X$. If $\alpha \otimes^{\mathsf{L}}_R X$ is an isomorphism, then α is an isomorphism.*

PROOF. There is a distinguished triangle in $\mathcal{D}(R)$,

$$(\flat) \qquad M \xrightarrow{\alpha} N \longrightarrow C \longrightarrow \Sigma M\,,$$

and it suffices by 6.5.20(c) to prove that the complex C is acyclic. Application of the triangulated functor $- \otimes^{\mathsf{L}}_R X$ to $(\flat)$ yields another distinguished triangle in $\mathcal{D}(R)$, and since $\alpha \otimes^{\mathsf{L}}_R X$ is an isomorphism, it follows from 6.5.20(c) that the complex $C \otimes^{\mathsf{L}}_R X$ is acyclic. Thus, 15.1.15 and the Support Formula 15.1.16 yield

$$\varnothing \;=\; \operatorname{supp}_R(C \otimes^{\mathsf{L}}_R X) \;=\; \operatorname{supp}_R C \cap \operatorname{supp}_R X\,.$$

From $(\flat)$ and 15.1.8 it follows that $\operatorname{supp}_R C$ is contained in $\operatorname{supp}_R M \cup \operatorname{supp}_R N$, so it follows from the assumption on X that $\operatorname{supp}_R C$ is empty, i.e. C is acyclic. ◻

Applied to the morphism $M \to 0$ the next result recovers the "in particular" statement in 2.5.7(c).

15.3.10 Corollary. *Let $\alpha \colon M \to N$ be a morphism in $\mathcal{D}(R)$ and F a faithfully flat R-module. If $\alpha \otimes^{\mathsf{L}}_R F$ is an isomorphism, then α is an isomorphism.*

Proof. By 15.1.18 one has $\operatorname{supp}_R F = \operatorname{Spec} R$, so 15.3.9 yields the claim. □

Applied to the morphism $M \to 0$, the equivalence of (*i*) and (*iv*) in next result recovers the fact that a complex with empty support is acyclic, cf. 15.1.15.

15.3.11 Corollary. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$. The following conditions are equivalent.*

(*i*) *α is an isomorphism.*

(*ii*) *$\alpha \otimes_R R_\mathfrak{p}$ is an isomorphism for every $\mathfrak{p} \in \operatorname{supp}_R M \cup \operatorname{supp}_R N$.*

(*iii*) *$\alpha \otimes^{\mathsf{L}}_R R/\mathfrak{p}$ is an isomorphism for every $\mathfrak{p} \in \operatorname{supp}_R M \cup \operatorname{supp}_R N$.*

(*iv*) *$\alpha \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p})$ is an isomorphism for every $\mathfrak{p} \in \operatorname{supp}_R M \cup \operatorname{supp}_R N$.*

Proof. Condition (*i*) evidently implies (*ii*) and (*iii*). Per the isomorphisms,

$$(\alpha \otimes_R R_\mathfrak{p}) \otimes^{\mathsf{L}}_R R/\mathfrak{p} \;\simeq\; \alpha \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p}) \;\simeq\; (\alpha \otimes^{\mathsf{L}}_R R/\mathfrak{p}) \otimes^{\mathsf{L}}_R R_\mathfrak{p}\,,$$

which hold by the definition, 15.1.1, of $\kappa(\mathfrak{p})$ combined with associativity 12.3.6 and commutativity 12.3.5, conditions (*ii*) and (*iii*) both imply (*iv*). To see that (*iv*) implies (*i*), set $U = \operatorname{supp}_R M \cup \operatorname{supp}_R N$ and $X = \coprod_{\mathfrak{p}\in U} \kappa(\mathfrak{p})$. By assumption $\alpha \otimes^{\mathsf{L}}_R X$ is an isomorphism as one has $\alpha \otimes^{\mathsf{L}}_R X = \coprod_{\mathfrak{p}\in U}(\alpha \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p}))$ by 7.4.5. By 15.1.7 and 15.1.23 the support of X is $\operatorname{supp}_R M \cup \operatorname{supp}_R N$, so it follows from 15.3.9 that α is an isomorphism. □

15.3.12 Proposition. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and X an R-complex with $\operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N \subseteq \operatorname{supp}_R X$. If $\mathsf{RHom}_R(X,\alpha)$ is an isomorphism, then α is an isomorphism.*

Proof. There is a distinguished triangle in $\mathcal{D}(R)$,

$$(\star) \qquad M \xrightarrow{\;\alpha\;} N \longrightarrow C \longrightarrow \Sigma M\,,$$

and it suffices by 6.5.20(c) to prove that the complex C is acyclic. Application of the triangulated functor $\mathsf{RHom}_R(X,-)$ to $(\star)$ yields another distinguished triangle, and since $\mathsf{RHom}_R(X,\alpha)$ is an isomorphism, it follows from 6.5.20(c) that the complex $\mathsf{RHom}_R(X,C)$ is acyclic. Thus, 15.2.8 and the Cosupport Formula 15.2.9 yield

$$\varnothing \;=\; \operatorname{cosupp}_R \mathsf{RHom}_R(X,C) \;=\; \operatorname{supp}_R X \cap \operatorname{cosupp}_R C\,.$$

By 15.2.4 applied to $(\star)$, the set $\operatorname{cosupp}_R C$ is contained in $\operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$, so it follows from the assumption on X that $\operatorname{cosupp}_R C$ is empty, i.e. C is acyclic. □

Applied to the morphism $M \to 0$ the next result recovers 15.2.10.

15.3.13 Corollary. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and F a faithfully flat R-module. If $\mathsf{RHom}_R(F,\alpha)$ is an isomorphism, then α is an isomorphism.*

Proof. By 15.1.18 one has $\operatorname{supp}_R F = \operatorname{Spec} R$, so 15.3.12 justifies the claim. □

Applied to the morphism $M \to 0$, the equivalence of (*i*) and (*iv*) in next result recovers the fact that a complex with empty cosupport is acyclic, cf. 15.2.8.

15.3.14 Corollary. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$. The following conditions are equivalent.*

(*i*) *α is an isomorphism.*
(*ii*) *$\mathsf{RHom}_R(R_\mathfrak{p}, \alpha)$ is an isomorphism for every $\mathfrak{p} \in \operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$.*
(*iii*) *$\mathsf{RHom}_R(R/\mathfrak{p}, \alpha)$ is an isomorphism for every $\mathfrak{p} \in \operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$.*
(*iv*) *$\mathsf{RHom}_R(\kappa(\mathfrak{p}), \alpha)$ is an isomorphism for every $\mathfrak{p} \in \operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$.*

Proof. Condition (*i*) evidently implies (*ii*) and (*iii*). Per the isomorphisms,

$$\begin{aligned}\mathsf{RHom}_R(R/\mathfrak{p}, \mathsf{RHom}_R(R_\mathfrak{p}, \alpha)) &\simeq \mathsf{RHom}_R(\kappa(\mathfrak{p}), \alpha)\\ &\simeq \mathsf{RHom}_R(R_\mathfrak{p}, \mathsf{RHom}_R(R/\mathfrak{p}, \alpha))\,,\end{aligned}$$

which hold by the definition, 15.1.1, of $\kappa(\mathfrak{p})$ combined with adjunction 12.3.8 and commutativity 12.3.5, conditions (*ii*) and (*iii*) both imply (*iv*). To see that (*iv*) implies (*i*), set $U = \operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$ and $X = \prod_{\mathfrak{p}\in U} \kappa(\mathfrak{p})$. By 7.3.6 one has $\mathsf{RHom}_R(X, \alpha) = \prod_{\mathfrak{p}\in U} \mathsf{RHom}_R(\kappa(\mathfrak{p}), \alpha)$, so $\mathsf{RHom}_R(X, \alpha)$ is an isomorphism. By 15.1.7 and 15.1.23 the support of X is $\operatorname{cosupp}_R M \cup \operatorname{cosupp}_R N$, so it follows from 15.3.12 that α is an isomorphism. ◻

15.3.15 Proposition. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and X an R-complex with $\operatorname{supp}_R M \cup \operatorname{supp}_R N \subseteq \operatorname{cosupp}_R X$. If $\mathsf{RHom}_R(\alpha, X)$ is an isomorphism, then α is an isomorphism.*

Proof. There is a distinguished triangle in $\mathcal{D}(R)$,

$$(\diamond) \qquad M \xrightarrow{\ \alpha\ } N \longrightarrow C \longrightarrow \Sigma M\,,$$

and it suffices by 6.5.20(c) to prove that the complex C is acyclic. Application of the triangulated functor $\mathsf{RHom}_R(-, X)$ to ($\diamond$) yields another distinguished triangle, and since $\mathsf{RHom}_R(\alpha, X)$ is an isomorphism, it follows from 6.5.20(c) that the complex $\mathsf{RHom}_R(C, X)$ is acyclic. Thus, 15.2.8 and the Cosupport Formula 15.2.9 yield

$$\varnothing = \operatorname{cosupp}_R \mathsf{RHom}_R(C, X) = \operatorname{supp}_R C \cap \operatorname{cosupp}_R X\,.$$

From 15.1.8 applied to ($\diamond$) one gets that $\operatorname{supp}_R C$ is contained in $\operatorname{supp}_R M \cup \operatorname{supp}_R N$, so it follows from the assumption on X that $\operatorname{supp}_R C$ is empty, i.e. C is acyclic. ◻

Applied to the morphism $M \to 0$ the next result recovers the “in particular” statement in 2.5.7(b).

15.3.16 Corollary. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$ and E a faithfully injective R-module. If $\mathsf{RHom}_R(\alpha, E)$ is an isomorphism, then α is an isomorphism.*

Proof. By 15.2.11 one has $\operatorname{cosupp}_R E = \operatorname{Spec} R$, so 15.3.15 justifies the claim. ◻

15.3.17 Corollary. *Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}(R)$. If $\mathsf{RHom}_R(\alpha, \kappa(\mathfrak{p}))$ is an isomorphism for every $\mathfrak{p}$ in $\operatorname{supp}_R M \cup \operatorname{supp}_R N$, then α is an isomorphism.*

Proof. Set $U = \operatorname{supp}_R M \cup \operatorname{supp}_R N$ and $X = \prod_{\mathfrak{p}\in U} \kappa(\mathfrak{p})$; by assumption the morphism $\mathsf{RHom}_R(\alpha, X)$ is an isomorphism as $\mathsf{RHom}_R(\alpha, X) = \prod_{\mathfrak{p}\in U}\mathsf{RHom}_R(\alpha, \kappa(\mathfrak{p}))$

holds by 7.3.6. By 15.2.3 and 15.2.7 the cosupport of X is $\operatorname{supp}_R M \cup \operatorname{supp}_R N$, so it follows from 15.3.15 that α is an isomorphism. □

COSUPPORT OF DERIVED COMPLETE COMPLEX

The next lemma is key to the characterization of derived $\mathfrak{a}$-complete complexes in 15.3.19. Once that has been proved, the lemma is a special case of 13.4.20(a).

15.3.18 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. If one has* $\operatorname{cosupp}_R N \subseteq \mathrm{V}(\mathfrak{a})$*, then there is an isomorphism,*

$$\mathsf{R}\mathrm{Hom}_R(\lambda^{\mathfrak{a}}_M, N) : \mathsf{R}\mathrm{Hom}_R(\mathsf{L}\Lambda^{\mathfrak{a}}(M), N) \longrightarrow \mathsf{R}\mathrm{Hom}_R(M, N) .$$

PROOF. By the assumption $\operatorname{cosupp}_R N \subseteq \mathrm{V}(\mathfrak{a})$ and the Cosupport Formula 15.2.9, both complexes $\mathsf{R}\mathrm{Hom}_R(\mathsf{L}\Lambda^{\mathfrak{a}}(M), N)$ and $\mathsf{R}\mathrm{Hom}_R(M, N)$ have cosupport contained in $\mathrm{V}(\mathfrak{a})$. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$ and recall from 15.1.10 that one has $\operatorname{supp}_R \mathrm{K}(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a})$. Thus, in order to see that $\mathsf{R}\mathrm{Hom}_R(\lambda^{\mathfrak{a}}_M, N)$ is an isomorphism it suffices by 15.3.12 to show that $\mathsf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), \mathsf{R}\mathrm{Hom}_R(\lambda^{\mathfrak{a}}_M, N))$ is an isomorphism. By adjunction 12.3.8 and commutativity 12.3.5 there is an isomorphism,

$$\mathsf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), \mathsf{R}\mathrm{Hom}_R(\lambda^{\mathfrak{a}}_M, N)) \simeq \mathsf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R \lambda^{\mathfrak{a}}_M, N) ,$$

so it is enough to argue that $\mathrm{K}(\boldsymbol{x}) \otimes^{\mathsf{L}}_R \lambda^{\mathfrak{a}}_M$ is an isomorphism, and in view of 13.3.31 this is merely a special case of 13.4.20(c). □

The next result adds to the description 13.4.4 of derived $\mathfrak{a}$-complete complexes.

15.3.19 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. The next conditions are equivalent*:

(*i*) *M is derived $\mathfrak{a}$-complete.*
(*ii*) $\operatorname{cosupp}_R M \subseteq \mathrm{V}(\mathfrak{a})$.

PROOF. It follows from 15.2.17 that (*i*) implies (*ii*). For the converse, consider the distinguished triangle in $\mathcal{D}(R)$,

$$(\diamond) \qquad M \xrightarrow{\lambda^{\mathfrak{a}}_M} \mathsf{L}\Lambda^{\mathfrak{a}}(M) \longrightarrow Z \longrightarrow \Sigma M .$$

If one has $\operatorname{cosupp}_R M \subseteq \mathrm{V}(\mathfrak{a})$, then 15.3.18 and 7.3.26 yield an isomorphism,

$$\mathcal{D}(R)(\lambda^{\mathfrak{a}}_M, M) : \mathcal{D}(R)(\mathsf{L}\Lambda^{\mathfrak{a}}(M), M) \longrightarrow \mathcal{D}(R)(M, M) ;$$

in particular, there exists a morphism $\varrho \colon \mathsf{L}\Lambda^{\mathfrak{a}}(M) \to M$ with $\varrho\lambda^{\mathfrak{a}}_M = 1^M$. Now E.22 shows that the distinguished triangle $(\diamond)$ is split and that M is a direct summand of $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$. By 13.4.2 the complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is derived $\mathfrak{a}$-complete, and it follows directly from the definition, 11.3.3, that the class of derived $\mathfrak{a}$-complete complexes is closed under direct summands. Consequently, M is derived $\mathfrak{a}$-complete. □

15.3.20 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. If every prime ideal in* $\operatorname{cosupp}_R M$ *is contained in $\mathfrak{p}$, in particular, if* $\operatorname{cosupp}_R M \subseteq \{\mathfrak{p}\}$ *holds, then there is an isomorphism* $\mathsf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M) \simeq M$ *in $\mathcal{D}(R)$.*

Proof. Let $\varphi\colon R \to R_{\mathfrak{p}}$ be the canonical map and α the composite morphism,

$$\mathsf{RHom}_R(R_{\mathfrak{p}}, M) \xrightarrow{\mathsf{RHom}(\varphi,M)} \mathsf{RHom}_R(R, M) \xrightarrow{(\epsilon^M)^{-1}} M\ ,$$

in $\mathcal{D}(R)$. One has $\operatorname{cosupp}_R \mathsf{RHom}_R(R_{\mathfrak{p}}, M) \subseteq \operatorname{cosupp}_R M$ by the Cosupport Formula 15.2.9. Thus, to see that α is an isomorphism, it suffices by 15.3.14 to show that $\mathsf{RHom}_R(R_{\mathfrak{q}}, \alpha)$ is an isomorphism for every $\mathfrak{q}$ in $\operatorname{cosupp}_R M$; and to that end it is enough to show that $\mathsf{RHom}_R(R_{\mathfrak{q}}, \mathsf{RHom}_R(\varphi, M))$ is an isomorphism. For every prime ideal $\mathfrak{q}$ in R, adjunction 12.3.8 and 14.1.10 yield

$$\mathsf{RHom}_R(R_{\mathfrak{q}}, \mathsf{RHom}_R(\varphi, M)) \simeq \mathsf{RHom}_R(\varphi_{\mathfrak{q}}, M)\ .$$

If $\mathfrak{q}$ is in $\operatorname{cosupp}_R M$, then one has $\mathfrak{q} \subseteq \mathfrak{p}$ by assumption, so the map $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}} \to (R_{\mathfrak{p}})_{\mathfrak{q}}$ is an isomorphism and the desired conclusion follows. □

15.3.21 Corollary. *Let $\mathfrak{m}$ be a maximal ideal in R and M an R-complex. If M is derived $\mathfrak{m}$-complete, then there is an isomorphism $\mathsf{RHom}_R(R_{\mathfrak{m}}, M) \simeq M$ in $\mathcal{D}(R)$.*

Proof. If M is derived $\mathfrak{m}$-complete, then $\operatorname{cosupp}_R M \subseteq \{\mathfrak{m}\}$ holds by 15.3.19, and the conclusion follows from 15.3.20. □

Support of Derived Torsion Complex

The next lemma is key to the characterization of derived $\mathfrak{a}$-torsion complexes in 15.3.23. Once that has been proved, the lemma is a special case of 13.4.20(b).

15.3.22 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes. If one has $\operatorname{supp}_R N \subseteq \mathrm{V}(\mathfrak{a})$, then there is an isomorphism,*

$$\mathsf{RHom}_R(N, \boldsymbol{\gamma}_{\mathfrak{a}}^M) : \mathsf{RHom}_R(N, \mathsf{R}\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathsf{RHom}_R(N, M)\ .$$

Proof. By the assumption $\operatorname{supp}_R N \subseteq \mathrm{V}(\mathfrak{a})$ and the Cosupport Formula 15.2.9, both complexes $\mathsf{RHom}_R(N, \mathsf{R}\Gamma_{\mathfrak{a}}(M))$ and $\mathsf{RHom}_R(N, M)$ have cosupport contained in $\mathrm{V}(\mathfrak{a})$. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$ and recall from 15.1.10 that one has $\operatorname{supp}_R \mathrm{K}(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a})$. Thus, in order to see that $\mathsf{RHom}_R(N, \boldsymbol{\gamma}_{\mathfrak{a}}^M)$ is an isomorphism it suffices by 15.3.12 to show that $\mathsf{RHom}_R(\mathrm{K}(\boldsymbol{x}), \mathsf{RHom}_R(N, \boldsymbol{\gamma}_{\mathfrak{a}}^M))$ is an isomorphism. By swap 12.3.7 there is an isomorphism,

$$\mathsf{RHom}_R(\mathrm{K}(\boldsymbol{x}), \mathsf{RHom}_R(N, \boldsymbol{\gamma}_{\mathfrak{a}}^M)) \simeq \mathsf{RHom}_R(N, \mathsf{RHom}_R(\mathrm{K}(\boldsymbol{x}), \boldsymbol{\gamma}_{\mathfrak{a}}^M))\ ,$$

so it is enough to argue that $\mathsf{RHom}_R(\mathrm{K}(\boldsymbol{x}), \boldsymbol{\gamma}_{\mathfrak{a}}^M)$ is an isomorphism. In view of 13.3.31 this is merely a special case of 13.4.20(b). □

The next result adds to the characterization 13.4.9 of derived $\mathfrak{a}$-torsion complexes.

15.3.23 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex.*

(a) *The following conditions are equivalent.*

  (*i*) *M is derived $\mathfrak{a}$-torsion.*

  (*ii*) $\operatorname{supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$.

(*iii*) $\operatorname{Supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$.

(b) *If M is derived $\mathfrak{a}$-torsion and not acyclic, then there are inequalities,*

$$-\inf M \leqslant \dim_R M \leqslant \dim R/\mathfrak{a} - \inf M .$$

Proof. The first inequality in part (b) holds by 14.2.4. If M is derived $\mathfrak{a}$-torsion, then it follows from part (a) that every homology module $\mathrm{H}_v(M)$ has Krull dimension at most $\dim R/\mathfrak{a}$, whence the second inequality in (b) holds by 14.2.1.

(a): In view of 13.4.9 and the definition, 14.1.4, of classic support for complexes, conditions (*i*) and (*iii*) are equivalent by 14.1.3. It follows from 15.1.9 that condition (*iii*) implies (*ii*). It remains to argue that (*ii*) implies (*i*).

As $\mathcal{D}(R)$ is triangulated, the morphism $\boldsymbol{\gamma}_{\mathfrak{a}}^M \colon \mathsf{R}\Gamma_{\mathfrak{a}}(M) \to M$ fits by axioms (TR1) and (TR2) in E.2 into a distinguished triangle,

$$(\diamond) \qquad Z \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\gamma_{\mathfrak{a}}^M} M \longrightarrow \Sigma Z .$$

If one has $\operatorname{supp}_R M \subseteq \mathrm{V}(\mathfrak{a})$, then 15.3.22 and 7.3.26 yield an isomorphism,

$$\mathcal{D}(R)(M, \boldsymbol{\gamma}_{\mathfrak{a}}^M) : \mathcal{D}(R)(M, \mathsf{R}\Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathcal{D}(R)(M, M) ;$$

in particular, there exists a morphism $\sigma \colon M \to \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ with $\boldsymbol{\gamma}_{\mathfrak{a}}^M \sigma = 1^M$. Now E.22 shows that the distinguished triangle $(\diamond)$ is split and that M is a direct summand of $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$. By 13.4.7 the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is derived $\mathfrak{a}$-torsion, and it follows directly from the definition, 11.3.17, that the class of derived $\mathfrak{a}$-torsion complexes is closed under direct summands. Consequently, M is derived $\mathfrak{a}$-torsion. □

15.3.24 Proposition. *Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. If every prime ideal in $\operatorname{supp}_R M$ is contained in $\mathfrak{p}$, in particular, if $\operatorname{supp}_R M \subseteq \{\mathfrak{p}\}$ holds, then there is an isomorphism $M_{\mathfrak{p}} \simeq M$ in $\mathcal{D}(R)$.*

Proof. We identify $M_{\mathfrak{p}}$ with $R_{\mathfrak{p}} \otimes_R^{\mathsf{L}} M$, see 14.1.10. Let $\varphi \colon R \to R_{\mathfrak{p}}$ be the canonical map and α the composite morphism,

$$M \xrightarrow{(\mu^M)^{-1}} R \otimes_R^{\mathsf{L}} M \xrightarrow{\varphi \otimes^{\mathsf{L}} M} R_{\mathfrak{p}} \otimes_R^{\mathsf{L}} M ,$$

in $\mathcal{D}(R)$. The Support Formula 15.1.16 yields $\operatorname{supp}_R(R_{\mathfrak{p}} \otimes_R^{\mathsf{L}} M) \subseteq \operatorname{supp}_R M$. Thus, to see that α is an isomorphism, it suffices by 15.3.11 and commutativity 12.3.5 to prove that $R_{\mathfrak{q}} \otimes_R^{\mathsf{L}} \alpha$ is an isomorphism for every $\mathfrak{q}$ in $\operatorname{supp}_R M$; and to that end it is enough to verify that $R_{\mathfrak{q}} \otimes_R^{\mathsf{L}} (\varphi \otimes_R^{\mathsf{L}} M)$ is an isomorphism. For every prime ideal $\mathfrak{q}$ in R, associativity 12.3.6 and 14.1.10 yield

$$R_{\mathfrak{q}} \otimes_R^{\mathsf{L}} (\varphi \otimes_R^{\mathsf{L}} M) \simeq \varphi_{\mathfrak{q}} \otimes_R^{\mathsf{L}} M .$$

If $\mathfrak{q}$ is in $\operatorname{supp}_R M$, then one has $\mathfrak{q} \subseteq \mathfrak{p}$ by assumption, so the map $\varphi_{\mathfrak{q}} \colon R_{\mathfrak{q}} \to (R_{\mathfrak{p}})_{\mathfrak{q}}$ is an isomorphism and the desired conclusion follows. □

15.3.25 Corollary. *Let $\mathfrak{m}$ be a maximal ideal in R and M an R-complex. If M is derived $\mathfrak{m}$-torsion, then there is an isomorphism $M_{\mathfrak{m}} \simeq M$ in $\mathcal{D}(R)$.*

Proof. If M is derived $\mathfrak{m}$-torsion, then $\operatorname{supp}_R M \subseteq \{\mathfrak{m}\}$ holds by 15.3.23, and the conclusion follows from 15.3.24. □

Derived Annihilator

15.3.26 Definition. Let M be an R-complex. The set

$$\operatorname{ann}_R M = \{x \in R \mid x^M = 0 \text{ in } \mathcal{D}(R)\}$$

is called the *derived annihilator* of M.

Remark. The derived annihilator was introduced by Apassov [6] as the 'homotopy annihilator'.

The next result makes it explicit that the derived annihilator is an invariant on the derived category; that is, isomorphic complexes in $\mathcal{D}(R)$ have the same annihilator.

15.3.27 Proposition. *Let M be an R-complex. There is an equality,*

$$\operatorname{ann}_R M = \operatorname{Ker} \mathrm{H}_0(\chi_R^M) ,$$

so $\operatorname{ann}_R M$ *is an ideal in R. Further, for $x \in R$ the next conditions are equivalent.*

(*i*) $x \in \operatorname{ann}_R M$.

(*ii*) *For some/every semi-projective replacement P of M the homothety x^P is null-homotopic.*

(*iii*) *For some/every semi-injective replacement I of M the homothety x^I is null-homotopic.*

Proof. By the final assertion in 6.4.8 and 6.1.6, the homothety x^M is the zero morphism in $\mathcal{D}(R)$ if and only if the homothety $x^{\mathrm{P}(M)}$ is null-homotopic. By 2.3.10 this happens if and only if x is in the kernel of the map

$$\mathrm{H}_0(\chi_R^{\mathrm{P}(M)}) : R \longrightarrow \mathrm{H}_0(\operatorname{Hom}_R(\mathrm{P}(M), \mathrm{P}(M))) .$$

Let P be any semi-projective replacement M and I a semi-injective replacement. By 7.3.17 and 7.3.19 there are quasi-isomorphisms $\pi\colon P \xrightarrow{\simeq} M$ and $\iota\colon M \xrightarrow{\simeq} I$, which induce a commutative diagram,

$$\begin{tikzcd} R \arrow[r, "\chi_R^I"] \arrow[d, "\chi_R^P"'] & \operatorname{Hom}_R(I,I) \arrow[d, "\operatorname{Hom}(\iota\pi,I)", "\simeq"'] \\ \operatorname{Hom}_R(P,P) \arrow[r, "\operatorname{Hom}(P,\iota\pi)", "\simeq"'] & \operatorname{Hom}_R(P,I) . \end{tikzcd}$$

The diagram shows that $\operatorname{Ker} \mathrm{H}_0(\chi_R^P) = \operatorname{Ker} \mathrm{H}_0(\chi_R^I)$ holds. As it applies, in particular, to $P = \mathrm{P}(M)$ it follows that conditions (*i*)–(*iii*) are equivalent. Finally, by the definition, 10.1.10, of the morphism χ_R^M, the set $\operatorname{ann}_R M$ is precisely the kernel of $\mathrm{H}_0(\chi_R^{\mathrm{I}(M)}) = \mathrm{H}_0(\chi_R^M)$ which, in particular, is an ideal in R. □

15.3.28 Proposition. *Let M be an R-complex; one has*

$$\operatorname{ann}_R M \;\subseteq\; \{x \in R \mid \mathrm{H}(x^M) = 0\} \;=\; (0 :_R \mathrm{H}(M)) \;=\; \bigcap_{v\in\mathbb{Z}} (0 :_R \mathrm{H}_v(M))\;.$$

Proof. Homology is by 6.5.17 an R-linear functor on $\mathcal{D}(R)$, so for $x \in \operatorname{ann}_R M$ one has $x^{\mathrm{H}(M)} = \mathrm{H}(x^M) = \mathrm{H}(0) = 0$. ◻

15.3.29 Example. For an R-module M and an element x in R one has $x^M = 0$ in $\mathcal{D}(R)$ if and only if x^M is the zero homomorphism, see 6.4.15, so $\operatorname{ann}_R M$ is simply the annihilator $(0 :_R M)$.

15.3.30 Example. Let x be an element in R and set $K = \mathrm{K}^R(x)$. Evidently, the homothety x^K is null-homotopic, so the ideal $\operatorname{ann}_R K$ contains (x). Per 2.2.9 one has

$$(0 :_R \mathrm{H}_1(K)) \;\supseteq\; (x) \;=\; (0 :_R \mathrm{H}_0(K)) \quad \text{and hence} \quad (0 :_R \mathrm{H}(K)) \;=\; (x)\;.$$

Now 15.3.28 yields $\operatorname{ann}_R K = (x)$.

15.3.31 Example. Consider the $\mathbb{Z}/4\mathbb{Z}$-complex

$$M \;=\; 0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0\;.$$

Evidently one has $(0 :_{\mathbb{Z}/4\mathbb{Z}} \mathrm{H}(M)) = 2\mathbb{Z}/4\mathbb{Z}$, but it is elementary to verify that the homothety on M induced by 2 is not null-homotopic. As M is semi-projective, and for that matter semi-injective, $\operatorname{ann}_{\mathbb{Z}/4\mathbb{Z}} M = 0$ holds by 15.3.27 and 15.3.28.

The next result applies, in particular, to the functors $\mathsf{R}\Gamma_{\mathfrak{a}}$ and $\mathsf{L}\Lambda^{\mathfrak{a}}$, and for every R-complex X to the functors $\mathsf{R}\mathrm{Hom}_R(X,-)$, $\mathsf{R}\mathrm{Hom}_R(-,X)$, $X \otimes^{\mathsf{L}}_R -$, and $- \otimes^{\mathsf{L}}_R X$.

15.3.32 Proposition. *Let $\mathrm{F}\colon \mathcal{D}(R) \to \mathcal{D}(R)$ be an R-linear functor and M an R-complex; there is an inclusion,*

$$\operatorname{ann}_R M \;\subseteq\; \operatorname{ann}_R \mathrm{F}(M)\;.$$

Proof. For $x \in R$ one has $x^{\mathrm{F}(M)} = \mathrm{F}(x^M)$ as F is R-linear, so for $x \in \operatorname{ann}_R M$ one has $x^{\mathrm{F}(M)} = \mathrm{F}(0) = 0$ in $\mathcal{D}(R)$. ◻

15.3.33 Proposition. *Let S be an R-algebra, flat as an R-module, and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. The derived annihilator $\operatorname{ann}_S(S \otimes_R M)$ is the extended ideal $(\operatorname{ann}_R M)S$.*

Proof. There is a commutative diagram in $\mathcal{D}(S)$,

$$\begin{tikzcd}
S \otimes_R R \arrow[r, "S\otimes\chi^M_R"] \arrow[d, "\mu^S_R"', "\simeq"] & S \otimes_R \mathsf{R}\mathrm{Hom}_R(M,M) \arrow[d, "\simeq"'] \\
S \arrow[r, "\chi^{S\otimes M}_S"] & \mathsf{R}\mathrm{Hom}_S(S \otimes_R M, S \otimes_R M)\;,
\end{tikzcd}$$

where the right-hand vertical isomorphism comes from 12.3.33(a). Passing to homology and applying the Five Lemma 2.1.41, it yields an isomorphism,

$$\operatorname{Ker}\mathrm{H}_0(S \otimes_R \chi^M_R) \;\cong\; \operatorname{Ker}\mathrm{H}_0(\chi^{S\otimes M}_S)\;.$$

As the functor $S \otimes_R -$ is exact, one has per 2.2.19 and 1.1.41 the isomorphisms,

$$\operatorname{Ker} \mathrm{H}_0(S \otimes_R \chi_R^M) \cong \operatorname{Ker}(S \otimes_R \mathrm{H}_0(\chi_R^M)) \cong S \otimes_R \operatorname{Ker} \mathrm{H}_0(\chi_R^M) \,.$$

Set $\mathfrak{a} = \operatorname{Ker} \mathrm{H}_0(\chi_R^M)$. As S is a flat R-module, the canonical map $S \otimes_R \mathfrak{a} \longrightarrow \mathfrak{a}S$ is an isomorphism, see 12.1.22. The assertion now follows from 15.3.27. □

15.3.34 Corollary. *Let S be an R-algebra, flat as an R-module, and M a finitely generated R-module. There is an equality,*

$$(0 :_S (S \otimes_R M)) = (0 :_R M)S \,.$$

Proof. The asserted equality follows from 15.3.29 and 15.3.33. □

Exercises

E 15.3.1 Let R be an integral domain with field of fractions Q and M an R-module. Show that the next conditions are equivalent: (*i*) $\operatorname{supp}_R M = \{0\}$; (*ii*) $\operatorname{cosupp}_R M = \{0\}$; (*iii*) $M \cong Q^{(U)}$ for some set U.

E 15.3.2 Show from the definition of the derived annihilator that if M and N are isomorphic complexes in $\mathcal{D}(R)$ then $\operatorname{ann}_R M = \operatorname{ann}_R N$ holds.

E 15.3.3 Show from the definition of the derived annihilator of an R-complex is an ideal in R.

15.4 Homological Dimensions

Synopsis. Projective dimension; injective dimension; ~ of derived $\mathfrak{a}$-torsion complex; rigidity of Ext; flat dimension; ~ of derived $\mathfrak{a}$-complete complex; rigidity of Tor; flat dimension vs. (faithfully flat) base change.

We continue the project of rewriting results from Part II in their more facile form for commutative Noetherian rings, now also taking into account that vanishing of Ext and Tor functors can be tested on prime ideals, see Sect. 12.4.

Projective Dimension

Let M be an R-complex. Recall from Sect. 8.1 that a semi-projective R-complex P that is isomorphic to M in $\mathcal{D}(R)$ is called a semi-projective replacement of M and that the projective dimension of M is defined as

$$\operatorname{pd}_R M = \inf\{\sup P^\natural \mid P \text{ is a semi-projective replacement of } M\} \,.$$

15.4.1 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.

(*ii*) $-\inf \operatorname{RHom}_R(M,N) \leqslant n - \inf N$ *holds for every R-complex N.*

(*iii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}_R^{n+1}(M,N) = 0$ *holds for every R-module N.*

(*iv*) $n \geqslant \sup M$ *and* $\operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P)) = 0$ *holds for some, equivalently every, semi-projective replacement* P *of* M.

(*v*) $n \geqslant \sup M$ *and for some, equivalently every, semi-projective replacement* P *of* M*, the module* $\mathrm{C}_n(P)$ *is projective.*

(*vi*) $n \geqslant \sup M$ *and for every semi-projective replacement* P *of* M*, there is a semi-projective resolution* $P_{\subseteq n} \xrightarrow{\simeq} M$.

(*vii*) *There is a semi-projective resolution* $P \xrightarrow{\simeq} M$ *with* $P_v = 0$ *for all* $v > n$ *and for all* $v < \inf M$.

In particular, there are equalities,

$$\begin{aligned}\operatorname{pd}_R M &= \sup\{-\inf \mathsf{RHom}_R(M, N) \mid N \text{ is an } R\text{-module}\}\\ &= \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_R(M, N) \neq 0 \text{ for some } R\text{-module } N\}\,.\end{aligned}$$

Proof. This is a restatement of 8.1.8. □

15.4.2 Theorem. *Let* M *be an* R*-complex and* n *an integer. If* M *belongs to* $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$*, then the following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.

(*ii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}^{n+1}_R(M, R/\mathfrak{p}) = 0$ *for every prime ideal* $\mathfrak{p}$ *in* R.

(*iii*) *There is a semi-projective resolution* $P \xrightarrow{\simeq} M$ *with* P *degreewise finitely generated and* $P_v = 0$ *for all* $v > n$ *and for all* $v < \inf M$.

In particular, there are equalities,

$$\begin{aligned}\operatorname{pd}_R M &= \sup\{-\inf \mathsf{RHom}_R(M, R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \operatorname{Ext}^m_R(M, R/\mathfrak{p}) \neq 0\}\,.\end{aligned}$$

Proof. The functors $\operatorname{Ext}^m_R(M, -)$ are half exact, see 7.3.35, so for every ideal $\mathfrak{a}$ in R with $\operatorname{Ext}^m_R(M, R/\mathfrak{a}) \neq 0$ it follows from 12.4.1 that there is a prime ideal $\mathfrak{p}$ with $\operatorname{Ext}^m_R(M, R/\mathfrak{p}) \neq 0$. In view of this, the claims follows from 8.1.14. □

15.4.3 Proposition. *Let* M *be a complex in* $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ *of finite projective dimension. The functors* $- \otimes^{\mathsf{L}}_R M$ *and* $\mathsf{RHom}_R(M, -)$ *restrict to endofunctors on* $\mathcal{D}^{\mathrm{f}}(R)$ *and to functors*:

$$\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R) \longrightarrow \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)\,,\quad \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R) \longrightarrow \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)\,,\quad \textit{and}\quad \mathcal{D}^{\mathrm{f}}_{\square}(R) \longrightarrow \mathcal{D}^{\mathrm{f}}_{\square}(R)\,,$$

and further to

$$\mathcal{P}^{\mathrm{f}}(R) \longrightarrow \mathcal{P}^{\mathrm{f}}(R) \quad \textit{and} \quad \mathcal{I}^{\mathrm{f}}(R) \longrightarrow \mathcal{I}^{\mathrm{f}}(R)\,.$$

Proof. As $\operatorname{pd}_R M$ is finite, also $\mathsf{RHom}_R(M, R)$ is a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension, and the functors $- \otimes^{\mathsf{L}}_R M$ and $\mathsf{RHom}_R(\mathsf{RHom}_R(M, R), -)$ are naturally isomorphic, see 12.3.20. It is, therefore, sufficient to prove the assertions for the functor $\mathrm{F} = \mathsf{RHom}_R(M, -)\colon \mathcal{D}(R) \to \mathcal{D}(R)$. As F is triangulated and bounded, see 7.3.6 and A.26(c), it follows from 12.2.6 and A.29(d), applied with $\mathcal{U} = \mathcal{D}^{\mathrm{f}}(R)$, that F restricts to a functor $\mathcal{D}^{\mathrm{f}}(R) \to \mathcal{D}^{\mathrm{f}}(R)$. Thus, F restricts by A.25 to functors $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R) \to \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R) \to \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, and $\mathcal{D}^{\mathrm{f}}_{\square}(R) \to \mathcal{D}^{\mathrm{f}}_{\square}(R)$. This final restriction in combination with 8.4.26 and 8.3.15(b) yields the restrictions of F to functors $\mathcal{P}^{\mathrm{f}}(R) \to \mathcal{P}^{\mathrm{f}}(R)$ and $\mathcal{I}^{\mathrm{f}}(R) \to \mathcal{I}^{\mathrm{f}}(R)$. □

15.4.4 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex.*

(a) *If* M *belongs to* $\mathcal{D}^{\mathrm{f}}(R)$*, then one has* $\operatorname{pd}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \leqslant \operatorname{pd}_R M + \operatorname{pd}_R \Lambda^{\mathfrak{a}}(R)$.
(b) *There is an inequality* $\operatorname{pd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{pd}_R M$.

Proof. Part (a) is trivial if M is acyclic; if not it follows from 13.2.5 and 8.3.15(d).

(b): Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. One can assume that M and the Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ are not acyclic, otherwise the inequality is trivial per 13.3.18. The projective dimension of $\check{\mathrm{C}}^R(\boldsymbol{x})$ is by 11.4.26 at most 0, so the inequality follows from 13.3.18 and 8.3.15(d). □

Restriction of Scalars

15.4.5 Proposition. *Let* S *be an* R*-algebra and* N *an* S*-complex. If* N *is not acyclic, then there is an inequality,*

$$\operatorname{pd}_R N \leqslant \operatorname{pd}_R S + \operatorname{pd}_S N .$$

In particular, if S *is projective as an* R*-module, then one has* $\operatorname{pd}_R N \leqslant \operatorname{pd}_S N$.

Proof. In view of the unitor 12.3.3, this holds by 8.3.15(d) applied with $X = S$. □

15.4.6 Proposition. *Let* $\mathfrak{a}$ *be an ideal in* R *and* M *an* R*-complex*; *one has*

$$\operatorname{pd}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{pd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{pd}_R \widehat{R}^{\mathfrak{a}} + \operatorname{pd}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) .$$

Proof. In view of 13.4.17 the first inequality is a special case of 8.1.4; the second inequality holds by 15.4.5. □

Remark. Let $\mathfrak{a}$ be an ideal in R and M and R-complex. If $\mathfrak{a}$ is contained in the Jacobson radical of R, then the R-algebra $\widehat{R}^{\mathfrak{a}}$ is faithfully flat as an R-module, see 15.3.6, and hence the equality $\operatorname{pd}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \operatorname{pd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds per the Remark after 15.4.19.

Injective Dimension

Let M be an R-complex. Recall from Sect. 8.2 that a semi-injective R-complex I that is isomorphic to M in $\mathcal{D}(R)$ is called a semi-injective replacement of M and that the injective dimension of M is defined as

$$\operatorname{id}_R M = \inf\{-\inf I^{\natural} \mid I \text{ is a semi-injective replacement of } M\} .$$

The injective dimension of R as an R-module, sometimes referred to as the self-injective dimension of R, is abbreviated $\operatorname{id} R$. It is an important invariant that gets investigated starting from Sect. 17.4.

Caveat. The notation $\operatorname{id} R$ is permissable as R is commutative and hence a symmetric R–R^{o}-bimodule, cf. the Remark before 9.4.15 but see also 8.5.30.

15.4.7 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{id}_R M \leqslant n$.

(*ii*) $-\inf \mathsf{RHom}_R(N,M) \leqslant n + \sup N$ *holds for every R-complex N.*

(*iii*) $n \geqslant -\inf M$ *and* $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p},M) = 0$ *holds for every prime ideal $\mathfrak{p}$ in R.*

(*iv*) $n \geqslant -\inf M$ *and one has* $\operatorname{Ext}_R^1(\mathrm{Z}_{-(n+1)}(I),\mathrm{Z}_{-n}(I)) = 0$ *for some, equivalently every, semi-injective replacement I of M.*

(*v*) $n \geqslant -\inf M$ *and for some, equivalently every, semi-injective replacement I of M, the module $\mathrm{Z}_{-n}(I)$ is injective.*

(*vi*) $n \geqslant -\inf M$ *and for every semi-injective replacement I of M, there is a semi-injective resolution* $M \xrightarrow{\simeq} I_{\sqsupseteq -n}$.

(*vii*) *There is a semi-injective resolution* $M \xrightarrow{\simeq} I$ *with $I_{-v} = 0$ for all $v > n$ and for all $v < -\sup M$.*

In particular, there are equalities,

$$\begin{aligned} \operatorname{id}_R M &= \sup\{-\inf \mathsf{RHom}_R(R/\mathfrak{p},M) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \operatorname{Ext}_R^m(R/\mathfrak{p},M) \neq 0\}\,. \end{aligned}$$

Proof. The functors $\operatorname{Ext}_R^m(-,M)$ are half exact, see 7.3.35, so for every ideal $\mathfrak{a}$ in R with $\operatorname{Ext}_R^m(R/\mathfrak{a},M) \neq 0$ it follows from 12.4.7 that there is a prime ideal $\mathfrak{p}$ with $\operatorname{Ext}_R^m(R/\mathfrak{p},M) \neq 0$. In view of this, the claims follows from 8.2.8. $\square$

15.4.8 Proposition. *Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite injective dimension. The functor* $\mathsf{RHom}_R(-,M)$ *restricts to a functor from $\mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}}$ to $\mathcal{D}^{\mathrm{f}}(R)$ and to functors*:

$$\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)\,, \quad \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)\,, \quad \textit{and} \quad \mathcal{D}^{\mathrm{f}}_{\square}(R)^{\mathrm{op}} \longrightarrow \mathcal{D}^{\mathrm{f}}_{\square}(R)\,,$$

and further to

$$\mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}} \longrightarrow \mathcal{I}^{\mathrm{f}}(R) \quad \textit{and} \quad \mathcal{I}^{\mathrm{f}}(R)^{\mathrm{op}} \longrightarrow \mathcal{P}^{\mathrm{f}}(R)\,.$$

Proof. By the assumptions on M, the functor $\mathrm{G} = \mathsf{RHom}_R(-,M)\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(R)$ is bounded by A.32(c). As G is triangulated, see 7.3.6, it follows from 12.2.6 and A.34(d), applied with $\mathcal{U} = \mathcal{D}^{\mathrm{f}}(R)$, that G restricts to a functor $\mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{f}}(R)$. Thus, G restricts by A.31 to functors $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, and $\mathcal{D}^{\mathrm{f}}_{\square}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{f}}_{\square}(R)$. This final restriction combined with 8.3.15(b), 8.4.27, and 8.3.19 yields the restrictions to functors $\mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}} \to \mathcal{I}^{\mathrm{f}}(R)$ and $\mathcal{I}^{\mathrm{f}}(R)^{\mathrm{op}} \to \mathcal{P}^{\mathrm{f}}(R)$. $\square$

Restriction of Scalars

15.4.9 Proposition. *Let S be an R-algebra and N an S-complex. If N is not acyclic, then there is an inequality,*

$$\operatorname{id}_R N \leqslant \operatorname{fd}_R S + \operatorname{id}_S N\,.$$

In particular, if S is flat as an R-module, then one has $\operatorname{id}_R N \leqslant \operatorname{id}_S N$.

Proof. In view of the counitor 12.3.4, this holds by 8.3.15(a) applied with the roles of R and S interchanged and $X = S$. □

Given an ideal $\mathfrak{a}$ in R and an R-complex M, the object $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is an $\widehat{R}^{\mathfrak{a}}$-complex, see 11.3.18. The next result compares the injective dimensions of $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ over R and $\widehat{R}^{\mathfrak{a}}$; it is complemented by 16.1.20 and 17.3.19.

15.4.10 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex; one has*

$$\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{id}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) .$$

Proof. As an R-module, $\widehat{R}^{\mathfrak{a}}$ is flat, see 13.1.27, so in view of 11.3.18 the inequality is a special case of 15.4.9. □

Rigidity of Ext

The phenomenon that vanishing of Ext^m for a single index implies vanishing for all subsequent indices is often referred to as "rigidity of Ext". It occurs for complexes that satisfy the assumption in 13.4.11. In the special case of the zero ideal, the equalities in the next result just recovers the equalities in 15.4.7.

15.4.11 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the complex $\mathsf{RHom}_R(R/\mathfrak{p}, M)$ is derived $\mathfrak{a}$-torsion for every prime ideal $\mathfrak{p}$ in R. One has*

$$\begin{aligned} \operatorname{id}_R M &= \sup\{-\inf \mathsf{RHom}_R(R/\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(R/\mathfrak{p}, M) \neq 0\} . \end{aligned}$$

Further, if for an integer $n \geqslant -\inf M$ one has $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{id}_R M \leqslant n$ holds.

Proof. Let $n \geqslant -\inf M$ be an integer. If $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ holds for all $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then 13.4.11 yields $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p}$ in R. The equivalence of (*i*) and (*iii*) in 15.4.7 now yields $\operatorname{id}_R M \leqslant n$. This proves the last assertion.

The second equality in the display holds by 7.3.24. Let s denote the supremum in the asserted equality. The inequality $\operatorname{id}_R M \geqslant s$ holds by 15.4.7, and 13.4.11 yields $s \geqslant -\inf M$. To prove that the opposite inequality, $\operatorname{id}_R M \leqslant s$, holds, one can now assume that s is an integer. As one has $\operatorname{Ext}_R^{s+1}(R/\mathfrak{p}, M) = 0$ for every $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, the desired inequality holds by the argument above. □

The gist of the next result is that vanishing of certain Ext modules of a complex M detects the injective dimension of the derived $\mathfrak{a}$-torsion complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$. The flat dimension of $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is computed in 17.3.5.

15.4.12 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the inequality $\mathfrak{a}\text{-}\operatorname{depth}_R M > -\infty$ holds. There are equalities,*

$$\begin{aligned} \operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) &= \sup\{-\inf \mathsf{RHom}_R(R/\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(R/\mathfrak{p}, M) \neq 0\} . \end{aligned}$$

Further, if for an integer $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ one has $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant n$ holds.

Proof. The complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is by 13.4.7 derived $\mathfrak{a}$-torsion. Per 14.4.3 the assumption $\mathfrak{a}\text{-depth}_R M > -\infty$ means that $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ belongs to $\mathcal{D}_{\sqsubset}(R)$, so 15.4.11 applies by 13.3.28(b) to $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$. It remains to recall that for $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ the cyclic module $R/\mathfrak{p}$ is $\mathfrak{a}$-torsion and, therefore, by 13.3.30 derived $\mathfrak{a}$-torsion, whence there is an isomorphism $\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, \mathsf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)$ by 13.4.20(b). □

15.4.13 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M a derived $\mathfrak{a}$-torsion complex in $\mathcal{D}_{\sqsubset}(R)$. There are equalities,*

$$\begin{aligned} \operatorname{id}_R M &= \sup\{-\inf \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(R/\mathfrak{p}, M) \neq 0\}\,. \end{aligned}$$

Further, if for an integer $n \geqslant -\inf M$ one has $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{id}_R M \leqslant n$ holds.

Proof. By assumption there is an isomorphism $\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq M$ in $\mathcal{D}(R)$, and 14.3.16 yields $\mathfrak{a}\text{-depth}_R M \geqslant -\sup M > -\infty$, so the result is a special case of 15.4.12. □

15.4.14 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an $\mathfrak{a}$-torsion R-module. One has*

$$\operatorname{id}_R M = \sup\{m \in \mathbb{N}_0 \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(R/\mathfrak{p}, M) \neq 0\}\,.$$

Further, if for an integer $n \geqslant 0$ one has $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{id}_R M \leqslant n$ holds.

Proof. In view of 13.3.30, this result is a special case of 15.4.13. □

The assumption in part (a) below is necessary; see 17.5.16. The injective dimensions of $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ and $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ are computed in 15.4.12 and 17.3.20.

15.4.15 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex.*

(a) *If $\mathfrak{a}\text{-depth}_R M > -\infty$ holds, then one has $\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{id}_R M$.*
(b) *There is an inequality $\operatorname{id}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \leqslant \operatorname{id}_R M$.*

Proof. Part (a) is an immediate consequence of 15.4.12 and 15.4.7.

(b): Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. One can assume that M and the Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ are not acyclic, otherwise the inequality is trivial per 13.1.15. The Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ has by 11.4.10(c) flat dimension at most 0, so the inequality follows from 13.1.15 and 8.3.15(a). □

For an R-complex M with $\mathfrak{a}\text{-depth}_R M > -\infty$ the next rigidity statement follows from 15.4.12.

15.4.16 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If for an integer $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M)$ one has $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{Ext}_R^m(R/\mathfrak{p}, M) = 0$ holds for all integers $m > n$ and all prime ideals $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$.*

Proof. Set $i = \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M)$, let I be a semi-injective replacement of M, and set $E = \Gamma_{\mathfrak{a}}(I)$. Note that $\mathrm{H}_v(E) = 0$ holds for $v < i$ as one has $E \simeq \mathsf{R}\Gamma_{\mathfrak{a}}(M)$. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ the R-module $R/\mathfrak{p}$ is $\mathfrak{a}$-torsion, so by 7.3.22, 11.2.22, and the definition of E, one has

$$(\diamond)\qquad \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M) \simeq \mathrm{Hom}_R(R/\mathfrak{p}, I) \cong \mathrm{Hom}_R(R/\mathfrak{p}, E)\ .$$

Now, for every integer $m > -i$ and prime ideal $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$ there are isomorphisms,

$$(\star)\qquad \begin{aligned} \mathrm{Ext}_R^m(R/\mathfrak{p}, M) &\cong \mathrm{H}_{-m}(\mathrm{Hom}_R(R/\mathfrak{p}, E)) \\ &\cong \mathrm{H}_{-m}(\mathrm{Hom}_R(R/\mathfrak{p}, E_{\leqslant i})) \\ &\cong \mathrm{H}_{-m-i}(\mathrm{Hom}_R(R/\mathfrak{p}, \Sigma^{-i} E_{\leqslant i})) \\ &\cong \mathrm{Ext}_R^{m+i}(R/\mathfrak{p}, \mathrm{Z}_i(E))\ . \end{aligned}$$

Indeed, the 1st isomorphism holds by 7.3.23 and $(\diamond)$, the 2nd holds as $m > -i$, and the 3rd follows from 2.3.16 and 2.2.15. By 13.3.4 each module E_v is injective, so per 5.3.12 the complex $J = \Sigma^{-i} E_{\leqslant i}$ is a semi-injective replacement of the module $\mathrm{Z}_i(E)$. This explains the 4th isomorphism. As $n + 1 > -i$, the assumption and $(\star)$ yield

$$(\dagger\dagger)\qquad 0 = \mathrm{Ext}_R^{n+1}(R/\mathfrak{p}, M) \cong \mathrm{Ext}_R^{(n+i)+1}(R/\mathfrak{p}, \mathrm{Z}_i(E)) \quad \text{for all} \quad \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\ .$$

The complex $E = \Gamma_{\mathfrak{a}}(I)$ is $\mathfrak{a}$-torsion by 11.2.18, and hence so is $J = \Sigma^{-i} E_{\leqslant i}$. Thus one has $\mathsf{R}\Gamma_{\mathfrak{a}}(\mathrm{Z}_i(E)) \simeq \Gamma_{\mathfrak{a}}(J) = J \simeq \mathrm{Z}_i(E)$ in $\mathcal{D}(R)$, so the module $\mathrm{Z}_i(E)$ is derived $\mathfrak{a}$-torsion, see 13.4.9. As one has $n + i \geqslant 0$, it now follows from 15.4.13 and $(\dagger\dagger)$ that $\mathrm{id}_R \mathrm{Z}_i(E) \leqslant n + i$ holds. Now, for every $m > n$ and $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$ one has

$$\mathrm{Ext}_R^m(R/\mathfrak{p}, M) \cong \mathrm{Ext}_R^{m+i}(R/\mathfrak{p}, \mathrm{Z}_i(E)) = 0$$

where the isomorphism follows from $(\star)$ and the equality holds by 15.4.7. □

Flat Dimension

Let M be an R-complex. Recall from Sect. 8.3 that a semi-flat R-complex F that is isomorphic to M in $\mathcal{D}(R)$ is called a semi-flat replacement of M and that the flat dimension of M is defined as

$$\mathrm{fd}_R M = \inf\{\sup F^{\natural} \mid F \simeq M \text{ is a semi-flat replacement of } M\}\ .$$

15.4.17 Theorem. *Let M be an R-complex and n an integer. The following conditions are equivalent.*

(*i*) $\mathrm{fd}_R M \leqslant n$.
(*ii*) $\sup (N \otimes^{\mathsf{L}}_R M) \leqslant n + \sup N$ *holds for every R-complex N.*
(*iii*) $n \geqslant \sup M$ *and* $\mathrm{Tor}^R_{n+1}(R/\mathfrak{p}, M) = 0$ *holds for every prime ideal $\mathfrak{p}$ in R.*
(*iv*) $n \geqslant \sup M$ *and* $\mathrm{Tor}^R_1(\mathrm{Hom}_R(\mathrm{C}_{n+1}(F), E), \mathrm{C}_n(F)) = 0$ *holds for some, equivalently every, faithfully injective R-module E and semi-flat replacement F of M.*
(*v*) $n \geqslant \sup M$ *and for some, equivalently every, semi-flat replacement F of M the module $\mathrm{C}_n(F)$ is flat.*

(*vi*) $n \geqslant \sup M$ *and for every semi-flat replacement* F *of* M *the complex* $F_{\subseteq n}$ *is a semi-flat replacement of* M.

(*vii*) *There exists a semi-flat replacement* F *of* M *with* $F_v = 0$ *for all* $v > n$ *and for all* $v < \inf M$.

In particular, there are equalities

$$\begin{aligned} \operatorname{fd}_R M &= \sup\{\sup (R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R \colon \operatorname{Tor}^R_m(R/\mathfrak{p}, M) \neq 0\}\,. \end{aligned}$$

Proof. The functors $\operatorname{Tor}^R_m(-, M)$ are half exact, see 7.4.29, so for every ideal $\mathfrak{a}$ in R with $\operatorname{Tor}^R_m(R/\mathfrak{a}, M) \neq 0$ it follows from 12.4.1 that there is a prime ideal $\mathfrak{p}$ with $\operatorname{Tor}^R_m(R/\mathfrak{p}, M) \neq 0$. In view of this, the claims follows from 8.3.11. ◻

15.4.18 Theorem. *Let* M *be an* R*-complex. There is an inequality,*

$$\operatorname{fd}_R M \leqslant \operatorname{pd}_R M\,,$$

and equality holds if M *is in* $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$.

Proof. The assertions are immediate from 8.3.6 and 8.3.19. ◻

If R has finite Krull dimension, then an R-complex has finite flat dimension if and only if it has finite projective dimension; see 17.4.26.

Flat Base Change and Restriction of Scalars

The next result sharpens 8.3.8. For a result parallel to 15.4.19 about the injective dimension of a derived cobase changed complex, see 17.3.16.

15.4.19 Theorem. *Let* S *be an* R*-algebra and* M *an* R*-complex. There is an inequality,*

$$\operatorname{fd}_S(S \otimes^{\mathsf{L}}_R M) \leqslant \operatorname{fd}_R M\,,$$

and equality holds if S *is faithfully flat as an* R*-module.*

Proof. The inequality holds by 8.3.8. Assume that S is faithfully flat as an R-module; to prove the inequality $\operatorname{fd}_R M \leqslant \operatorname{fd}_S(S \otimes^{\mathsf{L}}_R M)$ assume that $\operatorname{fd}_S(S \otimes^{\mathsf{L}}_R M) < \infty$ holds. If the complex $S \otimes^{\mathsf{L}}_R M = S \otimes_R M$ is acyclic, then M is acyclic, see 2.5.7(c); one can now assume that $\operatorname{fd}_S(S \otimes_R M) = n$ holds for some $n \in \mathbb{Z}$. Let F be a semi-flat replacement of M; by 5.4.18(a) the S-complex $S \otimes_R F$ is a semi-flat replacement of $S \otimes^{\mathsf{L}}_R M$. It follows from 8.3.11 that the S-module $\mathrm{C}_n(S \otimes_R F) \cong S \otimes_R \mathrm{C}_n(F)$, see 2.2.19, is flat and hence also flat as an R-module by 5.4.24(b). That is, the functor $(S \otimes_R \mathrm{C}_n(F)) \otimes_R - \cong S \otimes_R (\mathrm{C}_n(F) \otimes_R -)$, where the isomorphism comes from associativity 12.1.8, is exact. Since $S \otimes_R -$ is faithfully exact, see 1.3.41, it follows from 1.1.45 that the functor $\mathrm{C}_n(F) \otimes_R -$ is exact, i.e. $\mathrm{C}_n(F)$ is a flat R-module. Now 15.4.17 yields $\operatorname{fd}_R M \leqslant n$. ◻

REMARK. Let S be an R-algebra and M an R-module. The crux of the proof above is that if S is faithfully flat as an R-module, then flatness of $S \otimes_R M$ over S implies flatness of M over R. Raynaud and Gruson [207] and Perry [201] show that if S is faithfully flat as an R-module, then projectivity of $S \otimes_R M$ over S implies projectivity of M over R. It follows that the assumption on the R-complex M in the next result is superfluous.

15.4.20 Proposition. *Let S be an R-algebra and M an R-complex. There is an inequality,*

$$\operatorname{pd}_S(S \otimes^{\mathsf{L}}_R M) \leqslant \operatorname{pd}_R M ,$$

and equality holds if S is faithfully flat as an R-module and M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$.

PROOF. The inequality holds by 8.1.4. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then the complex $S \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$, see 12.2.12, so assuming that S is faithfully flat as an R-module, the asserted equality holds by 15.4.18 and 15.4.19. $\square$

15.4.21 Proposition. *Let S be an R-algebra and N an S-complex. If N is not acyclic, then there is an inequality,*

$$\operatorname{fd}_R N \leqslant \operatorname{fd}_R S + \operatorname{fd}_S N .$$

In particular, if S is flat as an R-module, then one has $\operatorname{fd}_R N \leqslant \operatorname{fd}_S N$.

PROOF. In view of the unitor 12.3.3, this holds by 8.3.15(c) applied with $X = S$. $\square$

Given an ideal $\mathfrak{a}$ in R and an R-complex M, the objects $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ are $\widehat{R}^{\mathfrak{a}}$-complexes, see 11.3.4 and 11.3.18. The next results compare the flat dimensions of $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ over R and $\widehat{R}^{\mathfrak{a}}$; the first result is complemented by 16.1.21.

15.4.22 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex*; *one has*

$$\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \leqslant \operatorname{fd}_{\widehat{R}^{\mathfrak{a}}} \mathsf{L}\Lambda^{\mathfrak{a}}(M) .$$

PROOF. As an R-module, $\widehat{R}^{\mathfrak{a}}$ is flat, see 13.1.27, so in view of 11.3.4 the inequalities are special cases of 15.4.21. $\square$

15.4.23 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex*; *one has*

$$\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \operatorname{fd}_{\widehat{R}^{\mathfrak{a}}} \mathsf{R}\Gamma_{\mathfrak{a}}(M) .$$

PROOF. Recall from 13.1.27 that $\widehat{R}^{\mathfrak{a}}$ is flat as an R-module. In view of 13.4.17 the inequality "$\geqslant$" follows from 15.4.19, and the inequality "$\leqslant$" holds by 15.4.21. $\square$

RIGIDITY OF TOR

The phenomenon that vanishing of Tor_m for a single index implies vanishing for all subsequent indices is often referred to as "rigidity of Tor". It occurs for complexes that satisfy the assumption in 13.4.11. In the special case of the zero ideal, the equalities in the next result just recovers the equalities in 15.4.17.

15.4.24 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the complex $R/\mathfrak{p} \otimes_R^{\mathsf{L}} M$ is derived $\mathfrak{a}$-complete for every prime ideal $\mathfrak{p}$ in R. One has*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\sup (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}\\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \colon\ \operatorname{Tor}_m^R(R/\mathfrak{p}, M) \neq 0\}\,.\end{aligned}$$

Further, if for an integer $n \geqslant \sup M$ one has $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{fd}_R M \leqslant n$ holds.

Proof. Let $n \geqslant \sup M$ be an integer. If $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ holds for all $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then 13.4.6 yields $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p}$ in R. The equivalence of (*i*) and (*iii*) in 15.4.17 now yields $\operatorname{fd}_R M \leqslant n$. This proves the last assertion.

The second equality in the display holds by 7.4.19. Let s denote the supremum in the asserted equality. The inequality $\operatorname{fd}_R M \geqslant s$ holds by 15.4.17, and 13.4.6 yields $s \geqslant \sup M$. To prove that the opposite inequality, $\operatorname{fd}_R M \leqslant s$, holds, one can now assume that s is an integer. As one has $\operatorname{Tor}_{s+1}^R(R/\mathfrak{p}, M) = 0$ for every $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, the desired inequality holds by the argument above. ◻

The gist of the next result is that vanishing of certain Tor modules of a complex M detects the flat dimension of the derived $\mathfrak{a}$-complete complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$. The injective dimension of $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is computed in 17.3.20.

15.4.25 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex such that the inequality $\mathfrak{a}\text{-width}_R M > -\infty$ holds. There are equalities,*

$$\begin{aligned}\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) &= \sup\{\sup (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}\\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \colon\ \operatorname{Tor}_m^R(R/\mathfrak{p}, M) \neq 0\}\,.\end{aligned}$$

Further, if for an integer $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ one has $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \leqslant n$ holds.

Proof. The complex $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ is by 13.4.2 derived $\mathfrak{a}$-complete. Per 14.4.8 the assumption $\mathfrak{a}\text{-width}_R M > -\infty$ means that $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ belongs to $\mathcal{D}_{\sqsupset}(R)$, so 15.4.24 applies by 13.1.31(b) to $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$. It remains to recall that for $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ the cyclic module $R/\mathfrak{p}$ is $\mathfrak{a}$-torsion and, therefore, by 13.3.30 derived $\mathfrak{a}$-torsion, whence there is an isomorphism $R/\mathfrak{p} \otimes_R^{\mathsf{L}} \mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq R/\mathfrak{p} \otimes_R^{\mathsf{L}} M$ by 13.4.20(c). ◻

15.4.26 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M a derived $\mathfrak{a}$-complete complex in $\mathcal{D}_{\sqsupset}(R)$. There are equalities,*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\sup (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}\\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \colon\ \operatorname{Tor}_m^R(R/\mathfrak{p}, M) \neq 0\}\,.\end{aligned}$$

Further, if for an integer $n \geqslant \sup M$ one has $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{fd}_R M \leqslant n$ holds.

Proof. By assumption there is an isomorphism $\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq M$ in $\mathcal{D}(R)$, and 14.3.28 yields $\mathfrak{a}\text{-width}_R M \geqslant \inf M > -\infty$, so the result is a special case of 15.4.25. ◻

15.4.27 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an $\mathfrak{a}$-complete R-module. One has*

$$\operatorname{fd}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\; \operatorname{Tor}_m^R(R/\mathfrak{p}, M) \neq 0\}\;.$$

Further, if for an integer $n \geqslant 0$ one has $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{fd}_R M \leqslant n$ holds.

Proof. If view of 13.1.33, this result is a special case of 15.4.26. □

The flat dimensions of $\mathsf{L}\Lambda^{\mathfrak{a}}(M)$ and $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ are computed in 15.4.25 and 17.3.5.

15.4.28 Proposition. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex.*

(a) *If $\mathfrak{a}$-$\operatorname{width}_R M > -\infty$ holds, then one has $\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) \leqslant \operatorname{fd}_R M$.*
(b) *There is an inequality $\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) \leqslant \operatorname{fd}_R M$.*

Proof. Part (a) is an immediate consequence of 15.4.25 and 15.4.17.

(b): Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{a}$. One can assume that M and the Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ are not acyclic, otherwise the inequality is trivial per 13.3.18. The Čech complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ has by 11.4.10(c) flat dimension at most 0, so the inequality follows from 13.3.18 and 8.3.15(c). □

Remark. An example by Christensen, Ferraro, and Thompson [58] shows that the assumption in 15.4.28(a) is necessary.

For an R-complex M with $\mathfrak{a}$-$\operatorname{width}_R M > -\infty$ the next rigidity statement follows from 15.4.25.

15.4.29 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If for an integer $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{a}}(M)$ one has $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, then $\operatorname{Tor}_m^R(R/\mathfrak{p}, M) = 0$ holds for all integers $m > n$ and all prime ideals $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$.*

Proof. Set $s = \sup \mathsf{L}\Lambda^{\mathfrak{a}}(M)$, let P be a semi-projective replacement of M, and set $F = \Lambda^{\mathfrak{a}}(P)$. Note that $\mathrm{H}_v(F) = 0$ holds for $v > s$ as one has $F \simeq \mathsf{L}\Lambda^{\mathfrak{a}}(M)$. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ the module $R/\mathfrak{p}$ is $\mathfrak{a}$-torsion, so by 7.4.9, 11.2.21, and the definition of F, one has

$$(\diamond)\qquad R/\mathfrak{p} \otimes^{\mathsf{L}}_R M \;\simeq\; R/\mathfrak{p} \otimes_R P \;\cong\; R/\mathfrak{p} \otimes_R F\;.$$

Now, for every integer $m > s$ and every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ there are isomorphisms,

$$(\star)\qquad \begin{aligned} \operatorname{Tor}_m^R(R/\mathfrak{p}, M) &\cong \mathrm{H}_m(R/\mathfrak{p} \otimes_R F) \\ &\cong \mathrm{H}_m(R/\mathfrak{p} \otimes_R F_{\geqslant s}) \\ &\cong \mathrm{H}_{m-s}(R/\mathfrak{p} \otimes_R \Sigma^{-s} F_{\geqslant s}) \\ &\cong \operatorname{Tor}_{m-s}^R(R/\mathfrak{p}, \mathrm{C}_s(F))\;. \end{aligned}$$

Indeed, the 1st isomorphism holds by 7.4.18 and ($\diamond$), the 2nd holds as $m > s$, and the 3rd follows from 2.4.13 and 2.2.15. By 13.1.26 each module F_v is flat, so per 5.4.8 the complex $G = \Sigma^{-s} F_{\geqslant s}$ is a semi-flat replacement of the module $\mathrm{C}_s(F)$; this explains the 4th isomorphism. As $n + 1 > s$ holds, the assumption and ($\star$) yield

$$(\dagger\dagger)\qquad 0 = \operatorname{Tor}^R_{n+1}(R/\mathfrak{p}, M) \cong \operatorname{Tor}^R_{(n-s)+1}(R/\mathfrak{p}, \mathrm{C}_s(F)) \quad \text{for all} \quad \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\ .$$

The complex $F = \Lambda^{\mathfrak{a}}(P)$ is $\mathfrak{a}$-complete by 11.1.38, and hence so is $G = \Sigma^{-s} F_{\geqslant s}$. In view of 13.1.15 there are now isomorphisms $\mathsf{L}\Lambda^{\mathfrak{a}}(\mathrm{C}_s(F)) \simeq \Lambda^{\mathfrak{a}}(G) = G \simeq \mathrm{C}_s(F)$ in $\mathcal{D}(R)$, whence the module $\mathrm{C}_s(F)$ is derived $\mathfrak{a}$-complete, see 13.4.4. As one has $n - s \geqslant 0$, it now follows from 15.4.26 and $(\dagger\dagger)$ that $\operatorname{fd}_R \mathrm{C}_s(F) \leqslant n - s$ holds. For every $m > n$ and every $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ one now has

$$\operatorname{Tor}^R_m(R/\mathfrak{p}, M) \cong \operatorname{Tor}^R_{m-s}(R/\mathfrak{p}, \mathrm{C}_s(F)) = 0\ ,$$

where the isomorphism follows from $(\star)$ and the equality holds by 15.4.17. □

Faithful Injectivity

15.4.30 Theorem. *Let M be an R-complex and E an injective R-module. There is an inequality,*

$$\operatorname{id}_R \operatorname{Hom}_R(M, E) \leqslant \operatorname{fd}_R M\ ,$$

and equality holds if E is faithfully injective.

Proof. The inequailty is trivial if M is acyclic or E is zero; otherwise it is, in view of 7.3.22, a special case of 8.3.15(a). If E is faithfully injective, then equality holds by 8.3.17 applied with $\Bbbk = R$ and $\mathbb{E} = E$. □

Example 17.5.15 shows that the boundedness condition in 15.4.31 is necessary.

15.4.31 Theorem. *Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$ and E an injective R-module. There is an inequality,*

$$\operatorname{fd}_R \operatorname{Hom}_R(M, E) \leqslant \operatorname{id}_R M\ ,$$

and equality holds if E is faithfully injective.

Proof. The claims are trivial if M is acyclic; otherwise invoke 8.4.27. □

Faithful Flatness

15.4.32 Theorem. *Let M be an R-complex and F a flat R-module. There is an inequality,*

$$\operatorname{fd}_R(F \otimes_R M) \leqslant \operatorname{fd}_R M\ ,$$

and equality holds if F is faithfully flat.

Proof. The R-module $\operatorname{Hom}_{\Bbbk}(F, \mathbb{E})$ is by 1.3.48 injective and faithfully injective if F is faithfully flat. By 8.3.17, adjunction 12.1.10, and 15.4.30 one has

$$\begin{aligned}
\operatorname{fd}_R(F \otimes_R M) &= \operatorname{id}_R \operatorname{Hom}_{\Bbbk}(F \otimes_R M, \mathbb{E}) \\
&= \operatorname{id}_R \operatorname{Hom}_R(M, \operatorname{Hom}_{\Bbbk}(F, \mathbb{E})) \\
&\leqslant \operatorname{fd}_R M\ ,
\end{aligned}$$

and equality holds if F is faithfully flat. □

Example 17.5.14 shows that the boundedness condition in 15.4.33 is necessary.

15.4.33 Theorem. *Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$ and F a flat R-module. There is an inequality,*

$$\mathrm{id}_R(F \otimes_R M) \leqslant \mathrm{id}_R M\,,$$

and equality holds if F is faithfully flat.

PROOF. The claims are trivial if M is acyclic; otherwise invoke 8.4.16(a). □

EXERCISES

E 15.4.1 Let S be an R-algebra that is faithfully projective as an R-module. Show that the equality $\mathrm{pd}_S(S \otimes_R M) = \mathrm{pd}_R M$ holds for every R-complex M.

E 15.4.2 Let $R \to S$ be a surjective ring homomorphism. Show that every injective S-module is isomorphic to $\mathrm{Hom}_R(S, E)$ for some injective R-module E. *Hint:* C.16.

E 15.4.3 Let M be an R-complex of finite injective dimension n. Show that there is a prime ideal $\mathfrak{p}$ in R with $\mathrm{Ext}^n_R(\mathrm{E}_R(R/\mathfrak{p}), M) \neq 0$.

E 15.4.4 Let M be an R-complex of finite flat dimension n. Show that there is a prime ideal $\mathfrak{p}$ in R with $\mathrm{Tor}^R_n(\mathrm{E}_R(R/\mathfrak{p}), M) \neq 0$.

E 15.4.5 Let S be an R-algebra and $M \in \mathcal{D}(R)$ and $N \in \mathcal{D}(S)$ be complexes that are not acyclic. Show that the following inequalities hold:

(a) $\mathrm{id}_S \,\mathsf{RHom}_R(N, M) \leqslant \mathrm{fd}_S N + \mathrm{id}_R M\,.$

(b) $\mathrm{id}_S \,\mathsf{RHom}_R(M, N) \leqslant \mathrm{pd}_R M + \mathrm{id}_S N\,.$

(c) $\mathrm{fd}_S(N \otimes^{\mathsf{L}}_R M) \leqslant \mathrm{fd}_S N + \mathrm{fd}_R M\,.$

(d) $\mathrm{pd}_S(N \otimes^{\mathsf{L}}_R M) \leqslant \mathrm{pd}_S N + \mathrm{pd}_R M\,.$

E 15.4.6 Let S be an R-algebra and $M \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$ and $N \in \mathcal{D}(S)$ be complexes that are not acyclic. Show that if $\mathrm{pd}_R M$ is finite, then the following inequalities hold:

(a) $\mathrm{pd}_S \,\mathsf{RHom}_R(M, N) \leqslant \mathrm{pd}_S N - \inf M\,.$

(b) $\mathrm{fd}_S \,\mathsf{RHom}_R(M, N) \leqslant \mathrm{fd}_S N - \inf M\,.$

(c) $\mathrm{id}_S(N \otimes^{\mathsf{L}}_R M) \leqslant \mathrm{id}_S N - \inf M\,.$

E 15.4.7 Let S be an R-algebra and $M \in \mathcal{D}(R)$ and $N \in \mathcal{D}(S)$ be complexes with that are not acyclic. Show that the following inequalities hold:

(a) $\mathrm{fd}_S \,\mathsf{RHom}_R(N, M) \leqslant \mathrm{id}_S N + \sup M$ if $\mathrm{id}_R M < \infty$ and $N \in \mathcal{D}_{\sqsubset}(S)\,.$

(b) $\mathrm{id}_S(N \otimes^{\mathsf{L}}_R M) \leqslant \mathrm{id}_S N - \inf M$ if $\mathrm{fd}_R M < \infty$ and $N \in \mathcal{D}_{\sqsubset}(S)\,.$

(c) $\mathrm{fd}_S \,\mathsf{RHom}_R(M, N) \leqslant \mathrm{fd}_S N - \inf M$ if $\mathrm{pd}_R M < \infty$ and $N \in \mathcal{D}_{\sqsupset}(S)\,.$

E 15.4.8 Let $\mathfrak{p}$ be a prime ideal in R and M an $R_\mathfrak{p}$-complex. Show that $\mathrm{fd}_R M = \mathrm{fd}_{R_\mathfrak{p}} M$ and $\mathrm{id}_R M = \mathrm{id}_{R_\mathfrak{p}} M$ hold.

E 15.4.9 Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and E a faithfully injective R-module. (a): Show that $\mathrm{id}_R \,\mathsf{L}\Lambda^{\mathfrak{a}}(\mathrm{Hom}_R(M, E)) = \mathrm{fd}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds. (b): Provided that one has $\mathfrak{a}\text{-depth}_R M > -\infty$, show that $\mathrm{fd}_R \,\mathsf{L}\Lambda^{\mathfrak{a}}(\mathrm{Hom}_R(M, E)) = \mathrm{id}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds.

E 15.4.10 Let $\mathfrak{a} \subseteq R$ be an ideal, M an R-complex, and F a faithfully flat R-module. (a): Show that $\mathrm{fd}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(F \otimes_R M) = \mathrm{fd}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ holds. (b): Provided that $\mathfrak{a}\text{-depth}_R M > -\infty$ holds, show that one has $\mathrm{id}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(F \otimes_R M) = \mathrm{id}_R \,\mathsf{R}\Gamma_{\mathfrak{a}}(M)$.

E 15.4.11 Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. Show that there is an inequality $\mathrm{pd}_S \,\mathsf{R}\Gamma_{\mathfrak{a}}(N) \leqslant \mathrm{pd}_S N$.

E 15.4.12 Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. (a) Show that there is an inequality $\operatorname{id}_S \mathsf{L}\Lambda^{\mathfrak{a}}(N) \leqslant \operatorname{id}_S N$. (b) Show that if $\mathfrak{a}\text{-width}_R N > -\infty$ holds, then one has $\operatorname{id}_S \mathsf{R}\Gamma_{\mathfrak{a}}(N) \leqslant \operatorname{id}_S N$.

E 15.4.13 Let $\mathfrak{a} \subseteq R$ be an ideal, S an R-algebra, and N an S-complex. (a) Show that there is an inequality $\operatorname{fd}_S \mathsf{R}\Gamma_{\mathfrak{a}}(N) \leqslant \operatorname{fd}_S N$. (b) Show that if $\mathfrak{a}\text{-depth}_R N > -\infty$ holds, then one has $\operatorname{fd}_S \mathsf{L}\Lambda^{\mathfrak{a}}(N) \leqslant \operatorname{fd}_S N$.

E 15.4.14 Let U be a multiplicative subset of R; show that $\operatorname{gldim} U^{-1}R \leqslant \operatorname{gldim} R$ holds.

E 15.4.15 Let U be a multiplicative subset of R; show that $\operatorname{FPD} U^{-1}R \leqslant \operatorname{FPD} R$ holds.

E 15.4.16 Let U be a multiplicative subset of R; show that $\operatorname{FFD} U^{-1}R \leqslant \operatorname{FFD} R$ holds.

Chapter 16
Homological Invariants over Local Rings

In this chapter we are concerned with homological invariants of complexes over local rings. The transition to commutative Noetherian rings allowed us in Sects. 12.1–12.3 and 15.4 to restate several results from Parts I and II in a simpler or stronger form. There are similar gains to be reaped when one focuses on local rings, and many of them are recorded in Sects. 16.1 and 16.2. Firstly though, we recall some terminology and facts that are particular to the local situation; again, they may be found in [182] or any other standard reference on commutative algebra.

To save space, we abbreviate the statement that R is local with unique maximal ideal $\mathfrak{m}$ to "$(R, \mathfrak{m})$ is local." When we need the simple and standard notation $\boldsymbol{k}$ for the residue field $\kappa(\mathfrak{m}) = R/\mathfrak{m}$, we say that "$(R, \mathfrak{m}, \boldsymbol{k})$ is local."

A local ring $(R, \mathfrak{m})$ has finite Krull dimension, and the process of adjoining a power series variable increases the Krull dimension by one: $R[\![x]\!]$ is a local ring with unique maximal ideal generated by $\mathfrak{m}$ and x, traditionally written $\mathfrak{m} + (x)$, see 12.1.25; further one has $\dim R[\![x]\!] = \dim R + 1$. In particular, for a field $\Bbbk$ the algebra $\Bbbk[\![x_1, \ldots, x_n]\!]$ of power series in n variables with coefficients in $\Bbbk$ is local of Krull dimension n. Where it causes no confusion, the coset of an indeterminate x_i in a quotient ring $\Bbbk[\![x_1, \ldots, x_n]\!]/\mathfrak{a}$ is simply denoted x_i rather than $[x_i]_{\mathfrak{a}}$.

Let $(R, \mathfrak{m})$ be local of Krull dimension d. By maximality of $\mathfrak{m}$ every ideal $\mathfrak{a}$ in R with $\sqrt{\mathfrak{a}} = \mathfrak{m}$ is $\mathfrak{m}$-primary. Every power of $\mathfrak{m}$ is $\mathfrak{m}$-primary, and by Krull's intersection theorem 15.3.7 one has $\bigcap_{u \geqslant 1} \mathfrak{m}^u = 0$. An $\mathfrak{m}$-primary ideal is generated by no less than d elements, this follows from the general version of Krull's principal ideal theorem, see also 18.4.19, and d-generated $\mathfrak{m}$-primary ideals do exist. Let $\boldsymbol{x} = x_1, \ldots, x_d$ be a sequence in $\mathfrak{m}$. If the ideal $(\boldsymbol{x})$ is $\mathfrak{m}$-primary, then $\boldsymbol{x}$ is called a *parameter sequence* for R, and $(\boldsymbol{x})$ is called a *parameter ideal*. Every R-regular sequence, see 14.4.16, is part of a parameter sequence.

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a finitely generated R-module. The rank of the $\boldsymbol{k}$-vector space $\boldsymbol{k} \otimes_R M \cong M/\mathfrak{m}M$ is the *minimal number of generators* of M; it is the least possible cardinality of a set of generators for M, and every minimal set of generators for M has this cardinality. Every element in $M \setminus \mathfrak{m}M$ is part of minimal set of generators for M.

16.1 Modules and Complexes over Local Rings

Synopsis. Socle; Nakayama's lemma, vanishing of functor on residue field; complete local ring; injective envelope of residue field; Matlis Duality functor; Artinian module; (derived) Matlis reflexive complex; module of finite length.

Three modules play central roles in homological studies of modules and complexes over a local ring $(R, \mathfrak{m}, \boldsymbol{k})$. They are the residue field $\boldsymbol{k}$, its injective envelope $\mathrm{E}_R(\boldsymbol{k})$, and the $\mathfrak{m}$-completion $\widehat{R}$ of R.

Socle of a Module

16.1.1 Definition. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module. The submodule $(0 :_M \mathfrak{m})$ is called the *socle* of M and is also denoted $\mathrm{Soc}_R M$.

16.1.2. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module. By 1.1.8 there is an isomorphism $\mathrm{Soc}_R M \cong \mathrm{Hom}_R(\boldsymbol{k}, M)$. In particular, $\mathrm{Soc}_R M$ is a $\boldsymbol{k}$-vector space.

16.1.3 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. By 16.1.2 and C.21(a) there are isomorphisms $\mathrm{Soc}_R \mathrm{E}_R(\boldsymbol{k}) \cong \mathrm{Hom}_R(\boldsymbol{k}, \mathrm{E}_R(\boldsymbol{k})) \cong \boldsymbol{k}$.

16.1.4 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module. If M is finitely generated or Artinian, then the $\boldsymbol{k}$-vector space $\mathrm{Soc}_R M$ has finite rank.*

Proof. If M is finitely generated or Artinian, then so is the submodule $\mathrm{Soc}_R M$. The R-action on $\mathrm{Soc}_R M$ factors through $\boldsymbol{k}$, so it is a $\boldsymbol{k}$-vector space of finite rank. □

Nakayama's Lemma

In commutative algebra it is custom to also refer to the next consequene of B.32 for finitely generated modules as Nakayama's lemma.

16.1.5 Lemma. *Let $(R, \mathfrak{m})$ be local, $\mathfrak{a}$ a proper ideal in R, and M an R-module. If M is finitely generated or $\mathfrak{a}$-complete, then the following conditions are equivalent.*

(*i*) $M \neq 0$.
(*ii*) $R/\mathfrak{a} \otimes_R M \neq 0$ *i.e.* $\mathfrak{a}M \neq M$.
(*iii*) $\mathfrak{a}\text{-width}_R M = 0$.

Proof. Conditions (*ii*) and (*iii*) are equivalent by 14.3.29. Every proper ideal $\mathfrak{a}$ in R satisfies condition (*ii*) in Nakayama's lemma B.32, so conditions (*i*) and (*ii*) are equivalent if M is finitely generated. If M is $\mathfrak{a}$-complete, then (*i*) and (*ii*) are equivalent by 11.1.30. □

The next lemma is dual to Nakayama's lemma. It implies by 16.1.29 in particular to Artinian modules and shows that a non-zero Artinian module has non-zero socle.

16.1.6 Lemma. *Let $(R, \mathfrak{m})$ be local, $\mathfrak{a}$ a proper ideal in R, and M an R-module. If M is $\mathfrak{a}$-torsion, then the following conditions are equivalent.*

(*i*) $M \neq 0$.
(*ii*) $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$ *i.e.* $(0 :_M \mathfrak{a}) \neq 0$.
(*iii*) $\mathfrak{a}\text{-depth}_R M = 0$.

Proof. As $\Gamma_{\mathfrak{a}}(M) = M$ by assumption, the conditions are equivalent by 14.3.17. □

Vanishing of Functors

For linear endofunctors on the category of modules over a local ring, vanishing on finitely generated modules can be detected on the simple module.

16.1.7 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{F}(R/\mathfrak{p})$ is finitely generated for every prime ideal $\mathfrak{p}$ in R. If there is a finitely generated R-module M with $\mathrm{F}(M) \neq 0$, then one has $\mathrm{F}(\boldsymbol{k}) \neq 0$.*

Proof. This is a special case of 12.4.5, as $\mathfrak{m}$ is the Jacobson radical of R. □

16.1.8 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and $\mathrm{F}\colon \mathcal{M}(R) \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{F}(R/\mathfrak{p})$ is $\mathfrak{m}$-complete for every prime ideal $\mathfrak{p}$ in R. If there is a finitely generated R-module M with $\mathrm{F}(M) \neq 0$, then one has $\mathrm{F}(\boldsymbol{k}) \neq 0$.*

Proof. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 12.4.6. □

16.1.9 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ a half exact and R-linear functor such that $\mathrm{G}(R/\mathfrak{p})$ is $\mathfrak{m}$-torsion for every prime ideal $\mathfrak{p}$ in R. If there is a finitely generated R-module M with $\mathrm{G}(M) \neq 0$, then one has $\mathrm{G}(\boldsymbol{k}) \neq 0$.*

Proof. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 12.4.11. □

Completion of a Local Ring

To parse the next statement recall from 5.5.15 the definition a of pure monomorphism.

16.1.10 Proposition. *Let R be local and $\mathfrak{a}$ a proper ideal in R.*

(a) *The R-algebra $\widehat{R}^{\mathfrak{a}}$ is a Noetherian local ring, and the structure map $R \to \widehat{R}^{\mathfrak{a}}$ is a pure monomorphism.*
(b) *As an R-module, $\widehat{R}^{\mathfrak{a}}$ is faithfully flat.*

Proof. That $\widehat{R}^{\mathfrak{a}}$ is a Noetherian local ring was proved in 11.1.22. As $\mathfrak{a}$ is contained in the maximal ideal of R, part (b) is a special case of 15.3.6, and it now follows from 12.1.23 that the structure map $R \to \widehat{R}^{\mathfrak{a}}$ is a pure monomorphism. □

16.1.11 Definition. A local ring $(R, \mathfrak{m})$ that is $\mathfrak{m}$-complete is just called *complete*.

16.1.12 Definition. Let $(R, \mathfrak{m})$ be local. One writes $\widehat{R}$ for the $\mathfrak{m}$-completion of R and refers to it as simply the *completion* of R. It is by 11.1.22 a commutative Noetherian local ring, and one denotes its maximal ideal $\widehat{\mathfrak{m}}$.

16.1.13 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. The local ring $(\widehat{R}, \widehat{\mathfrak{m}})$ is complete; it has maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field $\widehat{R}/\widehat{\mathfrak{m}} \cong \boldsymbol{k}$. As an R-module, $\widehat{R}$ is faithfully flat. Moreover, there is an isomorphism $\widehat{R}/\mathfrak{m}^u\widehat{R} \cong R/\mathfrak{m}^u$ for every $u \geqslant 1$.*

Proof. The last assertion is a special case of 11.1.37(a). For $u = 1$ it reads $\widehat{R}/\mathfrak{m}\widehat{R} \cong \boldsymbol{k}$, so $\mathfrak{m}\widehat{R}$ must be the unique maximal ideal $\widehat{\mathfrak{m}}$ of $\widehat{R}$. It now follows from 11.1.39 that $(\widehat{R}, \widehat{\mathfrak{m}})$ is complete per 16.1.11. As an R-module, $\widehat{R}$ is faithfully flat by 16.1.10(b). $\square$

16.1.14. Let $(R, \mathfrak{m})$ be local. Notice from 12.1.22 and 13.2.3 that the maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ of $\widehat{R}$ is the $\mathfrak{m}$-completion of $\mathfrak{m}$ as an R-module.

16.1.15. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and V a $\boldsymbol{k}$-vector space; it is by 16.1.13 an $\widehat{R}$-module, and by 3.1.13 and 1.1.10 there is an isomorphism $\widehat{R} \otimes_R V \cong V$ of $\widehat{R}$-modules. Thus, for every complex W of $\boldsymbol{k}$-vector spaces one has $\widehat{R} \otimes_R W \cong W$ in $\mathcal{C}(\widehat{R})$.

16.1.16 Proposition. *Let $(R, \mathfrak{m})$ be local; there is an inclusion,*

$$\mathcal{D}^{\mathrm{f}}(\widehat{R}) \subseteq \mathcal{D}^{\mathfrak{m}\text{-com}}(R)\,.$$

Proof. As $(\widehat{R}, \widehat{\mathfrak{m}})$ per 16.1.13 is complete, $\mathsf{L}\Lambda^{\widehat{\mathfrak{m}}}$ is the identity functor on $\mathcal{D}^{\mathrm{f}}(\widehat{R})$ by 13.2.7 and the unitor 12.3.3. Thus every complex in $\mathcal{D}^{\mathrm{f}}(\widehat{R})$ is derived $\widehat{\mathfrak{m}}$-complete. As $\widehat{\mathfrak{m}}$ is the extension $\mathfrak{m}\widehat{R}$, still per 16.1.13, it now follows from 13.1.21(a) that every complex in $\mathcal{D}^{\mathrm{f}}(\widehat{R})$ is derived $\mathfrak{m}$-complete as an R-complex. $\square$

16.1.17 Theorem. *Let $(R, \mathfrak{m})$ be local and M an R-complex. The following conditions are equivalent.*

(*i*) *M is derived $\mathfrak{m}$-complete.*
(*ii*) $\operatorname{cosupp}_R M \subseteq \{\mathfrak{m}\}$.

Proof. The statement is the special case $\mathfrak{a} = \mathfrak{m}$ of 15.3.19. $\square$

The two results above determine the cosupport of every finitely generated module over a complete local ring.

16.1.18 Corollary. *Let $(R, \mathfrak{m})$ be local. For every complex M in $\mathcal{D}^{\mathrm{f}}(\widehat{R})$ that is not acyclic one has* $\operatorname{cosupp}_R M = \{\mathfrak{m}\}$.

Proof. The assertion follows in view of 15.2.8 from 16.1.16 and 16.1.17. $\square$

For a complete local ring the support and cosupport differ as much as they possibly can, see 15.2.19.

16.1.19 Example. Let $(R, \mathfrak{m})$ be local. As $\widehat{R}$ by 16.1.13 is a faithfully flat R-module, one has $\operatorname{supp}_R \widehat{R} = \operatorname{Spec} R$ by 15.1.18 while 16.1.18 yields $\operatorname{cosupp}_R \widehat{R} = \{\mathfrak{m}\}$.

In particular, if R is complete, then one has $\operatorname{cosupp}_R R = \{\mathfrak{m}\}$.

Recall from 11.3.4 and 11.3.18 that for an R-complex M, both $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ and $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ are $\widehat{R}$-complexes. Natural partners to the next result are 15.4.23 and 17.3.19.

16.1.20 Theorem. *Let $(R, \mathfrak{m})$ be local and M an R-complex. If* $\operatorname{depth}_R M > -\infty$ *holds, then there is an equality,*

$$\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;=\; \operatorname{id}_{\widehat{R}} \mathsf{R}\Gamma_{\mathfrak{m}}(M)\,.$$

Proof. Let $\boldsymbol{k}$ be the residue field of R. Recall from 16.1.13 that the ring $\widehat{R}$ is local with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field $\boldsymbol{k}$. As an R-complex, $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ is derived $\mathfrak{m}$-torsion by 13.4.7, so it follows from 11.3.18 and 13.3.23(a) that $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ is derived $\widehat{\mathfrak{m}}$-torsion as an $\widehat{R}$-complex. Per 14.4.3 it follows from the assumption on M that $\mathrm{H}(\mathsf{R}\Gamma_{\mathfrak{m}}(M))$ is bounded above. Now 15.4.13 applies with $\mathfrak{a} = \mathfrak{m}$ to the R-complex $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$, and it also applies with $\mathfrak{a} = \widehat{\mathfrak{m}}$ to $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ viewed as an $\widehat{R}$-complex. That explains the first and last equalities in the computation below; the middle equality holds by 13.4.18(b):

$$\begin{aligned}
\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;&=\; -\inf \mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(M))\\
&=\; -\inf \mathsf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(M))\\
&=\; \operatorname{id}_{\widehat{R}} \mathsf{R}\Gamma_{\mathfrak{m}}(M)\,. \qquad \square
\end{aligned}$$

The theorem above is applied in the proof of 16.1.25 below; here we also record the accompanying result for flat dimension of derived $\mathfrak{m}$-complete complexes.

16.1.21 Theorem. *Let $(R, \mathfrak{m})$ be local and M an R-complex. If* $\operatorname{width}_R M > -\infty$ *holds, then there is an equality,*

$$\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) \;=\; \operatorname{fd}_{\widehat{R}} \mathsf{L}\Lambda^{\mathfrak{m}}(M)\,.$$

Proof. Let $\boldsymbol{k}$ be the residue field of R. Recall from 16.1.13 that the ring $\widehat{R}$ is local with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field $\boldsymbol{k}$. As an R-complex, $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ is derived $\mathfrak{m}$-complete by 13.4.2, so it follows from 11.3.4 and 13.1.21(a) that $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ is derived $\widehat{\mathfrak{m}}$-complete as an $\widehat{R}$-complex. Per 14.4.8 it follows from the assumption on M that $\mathrm{H}(\mathsf{L}\Lambda^{\mathfrak{m}}(M))$ is bounded below. Consequently, 15.4.26 applies with $\mathfrak{a} = \mathfrak{m}$ to the R-complex $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$, and it also applies with $\mathfrak{a} = \widehat{\mathfrak{m}}$ to $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ viewed as an $\widehat{R}$-complex. This explains the first and last equalities in the computation below; the middle equality holds by 13.4.18(c) and commutativity 12.3.5:

$$\begin{aligned}
\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) \;&=\; \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M))\\
&=\; \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_{\widehat{R}} \mathsf{L}\Lambda^{\mathfrak{m}}(M))\\
&=\; \operatorname{fd}_{\widehat{R}} \mathsf{L}\Lambda^{\mathfrak{m}}(M)\,. \qquad \square
\end{aligned}$$

The Matlis Duality Functor

The functor $\mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))$ is called the *Matlis Duality functor*, and for an R-complex M one refers to $\mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))$ as the *Matlis dual* of M. For ease of reference, we recall some frequetly used facts about this functor in 16.1.22; there is more than this to actual Matlis Duality, see 18.1.9.

16.1.22. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. Per 16.1.3 there is an isomorphism,

$$\mathrm{Hom}_R(\boldsymbol{k}, \mathrm{E}_R(\boldsymbol{k})) \;\cong\; \boldsymbol{k}\;. \tag{16.1.22.1}$$

The R-module $\mathrm{E}_R(\boldsymbol{k})$ is $\mathfrak{m}$-torsion per C.14 and hence derived $\mathfrak{m}$-torsion by 13.3.30. It is proved below in 16.1.26 that $\mathrm{E}_R(\boldsymbol{k})$ is Artinian, which is stronger, see 16.1.29.

Further, $\mathrm{E}_R(\boldsymbol{k})$ is a faithfully injective R-module, see C.3; that is, the functor $\mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))$ is faithfully exact. Let M and N be R-complexes. By 2.5.7(b) the next equalities hold,

$$\begin{aligned} -\sup \mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})) \;&=\; \inf M \quad \text{and} \\ -\inf \mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})) \;&=\; \sup M\;. \end{aligned} \tag{16.1.22.2}$$

Moreover, for every $m \in \mathbb{Z}$ there is by 8.3.1 an isomorphism,

$$\mathrm{Hom}_R(\mathrm{Tor}^R_m(M, N), \mathrm{E}_R(\boldsymbol{k})) \;\cong\; \mathrm{Ext}^m_R(N, \mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})))\;; \tag{16.1.22.3}$$

in particular, $\mathrm{Tor}^R_m(M, N) = 0$ if and only if $\mathrm{Ext}^m_R(N, \mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))) = 0$.

Finally, recall from 13.3.6 that $\mathrm{E}_R(\boldsymbol{k})$ is an $\widehat{R}$-module.

The Injective Envelope $\mathrm{E}_R(\boldsymbol{k})$ and its Endomorphisms

To parse the next result, recall from 4.5.5 and 10.1.1 the definition of the homothety formation map.

16.1.23 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. Homothety formation,*

$$\chi^{\mathrm{E}_R(\boldsymbol{k})}_{\widehat{R}R} : \widehat{R} \longrightarrow \mathrm{Hom}_R(\mathrm{E}_R(\boldsymbol{k}), \mathrm{E}_R(\boldsymbol{k}))\;,$$

is an isomorphism of R-algebras, and this map agrees with homothety formation

$$\chi^{\mathrm{E}_R(\boldsymbol{k})}_{\widehat{R}} : \widehat{R} \longrightarrow \mathrm{Hom}_{\widehat{R}}(\mathrm{E}_R(\boldsymbol{k}), \mathrm{E}_R(\boldsymbol{k}))\;.$$

Proof. Set $E = \mathrm{E}_R(\boldsymbol{k})$. By C.14 one has $\Gamma_{\mathfrak{m}}(E) = E$, so 11.2.26(a) yields

$$\mathrm{Hom}_R(E, E) \;=\; \mathrm{Hom}_{\widehat{R}}(E, E)\;,$$

and hence the two homotopy formation maps in question are identical. Set $\chi = \chi^E_{\widehat{R}R}$. It is known from 11.1.26 that χ is a morphism of R-algebras, so it suffices to argue that χ is bijective. With $E^u = (0 :_E \mathfrak{m}^u)$ as in C.20 one has, per 3.3.34,

$$E \;=\; \bigcup_{u\geqslant 1} E^u \;\cong\; \operatorname*{colim}_{u\geqslant 1} E^u\;. \tag{$\diamond$}$$

By definition, $\chi(r)(e) = r^u e$ for elements $r = ([r^v]_{\mathfrak{m}^v})_{v\geqslant 1} \in \widehat{R}$ and $e \in E^u \subseteq E$. Furthermore, for every $u \geqslant 1$ there is an isomorphism $\chi^u \colon R/\mathfrak{m}^u \to \mathrm{Hom}_R(E^u, E)$ given by $\chi^u([r']_{\mathfrak{m}^u})(e) = r'e$ for $[r']_{\mathfrak{m}^u} \in R/\mathfrak{m}^u$ and $e \in E^u$, see C.21(c). It is straightforward to verify that $\{\chi^u\}_{u\geqslant 1}$ is an isomorphism of towers which, therefore, induces the lower horizontal isomorphism in the following commutative diagram:

$$\begin{tikzcd} \widehat{R} \arrow[rr, "\chi"] \arrow[d, equal] & & \operatorname{Hom}_R(E,E) \arrow[d, "\cong"'] \\ \lim\limits_{u\geqslant 1} R/\mathfrak{m}^u \arrow[rr, "\lim_{u\geqslant 1}\chi^u", "\cong"'] & & \lim\limits_{u\geqslant 1} \operatorname{Hom}_R(E^u,E) \, . \end{tikzcd}$$

The vertical isomorphism in the diagram follows, in view of ($\diamond$), from 3.4.29. Consequently, χ is an isomorphism. $\square$

In view 14.2.19(d) the next result generalizes 10.1.8.

16.1.24 Corollary. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and complete. Homothety formation,*

$$\chi_R^{\mathrm{E}(\boldsymbol{k})} : R \longrightarrow \operatorname{Hom}_R(\mathrm{E}_R(\boldsymbol{k}),\mathrm{E}_R(\boldsymbol{k}))\,,$$

is an isomorphism.

Proof. As one has $\widehat{R} = R$, this is a special case of 16.1.23. $\square$

16.1.25 Proposition. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local; there are isomorphisms of $\widehat{R}$-modules,*

$$\mathrm{E}_R(\boldsymbol{k}) \cong \widehat{R}\otimes_R \mathrm{E}_R(\boldsymbol{k}) \qquad \textit{and} \qquad \mathrm{E}_R(\boldsymbol{k}) \cong \mathrm{E}_{\widehat{R}}(\boldsymbol{k})\,.$$

Proof. As one has $\Gamma_{\mathfrak{m}}(\boldsymbol{k}) = \boldsymbol{k}$ and $\Gamma_{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k})) = \mathrm{E}_R(\boldsymbol{k})$, see 13.3.3, it follows from 11.2.23 that $\boldsymbol{k} \rightarrowtail \mathrm{E}_R(\boldsymbol{k})$ is an embedding of $\widehat{R}$-modules. The first isomorphism is a special case of 13.3.7. Further, $\mathrm{E}_R(\boldsymbol{k})$ is by 16.1.20 injective as an $\widehat{R}$-module, and since as $\boldsymbol{k}$ is essential in $\mathrm{E}_R(\boldsymbol{k})$ as an R-submodule it is essential as an $\widehat{R}$-submodule. Thus, the second isomorphism follows from B.13 and B.16. $\square$

16.1.26 Proposition. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local. The module $\mathrm{E}_R(\boldsymbol{k})$ is Artinian both as R- and $\widehat{R}$-module.*

Proof. The R- and $\widehat{R}$-submodules of $\mathrm{E}_R(\boldsymbol{k})$ agree; see 13.3.6. Set $E = \mathrm{E}_R(\boldsymbol{k})$ and let $E \supseteq M^1 \supseteq M^2 \supseteq \cdots$ be a descending chain of submodules. As E is injective, the chain yields a sequence of surjective homomorphisms,

$$\operatorname{Hom}_R(E,E) \longrightarrow \operatorname{Hom}_R(M^1,E) \longrightarrow \operatorname{Hom}_R(M^2,E) \longrightarrow \cdots\,,$$

of $\widehat{R}$-modules. By 16.1.23 one has $\operatorname{Hom}_R(E,E) \cong \widehat{R}$, so the modules $\operatorname{Hom}_R(M^u,E)$ are cyclic. Thus, one has an ascending sequence of ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ with $\widehat{R}/\mathfrak{a}_u \cong \operatorname{Hom}_R(M^u,E)$. As $\widehat{R}$ is Noetherian, see 16.1.10, this sequence stabilizes. As E is faithfully injective, see C.3, it follows that the original descending sequence of submodules stabilizes. $\square$

ARTINIAN MODULES

16.1.27 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If M is degreewise finitely generated, then* $\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))$ *is degreewise Artinian.*

PROOF. By 2.3.1 one has $\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))_v = \operatorname{Hom}_R(M_{-v}, \mathrm{E}_R(\boldsymbol{k}))$ for every $v \in \mathbb{Z}$, so one can assume that M is a finitely generated R-module. By 1.3.12 there is a surjective homomorphism $R^m \twoheadrightarrow M$ for some $m \in \mathbb{N}$, which by the counitor 1.2.2 yields an injective homomorphism $\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})) \rightarrowtail \mathrm{E}_R(\boldsymbol{k})^m$. Hence, $\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))$ is Artinian by 16.1.26. □

16.1.28 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If M is in $\mathcal{D}^{\mathrm{f}}(R)$, then* $\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))$ *belongs to* $\mathcal{D}^{\mathrm{art}}(R)$.

PROOF. By 2.2.19 one has $\mathrm{H}(\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))) \cong \operatorname{Hom}_R(\mathrm{H}(M), \mathrm{E}_R(\boldsymbol{k}))$, so the assertion follows from 16.1.27. □

16.1.29 Proposition. *Let $(R, \mathfrak{m})$ be local. Every degreewise Artinian R-complex is $\mathfrak{m}$-torsion.*

PROOF. It suffices to prove the assertion for R-modules. Every Artinian R-module M has $\operatorname{Supp}_R M \subseteq \{\mathfrak{m}\}$ by 14.2.10 and is hence $\mathfrak{m}$-torsion by 14.1.3. □

We record the following counterpart to the inclusion in 16.1.16.

16.1.30 Corollary. *Let $(R, \mathfrak{m})$ be local; there is an inclusion,*

$$\mathcal{D}^{\mathrm{art}}(R) \subseteq \mathcal{D}^{\mathfrak{m}\text{-tor}}(R) .$$

PROOF. The inclusion follows from 16.1.29 and 13.4.9. □

16.1.31 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex.*

(a) *The following conditions are equivalent.*

  (*i*) *M is derived $\mathfrak{m}$-torsion.*

  (*ii*) $\operatorname{supp}_R M \subseteq \{\mathfrak{m}\}$.

  (*iii*) $\operatorname{Supp}_R M \subseteq \{\mathfrak{m}\}$.

(b) *If M is derived $\mathfrak{m}$-torsion, then* $\dim_R M = -\inf M$ *holds.*

PROOF. The statement is the special case $\mathfrak{a} = \mathfrak{m}$ of 15.3.23. □

By 16.1.30 the next result applies, in particular, to complexes in $\mathcal{D}^{\mathrm{art}}(R)$.

16.1.32 Corollary. *Let $(R, \mathfrak{m})$ be local and M an R-complex. If M is derived $\mathfrak{m}$-torsion and not acyclic, then there are equalities,*

$$\operatorname{supp}_R M = \{\mathfrak{m}\} = \operatorname{Supp}_R M .$$

PROOF. The equalities follow from 16.1.31(a), 15.1.15, and 14.2.4. □

The next equality compares to the last equality in 14.2.16 and to 16.1.30.

16.1.33 Corollary. *Let $(R, \mathfrak{m})$ be local; there is an equality,*

$$\mathcal{D}^{\ell}(R) = \mathcal{D}^{\mathfrak{m}\text{-tor}}(R) \cap \mathcal{D}^{\mathrm{f}}(R) .$$

PROOF. The equality follows from 13.4.9, 16.1.31, and 14.2.9. □

16.1.34 Example. Let $(R, \mathfrak{m})$ be local, $\boldsymbol{x}$ a sequence that generates $\mathfrak{m}$, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. By 14.3.4(a) the complex $\mathrm{K}^{R}(\boldsymbol{x}) \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$, and by 13.3.31 it is derived $\mathfrak{m}$-torsion, so it belongs to $\mathcal{D}^{\ell}(R)$ by 16.1.33.

MATLIS REFLEXIVITY

16.1.35 Definition. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex.

(1) If the biduality morphism in $\mathcal{C}(R)$ from 4.5.2,

$$\delta^{M}_{\mathrm{E}_R(\boldsymbol{k})} : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})), \mathrm{E}_R(\boldsymbol{k})) ,$$

is an isomorphism, then M is called *Matlis reflexive*.

(2) If the biduality morphism in $\mathcal{D}(R)$ from 8.4.2,

$$\boldsymbol{\delta}^{M}_{\mathrm{E}_R(\boldsymbol{k})} : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})), \mathrm{E}_R(\boldsymbol{k})) ,$$

is an isomorphism, then M is called *derived Matlis reflexive*.

16.1.36 Lemma. *Let R be local and M an R-complex.*

(a) *If M is Matlis reflexive, then M is derived Matlis reflexive.*
(b) *M is derived Matlis reflexive if and only if $\mathrm{H}(M)$ is Matlis reflexive.*
(c) *If M is a module, then it is Matlis reflexive if and only if it is derived Matlis reflexive.*

PROOF. Let $\boldsymbol{k}$ denote the residue field of R and set $E = \mathrm{E}_R(\boldsymbol{k})$. As E is an injective R-module, the biduality morphism $\boldsymbol{\delta}^{M}_{E}$ is induced by δ^{M}_{E}, see 8.4.3. This proves part (a). Further, one has

$$\mathrm{H}(\boldsymbol{\delta}^{M}_{E}) = \mathrm{H}(\delta^{M}_{E}) \cong \delta^{\mathrm{H}(M)}_{E} ,$$

where the isomorphism comes from 2.2.19. In view of 6.5.17 this proves part (b). If M is a module, then it is isomorphic to $\mathrm{H}(M)$, so part (c) follows from (b). □

The next proposition applies, in particular, to a short exact sequence of complexes, see 6.5.24.

16.1.37 Proposition. *Let R be local and $M' \to M \to M'' \to \Sigma M'$ a distinguished triangle of R-complexes. If two of the complexes M', M, and M'' are derived Matlis reflexive, then so is the third.*

PROOF. Let $\boldsymbol{k}$ be the residue field of R. The biduality morphism $\boldsymbol{\delta}_{\mathrm{E}_R(\boldsymbol{k})}$ from 16.1.35(2) is a triangulated natural transformation of triangulated endofunctors on $\mathcal{D}(R)$, see 12.3.9. Thus the assertion follows from E.19. □

REMARK. Notice from the proof of 16.1.37 that the full subcategory of $\mathcal{D}(R)$ whose objects are the derived Matlis reflexive complexes is a triangulated subcategory of $\mathcal{D}(R)$.

16.1.38 Proposition. *Let R be local and $0 \to M' \to M \to M'' \to 0$ an exact sequence of R-complexes. The complex M is Matlis reflexive if and only if M' and M'' are Matlis reflexive.*

PROOF. Let $\boldsymbol{k}$ denote the residuel field of R and set $E = \mathrm{E}_R(\boldsymbol{k})$. Since the functor $(-)^\vee = \mathrm{Hom}_R(-, E)$ is exact, there is a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\delta_E^{M'}"] & M \arrow[r] \arrow[d, "\delta_E^{M}"] & M'' \arrow[r] \arrow[d, "\delta_E^{M''}"] & 0 \\
0 \arrow[r] & (M')^{\vee\vee} \arrow[r] & (M)^{\vee\vee} \arrow[r] & (M'')^{\vee\vee} \arrow[r] & 0 .
\end{tikzcd}$$

As E is faithfully injective, see 16.1.22, the vertical morphisms are injective by 4.5.3; the assertion now follows from the Snake Lemma 2.1.45. □

REMARK. The proposition above expresses that the class of Matlis reflexive R-complexes constitute a Serre subcategory of $\mathcal{C}(R)$.

MODULES OF FINITE LENGTH

16.1.39 Corollary. *Let R be local. Every R-complex that is degreewise of finite length is Matlis reflexive.*

PROOF. By C.21(b) the residue field $\boldsymbol{k}$ of R is a Matlis reflexive module. It thus follows from 16.1.38 that every R-module of finite length is Matlis reflexive, and hence so is every complex M of R-modules of finite length as one has $(\delta^M_{\mathrm{E}_R(\boldsymbol{k})})_v = \delta^{M_v}_{\mathrm{E}_R(\boldsymbol{k})}$. □

16.1.40 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex that is degreewise of finite length. The canonical morphism $M \to \widehat{R} \otimes_R M$ is an isomorphism; in particular, M is $\mathfrak{m}$-complete.*

PROOF. It follows from 14.2.12 and 16.1.29 that M is degreewise $\mathfrak{m}$-torsion, so the canonical morphism is an isomorphism by 11.2.27. It now follows from 13.2.4 that it is $\mathfrak{m}$-complete. □

16.1.41 Corollary. *Let $(R, \mathfrak{m})$ be local and M a complex in $\mathcal{D}^\ell(R)$. The canonical morphism $M \to \widehat{R} \otimes_R M$ is an isomorphism in $\mathcal{D}(R)$; in particular, M is derived $\mathfrak{m}$-complete.*

PROOF. To see that the canonical morphism is an isomorphism in $\mathcal{D}(R)$, it suffices by 6.4.18 to verify that it is a quasi-isomorphism. By 16.1.10 the functor $\widehat{R} \otimes_R -$ is exact, so application of H to $M \to \widehat{R} \otimes_R M$ yields by 12.1.20(b) the canonical morphism $\mathrm{H}(M) \to \widehat{R} \otimes_R \mathrm{H}(M)$, which is an isomorphism by 16.1.40. Further, the complex $\mathrm{H}(M)$ is $\mathfrak{m}$-complete, so M is derived $\mathfrak{m}$-complete by 13.4.4. □

16.1.42 Corollary. *Let $(R, \mathfrak{m})$ be local. Every complex in $\mathcal{D}^{\ell}(R)$ is derived $\mathfrak{m}$-torsion and derived $\mathfrak{m}$-complete. That is, one has*

$$\mathcal{D}^{\ell}(R) \subseteq \mathcal{D}^{\mathfrak{m}\text{-com}}(R) \cap \mathcal{D}^{\mathfrak{m}\text{-tor}}(R) .$$

Proof. The claim is immediate from 16.1.41 and 16.1.33. □

16.1.43 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. The R-module $\boldsymbol{k}^{(\mathbb{N})}$ belongs by 13.1.22 and 13.3.24 to $\mathcal{D}^{\mathfrak{m}\text{-com}}(R) \cap \mathcal{D}^{\mathfrak{m}\text{-tor}}(R)$ but evidently not to $\mathcal{D}^{\ell}(R)$; thus the inclusion in 16.1.42 is strict.

16.1.44 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module; one has*

$$\operatorname{length}_R M = \operatorname{length}_R \operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})) .$$

Proof. Set $E = \mathrm{E}_R(\boldsymbol{k})$. Assume first that $l = \operatorname{length}_R M$ is finite and proceed by induction on l. If $l = 1$, then M is simple, whence $\operatorname{Hom}_R(M, E) \cong \operatorname{Hom}_R(\boldsymbol{k}, E) \cong \boldsymbol{k}$ by (16.1.22.1). If $l > 1$ holds, then a composition series of M yields an exact sequence $0 \to M' \to M \to \boldsymbol{k} \to 0$ with $\operatorname{length}_R M' = l - 1$. Now apply the exact functor $\operatorname{Hom}_R(-, E)$ to this sequence and use that length is additive on exact sequences.

Assume now that $\operatorname{length}_R \operatorname{Hom}_R(M, E)$ is finite; it follows that also the module $\operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)$ has finite length. The biduality map δ_E^M is injective by 4.5.3, so M is isomorphic to a submodule of a module of finite length and hence of finite length. □

Exercises

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 16.1.1 Show that for complexes $M \in \mathcal{D}(R)$ and $K \in \mathcal{D}(\boldsymbol{k})$ with $\mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \neq 0 \neq \mathrm{H}(K)$ there are equalities:

(a) $\sup (K \otimes^{\mathsf{L}}_R M) = \sup K + \sup (\boldsymbol{k} \otimes^{\mathsf{L}}_R M)$.

(b) $\inf (K \otimes^{\mathsf{L}}_R M) = \inf K + \inf (\boldsymbol{k} \otimes^{\mathsf{L}}_R M)$.

(c) $-\sup \mathsf{RHom}_R(K, M) = \inf K - \sup \mathsf{RHom}_R(\boldsymbol{k}, M)$.

(d) $-\inf \mathsf{RHom}_R(K, M) = \sup K - \inf \mathsf{RHom}_R(\boldsymbol{k}, M)$.

(e) $-\sup \mathsf{RHom}_R(M, K) = \inf (\boldsymbol{k} \otimes^{\mathsf{L}}_R M) - \sup K$.

(f) $-\inf \mathsf{RHom}_R(M, K) = \sup (\boldsymbol{k} \otimes^{\mathsf{L}}_R M) - \inf K$.

E 16.1.2 Show that if R is not complete, then $\mathrm{E}_R(\boldsymbol{k})$ is a proper direct summand of the R-module $\operatorname{Hom}_R(\widehat{R}, \mathrm{E}_R(\boldsymbol{k}))$.

E 16.1.3 (a) Show that every homomorphism $\widehat{R} \to R$ of R-modules is given by multiplication by an element $r \in R$. (b) Show that if R is an integral domain and not complete, then the zero map is the only homomorphism, $\widehat{R} \to R$, of R-modules. *Hint:* $\widehat{R}/R$ is torsion-free.

E 16.1.4 Show that R is complete if and only if $\widehat{R}$ is a finitely generated R-module.

E 16.1.5 Let F be a semi-flat R-complex. Show that if F is bounded below, then the $\widehat{R}$-complex $\Lambda^{\mathfrak{m}}(F)$ is semi-flat. Compare to 13.1.28.

E 16.1.6 Let I be a semi-injective R-complex. Show that if I is bounded above, then the $\widehat{R}$-complex $\Gamma_{\mathfrak{m}}(I)$ is semi-injective. Compare to 13.3.8.

E 16.1.7 Give an example of a complex that is derived Matlis reflexive though not Matlis reflexive.

E 16.1.8 Let R be an integral domain, not a field and not complete. Show that $\operatorname{Hom}_R(\widehat{R}, R)$ is zero and hence an injective $\widehat{R}$-module though R is not an injective R-module; cf. E 17.3.1.

E 16.1.9 Let M be an R-complex and set $(-)^\vee = \operatorname{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))$. (a) Show that if M is degreewise finitely generated, then there is an isomorphism, $M^{\vee\vee} \cong \widehat{R} \otimes_R M$, of $\widehat{R}$-complexes. (b) Show that if M is in $\mathcal{D}^{\mathrm{f}}(R)$, then one has $M^{\vee\vee} \simeq \widehat{R} \otimes_R M$ in $\mathcal{D}(\widehat{R})$.

E 16.1.10 Let F be a half exact functor from $\mathcal{M}(R)$ or $\mathcal{M}(R)^{\mathrm{op}}$ to $\mathcal{M}(S)$. Show that $\mathrm{F}(\boldsymbol{k}) = 0$ if and only if $\mathrm{F}(L) = 0$ holds for every R-module L of finite length.

E 16.1.11 Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in R. (a) Show that one has $\operatorname{Ext}_R^m(\widehat{R_\mathfrak{p}}, \widehat{R_\mathfrak{q}}) = 0$ for all $m > 0$. (b) Show that $\operatorname{Hom}_R(\widehat{R_\mathfrak{p}}, \widehat{R_\mathfrak{q}}) \neq 0$ holds if and only if $\mathfrak{p}$ contains $\mathfrak{q}$.

16.2 Local Theory of Depth and Width

Synopsis. Width; ~ of derived tensor product; depth; ~ of derived Hom; (co)support; regular sequence; local (co)homology.

For complexes over a local ring $(R, \mathfrak{m})$ the $\mathfrak{m}$-width and $\mathfrak{m}$-depth are invariants of particular interest.

Width

16.2.1 Definition. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. For the $\mathfrak{m}$-width of M, see 14.3.21, one uses the abbreviated notation $\operatorname{width}_R M$ and refers to it, accordingly, as simply the *width* of M.

Consistent with the definition of width in the local setting, one has:

16.2.2 Proposition. *Let* $\mathfrak{m}$ *be a maximal ideal in* R *and* M *an* R*-complex; one has*:

$$\operatorname{width}_{R_\mathfrak{m}} M_\mathfrak{m} \;=\; \mathfrak{m}\text{-}\operatorname{width}_R M \;=\; \operatorname{width}_{R_\mathfrak{m}} \mathsf{RHom}_R(R_\mathfrak{m}, M)\ .$$

Proof. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in R that generates $\mathfrak{m}$. Viewed as a sequence in $R_\mathfrak{m}$ it generates the unique maximal ideal $\mathfrak{m}_\mathfrak{m}$ of $R_\mathfrak{m}$, so the 1st and 5th equalities below hold by 16.2.1 and 14.4.8. The 2nd equality follows from 11.4.18 and 14.1.33(c), while the 3rd equality follows from swap 12.3.7. As $\mathsf{RHom}_R(\mathrm{K}^R(\boldsymbol{x}), M)$ is derived $\mathfrak{m}$-complete by 13.1.34 and 14.3.2, the 4th equality follows from 15.3.21.

$$\begin{aligned}
\operatorname{width}_{R_\mathfrak{m}} \mathsf{RHom}_R(R_\mathfrak{m}, M) &= n + \inf \mathsf{RHom}_{R_\mathfrak{m}}(\mathrm{K}^{R_\mathfrak{m}}(\boldsymbol{x}), \mathsf{RHom}_R(R_\mathfrak{m}, M)) \\
&= n + \inf \mathsf{RHom}_R(\mathrm{K}^R(\boldsymbol{x}), \mathsf{RHom}_R(R_\mathfrak{m}, M)) \\
&= n + \inf \mathsf{RHom}_R(R_\mathfrak{m}, \mathsf{RHom}_R(\mathrm{K}^R(\boldsymbol{x}), M)) \\
&= n + \inf \mathsf{RHom}_R(\mathrm{K}^R(\boldsymbol{x}), M) \\
&= \mathfrak{m}\text{-}\operatorname{width}_R M\ .
\end{aligned}$$

This is the second of the asserted equalities; to prove the first one, note that $R/\mathfrak{m}$ is the residue field of the local ring $R_\mathfrak{m}$, cf. 15.1.1. In the next computation, the first and last equalities hold by 14.4.8. The middle equality holds by 14.1.16(b).

$$\begin{aligned}\operatorname{width}_{R_\mathfrak{m}} M_\mathfrak{m} &= \inf\,(R/\mathfrak{m} \otimes^{\mathsf{L}}_{R_\mathfrak{m}} M_\mathfrak{m})\\ &= \inf\,(R/\mathfrak{m} \otimes^{\mathsf{L}}_{R} M)\\ &= \mathfrak{m}\text{-}\operatorname{depth}_R M\,. \end{aligned}$$ □

16.2.3 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local, K the Koszul complex on a sequence that generates $\mathfrak{m}$, and M an R-complex. The following quantities are equal.*

(i) $\operatorname{width}_R M$.
(ii) $n + \inf \operatorname{Hom}_R(K, M)$.
(iii) $\inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \inf\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}$.
(iv) $\inf \mathsf{L}\Lambda^\mathfrak{m}(M) = \inf\{n \in \mathbb{Z} \mid \mathrm{H}^\mathfrak{m}_n(M) \neq 0\}$.

Further, for every proper ideal $\mathfrak{a}$ in R, then quantities (i)–(iv) are equal to

(v) $\operatorname{width}_R \mathsf{L}\Lambda^\mathfrak{a}(M)$.
(vi) $\operatorname{width}_R \mathsf{R}\Gamma_\mathfrak{a}(M)$.

Proof. For $\mathfrak{a} = \mathfrak{m}$ these numbers (i)–(vi) agree by 14.4.1 and 14.4.8. Further, for every proper ideal $\mathfrak{a}$ in R the equalities

$$\operatorname{width}_R \mathsf{L}\Lambda^\mathfrak{a}(M) = \operatorname{width}_R \mathsf{L}\Lambda^\mathfrak{m}(\mathsf{L}\Lambda^\mathfrak{a}(M)) = \operatorname{width}_R \mathsf{L}\Lambda^\mathfrak{m}(M) = \operatorname{width}_R M$$

hold by 14.4.1 and 13.1.20. Similarly, 14.4.1 and 13.3.21 yield

$$\operatorname{width}_R \mathsf{R}\Gamma_\mathfrak{a}(M) = \operatorname{width}_R \mathsf{R}\Gamma_\mathfrak{m}(\mathsf{R}\Gamma_\mathfrak{a}(M)) = \operatorname{width}_R \mathsf{R}\Gamma_\mathfrak{m}(M) = \operatorname{width}_R M\,.$$ □

16.2.4 Corollary. *Let $(R, \mathfrak{m})$ be local and M an R-complex. One has $\operatorname{width}_R M < \infty$ if and only if $\mathfrak{m} \in \operatorname{supp}_R M$ holds.*

Proof. The assertion follows from 16.2.3 and the definition, 15.1.5, of support. □

16.2.5 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. One has*

$$\operatorname{width}_R M \geqslant \inf M\,,$$

and the following assertions hold.

(a) *If M is derived $\mathfrak{m}$-complete or in $\mathcal{D}^{\mathrm{f}}(R)$, then one has $\operatorname{width}_R M = \inf M$.*
(b) *If M is not acyclic and belongs to $\mathcal{D}_{\sqsupset}(R)$ with $w = \inf M$, then the following conditions are equivalent.*
    (i) $\operatorname{width}_R M = \inf M$.
    (ii) $\boldsymbol{k} \otimes_R \mathrm{H}_w(M) \neq 0$ *i.e.* $\mathfrak{m}\,\mathrm{H}_w(M) \neq \mathrm{H}_w(M)$.
    (iii) $\Lambda^\mathfrak{m}(\mathrm{H}_w(M)) \neq 0$.
    (iv) $\mathrm{H}^\mathfrak{m}_0(\mathrm{H}_w(M)) \neq 0$.

Proof. As the maximal ideal is the Jacobson radical of R, this is the special case $\mathfrak{a} = \mathfrak{m}$ of 14.3.28. □

The difference between the quantities compared in 16.2.5 may be infinite.

16.2.6 Example. Let R be local and $E \neq 0$ an R-module of infinite width; concrete examples of such modules are given in 16.2.29. Set $M = \coprod_{u \leqslant 0} \Sigma^u E$. Evidently one has $\inf M = -\infty$, while 14.3.22 and 14.3.25 yield $\operatorname{width}_R M = \infty$.

16.2.7 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module. The following conditions are equivalent.*

(*i*) $\operatorname{width}_R M = 0$.
(*ii*) $\boldsymbol{k} \otimes_R M \neq 0$ *i.e.* $\mathfrak{m}M \neq M$.
(*iii*) $\Lambda^{\mathfrak{m}}(M) \neq 0$.
(*iv*) $\mathrm{H}_0^{\mathfrak{m}}(M) \neq 0$.

Proof. As the maximal ideal is the Jacobson radical of R, this is the special case $\mathfrak{a} = \mathfrak{m}$ of 14.3.29. □

For a complex M in $\mathcal{D}_{\sqsupset}(R)$ such that $\mathrm{H}(M) \neq 0$ and the module $\mathrm{H}_{\inf M}(M)$ is $\mathfrak{m}$-complete, $\operatorname{width}_R M = \inf M$ holds by 16.2.5(b). Here is an unbounded analogue:

16.2.8 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex. If* $\inf M = -\infty$ *and* $\mathrm{H}_v(M)$ *is $\mathfrak{m}$-complete for $v \ll 0$, then* $\operatorname{width}_R M = -\infty$ *holds.*

Proof. Choose an integer n such that $\mathrm{H}_v(M)$ is $\mathfrak{m}$-complete for all $v \leqslant n$. By 7.6.6(c) there is a distinguished triangle,

$$M_{\supseteq n+1} \longrightarrow M \longrightarrow M_{\subseteq n} \longrightarrow \Sigma(M_{\supseteq n+1}) .$$

It follows from 2.5.24(b) and 13.4.4 that the complex $M_{\subseteq n}$ is derived $\mathfrak{m}$-complete, so 16.2.5(a) yields $\operatorname{width}_R M_{\subseteq n} = \inf M_{\subseteq n} = -\infty$. It also yields $\operatorname{width}_R M_{\supseteq n+1} \geqslant n+1$, so the asserted equality follows from 14.3.32. □

Remark. In 16.2.8 one can replace the maximal ideal by any ideal $\mathfrak{a}$ in R; see E 14.3.13.

16.2.9 Theorem. *Let R be local and M and N be R-complexes. The complex $M \otimes^{\mathsf{L}}_R N$ has finite width if and only if M and N have finite width, and in that case there is an equality,*

$$\operatorname{width}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{width}_R M + \operatorname{width}_R N .$$

Proof. The first assertion follows from 16.2.4 and the Support Formula 15.1.16. Let $\boldsymbol{k}$ be the residue field of R. Assuming that M and N have finite width, a computation based on 16.2.3, 12.3.30, and 7.6.12 yields the asserted equality,

$$\begin{aligned} \operatorname{width}_R(M \otimes^{\mathsf{L}}_R N) \;&=\; \inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N)) \\ &=\; \inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_{\boldsymbol{k}} (\boldsymbol{k} \otimes^{\mathsf{L}}_R N) \\ &=\; \inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) + \inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R N) \\ &=\; \operatorname{width}_R M + \operatorname{width}_R N. \end{aligned}$$ □

The next corollary compares to 7.6.8.

16.2.10 Corollary. *Let R be local and M and N be complexes in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$; one has*

$$\inf\,(M \otimes^{\mathsf{L}}_R N) \;=\; \inf M + \inf N\;.$$

PROOF. The complex $M \otimes^{\mathsf{L}}_R N$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ by 12.2.12. The equality now follows from 16.2.5(a) and 16.2.9. □

16.2.11 Proposition. *Let R be local, $\mathfrak{a}$ a proper ideal in R, and M an $R/\mathfrak{a}$-complex. There is an equality,*

$$\operatorname{width}_{R/\mathfrak{a}} M \;=\; \operatorname{width}_R M\;.$$

PROOF. Let $\mathfrak{m}$ be the maximal ideal of R. The quotient $R/\mathfrak{a}$ is local with maximal ideal $\mathfrak{m}/\mathfrak{a}$, so the equality holds by 14.3.31. □

DEPTH

16.2.12 Definition. Let $(R,\mathfrak{m})$ be local and M an R-complex. For the $\mathfrak{m}$-depth of M, see 14.3.10, one uses the abbreviated notation $\operatorname{depth}_R M$ and referes to it, accordingly, as simply the *depth* of M.

For the depth of the R-module R one uses the simplified notation $\operatorname{depth} R$.

Consistent with the definition of depth in the local setting, one has the equalities below, but see also 17.6.3 and 17.6.2.

16.2.13 Proposition. *Let $\mathfrak{m}$ be a maximal ideal in R and M an R-complex; one has:*

$$\operatorname{depth}_{R_\mathfrak{m}} M_\mathfrak{m} \;=\; \mathfrak{m}\text{-}\operatorname{depth}_R M \;=\; \operatorname{depth}_{R_\mathfrak{m}} \mathsf{RHom}_R(R_\mathfrak{m}, M)\;.$$

PROOF. Let $\boldsymbol{x} = x_1,\dots,x_n$ be a sequence in R that generates $\mathfrak{m}$. Viewed as a sequence in $R_\mathfrak{m}$ it generates the unique maximal ideal $\mathfrak{m}_\mathfrak{m}$ of $R_\mathfrak{m}$, so the 1st and 4th equalities below hold by 16.2.12 and 14.3.10. The 2nd equality follows from 11.4.18 and 14.1.15. As the complex $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M$ is derived $\mathfrak{m}$-torsion by 13.3.31, the 3rd equality holds by 15.3.25.

$$\begin{aligned}
\operatorname{depth}_{R_\mathfrak{m}} M_\mathfrak{m} &= n - \sup\,(\mathrm{K}^{R_\mathfrak{m}}(\boldsymbol{x}) \otimes_{R_\mathfrak{m}} M_\mathfrak{m})\\
&= n - \sup\,(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)_\mathfrak{m}\\
&= n - \sup\,(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)\\
&= \mathfrak{m}\text{-}\operatorname{depth}_R M\;.
\end{aligned}$$

This establishes the first of the asserted equalities; to prove the second one, note that $R/\mathfrak{m}$ is the residue field of the local ring $R_\mathfrak{m}$, cf. 15.1.1. In the computation below, the first and last equalities hold by 14.4.3. The middle equality holds by 14.1.33(d).

$$\begin{aligned}
\operatorname{depth}_{R_\mathfrak{m}} \mathsf{RHom}_R(R_\mathfrak{m}, M) &= -\sup \mathsf{RHom}_{R_\mathfrak{m}}(R/\mathfrak{m}, \mathsf{RHom}_R(R_\mathfrak{m}, M))\\
&= -\sup \mathsf{RHom}_R(R/\mathfrak{m}, M)\\
&= \mathfrak{m}\text{-}\operatorname{depth}_R M\;.
\end{aligned}$$ □

16.2.14 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, K the Koszul complex on a sequence that generates $\mathfrak{m}$, and M an R-complex. The following quantities are equal.*

(*i*) $\operatorname{depth}_R M$.
(*ii*) $-\sup \operatorname{Hom}_R(K,M)$.
(*iii*) $-\sup \mathsf{R}\operatorname{Hom}_R(\boldsymbol{k},M) = \inf\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(\boldsymbol{k},M) \neq 0\}$.
(*iv*) $-\sup \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \inf\{n \in \mathbb{Z} \mid \mathrm{H}_{\mathfrak{m}}^n(M) \neq 0\}$.

Further, for every proper ideal $\mathfrak{a}$ in R the quantities (i)–(iv) are equal to

(*v*) $\operatorname{depth}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M)$.
(*vi*) $\operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M)$.

Proof. For $\mathfrak{a} = \mathfrak{m}$ the numbers (*i*)–(*vi*) agree by 14.4.3 and 14.4.1. Further, for every proper ideal $\mathfrak{a}$ in R the equalities

$$\operatorname{depth}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \operatorname{depth}_R \mathsf{R}\Gamma_{\mathfrak{m}}(\mathsf{R}\Gamma_{\mathfrak{a}}(M)) = \operatorname{depth}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \operatorname{depth}_R M$$

hold by 14.4.1 and 13.3.21. Similarly, 14.4.1 and 13.1.20 yield

$$\operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{a}}(M) = \operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(\mathsf{L}\Lambda^{\mathfrak{a}}(M)) = \operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \operatorname{depth}_R M\,. \qquad \square$$

16.2.15 Corollary. *Let $(R,\mathfrak{m})$ be local and M an R-complex. One has* $\operatorname{depth}_R M < \infty$ *if and only if* $\mathfrak{m} \in \operatorname{cosupp}_R M$.

Proof. The claim follows from 16.2.14 and the definition, 15.2.1, of cosupport. $\square$

Part (a) below applies in particular to complexes with degreewise Artinian homology; see 16.1.30.

16.2.16 Proposition. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M an R-complex. One has*

$$\operatorname{depth}_R M \geqslant -\sup M\,,$$

and the following assertions hold.

(a) *If M is derived $\mathfrak{m}$-torsion, then one has* $\operatorname{depth}_R M = -\sup M$.
(b) *If M is not acyclic and belongs to $\mathcal{D}_{\sqsubset}(R)$ with $s = \sup M$, then the following conditions are equivalent.*
  (*i*) $\operatorname{depth}_R M = -\sup M$.
  (*ii*) $\operatorname{Hom}_R(\boldsymbol{k},\mathrm{H}_s(M)) \neq 0$ *i.e.* $(0 :_{\mathrm{H}_s(M)} \mathfrak{m}) \neq 0$.
  (*iii*) $\Gamma_{\mathfrak{m}}(\mathrm{H}_s(M)) \neq 0$.
  (*iv*) $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{H}_s(M)$.

Proof. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 14.3.16 when one takes into account that the only prime ideal that contains $\mathfrak{m}$ is the maximal ideal itself. $\square$

The difference between the quantities compared in 16.2.16 may be infinite.

16.2.17 Example. Let R be local and $E \neq 0$ an R-module of infinite depth; concrete examples of such modules are given in 16.2.29. Set $M = \coprod_{u \geqslant 0} \Sigma^u E$. Evidently one has $-\sup M = -\infty$ while 14.3.11 and 14.3.14 yield $\operatorname{depth}_R M = \infty$.

Per 16.1.2 condition (*ii*) below says that the socle of M is non-zero.

16.2.18 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-module. The following conditions are equivalent.*

(*i*) $\operatorname{depth}_R M = 0$.
(*ii*) $\operatorname{Hom}_R(\boldsymbol{k}, M) \neq 0$ *i.e.* $(0 :_M \mathfrak{m}) \neq 0$.
(*iii*) $\Gamma_{\mathfrak{m}}(M) \neq 0$.
(*iv*) $\mathfrak{m} \in \operatorname{Ass}_R M$.

Proof. The conditions are equivalent by 14.3.17 applied with $\mathfrak{a} = \mathfrak{m}$, as the only prime ideal that contains $\mathfrak{m}$ is the maximal ideal itself. □

16.2.19 Example. Let Q be a field and $M \neq 0$ a Q-vector space. Every element in M is annihilated by the maximal ideal 0, so 16.2.18 yields $\operatorname{depth}_Q M = 0$, which is already known from 14.3.12.

For a complex M in $\mathcal{D}_{\sqsubset}(R)$ such that $\mathrm{H}(M) \neq 0$ and the module $\mathrm{H}_{\sup M}(M)$ is $\mathfrak{m}$-torsion, the equality $\operatorname{depth}_R M = -\sup M$ holds by 16.2.16(b). Here is an unbounded analogue:

16.2.20 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex. If $\sup M = \infty$ and $\mathrm{H}_v(M)$ is $\mathfrak{m}$-torsion for $v \gg 0$, then $\operatorname{depth}_R M = -\infty$ holds.*

Proof. Choose an integer n such that $\mathrm{H}_v(M)$ is $\mathfrak{m}$-torsion for all $v \geqslant n$. By 7.6.6(c) there is a distinguished triangle,

$$M_{\supseteq n} \longrightarrow M \longrightarrow M_{\subseteq n-1} \longrightarrow \Sigma(M_{\supseteq n}) .$$

It follows from 2.5.25(b) and 13.4.9 that the complex $M_{\supseteq n}$ is derived $\mathfrak{m}$-torsion, so 16.2.16(a) yields $\operatorname{depth}_R M_{\supseteq n} = -\sup M_{\supseteq n} = -\infty$. It also yields $\operatorname{depth}_R M_{\subseteq n-1} \geqslant 1 - n$, so the asserted equality follows from 14.3.20. □

Remark. In 16.2.20 one can replace the maximal ideal by any ideal $\mathfrak{a}$ in R; see E 14.3.7.

For certain complexes one can also give an upper bound on the depth. Without conditions on the complex this bound may fail, see 16.2.29.

16.2.21 Proposition. *Let R be local and M an R-complex. If M is derived $\mathfrak{m}$-complete or belongs to $\mathcal{D}^{\mathrm{f}}(R)$, then the next inequalities hold,*

$$\dim R - \sup M \;\geqslant\; \operatorname{depth}_R M \;\geqslant\; -\sup M \;;$$

in particular, $\operatorname{depth}_R M = -\infty$ *holds if and only if one has* $\sup M = \infty$.

Proof. The second inequality holds by 16.2.16. To prove the first inequality, set $d = \dim R$ and let $\boldsymbol{x} = x_1, \ldots, x_d$ be a parameter sequence for R. With $\mathfrak{a} = (x_1, \ldots, x_d)$ one has $\mathfrak{m} = \sqrt{\mathfrak{a}}$ and, therefore, $\operatorname{depth}_R M = \mathfrak{a}\text{-}\operatorname{depth}_R M$ by 14.4.4 and 16.2.12. The definition, 14.3.10, of $\mathfrak{a}$-depth and 14.3.5 now yield $d - \sup M \geqslant \mathfrak{a}\text{-}\operatorname{depth}_R M$. □

16.2.22 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex*; *one has*

$$\operatorname{depth}_R M = \operatorname{width}_R \operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})) \quad \textit{and}$$
$$\operatorname{width}_R M = \operatorname{depth}_R \operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k}))\ .$$

PROOF. As $\mathrm{E}_R(\boldsymbol{k})$ is a faithfully injective R-module, see 16.1.22, the assertion is a special case of 14.4.14 with $\mathfrak{a} = \mathfrak{m}$. □

16.2.23 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. The following conditions are equivalent.*

(*i*) $\mathsf{RHom}_R(\boldsymbol{k}, M)$ *is not acyclic.*
(*ii*) $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ *is not acyclic.*
(*iii*) $\operatorname{depth}_R M$ *is finite.*
(*iv*) $\boldsymbol{k} \otimes^{\mathsf{L}}_R M$ *is not acyclic.*
(*v*) $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ *is not acyclic.*
(*vi*) $\operatorname{width}_R M$ *is finite.*

PROOF. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 14.4.12. □

16.2.24 Theorem. *Let R be local and M and N be R-complexes. The complex $\mathsf{RHom}_R(M, N)$ has finite depth if and only if M has finite width and N has finite depth, and in that case there is an equality,*

$$\operatorname{depth}_R \mathsf{RHom}_R(M, N) = \operatorname{width}_R M + \operatorname{depth}_R N\ .$$

PROOF. The first assertion follows from 16.2.4, 16.2.15, and the Cosupport Formula 15.2.9. Let $\boldsymbol{k}$ be the residue field of R. Assuming that $\operatorname{width}_R M$ and $\operatorname{depth}_R N$ are both finite, a straightforward computation based on 16.2.14, 12.3.35, 7.6.12, and 16.2.3 yields the asserted equality:

$$\begin{aligned}\operatorname{depth}_R \mathsf{RHom}_R(M, N) &= -\sup \mathsf{RHom}_R(\boldsymbol{k}, \mathsf{RHom}_R(M, N))\\ &= -\sup \mathsf{RHom}_{\boldsymbol{k}}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(\boldsymbol{k}, N))\\ &= \inf\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) - \sup \mathsf{RHom}_R(\boldsymbol{k}, N)\\ &= \operatorname{width}_R M + \operatorname{depth}_R N\ .\end{aligned}$$ □

16.2.25 Corollary. *Let R be local, M an R-complex that is derived $\mathfrak{m}$-complete or belongs to $\mathcal{D}^{\mathrm{f}}(R)$, and N an R-complex. If M is not acyclic and N has finite depth, then the next equality holds,*

$$\operatorname{depth}_R \mathsf{RHom}_R(M, N) = \inf M + \operatorname{depth}_R N\ .$$

PROOF. The equality follows from 16.2.5(a) and 16.2.24. □

16.2.26 Proposition. *Let R be local, $\mathfrak{a}$ a proper ideal in R, and M an $R/\mathfrak{a}$-complex. There is an equality,*

$$\operatorname{depth}_{R/\mathfrak{a}} M = \operatorname{depth}_R M\ .$$

PROOF. Let $\mathfrak{m}$ be the maximal ideal of R. The quotient $R/\mathfrak{a}$ is local with maximal ideal $\mathfrak{m}/\mathfrak{a}$, so the equality holds by 14.3.19. □

Support and Cosupport vs. Width and Depth

The theorem below identifies complexes whose homological properties align with the properties of complexes with non-zero and degreewise finitely generated homology; compare for example 17.2.1 to 18.3.31. In statements they are usually referred to as complexes of finite depth or complexes of finite width, but when neither of these descriptions is natural—or one is not more natural than the other—the condition gets expressed in terms of support or cosupport.

16.2.27 Theorem. *Let $(R, \mathfrak{m})$ be local and M an R-complex. The following conditions are equivalent.*

(*i*) $\mathfrak{m} \in \operatorname{supp}_R M$.
(*ii*) $\mathfrak{m} \in \operatorname{cosupp}_R M$.
(*iii*) $\operatorname{width}_R M$ *is finite.*
(*iv*) $\operatorname{depth}_R M$ *is finite.*

Moreover, if M is derived $\mathfrak{m}$-torsion, derived $\mathfrak{m}$-complete, or belongs to $\mathcal{D}^{\mathrm{f}}(R)$, then conditions (*i*)–(*iv*) *are equivalent to*

(*v*) *M is not acyclic.*

Proof. Conditions (*i*) and (*iii*) are equivalent by 16.2.4, while (*ii*) and (*iv*) are equivalent by 16.2.15. Finally, conditions (*iii*) and (*iv*) are equivalent per 14.3.27. If M is derived $\mathfrak{m}$-complete or belongs to $\mathcal{D}^{\mathrm{f}}(R)$, then conditions (*iii*) and (*v*) are equivalent by 16.2.5(a). If M is derived $\mathfrak{m}$-torsion, then conditions (*iv*) and (*v*) are equivalent by 16.2.16(a). □

16.2.28 Corollary. *Let $(R, \mathfrak{m})$ be local and M and N be complexes in one, not necessarily the same, of the categories $\mathcal{D}^{\mathrm{f}}(R)$, $\mathcal{D}^{\mathfrak{m}\text{-com}}(R)$, or $\mathcal{D}^{\mathfrak{m}\text{-tor}}(R)$. If M and N are not acyclic, then $\mathsf{RHom}_R(M, N)$ and $M \otimes^{\mathsf{L}}_R N$ are not acyclic.*

Proof. By 16.2.27 the maximal ideal $\mathfrak{m}$ belongs to both the support and the cosupport of M and N. It now follows from 15.2.8 and the Cosupport Formula 15.2.9 that $\mathsf{RHom}_R(M, N)$ is not acyclic, and it follows similarly from 15.1.15 and the Support Formula 15.1.16 that $M \otimes^{\mathsf{L}}_R N$ is not acyclic. □

16.2.29 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. For every prime ideal $\mathfrak{p} \neq \mathfrak{m}$ in R one has

$$\operatorname{depth}_R \mathrm{E}_R(R/\mathfrak{p}) \;=\; \infty \;=\; \operatorname{width}_R \mathrm{E}_R(R/\mathfrak{p})$$

by 15.1.12, 15.2.5, and 16.2.27. For the maximal ideal itself, 16.2.22 yields

$$\operatorname{depth}_R \mathrm{E}_R(\boldsymbol{k}) \;=\; 0 \qquad \text{and} \qquad \operatorname{width}_R \mathrm{E}_R(\boldsymbol{k}) \;=\; \operatorname{depth} R\,.$$

In particular, it follows that for every prime p one has $\operatorname{depth}_{\mathbb{Z}_{p\mathbb{Z}}} \mathbb{Q} = \infty = \operatorname{width}_{\mathbb{Z}_{p\mathbb{Z}}} \mathbb{Q}$, $\operatorname{depth}_{\mathbb{Z}_{p\mathbb{Z}}} \mathbb{Z}(p^\infty) = 0$, and $\operatorname{width}_{\mathbb{Z}_{p\mathbb{Z}}} \mathbb{Z}(p^\infty) = 1$, see B.15 and C.19.

Regular Sequences

Recall the definition of regular sequences from 14.4.16.

16.2.30. Let $(R, \mathfrak{m})$ be local and M an R-module. The elements in $R \setminus \mathfrak{m}$ are units in R, so an M-regular element must belong to $\mathfrak{m}$. Further, if $M \neq 0$ is finitely generated or derived $\mathfrak{m}$-complete, then it follows from Nakayama's lemma 16.1.5 or 13.1.35 that an element $x \in \mathfrak{m}$ is M-regular if and only if the homothety x^M is injective.

16.2.31 Proposition. *Let $(R, \mathfrak{m})$ be local, $\boldsymbol{x} = x_1, \dots, x_n$ a sequence in $\mathfrak{m}$, and $M \neq 0$ an R-module. If M is derived $\mathfrak{m}$-complete or finitely generated, then the following conditions are equivalent.*

(*i*) *$\boldsymbol{x}$ is M-regular.*
(*ii*) $\sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = 0$.
(*iii*) *There is an isomorphism* $\mathrm{K}^R(\boldsymbol{x}) \otimes_R M \simeq M/(\boldsymbol{x})M$.

Furthermore, if these conditions are satisfied, then one has

$$\operatorname{depth}_R M/(\boldsymbol{x})M \;=\; \operatorname{depth}_R M - n\,.$$

Proof. As $\mathfrak{m}$ is the Jacobson radical of R, conditions (*i*)–(*iii*) are equivalent by 14.4.28 applied with $\mathfrak{a} = \mathfrak{m}$, and the asserted equality holds by 14.4.18. □

16.2.32 Theorem. *Let $(R, \mathfrak{m})$ be local and $M \neq 0$ a finitely generated R-module. For an M-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ the following conditions are equivalent.*

(*i*) *$\boldsymbol{x}$ is a maximal M-regular sequence.*
(*ii*) $\mathfrak{m} \in \operatorname{Ass}_R(M/(\boldsymbol{x})M)$.
(*iii*) $\operatorname{depth}_R(M/(\boldsymbol{x})M) = 0$.
(*iv*) $\operatorname{depth}_R M = n$.

Proof. It follows from 16.2.30 that $\boldsymbol{x}$ is a sequence in $\mathfrak{m}$. By 16.2.27 the module M has finite depth, so the equivalence of the four conditions follows from 14.4.24 applied with $\mathfrak{a} = \mathfrak{m}$. □

16.2.33 Corollary. *Let R be local and $M \neq 0$ a finitely generated R-module. The depth, d, of M is finite and the following assertions hold.*

(a) *The maximal length of an M-regular sequence is d.*
(b) *There exists an M-regular sequence of length d.*
(c) *Every maximal M-regular sequence has length d.*

Proof. By 16.2.27 the module M has finite depth and the assertions follow from 14.4.25 applied with $\mathfrak{a} = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. □

Local (Co)homology

16.2.34 Proposition. *Let $(R, \mathfrak{m})$ be local of Krull dimension d and M an R-complex. There are inequalities,*

$$-\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;\leqslant\; d - \operatorname{width}_R M \quad \textit{i.e.} \quad \mathrm{H}^n_{\mathfrak{m}}(M) \;=\; 0 \;\textit{ for }\; n > d - \operatorname{width}_R M$$

and

$$\sup \mathsf{L}\Lambda^{\mathfrak{m}}(M) \;\leqslant\; d - \operatorname{depth}_R M \quad \textit{i.e.} \quad \mathrm{H}_n^{\mathfrak{m}}(M) \;=\; 0 \;\textit{ for }\; n > d - \operatorname{depth}_R M\,.$$

Proof. Both inequalities are trivial if M is acyclic, so assume that is not the case. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a parameter sequence for R. As $\sqrt{(\boldsymbol{x})} = \mathfrak{m}$ holds, 13.4.1(d), 13.3.2, and 13.3.18 yield $\mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{m}}(\mathsf{L}\Lambda^{\mathfrak{m}}(M)) \simeq \check{\mathrm{C}}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M)$. By 11.4.10(c) and 11.4.17 the complex $\check{\mathrm{C}}^R(\boldsymbol{x})$ is concentrated in degrees $0, \dots, -d$ and not acyclic. The first inequality now follows from 7.6.8 and 16.2.3. Similarly, 13.4.1(c), 13.1.3, and 13.1.15 yield $\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(\mathsf{R}\Gamma_{\mathfrak{m}}(M)) \simeq \mathsf{R}\mathrm{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}), \mathsf{R}\Gamma_{\mathfrak{m}}(M))$. The second inequality now follows from 7.6.7 and 16.2.14. □

The next result shows that the Krull dimension of a local ring R is an upper bound for the depth of an R-module of finite depth. It is also an upper bound for the Krull dimension of such a module, and it is shown in 17.2.1 and 18.3.31 that the Krull dimension is trapped between the depth and the Krull dimension of the ring.

16.2.35 Corollary. *Let $(R, \mathfrak{m})$ be local and M be an R-complex. If $\mathfrak{m} \in \mathrm{supp}_R M$ holds, then there is an inequality,*

$$\mathrm{depth}_R M + \mathrm{width}_R M \leqslant \dim R .$$

Proof. The assumption on M guarantees per 16.2.27 that $\mathrm{depth}_R M$ and $\mathrm{width}_R M$ are finite. By 16.2.14 one now has $\mathrm{depth}_R M = -\sup \mathsf{R}\Gamma_{\mathfrak{m}}(M) \leqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$, so the asserted inequality follows from 16.2.34. □

Exercises

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 16.2.1 Let M and N be R-complexes of finite width. Show that if M is derived $\mathfrak{m}$-complete, then one has

$$\inf M + \mathrm{width}_R N \geqslant \inf (M \otimes^{\mathsf{L}}_R N) \geqslant \mathrm{width}_R M + \inf N .$$

E 16.2.2 Let M and N be R-complexes with $\mathfrak{m} \in \mathrm{supp}_R M \cap \mathrm{supp}_R N$. (a) Show that if N is derived $\mathfrak{m}$-torsion, then one has

$$\mathrm{width}_R M - \sup N \geqslant -\sup \mathsf{R}\mathrm{Hom}_R(M, N) \geqslant \inf M + \mathrm{depth}_R N .$$

(b) Show that if M is derived $\mathfrak{m}$-complete, then one has

$$\inf M + \mathrm{depth}_R N \geqslant -\sup \mathsf{R}\mathrm{Hom}_R(M, N) \geqslant \mathrm{width}_R M - \sup N .$$

E 16.2.3 Let M and N be complexes that are not acyclic and belong to one, but not necessarily the same, of the categories $\mathcal{D}^{\mathrm{f}}(R)$ or $\mathcal{D}^{\mathfrak{m}\text{-com}}(R)$. Show that $M \otimes^{\mathsf{L}}_R N$ belongs to $\mathcal{D}_{\sqsupset}(R)$ only if M and N belong to $\mathcal{D}_{\sqsupset}(R)$.

E 16.2.4 Let M and N be complexes that are not acyclic. Show that if M belongs to $\mathcal{D}^{\mathfrak{m}\text{-com}}(R)$ or $\mathcal{D}^{\mathrm{f}}(R)$ and N belongs to $\mathcal{D}^{\mathfrak{m}\text{-tor}}(R)$, then $\mathsf{R}\mathrm{Hom}_R(M, N)$ belongs to $\mathcal{D}_{\sqsubset}(R)$ only if M is in $\mathcal{D}_{\sqsupset}(R)$ and N in $\mathcal{D}_{\sqsubset}(R)$.

E 16.2.5 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. Show that M is derived reflexive if and only if biduality $\delta^M_R \colon M \to \mathsf{R}\mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M, R), R)$ is an isomorphism in $\mathcal{D}(R)$.

E 16.2.6 Let M be an R-complex. (a) Let $K \neq 0$ be an R-module of finite length; show that $-\sup \mathsf{R}\mathrm{Hom}_R(K, M) = \mathrm{depth}_R M$ and $\inf (K \otimes^{\mathsf{L}}_R M) = \mathrm{width}_R M$ hold. (b) Let L be a complex in $\mathcal{D}^{\ell}_{\square}(R)$ with $\mathrm{H}(L) \neq 0$; show that one has $-\sup \mathsf{R}\mathrm{Hom}_R(L, M) = \inf L + \mathrm{depth}_R M$ and $\inf (L \otimes^{\mathsf{L}}_R M) = \inf L + \mathrm{width}_R M$. *Hint:* 7.6.9 and 7.6.10.

E 16.2.7 Let M and N be R-modules with $\mathrm{Tor}^R_m(M, N) = 0$ for all $m > 0$. Show that if M is finitely generated and non-zero, then one has $\mathrm{width}_R(M \otimes_R N) = \mathrm{width}_R N$.

E 16.2.8 Let M and N be R-modules with $\operatorname{Ext}_R^m(M,N) = 0$ for all $m > 0$. Show that if M is finitely generated and non-zero, then one has $\operatorname{depth}_R \operatorname{Hom}_R(M,N) = \operatorname{depth}_R N$.

E 16.2.9 Show that a sequence $\boldsymbol{x}$ in $\mathfrak{m}$ is R-regular if and only if the Koszul complex $\mathrm{K}^R(\boldsymbol{x})$ yields a free resolution of $R/(\boldsymbol{x})$.

E 16.2.10 Let M be an Artinian R-module. An element x in $\mathfrak{m}$ is called *M-coregular* if the homothety x^M is surjective. A sequence $x_1,\dots,x_n$ in $\mathfrak{m}$ is called *M-coregular* if x_1 is M-coregular and x_i is $\operatorname{Hom}_R(R/(x_1,\dots,x_{i-1}),M)$-coregular for $i \in \{2,\dots,n\}$. Show that $\operatorname{width}_R M$ is the maximal length of an M-coregular sequence in $\mathfrak{m}$.

16.3 Depth and Width vs. Homological Dimensions

Synopsis. Depth of derived tensor product; width of derived Hom; rigidity of Ext and Tor.

The width of a derived tensor product complex, $M \otimes_R^{\mathsf{L}} N$, and the depth of a derived Hom complex, $\mathsf{RHom}_R(M,N)$, can always be expressed in terms of the depth and width of M and N; see 16.2.9 and 16.2.24. To similarly express the depth of $M \otimes_R^{\mathsf{L}} N$ or the width of $\mathsf{RHom}_R(M,N)$ one needs assumptions on the homological dimensions of M or N. This is the beginning of an investigation of the relations between the homological dimensions and the invariants depth and width whose culmination comes in formulas of—or at least inspired by—Auslander, Bridger, Buchsbaum, Bass, and Chouinard; see 16.4.2, 16.4.11, 17.3.4, 17.3.14, 17.5.4, 17.5.7, 19.1.18, 19.4.24, 19.2.6, 19.2.4, 19.2.37, 19.3.3, and 19.3.8.

Depth of Derived Tensor Product

For complexes with degreewise finitely generated homology the next equality simplifies; see 16.4.34. Part (a) compares to 14.3.15.

16.3.1 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M and N be R-complexes. The complex $N \otimes_R^{\mathsf{L}} M$ has finite depth if and only if M and N have finite depth, and in that case the equalities*

$$\begin{aligned}\operatorname{depth}_R(N \otimes_R^{\mathsf{L}} M) &= \operatorname{depth}_R N - \sup(\boldsymbol{k} \otimes_R^{\mathsf{L}} M)\\ &= \operatorname{depth}_R N + \operatorname{depth}_R M - \operatorname{depth} R\end{aligned}$$

hold, provided that one of the next conditions is satisfied:

(a) $\operatorname{fd}_R M$ *is finite and* $\operatorname{depth}_R N > -\infty$ *holds.*
(b) M *belongs to* $\mathcal{D}_{\sqsubset}^{\mathrm{f}}(R)$ *and* $\operatorname{pd}_R M$ *is finite.*

Proof. The first assertion follows from 16.2.9 and 16.2.27. Assume that M and N have finite depth. In the next computation, the 1st and 4th isomorphisms hold by 13.4.20(b), as $\boldsymbol{k}$ per 13.3.24 is derived $\mathfrak{m}$-torsion. The 2nd isomorphism follows from 13.3.19. Notice that under the assumptions in (a), the complex $\mathsf{R}\Gamma_{\mathfrak{m}}(N)$ belongs to $\mathcal{D}_{\sqsubset}(R)$ by 16.2.14; in this case the 3rd isomorphism is tensor evaluation 12.3.23(b).

Under the assumptions in (b), the 3rd isomorphism is 12.3.23(c). The 5th isomorphism holds by 12.3.31.

$$\begin{aligned}\operatorname{RHom}_R(\boldsymbol{k}, N \otimes^{\mathsf{L}}_R M) &\simeq \operatorname{RHom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(N \otimes^{\mathsf{L}}_R M))\\ &\simeq \operatorname{RHom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(N) \otimes^{\mathsf{L}}_R M)\\ &\simeq \operatorname{RHom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(N)) \otimes^{\mathsf{L}}_R M\\ &\simeq \operatorname{RHom}_R(\boldsymbol{k}, N) \otimes^{\mathsf{L}}_R M\\ &\simeq \operatorname{RHom}_R(\boldsymbol{k}, N) \otimes^{\mathsf{L}}_{\boldsymbol{k}} (\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\,.\end{aligned}$$

Together with 16.2.14 and 7.6.12 this isomorphism yields

$$\begin{aligned}\operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M) &= -\sup\,(\operatorname{RHom}_R(\boldsymbol{k}, N) \otimes^{\mathsf{L}}_{\boldsymbol{k}} (\boldsymbol{k} \otimes^{\mathsf{L}}_R M))\\ &= -\sup \operatorname{RHom}_R(\boldsymbol{k}, N) - \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\\ &= \operatorname{depth}_R N - \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\,.\end{aligned}$$

This proves the first equality in the statement. The R-module $N = R$ has finite depth, and in this special case one gets $-\sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \operatorname{depth}_R M - \operatorname{depth} R$; the second equality now follows by substituting this expression into the first equality. □

16.3.2 Corollary. *Let $(R, \mathfrak{m})$ be local and M and N be R-complexes. If $\operatorname{fd}_R M$ is finite and $N \in \mathcal{D}_{\sqsubset}(R)$ is derived $\mathfrak{m}$-torsion and not acyclic, then one has*

$$\sup\,(N \otimes^{\mathsf{L}}_R M) - \sup N \;=\; \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Proof. As N is derived $\mathfrak{m}$-torsion one has $\operatorname{depth}_R N = -\sup N$ by 16.2.16(a). Further, the complex $N \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{m}$-torsion by 13.4.20(c); in particular, one has $\operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M) = -\sup\,(N \otimes^{\mathsf{L}}_R M)$ by another application of 16.2.16(a). As N is not acyclic, it has finite depth. If M has finite depth, then the equality follows from 16.3.1(a). Otherwise, also $N \otimes^{\mathsf{L}}_R M$ has infinite depth, again by 16.3.1, so the equality still holds as $\operatorname{depth} R$ is finite. □

16.3.3 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{fd}_R M$ is finite, then one has*

$$\sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Proof. As $\boldsymbol{k}$ by 13.3.24 is derived $\mathfrak{m}$-torsion, apply 16.3.2 with $N = \boldsymbol{k}$. □

For a complex M over a local ring R it is already clear from 15.4.17 and 16.3.3 that $\operatorname{fd}_R M \geqslant \operatorname{depth} R - \operatorname{depth}_R M$ holds; this inequality is also a special case of:

16.3.4 Proposition. *Let R be local, M an R-complex, and N a complex in $\mathcal{D}_{\sqsubset}(R)$ of finite depth. There is an inequality,*

$$\operatorname{fd}_R M \;\geqslant\; \operatorname{depth}_R N - \operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M)\,.$$

Proof. By 16.3.1 the quantities $\operatorname{depth}_R M$ and $\operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M)$ are simultaneously finite. Thus, assume that $\operatorname{depth}_R M$ and $\operatorname{fd}_R M$ are finite, otherwise the inequality is trivial. Let $\boldsymbol{k}$ denote the residue field of R. From 15.4.17 and 16.3.1(a) one now gets $\operatorname{fd}_R M \geqslant \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \operatorname{depth}_R N - \operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M)$. □

Width of Derived Hom

For complexes with degreewise finitely generated homology the next equality simplifies; see 16.4.19.

16.3.5 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M and N be R-complexes. The complex* $\mathsf{RHom}_R(M,N)$ *has finite width if and only if M has finite depth and N has finite width, and in that case the equalities*

$$\begin{aligned}\operatorname{width}_R \mathsf{RHom}_R(M,N) &= \operatorname{width}_R N - \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\\ &= \operatorname{depth}_R M + \operatorname{width}_R N - \operatorname{depth} R\end{aligned}$$

hold, provided that one of the next conditions is satisfied:

(a) $\operatorname{pd}_R M$ *is finite and* $\operatorname{width}_R N > -\infty$ *holds.*
(b) *M belongs to* $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ *and* $\operatorname{pd}_R M$ *is finite.*

Proof. The first assertion follows from 16.2.24 and 16.2.27. Assume that M has finite depth and N has finite width. In the next computation, the 1st and 4th isomorphisms hold by 13.4.20(c), as $\boldsymbol{k}$ per 13.3.24 is derived $\mathfrak{m}$-torsion. The 2nd isomorphism follows from 13.1.18. Notice that under the assumptions in (a), the complex $\mathsf{L}\Lambda^{\mathfrak{m}}(N)$ belongs to $\mathcal{D}_{\sqsupset}(R)$ by 16.2.3; in this case the 3rd isomorphism is tensor evaluation 12.3.23(d) combined with commutativity 12.3.5. Under the assumptions in (b), the 3rd isomorphism follows similarly from 12.3.23(a). The 5th isomorphism holds by 12.3.32.

$$\begin{aligned}\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M,N) &\simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(\mathsf{RHom}_R(M,N))\\ &\simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M, \mathsf{L}\Lambda^{\mathfrak{m}}(N))\\ &\simeq \mathsf{RHom}_R(M, \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(N))\\ &\simeq \mathsf{RHom}_R(M, \boldsymbol{k} \otimes^{\mathsf{L}}_R N)\\ &\simeq \mathsf{RHom}_{\boldsymbol{k}}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M, \boldsymbol{k} \otimes^{\mathsf{L}}_R N)\,.\end{aligned}$$

Together with 16.2.3 and 7.6.12 this isomorphism yields

$$\begin{aligned}\operatorname{width}_R \mathsf{RHom}_R(M,N) &= \inf \mathsf{RHom}_{\boldsymbol{k}}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M, \boldsymbol{k} \otimes^{\mathsf{L}}_R N)\\ &= \inf(\boldsymbol{k} \otimes^{\mathsf{L}}_R N) - \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\\ &= \operatorname{width}_R N - \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\,.\end{aligned}$$

This proves the first equality in the statement. The second now follows from 16.3.3, as M has finite flat dimension by 15.4.18. $\square$

16.3.6 Corollary. *Let $(R, \mathfrak{m})$ be local and M and N be R-complexes. If $\operatorname{pd}_R M$ is finite and $N \in \mathcal{D}_{\sqsupset}(R)$ is derived $\mathfrak{m}$-complete and not acyclic, then one has*

$$-\inf \mathsf{RHom}_R(M,N) + \inf N = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Proof. As N is derived $\mathfrak{m}$-complete one has $\operatorname{width}_R N = \inf N$ by 16.2.5(a). Further, the complex $\mathsf{RHom}_R(M,N)$ is derived $\mathfrak{m}$-complete by 13.4.20(a); in particular one has $\operatorname{width}_R \mathsf{RHom}_R(M,N) = \inf \mathsf{RHom}_R(M,N)$. As N is not acyclic, it has finite width. If M has finite depth, then the equality follows from 16.3.5(a). Otherwise, also $\mathsf{RHom}_R(M,N)$ has infinite width, again by 16.3.5, so the equality still holds as $\operatorname{depth} R$ is finite. □

Let $(R,\mathfrak{m},\boldsymbol{k})$ be local. For an R-complex M of finite projective dimension, a special case of 16.3.6 reads $-\inf \mathsf{RHom}_R(M,\boldsymbol{k}) = \operatorname{depth} R - \operatorname{depth}_R M$. However, in view of the next result, this already follows from 16.3.3.

16.3.7 Lemma. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M an R-complex. There is an equality,*

$$-\inf \mathsf{RHom}_R(M,\boldsymbol{k}) \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\;.$$

Proof. The first two equalities below hold by (16.1.22.1) and adjunction 12.3.8. As $\mathrm{E}_R(\boldsymbol{k})$ is faithfully injective, see 16.1.22, the third equality follows from 2.5.7(b).

$$\begin{aligned}
-\inf \mathsf{RHom}_R(M,\boldsymbol{k}) &= -\inf \mathsf{RHom}_R(M,\mathsf{RHom}_R(\boldsymbol{k},\mathrm{E}_R(\boldsymbol{k})))\\
&= -\inf \mathsf{RHom}_R(\boldsymbol{k}\otimes^{\mathsf{L}}_R M,\mathrm{E}_R(\boldsymbol{k}))\\
&= \sup\,(\boldsymbol{k}\otimes^{\mathsf{L}}_R M)\;.
\end{aligned}$$ □

16.3.8 Proposition. *Let R be local, M an R-complex, and N a complex in $\mathcal{D}_{\sqsupset}(R)$ of finite width. There is an inequality,*

$$\operatorname{pd}_R M \;\geqslant\; \operatorname{width}_R N - \operatorname{width}_R \mathsf{RHom}_R(M,N)\;.$$

Proof. By 16.3.5 the quantities $\operatorname{depth}_R M$ and $\operatorname{width}_R \mathsf{RHom}_R(N,M)$ are simultaneously finite. Thus, assume that $\operatorname{depth}_R M$ and $\operatorname{pd}_R M$ are finite, otherwise the inequality is trivial. Let $\boldsymbol{k}$ denote the residue field of R. Now 15.4.1, 16.3.7, and 16.3.5(a) yield

$$\begin{aligned}
\operatorname{pd}_R M &\geqslant -\inf \mathsf{RHom}_R(M,\boldsymbol{k})\\
&= \sup\,(\boldsymbol{k}\otimes^{\mathsf{L}}_R M)\\
&= \operatorname{width}_R N - \operatorname{width}_R \mathsf{RHom}_R(N,M)\;.
\end{aligned}$$ □

For complexes with degreewise finitely generated homology the next equality simplifies; see 16.4.33.

16.3.9 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M and N be R-complexes. The complex $\mathsf{RHom}_R(N,M)$ has finite width if and only if M has finite width and N has finite depth, and in that case the equalities*

$$\begin{aligned}
\operatorname{width}_R \mathsf{RHom}_R(N,M) &= \operatorname{depth}_R N + \inf \mathsf{RHom}_R(\boldsymbol{k},M)\\
&= \operatorname{depth}_R N + \operatorname{width}_R M - \operatorname{depth} R
\end{aligned}$$

hold, provided that one of the next conditions is satisfied:

(a) $\operatorname{id}_R M$ *is finite and* $\operatorname{depth}_R N > -\infty$ *holds.*
(b) *M belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\operatorname{id}_R M$ *is finite.*

Proof. The first assertion follows from 16.2.24 and 16.2.27. Assume henceforth that N has finite depth and that M has finite width and finite injective dimension.

(a): In the next computation, the 1st and 5th isomorphisms hold by 13.4.20(c,b), as $\boldsymbol{k}$ per 13.3.24 is derived $\mathfrak{m}$-torsion. The 2nd and 3rd isomorphisms hold by 13.1.18 and 13.4.12. Assuming that $\operatorname{depth}_R N > -\infty$ holds, the complex $\mathsf{R}\Gamma_{\mathfrak{m}}(N)$ belongs to $\mathcal{D}_{\sqsubset}(R)$ by 16.2.14, so homomorphism evaluation 12.3.27(b) accounts for the 4th isomorphism. The 6th isomorphism holds by 12.3.36.

$$
\begin{aligned}
\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(N,M) &\simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(\mathsf{RHom}_R(N,M))\\
&\simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(N,\mathsf{L}\Lambda^{\mathfrak{m}}(M))\\
&\simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(N),M)\\
&\simeq \mathsf{RHom}_R(\mathsf{RHom}_R(\boldsymbol{k},\mathsf{R}\Gamma_{\mathfrak{m}}(N)),M)\\
&\simeq \mathsf{RHom}_R(\mathsf{RHom}_R(\boldsymbol{k},N),M)\\
&\simeq \mathsf{RHom}_{\boldsymbol{k}}(\mathsf{RHom}_R(\boldsymbol{k},N),\mathsf{RHom}_R(\boldsymbol{k},M))\;.
\end{aligned}
$$

Together with 16.2.3 and 7.6.12 this isomorphism yields

$$
\begin{aligned}
\operatorname{width}_R \mathsf{RHom}_R(N,M) &= \inf \mathsf{RHom}_{\boldsymbol{k}}(\mathsf{RHom}_R(\boldsymbol{k},N),\mathsf{RHom}_R(\boldsymbol{k},M))\\
&= \inf \mathsf{RHom}_R(\boldsymbol{k},M) - \sup \mathsf{RHom}_R(\boldsymbol{k},N)\\
&= \operatorname{depth}_R N + \inf \mathsf{RHom}_R(\boldsymbol{k},M)\;.
\end{aligned}
$$

This proves the first equality in the statement. The R-module $N = R$ has finite depth, and in this special case one gets $\inf \mathsf{RHom}_R(\boldsymbol{k},M) = \operatorname{width}_R M - \operatorname{depth} R$; the second equality now follows by substituting this expression into the first equality.

(b): As established above, one has $\inf \mathsf{RHom}_R(\boldsymbol{k},M) = \operatorname{width}_R M - \operatorname{depth} R$, so it suffices to prove the equality

$$
\operatorname{width}_R \mathsf{RHom}_R(N,M) = \operatorname{depth}_R N + \operatorname{width}_R M - \operatorname{depth} R\;.
$$

Let $(-)^{\vee}$ be the Matlis Duality functor and K the Koszul complex on a sequence that generates $\mathfrak{m}$. By 16.1.34 and commutativity 12.3.5 the complex $M \otimes^{\mathsf{L}}_R K$ belongs to $\mathcal{D}^{\ell}(R)$, so it is derived Matlis reflexive by 16.1.39 and 16.1.36(b). This explains the second isomorphism in the computation below. The first isomorphism follows from tensor evaluation 12.3.23(c), which applies as K has finite projective dimension, see 11.4.3(c). The third isomorphism is swap 12.3.16.

$$
\begin{aligned}
\mathsf{RHom}_R(N,M) \otimes^{\mathsf{L}}_R K &\simeq \mathsf{RHom}_R(N, M \otimes^{\mathsf{L}}_R K)\\
(\dagger) \qquad &\simeq \mathsf{RHom}_R(N,\mathsf{RHom}_R((M \otimes^{\mathsf{L}}_R K)^{\vee},\mathrm{E}_R(\boldsymbol{k})))\\
&\simeq \mathsf{RHom}_R((M \otimes^{\mathsf{L}}_R K)^{\vee},N^{\vee})\;.
\end{aligned}
$$

From (†), commutativity 12.3.5, and 14.4.15 one gets

$$
\operatorname{width}_R \mathsf{RHom}_R(N,M) = \operatorname{width}_R \mathsf{RHom}_R((M \otimes^{\mathsf{L}}_R K)^{\vee},N^{\vee})\;.
$$

By 15.4.3 the complex $M \otimes^{\mathsf{L}}_R K$ has finite injective dimension. Per 2.2.19 the Matlis Duality functor commutes with homology, so it follows from 16.1.44 that $(M \otimes^{\mathsf{L}}_R K)^{\vee}$

belongs to $\mathcal{D}^{\ell}_{\square}(R)$. Further it follows from 15.4.31 and 15.4.18 that $(M \otimes^{\mathsf{L}}_R K)^{\vee}$ is a complex of finite projective dimension. Now 16.3.5(b) applies, and together with the equalities from 16.2.22 and 14.4.15 it yields

$$\begin{aligned}
\operatorname{width}_R \mathsf{RHom}_R((M \otimes^{\mathsf{L}}_R K)^{\vee}, N^{\vee}) \\
&= \operatorname{depth}_R (M \otimes^{\mathsf{L}}_R K)^{\vee} + \operatorname{width}_R N^{\vee} - \operatorname{depth} R \\
&= \operatorname{width}_R (M \otimes^{\mathsf{L}}_R K) + \operatorname{depth}_R N - \operatorname{depth} R \\
&= \operatorname{width}_R M + \operatorname{depth}_R N - \operatorname{depth} R \,. \qquad \square
\end{aligned}$$

16.3.10 Corollary. *Let $(R, \mathfrak{m})$ be local and M and N be R-complexes. If $\operatorname{id}_R M$ is finite and $N \in \mathcal{D}_{\sqsubset}(R)$ is derived $\mathfrak{m}$-torsion and not acyclic, then one has*

$$-\inf \mathsf{RHom}_R(N, M) - \sup N \;=\; \operatorname{depth} R - \operatorname{width}_R M \,.$$

Proof. As N is derived $\mathfrak{m}$-torsion one has $\operatorname{depth}_R N = -\sup N$ by 16.2.16(a). Further, the complex $\mathsf{RHom}_R(N, M)$ is derived $\mathfrak{m}$-complete by 13.4.20(b), whence $\operatorname{width}_R \mathsf{RHom}_R(N, M) = \inf \mathsf{RHom}_R(N, M)$ holds by 16.2.5(a). As N is not acyclic, it has finite depth. If M has finite width, then the equality follows from 16.3.9(a). Otherwise, also $\mathsf{RHom}_R(N, M)$ has infinite width, again by 16.3.9, so the equality still holds as $\operatorname{depth} R$ is finite. $\square$

16.3.11 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{id}_R M$ is finite, then one has*

$$-\inf \mathsf{RHom}_R(\boldsymbol{k}, M) \;=\; \operatorname{depth} R - \operatorname{width}_R M \,.$$

Proof. As $\boldsymbol{k}$ by 13.3.24 is derived $\mathfrak{m}$-torsion, 16.3.10 applies with $N = \boldsymbol{k}$. $\square$

For a complex M over a local ring R it is already clear from 15.4.7 and 16.3.11 that $\operatorname{id}_R M \geqslant \operatorname{depth} R - \operatorname{width}_R M$ holds; this inequality is also a special case of:

16.3.12 Proposition. *Let R be local, M an R-complex, and N a complex in $\mathcal{D}_{\sqsubset}(R)$ of finite depth. There is an inequality,*

$$\operatorname{id}_R M \;\geqslant\; \operatorname{depth}_R N - \operatorname{width}_R \mathsf{RHom}_R(N, M) \,.$$

Proof. By 16.3.9 the quantities $\operatorname{width}_R M$ and $\operatorname{width}_R \mathsf{RHom}_R(N, M)$ are simultaneously finite. Thus, assume that $\operatorname{width}_R M$ and $\operatorname{id}_R M$ are finite, otherwise the inequality is trivial. Let $\boldsymbol{k}$ denote the residue field of R. From 15.4.7 and 16.3.9(a) one now gets $\operatorname{id}_R M \geqslant -\inf \mathsf{RHom}_R(\boldsymbol{k}, M) = \operatorname{depth}_R N - \operatorname{width}_R \mathsf{RHom}_R(N, M)$. $\square$

Rigidity of Ext

16.3.13 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex such that the inequality $\operatorname{depth}_R M > -\infty$ holds. There are equalities,*

$$\operatorname{id}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;=\; -\inf \mathsf{RHom}_R(\boldsymbol{k}, M) \;=\; \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_R(\boldsymbol{k}, M) \neq 0\} \,.$$

Further, if $\operatorname{Ext}^{n+1}_R(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$, then one has

$$n \geqslant \mathrm{id}_R\, \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \operatorname{depth} R - \operatorname{width}_R M\,.$$

PROOF. Specialization of 15.4.12 to $\mathfrak{a} = \mathfrak{m}$ yields the equalities in the first display as well as the inequality $n \geqslant \mathrm{id}_R\, \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ under the assumption that $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$. Further, if $\mathrm{id}_R\, \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ is finite, then 16.3.11 yields

$$(\dagger) \qquad -\inf \mathsf{RHom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(M)) = \operatorname{depth} R - \operatorname{width}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M)\,.$$

As $\boldsymbol{k}$ by 13.3.24 is derived $\mathfrak{m}$-torsion, $\mathsf{RHom}_R(\boldsymbol{k}, \mathsf{R}\Gamma_{\mathfrak{m}}(M)) \simeq \mathsf{RHom}_R(\boldsymbol{k}, M)$ holds by 13.4.20(b). Moreover, one has $\operatorname{width}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \operatorname{width}_R M$ by 16.2.3, so the equality asserted in the second display now follows from (†) and the previously established equality $\mathrm{id}_R\, \mathsf{R}\Gamma_{\mathfrak{m}}(M) = -\inf \mathsf{RHom}_R(\boldsymbol{k}, M)$. □

16.3.14 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a derived $\mathfrak{m}$-torsion complex that belongs to $\mathcal{D}_{\sqsubset}(R)$. There are equalities,*

$$\mathrm{id}_R\, M = -\inf \mathsf{RHom}_R(\boldsymbol{k}, M) = \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(\boldsymbol{k}, M) \neq 0\}\,.$$

Further, if $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant -\inf M$, then one has

$$n \geqslant \mathrm{id}_R\, M = \operatorname{depth} R - \operatorname{width}_R M\,.$$

PROOF. By assumption there is an isomorphism $\mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq M$ in $\mathcal{D}(R)$, and 16.2.16 yields $\operatorname{depth}_R M \geqslant -\sup M > -\infty$, so this is a special case of 16.3.13. □

16.3.15 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an $\mathfrak{m}$-torsion R-module. One has*

$$\mathrm{id}_R\, M = -\inf \mathsf{RHom}_R(\boldsymbol{k}, M) = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(\boldsymbol{k}, M) \neq 0\}\,.$$

Further, if $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant 0$, then one has

$$n \geqslant \mathrm{id}_R\, M = \operatorname{depth} R - \operatorname{width}_R M\,.$$

PROOF. In view of 13.3.30, this assertion is a special case of 16.3.14. □

For an R-complex M with $\operatorname{depth}_R M > -\infty$ the following rigidity statement is part of 16.3.13.

16.3.16 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$, then one has*

$$n \geqslant \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}_R^m(\boldsymbol{k}, M) \neq 0\} = \operatorname{depth} R - \operatorname{width}_R M\,.$$

PROOF. Assuming that $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$, specialization of 15.4.16 to $\mathfrak{a} = \mathfrak{m}$ yields $\operatorname{Ext}_R^m(\boldsymbol{k}, M) = 0$ for all integers $m > n$.

Let X be any R-complex and K the Koszul complex on a sequence that generates $\mathfrak{m}$. Tensor evaluation 12.3.23(c) combined with commutativity 12.3.5 yields

$$(\dagger) \qquad \mathsf{RHom}_R(X, K \otimes_R^{\mathsf{L}} M) \simeq K \otimes_R^{\mathsf{L}} \mathsf{RHom}_R(X, M)\,.$$

This accounts for the first equality in the computation below, and the second equality holds by the definition of width 16.2.1. The complex $\mathsf{RHom}_R(\boldsymbol{k}, M)$ is derived $\mathfrak{m}$-complete by 13.1.23, so the last equality holds by 16.2.5(a).

$$(\ddagger)\qquad \begin{aligned} -\inf \mathsf{RHom}_R(\boldsymbol{k}, K \otimes^{\mathsf{L}}_R M) &= -\inf (K \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(\boldsymbol{k}, M)) \\ &= -\operatorname{width}_R \mathsf{RHom}_R(\boldsymbol{k}, M) \\ &= -\inf \mathsf{RHom}_R(\boldsymbol{k}, M)\,. \end{aligned}$$

For every prime ideal $\mathfrak{p}$ in R it follows from (†), applied with $X = R/\mathfrak{p}$, and 13.3.31 that the complex $\mathsf{RHom}_R(R/\mathfrak{p}, K \otimes^{\mathsf{L}}_R M)$ is derived $\mathfrak{m}$-torsion. Further, one has

$$n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) \geqslant -\operatorname{width}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) = -\operatorname{width}_R M = -\inf (K \otimes^{\mathsf{L}}_R M)$$

by 16.2.5, 16.2.3, and the definition of width. Thus 15.4.11 applies to the complex $K \otimes^{\mathsf{L}}_R M$ and yields $\operatorname{id}_R(K \otimes^{\mathsf{L}}_R M) \leqslant n$. Now (‡) and 16.3.11 yield

$$-\inf \mathsf{RHom}_R(\boldsymbol{k}, M) = \operatorname{depth} R - \operatorname{width}_R(K \otimes^{\mathsf{L}}_R M) = \operatorname{depth} R - \operatorname{width}_R M\,,$$

where the last equality follows from 14.4.15. □

Theorem 16.3.16 remains true with the bound on n lowered to $-\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) - 1$, see 17.5.10; it is a technicality, but it comes in handy in the proof of 17.5.11.

16.3.17 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{Ext}^{n+1}_R(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant \dim R - \operatorname{width}_R M$, then one has*

$$n \geqslant \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_R(\boldsymbol{k}, M) \neq 0\} = \operatorname{depth} R - \operatorname{width}_R M\,.$$

Proof. The assertion follows in view of 16.2.34 from 16.3.16. □

Rigidity of Tor

16.3.18 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex such that the inequality $\operatorname{width}_R M > -\infty$ holds. There are equalities,*

$$\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \sup (\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}\,.$$

Further, if $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M)$, then one has

$$n \geqslant \operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Proof. Specialization of 15.4.25 to $\mathfrak{a} = \mathfrak{m}$ yields the equalities in the first display as well as the inequality $n \geqslant \operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M)$ under the assumption that $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ holds for an $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M)$. Further, if $\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M)$ is finite, then 16.3.3 yields

$$(\dagger)\qquad \sup (\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M)) = \operatorname{depth} R - \operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M)\,.$$

As $\boldsymbol{k}$ by 13.3.24 is derived $\mathfrak{m}$-torsion, $\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R M$ holds by 13.4.20(c). Moreover, one has $\operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \operatorname{depth}_R M$ by 16.2.14, so the equality asserted in the second display now follows from (†) and the previously established equality $\operatorname{fd}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \sup (\boldsymbol{k} \otimes^{\mathsf{L}}_R M)$. □

16.3.19 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a derived $\mathfrak{m}$-complete complex that belongs to $\mathcal{D}_{\sqsupset}(R)$. There are equalities,*

$$\operatorname{fd}_R M \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}\;.$$

Further, if $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ *holds for an integer* $n \geqslant \sup M$*, then one has*

$$n \;\geqslant\; \operatorname{fd}_R M \;=\; \operatorname{depth} R - \operatorname{depth}_R M\;.$$

Proof. By assumption there is an isomorphism $\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq M$ in $\mathcal{D}(R)$, and 16.2.5 yields $\operatorname{width}_R M \geqslant \inf M > -\infty$, so this is a special case of 16.3.18. $\square$

16.3.20 Corollary. *Let* $(R, \mathfrak{m}, \boldsymbol{k})$ *be local and* M *an* $\mathfrak{m}$*-complete* R*-module. One has,*

$$\operatorname{fd}_R M \;=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; \sup\{m \in \mathbb{N}_0 \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}\;.$$

Further, if $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ *holds for an integer* $n \geqslant 0$*, then one has*

$$n \;\geqslant\; \operatorname{fd}_R M \;=\; \operatorname{depth} R - \operatorname{depth}_R M\;.$$

Proof. In view of 13.1.33, this assertion is a special case of 16.3.19. $\square$

For an R-complex M with $\operatorname{width}_R M > -\infty$ the following rigidity statement is part of 16.3.18.

16.3.21 Theorem. *Let* $(R, \mathfrak{m}, \boldsymbol{k})$ *be local and* M *an* R*-complex. If* $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ *holds for an integer* $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M)$*, then one has*

$$n \;\geqslant\; \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\} \;=\; \operatorname{depth} R - \operatorname{depth}_R M\;.$$

Proof. Assuming that $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M)$, specialization of 15.4.29 to $\mathfrak{a} = \mathfrak{m}$ yields of $\operatorname{Tor}^R_m(\boldsymbol{k}, M) = 0$ for all integers $m > n$.

Let X be any R-complex and K the Koszul complex on a sequence that generates $\mathfrak{m}$. Tensor evaluation 12.3.23(a) combined with commutativity 12.3.5 yields

$$(\dagger)\qquad X \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(K, M) \;\simeq\; \mathsf{RHom}_R(K, X \otimes^{\mathsf{L}}_R M)\;.$$

This accounts for the first equality in the computation below, and the second equality holds by 16.2.14. The complex $\boldsymbol{k} \otimes^{\mathsf{L}}_R M$ is derived $\mathfrak{m}$-torsion by 13.3.25, so the last equality holds by 16.2.16(a).

$$(\ddagger)\qquad \begin{aligned} \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(K, M)) \;&=\; \sup \mathsf{RHom}_R(K, \boldsymbol{k} \otimes^{\mathsf{L}}_R M) \\ &=\; -\operatorname{depth}_R(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \\ &=\; \sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)\;. \end{aligned}$$

For every prime ideal $\mathfrak{p}$ in R it follows from ($\dagger$), applied with $X = R/\mathfrak{p}$, and 13.1.34 combined with 14.3.2 that the complex $R/\mathfrak{p} \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(K, M)$ is derived $\mathfrak{m}$-complete. Further, by 16.2.16 and 16.2.14 one has

$$n \;\geqslant\; \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M) \;\geqslant\; -\operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) \;=\; -\operatorname{depth}_R M \;=\; \sup \mathsf{RHom}_R(K, M)\;.$$

Thus 15.4.24 applies to $\mathsf{RHom}_R(K, M)$ and yields $\operatorname{fd}_R \mathsf{RHom}_R(K, M) \leqslant n$. Now ($\ddagger$) and 16.3.3 yield

$$\sup\,(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; \operatorname{depth} R - \operatorname{depth}_R \mathsf{RHom}_R(K, M) \;=\; \operatorname{depth} R - \operatorname{depth}_R M$$

where the last equality follows from 14.3.2, 14.3.11, and 14.4.15. $\square$

REMARK. For a complex M over a local ring $(R, \mathfrak{m})$ with $s = \sup \mathsf{L}\Lambda^{\mathfrak{m}}(M) \in \mathbb{Z}$, Christensen, Ferraro, and Thompson [58] show that $\operatorname{Tor}_s^R(\boldsymbol{k}, M)$ is non-zero, which means that 16.3.21 remeains true with the bound on n lowered to $\sup \mathsf{L}\Lambda^{\mathfrak{m}}(M) - 1$; see also the comment after 16.3.16. It follows that the next corollary holds with the improved bound $n \geqslant \dim R - \operatorname{depth}_R M - 1$, see also E 16.3.8.

16.3.22 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{Tor}_{n+1}^R(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant \dim R - \operatorname{depth}_R M$, then one has*

$$n \;\geqslant\; \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}_m^R(\boldsymbol{k}, M) \neq 0\} \;=\; \operatorname{depth} R - \operatorname{depth}_R M\,.$$

PROOF. The assertion follows in view of 16.2.34 from 16.3.21. $\square$

EXERCISES

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 16.3.1 Let $L \in \mathcal{D}_{\square}^{f}(R)$ be a complex of finite projective dimension with $\mathrm{H}(L) \neq 0$. Show that the inequality $\operatorname{depth}_R M + \operatorname{width}_R M \leqslant \operatorname{depth} R + \operatorname{amp} L$ holds for every complex $M \in \mathcal{D}_{\square}(R)$ with $\mathfrak{m} \in \operatorname{supp}_R M$. *Hint:* E 16.2.6.

E 16.3.2 Let $M \neq 0$ be a finitely generated R-module. Show that if there exists a non-zero finitely generated R-module of finite injective dimension, then $\operatorname{depth}_R M \leqslant \operatorname{depth} R$ holds.

E 16.3.3 Let $\boldsymbol{x}$ be a sequence of generators for $\mathfrak{m}$. Show that $\operatorname{fd}_R \check{\mathrm{C}}^R(\boldsymbol{x}) = 0$ holds.

E 16.3.4 Let M be an R-complex. (a) Show that if $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ or $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ has finite flat dimension, then $\sup(\boldsymbol{k} \otimes_R^{\mathsf{L}} M) = \operatorname{depth} R - \operatorname{depth}_R M$ holds. (b) Show that if $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ or $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ has finite injective dimension, then $-\inf \mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) = \operatorname{depth} R - \operatorname{width}_R M$ holds.

E 16.3.5 Let $M \neq 0$ be an R-module. Show that if M is derived $\mathfrak{m}$-complete and $\operatorname{Tor}_1^R(\boldsymbol{k}, M) = 0$ holds, then one has $\operatorname{depth}_R M = \operatorname{depth} R$. (Under the weaker assumption that $\operatorname{width}_R M = 0 = \operatorname{Tor}_1^R(\boldsymbol{k}, M)$ holds, Iyengar and Bridgeland [44] obtain the inequality $\operatorname{depth}_R M \leqslant \operatorname{depth} R$, which compares to the one in 16.2.35.)

E 16.3.6 Let M be an R-complex and $n \in \mathbb{Z}$. Show that $\mathrm{H}_{\mathfrak{m}}^n(M) \neq 0$ implies $\operatorname{Ext}_R^n(\boldsymbol{k}, M) \neq 0$. Conclude that $-\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) \leqslant -\inf \mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ holds and that 16.3.16 remains true with the improved bound $n \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) - 1$. *Hint:* E 11.3.2.

E 16.3.7 Show that 16.3.17 remains true with the improved bound $n \geqslant \dim R - \operatorname{width}_R M - 1$.

E 16.3.8 Show that 16.3.22 remains true with the improved bound $n \geqslant \dim R - \operatorname{depth}_R M - 1$.

E 16.3.9 Let M and N be non-zero finitely generated R-modules. Show that if $\operatorname{pd}_R M$ or $\operatorname{id}_R N$ is finite, then one has $\sup\{m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M, N) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R M$.

16.4 Formulas of Auslander, Buchsbaum, and Bass

SYNOPSIS. Auslander–Buchsbaum Formula; Bass Formula; Betti number; Poincaré series; minimal semi-free resolution; Bass number; Bass series; minimal semi-injective resolution.

For finitely generated modules over a local ring, the relations from Sect. 16.3 between homological dimensions and the invariants depth and width simplify. Unsurprisingly, our order of presentation is reverse chronological: The case of finitely generated modules was investigated first. For example, the Auslander–Buchsbaum Formula 16.4.2 first appeared in the 1957 paper [11], while the more general formula 16.3.1 came two decades later in [94].

THE AUSLANDER–BUCHSBAUM FORMULA

A local ring is semi-perfect, see B.44, so the next result is essentially a special case of 8.1.18. Here we provide a different proof that does not rely on the existence of minimal semi-projective resolutions.

16.4.1 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, and n an integer. The following conditions are equivalent.*

(*i*) $\operatorname{pd}_R M \leqslant n$.

(*ii*) $-\inf \mathsf{RHom}_R(M, \boldsymbol{k}) \leqslant n$.

(*iii*) $n \geqslant \sup M$ *and* $\operatorname{Ext}^{n+1}_R(M, \boldsymbol{k}) = 0$.

(*iv*) $\operatorname{fd}_R M \leqslant n$.

(*v*) $\sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \leqslant n$.

(*vi*) $n \geqslant \sup M$ *and* $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$.

In particular, there are equalities,

$$\operatorname{pd}_R M \;=\; -\inf \mathsf{RHom}_R(M, \boldsymbol{k}) \;=\; \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \;=\; \operatorname{fd}_R M\,.$$

PROOF. The six conditions are evidently satisfied if M is acyclic, so assume that M is not acyclic and set $s = \sup M$. The implication $(iv) \Rightarrow (v)$ is immediate from 15.4.17. To see that (v) implies (vi), notice that each functor $\operatorname{Tor}^R_m(-, M)$ is R-linear and half exact; per 12.2.12 the module $\operatorname{Tor}^R_m(R/\mathfrak{p}, M)$ is finitely generated for every $\mathfrak{p} \in \operatorname{Spec} R$. Since $\operatorname{Tor}^R_s(R, M) \cong \mathrm{H}_s(M)$ is non-zero, it now follows from 16.1.7 that $\operatorname{Tor}^R_s(\boldsymbol{k}, M) \neq 0$ holds. Another application of 16.1.7 now shows that if $\operatorname{Tor}^R_{n+1}(\boldsymbol{k}, M) = 0$ holds, then one has $\operatorname{Tor}^R_{n+1}(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in R, whence (vi) implies (iv) by 15.4.17. By 15.4.18 one has $\operatorname{pd}_R M = \operatorname{fd}_R M$; in particular, conditions (i) and (iv) are equivalent. Let $m \in \mathbb{Z}$; by 16.1.22 the module $\operatorname{Tor}^R_m(\boldsymbol{k}, M)$ vanishes if and only if $\operatorname{Ext}^m_R(M, \boldsymbol{k})$ vanishes. This shows that conditions (v) and (vi) are equivalent to (ii) and (iii), respectively. □

The next result is known as the Auslander–Buchsbaum Formula.

16.4.2 Corollary. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. If M has finite projective dimension, then the next equality holds,*

$$\operatorname{pd}_R M \;=\; \operatorname{depth} R - \operatorname{depth}_R M\,.$$

PROOF. The equality follows immediately from 16.4.1 and 16.3.3. □

The equality in the next result can be rewritten to match the inequality in 16.3.4, provided that the complex N belongs to $\mathcal{D}_{\sqsubset}(R)$, cf. 16.2.21. The special case $N = R$ is then simply the Auslander–Buchsbaum Formula.

16.4.3 Theorem. *Let R be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, and N an R-complex. If M has finite projective dimension and is not acyclic, then there is an equality,*

$$\operatorname{pd}_R M + \operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M) \;=\; \operatorname{depth}_R N\,.$$

PROOF. It follows from 16.2.27 that M is a complex of finite depth. Per 16.3.1 the complex $N \otimes^{\mathsf{L}}_R M$ has infinite depth if and only if N has infinite depth, in which case the assered equality is trivial. Assuming now that N has finite depth, the asserted equality hold by 16.3.1(b) in view of 16.4.1. □

The equality in the next result can be rewritten to match the inequality in 16.3.8, provided that the complex N belongs to $\mathcal{D}_{\sqsupset}(R)$, cf. 16.2.5(a). The special case $N = R$ then recovers one of the equalities in 12.3.20.

16.4.4 Theorem. *Let R be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, and N an R-complex. If M has finite projective dimension and is not acyclic, then there is an equality,*

$$\operatorname{pd}_R M + \operatorname{width}_R \mathsf{RHom}_R(M,N) \;=\; \operatorname{width}_R N\,.$$

Proof. It follows from 16.2.27 that M is a complex of finite depth. Per 16.3.5 the complex $\mathsf{RHom}_R(M,N)$ has infinite width if and only if N has infinite width, in which case the asserted equality is trivial. Assuming now that N has finite width, the asserted equality holds, in view of 16.4.1, by 16.3.5(b). □

The Bass Formula

Bass' 1963 paper [32] on Gorenstein rings remains one of the most cited papers in commutative algebra. The next lemma is the first result proved in [32] and fundamental to the rest of the paper.

16.4.5 Lemma. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, $\mathfrak{p}$ a prime ideal in R, and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. For every integer m one has*

$$\operatorname{Ext}^m_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),M_\mathfrak{p}) \;\neq\; 0 \qquad\Longrightarrow\qquad \operatorname{Ext}^{m+\dim R/\mathfrak{p}}_R(\boldsymbol{k},M) \;\neq\; 0\,.$$

In particular, there are inequalities,

(a) $\;-\sup \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),M_\mathfrak{p}) + \dim R/\mathfrak{p} \;\geqslant\; -\sup \mathsf{RHom}_R(\boldsymbol{k},M)\,.$

(b) $\;-\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),M_\mathfrak{p}) + \dim R/\mathfrak{p} \;\leqslant\; -\inf \mathsf{RHom}_R(\boldsymbol{k},M)\,.$

Proof. Set $d = \dim R/\mathfrak{p}$ and proceed by induction on d. If $d = 0$ holds, then one has $\mathfrak{p} = \mathfrak{m}$ and the claim is trivial.

$d = 1$: By 14.1.23 there is an isomorphism $\operatorname{Ext}^m_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),M_\mathfrak{p}) \cong \operatorname{Ext}^m_R(R/\mathfrak{p},M)_\mathfrak{p}$, so by assumption one has $\operatorname{Ext}^m_R(R/\mathfrak{p},M) \neq 0$. Choose an element $x \in \mathfrak{m} \setminus \mathfrak{p}$. From the exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;x\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}+(x)) \longrightarrow 0$$

one gets per 7.3.35 and 12.2.6 an exact sequence of finitely generated R-modules,

$$\operatorname{Ext}^m_R(R/\mathfrak{p},M) \xrightarrow{\;x\;} \operatorname{Ext}^m_R(R/\mathfrak{p},M) \longrightarrow \operatorname{Ext}^{m+1}_R(R/(\mathfrak{p}+(x)),M)\,,$$

so Nakayama's lemma 16.1.5 yields $\operatorname{Ext}^{m+1}_R(R/(\mathfrak{p}+(x)),M) \neq 0$. As $d = 1$, the classic support of the R-module $R/(\mathfrak{p}+(x))$ is $\{\mathfrak{m}\}$, whence it follows from 12.4.7 that also $\operatorname{Ext}^{m+1}_R(\boldsymbol{k},M)$ is non-zero.

$d > 1$: Choose a chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = \mathfrak{m}$. Set $R' = R_{\mathfrak{p}_1}$, it is a local ring with maximal ideal $\mathfrak{p}' = \mathfrak{p}_1 R'$, and set $M' = M_{\mathfrak{p}_1}$; recall from 14.1.11 that M' belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R')$. Denote by $\mathfrak{q}$ the prime ideal $\mathfrak{p}R'$ in R'. The module

$$\operatorname{Ext}^m_{R'_{\mathfrak{q}}}(\kappa(\mathfrak{q}), M'_{\mathfrak{q}}) \;\cong\; \operatorname{Ext}^m_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}})\;,$$

see 15.1.3, is by assumption non-zero, and one has $\dim R'/\mathfrak{q} = 1$, so the induction base yields $\operatorname{Ext}^{m+1}_{R_{\mathfrak{p}_1}}(\kappa(\mathfrak{p}_1), M_{\mathfrak{p}_1}) = \operatorname{Ext}^{m+1}_{R'}(R'/\mathfrak{p}', M') \neq 0$. As one has $\dim R/\mathfrak{p}_1 = d-1$, it follows by the induction hypothesis that $\operatorname{Ext}^{m+d}_R(\boldsymbol{k}, M)$ is non-zero. □

The next inequality also holds for a derived $\mathfrak{m}$-complete complex over a local ring $(R,\mathfrak{m})$, see 18.3.26, but as illustrated by 16.4.7 it does not hold without assumptions on the complex.

16.4.6 Proposition. *Let R be local, $\mathfrak{p}$ a prime ideal in R, and M an R-complex. If M is in $\mathcal{D}^{\mathrm{f}}(R)$, then there is an inequality,*

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \;\geqslant\; \operatorname{depth}_R M\;.$$

Proof. If $\mathrm{H}(M)$ is not bounded above, then the inequality is trivial as 16.2.21 yields $\operatorname{depth}_R M = -\infty$. Assuming now that M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, the inequality follows immediately from 16.2.14 and 16.4.5(a). □

16.4.7 Example. Let $(R,\mathfrak{m})$ be local and $\mathfrak{p} \neq \mathfrak{m}$ a prime ideal in R. The $R_{\mathfrak{p}}$-module $\mathrm{E}_R(R/\mathfrak{p})_{\mathfrak{p}} = \mathrm{E}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}))$, see C.18, has depth 0, while it follows from 16.2.29 that the R-module $\mathrm{E}_R(R/\mathfrak{p})$ has depth ∞.

16.4.8 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$; one has*

$$\operatorname{id}_R M \;=\; -\inf \operatorname{RHom}_R(\boldsymbol{k}, M)\;.$$

Proof. Let $m \in \mathbb{Z}$ and notice that if $\operatorname{Ext}^m_R(R/\mathfrak{a}, M)$ is non-zero for some ideal $\mathfrak{a}$ in R, then one has $\operatorname{Ext}^m_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0$ for a prime ideal $\mathfrak{p}$ by 12.4.9. That is, one has $\operatorname{Ext}^m_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$, by 14.1.23, so 16.4.5 yields $\operatorname{Ext}^n_R(\boldsymbol{k}, M) \neq 0$ for some $n \geqslant m$. Thus one has

$$\sup\{-\inf \operatorname{RHom}_R(R/\mathfrak{a}, M) \mid \mathfrak{a} \text{ is an ideal in } R\} \;\leqslant\; -\inf \operatorname{RHom}_R(\boldsymbol{k}, M)$$

and the opposite inequality is trivial. Now invoke 15.4.7. □

16.4.9 Corollary. *Let R be local, $\mathfrak{p}$ a prime ideal in R, and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. There is an inequality,*

$$\operatorname{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \;\leqslant\; \operatorname{id}_R M\;.$$

Proof. By 14.1.11 the complex $M_{\mathfrak{p}}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R_{\mathfrak{p}})$, so the inequality follows immediately from 16.4.8 and 16.4.5(b). □

16.4.10 Corollary. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$; there is an inequality,*

$$\dim_R M \;\leqslant\; \operatorname{id}_R M\;.$$

Proof. Let $\mathfrak{p}$ be a prime idel in R. By 8.2.3 and 16.4.9 one has

$$\dim R/\mathfrak{p} - \inf M_{\mathfrak{p}} \;\leqslant\; \dim R/\mathfrak{p} + \operatorname{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \;\leqslant\; \operatorname{id}_R M\;;$$

now invoke 14.2.6. □

The next result, and the subsequent special case which first appeared in Bass' seminal paper [32], is known as the Bass Formula. The boundedness condition on the complex is superfluous, see 9.2.12 and 19.2.38.

16.4.11 Corollary. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. If M has finite injective dimension, then the next equality holds,*

$$\mathrm{id}_R M \;=\; \operatorname{depth} R - \inf M\,.$$

Proof. By 16.2.5(a) one has $\operatorname{width}_R M = \inf M$, so the equality is immediate from 16.4.8 and 16.3.11. □

16.4.12 Corollary. *Let R be local and $M \neq 0$ a finitely generated R-module. If M has finite injective dimension, then the next equality holds,*

$$\mathrm{id}_R M \;=\; \operatorname{depth} R\,.$$

Proof. This is a special case of 16.4.11 as a non-zero module has infimum 0. □

The equality in the next result can be rewritten to match the inequality in 16.3.12, provided that the complex N belongs to $\mathcal{D}_{\sqsubset}(R)$, cf. 16.2.21. The special case $N = R$ is then simply the Bass Formula.

16.4.13 Theorem. *Let R be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, and N and R-complex. If M has finite injective dimension and is not acyclic, then there is an equality,*

$$\mathrm{id}_R M + \operatorname{width}_R \mathsf{RHom}_R(N,M) \;=\; \operatorname{depth}_R N\,.$$

Proof. It follows from 16.2.27 that M is a complex of finite width. Per 16.3.9 the complex $\mathsf{RHom}_R(N,M)$ has infinite width if and only if N has infinite depth, in which case the asserted equality is trivial. Assuming now that N has finite depth, the asserted equality holds, in view of 16.4.8, by 16.3.9(b). □

Betti Numbers and Poincaré Series

Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. It follows from 12.2.12 and 12.2.6 that the complexes $\boldsymbol{k} \otimes^{\mathsf{L}}_R M$ and $\mathsf{RHom}_R(M,\boldsymbol{k})$ belong to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(\boldsymbol{k})$ and $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(\boldsymbol{k})$, respectively. In particular, the homology modules $\operatorname{Tor}^R_m(\boldsymbol{k},M)$ and $\operatorname{Ext}^m_R(M,\boldsymbol{k})$ are $\boldsymbol{k}$-vector spaces of finite rank for all $m \in \mathbb{Z}$ and vanish for $m \ll 0$. In particular, the Betti numbers defined below do indeed belong to $\mathbb{N}_0$. The Betti numbers are interpreted in 16.4.25.

16.4.14 Definition. Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. For $m \in \mathbb{Z}$ the m^{th} *Betti number* of M is defined by

$$\beta^R_m(M) \;=\; \operatorname{rank}_{\boldsymbol{k}} \operatorname{Tor}^R_m(\boldsymbol{k},M)\,.$$

The *Poincaré series* of M is the generating function

$$\mathrm{P}^R_M(t) \;=\; \sum_{m\in\mathbb{Z}} \beta^R_m(M)t^m.$$

It is standard to use the abbreviated notation $\mathrm{P}^R(t)$ for $\mathrm{P}^R_{\boldsymbol{k}}(t)$.

Notice from the comments before the definition that the Poincaré series of a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ is a Laurent series with coefficients in $\mathbb{N}_0$.

16.4.15. For a Laurent series in t with coefficients in $\Bbbk$, i.e. an expression of the form $\mathrm{L}(t) = \sum_{m\in\mathbb{Z}} a_m t^m$ with $a_m \in \Bbbk$ and $a_m = 0$ for $m \ll 0$, the order and degree are defined as for polynomials and power series:

$$\operatorname{ord}\mathrm{L}(t) \;=\; \inf\{m \in \mathbb{Z} \mid a_m \neq 0\} \quad \text{and} \quad \deg\mathrm{L}(t) \;=\; \sup\{m \in \mathbb{Z} \mid a_m \neq 0\}\,,$$

with the usual convention that zero series has order ∞ and degree $-\infty$. The Laurent series with coefficients in $\Bbbk$ form a ring, $\Bbbk(\!|t|\!)$, and if $\Bbbk$ is an integral domain, then so is $\Bbbk(\!|t|\!)$. For series $\mathrm{L}(t)$ and $\mathrm{S}(t)$ in $\mathbb{Z}(\!|t|\!)$ one, evidently, has

$$\operatorname{ord}(\mathrm{L}(t)\mathrm{S}(t)) \;=\; \operatorname{ord}\mathrm{L}(t) + \operatorname{ord}\mathrm{S}(t)\,;$$

assuming that both series have non-negative coefficients, one also has

$$\deg\,(\mathrm{L}(t)\mathrm{S}(t)) \;=\; \deg\mathrm{L}(t) + \deg\mathrm{S}(t)\,.$$

REMARK. Growth patterns in Betti numbers, indcluding rationality of Poincaré series, is a research topic of lasting interest in commutative algebra. See for example Avramov's surveys [15, 17, 18] for an introduction.

16.4.16 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. The degree and order of the Poincaré series $\mathrm{P}^R_M(t)$ satisfy the next equalities.*

$$\begin{aligned}\deg\mathrm{P}^R_M(t) \;&=\; \sup\{m \in \mathbb{Z} \mid \beta^R_m(M) \neq 0\} \;=\; \operatorname{pd}_R M \;=\; \operatorname{fd}_R M \quad \textit{and}\\ \operatorname{ord}\mathrm{P}^R_M(t) \;&=\; \inf\{m \in \mathbb{Z} \mid \beta^R_m(M) \neq 0\} \;=\; \inf M\,.\end{aligned}$$

Assume that M is not acyclic and set $w = \inf M$. The number $\beta^R_w(M)$ records the minimal number of generators of the module $\mathrm{H}_w(M)$.

PROOF. The equalities hold by 16.4.1 and 16.2.5(a). The last assertion follows as 7.6.8 yields $\beta^R_w(M) = \operatorname{rank}_{\boldsymbol{k}}(\boldsymbol{k} \otimes_R \mathrm{H}_w(M))$. □

The equality of projective dimensions in the next result compares to 8.3.15(d).

16.4.17 Proposition. *Let R be local and M and N be complexes in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. There is an equality of Laurent series,*

$$\mathrm{P}^R_{M\otimes^{\mathsf{L}}_R N}(t) \;=\; \mathrm{P}^R_M(t)\,\mathrm{P}^R_N(t)\,;$$

in particular, one has

$$\operatorname{pd}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{pd}_R M + \operatorname{pd}_R N\,.$$

PROOF. Recall from 12.2.12 that the complex $M \otimes^{\mathsf{L}}_R N$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$; in particular, 16.4.14 applies. Let $\boldsymbol{k}$ be the residue field of R. By 12.3.30 and 7.6.12 there are isomorphisms in $\mathcal{D}(\boldsymbol{k})$,

$$\boldsymbol{k} \otimes^{\mathsf{L}}_R (M \otimes^{\mathsf{L}}_R N) \;\simeq\; (\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \otimes^{\mathsf{L}}_{\boldsymbol{k}} (\boldsymbol{k} \otimes^{\mathsf{L}}_R N) \;\simeq\; \mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \otimes_{\boldsymbol{k}} \mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R N)\,.$$

For every $m \in \mathbb{Z}$ the definition, 2.1.14, of the graded tensor product yields

$$\begin{aligned}\beta_m^R(M \otimes_R^{\mathsf{L}} N) &= \operatorname{rank}_{\boldsymbol{k}}(\mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} M) \otimes_{\boldsymbol{k}} \mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} N))_m \\ &= \sum_{i\in\mathbb{Z}} \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_i(\boldsymbol{k} \otimes_R^{\mathsf{L}} M) \cdot \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{m-i}(\boldsymbol{k} \otimes_R^{\mathsf{L}} N) \\ &= \sum_{i+j=m} \beta_i^R(M)\beta_j^R(N)\,,\end{aligned}$$

and this is the degree m coefficient of the product series $\mathrm{P}_M^R(t)\,\mathrm{P}_N^R(t)$.

The Poincaré series have non-negative coefficients, so the equality of projective dimensions follows from 16.4.16, as per 16.4.15 one has

$$\deg \mathrm{P}_{M\otimes_R N}^R(t) \;=\; \deg\,(\mathrm{P}_M^R(t)\,\mathrm{P}_N^R(t)) \;=\; \deg \mathrm{P}_M^R(t) + \deg \mathrm{P}_N^R(t)\,. \qquad \square$$

16.4.18 Corollary. *Let R be local and M and N be finitely generated R-modules. If $\operatorname{Tor}_m^R(M,N) = 0$ holds for all $m > 0$, then one has $\operatorname{pd}_R(M \otimes_R N) = \operatorname{pd}_R M + \operatorname{pd}_R N$.*

PROOF. By 7.4.22 one has $M \otimes_R N \simeq M \otimes_R^{\mathsf{L}} N$ in $\mathcal{D}(R)$; now apply 16.4.17. $\square$

REMARK. E 16.4.11 illustrates how 16.4.18 may fail if the ring is not local.

The equalities in the next result compare to 8.4.26 and 16.3.5.

16.4.19 Proposition. *Let R be local, M a complex in $\mathcal{D}_{\square}^{\mathrm{f}}(R)$, and N a complex in $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$. If M has finite projective dimension, there is an equality of Laurent series,*

$$\mathrm{P}_{\mathsf{RHom}_R(M,N)}^R(t) \;=\; \mathrm{P}_M^R(t^{-1})\,\mathrm{P}_N^R(t)\,;$$

in particular, one has

$$\begin{aligned}\operatorname{pd}_R \mathsf{RHom}_R(M,N) &= \operatorname{pd}_R N - \inf M \quad \textit{and} \\ -\inf \mathsf{RHom}_R(M,N) &= \operatorname{pd}_R M - \inf N\,.\end{aligned}$$

PROOF. As $\operatorname{pd}_R M$ is finite, the complex $\mathsf{RHom}_R(M,N)$ belongs to $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$ by 15.4.3; in particular, 16.4.14 applies. Let $\boldsymbol{k}$ be the residue field of R. In the next chain, the first isomorphism follows from commutativity 12.3.5 and tensor evaluation 12.3.23(a), the second holds by 12.3.32, and the third follows from 7.6.12.

$$\begin{aligned}\boldsymbol{k} \otimes_R^{\mathsf{L}} \mathsf{RHom}_R(M,N) &\simeq \mathsf{RHom}_R(M, \boldsymbol{k} \otimes_R^{\mathsf{L}} N) \\ &\simeq \mathsf{RHom}_{\boldsymbol{k}}(\boldsymbol{k} \otimes_R^{\mathsf{L}} M, \boldsymbol{k} \otimes_R^{\mathsf{L}} N) \\ &\simeq \operatorname{Hom}_{\boldsymbol{k}}(\mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} M), \mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} N))\,.\end{aligned}$$

For every $m \in \mathbb{Z}$ the definition, 2.1.4, of the graded Hom yields

$$\begin{aligned}\beta_m^R(\mathsf{RHom}_R(M,N)) &= \operatorname{rank}_{\boldsymbol{k}}(\operatorname{Hom}_{\boldsymbol{k}}(\mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} M), \mathrm{H}(\boldsymbol{k} \otimes_R^{\mathsf{L}} N)))_m \\ &= \sum_{i\in\mathbb{Z}} \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{-i}(\boldsymbol{k} \otimes_R^{\mathsf{L}} M) \cdot \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{m-i}(\boldsymbol{k} \otimes_R^{\mathsf{L}} N) \\ &= \sum_{i+j=m} \beta_{-i}^R(M)\beta_j^R(N)\,,\end{aligned}$$

which is the degree m coefficient in the product series $\mathrm{P}_M^R(t^{-1})\,\mathrm{P}_N^R(t)$.

The Poincaré series have non-negative coefficients, so the last two statements follow from 16.4.16, as per 16.4.15 one has

$$\deg \mathrm{P}^R_{\mathsf{RHom}_R(M,N)}(t) \;=\; \deg \mathrm{P}^R_M(t^{-1}) + \deg \mathrm{P}^R_N(t) \;=\; -\operatorname{ord} \mathrm{P}^R_M(t) + \deg \mathrm{P}^R_N(t)$$

and

$$\operatorname{ord} \mathrm{P}^R_{\mathsf{RHom}_R(M,N)}(t) \;=\; \operatorname{ord} \mathrm{P}^R_M(t^{-1}) + \operatorname{ord} \mathrm{P}^R_N(t) \;=\; -\deg \mathrm{P}^R_M(t) + \operatorname{ord} \mathrm{P}^R_N(t)\,. \quad \square$$

In an important special case, equality holds in 8.3.15(d).

16.4.20 Theorem. *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality of Laurent series,*

$$\mathrm{P}^S_{S\otimes_R M}(t) \;=\; \mathrm{P}^R_M(t)\,;$$

in particular, one has

$$\operatorname{pd}_S(S\otimes_R M) \;=\; \operatorname{pd}_R M\,.$$

Proof. Let $\boldsymbol{k}$ and $\boldsymbol{K}$ be the residue fields of R and S. It follows from the assumption $\mathfrak{m}S \subseteq \mathfrak{M}$ that $\boldsymbol{K}$ is a $\boldsymbol{k}$-vector space. Thus the unitor 12.3.3 together with flatness of S over R accounts for the first isomorphism in the computation below. Commutativity 12.3.5 combined with 12.3.31 yields the second isomorphism, and the last isomorphism follows from associativity 12.3.6 and 7.6.12.

$$\begin{aligned} \boldsymbol{K}\otimes^{\mathsf{L}}_S (S\otimes_R M) &\simeq (\boldsymbol{k}\otimes^{\mathsf{L}}_{\boldsymbol{k}}\boldsymbol{K})\otimes^{\mathsf{L}}_S (S\otimes^{\mathsf{L}}_R M)\\ &\simeq (\boldsymbol{K}\otimes^{\mathsf{L}}_{\boldsymbol{k}}\boldsymbol{k})\otimes^{\mathsf{L}}_R M\\ &\simeq \boldsymbol{K}\otimes_{\boldsymbol{k}} \mathrm{H}(\boldsymbol{k}\otimes^{\mathsf{L}}_R M)\,. \end{aligned}$$

Recall from 12.1.20(b,c) that the complex $S\otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(S)$; in particular, 16.4.14 applies. For $m\in\mathbb{Z}$ one gets from the isomorphisms above

$$\beta^S_m(S\otimes_R M) \;=\; \operatorname{rank}_{\boldsymbol{K}}(\boldsymbol{K}\otimes_{\boldsymbol{k}}\mathrm{H}(\boldsymbol{k}\otimes^{\mathsf{L}}_R M))_m \;=\; \operatorname{rank}_{\boldsymbol{k}}\mathrm{H}_m(\boldsymbol{k}\otimes^{\mathsf{L}}_R M) \;=\; \beta^R_m(M)\,.$$

This shows the equality of Poincaré series, and the equality of projective dimensions follows from 16.4.16. $\square$

The assumptions on the R-algebra S in 16.4.20 actually imply that S is faithfully flat as an R-module.

16.4.21 Proposition. *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{M})$ be local rings such that S is an R-algebra and $\mathfrak{m}S \subseteq \mathfrak{M}$ holds. If S is flat as an R-module, then it is faithfully flat.*

Proof. If S is flat as an R-module, then one has $R/\mathfrak{m}\otimes^{\mathsf{L}}_R S \simeq R/\mathfrak{m}\otimes_R S \cong S/\mathfrak{m}S$, see 1.1.10, which is non-zero by the assumption $\mathfrak{m}S \subseteq \mathfrak{M}$. Thus $\mathfrak{m}$ is in $\operatorname{supp}_R S$, see 15.1.5, and it follows from 15.1.18 that S is a faithfully flat R-module. $\square$

MINIMAL SEMI-FREE RESOLUTIONS

While minimal semi-injective resolutions exist for complexes over any ring, minimal semi-projective resolutions are harder to come by. They exist by B.51 and B.60 for complexes over Artinian rings, and they exist for some complexes over commutative Noetherian local rings, as captured by the next restatement of facts from Appn. B.

16.4.22 Theorem. *Let $(R, \mathfrak{m})$ be local.*

(a) *A complex P of finitely generated projective R-modules is minimal if and only if $\partial^P(P) \subseteq \mathfrak{m}P$ holds.*
(b) *Every bounded below complex of finitely generated projective R-modules is semi-free. In particular, every finitely generated projective R-module is free.*
(c) *Every complex in M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ has a minimal semi-free resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated and $L_v = 0$ for all $v < \inf M$.*

PROOF. Part (b) follows from B.47 and 5.1.3. Being local, R is semi-perfect, see B.44, so by B.46 every degreewise finitely generated graded R-module is semi-perfect. Part (a) now follows from B.55(b). Part (c) holds by B.58 and B.63. □

REMARK. The study of free resolutions is both a classic and current research topic in commutative algebra. Over the years, surveys have been compiled by Avramov [16, 17, 18], Herzog [121], Northcott [194], and Peeva and McCullough [183], to name a few.

16.4.23 Example. Let $(R, \mathfrak{m})$ be local and $\boldsymbol{x} = x_1, \ldots, x_n$ a sequence in $\mathfrak{m}$. The Koszul complex $\mathrm{K}^R(\boldsymbol{x})$ is minimal by 11.4.3(c,d) and 16.4.22(a). If $\boldsymbol{x}$ is R-regular, then $\mathrm{K}^R(\boldsymbol{x})$ is a semi-free replacement of $R/(\boldsymbol{x})$, see 14.4.19, so the canonical map $\mathrm{K}^R(\boldsymbol{x}) \to R/(\boldsymbol{x})$ is a minimal free resolution, and $\mathrm{pd}_R\, R/(\boldsymbol{x}) = n$ holds by 8.1.16.

16.4.24 Proposition. *Let R be local and M a finitely generated R-module. If a seqeuence $\boldsymbol{x}$ in R is R- and M-regular, then $\mathrm{pd}_R\, M = \mathrm{pd}_{R/(\boldsymbol{x})}\, M/(\boldsymbol{x})M$ holds.*

PROOF. The Koszul complex $\mathrm{K}^R(\boldsymbol{x})$ is by 16.4.23 a semi-free replacement of $R/(\boldsymbol{x})$, and M-regularity of $\boldsymbol{x}$ yields per 16.2.31 isomorphisms,

$$M/(\boldsymbol{x})M \;\simeq\; \mathrm{K}^R(\boldsymbol{x}) \otimes_R M \;\simeq\; R/(\boldsymbol{x}) \otimes^{\mathsf{L}}_R M\,.$$

With $\boldsymbol{k}$ denoting the common residue field of R and $R/(\boldsymbol{x})$, the isomorphisms above conspire with 12.3.31 to yield $\boldsymbol{k} \otimes^{\mathsf{L}}_{R/(\boldsymbol{x})} M/(\boldsymbol{x})M \simeq \boldsymbol{k} \otimes^{\mathsf{L}}_R M$. Now invoke 16.4.1. □

The Betti numbers of a complex record the ranks of the free modules in a minimal semi-free resolution.

16.4.25 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ with minimal semi-free resolution $L \xrightarrow{\simeq} M$. The complexes $\boldsymbol{k} \otimes_R L$ and $\mathrm{Hom}_R(L, \boldsymbol{k})$ have zero differentials, and for every $m \in \mathbb{Z}$ there are equalities,*

$$\beta^R_m(M) \;=\; \mathrm{rank}_R\, L_m \;=\; \mathrm{rank}_{\boldsymbol{k}}\, \mathrm{Ext}^m_R(M, \boldsymbol{k})\,.$$

Proof. One has $\partial^L(L) \subseteq \mathfrak{m}L$ by 16.4.22(a), so it follows immediately from the definition, 2.4.1, of the differential on a tensor product complex that $\boldsymbol{k} \otimes_R L$ has zero differential. Therefore, one has

$$(\dagger) \qquad \operatorname{Tor}_m^R(\boldsymbol{k}, M) = \mathrm{H}_m(\boldsymbol{k} \otimes_R^{\mathsf{L}} M) = \mathrm{H}_m(\boldsymbol{k} \otimes_R L) = (\boldsymbol{k} \otimes_R L)_m = \boldsymbol{k} \otimes_R L_m \,.$$

It is similarly immediate from the definition, 2.3.1, of the differential on a Hom complex that $\operatorname{Hom}_R(L, \boldsymbol{k})$ has zero differential, whence one has

$$(\ddagger) \qquad \begin{aligned} \operatorname{Ext}_R^m(M, \boldsymbol{k}) &= \mathrm{H}_{-m}(\mathsf{R}\mathrm{Hom}_R(M, \boldsymbol{k})) \\ &= \mathrm{H}_{-m}(\operatorname{Hom}_R(L, \boldsymbol{k})) \\ &= \operatorname{Hom}_R(L, \boldsymbol{k})_{-m} \\ &= \operatorname{Hom}_R(L_m, \boldsymbol{k}) \,. \end{aligned}$$

Combining (†) and (‡) with $\operatorname{rank}_{\boldsymbol{k}}(\boldsymbol{k} \otimes_R L_m) = \operatorname{rank}_R L_m = \operatorname{rank}_{\boldsymbol{k}} \operatorname{Hom}_R(L_m, \boldsymbol{k})$ and the definition, 16.4.14, of Betti numbers one gets the asserted equalities. □

It is fairly elementary to observe that there are "no holes" in the sequence of Betti numbers of a finitely generated module.

16.4.26 Corollary. *Let R be local and M a complex in $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$. One has $\beta_m^R(M) \neq 0$ for every integer m in the range $\operatorname{pd}_R M \geqslant m \geqslant \sup M$.*

Proof. The statement is void if M is acyclic or $\mathrm{H}(M)$ is not bounded above, so assume that $s = \sup M$ is an integer. Let $L \xrightarrow{\simeq} M$ be a minimal semi-free resolution, see 16.4.22(c). As the module $\mathrm{H}_s(L) \cong \mathrm{H}_s(M)$ is a subquotient of L_s, one has $\beta_s^R(M) = \operatorname{rank}_R L_s \neq 0$ per 16.4.25. For $m > \sup M$ it follows from 16.4.25 that $\beta_m^R(M) = 0$ implies $L_m = 0$, which by 8.1.16 implies $m > \operatorname{pd}_R M$. □

For use in Chap. 18 we record an auxiliary result on Poincaré series.

16.4.27 Lemma. *Let R be local, M a complex in $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$, and n an integer. One has $\mathrm{P}_M^R(t) = t^n$ if and only there is an isomorphism $M \simeq \Sigma^n R$ in $\mathcal{D}(R)$.*

Proof. The "if" part is trivial as one has $\boldsymbol{k} \otimes_R^{\mathsf{L}} \Sigma^n R \simeq \Sigma^n \boldsymbol{k}$ by 12.2.8 and the unitor 12.3.3. For the converse, 16.4.16 and 8.1.3 yield $\inf M = n = \operatorname{pd}_R M \geqslant \sup M$, whence $\mathrm{H}(M)$ is concentrated in degree n. Consider the finitely generated R-module $H = \mathrm{H}_n(M)$. By 7.3.29 there is an isomorphism $M \simeq \Sigma^n H$ in $\mathcal{D}(R)$, so it suffices to show that H is isomorphic to R. By 8.1.3 one has $\operatorname{pd}_R H = \operatorname{pd}_R M - n = 0$, so H is projective by 8.1.19 and hence free by 16.4.22(b). The isomorphism $H \simeq \Sigma^{-n} M$ yields $\beta_0^R(H) = \beta_n^R(M) = 1$, which per 16.4.16 means that H is free of rank 1. □

Bass Numbers and Bass Series

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a complex in $\mathcal{D}_{\sqsubset}^{\mathrm{f}}(R)$. It follows from 12.3.34 that the complex $\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$ belongs to $\mathcal{D}_{\sqsubset}^{\mathrm{f}}(\boldsymbol{k})$. In particular, the homology modules $\operatorname{Ext}_R^m(\boldsymbol{k}, M)$ are $\boldsymbol{k}$-vector spaces of finite rank for all $m \in \mathbb{Z}$ and vanish for $m \ll 0$. In particular, the Bass numbers defined below do indeed belong to $\mathbb{N}_0$. The Bass numbers are interpreted in 16.4.37.

16.4.28 Definition. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. For $m \in \mathbb{Z}$ the m^{th} *Bass number* of M is defined by

$$\mu^m_R(M) = \operatorname{rank}_{\boldsymbol{k}} \operatorname{Ext}^m_R(\boldsymbol{k}, M) .$$

The *Bass series* of M is the generating function

$$\mathrm{I}^M_R(t) = \sum_{m \in \mathbb{Z}} \mu^m_R(M) t^m .$$

It is standard to use the abbreviated notation $\mathrm{I}_R(t)$ for $\mathrm{I}^R_R(t)$.

Notice from the comments before the definition that the Bass series of a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ is a Laurent series, see 16.4.15, with coefficients in $\mathbb{N}_0$.

16.4.29. Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. Notice that for $\mathfrak{p} \in \operatorname{Spec} R$ and $m \in \mathbb{Z}$ it follows from 16.4.5 that $\mu^m_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ implies $\mu^{m + \dim R/\mathfrak{p}}_R(M) \neq 0$.

16.4.30 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. The degree and order of the Bass series $\mathrm{I}^M_R(t)$ satisfy the next equalities,*

$$\deg \mathrm{I}^M_R(t) = \sup\{m \in \mathbb{Z} \mid \mu^m_R(M) \neq 0\} = \operatorname{id}_R M \quad \textit{and}$$
$$\operatorname{ord} \mathrm{I}^M_R(t) = \inf\{m \in \mathbb{Z} \mid \mu^m_R(M) \neq 0\} = \operatorname{depth}_R M .$$

Proof. The equalities hold by 16.4.8 and 16.2.14. □

Remark. Growth patterns in Bass numbers, indcluding rationality of Bass series, is a research topic in commutative algebra. See for example [68] or Avramov [19] for an introduction.

The equality of homological dimensions in the next result compares to 8.3.15(b).

16.4.31 Proposition. *Let R be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. There is an equality of Laurent series,*

$$\mathrm{I}^{\mathsf{RHom}_R(M,N)}_R(t) = \mathrm{P}^R_M(t)\, \mathrm{I}^N_R(t) ;$$

in particular, one has

$$\operatorname{id}_R \mathsf{RHom}_R(M,N) = \operatorname{pd}_R M + \operatorname{id}_R N .$$

Proof. Recall from 12.2.6 that the complex $\mathsf{RHom}_R(M,N)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$; in particular 16.4.28 applies. Let $\boldsymbol{k}$ be the residue field of R. By 12.3.35 and 7.6.12 there are isomorphisms,

$$\begin{aligned} \mathsf{RHom}_R(\boldsymbol{k}, \mathsf{RHom}_R(M,N)) &\simeq \mathrm{RHom}_{\boldsymbol{k}}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(\boldsymbol{k}, N)) \\ &\simeq \operatorname{Hom}_{\boldsymbol{k}}(\mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M), \mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, N))) , \end{aligned}$$

in $\mathcal{D}(\boldsymbol{k})$. For every $m \in \mathbb{Z}$ the definition, 2.1.4, of the graded Hom yields

$$\begin{aligned} \mu^m_R(\mathsf{RHom}_R(M,N)) &= \operatorname{rank}_{\boldsymbol{k}}(\operatorname{Hom}_{\boldsymbol{k}}(\mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R M), \mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, N)))_{-m} \\ &= \sum_{i \in \mathbb{Z}} \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_i(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) \cdot \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{i-m}(\mathsf{RHom}_R(\boldsymbol{k}, N)) \end{aligned}$$

$$= \sum_{i+j=m} \beta_i^R(M)\mu_R^j(N)\,,$$

and this is the degree m coefficient of the product series $\mathrm{P}_M^R(t)\,\mathrm{I}_R^N(t)$.

The Poincaré and Bass series have non-negative coefficients, so the equality of homological dimensions follows from 16.4.16 and 16.4.30, as per 16.4.15 one has

$$\deg \mathrm{I}_R^{\mathsf{RHom}_R(M,N)}(t) \;=\; \deg\left(\mathrm{P}_M^R(t)\,\mathrm{I}_R^N(t)\right) \;=\; \deg \mathrm{P}_M^R(t) + \deg \mathrm{I}_R^N(t)\,. \qquad \square$$

16.4.32 Corollary. *Let R be local and M and N be finitely generated R-modules. If $\mathrm{Ext}_R^m(M,N) = 0$ for all $m > 0$, then one has $\mathrm{id}_R \mathrm{Hom}_R(M,N) = \mathrm{pd}_R M + \mathrm{id}_R N$.*

Proof. By 7.3.30 one has $\mathrm{Hom}_R(M,N) \simeq \mathsf{RHom}_R(M,N)$; now apply 16.4.31. $\square$

The equalities in the next result compare to 8.4.27 and 16.3.9.

16.4.33 Proposition. *Let R be local, M a complex in $\mathcal{D}_{\sqsubset}^{\mathrm{f}}(R)$, and N a complex in $\mathcal{D}_{\square}^{\mathrm{f}}(R)$. If N has finite injective dimension, then there is an equality of Laurent series,*

$$\mathrm{P}^R_{\mathsf{RHom}_R(M,N)}(t) \;=\; \mathrm{I}_R^M(t)\,\mathrm{I}_R^N(t^{-1})\,;$$

in particular, one has

$$\begin{aligned} \mathrm{pd}_R \mathsf{RHom}_R(M,N) \;&=\; \mathrm{id}_R M - \mathrm{depth}_R N \quad \textit{and} \\ -\inf \mathsf{RHom}_R(M,N) \;&=\; \mathrm{id}_R N - \mathrm{depth}_R M\,. \end{aligned}$$

Proof. As $\mathrm{id}_R N$ is finite, the complex $\mathsf{RHom}_R(M,N)$ belongs to $\mathcal{D}_{\sqsupset}^{\mathrm{f}}(R)$ by 15.4.8; in particular, 16.4.14 applies. Let $\boldsymbol{k}$ be the residue field of R. In the next chain of isomorphisms, the first is homomorphism evaluation 12.3.27(b), the second holds by 12.3.36, and the third follows from 7.6.12.

$$\begin{aligned} \boldsymbol{k} \otimes_R^{\mathsf{L}} \mathsf{RHom}_R(M,N) \;&\simeq\; \mathsf{RHom}_R(\mathsf{RHom}_R(\boldsymbol{k},M),N) \\ &\simeq\; \mathsf{RHom}_{\boldsymbol{k}}(\mathsf{RHom}_R(\boldsymbol{k},M),\mathsf{RHom}_R(\boldsymbol{k},N)) \\ &\simeq\; \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k},M)),\mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k},N)))\,. \end{aligned}$$

For every $m \in \mathbb{Z}$ the definition, 2.1.14, of the graded tensor product yields

$$\begin{aligned} &\beta_m^R(\mathsf{RHom}_R(M,N)) \\ &\qquad = \mathrm{rank}_{\boldsymbol{k}}(\mathrm{Hom}_{\boldsymbol{k}}(\mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k},M)),\mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k},N))))_m \\ &\qquad = \sum_{i\in\mathbb{Z}} \mathrm{rank}_{\boldsymbol{k}}\,\mathrm{H}_{-i}(\mathsf{RHom}_R(\boldsymbol{k},M)) \cdot \mathrm{rank}_{\boldsymbol{k}}\,\mathrm{H}_{m-i}(\mathsf{RHom}_R(\boldsymbol{k},N)) \\ &\qquad = \sum_{i+j=m} \mu_R^i(M)\mu_R^{-j}(N)\,, \end{aligned}$$

which is the degree m coefficient of the product series $\mathrm{I}_R^M(t)\,\mathrm{I}_R^N(t^{-1})$.

The Poincaré and Bass series have non-negative coefficients, so the last two statements follow from 16.4.16 and 16.4.30, as per 16.4.15 one has

$$\deg \mathrm{P}^R_{\mathsf{RHom}_R(M,N)}(t) \;=\; \deg \mathrm{I}_R^M(t) + \deg \mathrm{I}_R^N(t^{-1}) \;=\; \deg \mathrm{I}_R^M(t) - \mathrm{ord}\,\mathrm{I}_R^N(t)$$

and

$$\operatorname{ord} \mathrm{P}^R_{\mathsf{RHom}_R(M,N)}(t) \;=\; \operatorname{ord} \mathrm{I}^M_R(t) + \operatorname{ord} \mathrm{I}^N_R(t^{-1}) \;=\; \operatorname{ord} \mathrm{I}^M_R(t) - \deg \mathrm{I}^N_R(t)\,. \qquad \square$$

The equalities in the next result compare to 8.4.16 and 16.3.1.

16.4.34 Proposition. *Let R be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. If N has finite projective dimension, there is an equality of Laurent series,*

$$\mathrm{I}_R^{M\otimes^{\mathsf{L}}_R N}(t) \;=\; \mathrm{I}^M_R(t)\,\mathrm{P}^R_N(t^{-1})\,;$$

in particular, one has

$$\operatorname{id}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{id}_R M - \inf N \quad \textit{and}$$
$$\operatorname{depth}_R(M \otimes^{\mathsf{L}}_R N) \;=\; \operatorname{depth}_R M - \operatorname{pd}_R N\,.$$

Proof. As $\operatorname{pd}_R N$ is finite, the complex $M \otimes^{\mathsf{L}}_R N$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ by 15.4.3; in particular, 16.4.28 applies. Let $\boldsymbol{k}$ be the residue field of R. In the next chain of isomorphisms in $\mathcal{D}(\boldsymbol{k})$, the first is tensor evaluation 12.3.23(c), the second holds by 12.3.31, and the third isomorphism comes from 7.6.12.

$$\begin{aligned} \mathsf{RHom}_R(\boldsymbol{k}, M \otimes^{\mathsf{L}}_R N) &\simeq \mathsf{RHom}_R(\boldsymbol{k}, M) \otimes^{\mathsf{L}}_R N \\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, M) \otimes^{\mathsf{L}}_{\boldsymbol{k}} (\boldsymbol{k} \otimes^{\mathsf{L}}_R N) \\ &\simeq \mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, M)) \otimes_{\boldsymbol{k}} \mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R N)\,. \end{aligned}$$

For every $m \in \mathbb{Z}$ the definition, 2.1.4, of the graded Hom yields

$$\begin{aligned} \mu^m_R(M \otimes^{\mathsf{L}}_R N) &= \operatorname{rank}_{\boldsymbol{k}} \left(\mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, M)) \otimes_{\boldsymbol{k}} \mathrm{H}(\boldsymbol{k} \otimes^{\mathsf{L}}_R N)\right)_{-m} \\ &= \sum_{i\in\mathbb{Z}} \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{-i}(\mathsf{RHom}_R(\boldsymbol{k}, M)) \cdot \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{i-m}(\boldsymbol{k} \otimes^{\mathsf{L}}_R N) \\ &= \sum_{i+j=m} \mu^i_R(M)\beta^R_{-j}(N)\,, \end{aligned}$$

which is the degree m coefficient of the product series $\mathrm{I}^M_R(t)\,\mathrm{P}^R_N(t^{-1})$.

The Poincaré and Bass series have non-negative coefficients, so the last two statements follow from 16.4.16 and 16.4.30, as per 16.4.15 one has

$$\deg \mathrm{I}_R^{M\otimes^{\mathsf{L}}_R N}(t) \;=\; \deg \mathrm{I}^M_R(t) + \deg \mathrm{P}^R_N(t^{-1}) \;=\; \deg \mathrm{I}^M_R(t) - \operatorname{ord} \mathrm{P}^R_N(t)$$

and

$$\operatorname{ord} \mathrm{I}_R^{M\otimes^{\mathsf{L}}_R N}(t) \;=\; \operatorname{ord} \mathrm{I}^M_R(t) + \operatorname{ord} \mathrm{P}^R_N(t^{-1}) \;=\; \operatorname{ord} \mathrm{I}^M_R(t) - \deg \mathrm{P}^R_N(t)\,. \qquad \square$$

16.4.35 Theorem. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality of Laurent series,*

$$\mathrm{I}_S^{S\otimes_R M}(t) \;=\; \mathrm{I}_{S/\mathfrak{m}S}(t)\,\mathrm{I}^M_R(t)\,;$$

in particular, one has

$$\operatorname{id}_S(S \otimes_R M) \;=\; \operatorname{id} S/\mathfrak{m}S + \operatorname{id}_R M \quad \textit{and}$$

$$\operatorname{depth}_R(S \otimes_R M) \;=\; \operatorname{depth} S/\mathfrak{m}S + \operatorname{depth}_R M\;.$$

Proof. Let $\boldsymbol{k}$ and $\boldsymbol{K}$ be the residue fields of R and S. It follows from the assumption $\mathfrak{m}S \subseteq \mathfrak{M}$ that $\boldsymbol{K}$ is the residue field of the local ring $S/\mathfrak{m}S$; thus 12.3.36 accounts for the first isomorphism in the computation below. The second isomorphism holds by 1.1.10, and the last isomorphism follows by flatness of S over R from 12.3.32.

$$(\dagger)\qquad \begin{aligned}\mathsf{RHom}_S(\boldsymbol{K}, S \otimes_R M) &\simeq \mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, \mathsf{RHom}_S(S/\mathfrak{m}S, S \otimes_R M))\\ &\simeq \mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, \mathsf{RHom}_S(S \otimes_R \boldsymbol{k}, S \otimes_R M))\\ &\simeq \mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, \mathsf{RHom}_R(\boldsymbol{k}, S \otimes_R M))\;.\end{aligned}$$

By flatness of S as an R-module, the first isomorphism in the next computation is tensor evaluation 12.3.22(b) combined with commutativity 12.3.5, while the second isomorphism follows from 12.1.18.

$$(\ddagger)\qquad \mathsf{RHom}_R(\boldsymbol{k}, S \otimes_R M) \;\simeq\; \mathsf{RHom}_R(\boldsymbol{k}, M) \otimes_R S \;\simeq\; \mathsf{RHom}_R(\boldsymbol{k}, M) \otimes_{\boldsymbol{k}} S/\mathfrak{m}S\;.$$

By 12.2.7 the complex $\mathsf{RHom}_R(\boldsymbol{k}, M)$ has bounded above homology, so as a $\boldsymbol{k}$-complex it has finite flat dimension by 7.6.11(c). Combining ($\dagger$) and ($\ddagger$) with commutativity 12.3.5, tensor evaluation 12.3.22(b), and 7.6.12 one now gets

$$(\flat)\qquad \begin{aligned}\mathsf{RHom}_S(\boldsymbol{K}, S \otimes_R M) &\simeq \mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, S/\mathfrak{m}S \otimes^{\mathsf{L}}_{\boldsymbol{k}} \mathsf{RHom}_R(\boldsymbol{k}, M))\\ &\simeq \mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, S/\mathfrak{m}S) \otimes^{\mathsf{L}}_{\boldsymbol{k}} \mathsf{RHom}_R(\boldsymbol{k}, M)\\ &\simeq \mathrm{H}(\mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, S/\mathfrak{m}S)) \otimes_{\boldsymbol{k}} \mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, M))\;.\end{aligned}$$

Recall from 12.1.20(b,c) that the complex $S \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(S)$; in particular, 16.4.28 applies. The residue field $\boldsymbol{K}$ is a $\boldsymbol{k}$-vector space, so for $m \in \mathbb{Z}$ one gets from ($\flat$) and the definition, 2.1.14, of the graded tensor product

$$\begin{aligned}&\mu^m_S(S \otimes_R M)\\ &\quad= \operatorname{rank}_{\boldsymbol{K}}(\mathrm{H}(\mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, S/\mathfrak{m}S)) \otimes_{\boldsymbol{k}} \mathrm{H}(\mathsf{RHom}_R(\boldsymbol{k}, M)))_{-m}\\ &\quad= \sum_{i\in\mathbb{Z}} \operatorname{rank}_{\boldsymbol{K}} \mathrm{H}_{-i}(\mathsf{RHom}_{S/\mathfrak{m}S}(\boldsymbol{K}, S/\mathfrak{m}S)) \cdot \operatorname{rank}_{\boldsymbol{k}} \mathrm{H}_{i-m}(\mathsf{RHom}_R(\boldsymbol{k}, M))\\ &\quad= \sum_{i+j=m} \mu^i_{S/\mathfrak{m}S}(S/\mathfrak{m}S)\mu^j_R(M)\;,\end{aligned}$$

which is the degree m coefficient of the product series $\mathrm{I}_{S/\mathfrak{m}S}(t)\,\mathrm{I}^M_R(t)$.

The Bass series have non-negative coefficients, so the last two statements follow from 16.4.30, as per 16.4.15 one has

$$\deg \mathrm{I}^{S\otimes_R M}_S(t) \;=\; \deg \mathrm{I}_{S/\mathfrak{m}S}(t) + \deg \mathrm{I}^M_R(t)$$

and

$$\operatorname{ord} \mathrm{I}^{S\otimes_R M}_S(t) \;=\; \operatorname{ord} \mathrm{I}_{S/\mathfrak{m}S}(t) + \operatorname{ord} \mathrm{I}^M_R(t)\;. \qquad \square$$

Remark. The equality of Bass series in 16.4.35 was proved by Foxby and Thorup [99]. It was subsequently generalized by Avramov, Foxby, and Lescot [26] and further by Avramov and Foxby [23] in their investigation of local ring homomorphisms of finite (Gorenstein) flat dimension.

MINIMAL SEMI-INJECTIVE RESOLUTIONS

The Bass numbers of a complex count the number of copies of $\mathrm{E}_R(\boldsymbol{k})$ in a minimal semi-injective resolution.

16.4.36 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and I a semi-injective R-complex. If I is minimal, then there is an isomorphism $\mathrm{Hom}_R(\boldsymbol{k}, \Gamma_{\mathfrak{m}}(I)) \cong \mathrm{Hom}_R(\boldsymbol{k}, I)$ and both complexes have zero differential.*

PROOF. The complex $\mathrm{Hom}_R(\boldsymbol{k}, I)$ has zero differential by 8.2.16 and the isomorphism $\mathrm{Hom}_R(\boldsymbol{k}, \Gamma_{\mathfrak{m}}(I)) \cong \mathrm{Hom}_R(\boldsymbol{k}, I)$ follows from 11.2.22 as $\boldsymbol{k}$ is $\mathfrak{m}$-torsion. □

16.4.37 Theorem. *Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ with minimal semi-injective resolution $M \xrightarrow{\simeq} I$. For every $m \in \mathbb{Z}$ there is an isomorphism,*

$$I_{-m} \;\cong\; \coprod_{\mathfrak{p} \in \operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{p})^{\mu^m_{R_\mathfrak{p}}(M_\mathfrak{p})} \,.$$

PROOF. Let $\mathfrak{p}$ be a prime ideal in R. By 14.1.11 the complex $M_\mathfrak{p}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R_\mathfrak{p})$. The induced morphism $M_\mathfrak{p} \to I_\mathfrak{p}$ is by 14.1.31 a minimal semi-injective resolution, and the complex $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ has zero differential by 16.4.36. Now one has

$$\begin{aligned} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (I_{-m})_\mathfrak{p}) &= \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})_{-m} \\ &= \mathrm{H}_{-m}(\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})) \\ &= \mathrm{Ext}^m_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \,. \end{aligned}$$

In particular, one has $\mathrm{rank}_{\kappa(\mathfrak{p})} \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (I_{-m})_\mathfrak{p}) = \mu^m_{R_\mathfrak{p}}(M_\mathfrak{p})$, see 16.4.28, so the asserted ismorphism follows from C.23. □

16.4.38 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local, M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, and $M \xrightarrow{\simeq} I$ a minimal semi-injective resolution. For every $m \in \mathbb{Z}$ one has*

$$\Gamma_{\mathfrak{m}}(I)_{-m} \;\cong\; \mathrm{E}_R(\boldsymbol{k})^{\mu^m_R(M)} \,.$$

PROOF. The equality follows immediately from 16.4.37 and 13.3.4. □

EXERCISES

In exercises E 16.4.3–16.4.10 let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 16.4.1 Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. Show that there is an inequality, $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q} \geqslant \mathrm{depth}_{R_\mathfrak{q}} M_\mathfrak{q}$.

E 16.4.2 Let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals in R and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. Show that there is an inequality, $\mathrm{id}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q} \leqslant \mathrm{id}_{R_\mathfrak{q}} M_\mathfrak{q}$.

E 16.4.3 Let M be a finitely generated R-module and x an M-regular element. Show that the equality $\mathrm{pd}_R M/xM = \mathrm{pd}_R M + 1$ holds.

E 16.4.4 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$. Set $s = \sup M$ and show that for prime ideals $\mathfrak{p}$ in $\mathrm{Ass}_R \mathrm{H}_s(M)$ there are inequalities

$$\mathrm{depth}_R M \;\leqslant\; \dim R/\mathfrak{p} - \sup M_\mathfrak{p} \;\leqslant\; \dim R/\mathfrak{p} - \inf M_\mathfrak{p} \;\leqslant\; \dim_R M \,.$$

E 16.4.5 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ with $\mathrm{H}(M) \neq 0$. Set $s = \sup M$ and show that one has $\operatorname{depth}_R M \leqslant \dim_R \mathrm{H}_s(M) - s$.

E 16.4.6 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $s \in \mathbb{Z}$. Show that $\mathrm{P}^R_{\Sigma^s M}(t) = t^s\, \mathrm{P}^R_M(t)$ holds.

E 16.4.7 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ and $s \in \mathbb{Z}$. Show that $\mathrm{I}_R^{\Sigma^s M}(t) = t^{-s}\, \mathrm{I}_R^M(t)$ holds.

E 16.4.8 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Show that if M has finite projective dimension, then the equality $\beta^R_m(\mathsf{R}\mathrm{Hom}_R(M,R)) = \beta^R_{-m}(M)$ holds for every $m \in \mathbb{Z}$.

E 16.4.9 Show that there is an equality $\mathrm{P}^R_{k}(t) = \mathrm{I}^{k}_R(t)$.

E 16.4.10 Let $M \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$ be a complex of finite projective dimension. Show that $\operatorname{pd}_R M = \sup\{m \in \mathbb{Z} \mid \operatorname{Ext}^m_R(M,N) \neq 0\}$ holds for every finitely generated R-module $N \neq 0$.

E 16.4.11 Show that $\operatorname{Tor}^{\mathbb{Z}}_m(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0$ holds for all $m \in \mathbb{Z}$ and conclude that 16.4.18 does not hold for $\mathbb{Z}$-modules.

Chapter 17
Going Local

Many questions in commutative algebra can be resolved locally, i.e. by localizing outside a prime and arguing in the resulting local ring. Kunz [165, 166, IV] calls it the "local-global principle in commutative algebra." This technique was already applied in the proofs of 14.1.18 and 15.1.9, and it is the recurring theme in this chapter.

17.1 Support and Cosupport

SYNOPSIS. Associated prime ideal; vanishing of Hom and tensor product; support; minimal semi-injective resolution; cosupport; guaranteed isomorphisms.

The first goal of this section is to strengthen the Hom Vanishing Lemma C.1 in the commutative Noetherian setting (17.1.3). To this end, the key is to understand the associated prime ideals of a Hom.

VANISHING OF HOM AND TENSOR PRODUCT

17.1.1 Lemma. *Let M and N be R-modules. If M is finitely generated, then one has*

$$\mathrm{Ass}_R \mathrm{Hom}_R(M,N) \;=\; \mathrm{Supp}_R M \cap \mathrm{Ass}_R N\,.$$

PROOF. A prime ideal $\mathfrak{p}$ is associated to $\mathrm{Hom}_R(M,N)$ if and only if the maximal ideal $\mathfrak{p}_\mathfrak{p}$ is associated to the $R_\mathfrak{p}$-module $\mathrm{Hom}_R(M,N)_\mathfrak{p}$. By 14.1.22 and 16.2.18 this means precisely that $\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}))$ is non-zero, and 12.1.27 yields

$$\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})) \;\cong\; \mathrm{Hom}_{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p})\otimes_{R_\mathfrak{p}} M_\mathfrak{p},\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}),N_\mathfrak{p}))\,.$$

A Hom of vector spaces is non-zero if and only if both spaces are non-zero. As $M_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$, see 14.1.11(c), it follows from Nakayama's lemma 16.1.5 that $\kappa(\mathfrak{p})\otimes_{R_\mathfrak{p}} M_\mathfrak{p}$ is non-zero if and only if $M_\mathfrak{p} \neq 0$. The assertion now follows from another application of 16.2.18. $\square$

17.1.2 Proposition. *Let M and N be R-modules with M finitely generated. The next conditions are equivalent.*

(i) $\operatorname{Hom}_R(M,N) \neq 0$.

(ii) $(0 :_R M)$ *is contained in a prime ideal* $\mathfrak{p} \in \operatorname{Ass}_R N$.

Further, if N is finitely generated, then conditions (i) *and* (ii) *are equivalent to*

(iii) $(0 :_R M) \subseteq \bigcup_{\mathfrak{p}\in\operatorname{Ass}_R N} \mathfrak{p}$.

Proof. By 14.1.1 one has $\operatorname{Supp}_R M = \mathrm{V}(0 :_R M)$, so the equivalence of conditions (i) and (ii) follows from 17.1.1. If N is finitely generated, then the set $\operatorname{Ass}_R N$ is finite, and the equivalence of (ii) and (iii) follows from Prime Avoidance. □

17.1.3 Corollary. *Let M and N be R-modules. A necessary condition for the module* $\operatorname{Hom}_R(M,N)$ *to be non-zero is the existence of elements m in M and $n \neq 0$ in N with*

$$(0 :_R m) \subseteq (0 :_R n)\,.$$

If M is finitely generated or N is injective, then this condition is also sufficient.

Proof. The necessity of the conditions was proved in C.1 and so was the sufficiency under the assumption that N is injective. Assume now that M is finitely generated and let m and $n \neq 0$ be elements with $(0 :_R m) \subseteq (0 :_R n)$. Evidently, one has $(0 :_R M) \subseteq (0 :_R m)$, and the annihilator $(0 :_R n)$ is contained in some $\mathfrak{p} \in \operatorname{Ass}_R N$. Thus 17.1.2 implies that $\operatorname{Hom}_R(M,N)$ is non-zero. □

17.1.4 Corollary. *Let M be a finitely generated R-module and $N \neq 0$ an R-module. If $(0 :_R M) \subseteq (0 :_R N)$ holds, then the module $M \otimes_R N$ is non-zero.*

Proof. Let $\mathfrak{p}$ be an associated prime ideal of the non-zero module $\operatorname{Hom}_R(N,N)$. Evidently one has $(0 :_R N) \subseteq (0 :_R \operatorname{Hom}_R(N,N))$, so by assumption $(0 :_R M)$ is contained in $\mathfrak{p}$. Now 17.1.2 implies that $\operatorname{Hom}_R(M,\operatorname{Hom}_R(N,N))$ is non-zero. As one has $\operatorname{Hom}_R(M,\operatorname{Hom}_R(N,N)) \cong \operatorname{Hom}_R(M \otimes_R N, N)$ by adjunction 12.1.10 and commutativity 12.1.7, it follows that $M \otimes_R N$ is non-zero. □

The next result compares to Nakayama's lemma B.32.

17.1.5 Proposition. *Let $\mathfrak{J}$ be the Jacobson radical of R. For an ideal $\mathfrak{a}$ in R the following conditions are equivalent.*

(i) *There is an inclusion* $\mathfrak{a} \subseteq \mathfrak{J}$.

(ii) *For each Artinian R-module M one has* $(0 :_M \mathfrak{a}) \neq 0$*, i.e.* $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$.

(iii) *For each simple R-module M one has* $(0 :_M \mathfrak{a}) \neq 0$*, i.e.* $\operatorname{Hom}_R(R/\mathfrak{a}, M) \neq 0$.

Proof. First assume that $\mathfrak{a}$ is contained in $\mathfrak{J}$ and let M be an Artinian R-module. By 14.2.10 one has $\operatorname{Ass}_R M \subseteq \mathrm{V}(\mathfrak{J})$, in particular, $\operatorname{Ass}_R M \subseteq \mathrm{V}(\mathfrak{a})$ holds, whence 17.1.1 yields $\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, M) = \operatorname{Ass}_R M$. As only the zero module has no associated prime ideals, it follows that condition (i) implies (ii).

A simple module is Artinian, so (ii) implies (iii). To prove that condition (iii) implies (i), let $\mathfrak{m}$ be a maximal ideal in R; the R-module $R/\mathfrak{m}$ is simple. By 17.1.1 one has $\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{m}) = \mathrm{V}(\mathfrak{a}) \cap \{\mathfrak{m}\}$. Now, if ($iii$) holds, then the module

$\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{m})$ is non-zero and, therefore, the set $\operatorname{V}(\mathfrak{a}) \cap \{\mathfrak{m}\}$ is non-empty. This means that $\mathfrak{a}$ is contained in $\mathfrak{m}$, and it follows that (*i*) holds. □

Support

17.1.6 Proposition. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned}\operatorname{supp}_R M &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty\}\,.\end{aligned}$$

Proof. By 15.1.22 a prime ideal $\mathfrak{p}$ in R belongs to $\operatorname{supp}_R M$ if and only if the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ belongs to $\operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p}$. The asserted equalities now follow from 16.2.27. □

17.1.7 Corollary. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned}\operatorname{supp}_R M &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M_\mathfrak{p}) \neq 0\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{H}(\mathsf{R}\!\operatorname{Hom}_R(\kappa(\mathfrak{p}), M_\mathfrak{p})) \neq 0\}\,.\end{aligned}$$

Proof. The first equality follows by 16.2.23 and 14.1.14(b) from the first equality in 17.1.6. Similarly, the second equality follows from the second equality in 17.1.6 by 16.2.23 and 14.1.21(b). □

17.1.8 Proposition. *Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$. If M is not acyclic, then one has*

$$\operatorname{Ass}_R \operatorname{H}_{\sup M}(M) \subseteq \operatorname{supp}_R M\,.$$

Proof. Set $s = \sup M$. For a prime ideal $\mathfrak{p}$ in $\operatorname{Ass}_R \operatorname{H}_s(M)$ the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ is associated to $\operatorname{H}_s(M_\mathfrak{p})$, so one has $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = -s$ by 16.2.16(b), whence $\mathfrak{p}$ belongs to $\operatorname{supp}_R M$ by 17.1.6. □

17.1.9 Corollary. *Let M be an R-module*; *there are inclusions,*

$$\operatorname{Ass}_R M \subseteq \operatorname{supp}_R M \subseteq \operatorname{Supp}_R M\,.$$

Proof. The second inclusion is known from 15.1.9. The first inclusion is trivial for $M = 0$, and for $M \neq 0$ it follows from 17.1.8. □

17.1.10 Proposition. *Let M and N be R-complexes. The equality*

$$\operatorname{supp}_R \mathsf{R}\!\operatorname{Hom}_R(M, N) = \operatorname{supp}_R M \cap \operatorname{supp}_R N$$

holds if one of the following conditions is satisfied.

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and* $\operatorname{pd}_R M$ *is finite.*
(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}_{\sqsubset}(R)$.*
(c) *M is in $\mathcal{D}^{\mathrm{f}}(R)$ and N is in $\mathcal{D}_{\square}(R)$ with* $\operatorname{id}_R N$ *finite.*

Proof. Let $\mathfrak{p}$ be a prime ideal in R. Consider the tensor evaluation morphism,

$$\theta^{MNR_\mathfrak{p}} : \mathsf{RHom}_R(M,N) \otimes^{\mathsf{L}}_R R_\mathfrak{p} \longrightarrow \mathsf{RHom}_R(M, N \otimes^{\mathsf{L}}_R R_\mathfrak{p})\,,$$

in $\mathcal{D}(R)$. Under the assumptions in (a), (b), and (c) it is an isomorphism by 12.3.23(a), 12.3.23(b), and 12.3.24(b), respectively. This accounts for the first isomorphism in the computation below; the subsequent isomorphisms follow 12.3.35 and 7.6.12.

$$\begin{aligned}
&\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{RHom}_R(M,N)_\mathfrak{p})\\
&\qquad\simeq \mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{RHom}_R(M, N_\mathfrak{p}))\\
&\qquad\simeq \mathsf{RHom}_{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M, \mathsf{RHom}_R(\kappa(\mathfrak{p}), N_\mathfrak{p}))\\
&\qquad\simeq \operatorname{Hom}_{\kappa(\mathfrak{p})}(\mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M), \mathrm{H}(\mathsf{RHom}_R(\kappa(\mathfrak{p}), N_\mathfrak{p})))\,.
\end{aligned}$$

The last complex is a Hom of $\kappa(\mathfrak{p})$-vector spaces and hence non-zero if and only if both spaces are non-zero. The asserted equality now follows from the definition, 15.1.5, of support and 17.1.7. □

17.1.11 Theorem. *Let M be an R-complex; there are equalities,*

$$\begin{aligned}
\operatorname{supp}_R M &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathsf{R}\Gamma_{\mathfrak{p}_\mathfrak{p}}(M_\mathfrak{p})) \neq 0\}\\
&= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathsf{R}\Gamma_\mathfrak{p}(M)_\mathfrak{p} \neq 0\}\,.
\end{aligned}$$

Proof. The first equality follows from 17.1.6 and 16.2.23:

$$\mathfrak{p} \in \operatorname{supp}_R M \iff \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty \iff \mathrm{H}(\mathsf{R}\Gamma_{\mathfrak{p}_\mathfrak{p}}(M_\mathfrak{p})) \neq 0\,.$$

The second equality now follows from 14.1.25. □

Recall from C.12 and Matlis' structure theorem C.23 that the indecomposable injective R-modules are precisely the injective envelopes $\mathrm{E}_R(R/\mathfrak{p})$, and that every injective R-module decomposes as a coproduct of indecomposables.

17.1.12 Lemma. *Let I be a complex of injective R-modules. One has*

$$\operatorname{supp}_R I \subseteq \bigcup_{v\in\mathbb{Z}} \operatorname{supp}_R I_v\,,$$

and equality holds if the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ is minimal and semi-injective for every prime ideal $\mathfrak{p}$ in R.

Proof. Let $\mathfrak{p} \in \operatorname{supp}_R I$; by 17.1.11 the complex $\mathsf{R}\Gamma_{\mathfrak{p}_\mathfrak{p}}(I_\mathfrak{p})$ is not acyclic. Further, as $I_\mathfrak{p}$ by C.24 is a complex of injective $R_\mathfrak{p}$-modules, 13.3.18 yields $\mathsf{R}\Gamma_{\mathfrak{p}_\mathfrak{p}}(I_\mathfrak{p}) \simeq \Gamma_{\mathfrak{p}_\mathfrak{p}}(I_\mathfrak{p})$. In particular, $\Gamma_{\mathfrak{p}_\mathfrak{p}}(I_\mathfrak{p})$ is not the zero complex, whence $\mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ is a direct summand of $(I_v)_\mathfrak{p}$ for some $v \in \mathbb{Z}$, see 13.3.5, and it follows that $\mathrm{E}_R(R/\mathfrak{p})$ is a direct summand of I_v. Thus, $\mathfrak{p}$ belongs to $\operatorname{supp}_R I_v$ by 15.1.14.

Now let $\mathfrak{p}$ be a prime ideal in R and assume that $I_\mathfrak{p}$ is minimal and semi-injective. If $\mathfrak{p}$ is not in $\operatorname{supp}_R I$, then the complex $\mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ is acyclic, see 17.1.6 and 16.2.23. As $I_\mathfrak{p}$ is semi-injective over $R_\mathfrak{p}$, this means that $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ is acyclic. By 16.4.36 this complex has zero differential, so it is the zero complex. Per (16.1.22.1) it follows that $\mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ is not a direct summand of any of the injective

$R_\mathfrak{p}$-modules $(I_v)_\mathfrak{p}$, so by 15.1.14 the maximal ideal $\mathfrak{p}_\mathfrak{p}$ is not in $\operatorname{supp}_{R_\mathfrak{p}}(I_v)_\mathfrak{p}$ for any module I_v, whence $\mathfrak{p}$ is not in any of the sets $\operatorname{supp}_R I_v$, cf. 15.1.22. $\square$

Remark. Chen and Iyengar [50] show that equality holds in 17.1.12 if and only if $I_\mathfrak{p}$ is minimal and K-injective for every prime ideal $\mathfrak{p}$, and they provide an example where that fails to be the case.

The next result says that one can read the support of a module off its minimal injective resolution.

17.1.13 Theorem. *Let M be a complex in $\mathcal{D}_\sqsubset(R)$ and I a semi-injective replacement of M. If I is minimal, then one has*

$$\operatorname{supp}_R M = \bigcup_{v\in\mathbb{Z}} \operatorname{supp}_R I_v \,.$$

Proof. By 7.3.19 and B.26 the complex I is bounded above. It thus follows from 14.1.31 that the $R_\mathfrak{p}$-complex $I_\mathfrak{p}$ is minimal and semi-injective for every prime ideal $\mathfrak{p}$ in R. As one has $\operatorname{supp}_R M = \operatorname{supp}_R I$ the claim follows from 17.1.12. $\square$

Remark. Also the cosupport of a module can be read off a certain associated complex; see Nakamura and Thompson [189, 243].

Cosupport

The next three characterizations of cosupport compare, in that order, to the characterizations of support in 17.1.6, 17.1.7, and 17.1.11.

17.1.14 Proposition. *Let M be an R-complex; there are equalities,*

$$\begin{aligned}\operatorname{cosupp}_R M &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) < \infty\}\\ &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) < \infty\}\,.\end{aligned}$$

Proof. By 15.2.14 a prime ideal $\mathfrak{p}$ in R belongs to $\operatorname{cosupp}_R M$ if and only if the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of $R_\mathfrak{p}$ belongs to $\operatorname{cosupp}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$. The asserted equalities now follow from 16.2.27. $\square$

17.1.15 Corollary. *Let M be an R-complex; there are equalities,*

$$\begin{aligned}\operatorname{cosupp}_R M &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \mathrm{H}(\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M))) \neq 0\}\\ &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \mathrm{H}(\kappa(\mathfrak{p})\otimes^{\mathsf{L}}_R \mathsf{RHom}_R(R_\mathfrak{p}, M)) \neq 0\}\,.\end{aligned}$$

Proof. The first equality follows by 16.2.23 and 14.1.21(b) from the first equality in 17.1.14. Similarly, the second equality follows from the second equality in 17.1.14 by 16.2.23 and 14.1.14(b). $\square$

17.1.16 Theorem. *Let M be an R-complex; there are equalities,*

$$\begin{aligned}\operatorname{cosupp}_R M &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \mathrm{H}(\mathsf{L}\Lambda^{\mathfrak{p}_\mathfrak{p}}(\mathsf{RHom}_R(R_\mathfrak{p}, M))) \neq 0\}\\ &= \{\mathfrak{p}\in\operatorname{Spec} R \mid \mathrm{H}(\mathsf{RHom}_R(R_\mathfrak{p}, \mathsf{L}\Lambda^{\mathfrak{p}}(M))) \neq 0\}\,.\end{aligned}$$

Proof. The first equality follows from 17.1.14 and 16.2.23:

$$\begin{aligned}\mathfrak{p} \in \operatorname{cosupp}_R M &\iff \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) < \infty \\ &\iff \mathrm{H}(\mathsf{L}\Lambda^{\mathfrak{p}_\mathfrak{p}}(\mathsf{RHom}_R(R_\mathfrak{p}, M))) \neq 0\,.\end{aligned}$$

The second equality now follows from 14.1.34. □

17.1.17 Proposition. *Let M and N be R-complexes. The equality*

$$\operatorname{cosupp}_R(N \otimes^{\mathsf{L}}_R M) = \operatorname{cosupp}_R N \cap \operatorname{supp}_R M$$

holds if

(a) *M is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ and $\operatorname{pd}_R M$ is finite.*

The equality also holds if every flat R-module has finite projective dimension and one of the next conditions is satisfied.

(b) *M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N in $\mathcal{D}_{\sqsupset}(R)$.*

(c) *M is in $\mathcal{D}^{\mathrm{f}}(R)$ and N is in $\mathcal{D}_{\square}(R)$ with $\operatorname{fd}_R N$ finite.*

Proof. Let $\mathfrak{p}$ be a prime ideal in R; by 1.3.42 the R-module $R_\mathfrak{p}$ is flat. Consider the tensor evaluation morphism in $\mathcal{D}(R)$,

$$\boldsymbol{\theta}^{R_\mathfrak{p}NM} : \mathsf{RHom}_R(R_\mathfrak{p}, N) \otimes^{\mathsf{L}}_R M \longrightarrow \mathsf{RHom}_R(R_\mathfrak{p}, N \otimes^{\mathsf{L}}_R M)\,.$$

Under the assumptions in (a) it is an isomorphism by 12.3.23(c). Assuming that every flat R-module has finite projective dimension, it follows from 12.3.23(d) and 12.3.24(a) that $\boldsymbol{\theta}^{R_\mathfrak{p}NM}$ is an isomorphism under the assumptions in (b) and (c). This accounts for the first isomorphism in the computation below; the subsequent isomorphisms follow from 12.3.30, and 7.6.12.

$$\begin{aligned}&\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(R_\mathfrak{p}, N \otimes^{\mathsf{L}}_R M) \\ &\qquad\simeq \kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R (\mathsf{RHom}_R(R_\mathfrak{p}, N) \otimes^{\mathsf{L}}_R M) \\ &\qquad\simeq (\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(R_\mathfrak{p}, N)) \otimes^{\mathsf{L}}_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \\ &\qquad\simeq \mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(R_\mathfrak{p}, N)) \otimes_{\kappa(\mathfrak{p})} \mathrm{H}(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M)\,.\end{aligned}$$

The last complex is a tensor product of $\kappa(\mathfrak{p})$-vector spaces and hence non-zero if and only if both spaces are non-zero. The asserted equality now follows from 17.1.15 and the definition, 15.1.5, of support. □

17.1.18 Example. Let $\mathfrak{a}$ be an ideal in R generated by a sequence $\boldsymbol{x}$. One has

$$\operatorname{cosupp}_R \mathrm{K}^R(\boldsymbol{x}) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R R$$

by 17.1.17(a) in view of the unitor 12.3.3, 11.4.3(c), and 15.1.10.

By 17.4.26 the next result applies, in particular, if R has finite Krull dimension.

17.1.19 Corollary. *Assume that every flat R-module has finite projective dimension and let M be a complex in $\mathcal{D}^{\mathrm{f}}(R)$. One has*

$$\operatorname{cosupp}_R M = \operatorname{cosupp}_R R \cap \operatorname{supp}_R M \subseteq \operatorname{supp}_R M\,.$$

Proof. Apply 17.1.17(c) with $N = R$ and invoke the unitor 12.3.3. □

For a ring R with full cosupport, i.e. $\operatorname{cosupp}_R R = \operatorname{Spec} R$, and finite $\operatorname{FPD} R$ it follows from 8.5.18 and 17.1.19 that the equality $\operatorname{cosupp}_R M = \operatorname{supp}_R M$ holds for every complex M in $\mathcal{D}^{\mathrm{f}}(R)$. The ring of integers is an example of such a ring, see 17.1.20 below, as it has finite global dimension, cf. 8.5.2. Thompson [244] has more examples of rings with full cosupport; the case of the integers goes back to Benson, Iyengar, and Krause [38]. See 16.1.19 for examples of otherwise well-behaved rings with $\operatorname{cosupp}_R R \neq \operatorname{Spec} R$.

17.1.20 Example. Let $p \in \mathbb{Z}$ be prime; one has $\operatorname{supp}_R \mathsf{RHom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \{p\mathbb{Z}\}$ by 17.1.10 and 15.1.10, so $p\mathbb{Z} \in \operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z}$ by 15.1.15. Also $\mathrm{H}(\mathsf{RHom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}))$ is non-zero, see 7.3.28, so 0 belongs to $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z}$. Thus $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$ holds.

Remark. Per the Remark after 9.3.30, the assumption in 17.1.19 about projective dimension of flat R-modules is equivalent to requiring the invariant $\operatorname{splf} R$ to be finite. In fact, one has $\operatorname{splf} R = \sup\{\operatorname{pd}_R R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$, see for example Nakamura and Thompson [189], while all that is used in the proof of 17.1.19 is that $\operatorname{pd}_R R_{\mathfrak{p}}$ is finite for every $\mathfrak{p} \in \operatorname{Spec} R$.

Guaranteed Isomorphisms

17.1.21 Lemma. *Let $\alpha \colon M \to N$ be a morphism of R-complexes and assume that M and N are both degreewise finitely generated or both degreewise Artinian. If the next two conditions are satisfied, then α is an isomorphism.*

(1) *The graded R-modules $M^{\natural}$ and $N^{\natural}$ are isomorphic.*
(2) *$\alpha \colon M^{\natural} \to N^{\natural}$ has a left inverse or a right inverse in $\mathcal{M}_{\mathrm{gr}}(R)$.*

Proof. It follows from the assumptions on M and N, combined with 14.1.11 and 14.2.18, that for every prime ideal $\mathfrak{p}$ in R the $R_{\mathfrak{p}}$-complexes $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are both degreewise finitely generated or both degreewise Artinian. Condition (1) implies that the graded $R_{\mathfrak{p}}$-modules $M_{\mathfrak{p}}^{\natural}$ and $N_{\mathfrak{p}}^{\natural}$ are isomorphic and condition (2) implies that the localized map $\alpha_{\mathfrak{p}} \colon M_{\mathfrak{p}}^{\natural} \to N_{\mathfrak{p}}^{\natural}$ has a left inverse or a right inverse in $\mathcal{M}_{\mathrm{gr}}(R_{\mathfrak{p}})$. Since α is an isomorphism in $\mathcal{C}(R)$ if and only if $\alpha_{\mathfrak{p}}$ is an isomorphism in $\mathcal{C}(R_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Spec} R$, see 15.3.8, one can assume that R is local; as usual we let $\boldsymbol{k}$ denote the residue field of R.

First assume that $\alpha \colon M^{\natural} \to N^{\natural}$ has a left inverse. If M and N are degreewise finitely generated, then so is $C = \operatorname{Coker} \alpha$ and the degreewise split exact sequence

$$0 \longrightarrow M \xrightarrow{\ \alpha\ } N \longrightarrow C \longrightarrow 0 \qquad (\flat)$$

induces for each $v \in \mathbb{Z}$ a split exact sequence of $\boldsymbol{k}$-vector spaces,

$$0 \longrightarrow \boldsymbol{k} \otimes_R M_v \xrightarrow{\boldsymbol{k} \otimes \alpha_v} \boldsymbol{k} \otimes_R N_v \longrightarrow \boldsymbol{k} \otimes_R C_v \longrightarrow 0\,,$$

which have finite rank by 12.1.20(a). By condition (1), the modules M_v and N_v are isomorphic, whence $\boldsymbol{k} \otimes_R M_v$ and $\boldsymbol{k} \otimes_R N_v$ have the same rank. Since $\boldsymbol{k} \otimes_R \alpha_v$ is injective, it must be an isomorphism. It follows that one has $\boldsymbol{k} \otimes_R C_v = 0$ and hence

$C_v = 0$ by Nakayama's lemma 16.1.5. This proves $C = 0$ and, consequently, α is an isomorphism. If M and N are degreewise Artinian, then so is the complex C and the degreewise split exact sequence ($\flat$) induces for each $v \in \mathbb{Z}$ a split exact sequence of $\boldsymbol{k}$-vector spaces,

$$0 \longrightarrow \operatorname{Hom}_R(\boldsymbol{k}, M_v) \xrightarrow{\operatorname{Hom}(\boldsymbol{k},\alpha_v)} \operatorname{Hom}_R(\boldsymbol{k}, N_v) \longrightarrow \operatorname{Hom}_R(\boldsymbol{k}, C_v) \longrightarrow 0\,,$$

which have finite rank by 16.1.4. By condition (1), the modules M_v and N_v are isomorphic, so $\operatorname{Hom}_R(\boldsymbol{k}, M_v)$ and $\operatorname{Hom}_R(\boldsymbol{k}, N_v)$ have the same rank. Since $\operatorname{Hom}_R(\boldsymbol{k}, \alpha_v)$ is injective, it must be an isomorphism. It follows that one has $\operatorname{Hom}_R(\boldsymbol{k}, C_v) = 0$ and hence $C_v = 0$ by 16.1.6. This proves $C = 0$ and, consequently, α is an isomorphism.

The case where $\alpha\colon M^\natural \to N^\natural$ has a right inverse is dealt with by a similar argument applied to the split exact sequence $0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow M \xrightarrow{\alpha} N \longrightarrow 0$. $\square$

17.1.22 Proposition. *Let M and X be R-complexes such that M and $\mathsf{R}\!\operatorname{Hom}_R(M, X)$ both belong to $\mathcal{D}^{\mathrm{f}}(R)$ or both belong to $\mathcal{D}^{\mathrm{art}}(R)$. If there is an isomorphism in $\mathcal{D}(R)$,*

$$M \simeq \mathsf{R}\!\operatorname{Hom}_R(\mathsf{R}\!\operatorname{Hom}_R(M, X), X)\,,$$

then the biduality morphism from 8.4.2,

$$\boldsymbol{\delta}_X^M : M \longrightarrow \mathsf{R}\!\operatorname{Hom}_R(\mathsf{R}\!\operatorname{Hom}_R(M, X), X)\,,$$

is an isomorphism as well.

Proof. We consider the case where M and $\mathsf{R}\!\operatorname{Hom}_R(M, X)$ belong to $\mathcal{D}^{\mathrm{f}}(R)$; the same arguments apply, mutatis mutandis, if the complexes belong to $\mathcal{D}^{\mathrm{art}}(R)$.

Let $(-)^\dagger$ denote the functor $\mathsf{R}\!\operatorname{Hom}_R(-, X)$. By assumption, the R-complex M satisfies the following conditions:

(1) There is an isomorphism $M \simeq M^{\dagger\dagger}$ in $\mathcal{D}(R)$.
(2) The complexes M and $M^\dagger$ belong to $\mathcal{D}^{\mathrm{f}}(R)$.

The zigzag identities related to the adjunction 10.1.22 yield the equality,

$$(\boldsymbol{\delta}_X^M)^\dagger \circ \boldsymbol{\delta}_X^{M^\dagger} = 1^{M^\dagger}\,.$$

As homology is a functor on $\mathcal{D}(R)$, see 6.5.17, this equality implies that one has

$$\mathrm{H}((\boldsymbol{\delta}_X^M)^\dagger) \circ \mathrm{H}(\boldsymbol{\delta}_X^{M^\dagger}) = 1^{\mathrm{H}(M^\dagger)}\,,$$

so $\mathrm{H}(\boldsymbol{\delta}_X^{M^\dagger})\colon \mathrm{H}(M^\dagger) \to \mathrm{H}(M^{\dagger\dagger\dagger})$ has a left inverse. Condition (1) implies that $\mathrm{H}(M^\dagger)$ and $\mathrm{H}(M^{\dagger\dagger\dagger})$ are isomorphic, and (2) implies that $\mathrm{H}(M^\dagger)$ is degreewise finitely generated. Thus it follows from 17.1.21 that $\mathrm{H}(\boldsymbol{\delta}_X^{M^\dagger})$ is an isomorphism in $\mathcal{C}(R)$, whence $\boldsymbol{\delta}_X^{M^\dagger}$ is an isomorphism in $\mathcal{D}(R)$ by 6.5.17.

Notice that conditions (1) and (2) are satisfied with M replaced by the complex $M^\dagger$, so $\boldsymbol{\delta}_X^{M^{\dagger\dagger}}$ is an isomorphism in $\mathcal{D}(R)$ by the argument above. By assumption there is an isomorphism $\varphi\colon M \to M^{\dagger\dagger}$ in $\mathcal{D}(R)$. As biduality $\boldsymbol{\delta}_X^-$ is a natural transformation, there is a commutative diagram in $\mathcal{D}(R)$,

$$\begin{tikzcd} M \arrow[r, "\varphi", "\simeq"'] \arrow[d, "\delta_X^M"'] & M^{\dagger\dagger} \arrow[d, "\delta_X^{M^{\dagger\dagger}}", "\simeq"'] \\ M^{\dagger\dagger} \arrow[r, "\varphi^{\dagger\dagger}", "\simeq"'] & M^{\dagger\dagger\dagger\dagger} . \end{tikzcd}$$

Since φ, $\varphi^{\dagger\dagger}$, and $\delta_X^{M^{\dagger\dagger}}$ are isomorphisms, so is δ_X^M. □

17.1.23 Corollary. *Let X be a complex in $\mathcal{D}^{\mathrm{f}}(R)$. If there is an isomorphism,*

$$R \simeq \mathrm{RHom}_R(X, X) ,$$

in $\mathcal{D}(R)$, then the homothety formation morphism,

$$\chi_R^X : R \longrightarrow \mathrm{RHom}_R(X, X) ,$$

from 10.1.10 *is an isomorphism as well.*

Proof. There is a commutative diagram,

$$\begin{tikzcd} R \arrow[rr, "\chi_R^X"] \arrow[dr, "\delta_X^R"'] & & \mathrm{RHom}_R(X, X) \\ & \mathrm{RHom}_R(\mathrm{RHom}_R(R, X), X) , \arrow[ur, "\simeq", "\mathrm{RHom}_R(\epsilon_R^X, X)"'] & \end{tikzcd}$$

where ϵ_R^X is the counitor 12.3.4. Thus, to prove that χ_R^X is an isomorphism it suffices to argue that δ_X^R is an isomorphism. An isomorphism from R to $\mathrm{RHom}_R(X, X)$ in $\mathcal{D}(R)$ can be composed with the inverse of $\mathrm{RHom}_R(\epsilon_R^X, X)$ to yield an isomorphism $R \simeq \mathrm{RHom}_R(\mathrm{RHom}_R(R, X), X)$. By assumption, the complex $\mathrm{RHom}_R(R, X) \simeq X$ is in $\mathcal{D}^{\mathrm{f}}(R)$, and hence 17.1.22 applied with $M = R$ shows that δ_X^R is an isomorphism. □

Exercises

E 17.1.1 Let M be a complex in $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$ and set $w = \inf M$. Show that $\mathrm{supp}_R\, M$ contains the set $\{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p}\, \mathrm{H}_w(M)_{\mathfrak{p}} \neq \mathrm{H}_w(M)_{\mathfrak{p}} \,\}$.

E 17.1.2 Let M be an R-module. Show that one has $\dim_R M = \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{supp}_R\, M\,\}$.

E 17.1.3 Let M be an R-complex. Show that there are equalities,

$$\begin{aligned} \mathrm{supp}_R\, M &= \{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(\mathrm{R}\Gamma_{\mathfrak{p}}(M_{\mathfrak{p}})) \neq 0 \,\} \\ &= \{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(\mathrm{L}\Lambda^{\mathfrak{p}}(M_{\mathfrak{p}})) \neq 0 \,\} . \end{aligned}$$

E 17.1.4 Let M be an R-complex. Show that there are equalities,

$$\begin{aligned} \mathrm{cosupp}_R\, M &= \{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(\mathrm{L}\Lambda^{\mathfrak{p}}(\mathrm{RHom}_R(R_{\mathfrak{p}}, M))) \neq 0 \,\} \\ &= \{\, \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathrm{H}(\mathrm{R}\Gamma_{\mathfrak{p}}(\mathrm{RHom}_R(R_{\mathfrak{p}}, M))) \neq 0 \,\} . \end{aligned}$$

E 17.1.5 Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. (a) Show that $\mathfrak{p}$ is in $\mathrm{supp}_R\, M$ if and only if it belongs to $\mathrm{cosupp}_R\, M_{\mathfrak{p}}$. (b) Show that $\mathfrak{p}$ is in $\mathrm{cosupp}_R\, M$ if and only if it belongs to $\mathrm{supp}_R\, \mathrm{RHom}_R(R_{\mathfrak{p}}, M)$.

E 17.1.6 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}_{\sqsupset}(R)$ and X a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ with $\mathrm{cosupp}_R\, M \cup \mathrm{cosupp}_R\, N \subseteq \mathrm{supp}_R\, X$. Show that if $\alpha \otimes^{\mathsf{L}}_R X$ is an isomorphism, then α is an isomorphism.

E 17.1.7 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and X a complex in $\mathcal{D}_{\sqsupset}(R)$ with $\operatorname{supp}_R M \cup \operatorname{supp}_R N \subseteq \operatorname{cosupp}_R X$. Show that if $X \otimes^{\mathsf{L}}_R \alpha$ is an isomorphism, then α is an isomorphism.

E 17.1.8 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}_{\sqsubset}(R)$ and X a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ with $\operatorname{supp}_R M \cup \operatorname{supp}_R N \subseteq \operatorname{supp}_R X$. Show that if $\mathsf{RHom}_R(X, \alpha)$ is an isomorphism, then α is an isomorphism.

E 17.1.9 Let $\alpha\colon M \to N$ be a morphism in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and X a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\operatorname{supp}_R M \cup \operatorname{supp}_R N \subseteq \operatorname{supp}_R X$. Show that if $\mathsf{RHom}_R(\alpha, X)$ is an isomorphism, then α is an isomorphism.

E 17.1.10 Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. Show that M is derived reflexive if and only if there is an isomorphism $M \simeq \mathsf{RHom}_R(\mathsf{RHom}_R(M, R), R)$ in $\mathcal{D}(R)$.

17.2 The Cohen–Macaulay Property

Synopsis. Krull dimenson vs. depth; Cohen–Macaulay defect; Cohen–Macaulay complex; equi(co)-dimensional ring; catenary ring; Cohen–Macaulay (local) ring.

For a local ring the depth is per 16.2.32 an algebraic invariant while the Krull dimension is geometric in nature. Local rings for which these invariants agree are of special interest because their prime ideal spectra are particularly well structured. For example, they exhibit a phenomenon known as 'unmixedness' (E 17.2.8). This was proved for power series algebras by Cohen [70], and Macaulay [176] had already thirty years earlier shown that polynomial algebras have the property.

The theory of these Cohen–Macaulay rings and the more general notion of Cohen–Macaulay modules is extremely rich and aspects of it is already consolidated in several monographs: "Cohen–Macaulay rings" [46] by Bruns and Herzog, "Maximal Cohen-Macaulay modules and Tate cohomology" [47] by Buchweitz, "Cohen–Macaulay representations" by Leuschke and Wiegand [171], and "Cohen–Macaulay modules over Cohen-Macaulay rings" by Yoshino [261].

Krull Dimenson vs. Depth

The first inequality below holds for all complexes of finite depth, see 18.3.31.

17.2.1 Theorem. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$ that is not acyclic. There is an inequality,*

$$\operatorname{depth}_R M \leqslant \dim_R M\,.$$

Moreover, if M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ with $s = \sup M$, then one has

$$\operatorname{depth}_R M \leqslant \dim_R \mathrm{H}_s(M) - s \leqslant \dim_R M\,.$$

Proof. If $\mathrm{H}(M)$ is not bounded above, then the inequality $\operatorname{depth}_R M \leqslant \dim_R M$ is trivial as 16.2.21 yields $\operatorname{depth}_R M = -\infty$. Assuming now that M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$, set $s = \sup M$ and choose $\mathfrak{p} \in \operatorname{Min}_R \mathrm{H}_s(M)$ with $\dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(M)$. It follows that the maximal ideal of $R_{\mathfrak{p}}$ is associated to $\mathrm{H}_s(M_{\mathfrak{p}})$, so $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = -s$ holds by 16.2.16(b). Now 16.4.6 and 14.2.1 yield

$$\operatorname{depth}_R M \leqslant \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} = \dim_R \mathrm{H}_s(M) - s \leqslant \dim_R M . \qquad \square$$

Cohen–Macaulay Complexes

For a complex M in $\mathcal{D}^\mathrm{f}(R)$ the difference $\dim_R M - \operatorname{depth}_R M$ is well-defined, see 14.2.4 and 16.2.27.

17.2.2 Definition. Let R be local and M a complex in $\mathcal{D}^\mathrm{f}(R)$. The *Cohen–Macaulay defect* of M, written $\operatorname{cmd}_R M$, is the difference

$$\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M .$$

The simpler notation $\operatorname{cmd} R$ deotes the Cohen–Macaulay defect of the R-module R.

17.2.3. Let R be local and M a complex in $\mathcal{D}^\mathrm{f}(R)$. By 14.2.2 and 14.3.11 one has

$$\operatorname{cmd}_R \Sigma^s M = \operatorname{cmd}_R M \quad \text{for every integer } s .$$

Moreover, it follows from 17.2.1 that one has $\operatorname{cmd}_R M \geqslant 0$ if M is not acyclic, while $\operatorname{cmd}_R M = -\infty$ holds if M is acyclic.

17.2.4 Lemma. *Let R be local and M a complex in $\mathcal{D}^\mathrm{f}(R)$. If $\operatorname{cmd}_R M$ is finite, then $\mathrm{H}(M)$ is bounded.*

Proof. The homology of an acyclic complex is trivially bounded, so assume that $\mathrm{H}(M)$ is non-zero. One thus has $\inf M < \infty$ and $\sup M > -\infty$. The assumption $\operatorname{cmd}_R M < \infty$ yields $\dim_R M < \infty$ and $\operatorname{depth}_R M > -\infty$ and hence one gets the inequalities $\inf M > -\infty$ from 14.2.4 and $\sup M < \infty$ from 16.2.21. $\square$

17.2.5 Definition. Let R be local and M a complex i $\mathcal{D}^\mathrm{f}(R)$. If $\operatorname{cmd}_R M \leqslant 0$ holds, then M is called *Cohen–Macaulay*.

17.2.6 Example. Let R be local. It follows from 14.2.10 and 16.2.18 that every Artinian R-module $M \neq 0$ has $\dim_R M = 0 = \operatorname{depth}_R M$. In particular, every R-module M of finite length is Cohen–Macaulay, see 14.2.12.

17.2.7 Proposition. *Let R be local and M a complex in $\mathcal{D}^\mathrm{f}(R)$. If M is not acyclic, then M is Cohen–Macaulay if and only if M is in $\mathcal{D}^\mathrm{f}_\square(R)$ and $\operatorname{cmd}_R M = 0$ holds.*

Proof. The assertion follows immediately from 17.2.4 and 17.2.3. $\square$

17.2.8 Proposition. *Let R be local, $\mathfrak{p}$ a prime ideal in R, and M an R-complex. If M is in $\mathcal{D}^\mathrm{f}(R)$, then there is an inequality*

$$\operatorname{cmd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{cmd}_R M ;$$

in particular, if M is Cohen–Macaulay, then the $R_\mathfrak{p}$-complex $M_\mathfrak{p}$ is Cohen–Macaulay.

Proof. Let $\mathfrak{p}$ be a prime ideal in R, by 14.1.11 the complex $M_\mathfrak{p}$ belongs to $\mathcal{D}^{\mathrm{f}}(R_\mathfrak{p})$. From 14.2.7 and 16.4.6 one gets

$$\begin{aligned}\operatorname{cmd}_{R_\mathfrak{p}} M_\mathfrak{p} &= \dim_{R_\mathfrak{p}} M_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}\\ &\leqslant (\dim_R M - \dim R/\mathfrak{p}) - (\operatorname{depth}_R M - \dim R/\mathfrak{p})\\ &= \operatorname{cmd}_R M\,.\end{aligned}$$

The last claim follows from the definition, 17.2.5, of a Cohen-Macaulay complex. □

A Cohen–Macaulay module has no embedded prime ideals.

17.2.9 Corollary. *Let R be local and M a finitely generated R-module. If M is Cohen–Macaulay, then one has*

$$\operatorname{Ass}_R M = \operatorname{Min}_R M\,.$$

Proof. Every prime ideal in $\operatorname{Min}_R M$ is associated to M. To prove the converse, let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass}_R M$. The maximal ideal $\mathfrak{p}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ is associated to $M_\mathfrak{p}$, so one has $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$ by 16.2.18. As M is Cohen–Macaulay, 17.2.8 yields $\dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0$, so $\mathfrak{p}_\mathfrak{p}$ is minimal in $\operatorname{Supp}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\mathfrak{p}$ hence minimal in $\operatorname{Supp}_R M$. □

For a Cohen–Macaulay complex, the support and classic support agree by 15.1.9, and for prime ideals in that set the inequalities in 14.2.7 and 16.4.6 are equalities.

17.2.10 Proposition. *Let R be local and M a Cohen–Macaulay R-complex. For every $\mathfrak{p} \in \operatorname{supp}_R M$ the following equalities hold.*

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} = \operatorname{depth}_R M = \dim_R M = \dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}\,.$$

Proof. By 17.2.7 the complex M belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. Let $\mathfrak{p} \in \operatorname{supp}_R M$; by 16.4.6, 17.2.1, and 14.2.7 there are inequalities,

$$\begin{aligned}\operatorname{depth}_R M &\leqslant \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}\\ &\leqslant \dim_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}\\ &\leqslant \dim_R M\,,\end{aligned}$$

and the assumption $\operatorname{depth}_R M = \dim_R M$ implies that equalities hold. □

17.2.11 Corollary. *Let R be local and M a Cohen–Macaulay R-complex. For prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{supp}_R M$ the next equality holds,*

$$\dim R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q} = \dim R/\mathfrak{p} - \dim R/\mathfrak{q}\,.$$

Proof. By 17.2.8 the $R_\mathfrak{q}$-complex $M_\mathfrak{q}$ is Cohen–Macaulay, and the first equality in the computation below follows from 17.2.10 applied to this complex. The second equality also follows from 17.2.10, but now applied to the R-complex M.

$$\begin{aligned}\dim R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q} &= \dim_{R_\mathfrak{q}} M_\mathfrak{q} - \dim_{R_\mathfrak{p}} M_\mathfrak{p}\\ &= (\dim_R M - \dim R/\mathfrak{q}) - (\dim_R M - \dim R/\mathfrak{p})\end{aligned}$$

$$= \dim R/\mathfrak{p} - \dim R/\mathfrak{q}\,. \qquad \square$$

Remark. In the literature one can find the statement in 17.2.11 formulated along the lines of "a Cohen–Macaulay complex has catenary support". We do not use that terminology, but the next definition could easily be generalized to sustain it.

Chain Conditions

The notions introduced below could be defined for subsets of $\operatorname{Spec} R$, but we only use them for the spectrum itself.

17.2.12 Definition. If $\dim R/\mathfrak{p} = \dim R/\mathfrak{p}'$ holds for all minimal prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ in R, then R is called *equidimensional*, and if $\dim R_{\mathfrak{m}} = \dim R_{\mathfrak{m}'}$ holds for all maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ in R, then R is called *equicodimensional*.

If given any two prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in R, all saturated chains

$$\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$$

in $\operatorname{Spec} R$ have the same length, then R is called *catenary*.

17.2.13. The following observations follow immediately from the definitions.

(a) R is equidimensional if and only if the equality $\dim R = \dim R/\mathfrak{p}$ holds for every minimal prime ideal $\mathfrak{p}$ in R.
(b) R is equicodimensional if and only if R has finite Krull dimension and the equality $\dim R = \dim R_{\mathfrak{m}}$ holds for every maximal ideal $\mathfrak{m}$ in R.
(c) If R is catenary and $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$ is a saturated chain in $\operatorname{Spec} R$, then $n = \dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}}$ holds.

Caveat. If all maximal chains of prime ideals in R have the same length, then R called *biequidimensional*, such a ring is evidently equidimensional, equicodimensional, and catenary. Contrary to a claim made in [111, 0.§14] the converse is not true. Heinrich [117] constructs a counterexample and shows that R is biequidimensional if it is equicodimensional and $R_{\mathfrak{m}}$ is equidimensional and catenary for every maximal ideal $\mathfrak{m}$ in R.

In the next result, the assumption that R is local is crucial; see 17.2.35.

17.2.14 Proposition. *Let R be an equidimensional catenary local ring. For every prime ideal $\mathfrak{p}$ in R the next equality holds,*

$$\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim R\,.$$

Proof. Fix a prime ideal $\mathfrak{p}$ in R. Choose a saturated chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ in $\operatorname{Spec} R$ with $n = \dim R_{\mathfrak{p}}$, and note that $\mathfrak{p}_0$ must be a minimal prime ideal in R. Let $\mathfrak{m}$ be the maximal ideal of R and choose a saturated chain $\mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_m = \mathfrak{m}$ in $\operatorname{Spec} R$ with $m = \dim R/\mathfrak{p}$. Now $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p} = \mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_m = \mathfrak{m}$ is a saturated chain in $\operatorname{Spec} R$ of length $n + m$, and as R is catenary one has $\dim R/\mathfrak{p}_0 = n + m$ by 17.2.13(c). As R is also equidimensional, $\dim R/\mathfrak{p}_0 = \dim R$ holds by 17.2.13(a). Consequently, one has $\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$, as asserted. $\square$

Remark. For a local integral domain R the equality in 17.2.14 holds for every prime ideal if and only if R is catenary. This was proved by Ratliff [206]; the proof is included in [182, §31].

Cohen–Macaulay Local Rings

17.2.15 Definition. A local ring R is called *Cohen–Macaulay* if per 17.2.5 it is Cohen–Macaulay as an R-module.

For ease of reference we record:

17.2.16 Proposition. *Let R be local. There is an inequality,*

$$\operatorname{depth} R \leqslant \dim R \,,$$

and equality holds, i.e. one has $\operatorname{cmd} R = 0$*, if and only if R is Cohen–Macaulay.*

Proof. The inequality is a special case of 17.2.1, and the last assertion is per 17.2.2 and 17.2.5 a restatement of 17.2.15. □

17.2.17 Example. An Artinian local ring is Cohen–Macaulay by 14.2.19 and 17.2.6.

17.2.18 Example. Let $\Bbbk$ be a field. In the local ring $R = \Bbbk[\![x_1, \ldots, x_n]\!]$ the indeterminates form a regular sequence, so the depth of R is per 16.2.33(a) at least $n = \dim R$, whence R is Cohen–Macaulay.

17.2.19 Proposition. *Let $(R, \mathfrak{m})$ be local. The following conditions are equivalent.*

(*i*) *R is Cohen–Macaulay.*
(*ii*) $\mathrm{H}^n_{\mathfrak{m}}(R) = 0$ *holds for all $n < \dim R$.*
(*iii*) $\mathrm{H}^n_{\mathfrak{m}}(R) = 0$ *holds for all $n \neq \dim R$.*

Proof. Being local, R has finite Krull dimension, and 16.2.34 yields $\mathrm{H}^n_{\mathfrak{m}}(R) = 0$ for $n > \dim R$. The equivalence of the three conditions now follows immediately from 16.2.14 and 17.2.16. □

17.2.20 Proposition. *Let R be a Cohen–Macaulay local ring. For every prime ideal $\mathfrak{p}$ in R the next equalities hold,*

$$\operatorname{depth} R_{\mathfrak{p}} + \dim R/\mathfrak{p} \;=\; \operatorname{depth} R \;=\; \dim R \;=\; \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} \,.$$

Proof. The equalities hold by 17.2.16 and by 17.2.10 applied to the R-module R. □

A local ring is trivially equicodimensional; a Cohen–Macaulay local ring also has the other properties from 17.2.12.

17.2.21 Theorem. *A Cohen–Macaulay local ring is equidimensional and catenary.*

Proof. Let R be local and Cohen–Macaulay. For every prime ideal $\mathfrak{p}$ in R the equality $\dim R = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$ holds by 17.2.20. In particular, $\dim R = \dim R/\mathfrak{p}$ holds for every minimal prime ideal, so R is equidimensional. To prove catenarity, fix prime ideals $\mathfrak{p} \subset \mathfrak{q}$ and consider a saturated chain $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$ in $\operatorname{Spec} R$. That the chain is saturated means that for each $i \in \{1, \ldots, n\}$ one has $\dim R_{\mathfrak{p}_i}/(\mathfrak{p}_{i-1})_{\mathfrak{p}_i} = 1$; at the same time, 17.2.11 yields $\dim R_{\mathfrak{p}_i}/(\mathfrak{p}_{i-1})_{\mathfrak{p}_i} = \dim R/\mathfrak{p}_{i-1} - \dim R/\mathfrak{p}_i$, thus:

$$n = \sum_{i=1}^{n} (\dim R/\mathfrak{p}_{i-1} - \dim R/\mathfrak{p}_i) = \dim R/\mathfrak{p} - \dim R/\mathfrak{q} .$$

That is, all saturated chains from $\mathfrak{p}$ to $\mathfrak{q}$ has the same length, i.e. R is catenary. □

An example of an equidimensional catenary local ring that is not Cohen–Macaulay is provided in 18.5.29. A quotient of a catenary ring is evidently catenary, but a quotient of an equidimensional ring, even a quotient of a Cohen–Macaulay local ring, need not be equidimensional.

17.2.22 Example. Let $\Bbbk$ be a field. The local ring $\Bbbk[\![x, y, z]\!]$ is Cohen–Macaulay and hence catenary, see 17.2.18 and 17.2.21. It follows that the quotient ring $R = \Bbbk[\![x, y, z]\!]/(xy, xz)$ is catenary as well. Both $\mathfrak{p} = (x)$ and $\mathfrak{q} = (y, z)$ are minimal prime ideals in R, so the isomorphisms $R/\mathfrak{p} \cong \Bbbk[\![y, z]\!]$ and $R/\mathfrak{q} \cong \Bbbk[\![x]\!]$ show that R is not equidimensional.

For a finitely generated module $M \neq 0$ over a Cohen–Macaulay local ring R the next result yields $\operatorname{depth}_R M \leqslant \operatorname{depth} R$. The example that follows shows that this inequality may fail if R is not Cohen–Macaulay.

17.2.23 Proposition. *Let R be a Cohen–Macaulay local ring and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. If M is not acyclic, then the next inequalities hold,*

$$\operatorname{depth} R - \inf M \geqslant \operatorname{depth}_R M \geqslant -\sup M .$$

Proof. The equality below holds holds as R is Cohen–Macaulay, see 17.2.16. The first inequality holds, in view of 16.2.27, by 16.2.35 and 16.2.5(a); the second inequality comes from 16.2.16,

$$\operatorname{depth} R - \inf M = \dim R - \inf M \geqslant \operatorname{depth}_R M \geqslant -\sup M . \qquad \square$$

17.2.24 Example. Let $\Bbbk$ be a field, set $R = \Bbbk[\![x, y]\!]/(x^2, xy)$, and $M = R/(x) \cong \Bbbk[\![y]\!]$. As $(0 :_R x) = (x, y)$ holds—that is, the maximal ideal of R is an associated prime ideal—one has $\operatorname{depth} R = 0$ by 16.2.18. On the other hand, y is evidently an M-regular element, so $\operatorname{depth}_R M$ is positive by 16.2.33(a).

Note from 17.2.8 and 17.2.15 that if R is a Cohen–Macaulay local ring, then so is $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. This justifies the next extension of 17.2.15 to non-local rings.

Cohen–Macaulay Rings

17.2.25 Definition. If the local ring $R_\mathfrak{p}$ is a Cohen–Macaulay for every prime ideal $\mathfrak{p}$ in R, then R is called *Cohen–Macaulay*.

17.2.26 Example. An Artinian ring is Cohen–Macaulay by 14.2.19(b) and 17.2.17.

The ring $\mathbb{Z}$ is an example of a Cohen–Macaulay ring of positive Krull dimension:

17.2.27 Example. Let R be an integral domain of Krull dimension 1. The field of fractions $R_{(0)}$ is an Artinian local ring and hence Cohen–Macaulay. For a non-zero prime ideal $\mathfrak{p}$ in R the local ring $R_\mathfrak{p}$ is an integral domain of Krull dimension 1. By 14.4.17 and 14.4.21(a) the depth of $R_\mathfrak{p}$ is at least 1, so $R_\mathfrak{p}$ is Cohen–Macaulay by 17.2.16. Thus R is Cohen–Macaulay. This applies, in particular, to $R = \mathbb{Z}$ and for a field $\Bbbk$ to $R = \Bbbk[x]$.

A product of Cohen–Macaulay local rings is Cohen–Macaulay, so such a ring need neither be equidimensional nor equicodimensional, but the third property from 17.2.12 does carry over from the local setting. For a less trivial example of a Cohen–Macaulay ring that is not equicodimensional see 17.2.35.

17.2.28 Theorem. *A Cohen–Macaulay ring is catenary.*

Proof. Let R be Cohen–Macaulay and $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals in R. Every chain of prime ideals $\mathfrak{p} \subset \cdots \subset \mathfrak{q}$ in R corresponds to a chain of prime ideals in the Cohen–Macaulay local ring $R_\mathfrak{q}$; as $R_\mathfrak{q}$ is catenary by 17.2.21, all saturated chains $\mathfrak{p} \subset \cdots \subset \mathfrak{q}$ in R have the same length. □

Extensions of Cohen–Macaulay Rings

We recall, say from [182, §15], a result from dimension theory.

Going Down Theorem. Let S be an R-algebra and $\mathfrak{P}$ a prime ideal in S. With $\mathfrak{p} = \mathfrak{P} \cap R$ there is an inequality,

$$\dim S_\mathfrak{P} \leqslant \dim S_\mathfrak{P}/\mathfrak{p}S_\mathfrak{P} + \dim R_\mathfrak{p} ,$$

and equality holds if S is flat as an R-module.

Of importance in the proof of the next theorem is the following corollary.

17.2.29 Corollary. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality,*

$$\dim S = \dim S/\mathfrak{m}S + \dim R .$$

Proof. Apply the Going Down Theorem to $\mathfrak{P} = \mathfrak{M}$ and notice that there are containments $\mathfrak{p} = \mathfrak{M} \cap R \supseteq \mathfrak{m}S \cap R \supseteq \mathfrak{m}$ and, therefore, $\mathfrak{p} = \mathfrak{m}$. □

17.2.30 Theorem. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality,*

$$\operatorname{cmd} S = \operatorname{cmd} S/\mathfrak{m}S + \operatorname{cmd} R .$$

In particular, the following conditions are equivalent.

(*i*) *S is Cohen–Macaulay.*
(*ii*) *R and $S/\mathfrak{m}S$ are Cohen–Macaulay.*

Proof. Applying 16.4.35 with $M = R$ one gets $\operatorname{depth} S = \operatorname{depth} S/\mathfrak{m}S + \operatorname{depth} R$, which combined with the equality of Krull dimensions from 17.2.29 yields the asserted equality of Cohen–Macaulay defects, see 17.2.2. The asserted equivalence now holds by 17.2.16. □

17.2.31 Lemma. *Let $(R, \mathfrak{m})$ be local and $x \in \mathfrak{m}$ an R-regular element. The ring $R/(x)$ is Cohen–Macaulay if and only if R is Cohen–Macaulay.*

Proof. One has $\operatorname{depth} R/(x) = \operatorname{depth}_R R/(x) = \operatorname{depth} R - 1$ by 16.2.26 and 16.2.31. As x is part of a parameter sequence, one has $\dim R/(x) = \dim R - 1$. The assertion now follows from 17.2.16. □

17.2.32 Proposition. *Let $\boldsymbol{x}$ be an R-regular sequence. If R is Cohen–Macaulay, then the ring $R/(\boldsymbol{x})$ is Cohen–Macaulay*; *the converse holds if $\boldsymbol{x}$ is contained in the Jacobson radical of R.*

Proof. By induction it suffices to handle the case $\boldsymbol{x} = x$ of a sequence of length one. Assume first that R is Cohen–Macaulay. A prime ideal in $R/(x)$ has the form $\mathfrak{p}/(x)$, where $\mathfrak{p}$ is a prime ideal in R that contains x. It follows from 14.4.27 that $\frac{x}{1}$ is an $R_\mathfrak{p}$-regular element, so by 17.2.31 the ring $R_\mathfrak{p}/(\frac{x}{1}) \cong (R/(x))_{\mathfrak{p}/(x)}$ is Cohen–Macaulay. Thus, $R/(x)$ is Cohen–Macaulay. Assume now that x is contained in the Jacobson radical of R and that $R/(x)$ is Cohen–Macaulay. Every prime ideal $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$, and 17.2.8 yields $\operatorname{cmd} R_\mathfrak{p} \leqslant \operatorname{cmd} R_\mathfrak{m}$, so it suffices to verify that $R_\mathfrak{m}$ is Cohen–Macaulay for every maximal ideal $\mathfrak{m}$ in R. As x belongs to every such ideal and $(R/(x))_{\mathfrak{m}/(x)} \cong R_\mathfrak{m}/(\frac{x}{1})$ is Cohen–Macaulay, it follows from 17.2.31 that $R_\mathfrak{m}$ is Cohen–Macaulay, since $\frac{x}{1}$ by 14.4.27 is an $R_\mathfrak{m}$-regular element. □

17.2.33 Proposition. *The following conditions are equivalent.*

(*i*) *R is Cohen–Macaulay.*
(*ii*) *The polynomial algebra $R[x_1, \dots, x_n]$ is Cohen–Macaulay.*
(*iii*) *The power series algebra $R[\![x_1, \dots, x_n]\!]$ is Cohen–Macaulay.*

Proof. By recursion, it suffices to handle the case $n = 1$; set $x = x_1$.

(*i*) ⇔ (*iii*): The indeterminate x is an $R[\![x]\!]$-regular element and it belongs by 12.1.25 to the Jacobson radical of $R[\![x]\!]$. Now invoke 17.2.32.

(*ii*) ⇒ (*i*): As above the indeterminate x is $R[x]$-regular, so $R[x]$ being Cohen–Macaulay implies by 17.2.32 that R is is Cohen–Macaulay.

(*i*) ⇒ (*ii*): Let $\mathfrak{P}$ be a prime ideal in $R[x]$; with $\mathfrak{p} = \mathfrak{P} \cap R$ one has $R[x]_\mathfrak{P} \cong (R_\mathfrak{p}[x])_\mathfrak{Q}$ where the prime ideal $\mathfrak{Q}$ is the contraction $\mathfrak{P}_\mathfrak{P} \cap R_\mathfrak{p}[x]$. The $R_\mathfrak{p}$-algebra $(R_\mathfrak{p}[x])_\mathfrak{Q}$ is flat as an $R_\mathfrak{p}$-module by 12.1.24, 1.3.42, and 5.4.24(b). To see that the local ring $R[x]_\mathfrak{P} \cong (R_\mathfrak{p}[x])_\mathfrak{Q}$ is Cohen–Macaulay it suffices, since $R_\mathfrak{p}$ is Cohen–Macaulay, to verify that the quotient ring $(R_\mathfrak{p}[x])_\mathfrak{Q}/\mathfrak{p}_\mathfrak{p}(R_\mathfrak{p}[x])_\mathfrak{Q}$ is Cohen–Macaulay, see 17.2.30. This quotient ring is a localization of the polynomial algebra $\kappa(\mathfrak{p})[x]$ and hence Cohen–Macaulay by 17.2.27. Thus $R[x]_\mathfrak{P}$ and, therefore, $R[x]$ is Cohen–Macaulay. □

17.2.34 Example. Let $\Bbbk$ be a field; it has Krull dimension 0, so it is Cohen–Macaulay and by 17.2.33 so are the $\Bbbk$-algebras $\Bbbk[x_1, \dots, x_n]$ and $\Bbbk[\![x_1, \dots, x_n]\!]$.

The next example shows that the local assumption in 17.2.20 and, therefore, also in 17.2.14, is crucial.

17.2.35 Example. Let $\Bbbk$ be a field and consider the algebra $R = \Bbbk[\![x]\!][y]$. It follows from 17.2.33 and 17.2.34 that R is Cohen–Macaulay and hence catenary, see 17.2.28. Further, R is an integral domain and hence trivially equidimensional. The quotient ring $R/(xy-1)$ is the field of fractions of $\Bbbk[\![x]\!]$ so $\mathfrak{m} = (xy-1)$ is a maximal ideal. As $\mathfrak{m}$ is principal, Krull's principal ideal theorem, see also 18.4.19, yields $\dim R_{\mathfrak{m}} = 1$, so R is not equicodimensional and one has $\dim R = 2 > 1 + 0 = \dim R_{\mathfrak{m}} + \dim R/\mathfrak{m}$.

Exercises

E 17.2.1 Let $(R, \mathfrak{m})$ be local and M a derived $\mathfrak{m}$-torsion R-complex. Show that the equality $\operatorname{amp} M = \dim_R M - \operatorname{depth}_R M$ holds.

E 17.2.2 Let R be Cohen–Macaulay of Krull dimension d. Show that $\operatorname{fd}_R \mathrm{H}^d_{\mathfrak{m}}(R) = d$ holds. (See E 18.5.3 for a stronger statement.)

E 17.2.3 Show that R is Cohen–Macaulay if and only if the local ring $R_{\mathfrak{m}}$ is Cohen–Macaulay for every maximal ideal $\mathfrak{m}$ in R.

E 17.2.4 Show that the product ring $R \times S$ is Cohen–Macaulay if and only if R and S are Cohen–Macaulay.

E 17.2.5 Let R and M be as in 17.2.24. (a) Theorem 16.2.14 yields $\operatorname{Hom}_R(\Bbbk, R) \neq 0$; find a non-zero homomorphism $\Bbbk \to R$. (b) Show that $\operatorname{depth}_R M = 1$ holds.

E 17.2.6 Let $(R, \mathfrak{m})$ and $(S, \mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. Show that if $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality $\operatorname{id} S = \operatorname{id} S/\mathfrak{m}S + \operatorname{id} R$.

E 17.2.7 Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and $\boldsymbol{x} = x_1, \dots, x_n$ a sequence in $\mathfrak{m}$. Show that $\boldsymbol{x}$ is R-regular if and only if $\dim R/(\boldsymbol{x}) = \dim R - n$ holds. *Hint:* 17.2.9.

E 17.2.8 Let R be a Cohen–Macaulay ring and $\mathfrak{a}$ an ideal in R generated by a sequence $x_1, \dots, x_n$. Assume that $\dim R_{\mathfrak{p}} = n$ holds for every prime ideal $\mathfrak{p}$ in $\operatorname{Min}_R R/\mathfrak{a}$. (a) For $\mathfrak{q}$ in $\operatorname{Ass}_R R/\mathfrak{a}$ show that $\frac{x_1}{1}, \dots, \frac{x_n}{1}$ is an $R_{\mathfrak{q}}$-regular sequence. (b) Show that the only associated prime ideals of the ring $R/\mathfrak{a}$ are its minimal prime ideals.

17.3 Homological Dimensions

Synopsis. Flat dimension; ~ vs. localization, ~ of derived $\mathfrak{a}$-torsion complex; Chouinard Formula for ~; injective dimension; ~ vs. colocalization; ~ vs. derived cobase change; ~ vs. localization; ~ of derived $\mathfrak{a}$-complete complex; projective dimension.

The injective and flat dimensions of an R-complex can by 15.4.7 and 15.4.17 be detected by vanishing of Ext and Tor with coefficients in integral domains $R/\mathfrak{p}$. When combined with the techniques of localization and colocalization, this yields additional ways to compute these dimensions. The projective dimension, on the other hand, is less amenable to these techniques; a simple example, 17.3.25, illustrates why.

FLAT DIMENSION

The flat dimension of a complex can be detected by vanishing of Tor with coefficients in residue fields. This has a certain advantage: Via 16.3.3 one can express the degree where vanishing starts in terms of the depth invariant, and that provides for 17.3.4.

17.3.1 Theorem. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \operatorname{supp}_R M\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \operatorname{Tor}^R_m(\kappa(\mathfrak{p}), M) \neq 0\}\,.\end{aligned}$$

PROOF. The second equality is immediate from 7.4.19 and the definition of support, 15.1.5. To see that the first equality holds, notice that the inequality "$\geqslant$" is evident from 15.4.17. To prove the opposite inequality, assume that there is an integer n such that $n \geqslant \sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M)$ holds for all $\mathfrak{p} \in \operatorname{supp}_R M$, and assume towards a contradiction that there exists an ideal $\mathfrak{a}$ in R with $\sup(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) \geqslant n+1$, cf. 15.4.17. Considered as a functor on the module category,

$$\mathrm{F} = \coprod_{v \geqslant n+1} \mathrm{H}_v(- \otimes^{\mathsf{L}}_R M) = \coprod_{v \geqslant n+1} \operatorname{Tor}^R_v(-, M)\,,$$

is half exact by 7.4.29 and 3.1.6. It follows from 12.4.2 that there is a prime ideal $\mathfrak{q}$ in R with $\mathrm{F}(R/\mathfrak{q}) \neq 0$ and $\mathrm{F}(R/\mathfrak{b}) = 0$ for every ideal $\mathfrak{b} \supset \mathfrak{q}$. To achieve a contradiction, it suffices to show that $\mathrm{F}(R/\mathfrak{q})$ vanishes, which per 2.5.25(b) means that the $R/\mathfrak{q}$-complex $(R/\mathfrak{q} \otimes^{\mathsf{L}}_R M)_{\supseteq n+1}$ is acyclic. By 15.1.15 this is tantamount to $\operatorname{supp}_{R/\mathfrak{q}}(R/\mathfrak{q} \otimes^{\mathsf{L}}_R M)_{\supseteq n+1}$ being empty.

Set $X = R/\mathfrak{q} \otimes^{\mathsf{L}}_R M$; by 7.6.6(c) there is a distingiushed triangle in $\mathcal{D}(R/\mathfrak{q})$,

$$(\diamond)\qquad X_{\supseteq n+1} \longrightarrow X \longrightarrow X_{\subseteq n} \longrightarrow \Sigma X_{\supseteq n+1}\,.$$

Recall that one has $\operatorname{Spec} R/\mathfrak{q} = \{\mathfrak{p}/\mathfrak{q} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}$ and let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal that contains $\mathfrak{q}$. The definition of X combined with 12.3.31 and the isomorphism $\kappa(\mathfrak{p}/\mathfrak{q}) \cong \kappa(\mathfrak{p})$ from 15.1.2 yields

$$\sup(\kappa(\mathfrak{p}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X) = \sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \leqslant n\,.$$

Applying $\kappa(\mathfrak{p}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} -$ to $(\diamond)$ and invoking 6.5.19 and 7.6.8 yields an isomorphism,

$$(\mathrm{H}(\kappa(\mathfrak{p}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X_{\subseteq n}))_{\geqslant n+2} \cong \Sigma\mathrm{H}(\kappa(\mathfrak{p}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X_{\supseteq n+1})\,.$$

The goal is now to prove that the right-hand complex is zero, which we accomplish by showing that the $R/\mathfrak{q}$-complex $X_{\subseteq n}$ has flat dimension at most n, cf. 15.4.17.

Let $\mathfrak{b}$ be an ideal in R that contains $\mathfrak{q}$. For $\mathfrak{b} = \mathfrak{q}$ one evidently has

$$(\star)\qquad \sup((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X_{\subseteq n}) = \sup(R/\mathfrak{q} \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X_{\subseteq n}) = \sup X_{\subseteq n} \leqslant n\,.$$

For $\mathfrak{q} \subset \mathfrak{b}$, the definition of X together with 12.3.31, the isomorphism $(R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}) \cong R/\mathfrak{b}$, and the assumption on $\mathfrak{q}$ yields

$$(\dagger\dagger)\qquad \sup((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}) \otimes^{\mathsf{L}}_{R/\mathfrak{q}} X) = \sup(R/\mathfrak{b} \otimes^{\mathsf{L}}_R M) \leqslant n\,.$$

In particular, one has $\operatorname{Tor}_{n+1}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X) = 0$. Associated to ($\diamond$) there is by 7.4.29 an exact sequence,

$$\operatorname{Tor}_{n+1}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X) \longrightarrow \operatorname{Tor}_{n+1}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\subseteq n}) \\ \longrightarrow \operatorname{Tor}_{n}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\supseteq n+1}) \,,$$

which yields

$$(\ddagger\ddagger) \qquad \operatorname{Tor}_{n+1}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\subseteq n}) \;=\; 0 \,,$$

as $\operatorname{Tor}_{n}^{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\supseteq n+1}) = 0$ holds by 7.6.8. Per 15.4.17 the desired inequality $\operatorname{fd}_{R/\mathfrak{q}} X_{\subseteq n} \leqslant n$ now follows from ($\star$) and ($\ddagger\ddagger$). □

REMARK. The equalities in 17.3.1 appeared already in Avramov and Foxby's 1991 paper [21] as did the companion result for injective dimension, 17.3.11, for complexes with bounded above homology. This boundedness condition was removed in a 2017 paper [64] by Christensen and Iyengar, which also removed assumptions on the ring from earlier versions of 17.3.15.

The flat dimension of a complex can be detected locally; in particular, it does not grow under localization.

17.3.2 Corollary. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned} \operatorname{fd}_R M \;&=\; \sup\{\operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &=\; \sup\{\operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} \,. \end{aligned}$$

PROOF. For prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ in R, the complexes $\kappa(\mathfrak{q}_\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\kappa(\mathfrak{q}) \otimes^{\mathsf{L}}_R M$ are isomorphic in $\mathcal{D}(R_\mathfrak{p})$ by 15.1.22. Now apply 17.3.1. □

17.3.3 Corollary. *Let $\mathfrak{p}$ be a prime ideal in R and N an $R_\mathfrak{p}$-complex*; *one has*

$$\operatorname{fd}_{R_\mathfrak{p}} N \;=\; \operatorname{fd}_R N \,.$$

PROOF. The inequality "$\leqslant$" holds by 17.3.2, per idempotence of localization, see 14.1.14(a). The opposite inequality holds by 15.4.21, since $R_\mathfrak{p}$ per 1.3.42 is flat as an R-module. □

The next result is reminiscent of the Auslander–Buchsbaum Formula 16.4.2 and originally due to Chouinard [51].

17.3.4 Corollary. *Let M be an R-complex. If $\operatorname{fd}_R M$ is finite, then one has*

$$\begin{aligned} \operatorname{fd}_R M \;&=\; \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &=\; \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} \,. \end{aligned}$$

PROOF. For every prime ideal $\mathfrak{p}$ in R the $R_\mathfrak{p}$-complex $M_\mathfrak{p}$ has finite flat dimension, see for example 17.3.2. By 14.1.16(b) and 16.3.3 one now has

$$\sup\,(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \;=\; \sup\,(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}) \;=\; \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \,,$$

and by 17.1.6 this quantity is $-\infty$ if $\mathfrak{p}$ is not in $\operatorname{supp}_R M$. Now apply 17.3.1. □

DERIVED TORSION COMPLEXES

The next theorem computes, in particular, the flat dimension of a derived $\mathfrak{a}$-torsion complex, which by 15.4.23 is the same over R and $\widehat{R}^{\mathfrak{a}}$. The injective dimension of such a complex over R is computed in 15.4.12 and compared to the injective dimension over $\widehat{R}^{\mathfrak{a}}$ in 15.4.10 and 16.1.20.

17.3.5 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex; there is an equality,*

$$\begin{aligned}\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) &= \sup\{\sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M\}\\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Tor}^R_m(\kappa(\mathfrak{p}), M) \neq 0\}\,.\end{aligned}$$

PROOF. By 15.1.27 one has $\operatorname{supp}_R \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M$. In view of 15.1.26 the asserted equalities hold by 17.3.1 applied to the R-complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$. □

17.3.6 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an $\mathfrak{a}$-torsion R-module. One has*

$$\operatorname{fd}_R M = \sup\{m \in \mathbb{N}_0 \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Tor}^R_m(\kappa(\mathfrak{p}), M) \neq 0\}\,.$$

PROOF. By 13.3.30 the module M is derived $\mathfrak{a}$-torsion, so 17.3.5 applies. □

The next corollary computes, in particular, the flat dimension of a derived $\mathfrak{m}$-torsion complex.

17.3.7 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex; there are equalities,*

$$\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M) = \sup\{m \in \mathbb{Z} \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}\,.$$

PROOF. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 17.3.5. □

17.3.8 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an $\mathfrak{m}$-torsion R-module. One has*

$$\operatorname{fd}_R M = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Tor}^R_m(\boldsymbol{k}, M) \neq 0\}\,.$$

PROOF. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 17.3.6. □

17.3.9 Proposition. *Let $(R, \mathfrak{m})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. One has:*

$$\operatorname{fd}_{\widehat{R}} \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \operatorname{pd}_R M\,.$$

PROOF. The first equality holds by 15.4.23. Since $\operatorname{fd}_R \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ and $\operatorname{pd}_R M$ are both equal to $\sup(\boldsymbol{k} \otimes^{\mathsf{L}}_R M)$, see 17.3.7 and 16.4.1, the second equality follows. □

The next example supplements 11.4.26.

17.3.10 Example. Let $(R, \mathfrak{m})$ be local and $\boldsymbol{x}$ a sequence that generates $\mathfrak{m}$. By 13.3.18 one has $\mathsf{R}\Gamma_{\mathfrak{m}}(R) \simeq \check{\mathrm{C}}^R(\boldsymbol{x})$ in $\mathcal{D}(R)$, so 17.3.9 yields the first equality below,

$$\operatorname{fd}_R \check{\mathrm{C}}^R(\boldsymbol{x}) = 0 = \operatorname{pd}_R \check{\mathrm{C}}^R(\boldsymbol{x})\,.$$

The second equality holds as $\operatorname{fd}_R \check{\mathrm{C}}^R(\boldsymbol{x}) \leqslant \operatorname{pd}_R \check{\mathrm{C}}^R(\boldsymbol{x}) \leqslant 0$, see 15.4.18 and 11.4.26.

INJECTIVE DIMENSION

The injective dimension of a complex is detected by vanishing of Ext with coefficients in residue fields. Via 16.3.11 one can express the degree where vanishing starts in terms of depth and width, which leads to 17.3.14.

17.3.11 Theorem. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned}\operatorname{id}_R M \;&=\; \sup\{-\inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\}\\ &=\; \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\; \operatorname{Ext}_R^m(\kappa(\mathfrak{p}), M) \neq 0\}\;.\end{aligned}$$

PROOF. The second equality follows immediately from 7.3.24 and the definition of cosupport, 15.2.1. To see that the first equality holds, notice that the inequality "$\geqslant$" is evident from 15.4.7. To prove the opposite inequality, assume that there is an integer n such that $n \geqslant -\inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M)$ holds for all $\mathfrak{p} \in \operatorname{cosupp}_R M$, and assume towards a contradiction that there is an ideal $\mathfrak{a}$ in R with $-\inf \mathsf{RHom}_R(R/\mathfrak{a}, M) \geqslant n+1$, cf. 15.4.7. Considered as a functor on the module category,

$$\mathrm{G} \;=\; \prod_{v \geqslant n+1} \mathrm{H}_{-v}(\mathsf{RHom}_R(-, M)) \;=\; \prod_{v \geqslant n+1} \operatorname{Ext}_R^v(-, M)\;,$$

is half exact by 7.3.35 and 3.1.18. It follows from 12.4.8 that there is a prime ideal $\mathfrak{q}$ in R with $\mathrm{G}(R/\mathfrak{q}) \neq 0$ and $\mathrm{G}(R/\mathfrak{b}) = 0$ for every ideal $\mathfrak{b} \supset \mathfrak{q}$. To achieve a contradiction, it suffices to show that $\mathrm{G}(R/\mathfrak{q})$ vanishes, which per 2.5.24(b) means that the $R/\mathfrak{q}$-complex $\mathsf{RHom}_R(R/\mathfrak{q}, M)_{\subseteq -(n+1)}$ is acyclic. By 15.2.8 this is tantamount to $\operatorname{cosupp}_{R/\mathfrak{q}} \mathsf{RHom}_R(R/\mathfrak{q}, M)_{\subseteq -(n+1)}$ being empty.

Set $X = \mathsf{RHom}_R(R/\mathfrak{q}, M)$; by 7.6.6(c) there is a distinguished triangle in $\mathcal{D}(R/\mathfrak{q})$,

$$(\diamond) \qquad X_{\supseteq -n} \longrightarrow X \longrightarrow X_{\subseteq -(n+1)} \longrightarrow \Sigma X_{\supseteq -n}\;.$$

Recall that one has $\operatorname{Spec} R/\mathfrak{q} = \{\mathfrak{p}/\mathfrak{q} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}$ and let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal that contains $\mathfrak{q}$. The definition of X combined with 12.3.36 and the isomorphism $\kappa(\mathfrak{p}/\mathfrak{q}) \cong \kappa(\mathfrak{p})$ of $R/\mathfrak{q}$-modules from 15.1.2 yields

$$-\inf \mathsf{RHom}_{R/\mathfrak{q}}(\kappa(\mathfrak{p}/\mathfrak{q}), X) \;=\; -\inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M) \;\leqslant\; n\;.$$

Applying the functor $\mathsf{RHom}_{R/\mathfrak{q}}(\kappa(\mathfrak{p}/\mathfrak{q}), -)$ to $(\diamond)$ and invoking 6.5.19, 7.6.7, and 2.2.15 one gets isomorphisms,

$$\Sigma^{-1}\mathrm{H}(\mathsf{RHom}_{R/\mathfrak{q}}(\kappa(\mathfrak{p}/\mathfrak{q}), X_{\subseteq -(n+1)})) \;\cong\; (\mathrm{H}(\mathsf{RHom}_{R/\mathfrak{q}}(\kappa(\mathfrak{p}/\mathfrak{q}), X_{\supseteq -n})))_{\leqslant -(n+2)}\;.$$

The goal is to prove that the left-hand complex is zero, which we accomplish by showing that the $R/\mathfrak{q}$-complex $X_{\supseteq -n}$ has injective dimension at most n.

Let $\mathfrak{b}$ be an ideal in R that contains $\mathfrak{q}$. For $\mathfrak{b} = \mathfrak{q}$ one evidently has

$$(\flat) \qquad \begin{aligned}-\inf \mathsf{RHom}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\supseteq -n}) \;&=\; -\inf \mathsf{RHom}_{R/\mathfrak{q}}(R/\mathfrak{q}, X_{\supseteq -n})\\ &=\; -\inf X_{\supseteq -n}\\ &\leqslant\; n\;.\end{aligned}$$

For $\mathfrak{q} \subset \mathfrak{b}$ the definition of X together with 12.3.36, the isomorphism $(R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}) \cong R/\mathfrak{b}$, and the assumption on $\mathfrak{q}$ yields

$$(\dagger) \qquad -\inf \mathsf{RHom}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X) \;=\; -\inf \mathsf{RHom}_R(R/\mathfrak{b}, M) \;\leqslant\; n\,.$$

In particular, one has $\operatorname{Ext}^{n+1}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X) = 0$. Associated to $(\diamond)$ there is by 7.3.35 an exact sequence,

$$\begin{aligned}\operatorname{Ext}^{n}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\subseteq -(n+1)}) \longrightarrow \operatorname{Ext}^{n+1}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\supseteq -n})&\\ \longrightarrow \operatorname{Ext}^{n+1}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X)\,,&\end{aligned}$$

which yields

$$(\ddagger) \qquad \operatorname{Ext}^{n+1}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\supseteq -n}) \;=\; 0\,,$$

as $\operatorname{Ext}^{n}_{R/\mathfrak{q}}((R/\mathfrak{q})/(\mathfrak{b}/\mathfrak{q}), X_{\subseteq -(n+1)}) = 0$ holds by 7.6.7. Per 15.4.7 the desired inequality $\operatorname{id}_{R/\mathfrak{q}} X_{\supseteq -n} \leqslant n$ now follows from $(\flat)$ and $(\ddagger)$. $\square$

Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. It is an elementary fact, see 14.1.16(b), that $\operatorname{Tor}^R_v(\kappa(\mathfrak{p}), M)$ can be computed locally as $\operatorname{Tor}^{R_\mathfrak{p}}_v(\kappa(\mathfrak{p}), M_\mathfrak{p})$ and, essentially, that is why it follows straight from 17.3.1 that flat dimension can be detected locally, as recorded in 17.3.2. One cannot in the same fashion compute $\operatorname{Ext}^v_R(\kappa(\mathfrak{p}), M)$ locally; for example one has $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq 0$ by 7.3.28 but $\operatorname{Ext}^1_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = 0$ as $\mathbb{Q}$ is a field and hence of global dimension 0, see 8.5.2. As a consequence, the injective dimension can not in general be detected locally the way flat dimension can; see 17.5.14 but also 17.3.18. Per 14.1.33(d) the next result is the conceptual companion to 17.3.2. It shows that the injective dimension of a complex can be detected colocally; in particular, it does not grow under colocalization.

17.3.12 Corollary. *Let M be an R-complex*; *there are equalities,*

$$\begin{aligned}\operatorname{id}_R M \;&=\; \sup\{\operatorname{id}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \sup\{\operatorname{id}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\}\,.\end{aligned}$$

Proof. For prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ the complexes $\mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{q}_\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M))$ and $\mathsf{RHom}_R(\kappa(\mathfrak{q}), M)$ are isomorphic in $\mathcal{D}(R_\mathfrak{p})$ by 15.2.14. Now apply 17.3.11. $\square$

17.3.13 Corollary. *Let $\mathfrak{p}$ be a prime ideal in R and N an $R_\mathfrak{p}$-complex*; *one has*

$$\operatorname{id}_{R_\mathfrak{p}} N \;=\; \operatorname{id}_R N\,.$$

Proof. The inequality "$\leqslant$" holds by 17.3.12 and the last assertion in 14.1.21(b); the opposite inequality holds by 15.4.9, as $R_\mathfrak{p}$ per 1.3.42 is flat as an R-module. $\square$

The next corollary is conceptually dual to the Chouinard Formula for flat dimension 17.3.4, but the formula for injective dimension found in [51] involves localizations, not colocalizations; it is proved in 17.5.7.

17.3.14 Corollary. *Let M be an R-complex. If* $\operatorname{id}_R M$ *is finite, then one has*

$$\begin{aligned}\operatorname{id}_R M \;&=\; \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\}\,.\end{aligned}$$

Proof. For every prime ideal $\mathfrak{p}$ in R the $R_\mathfrak{p}$-complex $\mathsf{RHom}_R(R_\mathfrak{p}, M)$ has finite injective dimension, see e.g. 17.3.12. By 14.1.33(d) and 16.3.11 one now has

$$\begin{aligned} -\inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M) &= -\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M)) \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)\,, \end{aligned}$$

and by 17.1.14 this quantity is $-\infty$ if $\mathfrak{p}$ is not in $\operatorname{cosupp}_R M$. Now apply 17.3.11. □

17.3.15 Theorem. *Let F be a flat R-module and M an R-complex. One has*

$$\operatorname{id}_R \mathsf{RHom}_R(F, M) \leqslant \operatorname{id}_R M\,,$$

and equality holds if F is faithfully flat.

Proof. The inequality holds by 8.3.15(a). Assume now that F is faithfully flat and let $\mathfrak{p}$ be a prime ideal in R. By adjunction 12.3.8 one has

$$\mathsf{RHom}_R(\kappa(\mathfrak{p}), \mathsf{RHom}_R(F, M)) \simeq \mathsf{RHom}_R(F \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p}), M)\,.$$

The module $F \otimes_R \kappa(\mathfrak{p}) \simeq F \otimes^{\mathsf{L}}_R \kappa(\mathfrak{p})$ is a non-zero $\kappa(\mathfrak{p})$-vector space, see 15.1.6, so one has $F \otimes_R \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^{(U)}$ for some set $U \neq \emptyset$. By 12.2.2 and 3.1.23 one has $\inf \mathsf{RHom}_R(\kappa(\mathfrak{p})^{(U)}, M) = \inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M)$, so the asserted equality follows from 17.3.11. □

17.3.16 Theorem. *Let S be an R-algebra and M an R-complex. One has*

$$\operatorname{id}_S \mathsf{RHom}_R(S, M) \leqslant \operatorname{id}_R M\,,$$

and equality holds if S is faithfully flat as an R-module.

Proof. The inequality is a special case of 8.2.4. Assuming that S is faithfully flat as an R-module, the (in)equalities $\operatorname{id}_R M = \operatorname{id}_R \mathsf{RHom}_R(S, M) \leqslant \operatorname{id}_S \mathsf{RHom}_R(S, M)$ hold by 17.3.15 and 15.4.9. □

Injective Dimension vs. Localization

17.3.17 Proposition. *Let M be an R-complex. If M is in $\mathcal{D}_{\sqsubset}(R)$, then one has*

$$\begin{aligned} \operatorname{id}_R M &= \sup\{-\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{supp}_R M\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \operatorname{Ext}^m_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0\}\,. \end{aligned}$$

Proof. The second equality follows from 17.1.6, 16.2.23, and 7.3.24. To prove the first equality, note that "$\geqslant$" holds by 15.4.7, as one has

$$-\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) = -\inf \mathsf{RHom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \leqslant -\inf \mathsf{RHom}_R(R/\mathfrak{p}, M)$$

by 14.1.23 and 14.1.11(c). For the converse inequality, let $\mathfrak{a}$ be an ideal in R and m an integer with $\operatorname{Ext}^m_R(R/\mathfrak{a}, M) \neq 0$. The functor $\operatorname{Ext}^m_R(-, M)$ is half exact and R-linear, see 12.2.5, so by 12.4.9 there is a prime ideal $\mathfrak{p}$ with $\operatorname{Ext}^m_R(R/\mathfrak{p}, M)_\mathfrak{p} \neq 0$, and there is an isomorphism $\operatorname{Ext}^m_R(R/\mathfrak{p}, M)_\mathfrak{p} \cong \operatorname{Ext}^m_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ by 14.1.23. In view of 15.4.7 this proves the opposite inequality. □

The injective dimension of a complex with appropriately bounded homology can be measured locally; in particular, it does not grow under localization. The boundedness condition is necessary, see 17.5.14 which also shows that the boundedness condition in 17.3.17 is needed.

17.3.18 Corollary. *Let M be an R-complex. If M is in $\mathcal{D}_{\sqsubset}(R)$, then one has*

$$\begin{aligned} \operatorname{id}_R M &= \sup\{\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\,. \end{aligned}$$

Proof. By 5.3.26 the complex M has a bounded above semi-injective replacement. Thus, for every prime ideal $\mathfrak{p}$ in R one has $\operatorname{id}_R M \geqslant \operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p}$ by 15.4.7 and 14.1.29(a). Further, $\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant -\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ holds by another application of 15.4.7. Now invoke 17.3.17. □

Derived Complete Complexes

For an ideal $\mathfrak{a}$ in R and an R-complex M, the object $\mathsf{L}\Lambda^\mathfrak{a}(M)$ is an $\widehat{R}^\mathfrak{a}$-complex, see 11.3.4. The next result is dual to 15.4.23.

17.3.19 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex*; *one has*

$$\operatorname{id}_R \mathsf{L}\Lambda^\mathfrak{a}(M) = \operatorname{id}_{\widehat{R}^\mathfrak{a}} \mathsf{L}\Lambda^\mathfrak{a}(M)\,.$$

Proof. Recall from 13.1.27 that $\widehat{R}^\mathfrak{a}$ is flat as an R-module. In view of 13.4.17 the inequality "$\geqslant$" follows from 17.3.16, and the inequality "$\leqslant$" holds by 15.4.9. □

The next theorem computes, in particular, the injective dimension of a derived $\mathfrak{a}$-complete complex, which by theorem above is the same over R and $\widehat{R}^\mathfrak{a}$. The flat dimension of such a complex over R is computed in 15.4.25 and compared to the flat dimension over $\widehat{R}^\mathfrak{a}$ in 15.4.22 and 16.1.21.

17.3.20 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex*; *there is an equality,*

$$\begin{aligned} \operatorname{id}_R \mathsf{L}\Lambda^\mathfrak{a}(M) &= \sup\{-\inf \mathsf{RHom}_R(\kappa(\mathfrak{p}), M) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R M\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(\kappa(\mathfrak{p}), M) \neq 0\}\,. \end{aligned}$$

Proof. By 15.2.17 one has $\operatorname{cosupp}_R \mathsf{L}\Lambda^\mathfrak{a}(M) = \mathrm{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R M$. Per 15.2.16 the asserted equalities hold by 17.3.11 applied to the R-complex $\mathsf{L}\Lambda^\mathfrak{a}(M)$. □

17.3.21 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M an $\mathfrak{a}$-complete R-module. One has*

$$\operatorname{id}_R M = \sup\{m \in \mathbb{N}_0 \mid \exists\, \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon\ \operatorname{Ext}_R^m(\kappa(\mathfrak{p}), M) \neq 0\}\,.$$

Proof. By 13.1.33 the module M is derived $\mathfrak{a}$-complete, so 17.3.20 applies. □

The next corollary computes, in particular, the injective dimension of a derived $\mathfrak{m}$-complete complex.

17.3.22 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex; there are equalities,*

$$\mathrm{id}_R \,\mathsf{L}\Lambda^{\mathfrak{m}}(M) \;=\; -\inf \mathsf{RHom}_R(\boldsymbol{k}, M) \;=\; \sup\{m \in \mathbb{Z} \mid \mathrm{Ext}_R^m(\boldsymbol{k}, M) \neq 0\}\,.$$

Proof. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 17.3.20. $\square$

17.3.23 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an $\mathfrak{m}$-complete R-module. One has*

$$\mathrm{id}_R M \;=\; \sup\{m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(\boldsymbol{k}, M) \neq 0\}\,.$$

Proof. This is the special case $\mathfrak{a} = \mathfrak{m}$ of 17.3.21. $\square$

17.3.24 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. One has:*

$$\mathrm{id}_{\widehat{R}}(\widehat{R} \otimes_R M) \;=\; \mathrm{id}_R(\widehat{R} \otimes_R M) \;=\; \mathrm{id}_R M\,.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal and $\boldsymbol{k}$ the residue field of R. By 13.2.7 one has $\widehat{R} \otimes_R M \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(M)$ in $\mathcal{D}(\widehat{R})$, so the first equality holds by 17.3.19. The quantities $\mathrm{id}_R(\widehat{R} \otimes_R M)$ and $\mathrm{id}_R M$ are both equal to $-\inf \mathsf{RHom}_R(\boldsymbol{k}, M)$ by 17.3.22 and 16.4.8, and hence the second equality follows. $\square$

Projective Dimension

By 17.3.2 the flat dimension of a complex can be computed locally; in particular it can not grow under localization. The projective dimension can not grow under localization either, see 8.1.4, but the next example shows that it may not be possible to compute it locally, not even for modules over a ring of global dimension 1.

17.3.25 Example. Let $\mathbb{P}$ denote the set of prime numbers. Let M be the submodule $\mathbb{Z}\langle \frac{1}{p} \mid p \in \mathbb{P}\rangle$ of the $\mathbb{Z}$-module $\mathbb{Q}$. For every $p \in \mathbb{P}$ the module $M_{p\mathbb{Z}} = \mathbb{Z}_{p\mathbb{Z}}\langle \frac{1}{p}\rangle$ is a free $\mathbb{Z}_{p\mathbb{Z}}$-module, and one has $M_0 = \mathbb{Q} = \mathbb{Z}_0$, so $M_{\mathfrak{p}}$ is a free $\mathbb{Z}_{\mathfrak{p}}$-module for every $\mathfrak{p} \in \operatorname{Spec}\mathbb{Z} = \{0\} \cup \{p\mathbb{Z} \mid p \in \mathbb{P}\}$, but M is not a free $\mathbb{Z}$-module and hence not projective, see 1.3.21.

For finitely generated modules the flat and projective dimensions agree, whence the latter can also be computed locally.

17.3.26 Corollary. *Let M be an R-complex. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then one has*

$$\begin{aligned}\mathrm{pd}_R M \;&=\; \sup\{\mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \sup\{\mathrm{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\,.\end{aligned}$$

Proof. In view of 14.1.11 the equality follows from 15.4.18 and 17.3.2. $\square$

Finiteness of a homological dimension locally at every prime does not per se guarantee global finiteness, but for the projective dimension of finitely generated modules it does. This is a theorem due to Bass and Murthy [33].

17.3.27 Theorem. *Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$; the next conditions are equivalent.*

(*i*) $\mathrm{pd}_R M$ *is finite.*

(*ii*) $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}$ *is finite for every* $\mathfrak{p} \in \operatorname{Spec} R$.
(*iii*) $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}$ *is finite for every* $\mathfrak{p} \in \operatorname{supp}_R M$.

Proof. The projective dimension of an acyclic complex is $-\infty$, so conditions (*ii*) and (*iii*) are equivalent by 14.1.12 and 15.1.9. Further, (*i*) implies (*ii*) by 17.3.26; we proceed to prove the converse. Assume without loss of generality that M is not acyclic, and let P be a degreewise finitely generated semi-projective replacement of M, see 5.2.16. For every integer n set

$$\mathfrak{a}_n = (0 :_R \operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P)))\,;$$

by 12.2.6 these Ext modules are finitely generated, so 14.1.1 yields

$$\operatorname{Supp}_R \operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P)) = \mathrm{V}(\mathfrak{a}_n)\,.$$

For every prime ideal $\mathfrak{p}$ in R the complex $P_\mathfrak{p}$ is a semi-projective replacement of $M_\mathfrak{p}$, see 14.1.27, and one has $\operatorname{Ext}^1_R(\mathrm{C}_n(P), \mathrm{C}_{n+1}(P))_\mathfrak{p} \cong \operatorname{Ext}^1_{R_\mathfrak{p}}(\mathrm{C}_n(P_\mathfrak{p}), \mathrm{C}_{n+1}(P_\mathfrak{p}))$ by 14.1.23 and 2.2.19. By assumption $s = \sup M$ is an integer. For $n \geqslant s$ it follows from 14.1.11(c) and 15.4.1 that one has $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}_n)$ if and only if $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p} > n$ holds. Subsequently, these classic support sets form a descending chain,

$$(\star)\qquad \mathrm{V}(\mathfrak{a}_s) \supseteq \mathrm{V}(\mathfrak{a}_{s+1}) \supseteq \cdots \supseteq \mathrm{V}(\mathfrak{a}_n) \supseteq \mathrm{V}(\mathfrak{a}_{n+1}) \supseteq \cdots\,,$$

and $\bigcap_{n\geqslant s} \mathrm{V}(\mathfrak{a}_n) = \varnothing$ holds by the assumption that $\mathrm{pd}_{R_\mathfrak{p}} M_\mathfrak{p}$ is finite for every $\mathfrak{p}$. For $n \geqslant s$ set $\mathfrak{r}_n = \bigcap_{\mathfrak{p}\in\mathrm{V}(\mathfrak{a}_n)} \mathfrak{p}$. The ascending chain $\mathfrak{r}_s \subseteq \mathfrak{r}_{s+1} \subseteq \cdots \subseteq \mathfrak{r}_n \subseteq \mathfrak{r}_{n+1} \subseteq \cdots$ of ideals becomes stationary; that is, there exists an m with $\mathfrak{r}_m = \bigcup_{n\geqslant s} \mathfrak{r}_n$. As $\mathfrak{r}_n$ is the radical of $\mathfrak{a}_n$ one has $\mathrm{V}(\mathfrak{a}_n) = \mathrm{V}(\mathfrak{r}_n)$, so also the sequence $(\star)$ becomes stationary at m. That is, one has $\mathrm{V}(\mathfrak{a}_m) = \varnothing$ and, therefore, $\operatorname{Ext}^1_R(\mathrm{C}_m(P), \mathrm{C}_{m+1}(P)) = 0$ and $\mathrm{pd}_R M \leqslant m$ by 15.4.1. □

Remark. As a topological space, $\operatorname{Spec} R$ is compact, i.e. if the intersection of a family $\{V^u\}_{u\in U}$ of closed subsets is empty, then already the intersection of a finite subfamily is empty. Eisenbud [78] and Matsumura [182] both leave this as an exercise; it is essentially solved in the proof above.

The boundedness condition in 17.3.27 is necessary.

17.3.28 Example. Consider the graded $\mathbb{Z}$-module M with

$$M_v = \begin{cases} \mathbb{Z}/v\mathbb{Z} & \text{if } v \text{ is a prime} \\ 0 & \text{if } v \text{ is not a prime}\,. \end{cases}$$

As there are infinitely many primes one has $\sup M = \infty$ and, therefore, $\mathrm{pd}_\mathbb{Z} M = \infty$ per 8.1.3. Nevertheless, for every prime p the complex $M_{p\mathbb{Z}}$ is concentrated in degree p, see 14.1.2, and hence $\mathrm{pd}_{\mathbb{Z}_{p\mathbb{Z}}} M_{p\mathbb{Z}} \leqslant p+1$ holds by 8.5.2 as $\mathbb{Z}_{p\mathbb{Z}}$ is a principal ideal domain.

17.3.29 Proposition. *Let M be an R-complex. If M is in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$, then one has*

$$\begin{aligned} \mathrm{pd}_R M &= \sup\{-\inf \mathsf{R}\mathrm{Hom}_R(M, \kappa(\mathfrak{p})) \mid \mathfrak{p} \in \operatorname{supp}_R M\} \\ &= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R\colon\ \operatorname{Ext}^m_R(M, \kappa(\mathfrak{p})) \neq 0\}\,. \end{aligned}$$

Proof. Let $\mathfrak{p}$ be a prime ideal in R. In the next computation the first equality holds by 14.1.16(b), the second and third equalities follow from 16.4.1 and 14.1.33(c).

$$\begin{aligned}\sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) &= \sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} M_\mathfrak{p}) \\ &= -\inf \mathsf{RHom}_{R_\mathfrak{p}}(M_\mathfrak{p}, \kappa(\mathfrak{p})) \\ &= -\inf \mathsf{RHom}_R(M, \kappa(\mathfrak{p}))\,.\end{aligned}$$

As $\mathrm{pd}_R M = \mathrm{fd}_R M$ holds by 15.4.18, the asserted equalities follow from 17.3.1. □

Exercises

E 17.3.1 Let S be an R-algebra and I an R-module. Show that if S is faithfully flat as an R-module, $\mathrm{Hom}_R(S, I)$ is an injective S-module, and $\mathrm{Ext}^m_R(S, I) = 0$ holds for all $m > 0$, then I is an injective R-module. See also E 16.1.8.

E 17.3.2 Let S be an R-algebra, F a flat S-module, and M an R-complex. Show that there is an inequality $\mathrm{id}_S \mathsf{RHom}_R(F, M) \leqslant \mathrm{id}_R M$ and equality holds if F is faithfully flat over R.

E 17.3.3 Let $(R, \mathfrak{m})$ be local and M a derived $\mathfrak{m}$-torsion complex of finite flat dimension. Show that there is an equality $\mathrm{fd}_R M = \operatorname{depth} R + \sup M$ holds.

E 17.3.4 Let M be an R-complex. Show that there are equalities,

$$\begin{aligned}\mathrm{id}_R M &= \sup\{-\inf \mathsf{RHom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{-\inf(\mathsf{RHom}_R(R/\mathfrak{p}, M) \otimes^{\mathsf{L}}_{R/\mathfrak{p}} \kappa(\mathfrak{p})) \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.\end{aligned}$$

Derive from these equalities 17.3.17 for M in to $\mathcal{D}_{\sqsubset}(R)$.

E 17.3.5 Let $(R, \mathfrak{m})$ be local and M a complex derived $\mathfrak{m}$-complete complex of finite injective dimension. Show that there is an equality $\mathrm{id}_R M = \operatorname{depth} R - \inf M$ holds.

E 17.3.6 Derive the Bass Formula 16.4.11 from the Chouinard Formula 17.3.4.

E 17.3.7 Show that one has $\mathrm{FFD}\, R = \sup\{\mathrm{FFD}\, R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$.

E 17.3.8 Show that one has $\mathrm{FFD}\, R = \sup\{\operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$.

17.4 Finitistic Dimensions and Gorenstein Rings

Synopsis. Gorenstein (local) ring; Bass series; finitistic injective/flat/projective dimension; ~ vs. depth; ~ vs. Krull dimension.

Recall from 8.5.16 that the finitistic dimensions capture the suprema of homological dimensions of modules of finite homological dimension. Recall also from Sect. 15.4 that $\mathrm{id}\, R$ is the notation for the injective dimension of R as an R-module.

17.4.1 Theorem. *There are (in)equalities,*

$$\mathrm{FID}\, R = \mathrm{FFD}\, R \leqslant \mathrm{FPD}\, R \leqslant \mathrm{id}\, R \leqslant \mathrm{gldim}\, R\,,$$

and if one of the quantities $\mathrm{id}\, R$ *or* $\mathrm{gldim}\, R$ *is finite, then equality holds everywhere to the left of it.*

Proof. The (in)equalities hold by 8.5.28, and the statements about equalities follow from 8.5.21 and 8.5.27. □

In this section we compare the invariants in 17.4.1 to depth and Krull dimension. We start with the injective dimension of the ring. Many of the results included here were proved in a few papers by Auslander and Buchsbaum [11, 12] and Bass [30].

Gorenstein Rings

17.4.2 Definition. A local ring R is called *Gorenstein* if it is Iwanaga–Gorenstein per 8.5.29; that is, if $\operatorname{id} R$ is finite.

Remark. Over a local ring R one can define that a complex M in $\mathcal{D}^{\mathrm{f}}(R)$ is Gorenstein if the equality $\operatorname{id}_R M = \operatorname{depth}_R M$ holds; in this way R is Gorenstein if it is Gorenstein as an R-module, cf. the Bass Formula 16.4.12. Such an approach would be analogous to 17.2.5/17.2.15.

The existence of Gorenstein modules over Cohen–Macaulay local rings was addressed in [90] and by Reiten [209] and Sharp [228]. When they exist, they are direct sums of copies of a dualizing module for R; see also the Remark after 18.2.2 and E 18.2.8.

17.4.3 Example. Let $n \in \mathbb{N}$. If p is a prime, then the Artinian local ring $\mathbb{Z}/p^n\mathbb{Z}$ is self-injective, see 8.2.10, and hence Gorenstein. Similarly, if $\Bbbk$ is a field, then the Artinian local ring $\Bbbk[x]/(x^n)$ is Gorenstein.

17.4.4 Proposition. *Let R be local. If R is Gorenstein, then R is Cohen–Macaulay and* $\operatorname{id} R = \dim R$ *holds.*

Proof. By 16.4.10 and the Bass Formula 16.4.12 one has $\dim R \leqslant \operatorname{id} R = \operatorname{depth} R$. The assertions now follow from 17.2.16. $\square$

17.4.5 Corollary. *A Gorenstein local ring is equidimensional and catenary.*

Proof. The assertions are immediate from 17.4.4 and 17.2.21. $\square$

Notice from 16.4.9 or 17.3.18 that if R is a Gorenstein local ring, then so is $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ in R. This justifies the next definition, which extends 17.4.2 and applies to non-local rings.

17.4.6 Definition. If the local ring $R_{\mathfrak{p}}$ is Gorenstein for every prime ideal $\mathfrak{p}$ in R, then R is called *Gorenstein*.

17.4.7 Proposition. *A Gorenstein ring is Cohen–Macaulay and hence catenary.*

Proof. The assertions follow from 17.4.6, 17.4.4, 17.2.25, and 17.2.28. $\square$

17.4.8 Example. If $\operatorname{id} R$ is finite, then it is immediate from 17.3.18 that R is Gorenstein. By 8.2.10 the ring $\mathbb{Z}/n\mathbb{Z}$ is, therefore, Gorenstein for every $n > 1$.

17.4.9 Proposition. *If R is Gorenstein, then the equality* $\operatorname{id} R = \dim R$ *holds.*

Proof. Per 17.3.18 and 14.2.7 the assertion follows from the local case 17.4.4. $\square$

While rings of finite self-injective dimension are Gorenstein, see 17.3.18, a Gorenstein ring may have infinite self-injective dimension, see 20.2.21.

17.4.10 Theorem. *If R is Gorenstein, then there are* (*in*)*equalities,*

$$\operatorname{FID} R \;=\; \operatorname{FFD} R \;=\; \operatorname{FPD} R \;=\; \operatorname{id} R \;=\; \dim R \;\leqslant\; \operatorname{gldim} R\;;$$

equality holds if gldim R *is finite.*

Proof. Assuming first that $\dim R$ is finite, the (in)equalities hold by 17.4.9 and 17.4.1; by another application of 17.4.1 equality holds if $\operatorname{gldim} R$ is finite. Assume now that R has infinite Krull dimension. For every prime ideal $\mathfrak{p}$ in R, one has $\dim R_\mathfrak{p} = \operatorname{id} R_\mathfrak{p} = \operatorname{id}_R R_\mathfrak{p} \leqslant \operatorname{FID} R$ by 17.4.9, 17.3.13, and 8.5.16. Thus, one has $\operatorname{FID} R = \infty$ and it follows from 17.4.1 that all six quantities are infinite. □

Bass Series of a Gorenstein Local Ring

The following characterization of Gorenstein local rings is complemented by 18.3.14 and 18.4.23.

17.4.11 Proposition. *Let R be local. The following conditions are equivalent.*

(*i*) *R is Gorenstein.*
(*ii*) *The Bass series $\mathrm{I}_R(t)$ is a monomial.*
(*iii*) *One has $\mathrm{I}_R(t) = t^{\dim R}$.*

Proof. If R is Gorenstein, then 17.4.4 and 17.2.16 yield $\operatorname{depth} R = \dim R = \operatorname{id} R$, so one has $\mathrm{I}_R(t) = t^{\dim R}$ by 16.4.30. In particular, $\mathrm{I}_R(t)$ is a monomial, and that implies by another application of 16.4.30 that $\operatorname{id} R$ is finite, so R is Gorenstein. □

The minimal injective resolution of a Gorenstein ring was first computed by Bass in the seminal paper [32].

17.4.12 Example. Let R be Gorenstein and $R \xrightarrow{\simeq} I$ a minimal injective resolution, see B.26. Given a prime ideal $\mathfrak{p}$ in R, the local ring $R_\mathfrak{p}$ has Bass series $\mathrm{I}_{R_\mathfrak{p}}(t) = t^{\dim R_\mathfrak{p}}$ by 17.4.11. Thus one has $\mu_{R_\mathfrak{p}}^{\dim R_\mathfrak{p}}(R_\mathfrak{p}) = 1$ and $\mu_{R_\mathfrak{p}}^{v}(R_\mathfrak{p}) = 0$ for $v \neq \dim R_\mathfrak{p}$. Now it follows from 16.4.37 that the modules in the complex I have the form

$$I_{-v} \;=\; \coprod_{\dim R_\mathfrak{p} = v} \mathrm{E}_R(R/\mathfrak{p})\;.$$

17.4.13 Theorem. *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{M})$ be local rings such that S is an R-algebra and flat as an R-module. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then there is an equality of Laurent series,*

$$\mathrm{I}_S(t) \;=\; \mathrm{I}_{S/\mathfrak{m}S}(t)\,\mathrm{I}_R(t)\;.$$

In particular, the following conditions are equivalent.

(*i*) *S is Gorenstein.*
(*ii*) *R and $S/\mathfrak{m}S$ are Gorenstein.*

Proof. The equality of Laurent series is the special case $M = R$ of 16.4.35. Since Bass series have non-negative coefficients, it follows that $\mathrm{I}_S(t)$ is a monomial if and only if $\mathrm{I}_{S/\mathfrak{m}S}(t)$ and $\mathrm{I}_R(t)$ are monomials, see 16.4.15. The asserted equivalence now holds by 17.4.11. □

EXTENSIONS OF GORENSTEIN RINGS

Gorenstein rings are special Cohen–Macaulay rings, and just like power series and polynomial extensions of Cohen–Macaulay rings are Cohen–Macualay, see 17.2.33, one can similarly generate Gorenstein rings from Gorenstein rings. We record these results right away and resume the study of finitistic dimensions in 17.4.18.

17.4.14 Lemma. *Let $(R,\mathfrak{m})$ be local and $x \in \mathfrak{m}$ an R-regular element. The ring $R/(x)$ is Gorenstein if and only if R is Gorenstein.*

PROOF. Let $\boldsymbol{k}$ be the common residue field of R and $R/(x)$. In $\mathcal{D}(R/(x))$ one has $R/(x) \otimes^{\mathsf{L}}_R \boldsymbol{k} \simeq \mathrm{K}^R(x) \otimes_R \boldsymbol{k} \simeq \Sigma\boldsymbol{k} \oplus \boldsymbol{k}$ by 16.4.23 and 16.4.25. This explains the first isomorphism in the computation below. The second isomorphism holds by 12.3.32, and the remaining isomorphisms come from 16.4.23, the unitor 12.3.3, and tensor evaluation 12.3.23(c).

$$\begin{aligned}\mathsf{RHom}_{R/(x)}(\Sigma\boldsymbol{k} \oplus \boldsymbol{k}, R/(x)) &\simeq \mathsf{RHom}_{R/(x)}(R/(x) \otimes^{\mathsf{L}}_R \boldsymbol{k}, R/(x))\\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, R/(x))\\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, \mathrm{K}^R(x))\\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, R) \otimes^{\mathsf{L}}_R \mathrm{K}^R(x)\,.\end{aligned}$$

As the functor RHom is additive and the complex $\mathsf{RHom}_R(\boldsymbol{k}, R)$ by 12.3.34 belongs to $\mathcal{D}^{\mathrm{f}}(R)$, it follows from 14.3.5(b) that $-\inf \mathsf{RHom}_R(\boldsymbol{k}, R)$ is finite if and only if $-\inf \mathsf{RHom}_{R/(x)}(\boldsymbol{k}, R/(x))$ is finite. Now invoke 16.4.8 and 17.4.2. □

17.4.15 Proposition. *Let $\boldsymbol{x}$ be an R-regular sequence. If R is Gorenstein, then the ring $R/(\boldsymbol{x})$ is Gorenstein*; *the converse holds if $\boldsymbol{x}$ is in the Jacobson radical of R.*

PROOF. By induction it suffices to handle the case $\boldsymbol{x} = x$ of a sequence of length one. Assume first that R is Gorenstein. A prime ideal in $R/(x)$ has the form $\mathfrak{p}/(x)$, where $\mathfrak{p}$ is a prime ideal in R that contains x. It follows from 14.4.27 that $\frac{x}{1}$ is an $R_{\mathfrak{p}}$-regular element, so by 17.4.14 the ring $R_{\mathfrak{p}}/(\frac{x}{1}) \cong (R/(x))_{\mathfrak{p}/(x)}$ is Gorenstein. Thus, $R/(x)$ is Gorenstein. Assume now that x is contained in the Jacobson radical of R and that $R/(x)$ is Gorenstein. Every prime ideal $\mathfrak{p}$ is contained in a maximal ideal $\mathfrak{m}$, and 17.3.18 yields $\operatorname{id} R_{\mathfrak{p}} \leqslant \operatorname{id} R_{\mathfrak{m}}$, so it suffices to verify that $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ in R. As x belongs to every such ideal and $(R/(x))_{\mathfrak{m}/(x)} \cong R_{\mathfrak{m}}/(\frac{x}{1})$ is Gorenstein, it follows from 17.4.14 that $R_{\mathfrak{m}}$ is Gorenstein, since $\frac{x}{1}$ by 14.4.27 is an $R_{\mathfrak{m}}$-regular element. □

17.4.16 Proposition. *The following conditions are equivalent.*

(*i*) *R is Gorenstein.*
(*ii*) *The polynomial algebra $R[x_1,\dots,x_n]$ is Gorenstein.*
(*iii*) *The power series algebra $R[\![x_1,\dots,x_n]\!]$ is Gorenstein.*

PROOF. By recursion, it suffices to handle the case $n = 1$; set $x = x_1$.

(*i*) ⇔ (*iii*): The indeterminate x is an $R[\![x]\!]$-regular element and it belongs by 12.1.25 to the Jacobson radical of $R[\![x]\!]$. Now invoke 17.4.15.

$(ii) \Rightarrow (i)$: As above the indeterminate x is $R[x]$-regular, so $R[x]$ being Gorenstein implies by 17.4.15 that R is Gorenstein.

$(i) \Rightarrow (ii)$: Let $\mathfrak{P}$ be a prime ideal in $R[x]$; with $\mathfrak{p} = \mathfrak{P} \cap R$ one has $R[x]_{\mathfrak{P}} \cong (R_{\mathfrak{p}}[x])_{\mathfrak{Q}}$ where the prime ideal $\mathfrak{Q}$ is the contraction $\mathfrak{P}_{\mathfrak{P}} \cap R_{\mathfrak{p}}[x]$. Notice that the $R_{\mathfrak{p}}$-algebra $(R_{\mathfrak{p}}[x])_{\mathfrak{Q}}$ is flat as an $R_{\mathfrak{p}}$-module by 12.1.24, 1.3.42, and 5.4.24(b). To see that the local ring $R[x]_{\mathfrak{P}} \cong (R_{\mathfrak{p}}[x])_{\mathfrak{Q}}$ is Gorenstein it suffices, since $R_{\mathfrak{p}}$ is Gorenstein, to verify that the quotient ring $(R_{\mathfrak{p}}[x])_{\mathfrak{Q}}/\mathfrak{p}_{\mathfrak{p}}(R_{\mathfrak{p}}[x])_{\mathfrak{Q}}$ is Gorenstein, see 17.4.13. This quotient ring is a localization of the polynomial algebra $Q = \kappa(\mathfrak{p})[x]$, which is a principal ideal domain and hence of global dimension 1, see 8.5.2. Thus, for every prime ideal $\mathfrak{q}$ in Q one has $\operatorname{id} Q_{\mathfrak{q}} \leqslant \operatorname{id} Q \leqslant 1$ by 17.3.18 and 8.5.3, so $(R_{\mathfrak{p}}[x])_{\mathfrak{Q}}/\mathfrak{p}_{\mathfrak{p}}(R_{\mathfrak{p}}[x])_{\mathfrak{Q}}$ is Gorenstein. This shows that $R[x]_{\mathfrak{P}}$ is Gorenstein, whence $R[x]$ is Gorenstein. □

17.4.17 Example. Let $\Bbbk$ be a field. Per 1.3.28 every $\Bbbk$-module is injective, so $\Bbbk$ is Gorenstein and by 17.4.16 so are the $\Bbbk$-algebras $\Bbbk[x_1, \dots, x_n]$ and $\Bbbk[\![x_1, \dots, x_n]\!]$.

Finitistic Injective Dimension and Finitistic Flat Dimension

17.4.18 Proposition. *There are equalities,*

$$\operatorname{FID} R = \sup\{\operatorname{FID} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \quad \textit{and} \quad \operatorname{FFD} R = \sup\{\operatorname{FFD} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.$$

Proof. Since the finitistic injective dimenson and finitistic flat dimension agree, see 17.4.1, it suffices to prove the second equality. Let $\mathfrak{p}$ be a prime ideal in R, for every $R_{\mathfrak{p}}$-module N one has $\operatorname{fd}_{R_{\mathfrak{p}}} N = \operatorname{fd}_R N$ by 17.3.3; this proves the inequality "$\geqslant$". For an R-module M of finite flat dimension, 17.3.2 yields a prime ideal $\mathfrak{p}$ in R with $\operatorname{fd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{fd}_R M$; this proves the opposite inequality. □

Remark. Example 17.3.25 shows that the argument above does not readily apply to show that the finitistic projective dimension can be computed locally. It actually can be computed locally, simply because it agrees with the Krull dimension, as discussed before 17.4.25.

17.4.19 Theorem. *There are equalities,*

$$\operatorname{FID} R \;=\; \operatorname{FFD} R \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.$$

Proof. The first equality is known from 17.4.1. For every R-module M and every prime ideal $\mathfrak{p}$ in R one has $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geqslant 0$ by 16.2.16. For an R-module M of finite flat dimension, 17.3.4 yields

$$\begin{aligned}\operatorname{fd}_R M \;&=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &\leqslant\; \sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.\end{aligned}$$

To prove the opposite inequality, fix a prime ideal $\mathfrak{p}$ in R and set $d = \operatorname{depth} R_{\mathfrak{p}}$. Choose a maximal $R_{\mathfrak{p}}$-regular sequence $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{p}_{\mathfrak{p}}$, see 16.2.33(b), and consider the finitely generated $R_{\mathfrak{p}}$-module $M = R_{\mathfrak{p}}/(\boldsymbol{x})$. It now follows from 17.3.3, 15.4.18, and 16.4.23 that one has $\operatorname{fd}_R M = \operatorname{fd}_{R_{\mathfrak{p}}} M = \operatorname{pd}_{R_{\mathfrak{p}}} M = \operatorname{depth} R_{\mathfrak{p}}$. □

17.4.20 Corollary. *If R is Cohen–Macaulay, then there are (in)equalities,*

$$\operatorname{FID} R \;=\; \operatorname{FFD} R \;=\; \dim R \;\leqslant\; \operatorname{FPD} R \;\leqslant\; \operatorname{id} R \;\leqslant\; \operatorname{gldim} R\;;$$

if one of the quantities $\operatorname{id} R$ *or* $\operatorname{gldim} R$ *is finite, then equality holds everywhere to the left of it.*

Proof. For every prime ideal $\mathfrak{p}$ in R one has $\operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$, so 17.4.19 yields $\operatorname{FFD} R = \dim R$. Given this equality, the assertion is a restatement of 17.4.1. □

The next result, as well as the proof we give here, is due to Bass [30].

17.4.21 Lemma. *For every integer d with $2 \leqslant d \leqslant \dim R$ there exist prime ideals $\mathfrak{p} \subset \mathfrak{m}$ in R, a sequence $x_1, \ldots, x_{d-1}$ in $\mathfrak{p}$, and an element $s \in \mathfrak{m} \setminus \mathfrak{p}$, such that the induced sequence $\frac{x_1}{1}, \ldots, \frac{x_{d-1}}{1}$ is $R_{\mathfrak{p}}$- and $\{s^n \mid n \geqslant 0\}^{-1}R$-regular.*

Proof. We proceed by induction on d. For $d = 2$ let $\mathfrak{Q}$ be a prime ideal in R with $\dim R_{\mathfrak{Q}} = 1$ and $\dim R/\mathfrak{Q} \geqslant 1$ and let $\mathfrak{P}$ a minimal prime ideal contained in $\mathfrak{Q}$. Let $\mathfrak{I}$ be the intersection of all the associated prime ideals $\mathfrak{q}$ of R with $\dim R_{\mathfrak{q}} \geqslant 1$. As $\mathfrak{I} \cap \mathfrak{Q}$ is an intersection of finitely many prime ideals not contained in $\mathfrak{P}$, one can choose an element $t \in (\mathfrak{I} \cap \mathfrak{Q}) \setminus \mathfrak{P}$. Let $\mathfrak{m}$ be a maximal ideal in R that contains $\mathfrak{Q}$; by the choices of $\mathfrak{Q}$ and $\mathfrak{P}$ one has $\dim R_{\mathfrak{m}} \geqslant 2$. Krull's principal ideal theorem, see also 18.4.19, applied to the elements of the maximal ideal $\mathfrak{m}/\mathfrak{P}$ of the integral domain $R/\mathfrak{P}$, implies that every element in $\mathfrak{m} \setminus \mathfrak{P}$ is contained in a prime ideal $\mathfrak{p}$ with $\mathfrak{P} \subset \mathfrak{p} \subset \mathfrak{m}$, so it follows from Prime Avoidance there are infinitely many such prime ideals in R. Choose a prime ideal $\mathfrak{p}$ with $\mathfrak{P} \subset \mathfrak{p} \subset \mathfrak{m}$ that does not contain t; this is possible as $t \notin \mathfrak{P}$ and $R/(t)$ has only finitely many minimal prime ideals. One has $\dim R_{\mathfrak{p}} \geqslant 1$, but $\mathfrak{p}$ is not one of the associated prime ideals $\mathfrak{q}$ described above, for they all contain t. It follows that the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ is not in $\operatorname{Ass} R_{\mathfrak{p}}$, so $R_{\mathfrak{p}}$ is by 16.2.18 a local ring of positive depth. Now 16.2.33(b) yields the existence of an element $x \in \mathfrak{p}$ such that $\frac{x}{1}$ is $R_{\mathfrak{p}}$-regular.

Let $\mathfrak{A}$ be the intersection of all the prime ideals $\mathfrak{r}$ in $\operatorname{Ass} R$ that contain x. No such prime can be contained in $\mathfrak{p}$, as that would make $\frac{x}{1}$ a zero divisor in $R_{\mathfrak{p}}$. Thus $\mathfrak{A}$ is not contained in $\mathfrak{p}$. Now, choose an element $u \in \mathfrak{A} \setminus \mathfrak{p}$ and set $s = tu$; the choices of t and u yield $s \in \mathfrak{m} \setminus \mathfrak{p}$. Set $S = \{s^n \mid n \geqslant 0\}^{-1}R$. As x is in $\mathfrak{p}$ and s is not, $\frac{x}{1}$ is not a unit in S; it is also not a zerodivisor as $\mathfrak{r}S = S$ holds for every prime ideal $\mathfrak{r} \in \operatorname{Ass} R$ that contains x. Thus $\frac{x}{1}$ is an S-regular element, see 14.4.17.

Now let $d \geqslant 3$ and choose a prime ideal $\mathfrak{Q}$ with $\dim R_{\mathfrak{Q}} = d-1$ and $\dim R/\mathfrak{Q} \geqslant 1$. By the induction hypothesis applied to $R_{\mathfrak{Q}}$ there is, in particular, a prime ideal $\mathfrak{P} \subset \mathfrak{Q}$ and a sequence $x_1, \ldots, x_{d-2}$ in $\mathfrak{P}$ that induces an $R_{\mathfrak{P}}$-regular sequence. Notice that $\dim R_{\mathfrak{P}} = d-2$ holds: The inequality "$\leqslant$" is clear from the choices of $\mathfrak{Q}$ and $\mathfrak{P}$; the existence of an $R_{\mathfrak{P}}$-regular sequence of length $d-2$ yields the opposite inequalty, see 16.2.33(a) and 17.2.16. For every $i \in \{1, \ldots, d-2\}$ let $\mathfrak{A}_i$ be the intersection of the prime ideals $\mathfrak{r}$ in $\mathrm{V}(x_1, \ldots, x_{i-1})$ such that $\mathfrak{r}/(x_1, \ldots, x_{i-1})$ belongs to $\operatorname{Ass} R/(x_1, \ldots, x_{i-1})$ and contains x_i. If such an ideal $\mathfrak{A}_i$ were contained in $\mathfrak{P}$, then one of the associated prime ideals $\mathfrak{r}$ described above would be contained in $\mathfrak{P}$. It follows that the element $\frac{x_i}{1} \in R_{\mathfrak{P}}$ whould be contained in an associated prime ideal of $R_{\mathfrak{P}}/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$, so multiplication by $\frac{x_i}{1}$ on $R_{\mathfrak{P}}/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$ would not be injective, which contradicts the $R_{\mathfrak{P}}$-regularity of the sequence $\frac{x_1}{1}, \ldots, \frac{x_{d-2}}{1}$. Thus, none of the ideals $\mathfrak{A}_i$ is contained in $\mathfrak{P}$. Let let $\mathfrak{I}$ be the intersection of all prime ideals $\mathfrak{q}$ in $\mathrm{V}(x_1, \ldots, x_{d-2})$ such that $\mathfrak{q}/(x_1, \ldots, x_{d-2})$ belongs to $\operatorname{Ass} R/(x_1, \ldots, x_{d-2})$ and $\dim R_{\mathfrak{q}} \geqslant d-1$ holds. As one has $\dim R_{\mathfrak{P}} = d-2 < \dim R_{\mathfrak{q}}$, no such prime ideal $\mathfrak{q}$ is contained in $\mathfrak{P}$, and it follows that $\mathfrak{I}$ is not contained in $\mathfrak{P}$. Now the ideal $(\bigcap_{i=1}^{d-2} \mathfrak{A}_i) \cap \mathfrak{I} \cap \mathfrak{Q}$ is an intersection of finitely many prime ideals none of which are contained in $\mathfrak{P}$, so one can choose an element $t \in ((\bigcap_{i=1}^{d-2} \mathfrak{A}_i) \cap \mathfrak{I} \cap \mathfrak{Q}) \setminus \mathfrak{P}$. The choices of $\mathfrak{Q}$ and $\mathfrak{P}$ yield $\dim R/\mathfrak{P} \geqslant 2$. Let $\mathfrak{m}$ be a maximal ideal that contains $\mathfrak{Q}$. As argued in the base case, one can now choose a prime ideal $\mathfrak{p}$ with $\mathfrak{P} \subset \mathfrak{p} \subset \mathfrak{m}$ such that $\mathfrak{p}$ does not contain t. One has $\dim R_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{P}} + 1 = d-1$, but $\mathfrak{p}$ is not one of the associated prime ideals $\mathfrak{q}$ described above, for they all contain t. It follows that the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$ is not an associated prime ideal of the $R_{\mathfrak{p}}$-module $(R/(x_1, \ldots, x_{d-2}))_{\mathfrak{p}} \cong R_{\mathfrak{p}}/(\frac{x_1}{1}, \ldots, \frac{x_{d-2}}{1})$. By 16.2.18 and 16.2.33(b) there is thus an element x_{d-1} in $\mathfrak{p}$ such that $\frac{x_{d-1}}{1}$ is regular for $R_{\mathfrak{p}}/(\frac{x_1}{1}, \ldots, \frac{x_{d-2}}{1})$. To see that the sequence $\frac{x_1}{1}, \ldots, \frac{x_{d-1}}{1}$ is $R_{\mathfrak{p}}$-regular, notice that $\frac{t}{1}$ is a unit in $R_{\mathfrak{p}}$. For $i \leqslant d-2$ it thus follows from the choice of t that for every prime ideal $\mathfrak{r}$ in $\mathrm{V}(x_1, \ldots, x_{i-1})$ such that $\mathfrak{r}/(x_1, \ldots, x_{i-1})$ belongs to $\operatorname{Ass} R/(x_1, \ldots, x_{i-1})$ and contains x_i one has $\mathfrak{r}R_{\mathfrak{p}} = R_{\mathfrak{p}}$, whence $\frac{x_i}{1}$ is regular for $R_{\mathfrak{p}}/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$.

Let $\mathfrak{A}_{d-1}$ be the intersection of all prime ideals $\mathfrak{r}$ in $\mathrm{V}(x_1, \ldots, x_{d-2})$ such that $\mathfrak{r}/(x_1, \ldots, x_{d-2})$ belongs to $\operatorname{Ass} R/(x_1, \ldots, x_{d-2})$ and contains x_{d-1}. As argued above, no such ideal $\mathfrak{r}$ can be contained in $\mathfrak{p}$, whence $\mathfrak{A}_{d-1}$ is not contained in $\mathfrak{p}$. Choose an element $u \in \mathfrak{A}_{v-1} \setminus \mathfrak{p}$ and set $s = tu$; it is an element in $\mathfrak{m} \setminus \mathfrak{p}$. In the ring $S = \{s^n \mid n \geqslant 0\}^{-1}R$, the elements $\frac{x_1}{1}, \ldots, \frac{x_{d-1}}{1}$ belong to the prime ideal $\mathfrak{p}S \neq S$, so to see that they form a regular sequence it suffices to verify that multiplication by $\frac{x_i}{1}$ on $S/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$ is injective. For $i \leqslant d-2$ the choice of t ensures that s belongs to $\mathfrak{A}_i$ and hence to every prime ideal $\mathfrak{r}$ in $\operatorname{Ass} R/(x_1, \ldots, x_{i-1})$ that contains x_i. For every such ideal one now has $\mathfrak{r}S = S$, so $\frac{x_i}{1}$ does not belong to any prime ideal in $\operatorname{Ass} S/(\frac{x_1}{1}, \ldots, \frac{x_{i-i}}{1})$, which means that multiplication by $\frac{x_i}{1}$ on $S/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$ is injective as desired. The choice of u similarly ensures that multiplication by $\frac{x_{d-1}}{1}$ on $S/(\frac{x_1}{1}, \ldots, \frac{x_{d-2}}{1})$ is injective. Thus, the sequence $\frac{x_1}{1}, \ldots, \frac{x_{d-1}}{1}$ is S-regular. □

The next result was first proved by Auslander and Buchsbaum [12]. It predates the lemma above by Bass but 17.4.21 facilitates a short proof.

17.4.22 Proposition. *For every integer d with $1 \leqslant d \leqslant \dim R$ there exists a prime ideal $\mathfrak{p}$ in R such that* $\operatorname{depth} R_{\mathfrak{p}} = d - 1 = \dim R_{\mathfrak{p}}$ *holds.*

Proof. For $d = 1$ every minimal prime ideal in R has the desired property. Now let $d \geqslant 2$ and $\mathfrak{m}$ be a prime ideal in R with $\dim R_{\mathfrak{m}} = d$. It follows from 17.4.21 applied to $R_{\mathfrak{m}}$ that there is a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that $R_{\mathfrak{p}}$ admits a regular sequence of length $d-1$. By 16.2.33(a) and 17.2.16 one now has $d-1 \leqslant \operatorname{depth} R_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} \leqslant d-1$. □

17.4.23 Theorem. *Let R be local*; *there are equalities,*

$$\operatorname{FID} R \;=\; \operatorname{FFD} R \;=\; \begin{cases} \dim R & \textit{if } R \textit{ is Cohen–Macaulay} \\ \dim R - 1 & \textit{if } R \textit{ is not Cohen–Macaulay}\,. \end{cases}$$

Proof. The first equality holds by 17.4.1. Set $d = \dim R$ and let $\mathfrak{m}$ be the unique maximal ideal of R. If R is Cohen–Macaulay, then one has $\operatorname{FFD} R = d$ by 17.4.20.

If R is not Cohen–Macaulay then, in particular, $d > 0$ holds by 17.2.16. If $d = 1$, then $\operatorname{depth} R = 0$ holds as R is not Cohen–Macaulay, and for a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ one has $\operatorname{depth} R_\mathfrak{p} \leqslant \dim R_\mathfrak{p} = 0$ by 17.2.16. Thus $\operatorname{FFD} R = 0 = d - 1$ holds by 17.4.19. Now, assume that d is at least 2. As R is not Cohen–Macaulay, one has $\operatorname{depth} R < d$ and for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$ one has $\operatorname{depth} R_\mathfrak{p} \leqslant \dim R_\mathfrak{p} < d$ by 17.2.16. By 17.4.22 there is a prime ideal $\mathfrak{p}$ in R with $\operatorname{depth} R_\mathfrak{p} = d - 1$, whence 17.4.19 yields $\operatorname{FFD} R = d - 1$. □

17.4.24 Corollary. *There are inequalities,*

$$\dim R - 1 \leqslant \operatorname{FID} R = \operatorname{FFD} R \leqslant \dim R .$$

Proof. The equality $\operatorname{FID} R = \operatorname{FFD} R$ is known from 17.4.1. For every prime ideal $\mathfrak{p}$ in R one has $\dim R_\mathfrak{p} - 1 \leqslant \operatorname{FFD} R_\mathfrak{p} \leqslant \dim R_\mathfrak{p}$ by 17.4.23. Now apply 17.4.18. □

Finitistic Projective Dimension

It is a fact that the equality $\operatorname{FPD} R = \dim R$ holds. The inequality "$\geqslant$", see 17.4.28, is due to Bass [30] and proofs of the opposite inequality have been given by Gruson and Raynaud [207] and by Thompson and Nakamura [189]. Their proofs rely on methods that are beyond the scope of this text, but 8.5.25 combined with 17.4.24 yields $\operatorname{FPD} R \leqslant \dim R + 1$. In 18.2.42 we establish the equality $\operatorname{FPD} R = \dim R$ in the special case of a local ring with a dualizing complex. For our exposition, however, it sufficies to know that the two quantities are simultaneously finite:

17.4.25 Proposition. *If one of the quantities* $\operatorname{FID} R$, $\operatorname{FFD} R$, $\operatorname{FPD} R$, *or* $\dim R$ *is finite, then they are all finite.*

Proof. The quantities $\operatorname{FID} R$, $\operatorname{FFD} R$, and $\dim R$ are simultaneously finite by 17.4.24, while $\operatorname{FFD} R$ and $\operatorname{FPD} R$ are simultaneously finite by 8.5.26. □

17.4.26 Corollary. *If R has finite Krull dimension, then an R-complex has finite flat dimension if and only if it has finite projective dimension.*

Proof. The assertion is immediate from 17.4.25 and 8.5.20. □

17.4.27 Lemma. *Assume that $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals in R. Let $s \in \mathfrak{q} \setminus \mathfrak{p}$ and set $S = \{s^n \mid n \geqslant 0\}^{-1}R$. There are equalities,*

$$\operatorname{pd}_R S = 1 = \operatorname{pd}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S) .$$

Further, there is an $R/\mathfrak{p}$-module X with $\operatorname{Ext}^1_R(S, X) \cong \operatorname{Ext}^1_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S, X) \neq 0$.

Proof. As an R-module, S is flat by 1.3.42. From 15.4.20 applied to the ring homomorphism $R \to R/\mathfrak{p}$ one gets $\operatorname{pd}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S) = \operatorname{pd}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes^{\mathsf{L}}_R S) \leqslant \operatorname{pd}_R S$. The ring homomorphism $R[x] \to S$ that evaluates a polynomial at $\frac{1}{s}$ is surjective with kernel $(sx - 1)$, so there is an exact sequence of R-modules,

$$(\star) \qquad 0 \longrightarrow R[x] \xrightarrow{sx-1} R[x] \longrightarrow S \longrightarrow 0\,.$$

As an R-module, $R[x]$ is free, see 12.1.24, so one has $\operatorname{pd}_R S \leqslant 1$. It remains to show that the $R/\mathfrak{p}$-module $R/\mathfrak{p} \otimes_R S$ is not projective. As S is flat over R, 15.4.17 yields $\operatorname{Tor}_1^R(R/\mathfrak{p}, S) = 0$, so the exact sequence from 7.4.29 induced by $(\star)$ reads in part

$$(\diamond) \qquad 0 \longrightarrow (R/\mathfrak{p})[x] \xrightarrow{\partial} (R/\mathfrak{p})[x] \longrightarrow R/\mathfrak{p} \otimes_R S \longrightarrow 0\,,$$

see 7.4.21. Here ∂ is multiplication by $[s]_\mathfrak{p}x - [1]_\mathfrak{p}$. Assume towards a contradiction that the $R/\mathfrak{p}$-module $R/\mathfrak{p} \otimes_R S$ is projective. By 1.3.17 the exact sequence $(\diamond)$ is split. Set $F = (R/\mathfrak{p})[x]$ and let ϱ be an endomorphism of F with $\varrho\partial = 1^F$, see 2.1.47. In terms of the basis $\{x^i \mid i \geqslant 0\}$ for F one has $\varrho(x^0) = a_0x^0 + \cdots + a_mx^m$ for some $m \in \mathbb{N}_0$ and $a_0, \ldots, a_m \in R/\mathfrak{p}$. Further,

$$x^i \;=\; \varrho\partial(x^i) \;=\; \varrho([s]_\mathfrak{p}x^{i+1} - x^i) \;=\; [s]_\mathfrak{p}\varrho(x^{i+1}) - \varrho(x^i)$$

holds for all $i \geqslant 0$, which yields a telescoping sum,

$$\sum_{i=0}^{m+1} [s]_\mathfrak{p}^i x^i \;=\; \sum_{i=0}^{m+1} \big([s]_\mathfrak{p}^{i+1}\varrho(x^{i+1}) - [s]_\mathfrak{p}^i\varrho(x^i)\big) \;=\; [s]_\mathfrak{p}^{m+2}\varrho(x^{m+2}) - \varrho(x^0)\,.$$

In the free $R/\mathfrak{p}$-module F one now has

$$(\flat) \qquad [s]_\mathfrak{p}^{m+2}\varrho(x^{m+2}) \;=\; [s]_\mathfrak{p}^{m+1}x^{m+1} + (a_m + [s]_\mathfrak{p}^m)x^m + \cdots + (a_0 + [1]_\mathfrak{p})x^0\,.$$

It follows that $[s]_\mathfrak{p}^{m+2}$ divides the coefficients of $x^0, \ldots, x^{m+1}$ in $(\flat)$; in particular, $[s]_\mathfrak{p}^{m+2}$ divides $[s]_\mathfrak{p}^{m+1}$, which as $R/\mathfrak{p}$ is a domain and $[s]_\mathfrak{p}$ is non-zero implies that $[s]_\mathfrak{p}$ is a unit. But this is absurd as $[s]_\mathfrak{p}$ belongs to the proper ideal $\mathfrak{q}/\mathfrak{p}$ in $R/\mathfrak{p}$.

Now it follows from 15.4.1 that $\operatorname{Ext}^1_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S, X) \neq 0$ holds for some $R/\mathfrak{p}$-module X. In the derived category there are isomorphisms,

$$\mathsf{RHom}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S, X) \;\simeq\; \mathsf{RHom}_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes^{\mathsf{L}}_R S, X) \;\simeq\; \mathsf{RHom}_R(S, X)\,,$$

by flatness of S over R and 12.3.32. By the definition, 7.3.23, of Ext one thus has $\operatorname{Ext}^1_{R/\mathfrak{p}}(R/\mathfrak{p} \otimes_R S, X) \cong \operatorname{Ext}^1_R(S, X)$. □

For a Cohen–Macaulay ring it is already known from 17.4.20 that the Krull dimension is a lower bound for the finitistic projective dimension. A result of Bass shows that it is so without assumptions on the ring:

17.4.28 Proposition. *For every integer d with $0 \leqslant d \leqslant \dim R$ there exists an R-module of projective dimension d. In particular, one has* $\operatorname{FPD} R \geqslant \dim R$.

PROOF. For $d = 0$ the R-module R has the desired property. For $d = 1$ apply 17.4.27.

For $d \geqslant 2$ there exist by 17.4.21 prime ideals $\mathfrak{p} \subset \mathfrak{m}$ in R, a sequence $x_1, \ldots, x_{d-1}$ in $\mathfrak{p}$, and an element $s \in \mathfrak{m} \setminus \mathfrak{p}$, such that the induced sequence $\frac{x_1}{1}, \ldots, \frac{x_{d-1}}{1}$ in the ring $S = \{s^n \mid n \geqslant 0\}^{-1}R$ is S-regular. For $i \in \{1, \ldots, d\}$ set $S_i = S/(\frac{x_1}{1}, \ldots, \frac{x_{i-1}}{1})$; per 14.4.19 and 11.4.3(c) one has $\operatorname{pd}_S S_i \leqslant i - 1$, and then 15.4.5 and 17.4.27 yield $\operatorname{pd}_R S_i \leqslant \operatorname{pd}_R S + \operatorname{pd}_S S_i \leqslant i$. Let X be an $R/\mathfrak{p}$-module as in 17.4.27. Each exact sequence $0 \longrightarrow S_i \xrightarrow{x_i} S_i \longrightarrow S_{i+1} \longrightarrow 0$ induces by 7.3.35 an exact sequence,

$$(\star) \qquad \mathrm{Ext}^i_R(S_i, X) \xrightarrow{x_i} \mathrm{Ext}^i_R(S_i, X) \longrightarrow \mathrm{Ext}^{i+1}_R(S_{i+1}, X) \longrightarrow \mathrm{Ext}^{i+1}_R(S_i, X)\ .$$

As $\mathrm{Ext}^i_R(S_i, X)$ is an $R/\mathfrak{p}$-module, see 12.2.2, and x_i belongs to $\mathfrak{p}$, the left-hand map in $(\star)$ is zero. Further, per 15.4.1 the inequality $\mathrm{pd}_R\, S_i \leqslant i$ yields $\mathrm{Ext}^{i+1}_R(S_i, X) = 0$, so there are isomorphisms $\mathrm{Ext}^i_R(S_i, X) \cong \mathrm{Ext}^{i+1}_R(S_{i+1}, X)$. Chaining these together one gets $\mathrm{Ext}^d_R(S_d, X) \cong \mathrm{Ext}^1_R(S_1, X)$, and since S_1 is S, one gets $\mathrm{Ext}^d_R(S_d, X) \neq 0$ from 17.4.27 and, therefore, $\mathrm{pd}_R\, S_d = d$ by 15.4.1. □

Remark. The polynomial ring $R = \mathbb{R}[x, y, z]$ is Gorenstein, see 17.4.17, of Krull dimension 3. The field of fractions, $Q = R_{(0)}$, is a flat R-module, so it follows from 17.4.10 and 8.5.20 that $\mathrm{pd}_R\, Q \leqslant \mathrm{FPD}\, R = 3$ holds. Osofsky [197] shows that if one assumes the Continuum Hypothesis, then $\mathrm{pd}_R\, Q = 2$ holds, and otherwise one has $\mathrm{pd}_R\, Q = 3$.

Exercises

E 17.4.1 Show that R is Gorenstein if and only if the local ring $R_\mathfrak{m}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ in R.

E 17.4.2 Let R be a local ring that is not Cohen–Macaulay, show that $\mathrm{FPD}\, R = \dim R$ holds.

E 17.4.3 Show that the product ring $R \times S$ is Gorenstein if and only if R and S are Gorenstein.

E 17.4.4 Let I be a bounded below complex of injective R-modules. Show that if I is acyclic, then it is contractible.

E 17.4.5 Let F be a bounded above complex of flat R-modules. Show that if F is acyclic, then it is pure acyclic.

17.5 Rigidity of Tor and Ext

Synopsis. Flat dimension vs. colocalization; rigidity of Tor; Chouinard Formula for injective dimension; rigidity of Ext; a menagerie of examples.

For a complex M in $\mathcal{D}^\mathrm{f}_\square(R)$ the flat dimension is the difference $\mathrm{depth}\, R - \mathrm{depth}_R\, M$, provided that it is finite; that is the Auslander–Buchsbaum Formula 16.4.2. By the Chouinard Formula 17.3.4, the flat dimension of any R-complex can similarly, if finite, be computed via such differences of depths measured locally at every prime ideal. By the Bass Formula 16.4.11 the injective dimension of a complex M in $\mathcal{D}^\mathrm{f}_\square(R)$ is the difference $\mathrm{depth}\, R - \mathrm{width}_R\, M$, provided that it is finite, and the injective dimension of any R-complex M can, if finite, similarly be computed via such difference measured locally at every prime ideal; see 17.5.7. Both results were proved by Chouinard in [51], but here they appear in seperate sections. The reason that while the formula for flat dimension, 17.3.4, is an immediate consequence of 17.3.1, it takes more work to derive the formula for injective dimension from 17.3.11. On the other hand, the immediate consequence of 17.3.11 that matches 17.3.4 is 17.3.14, which deals with colocalizations instead of localizations, and that result has a counterpart for flat dimension, see 17.5.4. Similarly, the fact that the injective dimension of an appropriately bounded complex can be detected locally, see 17.3.17, finds its counterpart in 17.5.2.

17.5.1 Lemma. *Assume that every flat R-module has finite projective dimension. Let $\mathfrak{p}$ a prime ideal in R and M a complex in $\mathcal{D}_{\sqsupset}(R)$. There is an isomorphism,*

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \simeq \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) .$$

Proof. In the computation below, the 1st isomorphism holds by commutativity 12.3.5 and 15.1.1, while the 2nd holds by 14.1.16(b). By assumption the flat R-module $R_\mathfrak{p}$, see 1.3.42, has finite projective dimension, so, the 3rd isomorphism holds by tensor evaluation 12.3.23(d) and commutativity 12.3.5. The 4th isomorphism follows from 12.3.32 and and 15.1.1.

$$\begin{aligned}
\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) &\simeq \mathsf{RHom}_R(R_\mathfrak{p}, M) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} (R/\mathfrak{p})_\mathfrak{p} \\
&\simeq \mathsf{RHom}_R(R_\mathfrak{p}, M) \otimes^{\mathsf{L}}_R R/\mathfrak{p} \\
&\simeq \mathsf{RHom}_R(R_\mathfrak{p}, R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \\
&\simeq \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) . \qquad \square
\end{aligned}$$

Flat Dimension vs. Colocalization

By 17.4.26 the next result applies, in particular, if R has finite Krull dimension.

17.5.2 Proposition. *Assume that every flat R-module has finite projective dimension and let M be a complex in $\mathcal{D}_{\sqsupset}(R)$. There are equalities,*

$$\begin{aligned}
\operatorname{fd}_R M &= \sup\{\sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\} \\
&= \sup\{m \in \mathbb{Z} \mid \exists\, \mathfrak{p} \in \operatorname{Spec} R \colon \operatorname{Tor}^{R_\mathfrak{p}}_m(\kappa(\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M)) \neq 0\} .
\end{aligned}$$

Proof. The second equality follows from 17.1.14, 16.2.23, and 7.4.19. To prove the first equality, let s denote the first supremum in the display. For every prime ideal $\mathfrak{p}$ in R one gets from 17.5.1, 7.6.7, and 15.4.17 (in)equalities,

$$\begin{aligned}
\sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)) &= \sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \\
&\leqslant \sup(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \\
&\leqslant \operatorname{fd}_R M ,
\end{aligned}$$

and hence one has $\operatorname{fd}_R M \geqslant s$. To prove that equality holds, it suffices to argue that for every integer n with $n \leqslant \operatorname{fd}_R M$ there exists a prime ideal $\mathfrak{p}$ in R that satisfies $n \leqslant \sup(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M))$, which by 17.5.1 is tantamount to

$$(\star) \qquad n \leqslant \sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) .$$

Let $n \leqslant \operatorname{fd}_R M$ be given. Each $\operatorname{Tor}^R_v(-, M)$ is a half exact endofunctor on $\mathcal{M}(R)$ by 7.4.29, and by 3.1.6 so is $\mathrm{F} = \coprod_{v \geqslant n} \operatorname{Tor}^R_v(-, M)$. As $n \leqslant \operatorname{fd}_R M$ holds, the "in particular" statement in 8.3.11 yields an ideal $\mathfrak{a}$ in R with $\mathrm{F}(R/\mathfrak{a}) \neq 0$. Now 12.4.2 yields a prime ideal $\mathfrak{p}$ in R with $\mathrm{F}(R/\mathfrak{p}) \neq 0$ such that $\mathrm{F}(R/\mathfrak{b}) = 0$ holds for every ideal $\mathfrak{b} \supset \mathfrak{p}$. We argue that this prime ideal satisfies the inequality in $(\star)$. Consider for each element $x \in R \setminus \mathfrak{p}$ the exact sequence,

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}+(x)) \longrightarrow 0 \,.$$

As one has $\mathrm{F}(R/(\mathfrak{p}+(x))) = 0$, it follows that $\mathrm{Tor}_v^R(R/(\mathfrak{p}+(x)), M) = 0$ holds for every $v \geqslant n$. As the functor $- \otimes_R^{\mathsf{L}} M$ is triangulated and R-linear, see 12.2.8, one gets for each $v \geqslant n$ an exact sequence,

$$0 \longrightarrow \mathrm{Tor}_v^R(R/\mathfrak{p}, M) \xrightarrow{x} \mathrm{Tor}_v^R(R/\mathfrak{p}, M) \longrightarrow 0 \,,$$

see 7.4.29; i.e. multiplication by x is an automorphism of $\mathrm{Tor}_v^R(R/\mathfrak{p}, M)$ for $v \geqslant n$. For the $R/\mathfrak{p}$-complex $X = (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M)_{\sqsupseteq n}$ one has

$$\mathrm{H}(X) \;=\; \coprod_{v \geqslant n} \Sigma^v \mathrm{Tor}_v^R(R/\mathfrak{p}, M) \,,$$

see 2.5.25(b) and 7.4.18, so the homothety $x^{\mathrm{H}(X)}$ is an isomorphism. Now 7.6.11(b,c) shows that $\mathrm{H}(X)$ is a $\kappa(\mathfrak{p})$-complex and, as an $R/\mathfrak{p}$-complex, $\mathrm{H}(X)$ is a semi-injective replacement of X. Thus, in view of 12.2.2 there is an isomorphism,

$$(\diamond) \qquad \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), X) \simeq \prod_{v \geqslant n} \Sigma^v \mathrm{Hom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Tor}_v^R(R/\mathfrak{p}, M)) \,.$$

As one has $\mathrm{F}(R/\mathfrak{p}) \neq 0$, the $\kappa(\mathfrak{p})$-vector space $\mathrm{Tor}_v^R(R/\mathfrak{p}, M)$ is non-zero for some $v \geqslant n$, and hence $\mathrm{Hom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Tor}_v^R(R/\mathfrak{p}, M)) \neq 0$ holds. It now follows from $(\diamond)$ that there is an inequality,

$$(\flat) \qquad n \;\leqslant\; \sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), X) \,.$$

By 7.6.6(c) there is a distinguished triangle in $\mathcal{D}(R/\mathfrak{p})$,

$$(\dagger) \qquad X \longrightarrow R/\mathfrak{p} \otimes_R^{\mathsf{L}} M \longrightarrow (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M)_{\subseteq n-1} \longrightarrow \Sigma X \,.$$

Note that 7.6.7 yields an inequality,

$$(\ddagger) \qquad \sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), (R/\mathfrak{p} \otimes_R^{\mathsf{L}} M)_{\subseteq n-1}) \;\leqslant\; n-1 \,.$$

Application of the triangulated functor $\mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), -)$ to $(\dagger)$ yields another distinguished triangle, to which 6.5.20, in view of $(\ddagger)$, applies to yield the second inequality in the computation below. The first inequality comes from $(\flat)$.

$$n \;\leqslant\; \sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), X) \;\leqslant\; \max\{\sup \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes_R^{\mathsf{L}} M), n-2\} \,.$$

It follows that $(\star)$ holds. ◻

Remark. Christensen, Ferraro, and Thompson [58] provide an example to show that the boundedness condition in 17.5.2 is necessary. They prove 17.5.2 via a rigidity result that applies to R and M as in that statement: If for an integer $n \geqslant \sup M$ one has $\mathrm{Tor}_{n+1}^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathsf{RHom}_R(R_\mathfrak{p}, M)) = 0$ for every prime ideal $\mathfrak{p}$ in R, then $\mathrm{fd}_R M \leqslant n$ holds. This result is a conceptual dual to 17.5.12.

It is known from 17.3.2 that the flat dimension of a complex can be detected locally. Under a boundedness condition it can also be detected colocally; in particular, it does not grow under colocalization.

17.5.3 Corollary. *Assume that every flat R-module has finite projective dimension and let M be a complex in $\mathcal{D}_{\sqsupset}(R)$. There are equalities,*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\operatorname{fd}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{fd}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\}\,.\end{aligned}$$

Proof. Fix a prime ideal $\mathfrak{p}$ in R. The prime ideals in $R_\mathfrak{p}$ have the form $\mathfrak{q}_\mathfrak{p}$ where $\mathfrak{q}$ is a prime ideal in R with $\mathfrak{q} \subseteq \mathfrak{p}$. Notice that for every such prime ideal, 12.3.36 yields

$$\mathsf{RHom}_{R_\mathfrak{p}}((R_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}, \mathsf{RHom}_R(R_\mathfrak{p}, M)) \simeq \mathsf{RHom}_R(R_\mathfrak{q}, M)\,,$$

which combined with 15.1.3 yields

$$\begin{aligned}&\kappa(\mathfrak{q}_\mathfrak{p}) \otimes^{\mathsf{L}}_{(R_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}} \mathsf{RHom}_{R_\mathfrak{p}}((R_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}, \mathsf{RHom}_R(R_\mathfrak{p}, M)) \\ &\qquad\qquad \simeq \kappa(\mathfrak{q}) \otimes^{\mathsf{L}}_{R_\mathfrak{q}} \mathsf{RHom}_R(R_\mathfrak{q}, M)\,.\end{aligned}$$

By assumption the flat R-module $R_\mathfrak{p}$, see 1.3.42, has finite projective dimension, so it follows from 15.4.1 that $\mathsf{RHom}_R(R_\mathfrak{p}, M)$ belongs to $\mathcal{D}_\sqsupset(R_\mathfrak{p})$. Now 17.5.2 applies to yield $\operatorname{fd}_R M \geqslant \operatorname{fd}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$. Further,

$$\operatorname{fd}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \geqslant \sup\,(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M))$$

holds by 15.4.17. Now invoke 17.5.2. ◻

By 17.4.26 the next result applies, in particular, if R has finite Krull dimension. It is reminiscent of the Chouinard Formula 17.3.4; comparing the two formulas one should keep in mind that the numbers $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$ do not immediately compare, see 17.5.5.

17.5.4 Theorem. *Assume that every flat R-module has finite projective dimension and let M be an R-complex. If $\operatorname{fd}_R M$ is finite, then there are equalities,*

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{cosupp}_R M\}\,.\end{aligned}$$

Proof. The equalities hold trivially if M is acyclic, so we may assume that $n = \operatorname{fd}_R M$ is an integer. Let F be a semi-flat replacement of M with $F_v = 0$ for all $v > n$. To prove the asserted equalities it suffices to show that the inequality

$$(\star)\qquad n \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F)$$

holds for every prime ideal $\mathfrak{p}$ in R with equality for some $\mathfrak{p} \in \operatorname{cosupp}_R F$.

For every $u \leqslant n$ there is an exact sequence of complexes of flat R-modules,

$$(\diamond)\qquad 0 \longrightarrow F_{\leqslant u-1} \longrightarrow F \longrightarrow F_{\geqslant u} \longrightarrow 0\,.$$

The complex $F_{\geqslant u}$ is semi-flat by 5.4.8, so $F_{\leqslant u-1}$ is semi-flat by 5.4.12. Now the definition of flat dimension, 8.3.3, and 8.3.12 yield

$$(\flat)\qquad \operatorname{fd}_R F_{\leqslant u-1} \leqslant u-1 \qquad\text{and}\qquad \operatorname{fd}_R F_{\geqslant u} = n\,.$$

The complex $F_{\geqslant u}$ is bounded, and for every prime ideal $\mathfrak{p}$ in R the $R_\mathfrak{p}$-complex $\mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant u})$ has finite flat dimension, see 17.5.3, so 16.3.3 yields

$$\sup\left(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant u})\right) \;=\; \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant u}) \;.$$

Per 17.5.2 one, therefore, has

$$(\dagger) \qquad n \;=\; \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant u}) \mid \mathfrak{p} \in \operatorname{Spec} R\} \;.$$

By 17.1.14 the inequality ($\star$) is trivial for $\mathfrak{p} \notin \operatorname{cosupp}_R F$, so let $\mathfrak{p}$ be in $\operatorname{cosupp}_R F$ and set $u = -\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F)$. By 16.2.16, 7.6.7, and 8.3.4 one has

$$u \;\leqslant\; \sup \mathsf{RHom}_R(R_\mathfrak{p}, F) \leqslant \sup F \;\leqslant\; n \;.$$

Further, 16.2.16 and 7.6.7 also yield $\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\leqslant u-1}) \;\geqslant\; -u+1$, so 14.3.20 applied to ($\diamond$) after colocalization, see 14.1.32, yields

$$\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F) \;=\; \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant u}) \;,$$

whence ($\star$) holds by ($\dagger$).

Choose by ($\dagger$) a prime ideal $\mathfrak{p}$ with $n = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant n-1})$. It was proved above that the inequality ($\star$) holds for every prime ideal and every complex of finite flat dimension. Applied to the complex $F_{\leqslant n-2}$ it yields the second inequality in the next display, where the first inequality comes from ($\flat$),

$$n-2 \;\geqslant\; \operatorname{fd}_R F_{\leqslant n-2} \;\geqslant\; \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\leqslant n-2}) \;.$$

By the choice of $\mathfrak{p}$ one now has

$$\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\leqslant n-2}) - 2 \;\geqslant\; \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant n-1}) \;.$$

Now 14.3.20 applied to ($\diamond$), with $u = n-1$ and colocalized at $R \setminus \mathfrak{p}$, yields

$$\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F) \;=\; \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F_{\geqslant n-1}) \;.$$

and, therefore, $n = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, F)$ as desired. Finally, it now follows from 17.1.14 that $\mathfrak{p}$ belongs to $\operatorname{cosupp}_R F$. □

17.5.5 Example. Let R be a domain and $\mathfrak{a}$-complete for some ideal $\mathfrak{a} \neq (0)$. By 15.2.17 the ideal (0) is not in $\operatorname{cosupp}_R R$, so the $R_{(0)}$-complex $\mathsf{RHom}_R(R_{(0)}, R)$ is acyclic and hence of infinite width and depth. On the other hand, $R_{(0)}$ is a field so one has $\operatorname{width}_{R_{(0)}} R_{(0)} = 0 = \operatorname{depth}_{R_{(0)}} R_{(0)}$.

Flat Dimension via Rigidity of Tor

For a complex M, vanishing of $\operatorname{Tor}^R_m(\kappa(\mathfrak{p}), M)$ for every prime ideal $\mathfrak{p}$ and a single sufficiently large index m guarantees that M has finite flat dimension. This follows from the rigidity property of Tor recorded in 16.3.22.

17.5.6 Theorem. *Assume that R has finite Krull dimension and let M be an R-complex. If for an integer $n \geqslant \dim R + \sup M$ one has $\operatorname{Tor}^R_{n+1}(\kappa(\mathfrak{p}), M) = 0$ for every prime ideal $\mathfrak{p}$ in R, then $\operatorname{fd}_R M \leqslant n$ holds.*

Proof. Fix a prime ideal $\mathfrak{p}$ in R. By 14.1.11(c) and 16.2.16 there are inequalities,

$$n \geqslant \dim R + \sup M \geqslant \dim R_\mathfrak{p} + \sup M_\mathfrak{p} \geqslant \dim R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} .$$

By 14.1.16(b) and the definition of Tor there is for every integer m an isomorphism $\operatorname{Tor}_m^R(\kappa(\mathfrak{p}), M) \cong \operatorname{Tor}_m^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$. Thus vanishing of $\operatorname{Tor}_{n+1}^R(\kappa(\mathfrak{p}), M)$ implies by 16.3.22 that $\operatorname{Tor}_m^R(\kappa(\mathfrak{p}), M) = 0$ holds for all $m > n$. Now invoke 17.3.1. $\square$

Remark. Theorem 17.5.6 remains valid with the improved bound $n \geqslant \dim R + \sup M - 1$; see E 17.5.1. Christensen, Iyengar, and Marley [66] show that this bound is optimal.

The Chouinard Formula for Injective Dimension

The next result is reminiscent of the Bass Formula 16.4.11 and originally due to Chouinard [51]. Comparing it to 17.3.14 one should bear in mind that the numbers $\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$ do not readily compare, see 17.5.8.

17.5.7 Theorem. *Let M be an R-complex. If $\operatorname{id}_R M$ is finite, then one has*

$$\begin{aligned} \operatorname{id}_R M &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} . \end{aligned}$$

Proof. The equalities hold trivially if M is acyclic, so we may assume that $n = \operatorname{id}_R M$ is an integer. Let I be a semi-injective replacement of M with $I_{-v} = 0$ for all $v > n$. To prove the asserted equalities it suffices to show that the inequality

$$(\dagger) \qquad \operatorname{id}_R I \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} I_\mathfrak{p}$$

holds for every prime ideal $\mathfrak{p}$ in R with equality for some $\mathfrak{p} \in \operatorname{supp}_R I$.

For every $u \leqslant n$ there is an exact sequence of complexes of injective R-modules

$$(\ddagger) \qquad 0 \longrightarrow I_{\leqslant -u} \longrightarrow I \longrightarrow I_{\geqslant -u+1} \longrightarrow 0 .$$

The complex $I_{\leqslant -u}$ is semi-injective by 5.3.12, so $I_{\geqslant -u+1}$ is semi-injective by 5.3.20. Now the definition of injective dimension, 8.2.2, and 8.2.9 yield

$$(\flat) \qquad \operatorname{id}_R I_{\geqslant -u+1} \leqslant u - 1 \qquad \text{and} \qquad \operatorname{id}_R I_{\leqslant -u} = n .$$

The complex $I_{\leqslant -u}$ is bounded, and for every $\mathfrak{p} \in \operatorname{Spec} R$ the $R_\mathfrak{p}$-complex $(I_{\leqslant -u})_\mathfrak{p}$ has finite injective dimension, see 17.3.18, so 16.3.11 yields

$$-\inf \mathsf{RHom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (I_{\leqslant -u})_\mathfrak{p}) = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -u})_\mathfrak{p} .$$

Per 17.3.17 one, therefore, has

$$(\diamond) \qquad n = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -u})_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} .$$

By 17.1.6 the inequality $(\dagger)$ is trivial for $\mathfrak{p} \notin \operatorname{supp}_R I$, so let $\mathfrak{p}$ be in $\operatorname{supp}_R I$ and set $u = -\operatorname{width}_{R_\mathfrak{p}} I_\mathfrak{p}$. By 16.2.5, 14.1.11(c), and 8.2.3 one has $u \leqslant -\inf I_\mathfrak{p} \leqslant -\inf I \leqslant n$. Further, 16.2.5 and 14.1.11(c) also yields

$$\operatorname{width}_{R_\mathfrak{p}} (I_{\geqslant -u+1})_\mathfrak{p} \geqslant -u + 1 ,$$

so 14.3.32 applied to (‡) after localization gives $\operatorname{width}_{R_\mathfrak{p}} I_\mathfrak{p} = \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -u})_\mathfrak{p}$, whence (†) holds by (⋄).

Choose by (⋄) a prime ideal $\mathfrak{p}$ with $n = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -n+1})_\mathfrak{p}$. It was proved above that the inequality (†) holds for every prime ideal and every complex of finite injective dimension. Applied to the complex $I_{\geqslant -n+2}$ it yields the second inequality in the next display, where the first inequality comes from (♭),

$$n - 2 \geqslant \operatorname{id}_R I_{\geqslant -n+2} \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} (I_{\geqslant -n+2})_\mathfrak{p} \, .$$

By the choice of $\mathfrak{p}$ one now has

$$\operatorname{width}_{R_\mathfrak{p}} (I_{\geqslant -n+2})_\mathfrak{p} - 2 \geqslant \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -n+1})_\mathfrak{p} \, .$$

Now 14.3.32 applied to (‡), with $u = n-1$ and localized at $R \setminus \mathfrak{p}$, yields $\operatorname{width}_{R_\mathfrak{p}} I_\mathfrak{p} = \operatorname{width}_{R_\mathfrak{p}} (I_{\leqslant -n+1})_\mathfrak{p}$ and, therefore, $n = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} I_\mathfrak{p}$ as desired. Finally, it now follows from 17.1.6 that $\mathfrak{p}$ belongs to $\operatorname{supp}_R I$. □

17.5.8 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local of positive Krull dimension. For every prime ideal $\mathfrak{p} \neq \mathfrak{m}$ in R the $R_\mathfrak{p}$-module $\mathrm{E}_R(\boldsymbol{k})_\mathfrak{p}$ is zero by 15.1.12 and hence of infinite depth and width. On the other hand, both $\operatorname{depth}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k}))$ and $\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k}))$ are finite by 17.1.14, as $\operatorname{cosupp}_R \mathrm{E}_R(\boldsymbol{k}) = \operatorname{Spec} R$ holds by 15.2.2.

Injective Dimension via Rigidity of Ext

The final theorem of this section, 17.5.11 below, is dual to 17.5.6, but the proof is more involved. It takes two auxiliary results, the second of which is only required because the best upper bound on $\operatorname{FPD} R$ established in this text is $\dim R + 1$, while it is known from work of Gruson and Raynaud [207] that $\operatorname{FPD} R = \dim R$ holds.

By 17.4.26 the next result applies, in particular, if R has finite Krull dimension.

17.5.9 Proposition. *Assume that every flat R-module has finite projective dimension. Let $\mathfrak{p}$ be a prime ideal in R and M an R-complex. If M is not acyclic, then there is an inequality,*

$$-\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) \leqslant \operatorname{FPD} R/\mathfrak{p} - \inf M \, .$$

Proof. One can assume that M belongs to $\mathcal{D}_{\sqsupset}(R)$, otherwise the inequality is trivial. In the computation below, the first equality holds by 16.2.3, and the second comes from 17.5.1. The first two inequalities hold by 15.4.1 and 7.6.8. As $\kappa(\mathfrak{p})$ is a flat $R/\mathfrak{p}$-module, see 15.1.1 and 1.3.42, the last inequality holds by 8.5.18.

$$\begin{aligned} -\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M) &= -\inf\,(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)) \\ &= -\inf \mathsf{RHom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \\ &\leqslant \operatorname{pd}_{R/\mathfrak{p}} \kappa(\mathfrak{p}) - \inf\,(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \\ &\leqslant \operatorname{pd}_{R/\mathfrak{p}} \kappa(\mathfrak{p}) - \inf M \\ &\leqslant \operatorname{FPD} R/\mathfrak{p} - \inf M \, . \end{aligned}$$ □

17.5.10 Lemma. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. If $\operatorname{Ext}_R^{n+1}(\boldsymbol{k}, M) = 0$ holds for an integer $n \geqslant \dim R - \operatorname{width}_R M - 1$, then one has $\operatorname{Ext}_R^m(\boldsymbol{k}, M) = 0$ for all $m > n$.*

Proof. Set $u = -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ and recall from 16.2.34 that $\dim R - \operatorname{width}_R M \geqslant u$ holds. By assumption one has $n \geqslant u - 1$. If the complex $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ is acyclic, i.e. $u = -\infty$, then by 16.2.23 so is $\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$, and the claim is trivial. One can now asume that u is an integer, and by 16.3.16 it suffices to consider the case $n = u - 1$.

To settle that case it suffics to argue that $\operatorname{Ext}_R^u(\boldsymbol{k}, M)$ is non-zero. To this end let I be a minimal semi-injective replacement of M. One has $\mathsf{R}\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(I)$ so it follows from 13.3.4 that $\mathrm{E}_R(\boldsymbol{k})$ is a summand of I_{-u}. By 16.4.36 the complex $\operatorname{Hom}_R(\boldsymbol{k}, I)$, which is $\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M)$, has zero differential, so $\operatorname{Ext}_R^u(\boldsymbol{k}, M) = \mathrm{H}_{-u}(\operatorname{Hom}_R(\boldsymbol{k}, I)) = \operatorname{Hom}_R(\boldsymbol{k}, I_{-u})$ is non-zero by (16.1.22.1). □

17.5.11 Theorem. *Assume that R has finite Krull dimension and let M be an R-complex. If for an integer $n \geqslant \dim R - \inf M$ one has $\operatorname{Ext}_R^{n+1}(\kappa(\mathfrak{p}), M) = 0$ for every prime ideal $\mathfrak{p}$ in R, then $\operatorname{id}_R M \leqslant n$ holds.*

Proof. Fix a prime ideal $\mathfrak{p}$ in R. By 8.5.25 and 17.4.24 there are inequalities $\operatorname{FPD} R/\mathfrak{p} \leqslant \operatorname{FFD} R/\mathfrak{p} + 1 \leqslant \dim R/\mathfrak{p} + 1$, which explains the third inequality in the computation below. The first inequality holds by assumption, and the second is standard. The last inequality follows from 17.5.9.

$$\begin{aligned} n &\geqslant \dim R - \inf M \\ &\geqslant \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} - \inf M \\ &\geqslant \dim R_{\mathfrak{p}} + \operatorname{FPD} R/\mathfrak{p} - \inf M - 1 \\ &\geqslant \dim R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} \mathsf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M) - 1\,. \end{aligned}$$

By the definition, 7.3.23, of Ext and 14.1.33(d) there is for every integer m an isomorphism $\operatorname{Ext}_R^m(\kappa(\mathfrak{p}), M) \cong \operatorname{Ext}_{R_{\mathfrak{p}}}^m(\kappa(\mathfrak{p}), \mathsf{R}\mathrm{Hom}_R(R_{\mathfrak{p}}, M))$. Thus it follows from 16.3.17 that vanishing of $\operatorname{Ext}_R^{n+1}(\kappa(\mathfrak{p}), M)$ implies that $\operatorname{Ext}_R^m(\kappa(\mathfrak{p}), M) = 0$ holds for all $m > n$. Now invoke 17.3.11. □

Remark. The bound on n in 17.5.11 is not optimal. Indeed, as mentioned before 17.5.9, the inequality $\operatorname{FPD} R/\mathfrak{p} \leqslant \dim R/\mathfrak{p}+1$ used in the proof could be replaced with the equality $\operatorname{FPD} R/\mathfrak{p} = \dim R/\mathfrak{p}$ to improve the bound to $n \geqslant \dim R - \inf M - 1$. That is the bound obtained by Christensen, Ferraro, and Thompson [58], who also proove that this bound is optimal. Per the Remark after 17.5.6 it matches the optimal bound in the corresponding statement for flat dimension. 17.5.6.

Under a boundedness assumption, the bound on n in 17.5.11 can be further improved.

17.5.12 Proposition. *Let M be a complex in $\mathcal{D}_{\sqsubset}(R)$. If for an integer $n \geqslant -\inf M$ one has $\operatorname{Ext}_{R_{\mathfrak{p}}}^{n+1}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ then $\operatorname{id}_R M \leqslant n$ holds.*

Proof. By 15.4.7 it is sufficient to show that $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$. Assume towards a contradiction that there is a prime ideal $\mathfrak{q}$ in R with $\operatorname{Ext}_R^{n+1}(R/\mathfrak{q}, M) \neq 0$. By 12.4.9 there exists a prime ideal $\mathfrak{p} \in \mathrm{V}(\mathfrak{q})$ with $\operatorname{Ext}_R^{n+1}(R/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0$ and 14.1.23 yields $\operatorname{Ext}_{R_{\mathfrak{p}}}^{n+1}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$; a contradiction. □

Remark. A longer but perhaps more conceptual proof of 17.5.12, based on minimal semi-injective resolutions, can be found in Christensen, Iyengar, and Marley [66]. Of course the statement remains true with the lower bound $n \geqslant -\inf M - 1$ but we find that too much of a red herring to pursue.

A Menagerie of Examples

If x is a non-zerodivisor in a ring Q, then $(0 :_R x) = (x)$ holds in the ring $R = Q/(x^2)$. In this situation the next construction, which generalizes E 5.2.5, applies.

17.5.13 Construction. Assume that $x \in R$ is an element with $(0 :_R x) = (x)$ and set $S = R/(x)$. The complex $L = \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow 0$, concentrated in non-negative degrees, is a semi-projective replacement of S in $\mathcal{D}(R)$ with $\mathrm{Z}_v(L) \cong S = \mathrm{C}_v(L)$ for all $v > 0$. The complex

$$P = \coprod_{u<0} \Sigma^u L$$

is semi-projective by 5.2.18. Since the truncated complex $P_{\geqslant 1}$ is semi-projective by 5.2.8, an application of 5.2.17 to the exact sequence $0 \to P_{\leqslant 0} \to P \to P_{\geqslant 1} \to 0$ shows that $P_{\leqslant 0}$ is semi-projective. Now 3.1.10(d) yields

$$\mathrm{H}_v(P_{\leqslant 0}) \cong \coprod_{u<0} \mathrm{H}_v((\Sigma^u L)_{\leqslant 0}) \cong \begin{cases} \coprod_{u<0} \mathrm{Z}_{-u}(L) \cong S^{(\mathbb{N})} & \text{for } v = 0 \\ \mathrm{H}_v(\Sigma^v L) = \mathrm{H}_0(L) \cong S & \text{for } v < 0\,. \end{cases}$$

In particular, one has $\operatorname{pd}_R P_{\leqslant 0} = 0$ as $\mathrm{H}_0(P_{\leqslant 0}) \neq 0$, see 8.1.3.

For every flat R-module F and semi-injective R-complex I the complex $F \otimes_R I$ consists by 8.4.17 of injective R-modules. If I is bounded above, then so is $F \otimes_R I$ whence it is semi-injective by 5.3.12. The next example shows that if I is not bounded above, then the complex $F \otimes_R I$ may not be semi-injective, not even in the special case $F = R_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ in R. The example also shows that the boundedness conditions in 14.1.29, 15.4.33, 17.3.17, and 17.3.18 are necessary.

17.5.14 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local, assume that there is an element $x \in R$ with $(0 :_R x) = (x)$, and set $S = R/(x)$. Let L and P be as in 17.5.13 and set

$$I = \operatorname{Hom}_R(L, \mathrm{E}_R(\boldsymbol{k})) \qquad \text{and} \qquad J = \operatorname{Hom}_R(P_{\leqslant 0}, \mathrm{E}_R(\boldsymbol{k}))\,;$$

they are by 5.3.17 semi-injective R-complexes. One has $\operatorname{Hom}_R(S, \mathrm{E}_R(\boldsymbol{k})) \cong \mathrm{E}_S(\boldsymbol{k})$ by C.16, so I is per 2.2.19 a semi-injective replacement in $\mathcal{D}(R)$ of $\mathrm{E}_S(\boldsymbol{k})$ and one has

$$\mathrm{H}_v(J) \cong \begin{cases} \operatorname{Hom}_R(\mathrm{H}_0(P_{\leqslant 0}), \mathrm{E}_R(\boldsymbol{k})) \cong \mathrm{E}_S(\boldsymbol{k})^{\mathbb{N}} & \text{for } v = 0 \\ \operatorname{Hom}_R(\mathrm{H}_{-v}(P_{\leqslant 0}), \mathrm{E}_R(\boldsymbol{k})) \cong \mathrm{E}_S(\boldsymbol{k}) & \text{for } v > 0\,. \end{cases}$$

As J is concentrated in non-negative degrees with $\mathrm{H}_0(J) \neq 0$, one has $\operatorname{id}_R J = 0$, see 8.2.3. Now assume that R has a prime ideal $\mathfrak{p} \neq \mathfrak{m}$. For $v > 0$ one has

$$\mathrm{H}_v(J_{\mathfrak{p}}) \cong \mathrm{H}_v(J)_{\mathfrak{p}} \cong \mathrm{E}_S(\boldsymbol{k})_{\mathfrak{p}} = 0$$

by 14.1.11(a) and 15.1.12, so $J_\mathfrak{p}$ is isomorphic to $(\mathrm{E}_S(\boldsymbol{k})^{\mathbb{N}})_\mathfrak{p}$ in $\mathcal{D}(R_\mathfrak{p})$; see 7.3.29. This explains the 2nd isomorphism in the computation below. The 1st, 3rd, and 4th isomorphisms hold by 14.1.21(b), 14.1.23, and 7.3.6, and the 5th comes from 12.2.2.

$$\begin{aligned}\mathsf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), J_\mathfrak{p}) &\simeq \mathsf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), J_\mathfrak{p})\\ &\simeq \mathsf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), (\mathrm{E}_S(\boldsymbol{k})^{\mathbb{N}})_\mathfrak{p})\\ &\simeq \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{E}_S(\boldsymbol{k})^{\mathbb{N}})_\mathfrak{p}\\ &\simeq (\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{E}_S(\boldsymbol{k}))^{\mathbb{N}})_\mathfrak{p}\\ &\simeq (\mathrm{Hom}_R(R/\mathfrak{p}, I)^{\mathbb{N}})_\mathfrak{p}\ .\end{aligned}$$

In the integral domain $R/\mathfrak{p}$ one has $[x]_\mathfrak{p} = 0$, as $x^2 = 0$ in R, so the complex $\mathrm{Hom}_R(R/\mathfrak{p}, I)$ has zero differential. For $m > 0$ one now has

$$\mathrm{Ext}_R^m(\kappa(\mathfrak{p}), J_\mathfrak{p}) \cong (\mathrm{Hom}_R(R/\mathfrak{p}, I_{-m})^{\mathbb{N}})_\mathfrak{p} \cong (\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k}))^{\mathbb{N}})_\mathfrak{p}\ .$$

By C.16 there is an isomorphism of $R/\mathfrak{p}$-modules,

$$(\mathrm{Hom}_R(R/\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k}))^{\mathbb{N}})_\mathfrak{p} \cong (\mathrm{E}_{R/\mathfrak{p}}(\boldsymbol{k})^{\mathbb{N}})_{(0)}\ ,$$

so by 14.1.8 the module $\mathrm{Ext}_R^m(\kappa(\mathfrak{p}), J_\mathfrak{p})$ is non-zero for every $m > 0$. In particular, one has $\mathrm{id}_{R_\mathfrak{p}} J_\mathfrak{p} = \mathrm{id}_R J_\mathfrak{p} = \infty$ by 17.3.13 and 17.3.11. Now, $J_\mathfrak{p}$ is by C.24 a complex of injective $R_\mathfrak{p}$-modules; since it is below it is, in particular, not semi-injective.

To conclude: Let $(R, \mathfrak{m})$ be local with a prime ideal $\mathfrak{p} \neq \mathfrak{m}$, i.e. not Artinian, and an element x with $(0 :_R x) = (x)$; a concrete example of such a ring is $R = \Bbbk[\![x, y]\!]/(x^2)$ where $\Bbbk$ is local for example a field. The R-complex J is concentrated in non-negative degrees and semi-injective with $\mathrm{id}_R J = 0$. The $R_\mathfrak{p}$-complex $J_\mathfrak{p}$ is not semi-injective, and one has $\mathrm{id}_{R_\mathfrak{p}} J_\mathfrak{p} = \mathrm{id}_R J_\mathfrak{p} = \infty$. More so, $\mathrm{Ext}_{R_\mathfrak{p}}^m(\kappa(\mathfrak{p}), J_\mathfrak{p}) \cong \mathrm{Ext}_R^m(\kappa(\mathfrak{p}), J_\mathfrak{p}) \neq 0$ holds for all $m > 0$; the isomorphism comes from 14.1.33(c).

For every injective R-module E and semi-injective R-complex I the complex $\mathrm{Hom}_R(I, E)$ consists of flat R-modules, see 8.4.28. If the complex I is bounded above, then $\mathrm{Hom}_R(I, E)$ is bounded below whence it is semi-flat by 5.4.8. The next example shows that if I is not bounded above, then $\mathrm{Hom}_R(I, E)$ may not be semi-flat. The example also shows that the boundedness condition in 15.4.31 is necessary.

17.5.15 Example. Let $(R, \mathfrak{m})$ be local with a prime ideal $\mathfrak{p} \neq \mathfrak{m}$ and an element x with $(0 :_R x) = (x)$; a concrete example of such a ring is $R = \Bbbk[\![x, y]\!]/(x^2)$ where $\Bbbk$ is local for example a field. Let J be the semi-injective R-complex with $\mathrm{id}_R J = 0$ from 17.5.14 and consider the $R_\mathfrak{p}$-complex

$$F = \mathrm{Hom}_R(J, \mathrm{E}_R(R/\mathfrak{p})) \cong \mathrm{Hom}_{R_\mathfrak{p}}(J_\mathfrak{p}, \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p})))\ ,$$

where the isomorphism holds by C.18 and 14.1.33(a). The complex $J_\mathfrak{p}$ has homology concentrated in degree 0, see 17.5.14, so homomorphism evaluation 12.3.27(b), and injectivity of $\mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ yield an isomorphism,

$$\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_{R_\mathfrak{p}} \mathrm{Hom}_{R_\mathfrak{p}}(J_\mathfrak{p}, \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(\mathsf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), J_\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p})))\ ,$$

in $\mathcal{D}(R_\mathfrak{p})$. From 14.1.14(b), the isomorphism above, 2.2.19, 17.5.14, and faithful injectivity of $\mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ one gets

$$\operatorname{Tor}_m^R(\kappa(\mathfrak{p}), F) \cong \operatorname{Tor}_m^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), F) \cong \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{Ext}_{R_\mathfrak{p}}^m(\kappa(\mathfrak{p}), J_\mathfrak{p}), \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))) \neq 0$$

for all $m > 0$. It follows from 17.3.1 that the R-complex F has infinite flat dimension, and by 8.4.28 it is a complex of flat R-modules. As F is bounded above it is, in particular, not semi-flat.

Let $\mathfrak{a}$ be an ideal in R and I a semi-injective R-complex. By 13.3.4 the complex $\Gamma_\mathfrak{a}(I)$ consists of injective R-modules, so if I is bounded above, then $\Gamma_\mathfrak{a}(I)$ is semi-injective, see 13.3.8. The next example shows that this boundedness condition is necessary; it also shows that the assumption in 15.4.15(a) is needed.

17.5.16 Example. Let $\Bbbk$ be a field and consider the local ring $R = \Bbbk[\![x, y]\!]/(x^2)$; it has only two prime ideals: $\mathfrak{p} = (x)$ and $\mathfrak{m} = (x, y)$. Notice that $(0 :_R x) = (x)$ holds and let J be as in 17.5.14. Each module J_v is by Matlis' structure theorem C.23 a coproduct of copies of $\mathrm{E}_R(R/\mathfrak{p})$ and $\mathrm{E}_R(R/\mathfrak{m})$. It thus follows from 13.3.4 and C.24 that the cokernel of the embedding of $\Gamma_\mathfrak{m}(J)$ into J is $J_\mathfrak{p}$; that is, there is an exact sequence of R-complexes,

$$0 \longrightarrow \Gamma_\mathfrak{m}(J) \longrightarrow J \longrightarrow J_\mathfrak{p} \longrightarrow 0\,.$$

By 17.5.14 the R-complex $J_\mathfrak{p}$ has infinite injective dimension while $\operatorname{id}_R J = 0$, so by 8.2.9 also the complex $\Gamma_\mathfrak{m}(J) = \mathsf{R}\Gamma_\mathfrak{m}(J)$ has infinite injective dimension. By 13.3.4 the complex $\Gamma_\mathfrak{m}(J)$ consists of injective R-modules; as it is bounded below it is, in particular, not semi-injective.

It now follows from 15.4.15(a) that one has $\operatorname{depth}_R J = -\infty$, but that can also be verified directly: Recall the definitions of the complexes L, P, and J from 17.5.13 and 17.5.14. As x belongs to $\mathfrak{m}$, the complex $R/\mathfrak{m} \otimes_R L$ and, therefore, also $R/\mathfrak{m} \otimes_R P$ has zero differential, see 3.1.13. Adjunction 12.1.10 yields

$$\operatorname{Hom}_R(R/\mathfrak{m}, J) \cong \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R P_{\geqslant 0}, \mathrm{E}_R(R/\mathfrak{m}))\,,$$

so also $\operatorname{Hom}_R(R/\mathfrak{m}, J)$ has zero differential. As $\mathrm{E}_R(R/\mathfrak{m})$ is a direct summand of J_v for every $v \geqslant 0$, one has in view of 16.2.14 and (16.1.22.1) the equalities

$$\operatorname{depth}_R J = -\sup \mathsf{R}\!\operatorname{Hom}_R(R/\mathfrak{m}, J) = -\sup \operatorname{Hom}_R(R/\mathfrak{m}, J) = -\sup J = -\infty\,.$$

Remark. For every projective R-module P and semi-flat R-complex F the complex $\operatorname{Hom}_R(P, F)$ consists by 8.4.15 of flat R-modules. If F is bounded below, then so is $\operatorname{Hom}_R(P, F)$ whence it is semi-flat by 5.4.8. An example by Christensen, Ferraro, and Thompson [58] shows that if F is not bounded below, then the complex $\operatorname{Hom}_R(P, F)$ may not be semi-flat.

Exercises

E 17.5.1 Show that 17.5.6 remains valid with the bound $n \geqslant \dim R + \sup M - 1$. *Hint:* E 16.3.8.

E 17.5.2 Derive the Auslander–Buchsbaum Formula 16.4.2 from the Chouinard Formula 17.5.7.

E 17.5.3 Let M be an R-module and $\iota\colon M \to I$ an embedding into an injective R-module. Show that I is the injective envelope of M if and only if $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \iota_\mathfrak{p})$ is an isomorphism for every $\mathfrak{p} \in \operatorname{Spec} R$.

E 17.5.4 Let M be an R-complex. Show that there are equalities,

$$\begin{aligned}\operatorname{fd}_R M &= \sup\{\sup \mathsf{R}\mathrm{Hom}_R(R_\mathfrak{p}, R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup\{\sup \mathsf{R}\mathrm{Hom}_{R/\mathfrak{p}}(\kappa(\mathfrak{p}), R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \mid \mathfrak{p} \in \operatorname{Spec} R\}\ .\end{aligned}$$

Derive from these equalities 17.5.2 for M in $\mathcal{D}_\sqsupset(R)$.

17.6 Depth and Krull Dimension

Synopsis. Depth; associated prime ideals of local cohomology module; $\mathfrak{a}$-depth; $\mathfrak{a}$-width; supremum; infimum; Krull dimension, prime ideals vs. faithfully flat ring extensions.

In the generality stated here, the first major result of this section, 17.6.3, was proved by Foxby and Iyengar [98]. It has the useful consequence, 17.6.4, that one can replace the classic support of a complex by its support in the supremum formula in 14.1.13. For the infimum one only gets a result of comparable utility for complexes over a ring of finite Krull dimension; see 17.6.11.

Depth and Localization

17.6.1 Lemma. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ there are inequalities,*

$$\mathfrak{a}\text{-depth}_R M \leqslant \mathfrak{a}_\mathfrak{p}\text{-depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}\ .$$

Proof. Let $\mathfrak{p}$ be a prime ideal that contains $\mathfrak{a}$. As $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ by 16.2.12 is $\mathfrak{p}_\mathfrak{p}$-$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$, the second of the asserted inequalities holds by 14.3.18. The first inequality follows from 14.4.3, 14.1.11, and 14.1.25:

$$\begin{aligned}\mathfrak{a}\text{-depth}_R M &= -\sup \mathsf{R}\Gamma_\mathfrak{a}(M)\\ &\leqslant -\sup \mathsf{R}\Gamma_\mathfrak{a}(M)_\mathfrak{p}\\ &= -\sup \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})\\ &= \mathfrak{a}_\mathfrak{p}\text{-depth}_{R_\mathfrak{p}} M_\mathfrak{p}\ . \qquad \square\end{aligned}$$

17.6.2 Example. Let $\Bbbk$ be a field and consider the ring $R = \Bbbk[\![x,y,z]\!]/(xz, yz, z^2)$ and the prime ideal $\mathfrak{p} = (y,z)$ in R. As $(0 :_R \mathfrak{p})$ contains z one has $\mathfrak{p}$-depth $R = 0$ by 14.3.17. The local ring $R_\mathfrak{p}$ is isomorphic to the ring of power series in y with coefficients in the field of fractions of $\Bbbk[\![x]\!]$, which is the ring of Laurent series in x; that is, $R_\mathfrak{p} \cong \Bbbk(\!(x)\!)[\![y]\!]$. In particular, $R_\mathfrak{p}$ is a domain and not a field, so it has positive depth, see 14.4.17 and 14.4.21(a). Thus one has $\mathfrak{p}$-depth $R < \operatorname{depth} R_\mathfrak{p}$. In fact, the depth of $R_\mathfrak{p}$ is 1 as, evidently, y is a maximal R-regular sequence.

The prime ideals that attain the infimum below are identified in 17.6.7. The “in particular” statement compares to 14.1.13.

17.6.3 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. There are equalities,*

$$\begin{aligned}\mathfrak{a}\text{-depth}_R M &= \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}\\ &= \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cap \operatorname{supp}_R M\}\,.\end{aligned}$$

In particular, one has

$$-\sup M = \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\,.$$

Proof. Per 14.3.12 the equalities in the second display are the special case $\mathfrak{a} = 0$ of the equalities in the first display. In the first display, the two infima are equal by 17.1.6. From 17.6.1 one gets $\mathfrak{a}\text{-depth}_R M \leqslant \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}$, and to prove the opposite inequality one can assume that $\mathfrak{a}\text{-depth}_R M < \infty$ holds. Let $\boldsymbol{x} = x_1,\dots,x_n$ be a sequence that generates $\mathfrak{a}$ and set $K = \mathrm{K}^R(\boldsymbol{x})$. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ the fractions $\frac{x_1}{1},\dots,\frac{x_n}{1}$ form a sequence in $\mathfrak{p}_\mathfrak{p}$, and $K_\mathfrak{p}$ is the Koszul complex on this sequence, see 11.4.18. Now 14.4.15 and 14.1.15 yield

$$(\diamond)\qquad \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = n + \operatorname{depth}_{R_\mathfrak{p}}(K_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p}) = n + \operatorname{depth}_{R_\mathfrak{p}}(K \otimes_R M)_\mathfrak{p}\,.$$

Notice from 14.1.5 and 14.1.17 that $\operatorname{Supp}_R(K \otimes_R M)$ is contained in $\mathrm{V}(\mathfrak{a})$. Assume first that $\mathrm{H}(K \otimes_R M)$ is bounded above, i.e. $\mathfrak{a}\text{-depth}_R M > -\infty$, and set $s = \sup(K \otimes_R M)$. The goal is to prove the existence of a prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ with $\mathfrak{a}\text{-depth}_R M = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$. As just noticed, one has $\operatorname{Supp}_R \mathrm{H}_s(K \otimes_R M) \subseteq \mathrm{V}(\mathfrak{a})$, so let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass}_R \mathrm{H}_s(K \otimes_R M)$. Now one has $\sup(K \otimes_R M)_\mathfrak{p} = s$, and the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of $R_\mathfrak{p}$ belongs to $\operatorname{Ass}_{R_\mathfrak{p}} \mathrm{H}_s((K \otimes_R M)_\mathfrak{p})$, whence $(\diamond)$ reads,

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = n - \sup(K \otimes_R M)_\mathfrak{p} = n - \sup(K \otimes_R M) = \mathfrak{a}\text{-depth}_R M\,,$$

by 16.2.16(b) and 14.3.10.

Assume now that $\mathfrak{a}\text{-depth}_R M = -\infty$ holds; this implies $\sup M = \infty$ by 16.2.16. The goal is to show that for every integer $v > 0$ there is a prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ with $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant -v$. First we reduce to the case where $\mathrm{H}(M)$ is annihilated by $\mathfrak{a}$: By 14.4.15 one has $\mathfrak{a}\text{-depth}_R(K \otimes_R M) = \mathfrak{a}\text{-depth}_R M - n = -\infty$, and $(\diamond)$ can be rewritten as $\operatorname{depth}_{R_\mathfrak{p}}(K \otimes_R M)_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} - n$, so one can replace M by $K \otimes_R M$, i.e. one can assume that $\mathrm{H}(M)$ is annihilated by $\mathfrak{a}$, see 11.4.6(a). For $v \in \mathbb{Z}$ set

$$U_v = \{\mathfrak{q} \in \mathrm{V}(\mathfrak{a}) \mid v \leqslant \sup M_\mathfrak{q} < \infty\}$$

and notice that these sets form a descending chain: $\cdots \supseteq U_v \supseteq U_{v+1} \supseteq \cdots$. Assume first that each set U_v is non-empty. Given $v \in \mathbb{Z}$ choose a prime ideal $\mathfrak{q}$ in U_v. Set $u = \sup M_\mathfrak{q}$ and let $\mathfrak{p}$ be an associated prime ideal of the R-module $\mathrm{H}_u(M_\mathfrak{q})$. By 16.2.16(b) one now has $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{q})_\mathfrak{p} = -\sup M_\mathfrak{q} = -u \leqslant -v$ as desired. Assume now that U_w is empty for some integer w and hence $U_v = \varnothing$ for all $v \geqslant w$. Since $\operatorname{Supp}_R M$ is contained in $\mathrm{V}(\mathfrak{a})$, see 14.1.3, one now has

$$\textstyle\bigcup_{v \geqslant w} \operatorname{Supp}_R \mathrm{H}_v(M) = \{\mathfrak{q} \in \mathrm{V}(\mathfrak{a}) \mid \sup M_\mathfrak{q} = \infty\}\,;$$

this set is non-empty as $\sup M = \infty$ holds. Let $\mathfrak{p}$ be a minimal element of the set; it follows that $\operatorname{Supp}_{R_\mathfrak{p}} \mathrm{H}_v(M_\mathfrak{p}) \subseteq \{\mathfrak{p}_\mathfrak{p}\}$ holds for all $v \geqslant w$. Per 14.1.3 the $R_\mathfrak{p}$-modules $\mathrm{H}_v(M_\mathfrak{p})$ for $v \geqslant w$ are now $\mathfrak{p}_\mathfrak{p}$-torsion, so $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = -\infty$ holds by 16.2.20. $\square$

17.6.4 Proposition. *Let M be an R-complex*; *one has*

$$\sup M \;=\; \sup\{\sup M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\;.$$

PROOF. By 17.6.3, 16.2.16, and 14.1.13 one has

$$\begin{aligned}\sup M \;&=\; -\inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\\ &=\; \sup\{-\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\\ &\leqslant\; \sup\{\sup M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\\ &\leqslant\; \sup\{\sup M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \sup M\;. \qquad\square\end{aligned}$$

For a prime ideal the next inequality is stronger than the inequality in 16.4.6, as illustrated by the example above. The inequality also holds for a derived $\mathfrak{m}$-complete complex over a local ring $(R,\mathfrak{m})$, see 18.3.27.

17.6.5 Corollary. *Let R be local, $\mathfrak{a}$ a proper ideal in R, and M an R-complex. If M is in $\mathcal{D}^{\mathrm{f}}(R)$, then there is an inequality,*

$$\mathfrak{a}\text{-}\operatorname{depth}_R M + \dim R/\mathfrak{a} \;\geqslant\; \operatorname{depth}_R M\;.$$

PROOF. One can assume that $\mathfrak{a}$-$\operatorname{depth}_R M$ is finite; in particular, M is not acyclic, see 14.3.11. Notice next from 16.2.21 that the inequality is trivial if $\mathrm{H}(M)$ is not bounded above, so one can further assume that M belongs to $\mathcal{D}_{\sqsubset}(R)$. Now it follows from 14.3.16 that $\mathfrak{a}$-$\operatorname{depth}_R M$ is an integer. By 17.6.3 there is a prime ideal $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$ with $\mathfrak{a}$-$\operatorname{depth}_R M = \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$, so via 16.4.6 one gets,

$$\mathfrak{a}\text{-}\operatorname{depth}_R M + \dim R/\mathfrak{a} \;\geqslant\; \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \;\geqslant\; \operatorname{depth}_R M\;. \qquad\square$$

17.6.6 Proposition. *Let $\mathfrak{a}$ be an ideal in R, generated by a sequence $\boldsymbol{x}$, and M be an R-complex. If $d = \mathfrak{a}$-$\operatorname{depth}_R M$ is an integer, then there are equalities,*

$$\operatorname{Ass}_R \mathrm{H}^d_{\mathfrak{a}}(M) \;=\; \operatorname{Ass}_R \operatorname{Ext}^d_R(R/\mathfrak{a}, M) \;=\; \operatorname{Ass}_R \operatorname{Ext}^d_R(\mathrm{K}^R(\boldsymbol{x}), M)\;.$$

Furthermore, this set is contained in $\mathrm{V}(\mathfrak{a})$.

PROOF. The module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion by 11.3.24, so its classic support, and thereby $\operatorname{Ass}_R \mathrm{H}^d_{\mathfrak{a}}(M)$, is contained in the set $\mathrm{V}(\mathfrak{a})$ by 14.1.3. This explains the second equality below. The first equality holds by 17.1.1 and 15.1.10.

$$\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^d_{\mathfrak{a}}(M)) \;=\; \mathrm{V}(\mathfrak{a}) \cap \operatorname{Ass}_R \mathrm{H}^d_{\mathfrak{a}}(M) \;=\; \operatorname{Ass}_R \mathrm{H}^d_{\mathfrak{a}}(M)\;.$$

The assertion now follows from 14.4.5. $\square$

REMARK. Understanding the associated prime ideals of local cohomology modules has been a problem of sustained interest in commutative algebra. We do not treat the problem within this text, but we notice that for an ideal $\mathfrak{a}$ in R and a complex M in $\mathcal{D}^{\mathrm{f}}(R)$ of $\mathfrak{a}$-depth d, it follows from 17.6.6 that the local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$ has only finitely many associated prime ideals. Indeed, with $\boldsymbol{x}$ a sequence that generates $\mathfrak{a}$, the module $\operatorname{Ext}^d_R(\mathrm{K}^R(\boldsymbol{x}), M)$ is finitely generated by 15.4.3. In [138] Huneke asked if every local cohomology module $\mathrm{H}^m_{\mathfrak{a}}(M)$ of a finitely generated R-module M has only finitely many associated prime ideals; a negative answer was provided by Singh [237].

17.6.7 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex. If $d = \mathfrak{a}\text{-depth}_R M$ is an integer, then one has*

$$\{\mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \mid \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = d\} \;=\; \operatorname{Ass}_R \mathrm{H}^d_\mathfrak{a}(M)\,.$$

Proof. It is known from 17.6.6 that the set $\operatorname{Ass}_R \mathrm{H}^d_\mathfrak{a}(M)$ is contained in $\mathrm{V}(\mathfrak{a})$. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, the (in)equalities below hold by 14.4.3, 14.1.11(c), 14.1.25, 16.2.16, and 16.2.14:

$$(\diamond)\qquad \begin{aligned} d &= -\sup \mathsf{R}\Gamma_\mathfrak{a}(M) \\ &\leqslant -\sup \mathsf{R}\Gamma_\mathfrak{a}(M)_\mathfrak{p} \\ &= -\sup \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}) \\ &\leqslant \operatorname{depth}_{R_\mathfrak{p}} \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}) \\ &= \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}\,. \end{aligned}$$

If $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = d$ holds then, in particular, the second inequality in $(\diamond)$ is an equality, so 16.2.16(b) yields that the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of $R_\mathfrak{p}$ is associated to the $R_\mathfrak{p}$-module $\mathrm{H}_{-d}(\mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})) \cong \mathrm{H}_{-d}(\mathsf{R}\Gamma_\mathfrak{a}(M))_\mathfrak{p}$; whence $\mathfrak{p}$ is associated to the R-module $\mathrm{H}_{-d}(\mathsf{R}\Gamma_\mathfrak{a}(M)) = \mathrm{H}^d_\mathfrak{a}(M)$. Conversely, if $\mathfrak{p}$ is associated to this module, then the first inequality in $(\diamond)$ is an equality, so $\sup \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}) = -d$ holds, and by 16.2.16(b) also the second inequality is an equality as $\mathfrak{p}_\mathfrak{p}$ is associated to $\mathrm{H}_{-d}(\mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}))$. □

17.6.8 Theorem. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R and M an R-complex. The following equalities hold.*

(a) $\mathfrak{ab}\text{-depth}_R M \;=\; (\mathfrak{a} \cap \mathfrak{b})\text{-depth}_R M \;=\; \min\{\mathfrak{a}\text{-depth}_R M, \mathfrak{b}\text{-depth}_R M\}\,.$

(b) $\mathfrak{ab}\text{-width}_R M \;=\; (\mathfrak{a} \cap \mathfrak{b})\text{-width}_R M \;=\; \min\{\mathfrak{a}\text{-width}_R M, \mathfrak{b}\text{-width}_R M\}\,.$

Proof. (a): One has $\mathrm{V}(\mathfrak{ab}) = \mathrm{V}(\mathfrak{a} \cap \mathfrak{b})$, so the first equality follows immediately from 17.6.3. As $\mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$ holds, further applications of 17.6.3 yield

$$\begin{aligned} &(\mathfrak{a} \cap \mathfrak{b})\text{-depth}_R M \\ &\qquad= \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})\} \\ &\qquad= \min\{\inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}, \inf\{\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{b})\}\} \\ &\qquad= \min\{\mathfrak{a}\text{-depth}_R M, \mathfrak{b}\text{-depth}_R M\}\,. \end{aligned}$$

(b): Let E be a faithfully injective R-module and invoke 14.4.14 and part (a). □

Width and Localization

In contrast to the last assertion in 17.6.3, the next example shows that one can not determine the infimum of a complex from the width of its localizations. For a complex over a ring of finite Krull dimension one can, however, infer from a uniform lower bound of the width of its localizations that the complex has bounded below homology, see 17.6.10.

17.6.9 Example. Let $(R, \mathfrak{m})$ be local of depth $d > 0$ and $\boldsymbol{x}$ be a sequence that generates $\mathfrak{m}$. By 15.1.27 one has $\operatorname{supp}_R \check{\mathrm{C}}^R(\boldsymbol{x}) = \{\mathfrak{m}\}$. From 13.3.18, 16.2.3, 16.2.5, and 16.2.14 one gets

$$\operatorname{width}_R \check{\mathrm{C}}^R(\boldsymbol{x}) \;=\; \operatorname{width}_R \mathsf{R}\Gamma_{\mathfrak{m}}(R) \;=\; \operatorname{width}_R R \;=\; 0 \quad \text{and} \quad \sup \check{\mathrm{C}}^R(\boldsymbol{x}) \;=\; -d\;.$$

In particular, one has $\inf \check{\mathrm{C}}^R(\boldsymbol{x}) \leqslant -d < 0$; in fact, $\inf \check{\mathrm{C}}^R(\boldsymbol{x}) = -\dim R$ by 18.3.23.

The example above shows that the inequality in the next lemma can be an equality.

17.6.10 Lemma. *Assume that R has finite Krull dimension and let M be an R-complex. There are* (*in*)*equalities,*

$$\begin{aligned}\inf M + \dim R \;&\geqslant\; \inf\{\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \inf\{\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\}\;.\end{aligned}$$

Proof. The equality holds by 17.1.6. To prove the inequality, one can assume that M is not acyclic, otherwise it is trivial. One can similarly, see 15.1.15, assume that $w = \inf\{\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$ is an integer. For every prime ideal $\mathfrak{p}$ in R one now has

$$(\diamond) \qquad \mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)_\mathfrak{p} \;=\; \mathrm{H}_v(\kappa(\mathfrak{p}) \otimes^{\mathsf{L}}_R M) \;=\; 0 \quad \text{for} \quad v < w - \dim R/\mathfrak{p}\;,$$

by 14.1.15, 14.1.16(b), and 16.2.1. We proceed to argue that the inequality

$$(\flat) \qquad \inf\,(K \otimes^{\mathsf{L}}_R M) \;\geqslant\; w - \dim_R K$$

holds for every finitely generated R-module K. The asserted inequality is, in view of the unitor 12.3.3, the special case $K = R$.

If $\dim_R K = 0$ holds, then every element $\mathfrak{p}$ in $\operatorname{Supp}_R K$ is a maximal ideal, whence $\kappa(\mathfrak{p}) = R/\mathfrak{p}$ and $\dim R/\mathfrak{p} = 0$ hold. For $v < w$ one now gets $\mathrm{H}_v(K \otimes^{\mathsf{L}}_R M) = 0$ from $(\diamond)$ and 12.4.1 applied to the functor $\mathrm{H}_v(- \otimes^{\mathsf{L}}_R M)$, that is, $(\flat)$ holds. Now let $d > 0$ and assume that $(\flat)$ holds for all finitely generated R-modules K of Krull dimension less than d. To prove that $(\flat)$ holds if $\dim_R K = d$, it suffices by 12.4.1 to show that for every prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R K$ one has $\mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) = 0$ for $v < w - d$. For such a prime ideal $\mathfrak{p}$ one has $\dim R/\mathfrak{p} \leqslant d$. For every element $x \in R \setminus \mathfrak{p}$ multiplication by x on $R/\mathfrak{p}$ is injective, so there is an exact sequence $0 \longrightarrow R/\mathfrak{p} \xrightarrow{\;x\;} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + (x)) \longrightarrow 0$ and per 6.5.24 and 12.2.8 an induced distinguished triangle,

$$R/\mathfrak{p} \otimes^{\mathsf{L}}_R M \xrightarrow{\;x\;} R/\mathfrak{p} \otimes^{\mathsf{L}}_R M \longrightarrow R/(\mathfrak{p} + (x)) \otimes^{\mathsf{L}}_R M \longrightarrow \Sigma(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)\;.$$

Since $\dim R/(\mathfrak{p}+(x)) < d$ holds, it follows from the induction hypothesis and 6.5.21 that for $v \leqslant w - d$ multiplication by x on the module $\mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)$ is injective, even an isomorphism. Thus the canonical map $\mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) \to \mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)_\mathfrak{p}$ is injective. Per $(\diamond)$ one has $\mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M)_\mathfrak{p} = 0$, so $\mathrm{H}_v(R/\mathfrak{p} \otimes^{\mathsf{L}}_R M) = 0$ holds for $v < w - d$ as desired. □

If R is an integral domain of finite Krull dimension and M the injective hull of R, which by 1.3.33 is the field of fractions of R, then it follows from 15.1.12 that both inequalities in the next result are equalities.

17.6.11 Proposition. *Assume that R has finite Krull dimension and let M be an R-complex. There are inequalities,*

$$\begin{aligned}\inf\{\inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\} &\geqslant \inf M \\ &\geqslant \inf\{\inf M_{\mathfrak{p}} + \dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} - \dim R\,.\end{aligned}$$

In particular, M is in $\mathcal{D}_{\sqsupset}(R)$ if and only if $\inf\{\inf M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M\} > -\infty$ holds.

Proof. The left-hand inequality follows from 14.1.13. The right-hand inequality comes from 17.6.10 and the inequality in 16.2.5. If M is acyclic, then $\inf M_{\mathfrak{p}} = \infty$ holds for every prime ideal $\mathfrak{p}$. The "in particular" statement is immediate from the displayed inequalities. □

Supremum and Depth

The remaining results in this section are of a technical nature; they find applications in Chap. 18.

17.6.12 Lemma. *For complexes $K \in \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and $M \in \mathcal{D}_{\sqsubset}(R)$ there are equalities,*

$$\begin{aligned}-\sup \mathsf{RHom}_R(K,M) &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \inf K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{supp}_R M \cap \operatorname{supp}_R K\} \\ &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \inf K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.\end{aligned}$$

Proof. By 17.1.10(b) one has $\operatorname{supp}_R \mathsf{RHom}_R(K,M) = \operatorname{supp}_R K \cap \operatorname{supp}_R M$, and for every prime ideal $\mathfrak{p}$ in this set, 14.1.23 yields the first equality in the next display. The second equality follows, in view of 17.1.6 from 16.2.25.

$$\operatorname{depth}_{R_{\mathfrak{p}}} \mathsf{RHom}_R(K,M)_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}} \mathsf{RHom}_{R_{\mathfrak{p}}}(K_{\mathfrak{p}},M_{\mathfrak{p}}) = \inf K_{\mathfrak{p}} + \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\,.$$

The first of the asserted equalities now holds by 17.6.3, and the second follows as one by 15.1.9, 14.1.12, and 17.1.6 has $\inf K_{\mathfrak{p}} = \infty$ or $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \infty$ for prime ideals $\mathfrak{p} \notin \operatorname{supp}_R K \cap \operatorname{supp}_R M$. □

17.6.13 Proposition. *Let K be a finitely generated R-module and M a complex in $\mathcal{D}_{\sqsubset}(R)$. There are equalities,*

$$\begin{aligned}-\sup \mathsf{RHom}_R(K,M) &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R K\} \\ &= \inf\{\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{V}(0 :_R K)\} \\ &= -\sup \mathsf{RHom}_R(R/(0 :_R K),M)\,.\end{aligned}$$

Proof. Recall from 15.1.9 that $\operatorname{supp}_R K = \operatorname{Supp}_R K$ holds. The first equality now follows from 17.6.12 as one has $\inf K_{\mathfrak{p}} = 0$ for $\mathfrak{p} \in \operatorname{Supp}_R K$. The second equality holds by 14.1.1, and the final equality follows from another application of 17.6.12, as one has $\mathrm{V}(0 :_R K) = \operatorname{Supp}_R(R/(0 :_R K))$. □

17.6.14 Proposition. *Let K be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and M a complex in $\mathcal{D}_{\sqsubset}(R)$. There is an equality,*

$$-\sup \mathsf{RHom}_R(K,M) \;=\; \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(K),M)+v \mid v\in\mathbb{Z}\}\;.$$

PROOF. Denote by s the quantity on the left-hand side of the asserted equality and by t the quantity on the right-hand side. To prove the inequality $s \geqslant t$ it suffices by 17.6.12 to show that $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \inf K_\mathfrak{p} \geqslant t$ holds for all $\mathfrak{p}\in\operatorname{supp}_R M\cap\operatorname{supp}_R K$. Given such a prime ideal $\mathfrak{p}$ set $w=\inf K_\mathfrak{p}$, it follows from 15.1.9 and 14.1.12 that w is an integer. Since $\mathfrak{p}$ is in $\operatorname{Supp}_R \mathrm{H}_w(K)$ one gets from 17.6.13 the desired inequality,

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \inf K_\mathfrak{p} \;\geqslant\; -\sup \mathsf{RHom}_R(\mathrm{H}_w(K),M)+w \;\geqslant\; t\;.$$

For the opposite inequality, $t\geqslant s$, it suffices to show that for every $v\in\mathbb{Z}$ with $\mathrm{H}(\mathsf{RHom}_R(\mathrm{H}_v(K),M))\neq 0$ one has $-\sup \mathsf{RHom}_R(\mathrm{H}_v(K),M)+v\geqslant s$. Given such a v, notice from 7.6.7 that $-\sup \mathsf{RHom}_R(\mathrm{H}_v(K),M)$ is an integer, whence 17.6.13 yields a prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R \mathrm{H}_v(K)$ with $-\sup \mathsf{RHom}_R(\mathrm{H}_v(K),M) = \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$. Now 17.6.12 yields

$$-\sup \mathsf{RHom}_R(\mathrm{H}_v(K),M)+v \;\geqslant\; \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \inf K_\mathfrak{p} \;\geqslant\; s\;. \qquad \square$$

The next inequality also holds, see 18.3.24, under the asumption that M is derived $\mathfrak{m}$-torsion or N is derived $\mathfrak{m}$-complete where, of course, $\mathfrak{m}$ denotes the maximal ideal of the local ring R.

17.6.15 Proposition. *Let R be local and M and N be complexes in $\mathcal{D}^{\mathrm{f}}(R)$ that are not acyclic. There is an inequality,*

$$-\sup \mathsf{RHom}_R(M,N) \;\geqslant\; \operatorname{depth}_R N - \dim_R M\;.$$

PROOF. First notice that if $\mathrm{H}(M)$ is not bounded below, then one has $\dim_R M=\infty$ by 14.2.4, and if $\mathrm{H}(N)$ is not bounded above, then one has $\operatorname{depth}_R N=-\infty$ by 16.2.21; in either case the inequality his trivial. Assume now that M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and N to $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. The first equality in the chain below comes from 17.6.12 and the last of them holds by 14.2.6. The inequality follows from 16.4.6.

$$\begin{aligned}
-\sup \mathsf{RHom}_R(M,N) &= \inf\{\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p}\in\operatorname{Spec} R\}\\
&\geqslant \inf\{\operatorname{depth}_R N - \dim R/\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p}\in\operatorname{Spec} R\}\\
&= \operatorname{depth}_R N - \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p}\in\operatorname{Spec} R\}\\
&= \operatorname{depth}_R N - \dim_R M\;. \qquad \square
\end{aligned}$$

INFIMUM AND KRULL DIMENSION

17.6.16 Proposition. *Let K be a finitely generated R-module and M a complex in $\mathcal{D}_{\sqsupset}(R)$. There is an equality,*

$$\inf\,(K\otimes^{\mathsf{L}}_R M) \;=\; \inf\,(R/(0:_R K)\otimes^{\mathsf{L}}_R M)\;.$$

Proof. Let E be a faithfully injective R-module. In the computation below, the 1st and 5th equalities hold by 2.5.7(b), while the 2nd and 4th both come from commutativity 12.3.5 and adjunction 12.3.18, and the 3rd holds by 17.6.13.

$$\begin{aligned}\inf(K \otimes^{\mathsf{L}}_R M) &= -\sup \mathsf{RHom}_R(K \otimes^{\mathsf{L}}_R M, E)\\ &= -\sup \mathsf{RHom}_R(K, \mathsf{RHom}_R(M,E))\\ &= -\sup \mathsf{RHom}_R(R/(0:_R K), \mathsf{RHom}_R(M,E))\\ &= -\sup \mathsf{RHom}_R(R/(0:_R K) \otimes^{\mathsf{L}}_R M, E)\\ &= \inf(R/(0:_R K) \otimes^{\mathsf{L}}_R M)\,. \qquad \square\end{aligned}$$

17.6.17 Proposition. *Let K be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and M a complex in $\mathcal{D}_{\sqsupset}(R)$. There is an equality,*

$$\inf(K \otimes^{\mathsf{L}}_R M) = \inf\{\inf(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M) + v \mid v \in \mathbb{Z}\}\,.$$

Proof. Let E be a faithfully injective R-module. In the next computation, the 1th and 5th equalities hold by 2.5.7(b), the 2nd and 4th equalities both follow from commutativity 12.3.5 and adjunction 12.3.18, and the 3rd holds by 17.6.14.

$$\begin{aligned}\inf(K \otimes^{\mathsf{L}}_R M) &= -\sup \mathsf{RHom}_R(K \otimes^{\mathsf{L}}_R M, E)\\ &= -\sup \mathsf{RHom}_R(K, \mathsf{RHom}_R(M,E))\\ &= \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(K), \mathsf{RHom}_R(M,E)) + v \mid v \in \mathbb{Z}\}\\ &= \inf\{-\sup \mathsf{RHom}_R(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M, E) + v \mid v \in \mathbb{Z}\}\\ &= \inf\{\inf(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M) + v \mid v \in \mathbb{Z}\}\,. \qquad \square\end{aligned}$$

17.6.18 Proposition. *Let K be a finitely generated R-module and M a complex in $\mathcal{D}_{\sqsupset}(R)$. There is an equality,*

$$\dim_R(K \otimes^{\mathsf{L}}_R M) = \dim_R(R/(0:_R K) \otimes^{\mathsf{L}}_R M)\,.$$

Proof. For every prime ideal $\mathfrak{p}$ in R there are isomorphisms,

$$(\dagger) \qquad (R/(0:_R K))_{\mathfrak{p}} \cong R_{\mathfrak{p}}/(0:_R K)_{\mathfrak{p}} \cong R_{\mathfrak{p}}/(0:_{R_{\mathfrak{p}}} K_{\mathfrak{p}})\,,$$

see 15.3.34. Thus it follows from 14.2.7 and 14.1.15 that it suffices to show the asserted equality under the assumption that R is local.

For every prime ideal $\mathfrak{p}$ in R one has

$$\begin{aligned}\inf(K \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} &= \inf(K_{\mathfrak{p}} \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}})\\ &= \inf(R_{\mathfrak{p}}/(0:_{R_{\mathfrak{p}}} K_{\mathfrak{p}}) \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}})\\ &= \inf(R/(0:_R K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}}\end{aligned}$$

by 14.1.15, 17.6.16, and (†). This explains the second equality in the computation below; the first and last equalities hold by 14.2.6 as $\dim R$ is finite.

$$\begin{aligned}\dim_R(K \otimes^{\mathsf{L}}_R M) &= \sup\{\dim R/\mathfrak{p} - \inf(K \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup\{\dim R/\mathfrak{p} - \inf(R/(0:_R K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\end{aligned}$$

$$= \dim_R(R/(0:_R K) \otimes^{\mathsf{L}}_R M) . \qquad \square$$

17.6.19 Proposition. *Let K be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and M a complex in $\mathcal{D}_{\sqsupset}(R)$. There is an equality,*

$$\dim_R(K \otimes^{\mathsf{L}}_R M) = \sup\{\dim_R(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M) - v \mid v \in \mathbb{Z}\} .$$

Proof. It follows from 14.2.7, 14.1.15, and 14.1.11(a) that it suffices to show the asserted equality under the assumption that R is local.

Let $\mathfrak{p}$ be a prime ideal in R. From 14.1.15, 17.6.17, and 14.1.11 one gets

$$\begin{aligned} \inf\,(K \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} &= \inf\,(K_{\mathfrak{p}} \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) \\ &= \inf\{\inf\,(\mathrm{H}_v(K_{\mathfrak{p}}) \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) + v \mid v \in \mathbb{Z}\} \\ &= \inf\{\inf\,(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} + v \mid v \in \mathbb{Z}\} . \end{aligned}$$

This explains the second equality in the next computation; the first and last equalities hold by 14.2.6 as R has finite Krull dimension.

$$\begin{aligned} &\dim_R(K \otimes^{\mathsf{L}}_R M) \\ &= \sup\{\dim R/\mathfrak{p} - \inf\,(K \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\dim R/\mathfrak{p} - \inf\{\inf\,(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} + v \mid v \in \mathbb{Z}\} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\dim R/\mathfrak{p} + \sup\{-\inf\,(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} - v \mid v \in \mathbb{Z}\} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\dim R/\mathfrak{p} - \inf\,(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} - v \mid \mathfrak{p} \in \operatorname{Spec} R,\ v \in \mathbb{Z}\} \\ &= \sup\{\dim_R(\mathrm{H}_v(K) \otimes^{\mathsf{L}}_R M) - v \mid v \in \mathbb{Z}\} . \qquad \square \end{aligned}$$

17.6.20 Proposition. *Let M and N be finitely generated R-modules. One has,*

$$\dim_R(M \otimes^{\mathsf{L}}_R M) = \dim_R(M \otimes_R M) .$$

Proof. As M and N are finitely generated modules, 14.1.18 yields

$$\operatorname{Supp}_R(N \otimes_R M) = \operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \operatorname{Supp}_R(N \otimes^{\mathsf{L}}_R M) ,$$

and for prime ideals in this set one has $\inf\,(N \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} = \inf\,(N_{\mathfrak{p}} \otimes^{\mathsf{L}}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) = 0$ by 14.1.15 and 16.2.10. Now 14.2.6 yields

$$\dim_R(N \otimes^{\mathsf{L}}_R M) = \sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(N \otimes_R M)\} = \dim_R(N \otimes_R M) . \quad \square$$

Ring Extensions and Prime Ideals

We close this section with a result that only finds applications later, from Chap. 19.

17.6.21 Theorem. *Let $\mathfrak{p}$ be a prime ideal in R and S an R algebra that is faithfully flat as an R-module. There exists a prime ideal $\mathfrak{q}$ in S with $\mathfrak{q} \cap R = \mathfrak{p}$, and for every such prime ideal the ring $S_{\mathfrak{q}}$ is an $R_{\mathfrak{p}}$-algebra with the following properties.*

(a) *The maximal ideals $\mathfrak{p}_{\mathfrak{p}}$ and $\mathfrak{q}_{\mathfrak{q}}$ of the local rings $R_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$ satisfy $\mathfrak{p}_{\mathfrak{p}} S_{\mathfrak{q}} \subseteq \mathfrak{q}_{\mathfrak{q}}$.*

(b) $S_{\mathfrak{q}}$ *is faithfully flat as an* $R_{\mathfrak{p}}$*-module.*

(c) *For every* R*-complex* M *there is an isomorphism in* $\mathcal{C}(S_{\mathfrak{q}})$*,*

$$S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \;\cong\; (S \otimes_R M)_{\mathfrak{q}} \;.$$

Moreover, one can choose $\mathfrak{q}$ *with* $\mathfrak{q} \cap R = \mathfrak{p}$ *such that the following assertions hold.*

(d) *For every* $S_{\mathfrak{q}}$*-complex* X *there are equalities,*

$$\operatorname{depth}_{S_{\mathfrak{q}}} X \;=\; \operatorname{depth}_{R_{\mathfrak{p}}} X \quad \textit{and} \quad \operatorname{width}_{S_{\mathfrak{q}}} X \;=\; \operatorname{width}_{R_{\mathfrak{p}}} X \;.$$

(e) *For every* $R_{\mathfrak{p}}$*-complex* N *there are equalities,*

$$\operatorname{depth}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N) \;=\; \operatorname{depth}_{R_{\mathfrak{p}}} N \quad \textit{and} \quad \operatorname{width}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} N) \;=\; \operatorname{width}_{R_{\mathfrak{p}}} N \;.$$

(f) *For every* R*-complex* M *there are equalities,*

$$\operatorname{depth}_{S_{\mathfrak{q}}}(S \otimes_R M)_{\mathfrak{q}} \;=\; \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \quad \textit{and} \quad \operatorname{width}_{S_{\mathfrak{q}}}(S \otimes_R M)_{\mathfrak{q}} \;=\; \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \;.$$

In particular, $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth} S_{\mathfrak{q}}$ *holds.*

PROOF. Set $T = \kappa(\mathfrak{p}) \otimes_R S$. In the commutative diagram of ring homomorphisms,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \kappa(\mathfrak{p}) \arrow[r] & T \,, \end{tikzcd}$$

the upper horizontal homomorphism is the structure map. As S is faithfully flat as an R-module, the ring T is non-zero, so there exists a prime ideal $\mathfrak{r}$ in T. The prime $\mathfrak{q} = \mathfrak{r} \cap S$ in S satisfies $\mathfrak{q} \cap R = \mathfrak{p}$; indeed, as $\kappa(\mathfrak{p})$ is a field, the commutative diagram above shows that one has

$$\mathfrak{q} \cap R \;=\; (\mathfrak{r} \cap S) \cap R \;=\; (\mathfrak{r} \cap \kappa(\mathfrak{p})) \cap R \;=\; (0) \cap R \;=\; \mathfrak{p} \;.$$

Now, let $\mathfrak{q}$ be any prime ideal in S with $\mathfrak{q} \cap R = \mathfrak{p}$; it follows that the structure map $R \to S$ maps $R \setminus \mathfrak{p}$ to $S \setminus \mathfrak{q}$, so $S_{\mathfrak{q}}$ is an $R_{\mathfrak{p}}$-algebra with the induced structure map.

(a): There is a commutative diagram of ring homomorphisms,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R_{\mathfrak{p}} \arrow[r] & S_{\mathfrak{q}} \,, \end{tikzcd}$$

which explains the second equality below. The first and last equalities are trivial, and the inclusion holds as one has $\mathfrak{p}S = (\mathfrak{q} \cap R)S \subseteq \mathfrak{q}$.

$$\mathfrak{p}_{\mathfrak{p}} S_{\mathfrak{q}} \;=\; (\mathfrak{p}R_{\mathfrak{p}})S_{\mathfrak{q}} \;=\; (\mathfrak{p}S)S_{\mathfrak{q}} \;\subseteq\; \mathfrak{q}S_{\mathfrak{q}} \;=\; \mathfrak{q}_{\mathfrak{q}} \;.$$

(c): For an R-complex M one gets from two applications of 2.1.50 and two applications of 12.1.18 the following isomorphisms in $\mathcal{C}(S_{\mathfrak{q}})$,

$$S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \;\cong\; S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} (R_{\mathfrak{p}} \otimes_R M) \;\cong\; S_{\mathfrak{q}} \otimes_R M \;\cong\; S_{\mathfrak{q}} \otimes_S (S \otimes_R M) \;\cong\; (S \otimes_R M)_{\mathfrak{q}} \;.$$

(b): The isomorphism from part (c) and idempotence of localization imply that on the category of $R_\mathfrak{p}$-modules, the functor $S_\mathfrak{q} \otimes_{R_\mathfrak{p}} -$ is naturally isomorphic to $(S \otimes_R -)_\mathfrak{q}$. This functor is exact as the tensor product functor $S \otimes_R -$ is exact, by flatness of S over R, and the localization functor $(-)_\mathfrak{q}$ is exact, see 2.1.50. Hence $S_\mathfrak{q}$ is flat as an $R_\mathfrak{p}$-module. The assertion now follows from part (a) and 16.4.21.

It remains to prove the "moreover" statement. First choose any prime ideal $\mathfrak{r}$ in S with $\mathfrak{r} \cap R = \mathfrak{p}$. Note that every prime ideal $\mathfrak{q}$ in S with $\mathfrak{p}S \subseteq \mathfrak{q} \subseteq \mathfrak{r}$ satisfies $\mathfrak{q} \cap R = \mathfrak{p}$. Now choose such a prime ideal $\mathfrak{q}$ such that $\mathfrak{q}/\mathfrak{p}S$ is a minimal prime ideal in $S/\mathfrak{p}S$. The prime ideals in $S_\mathfrak{q}$ are in order preserving one-to-one correspondence with prime ideals in S contained in $\mathfrak{q}$, so in the ring $S_\mathfrak{q}$, the only prime ideal that contains $(\mathfrak{p}S)_\mathfrak{q}$ is the unique maximal ideal $\mathfrak{q}_\mathfrak{q}$. It follows that in the ring $S_\mathfrak{q}$, the radical of the ideal $(\mathfrak{p}S)_\mathfrak{q}$ is $\mathfrak{q}_\mathfrak{q}$. As one has $\mathfrak{p}_\mathfrak{p}S_\mathfrak{q} = (\mathfrak{p}S)S_\mathfrak{q} = (\mathfrak{p}S)_\mathfrak{q}$, see the proof of part (a), the radical of $\mathfrak{p}_\mathfrak{p}S_\mathfrak{q}$ is $\mathfrak{q}_\mathfrak{q}$.

(d): As argued above, one has $\sqrt{(\mathfrak{p}_\mathfrak{p}S_\mathfrak{q})} = \mathfrak{q}_\mathfrak{q}$ in $S_\mathfrak{q}$. For an $S_\mathfrak{q}$-complex X, the equalities below hold by 16.2.12, 14.4.4 and 14.3.19.

$$\operatorname{depth}_{S_\mathfrak{q}} X = \mathfrak{q}_\mathfrak{q}\text{-}\operatorname{depth}_{S_\mathfrak{q}} X = \mathfrak{p}_\mathfrak{p}S_\mathfrak{q}\text{-}\operatorname{depth}_{S_\mathfrak{q}} X = \mathfrak{p}_\mathfrak{p}\text{-}\operatorname{depth}_{R_\mathfrak{p}} X = \operatorname{depth}_{R_\mathfrak{p}} X .$$

Similarly, 16.2.1, 14.4.9, and 14.3.31 yield:

$$\operatorname{width}_{S_\mathfrak{q}} X = \mathfrak{q}_\mathfrak{q}\text{-}\operatorname{width}_{S_\mathfrak{q}} X = \mathfrak{p}_\mathfrak{p}S_\mathfrak{q}\text{-}\operatorname{width}_{S_\mathfrak{q}} X = \mathfrak{p}_\mathfrak{p}\text{-}\operatorname{width}_{R_\mathfrak{p}} X = \operatorname{width}_{R_\mathfrak{p}} X .$$

(e): For an $R_\mathfrak{p}$-complex N one has

$$\operatorname{depth}_{S_\mathfrak{q}}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = \operatorname{depth}_{R_\mathfrak{p}}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = \operatorname{depth}_{R_\mathfrak{p}} N ,$$

where the first equality holds by part (d) and the second by 16.2.12 and 14.3.15 as $S_\mathfrak{q}$ per (b) is a faithfully flat $R_\mathfrak{p}$-module. Similarly, part (d), 16.2.1, and 14.3.26 yield:

$$\operatorname{width}_{S_\mathfrak{q}}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = \operatorname{width}_{R_\mathfrak{p}}(S_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = \operatorname{width}_{R_\mathfrak{p}} N .$$

(f): The asserted equalites follow from the equalities in part (e) applied to $N = M_\mathfrak{p}$, combined with the isomorphism in part (c). The final assertion follows from the unitor 12.1.5 and the first equality applied to $M = R$. □

Exercises

E 17.6.1 Let $\mathfrak{a}$ be an ideal in R and M an R-complex. Show that for every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ there are inequalities $\mathfrak{a}\text{-}\operatorname{width}_R M \leqslant \mathfrak{a}_\mathfrak{p}\text{-}\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$.

E 17.6.2 Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. Show that for every prime ideal $\mathfrak{p}$ in R one has

$$\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k})) = \operatorname{depth} R_\mathfrak{p} ,$$

and conclude that $\inf\{\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, \mathrm{E}_R(\boldsymbol{k})) \mid \mathfrak{p} \in \operatorname{cosupp}_R \mathrm{E}_R(\boldsymbol{k})\} = 0$ holds.

E 17.6.3 Let R be an equidimensional catenary local ring and M an R-complex. Show that there are inequalities,

$$\inf\{\inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} \geqslant \inf M \geqslant \inf\{\inf M_\mathfrak{p} - \dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{supp}_R M\} .$$

E 17.6.4 Let $K \in \mathcal{D}_\sqsubset(R)$ and M be a complex in $\mathcal{D}^\mathrm{f}_\square(R)$ with $\operatorname{pd}_R M$ finite. Show that

$$\sup(K \otimes^{\mathsf{L}}_R M) = \sup\{\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} K_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

holds. Conclude that if $(R, \mathfrak{m})$ is local and K and M finitely generated R-modules with $\operatorname{pd}_R M < \infty$ and $\operatorname{Supp}_R K \cap \operatorname{Supp}_R M = \{\mathfrak{m}\}$, then one has

$$\sup\{ m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(K, M) \neq 0\} = \operatorname{pd}_R M - \operatorname{depth}_R K .$$

E 17.6.5 Let $\mathfrak{p}$ be a prime ideal in R and S an R algebra. Show that there exists a prime ideal $\mathfrak{q}$ in S with $\mathfrak{q} \cap R = \mathfrak{p}$ if and only if $\kappa(\mathfrak{p}) \otimes_R S \neq 0$ holds

E 17.6.6 Let M and N be complexes in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Show that there is an equality $\dim_R(M \otimes^{\mathsf{L}}_R N) = \sup\{\dim_R(\mathrm{H}_v(M) \otimes_R \mathrm{H}_w(N)) - v - w \mid v, w \in \mathbb{Z}\}$.

E 17.6.7 Let S be an R-algebra that is faithfully flat as an R-module. (a) Show that for prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ in R there exist prime ideals $\mathfrak{q} \subset \mathfrak{q}'$ in S with $\mathfrak{q} \cap R = \mathfrak{p}$ and $\mathfrak{q}' \cap R = \mathfrak{p}'$. (b) Show that the inequality $\dim R \leqslant \dim S$ holds.

Chapter 18
Dualities and Cohen–Macaulay Rings

According to Hochster [127], Cohen–Macaulay rings are where "life is really worth living", and as a matter of fact classic results that come up in the last sections of this chapter say that finite length modules of finite projective dimension (18.4.16) and finitely generated modules of finite injective dimension (18.5.8) are only found over Cohen–Macaulay rings. The main results of the first two sections are the Matlis Duality Theorem 18.1.9 and the Grothendieck Duality Theorem 18.2.3. These two duality theories coincide for Artinian local rings (18.1.6) and come together in the Local Duality Theorem 18.3.18.

18.1 Matlis Duality

Synopsis. Minimal injective resolution of Artinian module; Artinian local ring; module of finite length; (derived) Matlis reflexive complex; Matlis Duality.

Matlis Duality is a duality of categories of complexes over a complete local ring. Up to terminology and generalizations from modules to complexes the result is there in Matlis first publication [180].

Recall from 1.1.21 that a direct sum of modules by convention is finite.

18.1.1 Lemma. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local, M an R-module, and $M \xrightarrow{\simeq} I$ a minimal injective resolution. If M is Artinian, then each module I_v is a direct sum of copies of* $\mathrm{E}_R(\boldsymbol{k})$.

Proof. The assertion is trivial for the zero module, so assume that M is non-zero. By 16.1.32 one has $\operatorname{supp}_R M = \{\mathfrak{m}\}$, so in view of Matlis' structure theorem C.23 it follows from 17.1.13 and 15.1.14 that every module I_v is a coproduct of copies of $\mathrm{E}_R(\boldsymbol{k})$. By 8.2.16 the complex $\operatorname{Hom}_R(\boldsymbol{k}, I)$ has zero differential, so one has

$$\operatorname{Hom}_R(\boldsymbol{k}, M) \cong \operatorname{Ext}^0_R(\boldsymbol{k}, M) = \mathrm{H}_0(\operatorname{Hom}_R(\boldsymbol{k}, I)) = \operatorname{Hom}_R(\boldsymbol{k}, I_0)\,.$$

As M is Artinian, $\operatorname{Hom}_R(\boldsymbol{k}, M) \cong \operatorname{Soc}_R M$ is by 16.1.4 a $\boldsymbol{k}$-vector space of finite rank, say m. Thus, as $\operatorname{Hom}_R(\boldsymbol{k}, –)$ is additive it follows per (16.1.22.1) that one has

$I_0 \cong \mathrm{E}_R(\boldsymbol{k})^m$. Let $n > 0$ be an integer and assume that I_{-v} is a direct sum of copies of $\mathrm{E}_R(\boldsymbol{k})$ for all $v < n$. In particular, I_{-n+1} is Artinian, see 16.1.26, and hence so is $\mathrm{B}_{-n}(I) = \mathrm{Z}_{-n}(I)$. As $\Sigma^n I_{\leqslant -n}$ yields an injective resolution of $\mathrm{Z}_{-n}(I)$ one has, as above, $\mathrm{Hom}_R(\boldsymbol{k}, \mathrm{Z}_{-n}(I)) \cong \mathrm{Hom}_R(\boldsymbol{k}, I_{-n})$, and it follows that I_{-n} is a direct sum of copies of $\mathrm{E}_R(\boldsymbol{k})$. □

18.1.2 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. An R-module is Artinian if and only if it can be embedded in $\mathrm{E}_R(\boldsymbol{k})^m$ for some $m \in \mathbb{N}$.*

Proof. The "if" part follows from 16.1.26. In view of 5.3.33 the "only if" part follows from B.26 and 18.1.1. □

Local cohomology modules are rarely finitely generated—see also the Remark after 17.6.7—but they may be Artinian.

18.1.3 Proposition. *Let $(R, \mathfrak{m})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. The complex $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ belongs to $\mathcal{D}^{\mathrm{art}}(R)$; that is, $\mathrm{H}^n_{\mathfrak{m}}(M)$ is Artinian for every $n \in \mathbb{Z}$.*

Proof. Let I be a minimal semi-injective replacement of M; one has $\mathsf{R}\Gamma_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(I)$. As Bass numbers are integers, 16.4.38 shows that each module $\Gamma_{\mathfrak{m}}(I)_v$ is a direct sum of copies of $\mathrm{E}_R(\boldsymbol{k})$, where $\boldsymbol{k}$ is the residue field of R. As $\mathrm{E}_R(\boldsymbol{k})$ by 16.1.26 is Artinian, the complex $\Gamma_{\mathfrak{m}}(I)$ is degreewise Artinian and hence so is the subquotient complex $\mathrm{H}(\Gamma_{\mathfrak{m}}(I))$ of local cohomology modules. □

Artinian Local Rings

18.1.4 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. The next conditions are equivalent.*

(*i*) *R is Artinian.*
(*ii*) *$\mathrm{E}_R(\boldsymbol{k})$ is finitely generated.*
(*iii*) *$\mathfrak{m}$ is nilpotent.*
(*iv*) *Every Artinian R-module has finite length.*
(*v*) *Every finitely generated R-module has finite length.*

Proof. The equivalence of (*i*) and (*v*) is part of 14.2.19. We argue the following implications (*i*) ⟹ (*iii*) ⟹ (*ii*) ⟹ (*iv*) ⟹ (*i*).

If R is Artinian, then $\mathfrak{m}$ is nilpotent by 14.2.19(a). If $\mathfrak{m}$ is nilpotent, then it follows from C.22(a) that $\mathrm{E}_R(\boldsymbol{k})$ has finite length, in particular it is finitely generated. By 18.1.2 every Artinian R-module is a submodule of a direct sum of $\mathrm{E}_R(\boldsymbol{k})^m$ for some $m \in \mathbb{N}$, so if $\mathrm{E}_R(\boldsymbol{k})$ is finitely generated, then so is every Artinian R-module, whence every Artinian module has finite length by 14.2.12. Finally, if every Artinian R-module has finite length, then $\mathrm{E}_R(\boldsymbol{k})$ has finite length by 16.1.26. It follows from 16.1.44 and 16.1.23 that $\mathrm{length}_R \widehat{R}$ is finite. As R is a submodule of $\widehat{R}$, see 16.1.13 and 12.1.23, also R has finite length and, therefore, R is Artinian by 14.2.19. □

18.1.5 Corollary. *Let $(R, \mathfrak{m})$ be Artinian and local; one has*

$$\mathcal{D}^{\mathrm{art}}(R) = \mathcal{D}^{\ell}(R) = \mathcal{D}^{\mathrm{f}}(R)\,.$$

Moreover, every R-complex is derived $\mathfrak{m}$-complete and derived $\mathfrak{m}$-torsion.

Proof. One has $\mathcal{D}^{\mathrm{art}}(R) = \mathcal{D}^{\ell}(R) = \mathcal{D}^{\mathrm{f}}(R)$ by 18.1.4. By 18.1.4 the maximal ideal $\mathfrak{m}$ is nilpotent, so the last assertion follows as $\Lambda^{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}}$, and hence the derived functors $\mathsf{L}\Lambda^{\mathfrak{m}}$ and $\mathsf{R}\Gamma_{\mathfrak{m}}$, are the identity functors; see 11.1.5, 11.2.2, and 7.2.11. □

Matlis Duality

In the special case of an Artinian local ring $(R, \mathfrak{m}, \boldsymbol{k})$ one has $\mathcal{D}^{\mathrm{f}}(R) = \mathcal{D}^{\ell}(R)$ by 18.1.5, and the injective envelope $\mathrm{E}_R(\boldsymbol{k})$ is by 10.1.8 a dualizing complex for R, so in that case the next theorem is simply a restatement of the Grothendieck Duality Theorem 10.1.23. In fact, the equivalence in the next theorem is, over every local ring with a dualizing complex, the restriction of Grothendieck Duality to complexes with homology degreewise of finite length, see 18.2.38(b).

18.1.6 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. Every complex in $\mathcal{D}^{\ell}(R)$ is derived Matlis reflexive. There is an adjoint equivalence of R-linear triangulated categories,*

$$\mathcal{D}^{\ell}(R) \underset{\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))}{\overset{\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\ell}(R)^{\mathrm{op}} .$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}^{\ell}_{\sqsubset}(R) \rightleftarrows \mathcal{D}^{\ell}_{\sqsupset}(R)^{\mathrm{op}} , \quad \mathcal{D}^{\ell}_{\sqsupset}(R) \rightleftarrows \mathcal{D}^{\ell}_{\sqsubset}(R)^{\mathrm{op}} , \quad \textit{and} \quad \mathcal{D}^{\ell}_{\square}(R) \rightleftarrows \mathcal{D}^{\ell}_{\square}(R)^{\mathrm{op}} ,$$

and further to

$$\mathcal{I}(R) \cap \mathcal{D}^{\ell}(R) \rightleftarrows (\mathcal{P}(R) \cap \mathcal{D}^{\ell}(R))^{\mathrm{op}} .$$

Proof. The first assertion is immediate from 16.1.36(b) and 16.1.39. Since $\mathrm{E}_R(\boldsymbol{k})$ is an injective R-module, it follows from 10.1.22 that the functors

$$\mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}} : \mathcal{D}(R) \rightleftarrows \mathcal{D}(R)^{\mathrm{op}} : \mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))$$

are adjoint, and they are R-linear and triangulated by 7.3.6. By 2.2.19 the functor $\mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))$ commutes with homology, so by 16.1.44 it restricts to a functor $\mathcal{D}^{\ell}(R)^{\mathrm{op}} \to \mathcal{D}^{\ell}(R)$, and $\mathrm{Hom}_R(-, \mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}}$ restricts to a functor in the opposite direction. By 10.1.22 biduality is the unit as well as the counit of the adjunction, and as already established it is an isomorphism for every complex in $\mathcal{D}^{\ell}(R)$, so these restrictions yield an adjoint equivalence. The asserted restrictions to bounded subcategories follow from (16.1.22.2), 15.4.30, 15.4.31, and 15.4.18. □

18.1.7 Proposition. *Let R be local and complete. Every degreewise finitely generated R-complex is Matlis reflexive.*

Proof. Let $\boldsymbol{k}$ denote the residue field of R and set $E = \mathrm{E}_R(\boldsymbol{k})$. Let M be a degreewise finitely generated R-complex. Biduality δ^M_E fits in a commutative diagram,

$$\begin{tikzcd} M \arrow[r, "\delta_E^M"] & \mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E) \\ M\otimes_R R \arrow[u, "\mu_R^M \upsilon^{MR}", "\cong"'] \arrow[r, "M\otimes\chi_R^E", "\cong"'] & M\otimes_R \mathrm{Hom}_R(E,E)\,, \arrow[u, "\cong", "\eta^{EEM}"'] \end{tikzcd}$$

where the lower horizontal isomorphism comes from 16.1.24, and homomorphism evaluation η^{EEM} is an isomorphism by 12.1.16(3,b). □

18.1.8 Proposition. *Let R be local and complete. Every degreewise Artinian R-complex is Matlis reflexive.*

Proof. Let $\boldsymbol{k}$ denote the residue field of R and set $E = \mathrm{E}_R(\boldsymbol{k})$. As one has $(\delta_E^M)_v = \delta_E^{M_v}$ for all $v \in \mathbb{Z}$, one can assume that M is an Artinian R-module. Let $M \xrightarrow{\simeq} I$ be a minimal injective resolution. Exactness of the Matlis Duality functor $(-)^\vee = \mathrm{Hom}_R(-,E)$ yields the following commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\delta_E^M"] & I_0 \arrow[r] \arrow[d, "\delta_E^{I_0}"] & I_{-1} \arrow[d, "\delta_E^{I_{-1}}"] \\ 0 \arrow[r] & M^{\vee\vee} \arrow[r] & (I_0)^{\vee\vee} \arrow[r] & (I_{-1})^{\vee\vee}\,. \end{tikzcd}$$

To prove that δ_E^M is an isomorphism, it suffices by the Five Lemma 1.1.2 to show that the biduality maps $\delta_E^{I_0}$ and $\delta_E^{I_{-1}}$ are isomorphisms. By 18.1.1 the modules I_0 and I_{-1} are direct sums af copies of E, so by additivity of Hom it is enough to show that δ_E^E is an isomorphism. To finish the proof, consider the commutative diagram,

$$\begin{tikzcd} & E \arrow[dl, "\delta_E^E"'] \arrow[dr, "\epsilon_R^E", "\cong"'] & \\ \mathrm{Hom}_R(\mathrm{Hom}_R(E,E),E) \arrow[rr, "\mathrm{Hom}(\chi_R^E,E)", "\cong"'] & & \mathrm{Hom}_R(R,E)\,, \end{tikzcd}$$

where the lower horizontal isomorphism comes from 16.1.24. □

We close the section with the actual Matlis Duality Theorem.

18.1.9 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and complete. Every complex that belongs to $\mathcal{D}^{\mathrm{f}}(R)$ or $\mathcal{D}^{\mathrm{art}}(R)$ is derived Matlis reflexive. Moreover, there is an adjoint equivalence of R-linear triangulated categories,*

$$\mathcal{D}^{\mathrm{f}}(R) \underset{\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))}{\overset{\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{art}}(R)^{\mathrm{op}}\,.$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R) \rightleftarrows \mathcal{D}^{\mathrm{art}}_{\sqsupset}(R)^{\mathrm{op}}\,,\quad \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R) \rightleftarrows \mathcal{D}^{\mathrm{art}}_{\sqsubset}(R)^{\mathrm{op}}\,,\quad \textit{and}\quad \mathcal{D}^{\mathrm{f}}_{\square}(R) \rightleftarrows \mathcal{D}^{\mathrm{art}}_{\square}(R)^{\mathrm{op}}\,.$$

Proof. Complexes in $\mathcal{D}^{\mathrm{f}}(R)$ are derived Matlis reflexive by 16.1.36(b) and 18.1.7. Similarly, complexes in $\mathcal{D}^{\mathrm{art}}(R)$ are derived Matlis reflexive by 16.1.36(b) and 18.1.8.

Since $\mathrm{E}_R(\boldsymbol{k})$ is an injective R-module, it follows from 10.1.22 that the functors

$$\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}} : \mathcal{D}(R) \rightleftarrows \mathcal{D}(R)^{\mathrm{op}} : \mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))$$

are adjoint, and they are R-linear and triangulated by 7.3.6. Let M be an R-complex. By exactness of the functor $(-)^{\vee} = \mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))$ one has $\mathrm{H}(M^{\vee}) \cong \mathrm{H}(M)^{\vee}$, see 2.2.19. To see that it restricts to a functor $\mathcal{D}^{\mathrm{art}}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{f}}(R)$, and that the opposite functor $\mathrm{Hom}_R(-,\mathrm{E}_R(\boldsymbol{k}))^{\mathrm{op}}$ restricts to a functor in the opposite direction, one can assume that M is a module. It remains to prove that $M^{\vee}$ is Artinian if M is finitely generated and that $M^{\vee}$ is finitely generated if M is Artinian.

If M is finitely generated, then $M^{\vee}$ is Artinian by 16.1.27. Let M be Artinian; by 18.1.2 there is an embedding $M \rightarrowtail \mathrm{E}_R(\boldsymbol{k})^m$ for some $m \in \mathbb{N}$. Applying $(-)^{\vee}$ yields a surjective homomorphism $(\mathrm{E}_R(\boldsymbol{k})^m)^{\vee} \twoheadrightarrow M^{\vee}$, and the module $(\mathrm{E}_R(\boldsymbol{k})^m)^{\vee}$ is, by 16.1.24 isomorphic to R^m. Thus, $M^{\vee}$ is finitely generated.

This establishes the asserted adjunction. By 10.1.22 biduality is the unit as well as the counit of the adjunction, and as already established it is an isomorphism for complexes in $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathcal{D}^{\mathrm{art}}(R)$, so these restrictions yield an adjoint equivalence. The asserted restrictions to bounded subcategories follow from (16.1.22.2). □

REMARK. Enochs [85] shows that over a complete local ring a module is Matlis reflexive if and only if it has a finitely generated submodule such that the quotient is Artinian. See also Belshoff [35].

As in the case of 18.1.6, the equivalence in 18.1.9 restricts to equivalences of subcategories of $\mathcal{P}(R)$ and $\mathcal{I}(R)$, see E 18.1.2.

EXERCISES

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 18.1.1 Let M be an R-module and $Z \subseteq M$ an essential submodule. Show that the canonical map $\mathrm{Hom}_R(\boldsymbol{k}, Z) \to \mathrm{Hom}_R(\boldsymbol{k}, M)$ is an isomorphism.

E 18.1.2 Show that if R is complete, there are two pairs of adjoint equivalences of triangulated subcategories: $\mathcal{I}^{\mathrm{f}}(R) \rightleftarrows (\mathcal{P}(R) \cap \mathcal{D}^{\mathrm{art}}(R))^{\mathrm{op}}$ and $\mathcal{P}^{\mathrm{f}}(R) \rightleftarrows (\mathcal{I}(R) \cap \mathcal{D}^{\mathrm{art}}(R))^{\mathrm{op}}$.

18.2 Grothendieck Duality

SYNOPSIS. Existence of dualizing complex; Gorenstein ring; spectrum of ring with dualizing complex; normalized dualizing complex; uniqueness of ~ for local ring; minimal semi-injective resolution of dualizing complex; projective dimension of flat module; finitistic dimensions.

The Grothendieck Duality Theorem was already established in Chap. 10. The main purpose of this section is to derive properties of dualizing complexes in the commutative setting and record their consequences for the Grothendieck Duality functor and for the rings that admit such complexes.

Dualizing Complexes

The notion of a dualizing complex is defined in 10.1.6. We start by restating some key results from Chap. 10 in the commutative case.

18.2.1 Proposition. *Let D be an R-complex; it is dualizing for R if and only if it meets the following conditions.*

(1) *D belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.*
(2) $\mathrm{id}_R\, D$ *is finite.*
(3) *There is an isomorphism* $\mathsf{RHom}_R(D,D) \simeq R$ *in* $\mathcal{D}(R)$.

Proof. This follows from 10.1.12 with $R = \Bbbk = S$ combined with 17.1.23. ◻

18.2.2 Example. Let R be Artinian with Jacobson radical $\mathfrak{J} = \bigcap_{u=1}^{n} \mathfrak{m}_u$ where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ are the finitely many maximal ideals of R, see 14.2.19. The direct sum of indecomposable injective R-modules $D = \bigoplus_{u=1}^{n} \mathrm{E}_R(R/\mathfrak{m}_u)$ is the injective envelope of the module $\bigoplus_{u=1}^{n} R/\mathfrak{m}_u$, which by the Chinese Remainder Theorem is isomorphic to $R/\mathfrak{J}$. By 14.2.22 the module D is finitely generated. By 14.2.19(c), 10.1.8, C.18, 14.1.21(a), and C.15(c) there are isomorphisms of rings,

$$\begin{aligned} R &\cong \bigtimes_{u=1}^{n} R_{\mathfrak{m}_u} \\ &\cong \bigtimes_{u=1}^{n} \mathrm{Hom}_{R_{\mathfrak{m}_u}}(\mathrm{E}_{R_{\mathfrak{m}_u}}(\kappa(\mathfrak{m}_u)), \mathrm{E}_{R_{\mathfrak{m}_u}}(\kappa(\mathfrak{m}_u))) \\ &\cong \bigtimes_{u=1}^{n} \mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{m}_u), \mathrm{E}_R(R/\mathfrak{m}_u)) \\ &\cong \mathrm{Hom}_R(D,D)\,. \end{aligned}$$

It now follows from 18.2.1 that D is a dualizing for R.

An R-module D that satisfies the conditions in 18.2.1 is, of course, called a *dualizing module* for R. By 18.2.28 the existence of such a module implies that R is Cohen–Macaulay, and by 18.2.2 every Artinian ring has a dualizing module.

Remark. It is not uncommon to see a dualizing module for a Cohen–Macaulay local ring referred to as a canonical module. By definition, a *canonical module* for a local ring $(R, \mathfrak{m}, k)$ of Krull dimension d is a finitely generated R-module K with $\mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{E}_R(k)) \cong \widehat{R} \otimes_R K$. It is a simple conseqeunce of Local Duality 18.3.18 that a dualizing module for a Cohen–Macaulay local ring is a canonical module; see E 18.3.3. In Herzog and Kunz's monograph [122, Vort. 6] it is shown that a canonical module for a Cohen–Macaulay local rings is dualizing; see also E 18.2.4.

18.2.3 Theorem. *Let D be a dualizing complex for R. For every complex M in $\mathcal{D}^{\mathrm{f}}(R)$ biduality*

$$\delta_D^M : M \longrightarrow \mathsf{RHom}_R(\mathsf{RHom}_R(M,D),D)$$

is an isomorphism in $\mathcal{D}(R)$. Moreover, there is an adjoint equivalence of R-linear triangulated categories,

$$\mathcal{D}^{\mathrm{f}}(R) \underset{\mathsf{RHom}_R(-,D)}{\overset{\mathsf{RHom}_R(-,D)^{\mathrm{op}}}{\rightleftarrows}} \mathcal{D}^{\mathrm{f}}(R)^{\mathrm{op}}\,.$$

It restricts to adjoint equivalences of triangulated subcategories:

$$\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)^{\mathrm{op}}\,,\quad \mathcal{D}^{\mathrm{f}}_{\sqsupset}(R) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)^{\mathrm{op}}\,,\quad \textit{and}\quad \mathcal{D}^{\mathrm{f}}_{\square}(R) \rightleftarrows \mathcal{D}^{\mathrm{f}}_{\square}(R)^{\mathrm{op}}\,,$$

and further to

$$\mathcal{I}^{\mathrm{f}}(R) \rightleftarrows \mathcal{P}^{\mathrm{f}}(R)^{\mathrm{op}}\,.$$

Proof. This follows from 10.1.19 and 10.1.23 with $R = \Bbbk = S$. □

By specialization of 10.2.10 to the setup $R = \Bbbk = S$, an R-complex U is invertible for R if it meets the following conditions.

(1) U belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$.
(2) $\mathrm{pd}_R\, U$ is finite.
(3) Homothety formation, $\chi^U_R \colon R \to \mathsf{RHom}_R(U,U)$, is an isomorphism in $\mathcal{D}(R)$.

18.2.4 Theorem. *Let D be a dualizing complex for R. There is a one-to-one correspondence of isomorphism classes in $\mathcal{D}(R)$,*

$$\{\text{invertible complexes for } R\} \longleftrightarrow \{\text{dualizing complexes for } R\}\,.$$

For every invertible complex U for R and every dualizing complex D' for R the correspondence is given by

$$U \longmapsto D \otimes^{\mathsf{L}}_R U \qquad \textit{and} \qquad \mathsf{RHom}_R(D,D') \longleftarrow D'\,.$$

Proof. Apply 10.3.17 to the setup $R = \Bbbk = S$. □

Existence of Dualizing Complexes

18.2.5 Theorem. *The following conditions are equivalent.*

(*i*) *R is Gorenstein of finite Krull dimension.*
(*ii*) *R is Iwanaga–Gorenstein.*
(*iii*) *R is a dualizing complex for R.*

Proof. If R is Gorenstein of finite Krull dimension, then it follows from 17.4.9 that R is Iwanaga–Gorenstein, whence R is a dualizing complex for R by 10.1.14 applied with $\Bbbk = R$. Thus (*i*) implies (*ii*) which implies (*iii*). To see that (*iii*) implies (*i*), notice that if R is a dualizing complex for R, then $\mathrm{id}\,R$ is finite by 18.2.1. For every prime ideal $\mathfrak{p}$ in R one has $\mathrm{id}\,R_{\mathfrak{p}} \leqslant \mathrm{id}\,R < \infty$ by 17.3.18, so $R_{\mathfrak{p}}$ is Gorenstein. Thus, R is Gorenstein and 17.4.9 yields $\dim R = \mathrm{id}\,R$, so R has finite Krull dimension. □

18.2.6 Theorem. *Let S be an R-algebra and D a dualizing complex for R. If S is finitely generated as an R-module, then* $\mathsf{RHom}_R(S,D)$ *is a dualizing complex for S.*

Proof. This is a special case of 10.1.15. □

18.2.7 Corollary. *If R is a homomorphic image of a Gorenstein ring of finite Krull dimension, then R has a dualizing complex.*

Proof. A surjective ring homomorphism $Q \to R$ makes R a Q-algebra, finitely generated as a Q-module. Thus, if Q is Gorenstein of finite Krull dimension, then the complex $\mathsf{RHom}_Q(R, Q)$ is a dualizing complex for R by 18.2.5 and 18.2.6. $\square$

Remark. It was conjectured by Sharp [229] and proved by Kawasaki [157] that every commutative Noetherian with a dualizing complex is a homomorphic image of a Gorenstein ring of finite Krull dimension. The case of Cohen–Macaulay rings was dealt with earlier in [90] and by Reiten [210].

18.2.8 Proposition. *Let D be a dualizing complex for R. For every prime ideal $\mathfrak{p}$ in R the complex $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$. Further, there are equalities,*

$$\operatorname{supp}_R D \;=\; \operatorname{Supp}_R D \;=\; \operatorname{Spec} R \qquad \textit{and} \qquad \operatorname{cosupp}_R D \;=\; \operatorname{cosupp}_R R\,.$$

Proof. The localized complex $D_\mathfrak{p}$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R_\mathfrak{p})$, see 14.1.11, and $\operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p}$ is finite by 17.3.18. By 14.1.23 one has $\mathsf{RHom}_{R_\mathfrak{p}}(D_\mathfrak{p}, D_\mathfrak{p}) \simeq \mathsf{RHom}_R(D, D)_\mathfrak{p} \simeq R_\mathfrak{p}$, whence $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$ by 18.2.1. In particular, $D_\mathfrak{p}$ is not acyclic, so $\operatorname{Supp}_R D$ is all of $\operatorname{Spec} R$. The equality $\operatorname{supp}_R D = \operatorname{Supp}_R D$ holds by 15.1.9, and then the Cosupport Formula 15.2.9 yields $\operatorname{cosupp}_R R = \operatorname{cosupp}_R D$. $\square$

A classic application of 18.2.7 is to combine it with Cohen's structure theorem [70] which says, in particular, that every complete local ring is a quotient of a Gorenstein local ring. Next we give a different argument for the existence of dualizing complexes for complete local rings; we learned it from Simon and Schenzel [224, 10.2].

Dualizing Complexes for Local Rings

18.2.9 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. There is an isomorphism in $\mathcal{D}(\widehat{R})$,*

$$\operatorname{Hom}_R(\mathsf{R}\Gamma_\mathfrak{m}(R), \mathrm{E}_R(\boldsymbol{k})) \;\simeq\; \mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k}))\,,$$

and this complex is dualizing for $\widehat{R}$.

Proof. Recall from 16.1.25 that $\mathrm{E}_R(\boldsymbol{k})$ is an $\widehat{R}$-module and isomorphic to $\mathrm{E}_{\widehat{R}}(\boldsymbol{k})$. By 13.1.15 and 13.3.18 there are isomorphisms in $\mathcal{D}(\widehat{R})$,

$$(\dagger)\qquad \mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k})) \;\simeq\; \operatorname{Hom}_R(\check{\mathrm{C}}^R(\boldsymbol{x}), \mathrm{E}_R(\boldsymbol{k})) \;\simeq\; \operatorname{Hom}_R(\mathsf{R}\Gamma_\mathfrak{m}(R), \mathrm{E}_R(\boldsymbol{k}))\,.$$

We proceed to show that $\mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k}))$ is dualizing for $\widehat{R}$. By 13.1.21(a) one has

$$\mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k})) \;\simeq\; \mathsf{L}\Lambda^{\mathfrak{m}\widehat{R}}(\mathrm{E}_{\widehat{R}}(\boldsymbol{k}))$$

in $\mathcal{D}(\widehat{R})$, and $\mathfrak{m}\widehat{R}$ is by 16.1.13 the maximal ideal of $\widehat{R}$. Thus one can without loss of generality assume that R is complete. Let $\boldsymbol{x}$ be a sequence that generates $\mathfrak{m}$. The complex $\mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k}))$ has finite injective dimension by 15.4.15(b). To see that it belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, recall first from 18.1.3 that $\mathsf{R}\Gamma_\mathfrak{m}(R)$ belongs to $\mathcal{D}^{\mathrm{art}}_{\square}(R)$. As the functor $\mathsf{R}\Gamma_\mathfrak{m}$ is bounded, see 13.3.18, it follows from $(\dagger)$ and Matlis Duality 18.1.9

that $\mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))$ belongs to $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$. In the next chain of isomorphisms, the 1st is the adjunction from 13.4.12, the 2nd follows from 13.4.1(d), the 3rd holds as $\mathrm{E}_R(\boldsymbol{k})$ is injective and derived $\mathfrak{m}$-torsion, see 16.1.22, and the 4th is 16.1.24.

$$\begin{aligned}\mathsf{RHom}_R(\mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k})), \mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))) &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(\mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))), \mathrm{E}_R(\boldsymbol{k}))\\ &\simeq \mathsf{RHom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k})), \mathrm{E}_R(\boldsymbol{k}))\\ &\simeq \mathrm{Hom}_R(\mathrm{E}_R(\boldsymbol{k}), \mathrm{E}_R(\boldsymbol{k}))\\ &\cong R\,.\end{aligned}$$

It now follows from 18.2.1 that the complex $\mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))$ is dualizing for R. □

Dualizing complexes for local rings have a simple characterization.

18.2.10 Theorem. *Let R be local and D a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. The Bass series $\mathrm{I}^D_R(t)$ is a monomial if and only if D is dualizing for R.*

Proof. If D is dualizing for R, then one has $R \simeq \mathsf{RHom}_R(D,D)$ in $\mathcal{D}(R)$ and D has finite injective dimension, so by 16.4.33 there are equalities of Laurent series,

$$1 \;=\; \mathrm{P}^R_R(t) \;=\; \mathrm{P}^R_{\mathsf{RHom}_R(D,D)}(t) \;=\; \mathrm{I}^D_R(t)\,\mathrm{I}^D_R(t^{-1})\,.$$

The Bass series has non-negative coefficients, so per 16.4.15 one has

$$0 \;=\; \operatorname{ord}(\mathrm{I}^D_R(t)\,\mathrm{I}^D_R(t^{-1})) \;=\; \operatorname{ord}\mathrm{I}^D_R(t) + \operatorname{ord}\mathrm{I}^D_R(t^{-1}) \;=\; \operatorname{ord}\mathrm{I}^D_R(t) - \deg\mathrm{I}^D_R(t)\,,$$

so $\mathrm{I}^D_R(t) = ct^n$ holds for some $c \in \mathbb{N}$ and $n \in \mathbb{Z}$. Now one has $1 = (ct^n)(ct^{-n}) = c^2$, so the coefficient c is 1, and $\mathrm{I}^D_R(t)$ is a monomial. For the converse assume that $\mathrm{I}^D_R(t) = t^n$ holds for some $n \in \mathbb{Z}$. It follows from 16.4.30 that D has finite injective dimension, n; in particular D belongs per 8.2.3 to $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$. Further, 16.4.33 yields $\mathrm{P}^R_{\mathsf{RHom}_R(D,D)}(t) = \mathrm{I}^D_R(t)\,\mathrm{I}^D_R(t^{-1}) = t^n t^{-n} = 1$. Thus there is by 16.4.27 an isomorphism $\mathsf{RHom}_R(D,D) \simeq R$ in $\mathcal{D}(R)$, so D is dualizing for R by 17.1.23. □

18.2.11 Corollary. *Let R be local and D a dualizing complex for R. For every prime ideal $\mathfrak{p}$ in R there are equalities,*

$$\operatorname{id}_R D \;=\; \operatorname{depth}_R D \;=\; \operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \dim R/\mathfrak{p}\,,$$

and D is Cohen–Macaulay.

Proof. By 18.2.10 the Bass series $\mathrm{I}^D_R(t)$ is a monomial, so 16.4.30 and 16.4.10 yield

$$(\dagger) \qquad \operatorname{depth}_R D \;=\; \operatorname{id}_R D \;\geqslant\; \dim_R D\,.$$

The opposite inequality holds by 17.2.1, so D is Cohen–Macaulay, see 17.2.5. By 18.2.8 one has $\operatorname{supp}_R D = \operatorname{Spec} R$, so the asserted equalities hold by the equality in (†) and 17.2.10. □

Remark. Let $\mathfrak{p}$ be a prime ideal in R and D a dualizing complex for R. By 18.2.17 the ring R, and hence also $R/\mathfrak{p}$, has finite Krull dimension, so there exists a maximal ideal $\mathfrak{q} \supseteq \mathfrak{p}$ with $\dim R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}} = \dim R/\mathfrak{p}$. Now 17.3.18, E 16.4.2, and 18.2.11 yield $\operatorname{id}_R D \geqslant \operatorname{id}_{R_{\mathfrak{q}}} D_{\mathfrak{q}} \geqslant \operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \dim R/\mathfrak{p}$. This is a version of 18.2.11 for non-local rings. The inequality $\operatorname{id}_R D \geqslant \operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} + \dim R/\mathfrak{p}$ may be strict. Indeed, let R and $\mathfrak{m}$ be as in 17.2.35. By 17.4.16 and 17.4.17 the ring R is Gorenstein ring, so one has $\operatorname{id} R = \dim R = 2$ by 17.4.9, and $D = R$ is a dualizing complex for R by 18.2.5. As $R_{\mathfrak{m}}$ is Gorenstein and hence Cohen–Macaulay, see 17.4.4, one has $\operatorname{depth} R_{\mathfrak{m}} = \dim R_{\mathfrak{m}} = 1$. Finally, $\dim R/\mathfrak{m} = 0$ holds as $\mathfrak{m}$ is a maximal ideal.

18.2.12 Corollary. *Let R be local and D a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. There is an equality of Bass series,*

$$\mathrm{I}_{\widehat{R}}^{\widehat{R}\otimes_R D}(t) \;=\; \mathrm{I}_R^D(t)\;.$$

In particular, D is dualizing for R if and only if $\widehat{R} \otimes_R D$ is dualizing for $\widehat{R}$.

Proof. Let $\mathfrak{m}$ and $\boldsymbol{k}$ be the maximal ideal and residue field of R. By 16.1.13 one has $\widehat{R}/\mathfrak{m}\widehat{R} \cong \boldsymbol{k}$ and $\widehat{R}$ is a flat R-module. Thus one has $\mathrm{I}_{\boldsymbol{k}}(t) = 1$ and the equality of Bass series follows from 16.4.35. The last assertion is now immediate from 18.2.10. □

18.2.13 Corollary. *Let R be local and D a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. Set $S = R[\![x_1,\dots,x_n]\!]$ and $E = S \otimes_R D$. There is an equality of Bass series,*

$$\mathrm{I}_S^E(t) \;=\; t^n\, \mathrm{I}_R^D(t)\;.$$

In particular, D is dualizing for R if and only if E is dualizing for S.

Proof. Let $\mathfrak{m}$ and $\boldsymbol{k}$ be the maximal ideal and residue field of R. The ring S is local with maximal ideal $\mathfrak{m} + (x_1,\dots,x_n)$, and S is flat as an R-module, see 12.1.24. The quotient ring $S/\mathfrak{m}S \cong \boldsymbol{k}[\![x_1,\dots,x_n]\!]$ is Gorenstein by 17.4.17 and has Krull dimension n, so the equality of Bass series follows from 16.4.35 and 17.4.11. The last assertion is now immediate from 18.2.10. □

In view of 18.2.5 the next result subsumes the equivalence of conditions (*i*) and (*ii*) in 17.4.13. It also subsumes the "in particular" statements in 18.2.12 and 18.2.13.

18.2.14 Theorem. *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{M})$ be local rings such that S is an R-algebra, flat as an R-module, and let D be an R-complex. If $\mathfrak{m}S \subseteq \mathfrak{M}$ holds, then the following conditions are equivalent.*

(*i*) *$S \otimes_R D$ is dualizing for S.*
(*ii*) *D is dualizing for R and the ring $S/\mathfrak{m}S$ is Gorenstein.*

Further, when these conditions are satisfied one has $\mathrm{I}_S^{S\otimes_R D}(t) = t^{(\dim S/\mathfrak{m}S)}\, \mathrm{I}_R^D(t)$.

Proof. If $S \otimes_R D$ is dualizing for S, then it follows from 18.2.1 and 12.1.20(a) that D is a complex in $\mathcal{D}^{\mathrm{f}}_{\Box}(R)$. By 16.4.35 there is an equality of Laurent series,

$$\mathrm{I}_S^{S\otimes_R D}(t) \;=\; \mathrm{I}_{S/\mathfrak{m}S}(t)\, \mathrm{I}_R^D(t)\;. \tag{†}$$

As Bass series have non-negative coefficients and $\mathrm{I}_S^{S\otimes_R D}(t)$ by 18.2.10 is a monomial, it follows that $\mathrm{I}_{S/\mathfrak{m}S}(t)$ and $\mathrm{I}_R^D(t)$ are monomials, see 16.4.15. Thus D is dualizing for R, and by 17.4.11 the local ring $S/\mathfrak{m}S$ is Gorenstein with $\mathrm{I}_{S/\mathfrak{m}S}(t) = t^{(\dim S/\mathfrak{m}S)}$. Conversely, if D is dualizing for R, then $S \otimes_R D$ is a complex in $\mathcal{D}^{\mathrm{f}}_{\Box}(S)$ by 12.1.20(b,c) and (†) holds by 16.4.35. If, further, $S/\mathfrak{m}S$ is Gorenstein, then $\mathrm{I}_{S/\mathfrak{m}S}(t)$ is a monomial by 17.4.11 and it follows from 18.2.10 and (†) that $S \otimes_R D$ is dualizing for S. □

18.2.15 Lemma. *Let R be local, D a dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. There is an equality,*

$$\sup \mathsf{RHom}_R(M, D) \;=\; \dim_R M - \operatorname{depth}_R D\;.$$

In particular, one has $\sup D = \dim R - \operatorname{depth}_R D$.

Proof. By 18.2.8 one has $\operatorname{supp}_R D = \operatorname{Spec} R$, so the equalities in the chain below hold by 17.6.12, 18.2.11, and 14.2.6.

$$\begin{aligned}\sup \mathsf{RHom}_R(M, D) \;&=\; -\inf\{\operatorname{depth}_{R_\mathfrak{p}} D_\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; -\inf\{\operatorname{depth}_R D - \dim R/\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \sup\{\dim R/\mathfrak{p} - \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} - \operatorname{depth}_R D\\ &=\; \dim_R M - \operatorname{depth}_R D\;.\end{aligned}$$

Set $M = R$ and apply the counitor 12.3.4 to get $\sup D = \dim R - \operatorname{depth}_R D$. □

18.2.16 Proposition. *Let R be local and D a dualizing complex for R. For every prime ideal $\mathfrak{p}$ in R there is an equality,*

$$\operatorname{depth}_R D + \sup D_\mathfrak{p} \;=\; \dim R_\mathfrak{p} + \dim R/\mathfrak{p}\;,$$

and one has

$$\dim_R \mathrm{H}_{\sup D}(D) \;=\; \dim R \;=\; \operatorname{id}_R D + \sup D\;.$$

Moreover, one has $\operatorname{Supp}_R \mathrm{H}_{\sup D}(D) = \operatorname{Spec} R$ *if and only if R is equidimensional.*

Proof. For a prime ideal $\mathfrak{p}$ in R the complex $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$ by 18.2.8. Now 18.2.15 and 18.2.11 yield the following equalities,

$$\operatorname{depth}_R D + \sup D_\mathfrak{p} \;=\; \dim R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} D_\mathfrak{p} + \operatorname{depth}_R D \;=\; \dim R_\mathfrak{p} + \dim R/\mathfrak{p}\;.$$

Applied to the maximal ideal of R and to a minimal prime ideal $\mathfrak{q}$, this equality reads

$$(\dagger)\qquad \operatorname{depth}_R D + \sup D \;=\; \dim R \quad\text{ and }\quad \operatorname{depth}_R D + \sup D_\mathfrak{q} \;=\; \dim R/\mathfrak{q}\;,$$

respectively. Thus $\dim R = \dim R/\mathfrak{q}$ holds if and only if one has $\sup D = \sup D_\mathfrak{q}$, which by 14.1.11(a,c) is tantamount to $\mathfrak{q} \in \operatorname{Supp}_R \mathrm{H}_{\sup D}(D)$. It is thus immediate that R is equidimensional, see 17.2.12, if and only if every minimal prime ideal in R belongs to $\operatorname{Supp}_R \mathrm{H}_{\sup D}(D)$. As the classic support is specialization closed, this is equivalent to the equality $\operatorname{Supp}_R \mathrm{H}_{\sup D}(D) = \operatorname{Spec} R$.

Let $\mathfrak{q}$ be a minimal prime ideal with $\dim R = \dim R/\mathfrak{q}$; as argued above one has $\mathfrak{q} \in \operatorname{Supp}_R \mathrm{H}_{\sup D}(D)$ and hence $\dim_R \mathrm{H}_{\sup D}(D) = \dim R$ holds. Further, the first equality in (†) combined with 18.2.11 yields $\operatorname{id}_R D + \sup D \;=\; \dim R$. □

Prime Ideal Spectrum of a Ring with a Dualizing Complex

18.2.17 Theorem. *If R has a dualizing complex, then R is catenary and of finite Krull dimension.*

Proof. Let D be a dualizing complex for R. Let $\mathfrak{p} \in \operatorname{Spec} R$ and choose a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ with $\dim R_\mathfrak{p} = \dim R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p}$. The complex $D_\mathfrak{p}$ is by 18.2.8 dualizing for the local ring $R_\mathfrak{p}$, so the equalities in the next computation follow from 18.2.11 and the choice of $\mathfrak{q}$, while the inequality holds by 16.2.16 and 14.1.11(c).

$$\operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p} \;=\; \operatorname{depth}_{R_\mathfrak{p}} D_\mathfrak{p} \;=\; \operatorname{depth}_{R_\mathfrak{q}} D_\mathfrak{q} + \dim R_\mathfrak{p} \;\geqslant\; \dim R_\mathfrak{p} - \sup D\;.$$

From 17.3.18 one now gets

$$\begin{aligned}\operatorname{id}_R D \;&=\; \sup\{\operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &\geqslant\; \sup\{\dim R_\mathfrak{p} - \sup D \mid \mathfrak{p} \in \operatorname{Spec} R\}\\ &=\; \dim R - \sup D\;.\end{aligned}$$

As $\operatorname{id}_R D$ and $\sup D$ by assumption are finite, it follows that $\dim R$ is finite.

To prove that R is catenary, fix prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in R and consider a saturated chain $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{q}$ in $\operatorname{Spec} R$, i.e. $\dim R_{\mathfrak{p}_i}/(\mathfrak{p}_{i-1})_{\mathfrak{p}_i} = 1$ holds for every $i \in \{1,\ldots,n\}$. By 18.2.8 and 18.2.11 the $R_\mathfrak{q}$-complex $D_\mathfrak{q}$ is Cohen–Macaulay with $\operatorname{Supp}_{R_\mathfrak{q}} D_\mathfrak{q} = \operatorname{Spec} R_\mathfrak{q}$. For prime ideals $\mathfrak{p}'$ contained in $\mathfrak{q}$ set $f(\mathfrak{p}') = \dim R_\mathfrak{q}/\mathfrak{p}'_\mathfrak{q}$; by 17.2.11 the equality $\dim R_{\mathfrak{p}_i}/(\mathfrak{p}_{i-1})_{\mathfrak{p}_i} = f(\mathfrak{p}_{i-1}) - f(\mathfrak{p}_i)$ holds for all $i \in \{1,\ldots,n\}$, so one has $n = \sum_{i=1}^n (f(\mathfrak{p}_{i-1}) - f(\mathfrak{p}_i)) = f(\mathfrak{p}) - f(\mathfrak{q})$. Thus, all saturated chains from $\mathfrak{p}$ to $\mathfrak{q}$ has the same length, i.e. R is catenary. □

18.2.18 Corollary. *A complete local ring is catenary.*

Proof. The assertion follows immediately from 18.2.9 and 18.2.17. □

18.2.19 Corollary. *If R has a dualizing complex, then an R-complex has finite flat dimension if and only if it has finite projective dimension.*

Proof. The assertion follows from 18.2.17 and 17.4.26. □

Grothendieck Duality preserves support and cosupport of complexes in $\mathcal{D}^\mathrm{f}(R)$; recall from 15.1.9 that the support agrees with the classic support for such complexes. Foxby–Sharp Equivalence has similar properties, see 19.1.3.

18.2.20 Theorem. *Let D be a dualizing complex for R and M a complex in $\mathcal{D}^\mathrm{f}(R)$. There are equalities,*

$$\begin{aligned}\operatorname{supp}_R M \;&=\; \operatorname{supp}_R \mathsf{RHom}_R(M,D) \quad \textit{and}\\ \operatorname{cosupp}_R M \;&=\; \operatorname{cosupp}_R \mathsf{RHom}_R(M,D)\;.\end{aligned}$$

Moreover, for every prime ideal $\mathfrak{p}$ in R there is an isomorphism,

$$\mathsf{RHom}_R(M,D)_\mathfrak{p} \;\simeq\; \mathsf{RHom}_{R_\mathfrak{p}}(M_\mathfrak{p},D_\mathfrak{p})\;.$$

Proof. The first equality follows from 17.1.10(c) and 18.2.8. Per 18.2.19 every flat R-module has finite projective dimension, so 17.1.19 conspires with 18.2.8 to yield

$$\operatorname{cosupp}_R M \;=\; \operatorname{cosupp}_R R \cap \operatorname{supp}_R M \;=\; \operatorname{cosupp}_R D \cap \operatorname{supp}_R M\;.$$

Now the second equality follows from the Cosupport Formula 15.2.9. The asserted isomorphism is a special case of 12.3.33(b). □

NORMALIZED DUALIZING COMPLEXES

Recall from 18.2.17 that a ring with a dualizing complex has finite Krull dimension.

18.2.21 Definition. A dualizing complex D for R is called *normalized* if the equality $\sup D = \dim R$ holds.

REMARK. For a local ring the definition above agrees with the standard definition used in the literature, see the commentary before 18.2.24. It is also the definition used by Avramov and Foxby in [23], but in [20, 22] the same authors work with a different definition, one that deems the dualizing module of a Cohen–Macaulay local ring a normalized dualizing complex.

18.2.22 Example. An Artinian ring R has Krull dimension 0, see 14.2.19, so the module D from 18.2.2 is a normalized dualizing complex for R.

Every ring that has a dualizing complex has a normalized one.

18.2.23 Lemma. *Let D be a dualizing complex for R. The complex $\Sigma^{\dim R - \sup D} D$ is a normalized dualizing complex for R.*

PROOF. As $\dim R$ is finite by 18.2.17, the claim follows from 2.5.5 and 10.1.3. □

The third condition in the next theorem, which translates to $\mathsf{RHom}_R(\boldsymbol{k}, D) \simeq \boldsymbol{k}$, is in the literature—from Hartshorne [114, Chap. V] to The Stacks Project [14, Chap. 47]—the prevailing definition of a normalized dualizing complex over a local ring, but see also the Remark after 18.2.21 right above.

18.2.24 Theorem. *Let R be local and D a dualizing complex for R. The following conditions are equivalent.*

(*i*) $\sup D = \dim R$*, i.e. D is normalized.*
(*ii*) $\inf D = \operatorname{depth} R$.
(*iii*) $\mathrm{I}_R^D(t) = 1$.
(*iv*) $\operatorname{id}_R D = 0$.
(*v*) $\operatorname{depth}_R D = 0$.

PROOF. Conditions (*iii*)–(*v*) are equivalent by 18.2.10 and 16.4.30. Conditions (*iv*) and (*ii*) are equivalent by the Bass Formula 16.4.11. The equivalence of conditions (*i*) and (*v*) follows from the equality $\sup D = \dim R - \operatorname{depth}_R D$, see 18.2.15. □

18.2.25 Corollary. *Let R be local, $\mathfrak{p}$ a prime ideal in R, and D a normalized dualizing complex for R. There is an equality,*

$$\dim R_{\mathfrak{p}} - \sup D_{\mathfrak{p}} \;=\; -\dim R/\mathfrak{p}\,,$$

and $\Sigma^{-\dim R/\mathfrak{p}} D_{\mathfrak{p}}$ is a normalized dualizing complex for $R_{\mathfrak{p}}$.

PROOF. By 18.2.24 one has $\operatorname{depth}_R D = 0$, so 18.2.16 yields the equality $\sup D_{\mathfrak{p}} = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$. The complex $D_{\mathfrak{p}}$ is dualizing for $R_{\mathfrak{p}}$ by 18.2.8, so by 18.2.23 the shifted complex $\Sigma^{-\dim R/\mathfrak{p}} D_{\mathfrak{p}}$ is a normalized dualizing complex for $R_{\mathfrak{p}}$. □

The assumption in 18.2.25 that R is local is necessary.

18.2.26 Example. Let $\Bbbk$ be a field and set $R = \Bbbk[\![x]\!][y]$. It follows from 17.4.16 and 17.4.17 that R is Gorenstein. As $\dim R = 2$ holds, $D = \Sigma^2 R$ is by 18.2.5 and 18.2.23 a normalized dualizing complex for R. The ideal $\mathfrak{m} = (xy-1)$ is maximal with $\dim R_\mathfrak{m} = 1$, see 17.2.35, so the dualizing complex $\Sigma^{-\dim R/\mathfrak{m}} D_\mathfrak{m} = \Sigma^2 R_\mathfrak{m}$ for $R_\mathfrak{m}$ is not normalized.

A dualizing complex for a local rings is unique up to shift and isomorphism in the derived category.

18.2.27 Theorem. *Let R be local and D a normalized dualizing complex for R. For an R-complex C the following conditions are equivalent.*

(*i*) *C is dualizing for R.*
(*ii*) *There is an isomorphism $C \simeq \Sigma^s D$ in $\mathcal{D}(R)$ for some integer s.*
(*iii*) *There is an isomorphism $\Sigma^{\operatorname{depth}_R C} C \simeq D$ in $\mathcal{D}(R)$.*

Proof. Condition (*iii*) evidently implies (*ii*), which by 10.1.3 implies (*i*). It remains to argue that (*i*) implies (*iii*). Set $d = \operatorname{depth}_R C$. By 18.2.10 and 16.4.30 one has $\mathrm{I}_R^C(t) = t^d$ while $\mathrm{I}_R^D(t) = 1$ holds by 18.2.24. As $\operatorname{id}_R C$ is finite, 16.4.33 yields

$$\mathrm{P}^R_{\mathsf{RHom}_R(D,C)}(t) = \mathrm{I}_R^D(t)\,\mathrm{I}_R^C(t^{-1}) = \mathrm{I}_R^C(t^{-1}) = t^{-d} .$$

By 16.4.27 there is thus an isomorphism $\mathsf{RHom}_R(D,C) \simeq \Sigma^{-d} R$ in $\mathcal{D}(R)$. Combined with Grothendieck Duality 18.2.3, the counitor 12.3.4, and the fact from 12.2.2 that RHom is triangulated this yields

$$D \simeq \mathsf{RHom}_R(\mathsf{RHom}_R(D,C),C) \simeq \mathsf{RHom}_R(\Sigma^{-d} R, C) \simeq \Sigma^d C . \qquad \square$$

18.2.28 Corollary. *Let R be local and D a dualizing complex for R; one has*

$$\operatorname{amp} D = \operatorname{cmd} R .$$

Proof. By 18.2.24 the amplitude of a normalized dualizing complex is $\dim R - \operatorname{depth} R = \operatorname{cmd} R$, so the claim is immediate from 18.2.27. $\square$

18.2.29 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. The dualizing complex for $\widehat{R}$ from* 18.2.9, $\operatorname{Hom}_R(\mathsf{R}\Gamma_\mathfrak{m}(R), \mathrm{E}_R(\boldsymbol{k})) \simeq \mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k}))$*, is normalized, and if D is a normalized dualizing complex for R, then $\widehat{R} \otimes_R D$ is isomorphic to this complex in $\mathcal{D}(\widehat{R})$.*

Proof. As the extended ideal $\mathfrak{m}\widehat{R}$ is the maximal ideal of the local ring $\widehat{R}$, see 16.1.13, the first equality in the computation below holds by 16.2.12 and 14.3.19. The remaining equalities hold by 16.2.14 and 16.2.29.

$$\operatorname{depth}_{\widehat{R}} \mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k})) = \operatorname{depth}_R \mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k})) = \operatorname{depth}_R \mathrm{E}_R(\boldsymbol{k}) = 0 .$$

Thus the dualizing complex $\mathsf{L}\Lambda^\mathfrak{m}(\mathrm{E}_R(\boldsymbol{k}))$ for $\widehat{R}$ is normalized by 18.2.24.

If D is a normalized dualizing complex for R, then $\widehat{R} \otimes_R D$ is a normalized dualizing for $\widehat{R}$ by 18.2.12 and 18.2.24; now invoke 18.2.27. $\square$

Grothendieck Duality and Homological Invariants

18.2.30 Proposition. *Let R be local, D a normalized dualizing complex for R, and M an R-complex. There are equalities,*

$$\operatorname{depth}_R M \;=\; \operatorname{width}_R \mathsf{RHom}_R(M,D) \quad \textit{and} \quad \operatorname{width}_R M \;=\; \operatorname{depth}_R \mathsf{RHom}_R(M,D)\;.$$

Proof. By 16.2.5(a) and 18.2.24 one has $\operatorname{width}_R D = \inf D = \operatorname{depth} R$, so the first equality follows from 16.3.9(b). By 18.2.24 one has $\operatorname{depth}_R D = 0$, so the second equality is a special case of 16.2.24. □

As the amplitude of a complex is invariant under shifts it follows in view of 18.2.27 that the equalities in 18.2.31(c) hold for any dualizing complex, normalized or not. A special case of this observation is recorded in 18.2.28.

18.2.31 Theorem. *Let R be local, D a normalized dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. The following equalities hold.*

(a) $\dim_R M = \sup \mathsf{RHom}_R(M,D)$ *and* $\sup M = \dim_R \mathsf{RHom}_R(M,D)\;.$

(b) $\operatorname{depth}_R M = \inf \mathsf{RHom}_R(M,D)$ *and* $\inf M = \operatorname{depth}_R \mathsf{RHom}_R(M,D)\;.$

(c) $\operatorname{cmd}_R M = \operatorname{amp} \mathsf{RHom}_R(M,D)$ *and* $\operatorname{amp} M = \operatorname{cmd}_R \mathsf{RHom}_R(M,D)\;.$

Proof. The equalities in part (c) follow from the equalities in (a) and (b).

(a): If M is acyclic, then the first equality holds as both sides equal $-\infty$, so one can assume that M is not acyclic. Assume first that $\inf M = -\infty$ holds. By 14.2.4 one has $\dim_R M = \infty$; to see that $\sup \mathsf{RHom}_R(M,D) = \infty$ holds, recall from 18.2.3 that there is an isomorphism,

$$(\dagger) \qquad M \;\simeq\; \mathsf{RHom}_R(\mathsf{RHom}_R(M,D),D)\;,$$

and that the Grothendieck Duality functor, $\mathsf{RHom}_R(-,D)$, maps $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Henceforth one can assume that M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. The first equality follows from 18.2.15, as $\operatorname{depth}_R D = 0$ holds by 18.2.24. The second equality follows by the isomorphism ($\dagger$) from the one already proved.

(b): These equalities are special cases of those in 18.2.30, as 16.2.5(a) and 18.2.3 yield $\inf M = \operatorname{width}_R M$ and $\inf \mathsf{RHom}_R(M,D) = \operatorname{width}_R \mathsf{RHom}_R(M,D)$. □

18.2.32 Theorem. *Let R be local and D a normalized dualizing complex for R.*

(a) *For every complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ there is an equality of Laurent series,*

$$\mathrm{I}_R^{\mathsf{RHom}_R(M,D)}(t) \;=\; \mathrm{P}_M^R(t)\;.$$

In particular, one has $\operatorname{id}_R \mathsf{RHom}_R(M,D) = \operatorname{pd}_R M$.

(b) *For every complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ there is an equality of Laurent series,*

$$\mathrm{P}^R_{\mathsf{RHom}_R(M,D)}(t) \;=\; \mathrm{I}_R^M(t)\;.$$

In particular, one has $\operatorname{pd}_R \mathsf{RHom}_R(M,D) = \operatorname{id}_R M$.

PROOF. By 18.2.24 one has $\mathrm{I}_R^D(t) = 1$ and $\operatorname{depth}_R D = 0 = \operatorname{id}_R D$. Thus part (a) is a special case of 16.4.31, and (b) is a special case of 16.4.33. □

18.2.33 Corollary. *Let R be local and D a normalized dualizing complex for R. There is an equality of Laurant series,*

$$\mathrm{I}_R(t) \;=\; \mathrm{P}_D^R(t)\;.$$

PROOF. The equality in 18.2.32(b) and the counitor 12.3.4 yield the assertion. □

MINIMAL SEMI-INJECTIVE RESOLUTION OF A DUALIZING COMPLEX

18.2.34 Example. Let R be a Gorenstein ring of finite Krull dimension d and $\mathfrak{a}$ a proper ideal in R. Set $S = R/\mathfrak{a}$; it follows from 18.2.5 and 18.2.6 that the complex $D = \mathsf{R}\mathrm{Hom}_R(S, R)$ is dualizing for S. Let $R \xrightarrow{\simeq} I$ be a minimal injective resolution; it follows from 17.4.12, 3.1.33, and C.16 that the modules in the complex $D = \mathrm{Hom}_R(S, I)$ have the form

$$D_{-v} \;=\; \coprod_{\substack{\dim R_{\mathfrak{p}} = v \\ \mathfrak{p}\in\operatorname{Spec} R}} \mathrm{Hom}_R(S, \mathrm{E}_R(R/\mathfrak{p})) \;=\; \coprod_{\substack{\dim R_{\mathfrak{p}} = v \\ \mathfrak{p}\in\mathrm{V}(\mathfrak{a})}} \mathrm{E}_S(R/\mathfrak{p})\;.$$

Notice that $D_{-v} = 0$ holds for $v < \min_{\mathfrak{p}\in\mathrm{V}(\mathfrak{a})}\{\dim R_{\mathfrak{p}}\}$ and $v > \max_{\mathfrak{p}\in\mathrm{V}(\mathfrak{a})}\{\dim R_{\mathfrak{p}}\}$.

Assume now that R is local and recall from 17.4.4 and 17.2.20 that one has $d = \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p}$ for every $\mathfrak{p}$ in $\operatorname{Spec} R$. Thus $\dim R_{\mathfrak{p}} \geqslant d - \dim S$ holds for all $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$, so in this case D is concentrated in degrees $-d + \dim S, \dots, -d$.

Recall from B.26 that every complex has a minimal semi-injective resolution which is unique up to isomorphism. For a dualizing complex this resolution can be described explicitly.

18.2.35 Theorem. *Let D be a dualizing complex for R. For every prime ideal $\mathfrak{p}$ in R there are equalities,*

$$\dim R_{\mathfrak{p}} - \sup D_{\mathfrak{p}} \;=\; \operatorname{id}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} \;=\; \operatorname{depth}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} \;=\; \operatorname{depth} R_{\mathfrak{p}} - \inf D_{\mathfrak{p}}\;.$$

Moreover, if D is normalized and $D \xrightarrow{\simeq} I$ is a minimal semi-injective resolution, then for every $v \in \mathbb{Z}$ one has

$$I_{-v} \;\cong\; \coprod_{\operatorname{id}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} = v} \mathrm{E}_R(R/\mathfrak{p})\,,$$

and I is concentrated in degrees $\dim R, \dots, -\operatorname{id}_R D$.

PROOF. Let $\mathfrak{p}$ be a prime ideal in R; by 18.2.8 the complex $D_{\mathfrak{p}}$ is dualizing for the local ring $R_{\mathfrak{p}}$, so the first and second equalities hold by 18.2.16 and 18.2.11; the last equality holds by the Bass Formula 16.4.11.

Now assume that D is normalized and let $D \xrightarrow{\simeq} I$ be a minimal semi-injective resolution. Fix a prime ideal $\mathfrak{p}$ in R and set $d = \operatorname{id}_{R_{\mathfrak{p}}} D_{\mathfrak{p}}$. As $d = \dim R_{\mathfrak{p}} - \sup D_{\mathfrak{p}}$ holds, it follows from 18.2.23 and 14.1.31 that the complex $\Sigma^d D_{\mathfrak{p}}$ is a normalized dualizing complex for the local ring $R_{\mathfrak{p}}$ with minimal semi-injective resolution $\Sigma^d D_{\mathfrak{p}} \xrightarrow{\simeq} \Sigma^d I_{\mathfrak{p}}$. By 18.2.24 the Bass series of $\Sigma^d D_{\mathfrak{p}}$ is 1, so by 7.3.31 one has

$$\mu^v_{R_{\mathfrak{p}}}(D_{\mathfrak{p}}) \;=\; \mu^{v-d}_{R_{\mathfrak{p}}}(\Sigma^d D_{\mathfrak{p}}) \;=\; \begin{cases} 1 & \text{for } \; d = v \\ 0 & \text{for } \; d \neq v\,. \end{cases}$$

Now invoke 16.4.37 to get the asserted expression for the module I_{-v}. As the complex I is minimal, the last assertion follows from B.26 and 8.2.15. □

18.2.36 Corollary. *Let R be local and D a normalized dualizing complex for R. If $D \xrightarrow{\simeq} I$ is a minimal semi-injective resolution, then one has*

$$I_{-v} \;\cong\; \coprod_{\dim R/\mathfrak{p} = -v} \mathrm{E}_R(R/\mathfrak{p})$$

for every $v \in \mathbb{Z}$. In particular, I is concentrated in degrees $\dim R, \dots, 0$.

Proof. Let $\mathfrak{p}$ be a prime ideal in R. It follows from 18.2.35 and 18.2.25 that the equality $\operatorname{id}_{R_{\mathfrak{p}}} D_{\mathfrak{p}} = -\dim R/\mathfrak{p}$ holds, and the asserted expression for I_{-v} follows, also from 18.2.35. Evidently, the complex I is concentrated in the asserted degrees. □

18.2.37 Corollary. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and D a normalized dualizing complex for R. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{R}\Gamma_{\mathfrak{m}}(D) \;\simeq\; \mathrm{E}_R(\boldsymbol{k})\,.$$

Proof. Let $D \xrightarrow{\simeq} I$ be a minimal semi-injective resolution, see B.26. By 18.2.36 and 13.3.4 one has $\mathsf{R}\Gamma_{\mathfrak{m}}(D) = \Gamma_{\mathfrak{m}}(I) \cong \mathrm{E}_R(R/\mathfrak{m}) = \mathrm{E}_R(\boldsymbol{k})$. □

18.2.38 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and D a normalized dualizing complex for R. There are natural isomorphisms in $\mathcal{D}(R)$:*

(a) $\mathsf{RHom}_R(D, Y) \simeq \mathsf{RHom}_R(\mathrm{E}_R(\boldsymbol{k}), Y)$ *if Y is a derived $\mathfrak{m}$-complete R-complex.*
(b) $\mathsf{RHom}_R(X, D) \simeq \mathsf{RHom}_R(X, \mathrm{E}_R(\boldsymbol{k}))$ *if X is a derived $\mathfrak{m}$-torsion R-complex.*
(c) $X \otimes^{\mathsf{L}}_R D \simeq X \otimes^{\mathsf{L}}_R \mathrm{E}_R(\boldsymbol{k})$ *if X is a derived $\mathfrak{m}$-torsion R-complex.*

Proof. By 18.2.37 and 13.3.3 one has $\mathsf{R}\Gamma_{\mathfrak{m}}(D) \simeq \mathrm{E}_R(\boldsymbol{k}) \simeq \mathsf{R}\Gamma_{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))$ in $\mathcal{D}(R)$, so the assertions follow immediately from 13.4.21. □

18.2.39 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and D a normalized dualizing complex for R. From 18.2.38(b) applied to $X = \mathrm{E}_R(\boldsymbol{k})$ and 16.1.23 one gets the isomorphism

$$\mathsf{RHom}_R(\mathrm{E}_R(\boldsymbol{k}), D) \;\simeq\; \widehat{R}\,.$$

18.2.40 Corollary. *Let R be local and D a normalized dualizing complex for R. Grothendieck Duality* 18.2.3 *restricts to an adjoint equivalence of $\Bbbk$-linear triangulated categories $\mathcal{D}^{\ell}(R) \leftrightarrows \mathcal{D}^{\ell}(R)^{\mathrm{op}}$ which coincides with Matlis Duality* 18.1.6.

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and recall, for example from 16.1.33, that $\mathcal{D}^{\ell}(R)$ is a subcategory of both $\mathcal{D}^{\mathrm{f}}(R)$ and $\mathcal{D}^{\mathfrak{m}\text{-tor}}(R)$. Now invoke 18.2.38(b). □

Finitistic Dimensions

In the local case, the next result refines the statement in 18.2.19. It implies the equality $\operatorname{FPD} R = \dim R$, see 18.2.42, and as such it is a special case of 8.5.19.

18.2.41 Theorem. *Let R be local, D a dualizing complex for R, and M an R-complex. If M has finite flat dimension, then there is an inequality,*

$$\operatorname{pd}_R M \leqslant \dim R + \sup M .$$

Proof. Let M be an R-complex of finite flat dimension. By 10.3.1(a) there is an isomorphism $M \simeq \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_R M)$ in $\mathcal{D}(R)$. This explains the first equality in the computation below. By 15.4.17 the complex $D \otimes^{\mathsf{L}}_R M$ belongs to $\mathcal{D}_{\sqsubset}(R)$ and the remaining two equalities hold by 17.6.12. By 18.2.8 one has $\operatorname{Supp}_R D = \operatorname{Spec} R$, so $\inf D_{\mathfrak{p}} \leqslant \sup D_{\mathfrak{p}} \leqslant \sup D$ holds per 14.1.11(c) for every prime ideal $\mathfrak{p}$, which explains the inequality below.

$$\begin{aligned}
\sup M &= \sup \mathsf{RHom}_R(D, D \otimes^{\mathsf{L}}_R M) \\
&= -\inf\{\operatorname{depth}_{R_{\mathfrak{p}}}(D \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} + \inf D_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\
&\geqslant -\inf\{\operatorname{depth}_{R_{\mathfrak{p}}}(D \otimes^{\mathsf{L}}_R M)_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} - \sup D \\
&= \sup (D \otimes^{\mathsf{L}}_R M) - \sup D .
\end{aligned}$$

The computation above combined with 18.2.16 yields

$$\operatorname{id}_R D + \sup (D \otimes^{\mathsf{L}}_R M) \leqslant \operatorname{id}_R D + \sup D + \sup M = \dim R + \sup M .$$

Now invoke 10.3.12 to get the desired inequality. □

The central inequality, $\sup (D \otimes^{\mathsf{L}}_R M) \leqslant \sup M + \sup D$, in the proof above is part of a family of inequalities that control suprema and infima of complexes in the Auslander and Bass categories; see 19.1.5.

18.2.42 Corollary. *Let R be local. If R has a dualizing complex, then there are (in)equalities,*

$$\operatorname{FID} R = \operatorname{FFD} R \leqslant \operatorname{FPD} R = \dim R \leqslant \operatorname{id} R \leqslant \operatorname{gldim} R ;$$

if one of the quantities $\operatorname{id} R$ *or* $\operatorname{gldim} R$ *is finite, then equality holds everywhere to the left of it.*

Proof. From 15.4.18 and 18.2.41 one gets $\operatorname{FPD} R \leqslant \dim R$, and 17.4.28 shows that equality holds. Given this equality, the assertion is a restatement of 17.4.1. □

Exercises

In exercises E 18.2.1–18.2.10 let R be local.

E 18.2.1 Let R be Artinian and assume that it contains the residue field $\boldsymbol{k}$ as a subring. Show that there is an isomorphism $\operatorname{Hom}_{\boldsymbol{k}}(R, \boldsymbol{k}) \cong \mathrm{E}_R(\boldsymbol{k})$ of R-modules.

E 18.2.2 Assume that R is a homomorphic image of a Gorenstein local ring Q. Prove that R is Gorenstein if and only if $\mathsf{RHom}_Q(R,Q) \simeq \Sigma^s R$ holds for some integer s.

E 18.2.3 Let $(R,\mathfrak{m},\boldsymbol{k})$ be a Cohen–Macaulay local ring of Krull dimension d and D a finitely generated R-module. Show that D is dualizing for R if and only if one has

$$\operatorname{Ext}_R^m(\boldsymbol{k},D) \;\cong\; \begin{cases} 0 & \text{for} \quad m \neq d \\ \boldsymbol{k} & \text{for} \quad m = d\,. \end{cases}$$

E 18.2.4 Let R be Cohen–Macaulay and K a finitely generated R-module. Show that K is a canonical module for R if and only if it is a dualizing module for R.

E 18.2.5 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset\!\sqsupset}(R)$ of finite projective dimension. Show that one has $\operatorname{amp} \mathsf{RHom}_R(M,R) \leqslant \operatorname{cmd}_R M$ and that equality holds if R is Cohen–Macaulay.

E 18.2.6 Let $\mathfrak{a}$ be an ideal in R and D a normalized dualizing complex for R. Show that $\mathsf{RHom}_R(R/\mathfrak{a},D)$ is a normalized dualizing complex for $R/\mathfrak{a}$.

E 18.2.7 Let M be an R-complex and D a normalized dualizing complex for $\widehat{R}$. Show that one has $\operatorname{depth}_R M = \operatorname{width}_R \mathsf{RHom}_R(M,D)$ and $\operatorname{width}_R M = \operatorname{depth}_R \mathsf{RHom}_R(M,D)$.

E 18.2.8 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset\!\sqsupset}(R)$ with $\operatorname{id}_R M = \operatorname{depth}_R M$. (a) Show that M is Cohen–Macaulay. (b) Show that if D is a dualizing complex for R, then M is up to a shift isomorphic in $\mathcal{D}(R)$ to a direct sum of copies of D.

E 18.2.9 Let D be a dualizing complex for R. Show that the equality $\dim_R D = \dim R - \inf D$ holds if and only if R is Cohen–Macaulay.

E 18.2.10 Let R be Cohen–Macaulay of Krull dimension d and D a normalized dualizing complex for R. (a) Show that the module $K = \mathrm{H}_d(D)$ is dualizing for R. (b) Show that an R-module M is Cohen–Macaulay of Krull dimension n if and only if $\operatorname{Ext}_R^m(M,K) = 0$ holds for $m \neq d-n$, in which case the module $\operatorname{Ext}_R^{d-n}(M,K)$ is Cohen–Macaulay of Krull dimension n. (c) Show that for a Cohen–Macaulay R-module of Krull dimension n there is an isomorphism $M \cong \operatorname{Ext}_R^{d-n}(\operatorname{Ext}_R^{d-n}(M,K),K)$.

E 18.2.11 Let $\mathfrak{a}$ be an ideal in R and assume that D is a normalized dualizing complex for R; describe the modules of the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(D)$.

E 18.2.12 Show that if R has a dualizing complex, then every complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$ is isomorphic in $\mathcal{D}(R)$ to a bounded above complex of finitely generated R-modules.

`add exercises about A and B closed under products and colimits`

18.3 Local Duality

Synopsis. Homological invariants under completion; Gorenstein ring; Local Duality Theorem; local cohomology vs. Krull dimension; Krull dimension vs. depth; non-zero Bass numbers.

Dualizing complexes are a powerful tool in homological computations, which makes it important to understand how a problem may be transplanted to a setting where this tool is available. The gist of 18.3.2–18.3.13 below is that questions about the homological nature of finitely generated modules over a local ring can be resolved over the completion of the ring, which by 18.2.9 has a dualizing complex.

Passing to the Completion

Let $(R,\mathfrak{m},\boldsymbol{k})$ be local and recall from 16.1.13 that the completion $\widehat{R}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field $\boldsymbol{k}$. The isomorphism $\boldsymbol{k} \cong \widehat{R} \otimes_R \boldsymbol{k}$

implies that basic homological invariants remain unchanged under base change from R to $\widehat{R}$, a process often referred to as "passing to the completion."

We open with a simpler consequence:

18.3.1 Proposition. *Let R be local and M an $\widehat{R}$-complex; there are equalities,*

$$\operatorname{width}_R M = \operatorname{width}_{\widehat{R}} M \qquad \text{and} \qquad \operatorname{depth}_R M = \operatorname{depth}_{\widehat{R}} M .$$

Proof. As $\mathfrak{m}\widehat{R}$ is the maximal ideal of $\widehat{R}$ the equalities are, in view of 16.2.12 and 16.2.1, special cases 14.3.19 and 14.3.31. □

18.3.2 Proposition. *Let R be local and M an R-complex. There are equalities,*

$$\begin{aligned} \inf M &= \inf (\widehat{R} \otimes_R M) , \\ \sup M &= \sup (\widehat{R} \otimes_R M) , \quad \text{and} \\ \operatorname{amp} M &= \operatorname{amp}(\widehat{R} \otimes_R M) . \end{aligned}$$

Moreover, the following assertions hold.

(a) *M belongs to $\mathcal{C}^{\mathrm{f}}(R)$ if and only if $\widehat{R} \otimes_R M$ belongs to $\mathcal{C}^{\mathrm{f}}(\widehat{R})$.*
(b) *There is an isomorphism $\mathrm{H}(\widehat{R} \otimes_R M) \cong \widehat{R} \otimes_R \mathrm{H}(M)$.*
(c) *M belongs to $\mathcal{D}^{\mathrm{f}}(R)$ if and only if $\widehat{R} \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}(\widehat{R})$.*

Proof. As an R-module the completion $\widehat{R}$ is faithfully flat, see 16.1.13, so the equalities follow straight from 2.5.7(c). The last assertions hold by 12.1.20. □

18.3.3 Lemma. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M an R-complex. There are isomorphisms in $\mathcal{D}(\boldsymbol{k})$,*

$$\boldsymbol{k} \otimes^{\mathsf{L}}_R M \simeq \boldsymbol{k} \otimes^{\mathsf{L}}_{\widehat{R}} (\widehat{R} \otimes_R M) \quad \text{and} \quad \mathsf{RHom}_R(\boldsymbol{k}, M) \simeq \mathsf{RHom}_{\widehat{R}}(\boldsymbol{k}, \mathsf{RHom}_R(\widehat{R}, M)) .$$

In particular, there are for all $m \in \mathbb{Z}$ isomorphisms of $\boldsymbol{k}$-vector spaces,

$$\operatorname{Tor}^R_m(\boldsymbol{k}, M) \cong \operatorname{Tor}^{\widehat{R}}_m(\boldsymbol{k}, \widehat{R} \otimes_R M) \quad \text{and} \quad \operatorname{Ext}^m_R(\boldsymbol{k}, M) \cong \operatorname{Ext}^m_{\widehat{R}}(\boldsymbol{k}, \mathsf{RHom}_R(\widehat{R}, M)) .$$

Proof. The isomorphisms in $\mathcal{D}(\boldsymbol{k})$ hold by 12.3.31 and 12.3.36. The isomorphisms of $\boldsymbol{k}$-vector spaces now follow from the definitions of Tor and Ext. □

18.3.4 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex. There is an isomorphism in $\mathcal{D}(\widehat{R})$,*

$$\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{m}\widehat{R}}(\mathsf{RHom}_R(\widehat{R}, M)) .$$

In particular, there are for all $n \in \mathbb{Z}$ isomorphisms of local homology modules,

$$\mathrm{H}^{\mathfrak{m}}_n(M) \cong \mathrm{H}^{\mathfrak{m}\widehat{R}}_n(\mathsf{RHom}_R(\widehat{R}, M)) .$$

Proof. By 13.4.17 and 13.1.21(b) there are isomorphisms in $\mathcal{D}(\widehat{R})$,

$$\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \mathsf{RHom}_R(\widehat{R}, \mathsf{L}\Lambda^{\mathfrak{m}}(M)) \simeq \mathsf{L}\Lambda^{\mathfrak{m}\widehat{R}}(\mathsf{RHom}_R(\widehat{R}, M)) .$$

The isomorphisms of local homology modules hold by the definition, 11.3.6. □

18.3.5 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-complex. There is an isomorphism in $\mathcal{D}(\widehat{R})$,*

$$\mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) .$$

In particular, there are for all $n \in \mathbb{Z}$ isomorphisms of local cohomology modules,

$$\mathrm{H}^n_{\mathfrak{m}}(M) \cong \mathrm{H}^n_{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) .$$

Proof. By 13.4.17 and 13.3.23(b) there are isomorphisms in $\mathcal{D}(\widehat{R})$,

$$\mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \widehat{R} \otimes_R \mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathsf{R}\Gamma_{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) .$$

The isomorphisms of local cohomology modules hold by the definition, 11.3.20. □

18.3.6 Theorem. *Let R be local and M an R-complex. There are equalities,*

$$\operatorname{depth}_{\widehat{R}}(\widehat{R} \otimes_R M) = \operatorname{depth}_R M = \operatorname{depth}_{\widehat{R}} \mathsf{R}\mathrm{Hom}_R(\widehat{R}, M)$$

and

$$\operatorname{width}_{\widehat{R}}(\widehat{R} \otimes_R M) = \operatorname{width}_R M = \operatorname{width}_{\widehat{R}} \mathsf{R}\mathrm{Hom}_R(\widehat{R}, M) .$$

Proof. The equalities of depths follow via 16.2.14 from 18.3.5 and 18.3.3. Similarly, the equalities of widths follow via 16.2.3 from 18.3.3 and 18.3.4. □

For finitely generated modules, local homology and Bass numbers are also invariant under passage to the completion.

18.3.7 Proposition. *Let $(R, \mathfrak{m})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. There is an isomorphism in $\mathcal{D}(\widehat{R})$,*

$$\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \mathsf{L}\Lambda^{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) .$$

In particular, there are for all $n \in \mathbb{Z}$ isomorphisms of local homology modules,

$$\mathrm{H}^{\mathfrak{m}}_n(M) \cong \mathrm{H}^{\mathfrak{m}\widehat{R}}_n(\widehat{R} \otimes_R M) .$$

Proof. The complex $\widehat{R} \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}(\widehat{R})$, see 18.3.2(c), so the first isomorphism in the display below holds by 13.2.5. The second isomorphism follows from 12.1.18 as $\widehat{R} = \Lambda^{\mathfrak{m}}(R)$ is $\mathfrak{m}\widehat{R}$-complete, see 11.1.39. The last isomorphism comes from 13.2.7.

$$\mathsf{L}\Lambda^{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) \simeq \Lambda^{\mathfrak{m}\widehat{R}}(\widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R M) \simeq \widehat{R} \otimes_R M \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(M) .$$

The isomorphisms of local homology modules now hold by the definition, 11.3.6. □

18.3.8 Lemma. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. There is an isomorphism in $\mathcal{D}(\boldsymbol{k})$,*

$$\mathsf{R}\mathrm{Hom}_R(\boldsymbol{k}, M) \simeq \mathsf{R}\mathrm{Hom}_{\widehat{R}}(\boldsymbol{k}, \widehat{R} \otimes_R M) .$$

In particular, there are for all $m \in \mathbb{Z}$ isomorphisms of $\boldsymbol{k}$-vector spaces,

$$\mathrm{Ext}^m_R(\boldsymbol{k}, M) \cong \mathrm{Ext}^m_{\widehat{R}}(\boldsymbol{k}, \widehat{R} \otimes_R M) .$$

Proof. In the computation below, the first isomorphism comes from 13.4.18(b), the second isomorphism holds in view of 11.1.19 by 13.2.5, and the last isomorphism holds by 13.4.20(b) as $\boldsymbol{k}$ by 13.3.24 is derived $\mathfrak{m}$-torsion.

$$\begin{aligned}\mathsf{RHom}_{\widehat{R}}(\boldsymbol{k}, \widehat{R} \otimes_R M) &\simeq \mathsf{RHom}_R(\boldsymbol{k}, \widehat{R} \otimes_R M)\\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, \mathsf{L}\Lambda^{\mathfrak{m}}(M))\\ &\simeq \mathsf{RHom}_R(\boldsymbol{k}, M)\;.\end{aligned}$$

The isomorphisms of $\boldsymbol{k}$-vector spaces now follow from the definition of Ext. □

18.3.9 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. There is an equality,*

$$\mathrm{P}^R_M(t) \;=\; \mathrm{P}^{\widehat{R}}_{\widehat{R}\otimes_R M}(t)\;,$$

of Laurent series. In particular, one has $\operatorname{pd}_R M = \operatorname{pd}_{\widehat{R}}(\widehat{R} \otimes_R M)$.

Proof. Per 18.3.3 the equalities are immediate from 16.4.14 and 16.4.16. □

The last assertion in the next result is known from 17.3.24.

18.3.10 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. There is an equality,*

$$\mathrm{I}^M_R(t) \;=\; \mathrm{I}^{\widehat{R}\otimes_R M}_{\widehat{R}}(t)\;,$$

of Laurent series. In particular, one has $\operatorname{id}_R M = \operatorname{id}_{\widehat{R}}(\widehat{R} \otimes_R M)$.

Proof. Per 18.3.8 the equalities are immediate from 16.4.28 and 16.4.30. □

18.3.11 Theorem. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$; there are equalities,*

$$\dim_R M \;=\; \dim_{\widehat{R}}(\widehat{R} \otimes_R M) \quad \textit{and} \quad \operatorname{cmd}_R M \;=\; \operatorname{cmd}_{\widehat{R}}(\widehat{R} \otimes_R M)\;.$$

Proof. It suffices to prove the first equality; the second then follows in view of 18.3.6 and 17.2.2. By 14.2.1 and the isomorphism $\mathrm{H}(\widehat{R} \otimes_R M) \cong \widehat{R} \otimes_R \mathrm{H}(M)$, see 18.3.2(b), one can assume that M is a finitely generated R-module. The equality is trivial for the zero module, so now let $M \neq 0$, set $\mathfrak{a} = (0 :_R M)$, and recall from 14.1.1 that $\dim_R M = \dim R/\mathfrak{a}$ holds. In view of 15.3.34 one similarly has $\dim_{\widehat{R}}(\widehat{R} \otimes_R M) = \dim \widehat{R}/\mathfrak{a}\widehat{R}$. The ring $R/\mathfrak{a}$ is local with maximal ideal $\mathfrak{m}/\mathfrak{a}$ and the quotient $\widehat{R}/\mathfrak{a}\widehat{R} \cong R/\mathfrak{a} \otimes_R \widehat{R}$ is flat as an $R/\mathfrak{a}$-module, see 1.1.10 and 5.4.24(a). To see that the extension $(\mathfrak{m}/\mathfrak{a})(\widehat{R}/\mathfrak{a}\widehat{R})$ is the maximal ideal of the local ring $\widehat{R}/\mathfrak{a}\widehat{R}$ notice that there is a commutative diagram of ring homomorphisms,

$$\begin{tikzcd} R \arrow[r] \arrow[d, two heads] & \widehat{R} \arrow[d, two heads] \\ R/\mathfrak{a} \arrow[r] & \widehat{R}/\mathfrak{a}\widehat{R}\;. \end{tikzcd}$$

Thus $(\mathfrak{m}/\mathfrak{a})(\widehat{R}/\mathfrak{a}\widehat{R})$ is the image in $\widehat{R}/\mathfrak{a}\widehat{R}$ of the maximal ideal $\mathfrak{m}\widehat{R}$ of $\widehat{R}$, see 16.1.13, so it is indeed the maximal ideal of $\widehat{R}/\mathfrak{a}\widehat{R}$. Now the desired equality follows, as 17.2.29 yields $\dim \widehat{R}/\mathfrak{a}\widehat{R} = \dim R/\mathfrak{a}$. □

18.3.12 Corollary. *Let R be local. There are equailities,*

$$\operatorname{depth} R \;=\; \operatorname{depth} \widehat{R} \qquad \textit{and} \qquad \dim R \;=\; \dim \widehat{R}\,.$$

Thus, R is a Cohen–Macaulay ring if and only if $\widehat{R}$ is a Cohen–Macaulay ring.

PROOF. The equalities hold by 18.3.6 and 18.3.11, and the last assertion is then immediate from 17.2.16. □

GORENSTEIN RINGS

18.3.13 Corollary. *Let R be local. There is an equality,*

$$\operatorname{id} R \;=\; \operatorname{id} \widehat{R}\,,$$

so R is a Gorenstein ring if and only if $\widehat{R}$ is a Gorenstein ring.

PROOF. The equality is an immediate consequence of 18.3.10, and the last assertion is then immediate form 17.4.2. □

The proof of the next result uses the technique of passing to the completion. It is the standard textbook characterization of Gorenstein local rings. In part (*ii*) the a priori assumption that R is Cohen–Macaulay is actually superfluous, see 18.4.23. This was conjectured by Vasconcelos [246, 2.Appn.] and proved by Roberts [214].

18.3.14 Theorem. *Let R be local of Krull dimension d. The following conditions are equivalent.*

(*i*) *R is Gorenstein.*
(*ii*) *R is Cohen–Macaulay with $\mu_R^d(R) = 1$.*
(*iii*) *$\Sigma^d R$ is a normalized dualizing complex for R.*
(*iv*) *One has $\mathrm{I}_R(t) = t^d$.*

PROOF. Conditions (*i*) and (*iv*) are equivalent by 17.4.11. As R has finite Krull dimension, conditions (*i*) and (*iii*) are equivalent by 18.2.5 and 18.2.23.

(*iii*) ⇒ (*ii*): It follows from 18.2.11 and 17.2.3 that R is Cohen–Macaulay. By 18.2.33 and 16.4.27 one has $\mathrm{I}_R(t) = \mathrm{P}^R_{\Sigma^d R}(t) = t^d$, whence $\mu_R^d(R) = 1$ holds.

(*ii*) ⇒ (*i*): By 18.3.12 the complete local ring $\widehat{R}$ is Cohen–Macaulay of Krull dimension $d = \dim R$. Per 18.2.29 let D be a normalized dualizing complex for $\widehat{R}$. By 18.2.28 the complex $\mathrm{H}(D)$ is concentrated in degree d. Set $K = \mathrm{H}_d(D)$; by 7.3.29 there is an isomorphism $D \simeq \Sigma^d K$ in $\mathcal{D}(\widehat{R})$, so K is per 10.1.3 a dualizing complex for $\widehat{R}$. In particular, one has $\widehat{R} \simeq \mathsf{R}\mathrm{Hom}_{\widehat{R}}(K,K)$ in $\mathcal{D}(\widehat{R})$, so by 7.3.27 there is an isomorphism of $\widehat{R}$-modules, $\widehat{R} \cong \mathrm{Hom}_{\widehat{R}}(K,K)$. By 18.3.10 and 18.2.33 one has $\mathrm{I}_R(t) = \mathrm{I}_{\widehat{R}}(t) = \mathrm{P}^{\widehat{R}}_D(t)$. In particular one gets via 7.4.24 and the definition, 16.4.14, of Betti numbers,

$$1 \;=\; \mu_R^d(R) \;=\; \beta_d^{\widehat{R}}(D) \;=\; \beta_0^{\widehat{R}}(K)\,.$$

It follows that K is cyclic. Thus there is an isomorphism $K \cong \widehat{R}/(0 :_{\widehat{R}} K)$, and the isomorphism $\widehat{R} \cong \operatorname{Hom}_{\widehat{R}}(K, K)$ forces $(0 :_{\widehat{R}} K) = 0$, see 1.1.8. Thus $\widehat{R} = K$ is a dualizing complex for $\widehat{R}$, which by 18.2.5 means that $\widehat{R}$ is Gorenstein, and then R is Gorenstein by 18.3.13. □

The characterization of Cohen–Macaulay rings in terms of vanishing of local cohomology 17.2.19 can be refined to a characterization of Gorenstein rings.

18.3.15 Proposition. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local of Krull dimension d. The following conditions are equivalent.*

(*i*) *R is Gorenstein.*
(*ii*) *R is Cohen–Macaulay and one has $\mathrm{H}^d_{\mathfrak{m}}(R) \cong \mathrm{E}_R(\boldsymbol{k})$.*
(*iii*) *$\mathrm{H}^n_{\mathfrak{m}}(R) = 0$ holds for $n \neq d$ and one has $\mathrm{H}^d_{\mathfrak{m}}(R) \cong \mathrm{E}_R(\boldsymbol{k})$.*

Proof. Conditions (*ii*) and (*iii*) are equivalent by 17.2.19.

(*i*) ⇒ (*iii*): Recall from 18.3.14 that $\Sigma^d R$ is a normalized dualizing complex for R. From 11.3.20 and 18.2.37 one now gets,

$$\mathrm{H}^n_{\mathfrak{m}}(R) = \mathrm{H}_{-n}(\mathsf{R}\Gamma_{\mathfrak{m}}(R)) \cong \mathrm{H}_{d-n}(\mathsf{R}\Gamma_{\mathfrak{m}}(\Sigma^d R)) \cong \begin{cases} 0 & \text{for } n \neq d \\ \mathrm{E}_R(\boldsymbol{k}) & \text{for } n = d\,. \end{cases}$$

(*ii*) ⇒ (*i*): As $\mathrm{H}^n_{\mathfrak{m}}(R) = 0$ holds for $n \neq d$ by 17.2.19 there is by 13.3.18 and 7.3.29 an isomorphism $\mathsf{R}\Gamma_{\mathfrak{m}}(R) \simeq \Sigma^{-d}\mathrm{E}_R(\boldsymbol{k})$ in $\mathcal{D}(R)$. The complex $\mathsf{L}\Lambda^{\mathfrak{m}}(\Sigma^{-d}\mathrm{E}_R(\boldsymbol{k}))$ is by 18.2.9 a dualizing complex for $\widehat{R}$, and there are isomorphisms in $\mathcal{D}(\widehat{R})$,

$$\mathsf{L}\Lambda^{\mathfrak{m}}(\Sigma^{-d}\mathrm{E}_R(\boldsymbol{k})) \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(\mathsf{R}\Gamma_{\mathfrak{m}}(R)) \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(R) \simeq \widehat{R}\,,$$

by 13.4.1(c) and 13.2.7. Thus, $\widehat{R}$ is a dualizing complex for $\widehat{R}$, whence R is Gorenstein by 18.2.5 and 18.3.13. □

18.3.16 Theorem. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local. The following conditions are equivalent.*

(*i*) *R is Gorenstein.*
(*ii*) $\operatorname{fd}_R \operatorname{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(R), \mathrm{E}_R(\boldsymbol{k}))$ *is finite.*
(*iii*) *There is an isomorphism,* $\operatorname{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(R), \mathrm{E}_R(\boldsymbol{k})) \simeq \Sigma^{\dim R}\widehat{R}$*, in* $\mathcal{D}(\widehat{R})$.

Proof. Set $D = \operatorname{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(R), \mathrm{E}_R(\boldsymbol{k}))$ and recall from 18.2.9 that it is a normalized dualizing complex for $\widehat{R}$. Further, recall from 17.2.29 that $\dim R = \dim \widehat{R}$ holds and denote this quantity by d.

(*i*) ⇒ (*iii*): If R is Gorenstein, then by 18.3.13 so is $\widehat{R}$, whence $D \simeq \Sigma^d \widehat{R}$ holds by 18.3.14 and 18.2.27.

(*iii*) ⇒ (*ii*): This implication is evident as $\widehat{R}$ is flat as an R-module, see 16.1.13.

(*ii*) ⇒ (*i*): Notice that 18.2.32(b), 15.4.18, and 16.1.21 yield

$$\operatorname{id}\widehat{R} = \operatorname{pd}_{\widehat{R}} D = \operatorname{fd}_{\widehat{R}} D = \operatorname{fd}_R \operatorname{Hom}_R(\mathsf{R}\Gamma_{\mathfrak{m}}(R), \mathrm{E}_R(\boldsymbol{k}))\,.$$

It follows that $\widehat{R}$ is Gorenstein, whence R is Gorenstein by 18.3.13. □

LOCAL DUALITY

18.3.17 Theorem. *Let $\mathfrak{a} \subseteq R$ be an ideal, D a dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathsf{R}\mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M,D),\mathsf{R}\Gamma_{\mathfrak{a}}(D)) .$$

PROOF. By Grothendieck Duality 18.2.3 the complex $\mathsf{R}\mathrm{Hom}_R(M,D)$ belongs to $\mathcal{D}^{\mathrm{f}}(R)$, and there is an isomorphism $M \simeq \mathsf{R}\mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M,D),D)$ in $\mathcal{D}(R)$. The asserted isomorphism now follows from 13.3.20(c). □

The following special case is known as the Local Duality Theorem.

18.3.18 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, D a normalized dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M,D),\mathrm{E}_R(\boldsymbol{k})) ;$$

in particular, for every $n \in \mathbb{Z}$ there is an isomorphism of R-modules,

$$\mathrm{H}^n_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{-n}(M,D),\mathrm{E}_R(\boldsymbol{k})) .$$

PROOF. In view of 18.2.37 the isomorphism in $\mathcal{D}(R)$ is a special case of 18.3.17. The isomorphisms of modules follows from 2.2.19 and the definitions of Ext and local cohomology, see 7.3.23 and 11.3.20. □

18.3.19 Example. Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, $\boldsymbol{x}$ a sequence that generates $\mathfrak{m}$, and D a normalized dualizing complex for R. By Local Duality 18.3.18, the counitor 12.3.4, and 13.3.18 there are isomorphisms in $\mathcal{D}(R)$,

$$\mathrm{Hom}_R(D,\mathrm{E}_R(\boldsymbol{k})) \simeq \mathsf{R}\Gamma_{\mathfrak{m}}(R) \simeq \check{\mathrm{C}}^R(\boldsymbol{x}) .$$

REMARK. As it is stated above in the language of the derived category, the Local Duality Theorem first appeared in Hartshorne's notes [114, V.§6] from Grothendieck's 1963/64 seminar at Harvard. However, Grothendieck had presented the Cohen–Macaulay case already in 1961, and that proof was written up by Hartshorne in [115], which appeared later. Without the language of the derived category, one has to settle for the "in particular" statement in 18.3.18 about isomorphisms of cohomology modules, and D has to be a dualizing module which introduces a correction factor in the cohomological degree. That is how the theorem appears in [115, §6]; see also E 18.3.3.

The next results are dual to 18.3.17 and 18.3.18.

18.3.20 Theorem. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, $\mathfrak{a}$ an ideal in R, and M a derived Matlis reflexive R-complex. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{L}\Lambda^{\mathfrak{a}}(M) \simeq \mathsf{R}\mathrm{Hom}_R(\mathrm{Hom}_R(M,\mathrm{E}_R(\boldsymbol{k})),\mathsf{L}\Lambda^{\mathfrak{a}}(\mathrm{E}_R(\boldsymbol{k}))) .$$

PROOF. By assumption one has $M \simeq \mathsf{R}\mathrm{Hom}_R(\mathrm{Hom}_R(M,\mathrm{E}_R(\boldsymbol{k})),\mathrm{E}_R(\boldsymbol{k}))$ in $\mathcal{D}(R)$, see 16.1.35. Now apply $\mathsf{L}\Lambda^{\mathfrak{a}}$ and invoke 13.1.18. □

18.3.21 Corollary. *Let $(R,\mathfrak{m},\boldsymbol{k})$ be local, D a normalized dualizing complex for $\widehat{R}$, and M a derived Matlis reflexive R-complex. There is an isomorphism in $\mathcal{D}(R)$,*

$$\mathsf{L}\Lambda^{\mathfrak{m}}(M) \simeq \mathsf{R}\mathrm{Hom}_R(\mathrm{Hom}_R(M,\mathrm{E}_R(\boldsymbol{k})),D) .$$

PROOF. Recall from 18.2.29 that $\mathsf{L}\Lambda^{\mathfrak{m}}(\mathrm{E}_R(\boldsymbol{k}))$ is a normalized dualizing complex for $\widehat{R}$, and hence it is by 18.2.27 and 18.2.21 isomorphic in $\mathcal{D}(R)$ to the given complex D. Thus, the statement is a special case of 18.3.20. □

KRULL DIMENSION AND VANISHING OF LOCAL COHOMOLOGY

18.3.22 Theorem. *Let R be local, $\mathfrak{a}$ an ideal in R, and M an R-complex. One has*

$$\dim_R M \;\geqslant\; -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) \;=\; \sup\{m \in \mathbb{Z} \mid \mathrm{H}^m_{\mathfrak{a}}(M) \neq 0\}\,,$$

and equality holds if $\mathfrak{a}$ is the maximal ideal of R and M belongs to $\mathcal{D}^{\mathrm{f}}(R)$.

PROOF. The asserted equality holds by the definition, 11.3.20, of local cohomology. If M is acyclic, then the inequality is trivial and, in fact, equality holds as both sides equal $-\infty$, so assume now that M is not acyclic.

First we prove that the (in)equality holds for $M \in \mathcal{D}^{\mathrm{f}}(R)$. Let $\mathfrak{m}$ be the maximal ideal of R and per 18.2.29 let D be a normalized dualizing complex for $\widehat{R}$. By 18.3.2 the base changed complex $\widehat{R} \otimes_R M$ belongs to $\mathcal{D}^{\mathrm{f}}(\widehat{R})$, and the first two equalities in the computation below hold by 18.3.11 and 18.2.31(a). As D belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(\widehat{R})$, 14.3.16, 15.4.15, and 18.2.24 yield

$$\operatorname{id}_{\widehat{R}} \mathsf{R}\Gamma_{\mathfrak{a}\widehat{R}}(D) \;\leqslant\; \operatorname{id}_{\widehat{R}} D \;=\; 0\,.$$

The inequality below thus follows from 15.4.7, and equality holds for $\mathfrak{a} = \mathfrak{m}$ by 16.1.13, (16.1.22.2) and the isomorphism $\mathsf{R}\Gamma_{\mathfrak{m}\widehat{R}}(D) \simeq \mathrm{E}_{\widehat{R}}(\boldsymbol{k})$ from 18.2.37. The last three equalities hold by 18.3.17, 13.3.23(b), and 18.3.2.

$$(\flat)\qquad \begin{aligned} \dim_R M \;&=\; \dim_{\widehat{R}}(\widehat{R} \otimes_R M)\\ &=\; \sup \mathsf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, D)\\ &\geqslant\; -\inf \mathsf{R}\mathrm{Hom}_{\widehat{R}}(\mathsf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, D), \mathsf{R}\Gamma_{\mathfrak{a}\widehat{R}}(D))\\ &=\; -\inf \mathsf{R}\Gamma_{\mathfrak{a}\widehat{R}}(\widehat{R} \otimes_R M)\\ &=\; -\inf\, (\widehat{R} \otimes_R \mathsf{R}\Gamma_{\mathfrak{a}}(M))\\ &=\; -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M)\,. \end{aligned}$$

It remains to prove that the asserted inequality holds for every R-complex M. Consider first the special case of a module. Let U be the filtered set of finitely generated submodules of M ordered under inclusion. By 3.3.5 one has $M \cong \operatorname{colim}_{M' \in U} M'$, and for every $m \in \mathbb{Z}$ there are by 13.3.22 isomorphisms,

$$\mathrm{H}^m_{\mathfrak{a}}(M) \;\cong\; \mathrm{H}^{\mathfrak{a}}_m\big(\operatorname*{colim}_{M' \in U} M'\big) \;\cong\; \operatorname*{colim}_{M' \in U} \mathrm{H}^m_{\mathfrak{a}}(M')\,.$$

For every finitely generated submodule M' of M one has $\operatorname{Supp}_R M' \subseteq \operatorname{Supp}_R M$ and, therefore, $\dim_R M' \leqslant \dim_R M$. Now it follows from $(\flat)$ that $\mathrm{H}^m_{\mathfrak{a}}(M') = 0$ holds for $m > \dim_R M$, so one has $\mathrm{H}^m_{\mathfrak{a}}(M) = 0$ for $m > \dim_R M$ as asserted.

Next we reduce the general case to the case of a complex with bounded homology. The asserted inequality is trivial if $\mathrm{H}(M)$ is not bounded below, see 14.2.4, so one can assume that M belongs to $\mathcal{D}_{\sqsupset}(R)$. For every $n \in \mathbb{Z}$ one gets from 7.6.6(c) and 11.3.15 a distinguished triangle,

$$\mathsf{R}\Gamma_{\mathfrak{a}}(M_{\supseteq n+1}) \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}(M) \longrightarrow \mathsf{R}\Gamma_{\mathfrak{a}}(M_{\subseteq n}) \longrightarrow \Sigma\mathsf{R}\Gamma_{\mathfrak{a}}(M_{\supseteq n+1})\;.$$

The asserted inequality is trivial if the complex $\mathsf{R}\Gamma_{\mathfrak{a}}(M)$ is acyclic, so assume that it is not acyclic. The functor $\mathsf{R}\Gamma_{\mathfrak{a}}$ is by 13.3.18 bounded, so for $n \gg 0$ one has $\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M_{\supseteq n+1}) \gg 0$ and, therefore, $\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathsf{R}\Gamma_{\mathfrak{a}}(M_{\subseteq n})$, see A.23 and 6.5.20. Further, notice from 14.2.1 that

$$\dim_R M \;=\; \dim_R M_{\subseteq n}$$

holds for $n \gg 0$. One can thus assume that M belongs to $\mathcal{D}_{\square}(R)$.

Now let M be a complex in $\mathcal{D}_{\square}(R)$ and $\boldsymbol{x}$ a sequence that generates $\mathfrak{a}$. The module case handled above justifies the last inequality in the next computation; the first inequality comes from 7.6.10, and the equalities hold by 13.3.18 and 14.2.1.

$$\begin{aligned} -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) &= -\inf\,(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R M) \\ &\leqslant -\inf\{\inf\,(\check{\mathrm{C}}^R(\boldsymbol{x}) \otimes^{\mathsf{L}}_R \mathrm{H}_v(M)) + v \mid v \in \mathbb{Z}\} \\ &= \sup\{-\inf \mathsf{R}\Gamma_{\mathfrak{a}}(\mathrm{H}_v(M)) - v \mid v \in \mathbb{Z}\} \\ &\leqslant \sup\{\dim_R \mathrm{H}_v(M) - v \mid v \in \mathbb{Z}\} \\ &= \dim_R M\;. \end{aligned}$$

□

18.3.23 Corollary. *Let $(R,\mathfrak{m})$ be local and M an R-complex. There is an inequality,*

$$\dim_R M \;\geqslant\; \dim_R \mathsf{R}\Gamma_{\mathfrak{m}}(M)\;,$$

and equality holds if M belongs to $\mathcal{D}^{\mathrm{f}}(R)$.

Proof. One has $\dim_R M \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) = \dim_R \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ by 18.3.22, 13.4.7, and 16.1.31(b), and equality holds if M belongs to $\mathcal{D}^{\mathrm{f}}(R)$. □

The next inequality is an important special case of 18.3.29; it also holds for complexes in $\mathcal{D}^{\mathrm{f}}(R)$, see 17.6.15.

18.3.24 Corollary. *Let $(R,\mathfrak{m})$ be local and M and N be R-complexes of finite depth. If M is derived $\mathfrak{m}$-torsion or N is derived $\mathfrak{m}$-complete, then the next inequality holds,*

$$-\sup \mathsf{R}\mathrm{Hom}_R(M,N) \;\geqslant\; \operatorname{depth}_R N - \dim_R M\;.$$

Proof. The inequality $-\sup \mathsf{R}\mathrm{Hom}_R(M,N) \;\geqslant\; \operatorname{depth}_R N + \inf \mathsf{R}\Gamma_{\mathfrak{m}}(M)$ holds by 14.4.6 applied with $\mathfrak{a} = \mathfrak{m}$, and it remains to invoke 18.3.22. □

The inequality in the next result is an important special case of 18.3.30. The result also compares to 14.2.8.

18.3.25 Corollary. *Let $(R,\mathfrak{m})$ be local and M and N be R-complexes of finite width. If M or N is derived $\mathfrak{m}$-torsion, then the next inequality holds,*

$$\dim_R(M \otimes^{\mathsf{L}}_R N) \;\leqslant\; \dim_R M - \operatorname{width}_R N\;.$$

Proof. The complex $M \otimes^{\mathsf{L}}_R N$ is by 13.4.20(c) derived $\mathfrak{m}$-torsion, so 16.1.31(b) yields $\dim_R(M \otimes^{\mathsf{L}}_R N) = -\inf(M \otimes^{\mathsf{L}}_R N)$. From 14.4.11 applied with $\mathfrak{a} = \mathfrak{m}$ one gets $-\inf(M \otimes^{\mathsf{L}}_R N) \leqslant -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) - \operatorname{width}_R M$, and it remains to invoke 18.3.22. □

The inequality in the next theorem also holds for complexes in $\mathcal{D}^{\mathrm{f}}(R)$, see 16.4.6.

18.3.26 Theorem. *Let $(R, \mathfrak{m})$ be local, $\mathfrak{p}$ a prime ideal in R, and M an R-complex. If M is derived $\mathfrak{m}$-complete, then the next inequality holds,*

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \;\geqslant\; \operatorname{depth}_R M\;.$$

Proof. If M is acylic, then the inequality is trivial as both sides equal ∞. Thus, assume that M derived $\mathfrak{m}$-complete and not acyclic. It follows from 16.2.27 that M and $R/\mathfrak{p}$ have finite depth. In the next computation the inequalities hold by 17.6.3 and 18.3.24 while the equality holds by 14.4.3.

$$\begin{aligned}\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &\;\geqslant\; \mathfrak{p}\text{-}\operatorname{depth}_R M\\ &\;=\; -\sup \mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)\\ &\;\geqslant\; \operatorname{depth}_R M - \dim R/\mathfrak{p}\;.\end{aligned}$$ □

The next inequality also holds for complexes in $\mathcal{D}^{\mathrm{f}}(R)$, see 17.6.5. Notice from 17.6.3 that, for a prime ideal, it is stronger than the inequality in 18.3.26.

18.3.27 Corollary. *Let $(R, \mathfrak{m})$ be local, $\mathfrak{a}$ a proper ideal in R, and M an R-complex. If M is derived $\mathfrak{m}$-complete, then the next inequality holds,*

$$\mathfrak{a}\text{-}\operatorname{depth}_R M + \dim R/\mathfrak{a} \;\geqslant\; \operatorname{depth}_R M\;.$$

Proof. For every prime ideal $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})$ one has

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{a} \;\geqslant\; \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \;\geqslant\; \operatorname{depth}_R M$$

by 18.3.26, so the asserted inequality follows from 17.6.3. □

Grothendieck's Vanishing Theorem

The next result is known as Grothendieck's vanishing theorem for local cohomology. Recall from 18.3.22 that equality holds if $\mathfrak{a}$ is the maximal ideal of a local ring and the complex has degreewise finitely generated homology.

18.3.28 Theorem. *Let $\mathfrak{a}$ be an ideal in R and M an R-complex*; *one has*

$$\dim_R M \;\geqslant\; -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) \;=\; \sup\{m \in \mathbb{Z} \mid \mathrm{H}^m_{\mathfrak{a}}(M) \neq 0\}\;,$$

Proof. The equality holds by the definition, 11.3.20, of local cohomology. Let $\mathfrak{p}$ be a prime ideal in R, and consider the ideal $\mathfrak{a}_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$. The inequality $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geqslant -\inf \mathsf{R}\Gamma_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}})$ holds by 18.3.22, and this justifies the inequality below; the equalities hold by 14.2.7, 14.1.25, and 14.1.13.

$$\dim_R M \;=\; \sup\{\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

$$\begin{aligned} &\geqslant \sup\{-\inf \mathsf{R}\Gamma_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= -\inf\{\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M)_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= -\inf \mathsf{R}\Gamma_{\mathfrak{a}}(M) . \end{aligned}$$ □

18.3.29 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes of finite $\mathfrak{a}$-depth. If M is derived $\mathfrak{a}$-torsion or N is derived $\mathfrak{a}$-complete, then one has*

$$-\sup \mathsf{R}\mathrm{Hom}_R(M,N) \;\geqslant\; \mathfrak{a}\text{-}\mathrm{depth}_R\, N - \dim_R M .$$

Proof. The inequality follows from 14.4.6 and 18.3.28. □

18.3.30 Corollary. *Let $\mathfrak{a}$ be an ideal in R and M and N be R-complexes of finite $\mathfrak{a}$-width. If M or N is derived $\mathfrak{a}$-torsion, then the next inequality holds,*

$$\inf\,(M \otimes^{\mathsf{L}}_R N) \;\geqslant\; \mathfrak{a}\text{-}\mathrm{width}_R\, N - \dim_R M .$$

Proof. The inequality follows from 14.4.11 and 18.3.28. □

Cohen–Macaulay Defect

For complexes of finite depth over a local ring R, the dimension and depth compare as they do for complexes in $\mathcal{D}^{\mathrm{f}}(R)$, cf. 17.2.1.

18.3.31 Theorem. *Let R be local and M an R-complex. If $\operatorname{depth}_R M$ is finite, then there is an inequality,*

$$\operatorname{depth}_R M \;\leqslant\; \dim_R M .$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. The assumption that $\operatorname{depth}_R M$ is finite ensures per 16.2.23 that the complex $\mathsf{R}\Gamma_{\mathfrak{m}}(M)$ is not acyclic, which justifies the first inequality in the next display. The equality comes from 16.2.14, and the second inequality holds by 18.3.22.

$$\operatorname{depth}_R M \;=\; -\sup \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;\leqslant\; -\inf \mathsf{R}\Gamma_{\mathfrak{m}}(M) \;\leqslant\; \dim_R M .$$ □

For a finitely generated module $M \neq 0$ over a local ring R, and more generally a non-acyclic complex in $\mathcal{D}^{\mathrm{f}}(R)$, the depth is guaranteed to be finite, see 16.2.27. This is why the inequality $\operatorname{depth}_R M \leqslant \dim_R M$ holds without further assumptions, see 17.2.1. On the other hand, the inequality fails for every R-module of infinite depth, such as $\mathrm{E}_R(R/\mathfrak{p})$ for a non-maximal prime ideal in R, see 16.2.29.

The special cases $N = R$ and $N \in \mathcal{D}^{\mathrm{f}}_{\square}(R)$ of the following result are improved in 18.4.13 and 18.5.3.

18.3.32 Corollary. *Let $(R,\mathfrak{m})$ be local, M an R-complex, and N a complex in $\mathcal{D}_{\sqsubset}(R)$. If one has $\mathfrak{m} \in \operatorname{supp}_R M \cap \operatorname{supp}_R N$, then the next inequality holds.*

$$\mathrm{fd}_R\, M \;\geqslant\; \operatorname{depth}_R N - \dim_R(M \otimes^{\mathsf{L}}_R N) .$$

Proof. The assumption $\mathfrak{m} \in \operatorname{supp}_R M \cap \operatorname{supp}_R N$ ensures by 16.2.27 and the Support Formula 15.1.16 that 18.3.31 applies to the complex $M \otimes^{\mathsf{L}}_R N$. Now combine it with 16.3.4 to get the asserted inequality. □

In view of 18.3.31 it is natural to extend the use of the invariant, cmd_R, introduced for complexes in $\mathcal{D}^{\mathrm{f}}(R)$ in 17.2.2. Per 16.2.27 the definition below essentially subsumes 17.2.2, but the latter also applies to acyclic complexes, and it remains the case that a complex has Cohen–Macaulay defect $-\infty$ if and only if it is acyclic. We stress that the term 'Cohen–Macaulay' remains reserved for complexes in $\mathcal{D}^{\mathrm{f}}(R)$.

18.3.33 Definition. Let $(R, \mathfrak{m})$ be local and M an R-complex. If $\operatorname{depth}_R M$ is finite, equivalently, $\mathfrak{m} \in \operatorname{supp}_R M$, see 16.2.27, then the *Cohen–Macaulay defect* of M is

$$\mathrm{cmd}_R M = \dim_R M - \operatorname{depth}_R M .$$

18.3.34 Proposition. *Let R be local, $\mathfrak{a}$ a proper ideal in R, and M an $R/\mathfrak{a}$-complex of finite depth. There is an equality,*

$$\mathrm{cmd}_{R/\mathfrak{a}} M = \mathrm{cmd}_R M .$$

Proof. The equality follows from 14.2.5 and 16.2.26. □

18.3.35 Example. Let R be local and $M \neq 0$ an Artinian R-module. By 14.2.10 and 16.2.18 one has $\dim_R M = 0 = \operatorname{depth}_R M$ and hence $\mathrm{cmd}_R M = 0$.

The equality $\mathrm{cmd}_R M = 0$ notwithstanding, the module M in 18.3.35 is only Cohen–Macaulay if it is finitely generated, i.e. of finite length, cf. 17.2.6.

No Holes in the Sequence of Bass Numbers

To parse the next statement recall the inequality $\dim_R M \leqslant \operatorname{id}_R M$ from 16.4.10.

18.3.36 Proposition. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. For every integer m in the range $\dim_R M \leqslant m \leqslant \operatorname{id}_R M$, one has $\mu^m_R(M) \neq 0$.*

Proof. By 16.4.28 and 18.3.10 one has $\mu^m_R(M) = \mu^m_{\widehat{R}}(\widehat{R} \otimes_R M)$, so in view of 18.3.11 one can assume that R is complete. Per 18.2.29 let D be a normalized dualizing complex for R. By Grothendieck Duality 18.2.3 the complex $\mathsf{RHom}_R(M, D)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ and 18.2.32(b) yields $\mu^m_R(M) = \beta^R_m(\mathsf{RHom}_R(M, D))$ for every $m \in \mathbb{Z}$. The assertion now follows from 16.4.26 as one has $\operatorname{pd}_R \mathsf{RHom}_R(M, D) = \operatorname{id}_R M$ and $\sup \mathsf{RHom}_R(M, D) = \dim_R M$ by 18.2.32(b) and 18.2.31(a). □

The upper bound on m in 18.3.36 is optimal by 16.4.30, and the next example shows that the lower bound is optimal too. For modules one can do better, see 18.3.38.

18.3.37 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be local and artinian. By 18.1.4 the complex

$$M = \mathrm{E}_R(\boldsymbol{k}) \oplus \Sigma^{-2} \mathrm{E}_R(\boldsymbol{k})$$

is in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, and 16.1.31 yields $\dim_R M = 2$. As M is semi-injective and minimal, see 5.3.12 and B.20, one has $\mu^0_R(M) = 1 = \mu^2_R(M)$ but $\mu^1_R(M) = 0$, see 16.4.38.

The next theorem is due to Roberts [213]. As the title of the paper suggests, it was an early application of dualizing complexes to answer a question in local algebra. Recall from 16.4.30 that Bass numbers vanish outside the depth–id range.

18.3.38 Theorem. *Let R be local and M a finitely generated R-module. For every integer m in the range* $\operatorname{depth}_R M \leqslant m \leqslant \operatorname{id}_R M$*, one has* $\mu_R^m(M) \neq 0$.

Proof. By 16.4.28 and 18.3.10 one has $\mu_R^m(M) = \mu_{\widehat{R}}^m(\widehat{R} \otimes_R M)$, so in view of 18.3.6 one can assume that R is complete. Per 18.2.29 let D be a normalized dualizing complex for R. Let $\mathfrak{m}$ be the maximal ideal of R. Let I be a minimal injective resolution of M and set $E = \Gamma_{\mathfrak{m}}(I)$. By the definition of $\mathsf{R}\Gamma_{\mathfrak{m}}$ and by Local Duality 18.3.18 one has

$$(\flat) \qquad E = \mathsf{R}\Gamma_{\mathfrak{m}}(M) \simeq \operatorname{Hom}_R(\mathsf{RHom}_R(M,D), \mathrm{E}_R(\boldsymbol{k})) \,.$$

Now $F = \operatorname{Hom}_R(E, \mathrm{E}_R(\boldsymbol{k}))$ is by 16.4.38 and 16.1.24 a complex of finitely generated free R-modules with $\operatorname{rank}_R F_m = \mu_R^m(M)$. By Grothendieck Duality 18.2.3 the complex $\mathsf{RHom}_R(M,D)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$, so Matlis Duality 18.1.9 and ($\flat$) yield

$$F \simeq \mathsf{RHom}_R(M,D) \,.$$

Assume towards a contradiction that $\mu_R^m(M) = 0$ holds for some integer m between $\operatorname{depth}_R M$ and $\operatorname{id}_R M$. By 16.4.30 and 18.3.36 on has

$$\operatorname{depth}_R M < m < \dim_R M \,.$$

As $F_m = 0$ holds, the complex F decomposes as a direct sum of $F' = F_{\geqslant m+1}$ and $F'' = F_{\leqslant m-1}$. To see that neither complex is acyclic, notice that 18.2.31(a) yields $\sup F = \dim_R M > m$ which forces $\mathrm{H}(F') \neq 0$; similarly 18.2.31(b) gives $\inf F = \operatorname{depth}_R M < m$, whence $\mathrm{H}(F'') \neq 0$. Grothendieck Duality 18.2.3 yields

$$M \simeq \mathsf{RHom}_R(F,D) \simeq \mathsf{RHom}_R(F',D) \oplus \mathsf{RHom}_R(F'',D) \,.$$

It follows that M is a direct sum of the non-zero modules $M' = \mathrm{H}(\mathsf{RHom}_R(F',D))$ and $M'' = \mathrm{H}(\mathsf{RHom}_R(F'',D))$. Notice that F'' is a bounded complex of free R-modules, in particular it is semi-projective by 5.2.8. By 7.3.29, 18.2.31(b), 8.1.2, 18.2.32(a), and the Bass Formula 16.4.12 one now has

$$\operatorname{depth}_R M' = \inf F' > m > \operatorname{pd}_R F'' = \operatorname{id}_R M'' = \operatorname{depth} R \,.$$

The complex $\mathsf{RHom}_R(M',M'')$ belongs by 12.2.6 to $\mathcal{D}^{\mathrm{f}}(R)$ and it is not acyclic, see 16.2.28. Now 7.6.7, 16.2.5(a), and 16.3.9(a) yield a contradiction,

$$\begin{aligned} 0 &\geqslant \sup \mathsf{RHom}_R(M',M'') \\ &\geqslant \inf \mathsf{RHom}_R(M',M'') \\ &= \operatorname{width}_R \mathsf{RHom}_R(M',M'') \\ &= \operatorname{depth}_R M' - \operatorname{depth} R \,. \end{aligned}$$

□

EXERCISES

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 18.3.1 Let R be Gorenstein of Krull dimension d and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. Show that there are isomorphisms $\mathrm{H}^n_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}^{d-n}_R(M, R), \mathrm{E}_R(\boldsymbol{k}))$ for all $n \in \mathbb{Z}$.

E 18.3.2 Let D be a normalized dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. Show that $\widehat{R} \otimes_R \mathrm{Ext}^n_R(M, D) \cong \mathrm{Hom}_R(\mathrm{H}^{-n}_{\mathfrak{m}}(M), \mathrm{E}_R(\boldsymbol{k}))$ holds for all $n \in \mathbb{Z}$.

E 18.3.3 Let R be Cohen–Macaulay, D a dualizing module for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. Show that with $d = \dim R$ one has $\mathrm{H}^n_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}^{d-n}_R(M, D), \mathrm{E}_R(\boldsymbol{k}))$ and $\mathrm{Hom}_R(\mathrm{H}^n_{\mathfrak{m}}(M), \mathrm{E}_R(\boldsymbol{k})) \cong \widehat{R} \otimes_R \mathrm{Ext}^{d-n}_R(M, D)$ for all $n \in \mathbb{Z}$. Conclude that there is an isomorphism $\mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{E}_R(\boldsymbol{k})) \cong \widehat{R} \otimes_R D$.

E 18.3.4 Let R be complete, D a normalized dualizing complex for R, and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. Show that $M \simeq \mathsf{R}\mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})), D)$ holds in $\mathcal{D}(R)$.

E 18.3.5 Let $\mathfrak{a}$ be a proper ideal in R and M an R-complex. Show that there is an inequality $\mathrm{depth}_R M \leqslant \mathfrak{a}\text{-}\mathrm{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) + \dim R/\mathfrak{a}$.

E 18.3.6 Let M be a derived $\mathfrak{m}$-complete R-complex. Show that the inequality $\mathrm{cmd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \mathrm{cmd}_R M$ holds for every prime ideal $\mathfrak{p}$ in $\mathrm{supp}_R M$.

E 18.3.7 Let R be Cohen–Macaulay and M an R-complex of finite depth. Show that the inequalities $\mathrm{depth}\, R - \inf M \geqslant \mathrm{depth}_R M \geqslant -\sup M$ hold.

18.4 Maximal Depth Modules and the New Intersection Theorem

SYNOPSIS. Module of maximal depth; maximal Cohen–Macaulay module; big Cohen–Macaulay module; Krull dimension vs. homological dimensions; Cohen–Macaulay ring; Krull's principal ideal theorem; (Improved) New Intersection Theorem.

We have in 16.4.2, 16.4.11, 17.3.4, and 17.5.7, to name a few results, established close connections between homological dimensions of complexes over local rings and the invariants depth and width. This should not be too surprising: Vanishing of homology characterizes homological dimensions, and the way they are introduced in this text, both depth and width are defined in terms of vanishing of homology. For finitely generated modules, the depth is by 16.2.33 even an algebraic invariant, while the width by 16.2.7 is trivial. The Krull dimension, on the other hand, is geometric in nature—not algebraic, not even for finitely generated modules—and comparing it to homological dimensions is significantly more difficult.

For a finitely generated module over a local ring, the Krull dimension is per 16.4.10 and 17.2.1 trapped between the depth and the injective dimension. However, 18.5.8 shows that finitely generated modules of finite injective dimension only exist over Cohen–Macaulay rings, so beyond that setting, this is not an interesting comparison of a homological dimension to the Krull dimension. The development of such comparisons, as covered in this section and the next, has largely been inspired by a paper by Peskine and Szpiro [202]; see also 18.5.13, 18.5.23, and 18.5.30.

MODULES OF MAXIMAL DEPTH

The Cohen–Macaulay defect of a module measures the difference between the Krull dimension and the depth, and modules for which these invariants agree and coincide with the Krull dimension of the ring play a central role in comparisons of homological dimensions to the Krull dimension.

18.4.1 Definition. Let R be local and M an R-module. If $\operatorname{depth}_R M = \dim R$ holds, then M is called an R-module *of maximal depth.*

For a local ring R and an R-module M of finite depth, one has $\operatorname{depth}_R M \leqslant \dim R$ by 16.2.35 or 18.3.31; this motivates the definition above.

18.4.2 Proposition. *Let $(R, \mathfrak{m})$ be local and M an R-module of maximal depth. The maximal ideal $\mathfrak{m}$ belongs to* $\operatorname{supp}_R M$ *and one has* $\operatorname{width}_R M = 0 = \operatorname{cmd}_R M$.

PROOF. It follows from 16.2.27 that $\mathfrak{m}$ belongs to $\operatorname{supp}_R M$, and 16.2.35 yields $\operatorname{width}_R M = 0$. Further, one has $\dim R \geqslant \dim_R M \geqslant \operatorname{depth}_R M = \dim R$ by 14.2.4 and 18.3.31, so $\operatorname{cmd}_R M = 0$ holds by 18.3.33. □

By 18.4.2 a finitely generated module of maximal depth is Cohen–Macaulay.

18.4.3 Definition. Let R be local. A finitely generated R-module of maximal depth is called *maximal Cohen–Macaulay*.

The existence of maximal Cohen–Macaulay modules is a wide open question—in [128] Hochster reflects the state of affairs and the level of expectations circa 2020—but over Cohen–Macaulay rings examples are easily available.

18.4.4 Example. Let R be a Cohen–Macaulay local ring. Every non-zero finitely generated free R-module is by 14.3.14 a maximal Cohen–Macaulay module. If D is a dualizing complex for R, then $\mathrm{H}(D)$ is by 18.2.28 concentrated in one degree, s, and the module $\mathrm{H}_s(D)$ is maximal Cohen–Macaulay by 7.3.29, 18.2.11, and 18.2.16.

Also 1-dimensional rings have conspicuous maximal Cohen–Macaulay modules.

18.4.5 Example. Let $(R, \mathfrak{m})$ be a local ring of Krull dimension 1. For every minimal prime ideal $\mathfrak{p}$ in R the quotient $R/\mathfrak{p}$ is a maximal Cohen–Macualy R-module: Every element in $\mathfrak{m} \setminus \mathfrak{p}$ is $R/\mathfrak{p}$-regular, see 14.4.17, so $\operatorname{depth}_R R/\mathfrak{p}$ is positive by 16.2.33 and hence equal to 1 as one has $\operatorname{depth}_R R/\mathfrak{p} \leqslant \dim_R R/\mathfrak{p} \leqslant \dim R = 1$, see 17.2.1.

18.4.6 Theorem. *Let $(R, \mathfrak{m})$ be local and M an R-module. If M is of maximal depth, then the homology of $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ is concentrated in degree 0 and $\mathrm{H}_0^{\mathfrak{m}}(M)$ is a derived $\mathfrak{m}$-complete R-module of maximal depth and an $\widehat{R}$-module of maximal depth.*

PROOF. Assume that M is of maximal depth and notice first that 16.2.3 and 18.4.2 yield $\inf \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \operatorname{width}_R M = 0$. By 16.2.34 and the assumption on M one has $\sup \mathsf{L}\Lambda^{\mathfrak{m}}(M) \leqslant \dim R - \operatorname{depth}_R M = 0$. Thus, $\mathsf{L}\Lambda^{\mathfrak{m}}(M)$ has homology concentrated in degree 0, and in $\mathcal{D}(R)$ it is per 7.3.29 isomorphic to the homology module in degree 0, i.e. the local homology module $\mathrm{H}_0^{\mathfrak{m}}(M)$. By 16.2.14 and the assumption on M one has $\operatorname{depth}_R \mathsf{L}\Lambda^{\mathfrak{m}}(M) = \operatorname{depth}_R M = \dim R$. This shows that $H = \mathrm{H}_0^{\mathfrak{m}}(M)$ is an R-module of maximal depth. It follows from 13.4.2 that $H \simeq \mathsf{L}\Lambda^{\mathfrak{m}}(M)$ is derived $\mathfrak{m}$-complete. Further, H is an $\widehat{R}$-module, see 11.3.4, and 18.3.1 yields $\operatorname{depth}_{\widehat{R}} H = \operatorname{depth}_R H$, so to see that it is an $\widehat{R}$-module of maximal depth it suffices to recall the equality $\dim \widehat{R} = \dim R$ from 18.3.12. □

18.4.7 Example. If $(R, \mathfrak{m})$ is Cohen–Macaulay, then $\widehat{R}$ is by 16.1.12, 13.2.5, and 18.4.6 a derived $\mathfrak{m}$-complete R-module of maximal depth.

18.4.8 Proposition. *Let R be local, $\boldsymbol{x}$ a parameter sequence for R, and M an R-module. If $\boldsymbol{x}$ is M-regular, then M is a module of maximal depth.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. As $\boldsymbol{x}$ is M-regular one has $(\boldsymbol{x})M \neq M$, so 14.3.29 and 14.4.9 yield $0 = (\boldsymbol{x})\text{-}\operatorname{width}_R M = \operatorname{width}_R M$, cf. 16.2.1. Now $\mathfrak{m}$ belongs to $\operatorname{supp}_R M$ by 16.2.27, so 16.2.35 yields the last inequality in the next display. The equality holds by 16.2.12 and 14.4.4, and the first inequality follows from 14.4.21(a) as a parameter sequence has $\dim R$ elements,

$$\dim R \leqslant (\boldsymbol{x})\text{-}\operatorname{depth}_R M = \operatorname{depth}_R M \leqslant \dim R . \qquad \square$$

Let R be local, $\boldsymbol{x}$ a parameter sequence for R, and M an R-module. If M is finitely generated and $\boldsymbol{x}$ is M-regular, then M is maximal Cohen–Macaulay by 18.4.3 and 18.4.8. If M is not finitely generated and $\boldsymbol{x}$ is M-regular, then M is called a *big Cohen–Macaulay module for $\boldsymbol{x}$*. An R-module that is a big Cohen–Macaulay module for an unspecified parameter sequence is simply called *big Cohen–Macaulay*.

It is a deep fact that big Cohen–Macaulay modules exist over every local ring; the only proof known at this time uses the theory of perfectoid spaces developed by Scholze [225]. As a prelude to the discussion below, we recall that a local ring $(R, \mathfrak{m}, \boldsymbol{k})$ is called *equicharacteristic* if R and $\boldsymbol{k}$ have the same characteristic; it follows that the common characteristic is 0 or a prime p. If the characteristics of R and $\boldsymbol{k}$ differ, then the ring is said to be of *mixed characteristic*; in this case, the characteristic of $\boldsymbol{k}$ is a prime p while the characteritic of R itself is 0 or a power of p.

Hochster [123] proved the existence of big Cohen–Macaulay modules in equicharacteristic $p > 0$ as early as 1973, and in equicharacteristic 0 the following year [126, §§4–5], but in mixed characteristic the matter was only settled in 2018 by André [4]. The theorem below, which in André's survey [5] is Theorem 3.2.1, goes further: One can even get an algebra that, as a module, is big Cohen–Macaulay. For some local rings, the existence of such algebras was proved in the early 1990s by Hochster and Huneke [129, 130].

André's Theorem. *Let R be local. There exists an R-algebra S such that S is a big Cohen–Macaulay R-module.*

18.4.9 Corollary. *Let $(R, \mathfrak{m})$ be local. There exists a derived $\mathfrak{m}$-complete R-module of maximal depth which is also an $\widehat{R}$-module of maximal depth.*

Proof. The assertion follows from the André's Theorem, 18.4.8, and 18.4.6. □

REMARK. In [96], modules of maximal depth are called 'Hochster modules' with a nod to the author of [123, 126] who proved the existence such modules, even of big Cohen–Macaulay modules, over equicharacteristic local rings. Here we opt for the more descriptive name, because these modules are exactly the modules that qualify as 'complexes of maximal depth' in the sense of Iyengar, Ma, Schwede, and Walker [144].

As is apparent in the proof of 18.4.25, it can be advantageous to work with a derived $\mathfrak{m}$-complete module of maximal depth rather than just any module of maximal depth. As discussed in the Remark after 13.1.17, a derived $\mathfrak{m}$-complete module need not be $\mathfrak{m}$-complete, so it is worth noting that André's theorem does imply the existence of $\mathfrak{m}$-complete modules of maximal depth. Indeed, the derived $\mathfrak{m}$-completion of the big Cohen–Macaulay R-algebra S can by 13.1.15 be computed as $\Lambda^{\mathfrak{m}}(S)$ and is hence $\mathfrak{m}$-complete, see 11.1.38, and in particular derived $\mathfrak{m}$-complete by 13.1.33.

18.4.10 Theorem. *Let $(R,\mathfrak{m})$ be local and M an R-module of maximal depth. If M is derived $\mathfrak{m}$-complete, then for every prime ideal $\mathfrak{p}$ in $\operatorname{supp}_R M$ the module $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module of maximal depth and* $\dim R = \dim R_\mathfrak{p} + \dim R/\mathfrak{p}$ *holds.*

PROOF. For $\mathfrak{p} \in \operatorname{supp}_R M$ it follows from 15.1.22 that the maximal ideal $\mathfrak{p}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ belongs to $\operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p}$, whence the first inequality below holds by 16.2.35. The second inequality is standard, cf. 14.2.7.

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \dim R_\mathfrak{p} \leqslant \dim R - \dim R/\mathfrak{p} .$$

As M is of maximal depth and derived $\mathfrak{m}$-complete, 18.3.26 yields

$$\dim R - \dim R/\mathfrak{p} = \operatorname{depth}_R M - \dim R/\mathfrak{p} \leqslant \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} .$$

The asserted equality, $\dim R = \dim R_\mathfrak{p} + \dim R/\mathfrak{p}$, follows from comparison of these two displays, and so does the equality $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \dim R_\mathfrak{p}$, whence $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module of maximal depth. □

18.4.11 Proposition. *Let $(R,\mathfrak{m})$ be local and M an R-module of maximal depth. If M is finitely generated or derived $\mathfrak{m}$-complete, then every parameter sequence for R is M-regular.*

PROOF. Set $d = \dim R$ and let $\boldsymbol{x} = x_1,\dots,x_d$ be a parameter sequence for R. As $\sqrt{(\boldsymbol{x})} = \mathfrak{m}$ holds, 14.4.4 and 16.2.12 yield $\operatorname{depth}_R M = (\boldsymbol{x})\text{-}\operatorname{depth}_R M$. That is, one has $d = d - \sup(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M)$ by 14.3.10, whence $\boldsymbol{x}$ is M-regular by 16.2.31. □

REMARK. Let $(R,\mathfrak{m})$ be local. An R-module that is a big Cohen–Macaulay module for every parameter sequence for R is called a *balanced big Cohen–Macaulay module*. Bruns and Herzog [46, 8.5] show that the $\mathfrak{m}$-completion of any big Cohen–Macaulay module is a balanced big Cohen–Macaulay module; such modules are by 13.1.33 and 18.4.8 examples of derived $\mathfrak{m}$-complete modules of maximal depth. Finally, a derived $\mathfrak{m}$-complete module of maximal depth that is not finitely generated is per 18.4.11 a balanced big Cohen–Macaulay module, and that connects these two rather obvious hierarchies:

$$\begin{array}{cc}
\{\text{modules of maximal depth}\} & \{\text{derived } \mathfrak{m}\text{-complete modules of maximal depth}\} \\
\cup\!\mid & \cup\!\mid \\
\{\text{big Cohen–Macaulay modules}\} & \{\mathfrak{m}\text{-complete modules of maximal depth}\} \\
\cup\!\mid & \cup\!\mid \\
\{\text{balanced big Cohen–Macaulay modules}\} & \{\mathfrak{m}\text{-complete big Cohen–Macaulay modules}\} .
\end{array}$$

For a balanced big Cohen–Macaulay module M the equality in 18.4.10 was proved by Sharp [230], who called the set $\operatorname{supp}_R M$ the 'supersupport' of M.

Krull Dimension vs. Homological Dimensions

18.4.12 Theorem. *Let $(R,\mathfrak{m})$ be local and M an R-complex of finite flat dimension. If one has $\mathfrak{m} \in \operatorname{supp}_R M$, then the following inequality holds*:

$$\operatorname{cmd}_R M \;\geqslant\; \operatorname{cmd} R\,.$$

Proof. Per 18.4.9 let H be an R-module of maximal depth. It follows from 16.2.27 and the Support Formula 15.1.16 that the complex $M \otimes^{\mathsf{L}}_R H$ has finite depth. Now 18.3.31 yields the second inequality in the computation below. The first inequality holds by 14.2.8, and the equality comes from 16.3.1(a).

$$\begin{aligned}\dim_R M \;&\geqslant\; \dim_R(M \otimes^{\mathsf{L}}_R H)\\ &\geqslant\; \operatorname{depth}_R(M \otimes^{\mathsf{L}}_R H)\\ &=\; \operatorname{depth}_R M + \operatorname{depth}_R H - \operatorname{depth} R\,.\end{aligned}$$

Per 18.3.33 the desired inequality now follows as $\operatorname{depth}_R H = \dim R$ holds. □

The next corollary compares to the special case $N = R$ of 16.3.4.

18.4.13 Corollary. *Let $(R,\mathfrak{m})$ be local and M be an R-complex. If $\mathfrak{m} \in \operatorname{supp}_R M$ holds, then there is an inequality,*

$$\operatorname{fd}_R M \;\geqslant\; \dim R - \dim_R M\,.$$

Proof. One can assume that $\operatorname{fd}_R M$ is finite, otherwise the inequality is trivial. By 16.3.4 one now has $\operatorname{fd}_R M \geqslant \operatorname{depth} R - \operatorname{depth}_R M$, and 18.4.12 can be rewritten $\operatorname{depth} R - \operatorname{depth}_R M \geqslant \dim R - \dim_R M$. □

It is immediate from the next corollary and 13.1.33 that only a Cohen–Macaulay ring can accomodate an $\mathfrak{m}$-torsion module of finite flat dimension.

18.4.14 Corollary. *Let $(R,\mathfrak{m})$ be local and M a derived $\mathfrak{m}$-torsion complex of finite flat dimension. If M is not acyclic, then the following inequality holds*:

$$\operatorname{amp} M \;\geqslant\; \operatorname{cmd} R\,.$$

Proof. As M is not acyclic, $\mathfrak{m}$ belongs to $\operatorname{supp}_R M$ by 16.2.27. By 16.1.31(b) and 16.2.16(a) one has $\dim_R M = -\inf M$ and $\operatorname{depth}_R M = -\sup M$ and, therefore, $\operatorname{cmd}_R M = \operatorname{amp} M$. Now apply 18.4.12. □

Finite Homology

18.4.15 Corollary. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$ of finite projective dimension. If M is not acyclic, then the following inequality holds*:

$$\operatorname{cmd}_R M \;\geqslant\; \operatorname{cmd} R\,.$$

Proof. The assumption $\mathrm{H}(M) \neq 0$ implies by 16.2.27 that the maximal ideal of R is in $\operatorname{supp}_R M$, so the inequality follows from 15.4.18 and 18.4.12. □

It is immediate from 18.4.15 that only a Cohen–Macaulay ring can accomodate a non-acyclic Cohen–Macaulay complex of finite projective dimension. A common specialization of this result and 18.4.14 is:

18.4.16 Corollary. *Let R be local. If there exists a non-zero R-module of finite length and finite projective dimension, then R is Cohen–Macaulay.*

Proof. A module of finite length is Cohen–Macaulay, see 17.2.6, so the assertion follows from 18.4.15. □

Every Cohen–Macaulay local ring admits a module of finite length and finite projective dimension.

18.4.17 Example. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and $\boldsymbol{x}$ a maximal R-regular sequence. As $\boldsymbol{x}$, in particular, is a parameter sequence, $\operatorname{Supp}_R R/(\boldsymbol{x}) = \{\mathfrak{m}\}$ holds, so $R/(\boldsymbol{x})$ has finite length, see 14.2.9, and $\operatorname{pd}_R R/(\boldsymbol{x})$ is finite by 16.4.23.

Remark. See E 16.3.1 for a direct proof of 18.4.16 that highlights the importance of existence of modules of maximal depth.

By 14.2.4 the inequality below is trivial unless $\mathrm{H}(M)$ is bounded below; it compares to the Auslander–Buchsbaum Formula 16.4.2.

18.4.18 Corollary. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$. If M is not acyclic, then the following inequality holds*:

$$\operatorname{pd}_R M \;\geqslant\; \dim R - \dim_R M \,.$$

Proof. The assumption $\mathrm{H}(M) \neq 0$ implies by 16.2.27 that the maximal ideal of R is in $\operatorname{supp}_R M$, so the inequality follows from 15.4.18 and 18.4.13. □

The general version of Krull's principal ideal theorem—some call it Krull's height theorem—is a special case of 18.4.18. The geometric interpretation of Krull's principal ideal theorem is that intersecting an irreducible variety by a hypersurface drops the dimension by at most 1. This is part of the explanation why results like 18.4.18 that compare projective dimension to Krull dimension are broadly referred to as "Intersection Theorems." More on this topic in 18.5.13.

18.4.19 Corollary. *Let $x_1, \dots, x_n$ be a sequence in R and $\mathfrak{p}$ a prime ideal in the set* $\operatorname{Min}_R R/(x_1, \dots, x_n)$*. The inequality* $\dim R_{\mathfrak{p}} \leqslant n$ *holds.*

Proof. Let $\boldsymbol{x}$ denote the sequence $\frac{x_1}{1}, \dots, \frac{x_n}{1}$ in $R_{\mathfrak{p}}$ and set $K = \mathrm{K}^{R_{\mathfrak{p}}}(\boldsymbol{x})$. The inequality $\operatorname{pd}_{R_{\mathfrak{p}}} K \leqslant n$ holds by 11.4.3(c), and 14.2.3 yields $\dim_{R_{\mathfrak{p}}} K = \dim R_{\mathfrak{p}}/(\boldsymbol{x}) = 0$. Now the asserted inequality follows from 18.4.18. □

The New Intersection Theorem

The special case of the next theorem where the homology complex $\mathrm{H}(F)$ is degreewise of finite length is the statement commonly known as the New Intersection

Theorem. It was proved in positive equicharateristic by Peskine and Szpiro [203] and in full generality by Roberts [215, 216]. The theorem stated here was proved, in the equicharacteristic case, in [92]. The history of this result and its extended family is briefly outlined after 18.5.14.

18.4.20 Theorem. *Let R be local and*

$$F \;=\; 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

a complex of finitely generated free R-modules. If $\dim_R \mathrm{H}_v(F) \leqslant v$ holds for all $v \in \mathbb{Z}$ and F is not acyclic, then the inequality $n \geqslant \dim R$ holds.

Proof. The complex F is semi-projective by 5.2.8, so $n \geqslant \mathrm{pd}_R F$ holds by 8.1.2. As $\dim_R \mathrm{H}_v(F) \leqslant v$ holds for all v on has $\dim_R F \leqslant 0$ by 14.2.1, so the asserted inequality follows from 18.4.18. □

18.4.21 Example. Let R be local and $M \neq 0$ be an R-module of finite length. The New Intersection Theorem 18.4.20 yields $\mathrm{pd}_R M \geqslant \dim R$. Combined with the Auslander–Buchsbaum Formula 16.4.2 this recovers 18.4.16.

Remark. Theorem 18.4.20 remains valid if F is a complex of flat modules with $\mathfrak{m}$ in $\mathrm{supp}_R F$; in the proof one just replaces the reference to 18.4.18 with a reference to 18.4.13. See also E 18.4.6.

18.4.22 Lemma. *Let R be local and*

$$F \;=\; 0 \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

an R-complex. If F is acyclic and the modules F_i for $i \in \{0,\dots,n\}$ are finitely generated free R-modules, then F_{n+1} is a free R-module of rank $\sum_{i=0}^{n}(-1)^{n-i}\,\mathrm{rank}_R F_i$.

Proof. Let $\boldsymbol{k}$ be the residue field of R. Recall from 1.3.18 and 16.4.22(b) that a finitely generated R-module is free if and only if it is projective; for such a module, F, one has $\mathrm{rank}_R F = \mathrm{rank}_{\boldsymbol{k}}(\boldsymbol{k} \otimes_R F)$. Proceed by induction on n. For $n = 0$ the assertion is trivial. For $n = 1$ the exact sequence $0 \to F_2 \to F_1 \to F_0 \to 0$ is split by 1.3.17. From the ensuing isomorphism, $F_1 \cong F_2 \oplus F_0$, it follows per 1.3.24 that F_2 is free. The induced isomorphism of $\boldsymbol{k}$-vector spaces yields the desired equality, $\mathrm{rank}_R F_2 = \mathrm{rank}_R F_1 - \mathrm{rank}_R F_0$. For $n > 1$ one gets two acyclic complexes,

$$(\flat) \qquad 0 \longrightarrow \mathrm{Z}_{n-1}(F) \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0 \quad \text{and}$$

$$(\diamond) \qquad 0 \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow \mathrm{Z}_{n-1}(F) \longrightarrow 0\,.$$

It follows from the induction hypothesis applied to $(\flat)$ that $\mathrm{Z}_{n-1}(F)$ is free of rank $\sum_{i=0}^{n-1}(-1)^{n-1-i}\,\mathrm{rank}_R F_i$. As in the case $n = 1$, it now follows from $(\diamond)$ that F_{n+1} is free with $\mathrm{rank}_R F_{n+1} = \mathrm{rank}_R F_n - \mathrm{rank}_R \mathrm{Z}_{n-1}(F) = \sum_{i=0}^{n}(-1)^{n-i}\,\mathrm{rank}_R F_i$. □

Roberts' original proof [214] of the next theorem does not rely on the New Intersection Theorem; the argument given here comes from [92].

18.4.23 Theorem. *Let R be local. If $\mu_R^{\dim R}(R) = 1$ holds, then R is Gorenstein.*

Proof. It follows from 18.3.12, 18.3.13, and 18.3.10 that one can assume that R is complete. Set $d = \dim R$ and $p = \operatorname{depth} R$. Per 18.2.29 let D be a normalized dualizing complex for R and recall from 18.2.24 that $\mathrm{H}_v(D) = 0$ holds for $v > d$ and for $v < p$. Let $F \xrightarrow{\simeq} D$ be a minimal semi-free resolution; in particular, one has $F_v = 0$ for $v < p$, see 16.4.22(c). As D is a Cohen–Macaulay complex, see 18.2.11, one has $\dim_R F = \dim_R D = 0$ by 18.2.24. In particular, it follows from 14.2.1 that $\dim_R \mathrm{H}_v(F) \leqslant v$ holds for all $v \in \mathbb{Z}$, and 18.2.16 yields $\dim_R \mathrm{H}_d(F) = d$.

For $n \geqslant p$ consider the complex,

$$F^{(n)} = 0 \longrightarrow \mathrm{Z}_n(F) \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_p \longrightarrow 0\,.$$

For prime ideals $\mathfrak{p}$ with $\dim R/\mathfrak{p} \geqslant n$ the complex $(F^{(n)})_\mathfrak{p}$ is acyclic: Indeed, one has $\mathrm{H}_{n+1}(F^{(n)}) = 0 = \mathrm{H}_n(F^{(n)})$ by construction, and for $v < n$ the module $\mathrm{H}_v(F^{(n)}) = \mathrm{H}_v(F)$ has Krull dimension less than n, whence one has $\mathrm{H}_v(F_\mathfrak{p}) = \mathrm{H}_v(F)_\mathfrak{p} = 0$. By 18.2.33 and 16.4.25 one has $\operatorname{rank}_R F_v = \mu_R^v(R)$, so it follows from 18.4.22 that $\mathrm{Z}_n(F)_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank $r_n = \sum_{v=p}^{n}(-1)^{n-v}\mu_R^v(R)$.

We proceed to argue that r_n is positive for $n \in \{d,\dots,p\}$. Notice, towards a contradiction, that $r_n = 0$ implies $\mathrm{Z}_n(F)_\mathfrak{p} = 0$ for all prime ideals with $\dim R/\mathfrak{p} \geqslant n$ and, therefore, $\dim_R \mathrm{Z}_n(F) < n$. For $n = d$ this is absurd as the quotient module $\mathrm{H}_d(F)$ has Krull dimension d. Now let $n < d$. One has $\mathrm{H}_n(F_{\leqslant n}) = \mathrm{Z}_n(F)$ and for $v < n$ one has $\mathrm{H}_v(F_{\leqslant n}) = \mathrm{H}_v(F)$, so the New Intersection Theorem 18.4.20 applies to the truncated complex $F_{\leqslant n}$, which has non-zero homology in degree p. As $n < d$ this yields a contradiction.

Finally, if $d > p$ holds, then one now has $\mu_R^d(R) > \mu_R^d(R) - r_{d-1} = r_d > 0$, so $\mu_R^d(R)$ is at least 2. This forces $d = p$, and then R is Gorenstein by 18.3.14. □

The Improved New Intersection Theorem

For an equicharacteristic local ring $(R,\mathfrak{m})$ the important special case $\mathfrak{a} = \mathfrak{m}$ of the next theorem, 18.4.25, was proved by Evans and Griffith [89]. As finitely generated $\mathfrak{m}$-torsion modules per 16.1.33 have finite length, 18.4.25 implies the classic version of the New Intersection Theorem where $\mathrm{H}(F)$ is assumed to be degreewise of finite length, and it is, therefore, known as the Improved New Intersection Theorem.

The following lemma also appeared in [89].

18.4.24 Lemma. *Let $(R,\mathfrak{m})$ be local, $\mathfrak{a} \subseteq R$ an ideal, $M \neq 0$ a finitely generated R-module, and H an R-module with $\mathfrak{m}H \neq H$. If an element in $M \setminus \mathfrak{m}M$ is $\mathfrak{a}$-torsion, then one has $\Gamma_\mathfrak{a}(M \otimes_R H) \neq 0$.*

Proof. Let m be an element in $M \setminus \mathfrak{m}M$. The canonical map $\pi\colon R \twoheadrightarrow \boldsymbol{k}$ factors through M as the map $\varphi\colon R \longrightarrow M$ given by $r \mapsto rm$ followed by the projection onto the rank 1 subspace of $\boldsymbol{k} \otimes_R M$ generated by $1 \otimes m$. By assumption $\boldsymbol{k} \otimes_R H$ is non-zero. As the induced map $\pi \otimes_R H$ is surjective, it follows that $\varphi \otimes_R H$ is non-zero. If m is $\mathfrak{a}$-torsion, then the image of $\varphi \otimes_R H$ in $M \otimes_R H$ is $\mathfrak{a}$-torsion. □

18.4.25 Theorem. *Let $(R,\mathfrak{m})$ be local, $\mathfrak{a}$ and ideal in R, and*

$$F = 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

a complex of finitely generated free R-modules. If $\mathrm{H}_v(F)$ is $\mathfrak{a}$-torsion for all $v > 0$ and an element in $\mathrm{H}_0(F) \setminus \mathfrak{m}\,\mathrm{H}_0(F)$ is $\mathfrak{a}$-torsion, then $n \geqslant \dim R - \dim R/\mathfrak{a}$ holds.

Proof. Per 18.4.9 let H be a derived $\mathfrak{m}$-complete R-module of maximal depth. The complex F is semi-free by 5.1.3, and the assumption on $\mathrm{H}_0(F)$ implies that F is not acyclic. It now follows from 16.2.27 that $\mathfrak{m}$ belongs to $\operatorname{supp}_R F$ and $\operatorname{supp}_R H$ and hence to $\operatorname{supp}_R(F \otimes_R H)$ by the Support Formula 15.1.16; in particular $F \otimes_R H$ is not acyclic, see 15.1.15. Set $s = \sup(F \otimes_R H)$; as the complex $F \otimes_R H$ is concentrated in non-negative degrees, one has $s \geqslant 0$.

Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass}_R \mathrm{H}_s(F \otimes_R H)$; it follows from 17.1.8 that $\mathfrak{p}$ belongs to $\operatorname{supp}_R(F \otimes_R H)$ and hence by the Support Formula 15.1.16 to $\operatorname{supp}_R F$ and $\operatorname{supp}_R H$. The maximal ideal $\mathfrak{p}_\mathfrak{p}$ of the local ring $R_\mathfrak{p}$ is associated to $\mathrm{H}_s((F \otimes_R H)_\mathfrak{p})$, so one has $s = -\operatorname{depth}_{R_\mathfrak{p}}(F \otimes_R H)_\mathfrak{p} = -\operatorname{depth}_{R_\mathfrak{p}}(F_\mathfrak{p} \otimes_{R_\mathfrak{p}} H_\mathfrak{p})$ by 16.2.16(b) and 14.1.15, which explains the second equality in the computation below. The first equality holds by 16.4.3, and the inequality holds by 18.3.26, as H is a derived $\mathfrak{m}$-complete R-module of maximal depth.

$$(\diamond) \qquad \begin{aligned} \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p} &= \operatorname{depth}_{R_\mathfrak{p}} H_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}}(F_\mathfrak{p} \otimes_{R_\mathfrak{p}} H_\mathfrak{p}) \\ &= \operatorname{depth}_{R_\mathfrak{p}} H_\mathfrak{p} + s \\ &\geqslant \dim R - \dim R/\mathfrak{p} + s\,. \end{aligned}$$

First consider the case $s > 0$. It suffices to argue that $\mathfrak{p}$ contains $\mathfrak{a}$, as that yields the final inequality in the following chain,

$$n \geqslant \operatorname{pd}_R F \geqslant \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p} > \dim R - \dim R/\mathfrak{p} \geqslant \dim R - \dim R/\mathfrak{a}\,,$$

where the first inequality holds by 8.1.2, the second holds by 17.3.26, and the third follows from $(\diamond)$. To see that $\mathfrak{p}$ contains $\mathfrak{a}$, assume towards a contradiction that $\mathfrak{p}$ does not contain $\mathfrak{a}$. As $\mathfrak{p}$ is in $\operatorname{supp}_R F$ the complex $\mathrm{H}(F_\mathfrak{p})$ is non-zero by 15.1.9, but the modules $\mathrm{H}_v(F)$ for $v > 0$ are $\mathfrak{a}$-torsion, so one has $\mathrm{H}_v(F_\mathfrak{p}) = \mathrm{H}_v(F)_\mathfrak{p} = 0$ for $v > 0$. It follows that $F_\mathfrak{p}$ is isomorphic to $\mathrm{H}_0(F)_\mathfrak{p} \neq 0$ in $\mathcal{D}(R_\mathfrak{p})$, see 7.3.29. Combining this with the Auslander–Buchsbaum Formula 16.4.2 and 16.2.16 one gets,

$$(\star) \qquad \operatorname{depth} R_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} F_\mathfrak{p} + \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p} \geqslant -\sup F_\mathfrak{p} + \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p} = \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p}\,.$$

At the same time, $(\diamond)$ and 14.2.7 yield

$$(\dagger\dagger) \qquad \operatorname{pd}_{R_\mathfrak{p}} F_\mathfrak{p} > \dim R - \dim R/\mathfrak{p} \geqslant \dim R_\mathfrak{p}\,.$$

Combining $(\star)$ and $(\dagger\dagger)$ one gets $\operatorname{depth} R_\mathfrak{p} > \dim R_\mathfrak{p}$, which contradicts 17.2.16.

Now assume that $s = 0$ holds. By 7.6.8 one has $\mathrm{H}_0(F \otimes_R H) \cong \mathrm{H}_0(F) \otimes_R H$. Since an element in $\mathrm{H}_0(F) \setminus \mathfrak{m}\,\mathrm{H}_0(F)$ is $\mathfrak{a}$-torsion and $\mathfrak{m}H \neq H$ by 18.4.2 and 16.2.7, it follows from 18.4.24 that $\Gamma_\mathfrak{a}(\mathrm{H}_0(F \otimes_R H))$ is non-zero, so $\mathfrak{a}\text{-}\operatorname{depth}_R(F \otimes_R H) = 0$ holds by 14.3.16(b). This together with the fact that H is of maximal depth explains the final equality in the next computation, and the first equality holds by 16.4.3. To justify the inequality, notice that the complex $F \otimes_R H$ by 13.1.31(b) is derived $\mathfrak{m}$-complete and invoke 18.3.27, which applies as $\mathfrak{a}$ is a proper ideal in R.

$$\begin{aligned}\operatorname{pd}_R F &= \operatorname{depth}_R H - \operatorname{depth}_R(F \otimes_R H) \\ &\geqslant \operatorname{depth}_R H - \mathfrak{a}\text{-}\operatorname{depth}_R(F \otimes_R H) - \dim R/\mathfrak{a} \\ &= \dim R - \dim R/\mathfrak{a}\ .\end{aligned}$$

Finally, $n \geqslant \operatorname{pd}_R F$ holds by 8.1.2. □

Exercises

In the following exercises let $(R, \mathfrak{m}, \boldsymbol{k})$ be local.

E 18.4.1 Let M be a finitely generated R-module. Show that the following conditions are equivalent: (*i*) M is maximal Cohen–Macaulay. (*ii*) $\mu_R^n(M) = 0$ holds for all $n < \dim R$. (*iii*) $\mathrm{H}_{\mathfrak{m}}^n(M) = 0$ holds for $n \neq \dim R$.

E 18.4.2 Let R be Cohen–Macaulay and M a finitely generated R-module. Show that M is maximal Cohen–Macaulay if and only if $\operatorname{Ext}_R^m(M, N) = 0$ holds for all $m > 0$ and all finitely generated R-modules N of finite injective dimension.

E 18.4.3 Let M an R-module of maximal depth. Show that $\operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathrm{E}_R(\boldsymbol{k})), \mathrm{E}_R(\boldsymbol{k}))$ is an R-module and an $\widehat{R}$-module of maximal depth.

E 18.4.4 Let $F = 0 \to F_n \to \cdots \to F_0 \to 0$ a complex of finitely generated free R-modules. Show that if F is not acyclic and for some integer d the inequality $\dim_R \mathrm{H}_v(F) \leqslant v + d$ holds for all $v \in \mathbb{Z}$, then the inequality $n \geqslant \dim R - d$ holds.

E 18.4.5 Let $F = 0 \to F_n \to \cdots \to F_0 \to 0$ a complex of finitely generated free R-modules. Show that if $F_{\mathfrak{p}}$ is acyclic for every prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} \geqslant \dim R - n$, then F is acyclic.

E 18.4.6 Let $0 \to F_n \to \cdots \to F_0 \to 0$ be a complex of flat R-modules. Show that if $\mathrm{H}(F)$ is non-zero and $\mathfrak{m}$-torsion, then $n \geqslant \dim R$ holds.

18.5 Intersection Theorems and Iversen's Amplitude Inequality

Synopsis. Intersection theorems; amplitude nequalities; Cohen–Macaulay ring; Auslander's zero-divisor theorem; classic Intersection Theorem; Acyclicity Lemma; grade; codimension.

Results that compare Krull dimension to projective dimension get referred to as "intersection theorems" for reasons laid out in 18.5.13. How the New Intersection Theorem qualifies as an intersection theorem is explained in 18.4.21.

Intersection Theorems

The next inequality compares to 16.3.1.

18.5.1 Proposition. *Let $(R, \mathfrak{m})$ be local, M a complex in $\mathcal{D}_{\square}(R)$ of finite flat dimension, and N a complex in $\mathcal{D}^{\mathrm{f}}(R)$. If $\mathfrak{m} \in \operatorname{supp}_R M$ holds, then one has*

$$\dim_R(N \otimes_R^{\mathsf{L}} M) \geqslant \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\ .$$

Proof. Notice that the inequality is trivial if N is acyclic and assume henceforth that N is not acyclic. First, we consider the case of a cyclic R-module, i.e. $N \cong R/\mathfrak{a}$ for some proper ideal $\mathfrak{a}$ in R. Now, $R/\mathfrak{a} \otimes^{\mathsf{L}}_R M$ is an $R/\mathfrak{a}$-complex of finite flat dimension and finite depth, see 15.4.19, 16.2.27, the Support Formula 15.1.16, and 16.2.26. Now 18.4.12 conspires with 18.3.34 to yield

$$\operatorname{cmd}_R(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) \;=\; \operatorname{cmd}_{R/\mathfrak{a}}(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) \;\geqslant\; \operatorname{cmd} R/\mathfrak{a} \;=\; \operatorname{cmd}_R R/\mathfrak{a}\;.$$

This inequality can be rewritten as follows:

$$(\star)\qquad \begin{aligned} \dim_R(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M) \;&\geqslant\; \dim_R R/\mathfrak{a} - \operatorname{depth}_R R/\mathfrak{a} + \operatorname{depth}_R(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)\\ &=\; \dim_R R/\mathfrak{a} + \operatorname{depth}_R M - \operatorname{depth} R\;, \end{aligned}$$

where the equality follows from 16.3.1(a).

Assume now that N is a finitely generated R-module and set $\mathfrak{a} = (0 :_R N)$. In the next computation, the inequality comes from $(\star)$ while the equalities hold by 17.6.18 and 14.1.1.

$$(\diamond)\qquad \begin{aligned} \dim_R(N \otimes^{\mathsf{L}}_R M) \;&=\; \dim_R(R/\mathfrak{a} \otimes^{\mathsf{L}}_R M)\\ &\geqslant\; \dim_R R/\mathfrak{a} + \operatorname{depth}_R M - \operatorname{depth} R\\ &=\; \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\;. \end{aligned}$$

Next assume that N belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. From 17.6.19, $(\diamond)$, and 14.2.1 one gets

$$\begin{aligned} \dim_R(N \otimes^{\mathsf{L}}_R M) \;&=\; \sup\{\dim_R(\mathrm{H}_v(N) \otimes^{\mathsf{L}}_R M) - v \mid v \in \mathbb{Z}\}\\ &\geqslant\; \sup\{\dim_R \mathrm{H}_v(N) - v \mid v \in \mathbb{Z}\} + \operatorname{depth}_R M - \operatorname{depth} R\\ &=\; \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\;. \end{aligned}$$

It remains to deal with the case $\inf N = -\infty$. By 16.2.27, 16.2.5, and 16.2.9 one has

$$-\infty \;=\; \inf N + \operatorname{width}_R M \;=\; \operatorname{width}_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \inf\,(N \otimes^{\mathsf{L}}_R M)\;.$$

Now 14.2.4 yields $\dim_R(N \otimes^{\mathsf{L}}_R M) = \infty$, so the asserted inequality is trivial. $\square$

18.5.2 Theorem. *Let $(R,\mathfrak{m})$ be local, M a complex in $\mathcal{D}_{\Box}(R)$ of finite flat dimension, and N a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. If one has $\mathfrak{m} \in \operatorname{supp}_R M$, then the next inequality holds.*

$$\operatorname{cmd}_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \operatorname{cmd}_R N\;.$$

Proof. The asserted inequality is trivial if N is acyclic. Assuming that N is not acyclic, $\mathfrak{m}$ belongs to $\operatorname{supp}_R(N \otimes^{\mathsf{L}}_R M)$ by 16.2.27 and the Support Formula 15.1.16. Thus, $\operatorname{cmd}_R(N \otimes^{\mathsf{L}}_R M)$ is defined, see 18.3.33, and the asserted inequality follows from 18.5.1 and 16.3.1(a), as $\operatorname{depth}_R N > -\infty$ holds by 16.2.16. $\square$

The next result compares to 16.3.4.

18.5.3 Theorem. *Let $(R,\mathfrak{m})$ be local, M a complex in $\mathcal{D}_{\sqsupset}(R)$, and N a complex in $\mathcal{D}^{\mathrm{f}}(R)$. If one has $\mathfrak{m} \in \operatorname{supp}_R M$ and N is not acyclic, then the next inequality holds.*

$$\operatorname{fd}_R M + \dim_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \dim_R N\;.$$

Proof. The inequality is trivial if $\operatorname{fd}_R M = \infty$ holds, so assume that is not the case. Now M belongs to $\mathcal{D}_{\sqsubset\!\sqsupset}(R)$, and the inequality follows from 16.3.4, applied with $N = R$, and 18.5.1. $\square$

Finite Homology

18.5.4 Theorem. *Let R be local and M and N be complexes in $\mathcal{D}^{\mathrm{f}}(R)$. If M is not acyclic and has finite projective dimension, then the next inequality holds.*

$$\operatorname{cmd}_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \operatorname{cmd}_R N\,.$$

Proof. The inequality is trivial if N is acyclic, so assume that is not the case. It now follows from 16.2.27 and the Support Formula 15.1.16 that the complexes N and $N \otimes^{\mathsf{L}}_R M$ have finite depth; in particular $\operatorname{cmd}_R(N \otimes^{\mathsf{L}}_R M)$ is defined, see 18.3.33. From 16.2.5 and 16.2.9 one gets,

$$\inf N + \inf M \;=\; \operatorname{width}_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \inf\,(N \otimes^{\mathsf{L}}_R M)\,.$$

Per 14.2.4 one, therefore, has $\dim_R(N \otimes^{\mathsf{L}}_R M) \geqslant -\inf N - \inf M$, so the asserted inequality holds trivially if $\inf M = -\infty$. The assumption that $\operatorname{pd}_R M$ is finite already means that $\mathrm{H}(M)$ is bounded above, see 8.1.3, so one can assume that M belongs to $\mathcal{D}^{\mathrm{f}}_{\sqsubset\!\sqsupset}(R)$. Next, recall from 16.2.21 that $\sup N = \infty$ implies $\operatorname{depth}_R N = -\infty$, and then $\operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M) = -\infty$ holds by 16.3.1(b). Thus, the asserted inequality holds trivially if $\sup N = \infty$. Now one can assume that N is in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. Since M is not acyclic it follows from 16.2.27 that the maximal ideal of R is in $\operatorname{supp}_R M$, so the asserted inequality follows from 15.4.18 and 18.5.2. $\square$

The inequality below compares to the equality in 16.4.3. For M in $\mathcal{D}^{\mathrm{f}}_{\sqsubset\!\sqsupset}(R)$ it is a special case of 18.5.3.

18.5.5 Corollary. *Let R be local and M and N be complexes in $\mathcal{D}^{\mathrm{f}}_{\sqsubset}(R)$. If M and N are not acyclic, then the next inequality holds.*

$$\operatorname{pd}_R M + \dim_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \dim_R N\,.$$

Proof. The inequality is trivial if $\operatorname{pd}_R M = \infty$ holds, so assume that is not the case. It follows from 16.2.27 and 16.2.21 that $\operatorname{depth}_R M$ and $\operatorname{depth}_R N$ are integers, so via 16.3.1(a) one can rewrite the inequality from 18.5.4 as

$$\dim_R(N \otimes^{\mathsf{L}}_R M) \;\geqslant\; \operatorname{cmd}_R N + \operatorname{depth}_R(N \otimes^{\mathsf{L}}_R M) \;=\; \dim_R N + \operatorname{depth}_R M - \operatorname{depth} R\,.$$

The Auslander–Buchsbaum Formula 16.4.2 now yields the asserted inequality. $\square$

18.5.6 Corollary. *Let R be local and M and N be finitely generated R-modules. If M and N are non-zero, then the next inequality holds*

$$\operatorname{pd}_R M + \dim_R(N \otimes_R M) \;\geqslant\; \dim_R N\,.$$

Proof. Per 17.6.20 the asserted inequality follows immediately from 18.5.5. $\square$

Amplitude Inequalities

18.5.7 Theorem. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}(R)$ of finite injective dimension. If M is not acyclic, then the following inequality holds.*

$$\operatorname{amp} M \geqslant \operatorname{cmd} R .$$

Proof. The inequality is trivial in the case $\sup M = \infty$ holds, so one can per 8.2.3 assume that M belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(R)$. It follows from 18.3.2 and 18.3.10 that $\widehat{R} \otimes_R M$ is a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(\widehat{R})$ of finite injective dimension. Per 18.2.29, let D be a normalized dualizing complex for $\widehat{R}$. By Grothendieck Duality 18.2.3 the complex $\mathsf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, D)$ belongs to $\mathcal{D}^{\mathrm{f}}_{\square}(\widehat{R})$, and it has finite projective dimension and is not acyclic. Now 18.4.15 applies and conspires with 18.2.31 and 18.3.12 to yield

$$\operatorname{amp} M = \operatorname{amp}(\widehat{R} \otimes_R M) = \operatorname{cmd}_{\widehat{R}} \mathsf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, D) \geqslant \operatorname{cmd} \widehat{R} = \operatorname{cmd} R . \qquad \square$$

In the seminal paper [32] Bass found it “concievable” that the next result would hold. It was first proved, in positive equicharacteristic, by Peskine and Szpiro [202] who also showed that existence of a cyclic module of finite injective dimension warrants the stronger conclusion that the ring is Gorenstein. This is left as an exercise since a stronger result is proved in 19.5.8.

18.5.8 Corollary. *Let R be local. If there exists a non-zero finitely generated R-module of finite injective dimension, then R is Cohen–Macaulay.*

Proof. Let $M \neq 0$ be a finitely generated R-module of finite injective dimension. By 18.5.7 one has $0 \geqslant \operatorname{cmd} R$, so R is per 17.2.16 Cohen–Macaulay. $\square$

Existence of a finitely generated module of finite injective dimension implies the existence of a finite length module of finite injective dimension.

18.5.9 Example. Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a Cohen–Macaulay local ring and $\boldsymbol{x}$ a maximal R-regular sequence. The module $\operatorname{Hom}_R(R/(\boldsymbol{x}), \mathrm{E}_R(\boldsymbol{k}))$ has finite length and finite injective dimension, see 18.4.17 and 18.1.6.

Under the extra assumption that the homology of the complex N is bounded, the next theorem was proved by Iversen [141], and it is often referred to as Iversen’s amplitude inequality. The boundedness condition was lifted by Foxby and Iyengar [98]. In the special case where the complex M is a Koszul complex, the (in)equalities are known from 14.3.5.

18.5.10 Theorem. *Let R be local, N a complex in $\mathcal{D}^{\mathrm{f}}(R)$, and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension. If M is not acyclic, then one has*

$$\sup (N \otimes^{\mathsf{L}}_R M) \geqslant \sup N + \inf M \quad \text{and} \quad \inf (N \otimes^{\mathsf{L}}_R M) = \inf N + \inf M ;$$

in particular, one has

$$\operatorname{amp}(N \otimes^{\mathsf{L}}_R M) \geqslant \operatorname{amp} N .$$

Proof. The (in)equalities are trivial if N is acyclic, so assume that is not the case. By 16.2.27 the complexes N and M are of finite width. The complex $N \otimes^{\mathsf{L}}_R M$ belongs by 15.4.3 to $\mathcal{D}^{\mathrm{f}}(R)$, so the equality of infima follows from 16.2.5(a) and 16.2.9.

The inequality of suprema is now equivalent to the inequality of amplitudes, so it suffices to prove the latter. From 18.3.2 and 12.3.30 one gets

$$\operatorname{amp}(N \otimes^{\mathsf{L}}_R M) = \operatorname{amp}((\widehat{R} \otimes_R N) \otimes^{\mathsf{L}}_{\widehat{R}} (\widehat{R} \otimes_R M)) \quad \text{and} \quad \operatorname{amp} N = \operatorname{amp}(\widehat{R} \otimes_R N)\,.$$

The complexes $\widehat{R} \otimes_R N$ and $\widehat{R} \otimes_R M$ belong to $\mathcal{D}^{\mathrm{f}}(\widehat{R})$ by 18.3.2(c), and 18.3.9 yields $\operatorname{pd}_{\widehat{R}}(\widehat{R} \otimes_R M) = \operatorname{pd}_R M < \infty$. Thus, one can assume that R is complete. Per 18.2.29 let D be a normalized dualizing complex for R. The following isomorphisms follow from 12.3.20, homomorphism evaluation 12.3.27(a), and commutativity 12.3.5.

$$\begin{aligned}
\mathsf{RHom}_R(N \otimes^{\mathsf{L}}_R M, D) &\simeq \mathsf{RHom}_R(\mathsf{RHom}_R(\mathsf{RHom}_R(M,R),N),D) \\
&\simeq \mathsf{RHom}_R(M,R) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(N,D) \\
&\simeq \mathsf{RHom}_R(N,D) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M,R)\,.
\end{aligned}$$

This isomorphism in $\mathcal{D}(R)$ explains the second equality in the next computation; the remaining equalities hold by 18.2.31(c). The inequality is from 18.5.4, as the complex $\mathsf{RHom}_R(M,R)$ per 12.3.20 is not acyclic and has finite projective dimension.

$$\begin{aligned}
\operatorname{amp}(N \otimes^{\mathsf{L}}_R M) &= \operatorname{cmd}_R \mathsf{RHom}_R(N \otimes^{\mathsf{L}}_R M, D) \\
&= \operatorname{cmd}_R(\mathsf{RHom}_R(N,D) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(M,R)) \\
&\geqslant \operatorname{cmd}_R \mathsf{RHom}_R(N,D) \\
&= \operatorname{amp} N\,.
\end{aligned}$$

□

Before it was proved—first in positive equicharacteristic by Peskine and Szpiro [202]—the next result was known as Auslander's zerodivisor conjecture.

18.5.11 Corollary. *Let $(R,\mathfrak{m})$ be local, $\boldsymbol{x}$ a sequence in $\mathfrak{m}$, and M a finitely generated R-module. If M has finite projective dimension and $\boldsymbol{x}$ is M-regular, then $\boldsymbol{x}$ is R-regular.*

Proof. One has $\operatorname{amp}(\mathrm{K}^R(\boldsymbol{x}) \otimes_R M) = 0$ by 16.2.31. Iversen's amplitude inequality 18.5.10 now yields $\operatorname{amp} \mathrm{K}^R(\boldsymbol{x}) = 0$, which by 16.2.31 means that $\boldsymbol{x}$ is R-regular. □

The Classic Intersection Theorem of Peskine and Szpiro

Notice from 14.1.18 and 14.2.9 that the assumptions on M and N in the next result imply that $M \otimes_R N$ is a module of finite length.

18.5.12 Theorem. *Let $(R,\mathfrak{m})$ be local and M and N be finitely generated R-modules. If one has $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\mathfrak{m}\}$, and M has finite projective dimension, then the next inequality holds.*

$$\dim_R N \leqslant \operatorname{pd}_R M\,.$$

PROOF. In view of 14.1.18, the assumption $\operatorname{Supp}_R N \cap \operatorname{Supp}_R M = \{\mathfrak{m}\}$ implies $\dim_R(N \otimes_R M) = 0$, so the inequality is a special case of 18.5.6. □

18.5.13 An Intersection Conjecture. For subspaces U and W of a finite dimensional vector space V, the dimension of their intersection, $U \cap W$, is at least the sum of their dimensions minus the dimension of V. In particular, if the intersection is the zero subspace, i.e. a single point, then the dimensions of U and W add up to at most the dimension of V. The same is true for affine algebraic varieties, see the Remark after 20.1.26. This fact, proved by Serre [227, III.D.5], inspired the conjecture that for modules M and N as in the Intersection Theorem 18.5.12 one has $\dim_R N + \dim_R M \leqslant \dim R$ or, equivalently,

$$\dim_R N \leqslant \dim R - \dim_R M \,. \tag{18.5.13.1}$$

Notice that this inequality by 18.4.18 is stonger than the one in 18.5.12. A special case of the conjecture was proved by Serre, see 20.1.27.

While (18.5.13.1) is stronger than the inequality in 18.5.12, the conjecture that it holds for R, M, and N as in 18.5.12 has not acquired a lasting name to reflect that fact—possibly because there is an even stronger conjecture around, see 18.5.23.

The next example shows that the assumption of finite projective dimension is necessary in the conjecture discussed above.

18.5.14 Example. Let $\Bbbk$ be a field and set $R = \Bbbk[\![x, y]\!]/(xy)$. The R-complex,

$$\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} \cdots \,,$$

is evidently acyclic, and by 16.4.22(a) it is minimal. It follows from 8.1.16 that the cokernels $M = R/(x)$ and $N = R/(y)$ of the differential on the complex are modues of infinite projective dimension. Evidently, the intersection $\operatorname{Supp}_R N \cap \operatorname{Supp}_R M$ contains only the maximal ideal (x, y) of R, and the isomorphisms $M \cong \Bbbk[\![y]\!]$ and $N \cong \Bbbk[\![x]\!]$ yield $\dim_R M = 1 = \dim_R N$. However, one also has $\dim R = 1$.

18.5.15 More Conjectures. In a landmark paper [202] from 1973, Peskine and Szpiro considered the conjecture discussed in 18.5.13 above together with six other conjectures. They established connections among these conjectures and proved three of them in positive equicharacteristic; all three are now established facts in any characteristic, see 18.5.8, 18.5.11, and 18.5.12. One of the six conjectures, known as Auslander's rigidity conjecture, was disproved by Heitman [119] in 1993. The remaining two conjectures, as well as the one stated in 18.5.13, remain open and are discussed in 18.5.23 and 18.5.30.

REMARK. In [202] the Intersection Theorem 18.5.12 is proved first and in turn shown to imply 18.5.8 and 18.5.11; this is all in positive equicharacteristic. Work of Hochster [124, 125] established the Intersection Theorem in equicharacteristic zero. Peskine and Szpiro's paper [202] was submitted before Hochster proved the existence of big Cohen–Macaulay modules, so the original proof of 18.5.12 is very different from the one presented here: key ingredients are the Frobenius functor and the Acyclicity Lemma, see 18.5.16. The relevance of big Cohen–Macaulay modules to the Intersection Theorem was established in [124].

The New Intersection Theorem 18.4.20 was proved in positive equicharacteristic by Peskine and Szpiro [203] and, independently, by Roberts [213]. Hochster [125] proved that the New Intersection Theorem holds in the presence of big Cohen–Macaulay modules, which made it a theorem in equicharacteristic zero. Finally, Roberts [215, 216] proved the theorem in full generality in 1987. The New Intersection Theorem implies the Intersection Theorem 18.5.12, see for example [92], so 18.5.8, 18.5.11, and 18.5.12 have been known to hold for all local rings since 1987.

The quest for big Cohen–Macaulay modules led to the formulation of further conjectures–the Direct Summand Conjecture and the Monomial Conjecture—which would follow from the existence of big Cohen–Macaulay modules. Together with the conjectures from [202], the Monomial and Direct Summand Conjectures were referred to as the Homological Conjectures; their statements, though, are not homological in nature, so we do not treat them bere, but due to André's [4, 5] work they are now theorems.

The Acyclicity Lemma

The next result is a variation on the Acyclicity Lemma of Peskine and Szpiro [202]. It is an integral part of their proof of the Intersection Theorem 18.5.12 in positive equicharacteristic, which merits including it in this text, though it won't be referenced.

18.5.16 Lemma. *Let R be local and*

$$M = 0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

an R-complex. If the conditions

(1) $\operatorname{depth}_R M_v > v$ *and*
(2) $\operatorname{depth}_R \mathrm{H}_v(M) = 0$ *or* $\mathrm{H}_v(M) = 0$

are satisfied for every $v \in \{0, \dots, n\}$, then M is acyclic.

Proof. Notice first that if $n = 0$, then one has $M_0 = \mathrm{H}_0(M)$, so (1) and (2) yield $M_0 = 0$, i.e. M is the zero complex. Notice also that siince M_v has positive depth, 14.3.20 applied to the exact sequence $0 \to \mathrm{Z}_v(M) \to M_v \to \mathrm{B}_{v-1}(M) \to 0$ yields $\operatorname{depth}_R \mathrm{Z}_v(M) \geqslant 1$ for every v in $\{0, \dots, n\}$. Assume towards a contradiction that M is not acyclic and set $u = \sup M$. As $\mathrm{H}_n(M) = \mathrm{Z}_n(M)$ has positive depth, (2) yields $u < n$. Thus one has $\mathrm{B}_{n-1}(M) \cong M_n$ and, therefore, $\operatorname{depth}_R \mathrm{B}_{n-1}(M) > n$ by (1). Now, if $u < n-1$ holds, then the acyclic complex,

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_{u+1} \longrightarrow \mathrm{B}_u(M) \longrightarrow 0\,,$$

breaks into exact sequences $0 \to B_{v+1} \to M_{v+1} \to B_v \to 0$ for $n-1 \geqslant v \geqslant u$, and repeated applications of 14.3.20 yield $\operatorname{depth}_R \mathrm{B}_u(M) > u+1$. Now consider the exact sequence $0 \to \mathrm{B}_u(M) \to \mathrm{Z}_u(M) \to \mathrm{H}_u(M) \to 0$. From 14.3.20 one gets $\operatorname{depth}_R \mathrm{H}_u(M) \geqslant 1$ which per (2) means $\mathrm{H}_u(M) = 0$; a contradiction. □

Remark. Peskine and Szpiro's original Lemme d'Acyclicité from [202] deals with a slightly different notion of acyclicity that one can still come across in commutative algebra: An R-complex M concentrated in non-negative degrees is deemed to be 'acyclic' if $\mathrm{H}_v(M) = 0$ holds for $v > 0$, i.e. allowing for non-zero homology in degree 0. If M is a complex of free R-modules, then acyclicity in this sense simply means that the complex is a free resolution of $\mathrm{H}_0(M)$, and that is the typical use of this terminology. The original result is easily derived from 18.5.16; see E 18.5.9.

CODIMENSION AND GRADE

The next definition compares to the expression for Krull dimension in 14.2.6.

18.5.17 Definition. Let M be an R-complex. The *grade* of M, written $\operatorname{grade}_R M$, and the *codimension* of M, written $\operatorname{codim}_R M$, are defined as

$$\begin{aligned}\operatorname{grade}_R M &= \inf\{\operatorname{depth} R_\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\} \quad \text{and} \\ \operatorname{codim}_R M &= \inf\{\dim R_\mathfrak{p} + \inf M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M\}\,,\end{aligned}$$

with the convention $\inf \varnothing = \infty$.

18.5.18. Let M be an R-complex. From 2.5.5 one gets

$$\begin{aligned}\operatorname{grade}_R \Sigma^s M &= \operatorname{grade}_R M + s \quad \text{and} \\ \operatorname{codim}_R \Sigma^s M &= \operatorname{codim}_R M + s\end{aligned}$$

for every integer s. Moreover, if M is not acyclic then $\operatorname{grade}_R M$ and $\operatorname{codim}_R M$ are finite while $\operatorname{grade}_R M = \infty = \operatorname{codim}_R M$ holds if M is acyclic.

REMARK. We do not use this terminology, but for an ideal $\mathfrak{a}$ in R it is standard to refer to the invariant $\inf\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\}$ as the *height* of $\mathfrak{a}$. For a finitely generated R-module M one has $\operatorname{Supp}_R M = \mathrm{V}(0 :_R M)$ by 14.1.1, so the codimension of M as defined above is the height of the annihilator ideal $\mathfrak{a} = (0 :_R M)$. This is the classical definition of the codimension of a finitely generated module. By 17.6.3 one has $\operatorname{grade}_R M = \mathfrak{a}\text{-depth}\, R$, so the grade of M is by 14.4.25 the maximal length of an R-regular sequence in the annihilator ideal $\mathfrak{a} = (0 :_R M)$. This is the classical definition of the grade of a finitely generated R-module.

Bruns and Herzog [46, 9.1] work with a notion of codimension for bounded complexes of finitely generated free modules, which differs from the one defined above. Christensen and Iyengar [65] show that the codimension, in the sense of [46], of a bounded complex F of finitely generated free R-modules can be recast in terms of the Krull dimension of the complex $\operatorname{Hom}_R(F, R)$.

18.5.19 Proposition. *Let M be an R-complex. There are equalities,*

$$\begin{aligned}\operatorname{grade}_R M &= \inf\{\operatorname{grade}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \quad \textit{and} \\ \operatorname{codim}_R M &= \inf\{\operatorname{codim}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.\end{aligned}$$

PROOF. For prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$ in R one has $(R_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}} \cong R_\mathfrak{q}$ and $(M_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}} \cong M_\mathfrak{q}$, so in view of 14.1.11(b) the equalities hold by 18.5.17, as $\operatorname{grade}_{R_\mathfrak{p}} M_\mathfrak{p} = \infty = \operatorname{codim}_{R_\mathfrak{p}} M_\mathfrak{p}$ holds for all prime ideals $\mathfrak{p}$ in $\operatorname{Spec} R \setminus \operatorname{Supp}_R M$. □

18.5.20 Proposition. *Let M be an R-complex*; *the following inequalities hold*:

$$\inf M \leqslant \operatorname{grade}_R M \leqslant \operatorname{codim}_R M\,.$$

Moreover, if $\dim R$ *is finite, then one has*

$$\operatorname{codim}_R M \leqslant \dim R - \dim_R M\,.$$

PROOF. If M is acyclic, then $\operatorname{Supp}_R M$ is the empty set, see 14.2.4, so by convention all the quantities compared in the two displays ∞. Assume now that M is not acyclic. For every prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R M$, which is a nonempty set, one has

$$\inf M \leqslant \inf M_\mathfrak{p} \leqslant \operatorname{depth} R_\mathfrak{p} + \inf M_\mathfrak{p} \leqslant \dim R_\mathfrak{p} + \inf M_\mathfrak{p}$$

by 14.1.11(c) and 17.2.16. Per 18.5.17 this justifies the inequalities in the first display.

Assume that $\dim R$ is finite and note that for every prime ideal $\mathfrak{p}$ in R one has

$$\dim R_\mathfrak{p} + \inf M_\mathfrak{p} \leqslant \dim R - (\dim R/\mathfrak{p} - \inf M_\mathfrak{p}) .$$

The inequality in the second display now follows from 18.5.17 and 14.2.6. □

18.5.21 Theorem. *Let R be local and M an R-complex. The next inequalities hold.*

$$\operatorname{depth} R - \dim_R M \leqslant \operatorname{grade}_R M \leqslant \operatorname{codim}_R M \leqslant \dim R - \dim_R M .$$

Proof. The second and third inequalities hold by 18.5.20. For every prime ideal $\mathfrak{p}$ in R the next inequality holds by 16.4.6,

$$\operatorname{depth} R - (\dim R/\mathfrak{p} - \inf M_\mathfrak{p}) \leqslant \operatorname{depth} R_\mathfrak{p} + \inf M_\mathfrak{p} .$$

In view of 14.2.6 and 18.5.17 this explains the first inequality. □

Remark. In texts dealing with modules over a Cohen–Macaulay local ring R it is not uncommon to see the codimension and/or grade of an R-module M defined as the difference $\dim R - \dim_R M$; per 18.5.21 and 17.2.16 this is still the same invariant.

For a complex M in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ the lemma above, together with 12.3.20, yields $\operatorname{grade}_R M \leqslant \operatorname{pd}_R M$, but one can do better:

18.5.22 Proposition. *Let M be an R-complex. If M is not acyclic, then one has*

$$\operatorname{grade}_R M \leqslant \operatorname{codim}_R M \leqslant \operatorname{fd}_R M \leqslant \operatorname{pd}_R M .$$

Proof. The first and third inequalities are know to hold from 18.5.20 and 15.4.18, so it suffices to show that $\operatorname{codim}_R M \leqslant \operatorname{fd}_R M$ holds. Let $\mathfrak{p}$ be minimal in $\operatorname{Supp}_R M$; by 15.1.9 and 15.1.15 one has $\operatorname{Supp}_{R_\mathfrak{p}} M_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\} = \operatorname{supp}_{R_\mathfrak{p}} M_\mathfrak{p}$. Now 18.5.17, 14.2.6, and 18.4.13 combine to yield

$$\operatorname{codim}_{R_\mathfrak{p}} M_\mathfrak{p} = \dim R_\mathfrak{p} + \inf M_\mathfrak{p} = \dim R_\mathfrak{p} - \dim_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p} .$$

Finally, recall the inequalities $\operatorname{codim}_R M \leqslant \operatorname{codim}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\operatorname{fd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{fd}_R M$ from 18.5.19 and 17.3.2. □

18.5.23 The Strong Intersection Conjecture. Three of the conjectures considered by Peskine and Szpiro in [202] remain open, cf. 18.5.15. The strongest of the three asserts that for finitely generated modules M and N over a local ring $(R, \mathfrak{m})$ such that M has finite projective dimension and $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\mathfrak{m}\}$ holds—that is, for R, M, and N exactly as in the Intersection Theorem 18.5.12—one has

(18.5.23.1) $$\dim_R N \leqslant \operatorname{grade}_R M .$$

In view of 18.5.20, this inequality implies the inequality $\dim_R N \leqslant \dim R - \dim_R M$ from 18.5.13, which as discussed there is stronger than the inequality in the Intersection Theorem; the conjecture is hence known as the Strong Intersection Conjecture. It is known to hold in an important special case, see 20.1.27.

18.5.24 Proposition. *Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension. If M is not acyclic, then the next inequality holds*:

$$\operatorname{grade}_R M \leqslant \operatorname{codim}_R M \leqslant \operatorname{grade}_R M + \operatorname{amp} M .$$

PROOF. The left-hand inequality is known from 18.5.22. For every prime ideal $\mathfrak{p}$ in R, the (in)equalities below hold by 18.5.22, 17.3.26, the Auslander–Buchsbaum Formula 16.4.2, 16.2.16, and 14.1.11(c).

$$\begin{aligned}\operatorname{codim}_{R_\mathfrak{p}} M_\mathfrak{p} &\leqslant \operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p}\\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}\\ &\leqslant \operatorname{depth} R_\mathfrak{p} + \sup M_\mathfrak{p}\\ &= \operatorname{depth} R_\mathfrak{p} + \inf M_\mathfrak{p} + \operatorname{amp} M_\mathfrak{p}\\ &\leqslant \operatorname{depth} R_\mathfrak{p} + \inf M_\mathfrak{p} + \operatorname{amp} M .\end{aligned}$$

The right-hand inequality now follows from 18.5.17 and 18.5.19. □

18.5.25 Corollary. *Let M be a finitely generated R-module. If M has finite projective dimension, then* $\operatorname{grade}_R M = \operatorname{codim}_R M$ *holds.*

PROOF. For $M \neq 0$ the equality holds by 18.5.24; for $M = 0$ it holds by 18.5.18. □

18.5.26 Lemma. *Assume that R has finite Krull dimension and let M be an R-complex. If* $\dim R_\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ *holds for all* $\mathfrak{p} \in \operatorname{Supp}_R M$*, then one has*

$$\operatorname{codim}_R M = \dim R - \dim_R M .$$

PROOF. For every prime ideal $\mathfrak{p}$ in $\operatorname{Supp}_R M$, the assumption yields

$$\dim R_\mathfrak{p} + \inf M_\mathfrak{p} = \dim R - (\dim R/\mathfrak{p} - \inf M_\mathfrak{p}) .$$

Now the asserted equality follows from 18.5.17 and 14.2.6. □

18.5.27 Theorem. *Let R be local and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension. If R is equidimensional and catenary, then the next inequalities hold.*

$$\dim R - \dim_R M - \operatorname{amp} M \leqslant \operatorname{grade}_R M \leqslant \dim R - \dim_R M .$$

PROOF. The inequalities are trivial if M is acyclic, as the quantities compared are ∞; see 2.5.4, 14.2.4, and 18.5.17. Assume now that M is not acyclic. The right-hand inequality holds by 18.5.21. By 18.5.24 one has $\operatorname{codim}_R M - \operatorname{amp} M \leqslant \operatorname{grade}_R M$. As R is equidimensional and catenary, the equality $\dim R_\mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$, see 17.2.14, so the left-hand inequality holds by 18.5.26. □

18.5.28 Corollary. *Let R be local and M a finitely generated R-module of finite projective dimension. If R is equidimensional and catenary, then there is an equality,*

$$\operatorname{grade}_R M = \dim R - \dim_R M .$$

PROOF. For $M = 0$ the equality holds by convention; see 14.2.4 and 18.5.20. For $M \neq 0$ the equality is a special case of 18.5.27. □

Cohen–Macaulay local rings are equidimensional and catenary, see 17.2.21, but 18.5.28 has per 18.5.21 nothing new to say about their modules. This calls for an example of an equidimensional catenary local ring that is not Cohen–Macaulay.

18.5.29 Example. The local ring $\Bbbk[\![x,y]\!]$ is Cohen–Macaulay and hence catenary, see 17.2.18 and 17.2.21. The quotient ring $R = \Bbbk[\![x,y]\!]/(x^2,xy)$ has exactly two prime ideals: $(x) \subset (x,y)$, so it is trivially both catenary and equidimensional. The Krull dimension of R is evidently 1, but the depth is 0 by 16.2.18 as $(0 :_R x) = (x,y)$, so R is not Cohen–Macaulay.

18.5.30 The Codimension Conjecture. The Codimension Conjecture, which Peskine and Szpiro [202] ascribe to Auslander, says that for a finitely generated module M of finite projective dimension over a local ring $(R,\mathfrak{m})$ the equality

$$\operatorname{grade}_R M = \dim R - \dim_R M \tag{18.5.30.1}$$

holds without the assumption, imposed in 18.5.28, that R is equidimensional and catenary. It is the last conjecture from [202] to be discussed here, but as implied in 18.5.23 it is weaker than the Strong Intersection Conjecture. Indeed, assume that M is non-zero and let $\boldsymbol{x}$ be a sequence in $\mathfrak{m}$ whose image in the local ring $R/(0 :_R M)$ is a parameter sequence. Thus $\operatorname{Supp}_R M \cap \operatorname{Supp}_R R/(\boldsymbol{x}) = \mathrm{V}((\boldsymbol{x}) + (0 :_R M)) = \{\mathfrak{m}\}$ holds. The length of $\boldsymbol{x}$ is $\dim_R M$, see 14.1.1, so $\dim R - \dim_R M \leqslant \dim R/(\boldsymbol{x})$ holds. This inequality together with (18.5.23.1) yields $\dim R - \dim_R M \leqslant \operatorname{grade}_R M$, and the opposite inequality is known from 18.5.21.

Thus, the Strong Intersection Conjecture implies both the Codimension Conjecture and the conjecture discussed in 18.5.13. On the other hand, assuming these two conjectures, also the Stong Intersection Conjecture holds: Indeed, (18.5.13.1) and (18.5.30.1) string together to yield $\dim_R N \leqslant \dim R - \dim_R M = \operatorname{grade}_R M$.

Summing up, for finitely generated modules M and N over a local ring R such that M has finite projective dimension and $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \{\mathfrak{m}\}$ holds, there are by 18.5.21 and 18.4.18 inequalities,

$$\operatorname{grade}_R M \leqslant \dim R - \dim_R M \leqslant \operatorname{pd}_R M\ .$$

By the Intersection Theorem 18.5.12 the right-hand quantity, $\operatorname{pd}_R M$, dominates $\dim_R N$. The middle quantity is conjectured to dominate $\dim_R N$, see (18.5.13.1), and that conjecture has been verified in a special case, see 20.1.27. Even the left-hand quantity, $\operatorname{grade}_R M$, is conjectured to dominate $\dim_R N$, see (18.5.23.1); that is the Strong Intersection Conjecture. Assuming the Codimension Conjecture, which has been verified for equidimensional catenary rings, see 18.5.28, the Strong Intersection Conjecture, see 18.5.23, is equivalent to the conjecture discussed in 18.5.13.

From 18.4.18 one can derive another special case of the Codimension Conjecture.

18.5.31 Example. Let R be local and M a finitely generated R-module. If the equality $\operatorname{grade}_R M = \operatorname{pd}_R M$ holds, cf. 18.5.22, then 18.5.21 and 18.4.18 yield

$$\operatorname{grade}_R M = \dim R - \dim_R M\ .$$

Notice that the assumption on M implies that it has finite projective dimension.

An module that satisfies the assumption in 18.5.31 is called *perfect*.

Exercises

In the following exercises let $(R, \mathfrak{m}, k)$ be local.

E 18.5.1 Let N be a complex in $\mathcal{D}^{\mathrm{f}}(R)$ and M a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite injective dimension. Show that if M is not acyclic, then one has $\operatorname{amp} \mathsf{RHom}_R(N, M) \geqslant \operatorname{cmd}_R N$.

E 18.5.2 Let $M \neq 0$ be a finitely generated R-module of finite injective dimension. Show that $\mu_R^{\operatorname{depth} R}(R)$ divides $\beta_0^R(M)$ and conclude that R is Gorenstein if M is cyclic.

E 18.5.3 Let $(R, \mathfrak{m})$ be a local ring and $d \geqslant 0$ an integer. Show that R is Cohen–Macaulay of Krull dimension d if and only if $\operatorname{fd}_R \mathrm{H}^d_{\mathfrak{m}}(R) = d$ holds.

E 18.5.4 Let M be an R-complex. Prove the equalities

$$\operatorname{grade}_R M = \inf\{\operatorname{grade}_R \mathrm{H}_v(M) + v \mid v \in \mathbb{Z}\} \quad \text{and}$$
$$\operatorname{codim}_R M = \inf\{\operatorname{codim}_R \mathrm{H}_v(M) + v \mid v \in \mathbb{Z}\}\,.$$

Conclude that for a complex M in $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$ and $w = \inf M$ one has (a) $w = \operatorname{grade}_R M$ if and only if $\operatorname{grade}_R \mathrm{H}_w(M) = 0$ and (b) $w = \operatorname{codim}_R M$ if and only if $\operatorname{codim}_R \mathrm{H}_w(M) = 0$.

E 18.5.5 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$. Show that $\operatorname{grade}_R M = -\sup \mathsf{RHom}_R(M, R)$ holds.

E 18.5.6 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\sqsupset}(R)$ with $\mathrm{H}(M) \neq 0$. Set $g = \operatorname{grade}_R M$ and show that $\mathfrak{m}$ is associated to $\operatorname{Ext}^g_R(M, R)$ if and only if $g = \operatorname{depth} R + \inf M$ holds.

E 18.5.7 Let M be a complex in $\mathcal{D}^{\mathrm{f}}_{\square}(R)$ of finite projective dimension. Show that if M is not acyclic, then one has $\dim R + \inf M \leqslant \dim_R \mathsf{RHom}_R(M, R) \leqslant \dim R + \sup M$.

E 18.5.8 Let $M \neq 0$ be a finitely generated R-module of finite projective dimension. Show that one has $\dim R = \dim_R \mathsf{RHom}_R(M, R) = \sup\{\dim_R \operatorname{Ext}^m_R(M, R) + m \mid m \in \mathbb{Z}\}$. Conclude that with $p = \operatorname{pd}_R M$ one has

$$\dim R - \dim_R M \leqslant \operatorname{pd}_R M \leqslant \dim R - \dim_R \operatorname{Ext}^p_R(M, R)\,.$$

E 18.5.9 Let M be an R-complex concentrated in degrees $n, \ldots, 0$. Show that $\mathrm{H}_v(M) = 0$ holds for $v > 0$ if the next two conditions are satisfied. (1) $\operatorname{depth}_R M_v \geqslant v$ holds for $v > 0$ and (2) $\operatorname{depth}_R \mathrm{H}_v(M) = 0$ or $\mathrm{H}_v(M) = 0$ holds for $v > 0$.

Appendices

Appendix A
Acyclicity and Boundedness

SYNOPSIS. Acyclicity of Hom complex; acyclicity of tensor product complex; bounded functor.

We collect here some technical results about acyclicity of Hom and tensor product complexes (A.4–A.12), and about bounded functors on categories of complexes (A.13–A.22) and on derived categories (A.23–A.34). Loosely speaking, the common theme of these results is that they allow one to bootstrap statements about complexes from statements about modules.

ACYCLICITY OF HOM COMPLEXES

Condition (2) in the next lemma is trivially satisfied if M is bounded below.

A.1 Lemma. *Let M be an acyclic R-complex and N an R-module. The complex $\operatorname{Hom}_R(M,N)$ is acyclic if the following conditions are satisfied.*

(1) $\operatorname{Ext}_R^m(M_v,N)=0$ *for every* $v\in\mathbb{Z}$ *and all* $m>0$.
(2) $\operatorname{Ext}_R^m(\mathrm{C}_v(M),N)=0$ *for every* $v\ll 0$ *and all* $m>0$.

PROOF. As M is acyclic, the sequence $0\to\mathrm{C}_{v+1}(M)\to M_v\to\mathrm{C}_v(M)\to 0$ is exact for every $v\in\mathbb{Z}$, and by 7.3.35 and 7.3.27 it is enough to argue that one has $\operatorname{Ext}_R^m(\mathrm{C}_v(M),N)=0$ for every $v\in\mathbb{Z}$ and all $m>0$. In view of (2) one can procede by induction on v. Fix $v\in\mathbb{Z}$ and assume that $\operatorname{Ext}_R^m(\mathrm{C}_v(M),N)=0$ holds for all $m>0$. As $\operatorname{Ext}_R^m(M_v,N)=0$ holds for all $m>0$ by (1), another application of 7.3.35 yields $\operatorname{Ext}_R^m(\mathrm{C}_{v+1}(M),N)=0$ for all $m>0$. □

The prototypical application of the next result is to an acyclic complex M and a complex N of injective modules; see for example 5.3.12.

A.2 Proposition. *Let M and N be R-complexes such that M or N is bounded above. If the complex $\operatorname{Hom}_R(M,N_v)$ is acyclic for every $v\in\mathbb{Z}$, then $\operatorname{Hom}_R(M,N)$ is acyclic.*

Proof. The assumptions and conclusion are invariant under shift, see 2.2.15 and 2.3.14, so it suffices to show that $\mathrm{H}_0(\mathrm{Hom}_R(M,N)) = 0$ holds. That is, it suffices to show that every morphism $\alpha\colon M \to N$ is null-homotopic; see 2.3.10. Given a morphism α, the goal is to construct a degree 1 homomorphism $\sigma\colon M \to N$ with

$$(\star) \qquad \alpha_v \;=\; \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v$$

for every $v \in \mathbb{Z}$. As M or N is bounded above, $\alpha_v = 0$ holds for $v \gg 0$. Thus, one can choose $\sigma_v = 0$ for $v \gg 0$. Now, proceed by descending induction. Given that $(\star)$ holds for v, it follows that $\alpha_{v-1} - \partial^N_v\sigma_{v-1}$ is a cycle in the complex $\mathrm{Hom}_R(M, N_{v-1})$; indeed, one has

$$(\alpha_{v-1} - \partial^N_v\sigma_{v-1})\partial^M_v \;=\; \partial^N_v(\alpha_v - \sigma_{v-1}\partial^M_v) \;=\; \partial^N_v(\partial^N_{v+1}\sigma_v) \;=\; 0\,.$$

As the complex $\mathrm{Hom}_R(M, N_{v-1})$ is acyclic, $\alpha_{v-1} - \partial^N_v\sigma_{v-1}$ is a boundary. I.e. there exists an element σ_{v-2} in $\mathrm{Hom}_R(M_{v-2}, N_{v-1})$ with $\sigma_{v-2}\partial^M_{v-1} = \alpha_{v-1} - \partial^N_v\sigma_{v-1}$. □

Condition (2) in the next proposition is trivially satisfied if the complex N is bounded above; that special case is already covered by A.2.

A.3 Proposition. *Let M and N be R-complexes. The complex* $\mathrm{Hom}_R(M,N)$ *is acyclic if the following conditions are satisfied.*

(1) $\mathrm{Hom}_R(M, N_v)$ *is acyclic for every* $v \in \mathbb{Z}$.
(2) $\mathrm{Hom}_R(M, \mathrm{Z}_v(N))$ *is acyclic for every* $v \gg 0$.

Proof. Condition (1) implies by A.2 that the complex $\mathrm{Hom}_R(M, N_{\leqslant n})$ is acyclic for every $n \in \mathbb{Z}$. By 2.3.12 and 2.5.6, an application of $\mathrm{Hom}_R(M,-)$ to the degreewise split exact sequence $0 \to N_{\leqslant n} \to N \to N_{\geqslant n+1} \to 0$ shows that it suffices to prove that $\mathrm{Hom}_R(M, N_{\geqslant w})$ is acyclic for some $w \in \mathbb{Z}$. Thus, one can assume that N is bounded below and $\mathrm{Hom}_R(M, \mathrm{Z}_v(N))$ is acyclic for every $v \in \mathbb{Z}$.

The assumptions and conclusion are invariant under shift, see 2.2.15 and 2.3.14, so it suffices to show that $\mathrm{H}_0(\mathrm{Hom}_R(M,N)) = 0$ holds. That is, it suffices to show that every morphism $\alpha\colon M \to N$ is null-homotopic, see 2.3.10. Let a morphism α be given; the goal is to construct a degree 1 homomorphism $\sigma\colon M \to N$ such that $\alpha_v = \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v$ holds for all $v \in \mathbb{Z}$. As N is bounded below, one can take $\sigma_v = 0$ for $v \ll 0$ and proceed by induction. Fix n and assume that the desired homomorphisms σ_v have been constructed for $v \leqslant n-2$; assume further that a homomorphism $\varrho_{n-1}\colon M_{n-1} \to N_n$ with $\alpha_{n-1} = \partial^N_n\varrho_{n-1} + \sigma_{n-2}\partial^M_{n-1}$ has been constructed. This map may not have all the properties required of σ_{n-1}, but in the induction step it is modified to yield the desired σ_{n-1}. For $v \ll 0$ one takes $\varrho_v = 0$. The next diagram captures the data from the induction hypothesis.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & M_n \arrow[r] \arrow[d, "\alpha_n"] & M_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] \arrow[dl, "\varrho_{n-1}"'] & M_{n-2} \arrow[r] \arrow[d, "\alpha_{n-2}"] \arrow[dl, "\sigma_{n-2}"'] & M_{n-3} \arrow[r] \arrow[d, "\alpha_{n-3}"] \arrow[dl, "\sigma_{n-3}"'] & \cdots \\
\cdots \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r] & N_{n-1} \arrow[r] & N_{n-2} \arrow[r] & N_{n-3} \arrow[r] & \cdots
\end{tikzcd}$$

In the induction step one constructs homomorphisms

$$(\flat) \qquad \varrho_n : M_n \longrightarrow N_{n+1} \qquad \text{and} \qquad \vartheta_{n-1} : M_{n-1} \longrightarrow N_n$$

such that ϱ_n and $\sigma_{n-1} = \varrho_{n-1} + \vartheta_{n-1}$ satisfy

$$(\diamond) \qquad \alpha_n = \partial^N_{n+1}\varrho_n + \sigma_{n-1}\partial^M_n \qquad \text{and} \qquad \alpha_{n-1} = \partial^N_n\sigma_{n-1} + \sigma_{n-2}\partial^M_{n-1} .$$

The computation

$$\partial^N_n(\alpha_n - \varrho_{n-1}\partial^M_n) = (\alpha_{n-1} - \partial^N_n\varrho_{n-1})\partial^M_n = (\sigma_{n-2}\partial^M_{n-1})\partial^M_n = 0$$

shows that $\alpha_n - \varrho_{n-1}\partial^M_n$ is a homomorphism from M_n to $\mathrm{Z}_n(N)$. The diagram,

$$\begin{tikzcd}
M_{n+3} \arrow[r, "\partial^M_{n+3}"] \arrow[d] & M_{n+2} \arrow[r, "\partial^M_{n+2}"] \arrow[d, "\alpha_{n+1}\partial^M_{n+2}"] & M_{n+1} \arrow[r, "\partial^M_{n+1}"] \arrow[d, "\alpha_{n+1}"] & M_n \arrow[r, "\partial^M_n"] \arrow[d, "\alpha_n - \varrho_{n-1}\partial^M_n"] & M_{n-1} \arrow[d] \\
0 \arrow[r] & \mathrm{Z}_{n+1}(N) \arrow[r] & N_{n+1} \arrow[r, "\partial^N_{n+1}"] & \mathrm{Z}_n(N) \arrow[r] & 0 ,
\end{tikzcd}$$

is commutative. Denoting the bottom row by N', the vertical maps in the diagram form a morphism of complexes $\alpha' : M \to N'$. By the assumptions and A.2 the complex $\mathrm{Hom}_R(M, N')$ is acyclic, so α' is null-homotopic. In particular, there exist homomorphisms ϱ_n and ϑ_{n-1} as in $(\flat)$ with

$$\alpha_n - \varrho_{n-1}\partial^M_n = \partial^N_{n+1}\varrho_n + \vartheta_{n-1}\partial^M_n \qquad \text{and} \qquad \partial^N_n\vartheta_{n-1} = 0 .$$

It is now straightforward to verify that the identities in $(\diamond)$ hold. □

Remark. The proof of A.2 is a folklore argument. The proof of A.3 is a second distillation by Christensen and Thompson [69] of an argument used by Neeman [192] and first distilled by Emmanouil [83].

Condition (2) in the next lemma is trivially satisfied if N is bounded above.

A.4 Lemma. *Let M be an R-module and N an acyclic R-complex. The complex $\mathrm{Hom}_R(M, N)$ is acyclic if the following conditions are satisfied.*

(1) $\mathrm{Ext}^m_R(M, N_v) = 0$ *for every $v \in \mathbb{Z}$ and all $m > 0$.*
(2) $\mathrm{Ext}^m_R(M, \mathrm{Z}_v(N)) = 0$ *for every $v \gg 0$ and all $m > 0$.*

Proof. As N is acyclic, the sequence $0 \to \mathrm{Z}_v(N) \to N_v \to \mathrm{Z}_{v-1}(N) \to 0$ is exact for every $v \in \mathbb{Z}$, and by 7.3.35 and 7.3.27 it is enough to show that one has $\mathrm{Ext}^m_R(M, \mathrm{Z}_v(N)) = 0$ for every $v \in \mathbb{Z}$ and all $m > 0$. In view of (2) one can procede by descending induction on v. Fix $v \in \mathbb{Z}$ and assume that $\mathrm{Ext}^m_R(M, \mathrm{Z}_v(N)) = 0$ holds for all $m > 0$. As $\mathrm{Ext}^m_R(M, N_v) = 0$ holds for all $m > 0$ by (1), another application of 7.3.35 yields $\mathrm{Ext}^m_R(M, \mathrm{Z}_{v-1}(N)) = 0$ for all $m > 0$. □

A standard application of the next result is to an acyclic complex N and a complex M of projective R-modules; see for example 5.2.8.

A.5 Proposition. *Let M and N be R-complexes such that M or N is bounded below. If the complex $\mathrm{Hom}_R(M_v, N)$ is acyclic for every $v \in \mathbb{Z}$, then $\mathrm{Hom}_R(M, N)$ is acyclic.*

Proof. The assumptions and conclusion are invariant under shift, see 2.2.15 and 2.3.16, so it suffices to show that $\mathrm{H}_0(\operatorname{Hom}_R(M,N)) = 0$ holds. That is, it suffices to show that every morphism $\alpha\colon M \to N$ is null-homotopic, see 2.3.10. Given a morphism α, the goal is to construct a degree 1 homomorphism $\sigma\colon M \to N$ with

$$(\star) \qquad \alpha_v \;=\; \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v$$

for every $v \in \mathbb{Z}$. As M or N is bounded below, $\alpha_v = 0$ holds for $v \ll 0$. Thus, one can choose $\sigma_v = 0$ for $v \ll 0$. Now, proceed by induction. Given that $(\star)$ holds for v, it follows that $\alpha_{v+1} - \sigma_v\partial^M_{v+1}$ is a cycle in the complex $\operatorname{Hom}_R(M_{v+1},N)$; indeed, one has

$$\partial^N_{v+1}(\alpha_{v+1} - \sigma_v\partial^M_{v+1}) \;=\; (\alpha_v - \partial^N_{v+1}\sigma_v)\partial^M_{v+1} \;=\; (\sigma_{v-1}\partial^M_v)\partial^M_{v+1} \;=\; 0\,.$$

As the complex $\operatorname{Hom}_R(M_{v+1},N)$ is acyclic, $\alpha_{v+1} - \sigma_v\partial^M_{v+1}$ is a boundary. Thus, there exists an element σ_{v+1} in $\operatorname{Hom}_R(M_{v+1},N_{v+2})$ with $\partial^N_{v+2}\sigma_{v+1} = \alpha_{v+1} - \sigma_v\partial^M_{v+1}$. □

Condition (2) in the next proposition is trivially satisfied if the complex M is bounded below; that special case is already covered by A.5.

A.6 Proposition. *Let M and N be R-complexes. The complex* $\operatorname{Hom}_R(M,N)$ *is acyclic if the following conditions are satisfied.*

(1) $\operatorname{Hom}_R(M_v,N)$ *is acyclic for every* $v \in \mathbb{Z}$.
(2) $\operatorname{Hom}_R(\mathrm{C}_v(M),N)$ *is acyclic for every* $v \ll 0$.

Proof. Condition (1) implies by A.5 that the complex $\operatorname{Hom}_R(M_{\geqslant n},N)$ is acyclic for every $n \in \mathbb{Z}$. By 2.3.13 and 2.5.6, an application of $\operatorname{Hom}_R(-,N)$ to the degreewise split exact sequence $0 \to M_{\leqslant n-1} \to M \to M_{\geqslant n} \to 0$ shows that it suffices to prove that $\operatorname{Hom}_R(M_{\leqslant u},N)$ is acyclic for some $u \in \mathbb{Z}$. Thus, one can assume that M is bounded above and $\operatorname{Hom}_R(\mathrm{C}_v(M),N)$ is acyclic for all $v \in \mathbb{Z}$.

The assumptions and conclusion are invariant under shift, see 2.2.15 and 2.3.16, so it suffices to show that $\mathrm{H}_0(\operatorname{Hom}_R(M,N)) = 0$ holds. That is, it suffices to show that every morphism $\alpha\colon M \to N$ is null-homotopic, see 2.3.10. Let a morphism α be given; the goal is to construct a degree 1 homomorphism $\sigma\colon M \to N$ such that $\alpha_v = \partial^N_{v+1}\sigma_v + \sigma_{v-1}\partial^M_v$ holds for all $v \in \mathbb{Z}$. As $M_v = 0$ holds for $v \gg 0$, one can take $\sigma_v = 0$ for $v \gg 0$ and proceed by descending induction. Fix n and assume that the desired homomorphisms σ_v have been constructed for $v \geqslant n+1$; assume further that a homomorphism $\varrho_n\colon M_n \to N_{n+1}$ with $\alpha_{n+1} = \partial^N_{n+2}\sigma_{n+1} + \varrho_n\partial^M_{n+1}$ has been constructed. This map may not have all the properties required of σ_n, but in the induction step it is modified to yield the desired σ_n. For $v \gg 0$ one takes $\varrho_v = 0$. The next diagram captures the data from the induction hypothesis.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n+3} \arrow[r] \arrow[d, "\alpha_{n+3}"] \arrow[dl] & M_{n+2} \arrow[r] \arrow[d, "\alpha_{n+2}"] \arrow[dl, "\sigma_{n+2}"'] & M_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] \arrow[dl, "\sigma_{n+1}"'] & M_n \arrow[r] \arrow[d, "\alpha_n"] \arrow[dl, "\varrho_n"'] & M_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \\
\cdots \arrow[r] & N_{n+3} \arrow[r] & N_{n+2} \arrow[r] & N_{n+1} \arrow[r] & N_n \arrow[r] & N_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

In the induction step one constructs homomorphisms

$$(\star) \qquad \vartheta_n : M_n \longrightarrow N_{n+1} \qquad \text{and} \qquad \varrho_{n-1} : M_{n-1} \longrightarrow N_n$$

such that $\sigma_n = \varrho_n + \vartheta_n$ and ϱ_{n-1} satisfy

$$(\diamond) \qquad \alpha_{n+1} \;=\; \partial^N_{n+2}\sigma_{n+1} + \sigma_n\partial^M_{n+1} \quad \text{and} \quad \alpha_n \;=\; \partial^N_{n+1}\sigma_n + \varrho_{n-1}\partial^M_n \,.$$

The computation

$$(\alpha_n - \partial^N_{n+1}\varrho_n)\partial^M_{n+1} \;=\; \partial^N_{n+1}(\alpha_{n+1} - \varrho_n\partial^M_{n+1}) \;=\; \partial^N_{n+1}(\partial^N_{n+2}\sigma_{n+1}) \;=\; 0$$

shows that the homomorphism $\alpha_n - \partial^N_{n+1}\varrho_n \colon M_n \to N_n$ factors through the cokernel $\mathrm{C}_n(M)$. Similarly, $\partial^N_{n-1}\alpha_{n-1}$ factors through $\mathrm{C}_{n-1}(M)$ as one has $(\partial^N_{n-1}\alpha_{n-1})\partial^M_n = \partial^N_{n-1}(\partial^N_n\alpha_n) = 0$. That is, with π_v denoting the quotient map $M_v \twoheadrightarrow \mathrm{C}_v(M)$ there are morphisms

$$(\flat) \qquad \widetilde{\alpha}_n \,:\, \mathrm{C}_n(M) \longrightarrow N_n \qquad \text{with} \qquad \widetilde{\alpha}_n\pi_n \;=\; \alpha_n - \partial^N_{n+1}\varrho_n$$

and

$$(\dagger) \qquad \widetilde{\alpha}_{n-2} \,:\, \mathrm{C}_{n-1}(M) \longrightarrow N_{n-2} \quad \text{with} \quad \widetilde{\alpha}_{n-2}\pi_{n-1} \;=\; \partial^N_{n-1}\alpha_{n-1} \,.$$

Let $\widetilde{\partial} \colon \mathrm{C}_n(M) \to M_{n-1}$ be the map induced by the differential, i.e. $\widetilde{\partial}\pi_n = \partial^M_n$. As α is a morphism of complexes, ($\flat$) yields $(\partial^N_n\widetilde{\alpha}_n - \alpha_{n-1}\widetilde{\partial})\pi_n = 0$, and since π is surjective this together with ($\dagger$) shows that the following diagram is commutative,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{C}_n(M) \arrow[r, "\widetilde{\partial}"] \arrow[d, "\widetilde{\alpha}_n"] & M_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d, "\alpha_{n-1}"] & \mathrm{C}_{n-1}(M) \arrow[r] \arrow[d, "\widetilde{\alpha}_{n-2}"] & 0 \arrow[d] \\
N_{n+1} \arrow[r, "\partial^N_{n+1}"] & N_n \arrow[r, "\partial^N_n"] & N_{n-1} \arrow[r, "\partial^N_{n-1}"] & N_{n-2} \arrow[r, "\partial^N_{n-2}"] & N_{n-3} \,.
\end{tikzcd}$$

Denoting the top row by $\widetilde{M}$, the vertical maps in the diagram form a morphism of complexes $\widetilde{\alpha} \colon \widetilde{M} \to N$. By the assumptions and A.5 the complex $\operatorname{Hom}_R(\widetilde{M}, N)$ is acyclic, so $\widetilde{\alpha}$ is null-homotopic. In particular, there exist homomorphisms $\widetilde{\vartheta} \colon \mathrm{C}_n(M) \to N_{n+1}$ and $\varrho_{n-1} \colon M_{n-1} \to N_n$ with $\widetilde{\alpha}_n = \partial^N_{n+1}\widetilde{\vartheta} + \varrho_{n-1}\widetilde{\partial}$ and, thus,

$$\alpha_n - \partial^N_{n+1}\varrho_n \;=\; \partial^N_{n+1}\widetilde{\vartheta}\pi_n + \varrho_{n-1}\partial^M_n$$

by ($\flat$). With $\vartheta_n = \widetilde{\vartheta}\pi_n$ one has $\vartheta_n\partial^M_{n+1} = 0$, and it is now straightforward to verify that the identities in ($\diamond$) hold. □

Acyclicity of Tensor Product Complexes

Condition (2) in the next lemma is trivially satisfied if M is bounded below.

A.7 Lemma. *Let M be an acyclic R°-complex and N an R-module. The complex $M \otimes_R N$ is acyclic if the following conditions are satisfied.*

(1) $\operatorname{Tor}^R_m(M_v, N) = 0$ *for every $v \in \mathbb{Z}$ and all $m > 0$.*
(2) $\operatorname{Tor}^R_m(\mathrm{C}_v(M), N) = 0$ *for every $v \ll 0$ and all $m > 0$.*

Proof. As M is acyclic, the sequence $0 \to \mathrm{C}_{v+1}(M) \to M_v \to \mathrm{C}_v(M) \to 0$ is exact for every $v \in \mathbb{Z}$, and by 7.4.29 and 7.4.21 it is enough to show that one has $\mathrm{Tor}_1^R(\mathrm{C}_v(M), N) = 0$. Part (2) yields the base case for a proof by induction on v that $\mathrm{Tor}_m^R(\mathrm{C}_v(M), N) = 0$ holds for all $v \in \mathbb{Z}$ and all $m > 0$. Fix $v \in \mathbb{Z}$ and assume that $\mathrm{Tor}_m^R(\mathrm{C}_v(M), N) = 0$ holds for all $m > 0$. As $\mathrm{Tor}_m^R(M_v, N) = 0$ holds for all $m > 0$ by (1), another application of 7.4.29 yields $\mathrm{Tor}_m^R(\mathrm{C}_{v+1}(M), N) = 0$ for all $m > 0$. □

A.8 Lemma. *Let M be an R^o-module and N an acyclic R-complex. The complex $M \otimes_R N$ is acyclic if the following conditions are satisfied.*

(1) $\mathrm{Tor}_m^R(M, N_v) = 0$ *for every* $v \in \mathbb{Z}$ *and all* $m > 0$.
(2) $\mathrm{Tor}_m^R(M, \mathrm{C}_v(N)) = 0$ *for every* $v \ll 0$ *and all* $m > 0$.

Proof. By commutativity 4.4.4 there is an isomorphism $M \otimes_R N \cong N \otimes_{R^\mathrm{o}} M$, and the right-hand complex is acyclic by A.7 and 7.4.23. □

A standard application of the next result is to an acyclic complex M and a complex N of flat R-modules; see for example 5.4.8.

A.9 Proposition. *Let M be an R^o-complex and N an R-complex, such that M is bounded above or N is bounded below. If the complex $M \otimes_R N_v$ is acyclic for every $v \in \mathbb{Z}$, then $M \otimes_R N$ is acyclic.*

Proof. By adjunction 4.4.12 there is an isomorphism of $\Bbbk$-complexes

$$\mathrm{Hom}_\Bbbk(M \otimes_R N, \mathbb{E}) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_\Bbbk(M, \mathbb{E}))\,.$$

Recall from 2.5.7(b) that $\mathrm{Hom}_\Bbbk(M, \mathbb{E})$ is bounded below if M is bounded above. As $\mathbb{E}$ is a faithfully injective $\Bbbk$-module, the assumptions imply that the complex $\mathrm{Hom}_R(N_v, \mathrm{Hom}_\Bbbk(M, \mathbb{E}))$ is acyclic for every $v \in \mathbb{Z}$, so $\mathrm{Hom}_R(N, \mathrm{Hom}_\Bbbk(M, \mathbb{E}))$ is acyclic by A.5 and hence $M \otimes_R N$ is acyclic. □

A.10 Proposition. *Let M be an R^o-complex and N an R-complex, such that M is bounded below or N is bounded above. If the complex $M_v \otimes_R N$ is acyclic for every $v \in \mathbb{Z}$, then $M \otimes_R N$ is acyclic.*

Proof. By commutativity 4.4.4 there is an isomorphism $M \otimes_R N \cong N \otimes_{R^\mathrm{o}} M$, and the right-hand complex is acyclic by A.9. □

A.11 Proposition. *Let M be an R^o-complex and N an R-complex. The complex $M \otimes_R N$ is acyclic if the next conditions are satisfied.*

(1) $M \otimes_R N_v$ *is acyclic for every* $v \in \mathbb{Z}$.
(2) $M \otimes_R \mathrm{C}_v(N)$ *is acyclic for every* $v \ll 0$.

Proof. This follows from A.6 by way of adjunction as in the proof of A.9. □

A.12 Proposition. *Let M be an R^o-complex and N an R-complex. The complex $M \otimes_R N$ is acyclic if the next conditions are satisfied.*

(1) $M_v \otimes_R N$ *is acyclic for every* $v \in \mathbb{Z}$.
(2) $\mathrm{C}_v(M) \otimes_R N$ *is acyclic for every* $v \ll 0$.

PROOF. By commutativity 4.4.4 there is an isomorphism $M \otimes_R N \cong N \otimes_{R^\circ} M$, and the right-hand complex is acyclic by A.11. □

BOUNDED FUNCTORS

A bounded functor $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ is one where the shape of $\mathrm{F}(M)$ depends on the shape of M in a controlled manner.

A.13 Definition. Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor.

The functor F is called *bounded above* if for every $n \in \mathbb{Z}$ there is a $u \in \mathbb{Z}$ such that $\sup(\mathrm{F}(M))^\natural \leqslant n$ holds for every R-complex M with $\sup M^\natural \leqslant u$.

The functor F is called *bounded below* if for every $n \in \mathbb{Z}$ there is a $w \in \mathbb{Z}$ such that $\inf(\mathrm{F}(M))^\natural \geqslant n$ holds for every R-complex M with $\inf M^\natural \geqslant w$.

The functor F is called *bounded* if it is bounded below and bounded above.

REMARK. Other terms for bounded above, bounded below, and bounded are 'way-out right', 'way-out left', and 'way-out in both directions'; see Hartshorne [114, I.§7]. In [174] Lipman uses the boundedness terminology but with "above" and "below" interchanged as he uses cohomological grading. His definitions assume that there are uniform bounds on the differences $n - u$ and $w - n$ in the definition above, but for functors that commute with shift this is automatic; see E A.14.

A.14 Example. A functor $\mathcal{C}(R) \to \mathcal{C}(S)$ that is extended from an additive functor on R-modules, as described in 2.1.48, is per construction bounded.

A.15 Proposition. *Let M be an R-complex.*

(a) *If M is bounded below, then the functor* $\mathrm{Hom}_R(M,-)$ *is bounded above.*
(b) *If M is bounded above, then the functor* $\mathrm{Hom}_R(M,-)$ *is bounded below.*
(c) *If M is bounded, then* $\mathrm{Hom}_R(M,-)$ *is bounded.*

PROOF. Part (a) is immediate from 2.5.12(a), and part (b) follows from an argument similar to the proof of loc. cit. Part (c) follows from (a) and (b). □

A.16 Proposition. *Let M be an R-complex.*

(a) *If M is bounded below, then the functor $-\otimes_R M$ is bounded below.*
(b) *If M is bounded above, then the functor $-\otimes_R M$ is bounded above.*
(c) *If M is bounded, then $-\otimes_R M$ is bounded.*

PROOF. Part (a) is immediate from 2.5.18(a), and part (b) follows from an argument similar to the proof of loc. cit. Part (c) follows from (a) and (b). □

A class $\mathcal{U}$ of R-complexes that satisfies the assumptions in the next lemma is, for example, the class of all complexes of projective R-modules.

A.17 Lemma. *Let* $\mathrm{E},\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ *be $\natural$- and Σ-functors and $\tau\colon \mathrm{E} \to \mathrm{F}$ a Σ-transformation. Let $\mathcal{U}$ be a class of R-complexes that is closed under shifts and hard truncations. If τ^M is a quasi-isomorphism for every module M in $\mathcal{U}$, then the following assertions hold.*

(a) *The morphism τ^M is a quasi-isomorphism for every bounded complex M in $\mathcal{U}$.*
(b) *If* E *and* F *are bounded below, then τ^M is a quasi-isomorphism for every bounded below complex M in $\mathcal{U}$.*
(c) *If* E *and* F *are bounded above, then τ^M is a quasi-isomorphism for every bounded above complex M in $\mathcal{U}$.*
(d) *If* E *and* F *are bounded, then τ^M is a quasi-isomorphism for every M in $\mathcal{U}$.*

Proof. (a): After a shift, M is concentrated in degrees $u, u-1, \dots, 0$. If M is a module, i.e. $u = 0$, then τ^M is a quasi-isomorphism. Let $u > 0$ and assume that τ is a quasi-isomorphism when evaluated at a complex in $\mathcal{U}$ concentrated u degrees. Consider the degreewise split exact sequence $0 \to M_{\leqslant u-1} \to M \to \Sigma^u M_u \to 0$; cf. 2.5.22. All three complexes in the sequence belong to $\mathcal{U}$, and together with τ it induces by 2.1.54 an obvious commutative diagram in which the morphisms $\tau^{M_{\leqslant u-1}}$ and $\tau^{\Sigma^u M_u} \cong \Sigma^u \tau^M$ are quasi-isomorphisms by assumption. It follows from 4.2.5 that the third map, τ^M, is a quasi-isomorphism. Thus the assertion follows by induction.

(b): To prove that τ^M is a quasi-isomorphism we fix an integer n and show that $\mathrm{H}_n(\tau^M)$ is an isomorphism. Choose $w \in \mathbb{Z}$ such that one has $\inf(\mathrm{E}(X))^\natural \geqslant n+2$ and $\inf(\mathrm{F}(X))^\natural \geqslant n+2$ for all complexes X with $\inf X^\natural \geqslant w$. All three complexes in the degreewise split exact sequence $0 \to M_{\leqslant w-1} \to M \to M_{\geqslant w} \to 0$ from 2.5.22 belong to $\mathcal{U}$, and it induces by 2.1.54 and 2.2.21 a commutative diagram,

$$\begin{tikzcd}
\mathrm{H}_{n+1}(\mathrm{E}(M_{\geqslant w})) \arrow[r] & \mathrm{H}_n(\mathrm{E}(M_{\leqslant w-1})) \arrow[r, "\cong"'] \arrow[d, "\cong"', "\mathrm{H}_n(\tau^{M_{\leqslant w-1}})"] & \mathrm{H}_n(\mathrm{E}(M)) \arrow[r] \arrow[d, "\mathrm{H}_n(\tau^M)"] & \mathrm{H}_n(\mathrm{E}(M_{\geqslant w})) \\
\mathrm{H}_{n+1}(\mathrm{F}(M_{\geqslant w})) \arrow[r] & \mathrm{H}_n(\mathrm{F}(M_{\leqslant w-1})) \arrow[r, "\cong"'] & \mathrm{H}_n(\mathrm{F}(M)) \arrow[r] & \mathrm{H}_n(\mathrm{F}(M_{\geqslant w}))
\end{tikzcd}$$

where the horizontal isomorphisms are forced by the vanishing of $\mathrm{E}(M_{\geqslant w})_v$ and $\mathrm{F}(M_{\geqslant w})_v$ for $v \leqslant n+1$. As $M_{\leqslant w-1}$ is a bounded complex in $\mathcal{U}$, it follows from part (a) that $\tau^{M_{\leqslant w-1}}$ is a quasi-isomorphism, which explains the vertical isomorphism in the diagram. By commutativity of the diagram, $\mathrm{H}_n(\tau^M)$ is an isomorphism.

(c): An argument similar to the proof of part (b) applies.

(d): The degreewise split exact sequence $0 \to M_{\leqslant 0} \to M \to M_{\geqslant 1} \to 0$ in $\mathcal{U}$ and the natural transformation τ induce by 2.1.54 an obvious commutative diagram. It follows from parts (b) and (c) that the morphisms $\tau^{M_{\geqslant 1}}$ and $\tau^{M_{\leqslant 0}}$ are quasi-isomorphisms, and 4.2.5 then implies that τ^M is a quasi-isomorphism. □

A.18 Definition. Let $\mathrm{G}\colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ be a functor.

The functor G is called *bounded above* if for every $n \in \mathbb{Z}$ there is a $w \in \mathbb{Z}$ such that $\sup(\mathrm{G}(M))^\natural \leqslant n$ holds for every R-complex M with $\inf M^\natural \geqslant w$.

The functor G is called *bounded below* if for every $n \in \mathbb{Z}$ there is a $u \in \mathbb{Z}$ such that $\inf(\mathrm{G}(M))^\natural \geqslant n$ holds for every R-complex M with $\sup M^\natural \leqslant u$.

The functor G is called *bounded* if it is bounded below and bounded above.

A.19 Example. A functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ that is extended from an additive functor on R-modules, as described in 2.1.48, is per construction bounded.

A.20 Proposition. *Let N be an R-complex.*

(a) *If N is bounded above, then* $\operatorname{Hom}_R(-,N)$ *is bounded above.*
(b) *If N is bounded below, then* $\operatorname{Hom}_R(-,N)$ *is bounded below.*
(c) *If N is bounded, then* $\operatorname{Hom}_R(-,N)$ *is bounded.*

Proof. Part (a) is immediate from 2.5.12(a), and part (b) follows from an argument similar to the proof of loc. cit. Part (c) follows from (a) and (b). □

A.21. A functor $\mathcal{C}(R) \to \mathcal{C}(S)^{\mathrm{op}}$ is called bounded above/below if and only if the opposite functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is bounded below/above according to A.18. For such functors A.20 holds with "bounded below" and "bounded above" interchanged.

A.22 Lemma. *Let* $\mathrm{G}, \mathrm{J} \colon \mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ *be* $\natural$- *and* Σ-*functors and* $\tau \colon \mathrm{G} \to \mathrm{J}$ *a* Σ-*transformation. Let* $\mathcal{U}$ *be a class of R-complexes that is closed under shifts and hard truncations. If* τ^M *is a quasi-isomorphism for every module M in* $\mathcal{U}$, *then the following assertions hold.*

(a) *The morphism* τ^M *is a quasi-isomorphism for every bounded complex M in* $\mathcal{U}$.
(b) *If* G *and* J *are bounded below, then* τ^M *is a quasi-isomorphism for every bounded above complex M in* $\mathcal{U}$.
(c) *If* G *and* J *are bounded above, then* τ^M *is a quasi-isomorphism for every bounded below complex M in* $\mathcal{U}$.
(d) *If* G *and* J *are bounded, then* τ^M *is a quasi-isomorphism for every M in* $\mathcal{U}$.

Proof. The claims follow from arguments parallel to those in the proof of A.17. □

Bounded Functors on the Derived Category

A bounded functor $\mathrm{F} \colon \mathcal{D}(R) \to \mathcal{D}(S)$ is one where the shape of the homology of $\mathrm{F}(M)$ depends on the shape of the homology of M in a controlled manner.

A.23 Definition. Let $\mathrm{F} \colon \mathcal{D}(R) \to \mathcal{D}(S)$ be a functor.

The functor F is called *bounded above* if for every $n \in \mathbb{Z}$ there is a $u \in \mathbb{Z}$ such that $\sup \mathrm{F}(M) \leqslant n$ holds for every R-complex M with $\sup M \leqslant u$.

The functor F is called *bounded below* if for every $n \in \mathbb{Z}$ there is a $w \in \mathbb{Z}$ such that $\inf \mathrm{F}(M) \geqslant n$ holds for every R-complex M with $\inf M \geqslant w$.

The functor F is called *bounded* if it is bounded below and bounded above.

A.24 Example. Let $\mathrm{F} \colon \mathcal{M}(R) \to \mathcal{M}(S)$ be additive functor. The left derived functor $\mathsf{L}\mathrm{F}$ is bounded below and the right derived functor $\mathsf{R}\mathrm{F}$ is bounded above; see 7.2.15.

A.25 Proposition. *Let* $\mathrm{F} \colon \mathcal{D}(R) \to \mathcal{D}(S)$ *be a triangulated functor.*

(a) *If* F *is bounded above, then it restricts to a functor* $\mathcal{D}_{\sqsubset}(R) \to \mathcal{D}_{\sqsubset}(S)$.
(b) *If* F *is bounded below, then it restricts to a functor* $\mathcal{D}_{\sqsupset}(R) \to \mathcal{D}_{\sqsupset}(S)$.
(c) *If* F *is bounded, then it restricts to a functor* $\mathcal{D}_{\square}(R) \to \mathcal{D}_{\square}(S)$.

Proof. (a): By assumption there is a $u \in \mathbb{Z}$ such that $\sup \mathrm{F}(M) \leqslant 0$ holds for all complexes M with $\sup M \leqslant u$. Let X be a complex in $\mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(X) \neq 0$ and set $s = \sup X$. As F is triangulated one has $\mathrm{F}(X) \simeq \Sigma^{s-u}\mathrm{F}(\Sigma^{-s+u}X)$ in $\mathcal{D}(S)$ and, therefore, $\sup \mathrm{F}(X) = s - u + \sup \mathrm{F}(\Sigma^{-s+u}X) \leqslant s - u$.

(b): There is a $w \in \mathbb{Z}$ such that $\inf \mathrm{F}(M) \geqslant 0$ holds for all complexes M with $\inf M \geqslant w$. Let $X \in \mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(X) \neq 0$ and set $i = \inf X$. As F is triangulated one has $\mathrm{F}(X) \simeq \Sigma^{i-w}\mathrm{F}(\Sigma^{-i+w}X)$ in $\mathcal{D}(S)$ and, therefore, $\inf \mathrm{F}(X) \geqslant i - w$.

(c): Combine parts (a) and (b). □

A.26 Proposition. *Let M be an R-complex.*

(a) *If M is in $\mathcal{D}_{\sqsupset}(R)$, then the functor $\mathsf{RHom}_R(M,-)$ is bounded above.*
(b) *If M has finite projective dimension, then $\mathsf{RHom}_R(M,-)$ is bounded below.*
(c) *If M in $\mathcal{D}_{\square}(R)$ has finite projective dimension, then $\mathsf{RHom}_R(M,-)$ is bounded.*

Proof. Parts (a) and (b) follow from 7.6.7 and 8.1.8, and they imply (c). □

A.27 Proposition. *Let M be an R-complex.*

(a) *If M is in $\mathcal{D}_{\sqsupset}(R)$, then the functor $-\otimes^{\mathsf{L}}_R M$ is bounded below.*
(b) *If M has finite flat dimension, then $-\otimes^{\mathsf{L}}_R M$ is bounded above.*
(c) *If M is in $\mathcal{D}_{\square}(R)$ and has has finite flat dimension, then $-\otimes^{\mathsf{L}}_R M$ is bounded.*

Proof. Parts (a) and (b) follow from 7.6.8 and 8.3.11, and they imply (c). □

If R is Noetherian, then the class $\mathcal{U} = \mathcal{D}^{\mathrm{f}}(R)$ satisfies the assumptions in the next lemmas; cf. 7.6.3.

A.28 Lemma. *Let $\mathrm{E}, \mathrm{F}\colon \mathcal{D}(R) \to \mathcal{D}(S)$ be triangulated functors and $\tau\colon \mathrm{E} \to \mathrm{F}$ a triangulated natural transformation. Let $\mathcal{U}$ be a class of R-complexes that is closed under shifts and soft truncations. If τ^M is an isomorphism for every module M in $\mathcal{U}$, then the following assertions hold.*

(a) *The morphism τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\square}(R)$.*
(b) *If E and F are bounded below, then τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\sqsupset}(R)$.*
(c) *If E and F are bounded above, then τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\sqsubset}(R)$.*
(d) *If E and F are bounded, then τ^M is an isomorphism for every M in $\mathcal{U}$.*

Proof. (a): Because of the isomorphism $M \simeq M_{\subseteq \sup M}$, see 4.2.4, it suffices to show that $\tau^{M_{\subseteq n}}$ is an isomorphism for every $n \in \mathbb{Z}$. For $n \ll 0$ one has $M_{\subseteq n} \simeq 0$ in $\mathcal{D}(R)$, and hence $\tau^{M_{\subseteq n}}$ is an isomorphism. Now proceed by induction on n. The complex $M_{\subseteq n-1}$ belongs to $\mathcal{U}$; assume that $\tau^{M_{\subseteq n-1}}$ is an isomorphism and consider the distinguished triangle from 7.6.6(a),

$$(\diamond) \qquad \Sigma^n \mathrm{H}_n(M) \longrightarrow M_{\subseteq n} \longrightarrow M_{\subseteq n-1} \longrightarrow \Sigma(\Sigma^n \mathrm{H}_n(M)) \,.$$

One has $\Sigma^n \mathrm{H}_n(M) \simeq (M_{\supseteq n})_{\subseteq n}$, in particular the module $\mathrm{H}_n(M)$ belongs to $\mathcal{U}$, so the morphism $\tau^{\Sigma^n \mathrm{H}_n(M)} \cong \Sigma^n \tau^{\mathrm{H}_n(M)}$ is an isomorphism by assumption. Now it follows from 6.5.19, applied to the commutative diagram induced by the natural transformation τ and ($\diamond$), that $\tau^{M_{\subseteq n}}$ is an isomorphism.

(b): The morphism τ^M is an isomorphism in $\mathcal{D}(S)$ if and only if $\mathrm{H}_n(\tau^M)$ is an isomorphism for all $n \in \mathbb{Z}$; cf. 6.5.17. Fix an integer n; choose $w \in \mathbb{Z}$ such that one has $\inf \mathrm{E}(X) \geqslant n+1$ and $\inf \mathrm{F}(X) \geqslant n+1$ for all complexes X with $\inf X \geqslant w$. Together with τ, the distinguished triangle from 7.6.6(c),

$$M_{\supseteq w} \longrightarrow M \longrightarrow M_{\subseteq w-1} \longrightarrow \Sigma(M_{\supseteq w})\,,$$

induces by 6.5.19 the following commutative diagram in $\mathcal{D}(S)$,

$$\begin{tikzcd}
\mathrm{H}_n(\mathrm{E}(M_{\supseteq w})) \arrow[r] & \mathrm{H}_n(\mathrm{E}(M)) \arrow[r, "\cong"'] \arrow[d, "\mathrm{H}_n(\tau^M)"] & \mathrm{H}_n(\mathrm{E}(M_{\subseteq w-1})) \arrow[r] \arrow[d, "\mathrm{H}_n(\tau^{(M_{\subseteq w-1})})", "\cong"'] & \mathrm{H}_{n-1}(\mathrm{E}(M_{\supseteq w})) \\
\mathrm{H}_n(\mathrm{F}(M_{\supseteq w})) \arrow[r] & \mathrm{H}_n(\mathrm{F}(M)) \arrow[r, "\cong"'] & \mathrm{H}_n(\mathrm{F}(M_{\subseteq w-1})) \arrow[r] & \mathrm{H}_{n-1}(\mathrm{F}(M_{\supseteq w}))\,.
\end{tikzcd}$$

The horizontal isomorphisms are forced by the vanishing of $\mathrm{H}_v(\mathrm{E}(M_{\supseteq w}))$ and $\mathrm{H}_v(\mathrm{F}(M_{\supseteq w}))$ for all $v \leqslant n$. The complex $M_{\subseteq w-1}$ is in $\mathcal{U} \cap \mathcal{D}_{\square}(R)$, so $\tau^{(M_{\subseteq w-1})}$ is an isomorphism by part (a), and it follows that $\mathrm{H}_n(\tau^M)$ is an isomorphism.

(c): An argument similar to the proof of part (b) applies.

(d): By parts (b) and (c) the morphisms $\tau^{M_{\supseteq 1}}$ and $\tau^{M_{\subseteq 0}}$ are isomorphisms. Now it follows from 6.5.19, applied to the commutative diagram induced by τ and the triangle $M_{\supseteq 1} \to M \to M_{\subseteq 0} \to \Sigma(M_{\supseteq 1})$, that also τ^M is an isomorphism. $\square$

A.29 Lemma. *Let* $\mathrm{F}\colon \mathcal{D}(R) \to \mathcal{D}(S)$ *be a triangulated functor and* $\mathcal{U}$ *a class of* R*-complexes that is closed under shifts and soft truncations. If* S *is left Noetherian and* $\mathrm{F}(M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}(S)$ *for every module* M *in* $\mathcal{U}$*, then the next assertions hold.*

(a) *The complex* $\mathrm{F}(M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}(S)$ *for every complex* M *in* $\mathcal{U} \cap \mathcal{D}_{\square}(R)$.
(b) *If* F *is bounded below, then* $\mathrm{F}(M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}(S)$ *for every* M *in* $\mathcal{U} \cap \mathcal{D}_{\sqsupset}(R)$.
(c) *If* F *is bounded above, then* $\mathrm{F}(M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}(S)$ *for every* M *in* $\mathcal{U} \cap \mathcal{D}_{\sqsubset}(R)$.
(d) *If* F *is bounded, then* $\mathrm{F}(M)$ *belongs to* $\mathcal{D}^{\mathrm{f}}(S)$ *for every* M *in* $\mathcal{U}$.

Proof. The assertions follow from arguments similar to those in the proof of A.28; see also the proof of A.34. $\square$

A.30 Definition. Let $\mathrm{G}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ be a functor.

The functor G is called *bounded above* if for every $n \in \mathbb{Z}$ there is a $w \in \mathbb{Z}$ such that $\sup \mathrm{G}(M) \leqslant n$ holds for every R-complex M with $\inf M \geqslant w$.

The functor G is called *bounded below* if for every $n \in \mathbb{Z}$ there is a $u \in \mathbb{Z}$ such that $\inf \mathrm{G}(M) \geqslant n$ holds for every R-complex M with $\sup M \leqslant u$.

The functor G is called *bounded* if it is bounded below and bounded above.

A.31 Proposition. *Let* $\mathrm{G}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ *be a triangulated functor.*

(a) *If* G *is bounded above, then it restricts to a functor* $\mathcal{D}_{\sqsupset}(R)^{\mathrm{op}} \to \mathcal{D}_{\sqsubset}(S)$.
(b) *If* G *is bounded below, then it restricts to a functor* $\mathcal{D}_{\sqsubset}(R)^{\mathrm{op}} \to \mathcal{D}_{\sqsupset}(S)$.
(c) *If* G *is bounded, then it restricts to a functor* $\mathcal{D}_{\square}(R)^{\mathrm{op}} \to \mathcal{D}_{\square}(S)$.

Proof. (a): By assumption there is a $w \in \mathbb{Z}$ such that $\sup \mathrm{G}(M) \leqslant 0$ holds for all complexes M with $\inf M \geqslant w$. Let X be a complex in $\mathcal{D}_{\sqsupset}(R)$ with $\mathrm{H}(X) \neq 0$ and set $i = \inf X$. As G is triangulated one has $\mathrm{G}(X) \simeq \Sigma^{w-i}\,\mathrm{G}(\Sigma^{w-i} X)$ in $\mathcal{D}(S)$ and, therefore, $\sup \mathrm{G}(X) = w - i + \sup \mathrm{G}(\Sigma^{w-i} X) \leqslant w - i$.

(b): There is a $u \in \mathbb{Z}$ such that $\inf \mathrm{G}(M) \geqslant 0$ holds for all complexes M with $\sup M \leqslant u$. Let $X \in \mathcal{D}_{\sqsubset}(R)$ with $\mathrm{H}(X) \neq 0$ and set $s = \sup X$. As G is triangulated one has $\mathrm{G}(X) \simeq \Sigma^{u-s}\,\mathrm{G}(\Sigma^{u-s} X)$ in $\mathcal{D}(S)$ and, therefore, $\inf \mathrm{G}(X) \geqslant u - s$.

(c): Combine parts (a) and (b). □

A.32 Proposition. *Let N be an R-complex.*

(a) *If N is in $\mathcal{D}_{\sqsubset}(R)$, then the functor $\mathsf{RHom}_R(-, N)$ is bounded above.*
(b) *If N has finite injective dimension, then $\mathsf{RHom}_R(-, N)$ is bounded below.*
(c) *If N in $\mathcal{D}_{\square}(R)$ has finite injective dimension, then $\mathsf{RHom}_R(-, N)$ is bounded.*

Proof. Parts (a) and (b) follow from 7.6.7 and 8.2.8, and they imply (c). □

A.33 Lemma. *Let $\mathrm{G}, \mathrm{J} \colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ be triangulated functors and $\tau \colon \mathrm{G} \to \mathrm{J}$ a triangulated natural transformation. Let $\mathcal{U}$ be a class of R-complexes that is closed under shifts and soft truncations. If τ^M is an isomorphism for every module M in $\mathcal{U}$, then the following assertions hold.*

(a) *The morphism τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\square}(R)^{\mathrm{op}}$.*
(b) *If G and J are bounded below, then τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\sqsubset}(R)^{\mathrm{op}}$.*
(c) *If G and J are bounded above, then τ^M is an isomorphism for every complex M in $\mathcal{U} \cap \mathcal{D}_{\sqsupset}(R)^{\mathrm{op}}$.*
(d) *If G and J are bounded, then τ^M is an isomorphism for every M in $\mathcal{U}$.*

Proof. The claims follow from arguments similar to those in the proof of A.28. □

A.34 Lemma. *Let $\mathrm{G} \colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ be a triangulated functor and $\mathcal{U}$ a class of R-complexes that is closed under shifts and soft truncations. If S is left Noetherian and $\mathrm{G}(M)$ belongs to $\mathcal{D}^{\mathrm{f}}(S)$ for every module M in $\mathcal{U}$, then the next assertions hold.*

(a) *The complex $\mathrm{G}(M)$ is in $\mathcal{D}^{\mathrm{f}}(S)$ for every complex M in $\mathcal{U} \cap \mathcal{D}_{\square}(R)^{\mathrm{op}}$.*
(b) *If G is bounded below, then $\mathrm{G}(M)$ is in $\mathcal{D}^{\mathrm{f}}(S)$ for every M in $\mathcal{U} \cap \mathcal{D}_{\sqsubset}(R)^{\mathrm{op}}$.*
(c) *If G is bounded above, then $\mathrm{G}(M)$ is in $\mathcal{D}^{\mathrm{f}}(S)$ for every M in $\mathcal{U} \cap \mathcal{D}_{\sqsupset}(R)^{\mathrm{op}}$.*
(d) *If G is bounded, then $\mathrm{G}(M)$ is in $\mathcal{D}^{\mathrm{f}}(S)$ for every M in $\mathcal{U}$.*

Proof. (a): Because of the isomorphism $M \simeq M_{\subseteq \sup M}$ in $\mathcal{D}(R)$, see 4.2.4, it suffices to show that $\mathrm{G}(M_{\subseteq n})$ is in $\mathcal{D}^{\mathrm{f}}(S)$ for every $n \in \mathbb{Z}$. For $n \ll 0$ one has $M_{\subseteq n} \simeq 0$ in $\mathcal{D}(R)$ and, therefore, $\mathrm{G}(M_{\subseteq n}) \simeq 0 \in \mathcal{D}^{\mathrm{f}}(S)$. Now proceed by induction on n. The complex $M_{\subseteq n-1}$ is in $\mathcal{U}$; assume that $\mathrm{G}\,(M_{\subseteq n-1})$ belongs to $\mathcal{D}^{\mathrm{f}}(S)$. One has $\Sigma^n \mathrm{H}_n(M) \simeq (M_{\supseteq n})_{\subseteq n}$, so the module $\mathrm{H}_n(M)$ belongs to $\mathcal{U}$ and thus $\mathrm{G}(\Sigma^n \mathrm{H}_n(M)) \cong \Sigma^{-n}\,\mathrm{G}(\mathrm{H}_n(M))$ is in $\mathcal{D}^{\mathrm{f}}(S)$ by assumption. Apply the triangulated functor G to the distinguished triangle 7.6.6(a),

$$\Sigma^n \mathrm{H}_n(M) \longrightarrow M_{\subseteq n} \longrightarrow M_{\subseteq n-1} \longrightarrow \Sigma(\Sigma^n \mathrm{H}_n(M))\,.$$

As the category $\mathcal{D}^{\mathrm{f}}(S)$ is triangulated, see 7.6.3, it follows that $\mathrm{G}(M_{\subseteq n})$ is in $\mathcal{D}^{\mathrm{f}}(S)$.

(b): Fix $n \in \mathbb{Z}$ and choose $u \in \mathbb{Z}$ such that $\inf \mathrm{G}(X) \geqslant n+1$ holds for all X with $\sup X \leqslant u$. Consider the distinguished triangle $M_{\supseteq u+1} \to M \to M_{\subseteq u} \to \Sigma(M_{\supseteq u+1})$; cf. 7.6.6(c). It induces by 6.5.19 an exact sequence of S-modules

$$\mathrm{H}_n(\mathrm{G}(M_{\subseteq u})) \longrightarrow \mathrm{H}_n(\mathrm{G}(M)) \xrightarrow{\cong} \mathrm{H}_n(\mathrm{G}(M_{\supseteq u+1})) \longrightarrow \mathrm{H}_{n-1}(\mathrm{G}(M_{\subseteq u}))\,,$$

where the isomorphism is forced by the vanishing of $\mathrm{H}_v(\mathrm{G}(M_{\subseteq u}))$ for $v \leqslant n$. As the complex $M_{\supseteq u+1}$ is in $\mathcal{U} \cap \mathcal{D}_{\square}(R)^{\mathrm{op}}$, it follows from part (a) that $\mathrm{G}(M_{\supseteq u+1})$ is in $\mathcal{D}^{\mathrm{f}}(S)$. In particular, $\mathrm{H}_n(\mathrm{G}(M_{\supseteq u+1}))$ is finitely generated, and hence so is $\mathrm{H}_n(\mathrm{G}(M))$.

(c): An argument similar to the proof of part (b) applies.

(d): Consider the distinguished triangle $M_{\supseteq 1} \to M \to M_{\subseteq 0} \to \Sigma(M_{\supseteq 1})$ in $\mathcal{U}$ from 7.6.6(c). It follows from parts (b) and (c) that $\mathrm{G}(M_{\subseteq 0})$ and $\mathrm{G}(M_{\supseteq 1})$ belong to $\mathcal{D}^{\mathrm{f}}(S)$. Since G is a triangulated functor, and $\mathcal{D}^{\mathrm{f}}(S)$ is a triangulated category, it follows that $\mathrm{G}(M)$ belongs to $\mathcal{D}^{\mathrm{f}}(S)$. □

A.35. A functor $\mathcal{D}(R) \to \mathcal{D}(S)^{\mathrm{op}}$ is called bounded above/below if and only if the opposite functor $\mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ is bounded below/above according to A.30. One gets results about such functors by interchanging "bounded above" and "bounded below" in A.33 and A.34.

Exercises

E A.1 Let M and N be R-complexes one of which is bounded above. Show that if one has $\mathrm{H}_{-v}(\mathrm{Hom}_R(M, N_v)) = 0$ for all $v \in \mathbb{Z}$, then $\mathrm{H}_0(\mathrm{Hom}_R(M, N)) = 0$ holds.

E A.2 Let M and N be R-complexes one of which is bounded below. Show that if one has $\mathrm{H}_v(\mathrm{Hom}_R(M_v, N)) = 0$ for all $v \in \mathbb{Z}$, then $\mathrm{H}_0(\mathrm{Hom}_R(M, N)) = 0$ holds.

E A.3 Let M be an R^{o}-complex and N an R-complex such that M is bounded above or N is bounded below. Show that if one has $\mathrm{H}_{-v}(M \otimes_R N_v) = 0$ for all $v \in \mathbb{Z}$, then $\mathrm{H}_0(M \otimes_R N) = 0$ holds.

E A.4 Let M be an acyclic R-complex and N an R-module of finite injective dimension. Show that $\mathrm{Hom}_R(M, N)$ is acyclic if $\mathrm{Ext}_R^m(M_v, N) = 0$ holds for every $v \in \mathbb{Z}$ and all $m > 0$.

E A.5 Let N be an acyclic R-complex and M an R-module of finite projective dimension. Show that $\mathrm{Hom}_R(M, N)$ is acyclic if $\mathrm{Ext}_R^m(M, N_v) = 0$ holds for every $v \in \mathbb{Z}$ and all $m > 0$.

E A.6 Let M be an acyclic R^{o}-complex and N an R-module of finite flat dimension. Show that $M \otimes_R N$ is acyclic if $\mathrm{Tor}_m^R(M_v, N) = 0$ holds for every $v \in \mathbb{Z}$ and all $m > 0$.

E A.7 Let M be an R-complex and N an R-module. Show that if $\mathrm{Hom}_R(N, M)$ is acyclic and $\mathrm{Ext}_R^1(N, M_v) = 0$ holds for all $v \in \mathbb{Z}$, then $\mathrm{Ext}_R^1(N, \mathrm{Z}_v(M)) = 0$ holds for all $v \in \mathbb{Z}$; show that the converse holds if M is acyclic.

E A.8 Let M be an R-complex and N an R-module. Show that if $\mathrm{Hom}_R(M, N)$ is acyclic and $\mathrm{Ext}_R^1(M_v, N) = 0$ holds for all $v \in \mathbb{Z}$, then $\mathrm{Ext}_R^1(\mathrm{B}_v(M), N) = 0$ holds for all $v \in \mathbb{Z}$; show that the converse holds if M is acyclic.

E A.9 Let M be an R-complex and N an R^{o}-module. Show that if $N \otimes_R M$ is acyclic and $\mathrm{Tor}_1^R(N, M_v) = 0$ holds for all $v \in \mathbb{Z}$, then $\mathrm{Tor}_1^R(N, \mathrm{B}_v(M)) = 0$ holds for all $v \in \mathbb{Z}$; show that the converse holds if M is acyclic.

E A.10 Give a proof of A.6.

E A.11 Show that A.9 is also a consequence of A.2.

E A.12 Show that A.11 is also a consequence of A.3.

E A.13 Give a direct proof of A.9, i.e. one that does not invoke A.5 or A.2.

E A.14 Let $\mathrm{F}\colon \mathcal{C}(R) \to \mathcal{C}(S)$ be a functor such that there is an isomorphism $\mathrm{F}(\Sigma M) \cong \Sigma\mathrm{F}(M)$ for every R-complex M. (a) Show that F is bounded above if and only if there exists an integer d with $\sup \mathrm{F}(M)^{\natural} \leqslant \sup M^{\natural} + d$ for all $M \in \mathcal{C}(R)$. (b) Show that F is bounded below if and only if there exists an integer d with $\inf \mathrm{F}(M)^{\natural} \geqslant \inf M^{\natural} - d$ for all $M \in \mathcal{C}(R)$.

E A.15 (Cf. A.21) Let $\mathrm{G}\colon \mathcal{C}(R) \to \mathcal{C}(S)^{\mathrm{op}}$ be a functor and $\mathcal{U}$ a class of R-complexes. Define G be bounded above/below on $\mathcal{U}$ if and only if the opposite functor $\mathcal{C}(R)^{\mathrm{op}} \to \mathcal{C}(S)$ is bounded below/above on $\mathcal{U}$. State and prove the result that parallels A.22.

E A.16 Let $\mathrm{G}\colon \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(S)$ be additive functor. Show that left derived functor $\mathsf{L}\mathrm{G}$ is bounded below and the right derived functor $\mathsf{R}\mathrm{G}$ is bounded above.

E A.17 (Cf. A.35) Let $\mathrm{G}\colon \mathcal{D}(R) \to \mathcal{D}(S)^{\mathrm{op}}$ be a functor. Define G to be bounded above/below if and only if the opposite functor $\mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ is bounded below/above. State and prove the results that correspond to A.33 and A.34.

E A.18 Let $\mathrm{E}\colon \mathcal{D}(R) \to \mathcal{D}(S)$ and $\mathrm{F}\colon \mathcal{D}(S) \to \mathcal{D}(T)$ be functors. Show that if E and F both are bounded (above/below), then the composite FE is bounded (above/below).

E A.19 Let $\mathrm{G}\colon \mathcal{D}(R)^{\mathrm{op}} \to \mathcal{D}(S)$ and $\mathrm{F}\colon \mathcal{D}(S) \to \mathcal{D}(T)$ be functors. Show that if F and G both are bounded (above/below), then the composite FG is bounded (above/below).

Appendix B
Minimality

Synopsis. Minimal complex; injective envelope; minimal semi-injective resolution; Nakayama's lemma; projective cover; minimal semi-projective resolution; semi-perfect ring; perfect ring.

If I is a semi-injective complex, then every quasi-isomorphism $I \to I$ is a homotopy equivalence by 5.3.24; minimality of I would imply that every such quasi-isomorphism would even be an isomorphism. After giving a few characterizations of minimal complexes, we prove existence of minimal semi-injective resolutions for all complexes over any ring. In contrast, existence of minimal semi-projective resolutions for all complexes holds only over perfect rings. For complexes with bounded above and degreewise finitely generated homology, minimal semi-projective resolutions are more accessible: They can be constructed for every such complex over a Noetherian semi-perfect ring, in particular over any Noetherian local ring.

The categorical approach to minimality taken here is already present in works of Roig [217, 218] from the early 1990s. In the context of simplicial complexes, the idea of minimal subobjects goes back to Eilenberg and Zilber [77] in the 1950s, and in the same decade Eilenberg [75] also considered minimal resolutions of modules.

Our exposition is heavily influenced by Avramov, Foxby, and Halperin [25]; parts of the material can be found in works of Avramov and Martsinkovsky [27], García Rozas [104], and Krause and Saorín [161].

B.1 Lemma. *Let* $0 \longrightarrow M' \xrightarrow{\alpha'} M \xrightarrow{\alpha} M'' \longrightarrow 0$ *be a degreewise split exact sequence of* R*-complexes.*

(a) *The complex* M' *is contractible if and only if* α *is a homotopy equivalence, and in that case the sequence is split in* $\mathcal{C}(R)$.

(b) *The complex* M'' *is contractible if and only if* α' *is a homotopy equivalence, and in that case the sequence is split in* $\mathcal{C}(R)$.

Proof. It follows from 2.3.12 and 2.3.13 that the following sequences are exact.

$$(\dagger)\quad 0 \to \operatorname{Hom}_R(M',M') \to \operatorname{Hom}_R(M',M) \xrightarrow{\operatorname{Hom}(M',\alpha)} \operatorname{Hom}_R(M',M'') \to 0\,,$$

$$(\ddagger)\quad 0 \to \operatorname{Hom}_R(M'',M) \xrightarrow{\operatorname{Hom}(\alpha,M)} \operatorname{Hom}_R(M,M) \to \operatorname{Hom}_R(M',M) \to 0\,,$$

($\flat$) $0 \to \operatorname{Hom}_R(M'', M') \to \operatorname{Hom}_R(M'', M) \xrightarrow{\operatorname{Hom}(M'',\alpha)} \operatorname{Hom}_R(M'', M'') \to 0\,.$

If α is a homotopy equivalence, then so is $\operatorname{Hom}_R(M', \alpha)$ by 4.3.19; in particular, it is a quasi-isomorphism. It follows from (†) and 4.2.6 that $\operatorname{Hom}_R(M', M')$ is acyclic, whence M' is contractible by 4.3.29.

Conversely, if M' is contractible, then $\operatorname{Hom}_R(M', M)$ and $\operatorname{Hom}_R(M'', M')$ are acyclic by 4.3.29. It follows from ($\flat$) that $\operatorname{Hom}_R(M'', \alpha)$ is a quasi-isomorphism. It is surjective on cycles by 4.2.7, so there exists a morphism $\beta\colon M'' \to M$ with $\alpha\beta = 1^{M''}$; see 2.3.10. In particular, the original sequence is split in $\mathcal{C}(R)$. Moreover, it follows from (‡) that $\operatorname{Hom}_R(\alpha, M)$ is a quasi-isomorphism, whence there exists a morphism $\beta'\colon M'' \to M$ such that $\beta'\alpha \sim 1^M$. Thus α is a homotopy equivalence; see 4.3.3. This proves part (a), and a similar argument proves (b). □

B.2 Definition. An R-complex M is called *minimal* if every homotopy equivalence $\varepsilon\colon M \to M$ is an isomorphism.

Minimality for complexes extends no interesting notion for modules; indeed, every module is minimal when viewed as a complex. Here are a couple less trivial examples of minimal complexes.

B.3 Example. The $\mathbb{Z}$-complex $I = 0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is minimal, as there are no non-zero homomorphisms $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ and hence no non-zero homomorphisms $I \to I$ of degree 1. It follows that every homotopy equivalence $I \to I$ is an isomorphism.

B.4 Example. Let $\Bbbk$ be a field and consider the local ring $R = \Bbbk[x]/(x^2)$ of dual numbers. For simplicity, let x denote the coset of x in R. The R-complex

$$X = \cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$$

is minimal. Indeed, every homomorphism $R \to R$ is given by multiplication by an element in R. A homotopy equivalence $\alpha\colon X \to X$ is, therefore, given by a sequence of ring elements $(a_v)_{v\in\mathbb{Z}}$; so is its homotopy inverse $\beta = (b_v)_{v\in\mathbb{Z}}$ and the homotopy $\varrho = (r_v)_{v\in\mathbb{Z}}$ from 1^X to $\alpha\beta$. For every $v \in \mathbb{Z}$ one has $1 - a_v b_v = x r_v + r_{v-1} x$. The element $a_v b_v = 1 - x(r_v + r_{v-1})$ is not in the maximal ideal (x) of R, so it is invertible, whence α_v is invertible and α is an isomorphism.

B.5 Lemma. *Let M be an R-complex. If M is minimal, then the zero complex is the only contractible direct summand of M.*

Proof. Let M be minimal and assume that it is a direct sum $M = M' \oplus C$ where C is contractible. The morphism $1^{M'} \oplus 0^C$ is a homotopy equivalence $M \to M$; see 4.3.6. As M is minimal, $1^{M'} \oplus 0^C$ is an isomorphism whence C is the zero complex. □

Minimality can be characterized in several different ways.

B.6 Proposition. *For an R-complex M, the following conditions are equivalent.*

(*i*) *M is minimal.*

(ii) *Every morphism $\varepsilon\colon M \to M$ with $\varepsilon \sim 1^M$ is an isomorphism.*
(iii) *Every homotopy equivalence $\alpha\colon K \to M$ has a right inverse.*
(iv) *Every homotopy equivalence $\varepsilon\colon M \to M$ has a right inverse.*
(v) *Every homotopy equivalence $\beta\colon M \to N$ has a left inverse.*
(vi) *Every homotopy equivalence $\varepsilon\colon M \to M$ has a left inverse.*

If these conditions hold, then the complexes $\operatorname{Ker}\alpha$ *and* $\operatorname{Coker}\beta$ *are contractible for all homotopy equivalences $\alpha\colon K \to M$ and $\beta\colon M \to N$.*

Proof. First notice that if $\alpha\colon K \to M$ is a homotopy equivalence and (iii) holds, then there is a split exact sequence $0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow K \xrightarrow{\alpha} M \longrightarrow 0$, and $\operatorname{Ker}\alpha$ is contractible by B.1. Similarly, if $\beta\colon M \to N$ is a homotopy equivalence and (v) holds, then $\operatorname{Coker}\beta$ is contractible.

The implications (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are evident.

(i) $\Rightarrow$ (ii): If the endomorphisms $\varepsilon\colon M \to M$ and 1^M are homotopic, then one has $\varepsilon^2 \sim \varepsilon \sim 1^M$; see 4.3.3. It follows that ε is its own homotopy inverse; in particular, ε is a homotopy equivalence, so by (i) it is an isomorphism.

(ii) $\Rightarrow$ (iii): Let $\alpha\colon K \to M$ be a homotopy equivalence with homotopy inverse $\gamma\colon M \to K$. By (ii) the morphism $\varepsilon = \alpha\gamma$ is an isomorphism, and it follows that $\gamma\varepsilon^{-1}$ is a right inverse of α.

(iv) $\Rightarrow$ (i): Let $\varepsilon\colon M \to M$ be a homotopy equivalence. It has a right inverse, so the sequence $0 \longrightarrow \operatorname{Ker}\varepsilon \xrightarrow{\iota} M \xrightarrow{\varepsilon} M \longrightarrow 0$ is split exact. Consider the split exact sequence $0 \longrightarrow M \xrightarrow{\sigma} M \xrightarrow{\tau} \operatorname{Ker}\varepsilon \longrightarrow 0$, where $\tau\iota = 1^{\operatorname{Ker}\varepsilon}$ and $\varepsilon\sigma = 1^M$. As already noted, $\operatorname{Ker}\varepsilon$ is contractible, whence σ is a homotopy equivalence by B.1 and surjective by (iv). Thus σ is an isomorphism and, therefore, ε is an isomorphism.

(ii) $\Rightarrow$ (v): Let $\beta\colon M \to N$ be a homotopy equivalence with homotopy inverse $\gamma\colon N \to M$. By (ii) the morphism $\varepsilon = \gamma\beta$ is an isomorphism, and it follows that $\varepsilon^{-1}\gamma$ is a left inverse of β.

(vi) $\Rightarrow$ (i): Let $\varepsilon\colon M \to M$ be a homotopy equivalence. It has a left inverse, so the sequence $0 \longrightarrow M \xrightarrow{\varepsilon} M \xrightarrow{\pi} \operatorname{Coker}\varepsilon \longrightarrow 0$ is split exact. Consider the split exact sequence $0 \longrightarrow \operatorname{Coker}\varepsilon \xrightarrow{\sigma} M \xrightarrow{\tau} M \longrightarrow 0$, where $\pi\sigma = 1^{\operatorname{Coker}\varepsilon}$ and $\tau\varepsilon = 1^M$. As noted, $\operatorname{Coker}\varepsilon$ is contractible, whence τ is a homotopy equivalence by B.1 and injective by (vi). Thus τ is an isomorphism and, therefore, ε is an isomorphism. □

B.7 Corollary. *Let M and M' be minimal R-complexes.*

(a) *Every homotopy equivalence $\alpha\colon M \to M'$ is an isomorphism.*
(b) *If there exist contractible R-complexes C and C' such that $M \oplus C$ and $M' \oplus C'$ are homotopy equivalent, then the complexes M and M' are isomorphic.*

Proof. (a): It follows from B.6 that α has a right inverse as well as a left inverse, whence it is an isomorphism.

(b): It follows from B.1 that the injection $M \rightarrowtail M \oplus C$ and the projection $M' \oplus C' \twoheadrightarrow M'$ are homotopy equivalences. Therefore, the composite map

$$M \rightarrowtail M \oplus C \xrightarrow{\cong} M' \oplus C' \twoheadrightarrow M'$$

is a homotopy equivalence, and hence an isomorphism by (a). □

Injective Envelopes

Every graded module can by 5.3.4 be embedded into a graded-injective one; the immediate goal is to show that there is a unique way to do this minimally.

B.8 Definition. Let M be a graded R-module. A graded submodule N of M is called *essential* if $M' \cap N \neq 0$ holds for every graded submodule $M' \neq 0$ of M.

Evidently, a graded R-submodule $N \subseteq M$ is essential if and only if it intersects every cyclic graded submodule non-trivially, i.e. $R\langle m\rangle \cap N \neq 0$ for every homogeneous element $m \neq 0$ in M.

Remark. Another word for essential submodule is 'large' submodule.

B.9 Example. Every non-zero ideal in $\mathbb{Z}$ is essential.

B.10. A graded direct summand N of a graded R-module M is essential if and only if $N = M$.

B.11 Lemma. *Let M be a graded R-module with graded submodules N and N'.*

(a) *Assume that there is an inclusion $N \subseteq N'$. If N is essential in N', and N' is essential in M, then N is essential in M.*
(b) *Let X and X' be graded submodules of N and N'. If X is essential in N and X' is essential in N', then $X \cap X'$ is essential in $N \cap N'$.*
(c) *Let $\alpha\colon M \to X$ be a homomorphism of graded R-modules with $N \cap \operatorname{Ker}\alpha = 0$. If N is essential in M, then $\alpha(N)$ is essential in $\alpha(M)$.*

Proof. (a): Let $M' \neq 0$ be a graded submodule of M. The graded submodule $N \cap M' = N \cap (N' \cap M')$ is non-zero, as N' is essential in M and N is essential in N'.

(b): Let $M' \neq 0$ be a graded submodule of $N \cap N'$. The graded submodule $(X \cap X') \cap M' = X \cap (X' \cap M')$ is non-zero, as X' is essential in N' and X is essential in N.

(c): Let $X' \neq 0$ be a graded submodule of $\alpha(M)$. As $\alpha(\alpha^{-1}(X')) = X'$ is non-zero one has $\alpha^{-1}(X') \neq 0$ and thus $\alpha^{-1}(X') \cap N \neq 0$ holds. By assumption, $N \cap \operatorname{Ker}\alpha = 0$, and hence $\alpha(\alpha^{-1}(X') \cap N) \neq 0$. Since there is an inclusion $\alpha(\alpha^{-1}(X') \cap N) \subseteq X' \cap \alpha(N)$, it follows that $X' \cap \alpha(N)$ is non-zero, whence $\alpha(N)$ is essential in $\alpha(M)$. □

B.12 Lemma. *Assume that R is commutative and Noetherian and let U be a multiplicative subset of R. Let M be a graded R-module and $N \subseteq M$ a graded submodule. If N is essential in M, then $U^{-1}N$ is an essential graded submodule of the graded $U^{-1}R$-module $U^{-1}M$.*

Proof. It must be argued that for every non-zero homogeneous element $x \in U^{-1}M$ one has $(U^{-1}R)\langle x\rangle \cap U^{-1}N \neq 0$. We can assume that x has the form $x = \frac{m}{1}$ with m homogeneous in M. Consider the set $\{\operatorname{Ann}_R(um) \mid u \in U\}$ of ideals in R. As R is Noetherian, it has a maximal element, say, $\operatorname{Ann}_R(vm)$, with $v \in U$. Note that $vm \neq 0$ in M as $\frac{m}{1} \neq 0$ in $U^{-1}M$. As N is essential in M, one has $R\langle vm\rangle \cap N \neq 0$.

The ideal $\mathfrak{a} = (N :_R vm)$ satisfies $\mathfrak{a}(vm) = R\langle vm\rangle \cap N$, so $\mathfrak{a}(vm) \neq 0$ holds. As R is Noetherian, the ideal $\mathfrak{a}$ is finitely generated: $\mathfrak{a} = (a_1, \dots, a_n)$. Suppose that $\frac{a_i vm}{1} = 0$ holds in $U^{-1}M$ for every $i \in \{1, \dots, n\}$. It follows that there exists a $w \in U$ with $wa_i vm = 0$ in M, that is, $a_i \in \operatorname{Ann}_R(wvm)$ for every $i \in \{1, \dots, n\}$. Evidently, there is an inclusion $\operatorname{Ann}_R(vm) \subseteq \operatorname{Ann}_R(wvm)$, and since $wv \in U$ the maximality of $\operatorname{Ann}_R(vm)$ implies that one has $\operatorname{Ann}_R(vm) = \operatorname{Ann}_R(wvm)$. Consequently, $a_i \in \operatorname{Ann}_R(vm)$ for every $i \in \{1, \dots, n\}$, which contradicts the fact that $\mathfrak{a}(vm) \neq 0$. It follows that $\frac{a_i vm}{1} \neq 0$ holds in $U^{-1}M$ for some $i \in \{1, \dots, n\}$. The non-zero element $\frac{a_i vm}{1}$ certainly belongs to $(U^{-1}R)\langle \frac{m}{1}\rangle$. As $a_i vm \in \mathfrak{a}(vm) = R\langle vm\rangle \cap N \subseteq N$ one also has $\frac{a_i vm}{1} \in U^{-1}N$, so the intersection $(U^{-1}R)\langle \frac{m}{1}\rangle \cap U^{-1}N$ is non-zero. □

B.13 Definition. An *injective envelope* of a graded R-module M is an injective morphism $\iota\colon M \to E$ of graded R-modules where E is graded-injective and $\operatorname{Im}\iota$ is essential in E. The module E alone is also referred to as an injective envelope of M.

Notice that an injective envelope is an injective preenvelope.

Remark. Another word for injective envelope is 'injective hull'.

B.14 Example. Recall from 8.2.10 that the ring $\mathbb{Z}/4\mathbb{Z}$ is self-injective. Evidently $2\mathbb{Z}/4\mathbb{Z}$ is the only prime ideal in $\mathbb{Z}/4\mathbb{Z}$, and it is the kernel of multiplication by 2 on $\mathbb{Z}/4\mathbb{Z}$. Thus there is an injective homomorphism $\iota\colon (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}$ whose image is clearly essential. It follows that ι is an injective envelope.

B.15 Example. The $\mathbb{Z}$-module $\mathbb{Q}$ is divisible and hence injective by 1.3.32. As $\mathbb{Z}$ is an essential submodule of $\mathbb{Q}$, the embedding $\mathbb{Z} \rightarrowtail \mathbb{Q}$ is an injective envelope.

For every prime p, the Prüfer p-group $\mathbb{Z}(p^\infty) = (\{1, p, p^2, \dots\}^{-1}\mathbb{Z})/\mathbb{Z}$ is divisible, and hence injective. To see this, it suffices to show that multiplication by q is surjective on $\mathbb{Z}(p^\infty)$ for every prime q. For $q = p$ this is clear as $[\frac{x}{p^n}]_{\mathbb{Z}} = p\,[\frac{x}{p^{n+1}}]_{\mathbb{Z}}$. Given $q \neq p$ and $[\frac{x}{p^n}]_{\mathbb{Z}}$ in $\mathbb{Z}(p^\infty)$, choose integers a and b with $aq + bp^n = 1$; one then has $[\frac{x}{p^n}]_{\mathbb{Z}} = q\,[\frac{ax}{p^n}]_{\mathbb{Z}}$. The homomorphism $\iota\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}(p^\infty)$ given by $[x]_{p\mathbb{Z}} \mapsto [\frac{x}{p}]_{\mathbb{Z}}$ is evidently injective. To see that its image is essential in $\mathbb{Z}(p^\infty)$, let $y = [\frac{x}{p^n}]_{\mathbb{Z}}$ be a non-zero element in $\mathbb{Z}(p^\infty)$. One can assume that x is not divisible by p, and thus $[\frac{x}{p}]_{\mathbb{Z}} = p^{n-1}y$ is a non-zero element in $\operatorname{Im}\iota \cap \mathbb{Z}\langle y\rangle$. It follows that ι is an injective envelope.

We shall see that injective envelopes always exist and that they are unique up to isomorphism. We begin by proving uniqueness, which is the easiest.

B.16 Proposition. *Let M be a graded R-module. If $\iota\colon M \rightarrowtail E$ and $\iota'\colon M \rightarrowtail E'$ are injective envelopes, then there exists an isomorphism $\gamma\colon E \to E'$ with $\gamma\iota = \iota'$.*

Proof. By graded-injectivity of E' there is a morphism $\gamma\colon E \to E'$ with $\gamma\iota = \iota'$; see 5.3.6. Since $\operatorname{Ker}\gamma \cap \operatorname{Im}\iota = 0$ holds and $\operatorname{Im}\iota$ is essential in E, the map γ is injective. In particular, $\operatorname{Im}\gamma \cong E$ is graded-injective, whence there is an equality $E' = \operatorname{Im}\gamma \oplus C$ by 5.3.6. As $\operatorname{Im}\iota'$ is contained in $\operatorname{Im}\gamma$ one has $\operatorname{Im}\iota' \cap C = 0$; consequently, C is zero as $\operatorname{Im}\iota'$ is essential in E'. It follows that γ is surjective and hence an isomorphism. □

B.17 Theorem. *Every graded R-module has an injective envelope.*

Proof. Let M be a graded R-module. By 5.3.4 there exists an injective morphism of graded R-modules $\iota\colon M \to I$, where I is graded-injective. The set of graded submodules of I that contains $\iota(M)$ as an essential submodule is inductively ordered by inclusion. Hence, by Zorn's lemma, it has a maximal element E. It remains to prove that E is graded-injective. By another application of Zorn's lemma, choose a graded submodule Z of I maximal with the property $Z \cap E = 0$. Denote by ζ the morphism $E \rightarrowtail I \twoheadrightarrow I/Z$. It is sufficient to prove that ζ is an isomorphism. Indeed, the composite of $I \twoheadrightarrow I/Z$ and $\zeta^{-1}\colon I/Z \to E$ is then a left inverse of the embedding $E \rightarrowtail I$. As I is graded-injective, so is the graded direct summand E.

To prove that ζ is an isomorphism, notice first that it is injective as one has $\operatorname{Ker}\zeta = Z \cap E = 0$. It follows from maximality of Z that $\operatorname{Im}\zeta$ is essential in I/Z. By the lifting property of the graded-injective module I there is a morphism $\alpha\colon I/Z \to I$ such that $\alpha\zeta\colon E \to I$ is the inclusion; see 5.3.6. As $\operatorname{Ker}\alpha \cap \operatorname{Im}\zeta = 0$ and $\operatorname{Im}\zeta$ is essential in I/Z, it follows that α is injective. Thus B.11 yields that $\alpha\zeta(E) = E$ is essential in $\alpha(I/Z)$, whence $\iota(M)$ is essential in $\alpha(I/Z)$. By maximality of E one gets $\alpha(I/Z) = E$, and thus $\alpha\colon I/Z \to E$ satisfies $\alpha\zeta = 1^E$. It follows that the essential submodule $\operatorname{Im}\zeta$ is a direct summand of I/Z, so one has $\operatorname{Im}\zeta = I/Z$. $\square$

B.18 Definition. Let M be a graded R-module. The injective module in an injective envelope of M, which by B.16 is unique up to isomorphism, is denoted $\mathrm{E}_R(M)$.

Remark. In spite of what the notation might suggest, $\mathrm{E}_R(M)$ is not natural in M; that is, the assignment of injective envelopes is not a functor. The obstruction is the non-uniqueness of lifts of morphisms $M \to N$ to $\mathrm{E}_R(M) \to \mathrm{E}_R(N)$; Goodearl makes it explicit in [106, 1.B].

B.19 Corollary. *Let I be a graded-injective R-module. For every graded submodule Z of I there exist graded-injective submodules E and V of I, such that Z is essential in E and there is an equality $I = E \oplus V$ of graded R-modules. In particular, E is an injective envelope of Z.*

Proof. By B.17 the graded module Z has an injective envelope $\iota\colon Z \rightarrowtail E'$. By the lifting property of I there is a morphism $\alpha\colon E' \to I$ such that $\alpha\iota\colon Z \to I$ is the inclusion; see 5.3.6. As one has $\operatorname{Ker}\alpha \cap \operatorname{Im}\iota = 0$ and $\operatorname{Im}\iota$ is essential in E', it follows that α is injective. Set $E = \operatorname{Im}\alpha$; note that $E \cong E'$ is graded-injective and hence a direct summand of I by 5.3.6. It follows from B.11 that $\alpha\iota(Z) = Z$ is essential in E, so the assertion holds with $V = I/E$. $\square$

Minimal Complexes of Injective Modules

For a complex of injective modules, minimality means that the subcomplex of cycles is essential.

B.20 Lemma. *Let I be a complex of injective R-modules. If the graded submodule $\mathrm{Z}(I)^\natural$ of $I^\natural$ is essential, then I is minimal.*

Proof. Assume that $\mathrm{Z}(I)^\natural$ is essential in $I^\natural$. Let $\varepsilon\colon I \to I$ be an endomorphism homotopic to 1^I; to prove that I is minimal, it suffices by B.6 to show that ε is an isomorphism. By assumption there exists a homomorphism $\sigma\colon I \to I$ of degree 1 with $1^I - \varepsilon = \partial^I\sigma + \sigma\partial^I$. Set $X = \mathrm{Z}(I) \cap \operatorname{Ker}\varepsilon$; for every x in X one has $x = \partial^I\sigma(x)$ and, therefore, $\sigma(X) \cap \mathrm{Z}(I) = 0$. As $\mathrm{Z}(I)^\natural$ is essential in $I^\natural$, it follows that $\sigma(X)$ is 0, and then X is zero. From the definition of X it follows that ε is injective.

The exact sequence $0 \longrightarrow I \xrightarrow{\varepsilon} I \longrightarrow \operatorname{Coker}\varepsilon \longrightarrow 0$ is degreewise split by 5.3.6. Since ε is homotopic to 1^I, it is a homotopy equivalence, and it follows from B.1 that the sequence is split in $\mathcal{C}(R)$. Let $\varrho\colon I \to I$ be a morphism such that $\varrho\varepsilon = 1^I$. It follows that ϱ is homotopic to 1^I, so by the argument above ϱ is injective and, therefore, it is an isomorphism. Hence, also ε is an isomorphism. ◻

B.21 Theorem. *Let I be a complex of injective R-modules. There is an equality $I = I' \oplus I''$, where I' and I'' are complexes of injective R-modules, I' is minimal, and I'' is contractible. Moreover, the following assertions hold.*

(a) *The complex I' is unique in the following sense: if one has $I = J' \oplus J''$, where J' is minimal and J'' is contractible, then J' is isomorphic to I'.*
(b) *I is minimal if and only if $\mathrm{Z}(I)^\natural$ is essential in $I^\natural$.*
(c) *If I is semi-injective, then I' and I'' are semi-injective.*

Proof. By B.19 there is an equality $I^\natural = E \oplus V$ of graded R-modules, where $\mathrm{Z}(I)^\natural$ is essential in E. As one has $V \cap \mathrm{Z}(I)^\natural = 0$, the differential induces an isomorphism $V \cong \Sigma^1\partial^I(V)$, in particular $\partial^I(V)$ is a graded-injective R-module. As $\partial^I(V)$ is contained in $\mathrm{Z}(I)^\natural$ and hence in E, there is a graded R-module U such that $E = U \oplus \partial^I(V)$; see 5.3.6. The differential ∂^I restricts to a homomorphism from the graded-injective module $V \oplus \partial^I(V)$ to itself. Denote by I'' the subcomplex of I given by $V \oplus \partial^I(V)$ and notice that it is contractible with contraction given by the inverse of the isomorphism $\partial\colon V \to \Sigma\partial^I(V)$. Now it follows from B.1 that there is an equality $I = I' \oplus I''$in $\mathcal{C}(R)$, where I' is isomorphic to the quotient complex $\bar{I} = I/I''$. Since I is a complex of injective modules, so is I'. To prove that the complex $I' \cong \bar{I}$ is minimal, it suffices by B.20 to prove that $\mathrm{Z}(\bar{I})^\natural$ is essential in $\bar{I}^\natural$. Denote by π the canonical map $I \twoheadrightarrow \bar{I}$ and note that its restriction to U is injective. Let $x \neq 0$ be an element in $\bar{I}$ and choose an element $u \in U$ with $\pi(u) = x$. As $\mathrm{Z}(I)^\natural$ is essential in $E = U \oplus \partial^I(V)$, there exists an element r in R such that $ru \neq 0$ is in $\mathrm{Z}(I)$ and, therefore $rx = \pi(ru) \neq 0$ is in $\mathrm{Z}(\bar{I})$. Thus $\mathrm{Z}(\bar{I})$ is essential in $\bar{I}$. This proves the existence of complexes I' and I'' with the desired properties. The assertions (a) and (c) follow from B.7 and 5.3.21, respectively.

(b): The "if" part is B.20. For the converse, assume that I is minimal. By the arguments above, there is an equality of R-complexes $I = I' \oplus I''$, where I'' is contractible and $\mathrm{Z}(I')^\natural$ is essential in $I'^\natural$; in particular, I' is minimal. The surjection $I \twoheadrightarrow I'$ is a homotopy equivalence by B.1 and hence an isomorphism by B.7. Thus, one has $I = I'$ so $\mathrm{Z}(I)^\natural$ is essential in $I^\natural$. ◻

B.22 Corollary. *A complex I of injective R-modules is minimal if and only if the zero complex is the only contractible direct summand of I.*

Proof. The “if” part is immediate from the decomposition $I = I' \oplus I''$ from B.21. and the “only if” follows from B.5. □

Minimal Semi-Injective Resolutions

Every quasi-isomorphism with a semi-injective domain has by 5.3.23/6.3.6 a homotopy left inverse. Under the additional assumption that the domain complex is minimal, such a quasi-isomorphism has a genuine left inverse.

B.23 Proposition. *For an R-complex I, the following conditions are equivalent.*

(*i*) *I is semi-injective and minimal.*

(*ii*) *Every quasi-isomorphism $I \longrightarrow M$ has a left inverse.*

Proof. Assume that I is semi-injective and minimal. If $\beta\colon I \longrightarrow M$ is a quasi-isomorphism, then there exists by 5.3.23 a morphism $\gamma\colon M \longrightarrow I$ with $\gamma\beta \sim 1^I$. Set $\varepsilon = \gamma\beta$; by assumption ε has an inverse, so $\varepsilon^{-1}\gamma$ is a left inverse for β.

Assume that every quasi-isomorphism $I \longrightarrow M$ has a left inverse. In particular, every homotopy equivalence $I \longrightarrow I$ has a left inverse and hence I is minimal by B.6. It follows from 5.3.16 that I is semi-injective. □

B.24 Corollary. *A semi-injective R-complex I is minimal if and only if the zero complex is the only acyclic subcomplex of I.*

Proof. A semi-injective complex I with no non-zero acyclic subcomplex is minimal per B.22. On the other hand, if A is an acyclic subcomplex of I and I is minimal, then the quasi-isomorphism $\pi\colon I \twoheadrightarrow I/A$ has a left inverse by B.23, so π is an isomorphism and $A = 0$. □

B.25 Definition. Let M be an R-complex. A semi-injective resolution $M \xrightarrow{\simeq} I$ is called *minimal* if the semi-injective complex I is minimal.

B.26 Theorem. *Let M be an R-complex. There exists a minimal semi-injective resolution $M \xrightarrow{\simeq} E$; further, E is unique up to isomorphism and has the next properties.*

(a) *$E_v = 0$ holds for all $v > \sup M$.*

(b) *For every semi-injective resolution $M \xrightarrow{\simeq} I$ the complex E is a direct summand of I.*

Proof. By 5.3.26 there is a semi-injective resolution $\iota\colon M \xrightarrow{\simeq} I$ with $I_v = 0$ for $v > \sup M$. By B.21 one has $I = E \oplus I''$, where E is semi-injective and minimal, and I'' is contractible. Let π be the projection $I \twoheadrightarrow E$; by 4.2.6 it is a quasi-isomorphism, as I'' is acyclic. Now the composite $\pi\iota\colon M \xrightarrow{\simeq} E$ is a minimal semi-injective resolution with $E_v = 0$ for all $v > \sup M$. This proves existence and part (a).

Let $M \xrightarrow{\simeq} I$ be any semi-injective resolution. It follows from 5.3.22 that there is a quasi-isomorphism $\alpha\colon E \longrightarrow I$, and by B.23 it has left inverse β. Thus E is a direct summand of I; this proves part (b). The morphism β is a quasi-isomorphism, so if I is minimal, then β has a left inverse by the same argument, and then it is an isomorphism with $\beta^{-1} = \alpha$. This proves the uniqueness statement. □

B.27 Example. If M is an acyclic R-complex, then $M \xrightarrow{\simeq} 0$ is the minimal semi-injective resolution; the morphism is only injective if M is the zero complex.

Nakayama's Lemma

Intersection and sum of submodules are categorically dual notions. Indeed, if M and N are submodules of same module, then the commutative diagram

$$\begin{tikzcd} M \cap N \arrow[r, tail] \arrow[d, tail] & N \arrow[d, tail] \\ M \arrow[r, tail] & N + M \end{tikzcd}.$$

is both a pushout and a pullback square. Thus, the following notion of a superfluous submodule is dual to that of an essential submodule from B.8.

B.28 Definition. Let M be a graded R-module. A graded submodule N of M is called *superfluous* if $N + M' \neq M$ holds for every graded submodule $M' \neq M$ of M.

Remark. Another word for superfluous submodule is 'small' submodule.

B.29 Example. The only superfluous ideal in $\mathbb{Z}$ is 0. The maximal ideal $2\mathbb{Z}/4\mathbb{Z}$ is superfluous in $\mathbb{Z}/4\mathbb{Z}$.

B.30. A graded direct summand N of a graded R-module M is superfluous if and only if $N = 0$.

B.31 Lemma. *Let M be a graded R-module with graded submodules N and N'.*

(a) *Assume that there is an inclusion $N \subseteq N'$. If N is superfluous in N', then N is superfluous in M.*
(b) *If N and N' are superfluous in M, then $N + N'$ is superfluous in M.*
(c) *If $\alpha\colon M \to X$ is a homomorphism of graded R-modules and N is superfluous in M, then $\alpha(N)$ is superfluous in X.*

Proof. (a): If M' is a graded submodule of M such that $N + M' = M$ holds, then one has $N' = (N + M') \cap N' = N + (M' \cap N')$. Thus, if N is superfluous in N', then one has $M' \cap N' = N'$ and, therefore, $N' \subseteq M'$. In particular, N is a submodule of M', whence $M' = N + M' = M$ holds.

(b): If M' is a graded submodule of M such that $(N + N') + M' = N + (N' + M')$ is M, then one has $N' + M' = M$ because N is superfluous in M, and then $M = M'$ because N' is superfluous in M as well.

(c): By part (a) it suffices to show that $\alpha(N)$ is superfluous in $\alpha(M)$, so assume without loss of generality that α is surjective. If X' is a submodule of X such that $\alpha(N) + X' = X$ holds, then one has $N + \alpha^{-1}(X') = M$ and, therefore, $\alpha^{-1}(X') = M$ as N is superfluous in M. Thus, $X' = \alpha(\alpha^{-1}(X')) = \alpha(M) = X$ holds. □

The next result is known as Nakayama's lemma.

B.32 Lemma. *Let $\mathfrak{J}$ denote the Jacobson radical of R. For a left ideal $\mathfrak{a}$ in R the following conditions are equivalent.*

(i) *The left ideal $\mathfrak{a}$ is a superfluous submodule of R.*

(ii) *There is an inclusion $\mathfrak{a} \subseteq \mathfrak{J}$.*

(iii) *For every finitely generated R-module $M \neq 0$ one has $\mathfrak{a}M \neq M$, that is, $R/\mathfrak{a} \otimes_R M \neq 0$.*

(iv) *For every graded R-module M and every graded submodule $N \subseteq M$ such that the quotient $(M/N)_v$ is non-zero and finitely generated for some $v \in \mathbb{Z}$, one has $N + \mathfrak{a}M \neq M$.*

(v) *For every degreewise finitely generated graded R-module M, the submodule $\mathfrak{a}M$ is superfluous.*

Proof. Condition (i) is a special case of (v).

(i) $\Rightarrow$ (ii): If $\mathfrak{a}$ is not contained in $\mathfrak{J}$, then $\mathfrak{a} \not\subseteq \mathfrak{M}$ holds for at least one maximal left ideal $\mathfrak{M}$. Thus one has $\mathfrak{a} + \mathfrak{M} = R$, and hence $\mathfrak{a}$ can not be superfluous in R.

(ii) $\Rightarrow$ (iii): Let $M \neq 0$ be a finitely generated R-module and choose a set of generators $\{m_1, \dots, m_t\}$ for M with t least possible. Assume towards a contradiction that one has $\mathfrak{a}M = M$. Then there exist elements $a_1, \dots, a_t$ in $\mathfrak{a}$ such that $m_1 = \sum_{i=1}^{t} a_i m_i$ holds. Since a_1 is in $\mathfrak{J}$, the element $1 - a_1$ is invertible, whence m_1 is a linear combination of $m_2, \dots, m_t$, which contradicts the minimality of t.

(iii) $\Rightarrow$ (iv): Assume that $(M/N)_v$ is non-zero and finitely generated. By (iii) one has $\mathfrak{a}(M/N)_v \neq (M/N)_v$ and, therefore $(N + \mathfrak{a}M)_v \neq M_v$.

(iv) $\Rightarrow$ (v): For every proper graded submodule $N \subset M$ it follows from (iv) that $N + \mathfrak{a}M \neq M$ holds. Thus, $\mathfrak{a}M$ is superfluous in M. □

Projective Covers

B.33 Definition. A *projective cover* of a graded R-module M is a surjective morphism $\pi\colon P \to M$ of graded R-modules where P is graded-projective and $\operatorname{Ker}\pi$ is superfluous in P.

B.34 Example. Let P be a graded-projective R-module. The identity morphism 1^P is a projective cover of P. Moreover, if P is degreewise finitely generated, then it follows from Nakayama's lemma B.32 that the canonical map $P \twoheadrightarrow P/\mathfrak{a}P$ is a projective cover for every left ideal $\mathfrak{a}$ contained in the Jacobson radical of R.

The notion of a projective cover is dual to that of an injective envelope. As established in B.17, injective envelopes exist for all graded modules, however, projective covers do not always exist. In fact, a ring over which every graded module has a projective cover is perfect and vice versa; see B.53.

B.35 Lemma. *Let M be a graded R-module with a projective cover $\pi\colon P \twoheadrightarrow M$ and $\pi'\colon P' \to M$ a surjective morphism with P' graded-projective. There is a morphism $\gamma\colon P \to P'$ with $\pi = \pi'\gamma$, and for every such morphism the following hold.*

(a) *The morphism γ has a left inverse.*

(b) *If π' is a projective cover, then γ is an isomorphism.*

For the direct summand $P'' = \gamma(P)$ of P', the restriction $\pi'|_{P''} \colon P'' \twoheadrightarrow M$ is a projective cover, and there is an equality of graded modules, $P' = P'' \oplus K$, where K is contained in $\operatorname{Ker}\pi'$.

PROOF. By 5.2.2 there exist morphisms $\gamma \colon P \to P'$ and $\gamma' \colon P' \to P$ with $\pi = \pi'\gamma$ and $\pi' = \pi\gamma'$. Set $\varepsilon = \gamma'\gamma$. Then one has $\pi\varepsilon = \pi$, so there is an equality $P = \varepsilon(P) + \operatorname{Ker}\pi$. As $\operatorname{Ker}\pi$ is superfluous in P, it follows that ε is surjective. Since P is graded-projective, $\operatorname{Ker}\varepsilon$ is a direct summand of P; again by 5.2.2. By B.31, the module $\operatorname{Ker}\varepsilon$ is superfluous as it is contained in $\operatorname{Ker}\pi$, whence it is zero. Thus, ε is an isomorphism, and $\varepsilon^{-1}\gamma'$ is a left inverse of γ. This proves part (a), and with $P'' = \gamma(P)$ and $K = \operatorname{Ker}\gamma'$ it follows that there is an equality $P' = P'' \oplus K$. The submodule $P'' \cap \operatorname{Ker}\pi'$ is superfluous in P'', because the isomorphism $\gamma'|_{P''} \colon P'' \to P$ maps it to $\operatorname{Ker}\pi$, which is superfluous in P. Thus, $\pi'|_{P''}$ is a projective cover. Moreover, the equality $\pi' = \pi\gamma'$ implies that K is contained in $\operatorname{Ker}\pi'$.

To prove part (b), note that one has $P'' + \operatorname{Ker}\pi' = P'$. Thus, if π' is a projective cover, then $P' = P''$ holds, whence γ is an isomorphism. ◻

As already mentioned, a module might not have a projective cover, however, any two covers of a given module are necessarily isomorphic.

B.36 Proposition. *Let M be a graded R-module. If $\pi \colon P \twoheadrightarrow M$ and $\pi' \colon P' \twoheadrightarrow M$ are projective covers, then there exists an isomorphism $\gamma \colon P \to P'$ with $\pi'\gamma = \pi$. Moreover, if M is degreewise finitely generated, then so are P and P'.*

PROOF. The existence of an isomorphism γ with $\pi'\gamma = \pi$ is part of B.35. It also follows from B.35 that if $P \twoheadrightarrow M$ is a projective cover and $P' \to M$ is any surjective morphism with P' graded-projective, then P is isomorphic to a graded direct summand of P'. If M is degreewise finitely generated, then P' can be chosen degreewise finitely generated by 2.5.28, and hence the direct summand P is degreewise finitely generated as well. ◻

B.37 Lemma. *Let $\mathfrak{J}$ denote the Jacobson radical of R and $P \neq 0$ be a graded-projective R-module. One has $\mathfrak{J}P \neq P$, and every superfluous submodule of P is contained in $\mathfrak{J}P$. In particular, if M is a graded R-module and $\pi \colon P \twoheadrightarrow M$ is a projective cover, then the induced morphism $\bar{\pi} \colon P/\mathfrak{J}P \to M/\mathfrak{J}M$ is an isomorphism.*

PROOF. By 5.2.2 the module P is a graded direct summand of a graded free R-module L. Let E be a graded basis for L. Fix a homogeneous element $p \neq 0$ in P; it is a unique linear combination of basis elements: $p = \sum_{i=1}^{m} r_i e_i$. Let ε be the composition of canonical morphisms $L \twoheadrightarrow P \rightarrowtail L$. For each $i \in \{1,\dots,m\}$ write $\varepsilon(e_i) = \sum_{j=1}^{n} a_{ij} e_j$, where also $e_{m+1},\dots,e_n$ are elements in E. Suppose the equality $P = \mathfrak{J}P$ holds, then all the elements a_{ij} belong to $\mathfrak{J}$. The equality $p = \varepsilon(p)$ yields

$$(\star) \qquad \sum_{i=1}^{m} r_i e_i = \sum_{i=1}^{m} r_i \Big(\sum_{j=1}^{n} a_{ij} e_j \Big) .$$

Let I_m denote the $m \times m$ identity matrix and A be the $m \times m$ matrix with entries a_{ij} for $i, j \in \{1,\dots,m\}$. From $(\star)$ one gets the equality $(r_1,\dots,r_m)(I_m - A) = 0$. It is elementary to verify that the Jacobson radical of the matrix ring $\mathrm{M}_{m\times m}(R)$ contains (in fact, equality holds) the ideal $\mathrm{M}_{m\times m}(\mathfrak{J})$. Since A has entries in $\mathfrak{J}$, it follows that $I_m - A$ is invertible in $\mathrm{M}_{m\times m}(R)$. Hence $(r_1,\dots,r_m)$ is the zero row and one has $p = 0$, a contradiction.

Let N be a superfluous graded submodule of P and thereby of L; cf. B.31. It follows that N is contained in every maximal submodule of L. In particular, N is contained in the module $\mathfrak{J}e' + R\langle E \setminus \{e'\}\rangle$ for every $e' \in E$. Indeed, this is the intersection of the maximal submodules $\mathfrak{M}e' + R\langle E \setminus \{e'\}\rangle$, where $\mathfrak{M}$ is a maximal left ideal in R. Thus, for an element $x = \sum_{e\in E} r_e e$ in N one has $r_e \in \mathfrak{J}$ for all $e \in E$. It follows that N is contained in $\mathfrak{J}L \cap P = \mathfrak{J}P$, where the equality holds because P is a direct summand of L.

The last assertion is now immediate as the kernel of a cover $P \twoheadrightarrow M$ is superfluous in P and hence contained in $\mathfrak{J}P$. $\square$

REMARK. For a graded-projective R-module P, the submodule $\mathfrak{J}P$ itself may not be superfluous; see E B.16.

SEMI-PERFECT MODULES

B.38 Definition. A graded R-module M is called *semi-perfect* if every homomorphic image of M has a projective cover.

The notion of semi-perfect modules is an auxiliary; its utility comes to fore in Theorems B.46 and B.53 and, implicitly, in Theorems B.60 and B.61.

B.39 Example. If R is semi-simple, then every R-module is projective by 1.3.28, and it follows that every R-module is semi-perfect.

The $\mathbb{Z}$-module $\mathbb{Z}$ has a projective cover, namely $1^{\mathbb{Z}}$, but it is not semi-perfect, since the quotient $\mathbb{Z}/n\mathbb{Z}$ has no projective cover for $n > 1$. Indeed, suppose that $\pi\colon P \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is a projective cover and let $\pi'\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ be the canonical map. By B.35 there is an isomorphism of $\mathbb{Z}$-modules, $\mathbb{Z} \cong P \oplus K$, and since $\mathbb{Z}$ is indecomposable, it follows that $K = 0$, $P = \mathbb{Z}$, and $\pi = \pi'$. However, $\pi'\colon \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is not a projective cover since its kernel is not superfluous; see B.29.

B.40 Lemma. *Let $\mathfrak{J}$ denote the Jacobson radical of R and M be a graded R-module. If M is semi-perfect, then the following assertions hold.*

(a) *If M is non-zero, then $\mathfrak{J}M \neq M$ holds.*
(b) *The submodule $\mathfrak{J}M$ is superfluous in M.*
(c) *If $P \twoheadrightarrow M$ is a projective cover, then P is semi-perfect.*
(d) *The graded $R/\mathfrak{J}$-module $M/\mathfrak{J}M$ is semi-simple.*

Let $\alpha\colon L \longrightarrow M$ be a morphism of graded R-modules and denote by $\bar{\alpha}$ the induced morphism $L/\mathfrak{J}L \longrightarrow M/\mathfrak{J}M$ of $R/\mathfrak{J}$-modules.

(e) *If $\bar{\alpha}$ is surjective, then α is surjective.*
(f) *If $\bar{\alpha}$ is bijective and L and M are graded-projective, then α is bijective.*

PROOF. Let $\pi\colon P \twoheadrightarrow M$ be a projective cover.

(a): If $\mathfrak{J}M = M$ holds, then one has $\pi(\mathfrak{J}P) = M$, whence there is an equality $\operatorname{Ker}\pi + \mathfrak{J}P = P$. As P is graded-projective and $\operatorname{Ker}\pi$ is superfluous in P, it follows from B.37 that P is zero, whence $M = 0$.

(b): Let M' be a graded submodule of M such that the equality $\mathfrak{J}M + M' = M$ holds. The quotient module $N = M/M'$ then satisfies $\mathfrak{J}N = N$. By assumption the module N is semi-perfect, so part (a) yields $N = 0$, whence one has $M' = M$.

(c): To prove that P is semi-perfect, it must be show that P/N has a projective cover for every graded submodule N of P. Let N be such a submodule and set $K = \operatorname{Ker}\pi$, then $P/(N+K)$ is a homomorphic image of $P/K \cong M$, so there is a projective cover $\pi'\colon P' \twoheadrightarrow P/(N+K)$. As P' is graded-projective, there exists by 5.2.2 a morphism $\gamma\colon P' \longrightarrow P/N$ with $\pi' = \beta\gamma$, where β is the canonical morphism $P/N \twoheadrightarrow P/(N+K)$. To see that γ is a projective cover, notice first that $\operatorname{Ker}\beta = (N+K)/N$ is a superfluous submodule of P/N, because it is the image of the superfluous submodule K of P; see B.31. On the other hand, since π' is surjective, one has $\gamma(P') + \operatorname{Ker}\beta = P/N$, so γ is surjective. Finally, $\operatorname{Ker}\gamma$ is contained in $\operatorname{Ker}\pi'$, which is superfluous in P'.

(d): The morphism of graded $R/\mathfrak{J}$-modules $P/\mathfrak{J}P \longrightarrow M/\mathfrak{J}M$, induced by π, is surjective; it is, therefore, sufficient to prove that $P/\mathfrak{J}P$ is semi-simple. To that end, let $X/\mathfrak{J}P$ be a proper graded submodule of $P/\mathfrak{J}P$; the goal is to show that this submodule is a graded direct summand. The module P/X has a projective cover $\varkappa\colon F \twoheadrightarrow P/X$ by part (c). Let β be the canonical morphism $P \twoheadrightarrow P/X$. By B.35 there is a morphism $\gamma\colon F \longrightarrow P$ with $\varkappa = \beta\gamma$ and $P = \gamma(F) \oplus K$, where K is contained in $\operatorname{Ker}\beta = X$ and the restriction of β to $\gamma(F)$ is a projective cover. Replacing $\varkappa$ with $\beta|_{\gamma(F)}$ one has $P = F \oplus K$ and, therefore,

$$\frac{P}{\mathfrak{J}P} = \frac{F + \mathfrak{J}P}{\mathfrak{J}P} + \frac{X}{\mathfrak{J}P}\,. \tag{$\mathfrak{b}$}$$

The submodule $F \cap X$ is contained in $\operatorname{Ker}\varkappa$, so it is superfluous in F and hence contained in $\mathfrak{J}F \subseteq \mathfrak{J}P$ by B.37. Thus, the sum in ($\mathfrak{b}$) is direct; indeed, one has

$$\frac{F + \mathfrak{J}P}{\mathfrak{J}P} \cap \frac{X}{\mathfrak{J}P} = \frac{(F + \mathfrak{J}P) \cap X}{\mathfrak{J}P} = \frac{(F \cap X) + \mathfrak{J}P}{\mathfrak{J}P} = 0\,.$$

(e): Set $C = \operatorname{Coker}\alpha$ and consider the exact sequence $L \xrightarrow{\alpha} M \longrightarrow C \longrightarrow 0$ of graded R-modules. It induces an exact sequence $L/\mathfrak{J}L \xrightarrow{\bar{\alpha}} M/\mathfrak{J}M \longrightarrow C/\mathfrak{J}C \longrightarrow 0$ with $C/\mathfrak{J}C = 0$ by assumption. The graded module C is semi-perfect, because it is a homomorphic image of M. Thus, part (a) yields $C = 0$, whence α is surjective.

(f): It follows from part (e) that α is surjective. Set $K = \operatorname{Ker}\alpha$ and consider the exact sequence $0 \longrightarrow K \longrightarrow L \xrightarrow{\alpha} M \longrightarrow 0$ of graded R-modules. By assumption, L and M are graded-projective, so the sequence is split and K is graded-projective by 5.2.2 and 5.2.3. Now the induced sequence $0 \longrightarrow K/\mathfrak{J}K \longrightarrow L/\mathfrak{J}L \xrightarrow{\bar{\alpha}} M/\mathfrak{J}M \longrightarrow 0$ is exact. By assumption $\bar{\alpha}$ is injective, so one has $K/\mathfrak{J}K = 0$. Lemma B.37 now yields $K = 0$, so α is injective. □

B.41 Lemma. *Let $\mathfrak{J}$ denote the Jacobson radical of R and set $\boldsymbol{k} = R/\mathfrak{J}$. Consider every $\boldsymbol{k}$-module as an R-module via the canonical homomorphism $\kappa\colon R \twoheadrightarrow \boldsymbol{k}$.*

(a) *Let $e \in R$ be an idempotent and set $u = \kappa(e)$. The map $\varkappa^u : Re \twoheadrightarrow \boldsymbol{k}u$ given by $re \mapsto \kappa(r)u$ is a projective cover of the R-module $\boldsymbol{k}u$, and* $\operatorname{Ker} \varkappa^u = \mathfrak{J}e$ *holds.*

(b) *If a graded $\boldsymbol{k}$-module generated by a homogeneous element u has a projective cover as a graded R-module, then it has one of the form $\varkappa^u : \Sigma^{|u|} Re_u \twoheadrightarrow \boldsymbol{k}u$, where e_u is an idempotent in R.*

(c) *Let M be a graded R-module such that the graded $\boldsymbol{k}$-module $M/\mathfrak{J}M$ is semi-simple. Let U be a set of homogeneous elements with $M/\mathfrak{J}M = \coprod_{u \in U} \boldsymbol{k}u$. Assume that each R-module $\boldsymbol{k}u$ has a projective cover $\varkappa^u : \Sigma^{|u|} Re_u \twoheadrightarrow \boldsymbol{k}u$ with e_u an idempotent in R. The R-module $F = \coprod_{u \in U} \Sigma^{|u|} Re_u$ is graded-projective, and the morphism $\varkappa = \coprod_{u \in U} \varkappa^u : F \longrightarrow M/\mathfrak{J}M$ is surjective with* $\operatorname{Ker} \varkappa = \mathfrak{J}F$.

Proof. (a): Let e be an idempotent in R. The ideal Re is a projective R-module as one has $R = Re \oplus R(1-e)$. The inclusion $\mathfrak{J}e \subseteq \operatorname{Ker} \varkappa^u$ holds by the definition of $\varkappa^u$. To prove the reverse inclusion, let re be an element in $\operatorname{Ker} \varkappa^u$. The equalities $0 = \kappa(r)u = \kappa(re)$ in $\boldsymbol{k}u$ show that re is in $\mathfrak{J}$, and since e is an idempotent one has $re = (re)e \in \mathfrak{J}e$. By Nakayama's lemma B.32, the Jacobson radical $\mathfrak{J}$ is superfluous in R, so $\mathfrak{J}e = \operatorname{Ker} \varkappa^u$ is superfluous in Re by B.31. Thus, $\varkappa^u$ is a projective cover.

(b): Let $\pi : P \twoheadrightarrow \boldsymbol{k}u$ be a projective cover of $\boldsymbol{k}u$ as a graded R-module. Set $P' = \Sigma^{|u|} R$ and let p denote the generator 1 in P'. Let $\pi' : P' \longrightarrow \boldsymbol{k}u$ be the surjective morphism of graded R-modules that maps p to u. By B.35 there is a morphism $\gamma : P \longrightarrow P'$ with $\pi = \pi'\gamma$, $P'' = \gamma(P)$, and $P' = P'' \oplus K$, such that $\varkappa^u = \pi'|_{P''} : P'' \twoheadrightarrow \boldsymbol{k}u$ is a projective cover. Restrict the codomain of γ so that it becomes an isomorphism $\gamma : P \longrightarrow P''$ and let $\upsilon : P' \longrightarrow P$ denote the canonical morphism with $\upsilon\gamma = 1^P$. One has $\gamma\upsilon(p) = ep$ for some $e \in R$. The identity $\gamma\upsilon = \gamma 1^P \upsilon = (\gamma\upsilon)(\gamma\upsilon)$ yields $ep = e^2 p$, so e is an idempotent. The equality $R = Re \oplus R(1-e)$ now yields $P'' = \Sigma^{|u|} Re$, so $\varkappa^u$ is the desired projective cover.

(c): Each graded R-module $\Sigma^{|u|} Re_u$ is graded-projective as e_u is an idempotent. It follows from 5.2.12 and 5.2.18 that F is graded-projective, and $\varkappa$ is surjective by construction. By 3.1.6 and (a) one has $\operatorname{Ker} \varkappa = \coprod_{u \in U} \operatorname{Ker} \varkappa^u = \coprod_{u \in U} \mathfrak{J}e_u = \mathfrak{J}F$. □

B.42 Theorem. *Let M be a semi-perfect graded R-module. If $P \twoheadrightarrow M$ is a projective cover, then P is isomorphic to a module of the form $\coprod_{u \in U} \Sigma^{n_u} Re_u$, where each e_u is an idempotent in R.*

Proof. Let $\mathfrak{J}$ denote the Jacobson radical of R and set $\boldsymbol{k} = R/\mathfrak{J}$. By parts (c) and (d) in B.40 the graded R-module P is semi-perfect, and the $\boldsymbol{k}$-module $P/\mathfrak{J}P$ is semi-simple. By a standard application of Zorn's lemma, choose a set U of homogeneous elements with $P/\mathfrak{J}P = \coprod_{u \in U} \boldsymbol{k}u$. Each graded module $\boldsymbol{k}u$ is a homomorphic image of P and, therefore, it has a projective cover. By B.41 it then has a projective cover of the form $\Sigma^{|u|} Re_u \twoheadrightarrow \boldsymbol{k}u$, where e_u is an idempotent in R. Now it follows, still from B.41, that there is a surjective morphism $\varkappa : F \longrightarrow P/\mathfrak{J}P$ with $\operatorname{Ker} \varkappa = \mathfrak{J}F$, where F is the graded-projective R-module $\coprod_{u \in U} \Sigma^{|u|} Re_u$. By graded-projectivity of F, there is a morphism $\gamma : F \longrightarrow P$ such that the composite $F \xrightarrow{\gamma} P \twoheadrightarrow P/\mathfrak{J}P$ equals $\varkappa$; see 5.2.2. The induced morphism $\bar{\varkappa} : F/\mathfrak{J}F \longrightarrow P/\mathfrak{J}P$ is an isomorphism; it follows that also $\bar{\gamma} : F/\mathfrak{J}F \longrightarrow P/\mathfrak{J}P$ is an isomorphism, so γ is an isomorphism by B.40(f). □

SEMI-PERFECT RINGS

B.43 Definition. Let $\mathfrak{J}$ denote the Jacobson radical of R. The ring R is called *semi-perfect* if $R/\mathfrak{J}$ is semi-simple and idempotents lift from $R/\mathfrak{J}$ to R.

REMARK. Since semi-simplicity is a left–right symmetric property so is semi-perfection. A ring R with $R/\mathfrak{J}$ semi-simple is called *semi-local*. A commutative ring is semi-local if and only it it has finitely many maximal ideals.

B.44 Example. If R is local with unique maximal ideal $\mathfrak{m}$, then $\boldsymbol{k} = R/\mathfrak{m}$ is a division ring, so 1 and 0 are the only idempotents in $\boldsymbol{k}$. Moreover, $\boldsymbol{k}$ is simple, so R is semi-perfect.

If R is left (or right) Artinian with Jacobson radical $\mathfrak{J}$, then $R/\mathfrak{J}$ is semi-simple. Moreover, if r is in R and $[r]_{\mathfrak{J}}$ is an idempotent, then $r - r^2$ is in $\mathfrak{J}$. Since $\mathfrak{J}$ is nilpotent, there is an integer $n \geqslant 1$ with $(r-r^2)^n = 0$. Powers of r commute, so one has $0 = (r(1-r))^n = r^n(1-r)^n = r^n - r^{n+1}x$ for an element x with $rx = xr$. It follows that $(rx)^n = r^n x^n = r^{n+1}x^{n+1} = (rx)^{n+1}$ holds, so the element $e = (rx)^n$ is an idempotent in R. One has $[r]_{\mathfrak{J}} = [r^n]_{\mathfrak{J}} = [r^{n+1}x]_{\mathfrak{J}} = [r^{n+1}]_{\mathfrak{J}}[x]_{\mathfrak{J}} = [rx]_{\mathfrak{J}}$ and hence $[e]_{\mathfrak{J}} = [rx]_{\mathfrak{J}}^n = [r]_{\mathfrak{J}}^n = [r]_{\mathfrak{J}}$. Thus R is semi-perfect.

REMARK. A ring R is semi-perfect with $R/\mathfrak{J}$ simple if and only if it is isomorphic to a matrix ring $\mathrm{M}_{n\times n}(S)$ where S is local; see [168, §23]. A commutative ring is semi-perfect if and only if it is a finite product of commutative local rings; see [168, §23].

B.45 Lemma. *Let $\mathfrak{J}$ denote the Jacobson radical of R and M be a graded R-module. If R is semi-perfect, then there exists a graded-projective R-module F and a surjective morphism $\varkappa\colon F \to M/\mathfrak{J}M$ with $\operatorname{Ker}\varkappa = \mathfrak{J}F$. Moreover, if M is degreewise finitely generated, then one can choose F degreewise finitely generated.*

PROOF. Set $\boldsymbol{k} = R/\mathfrak{J}$ and consider the graded $\boldsymbol{k}$-module $M/\mathfrak{J}M$. By assumption, $\boldsymbol{k}$ is semi-simple, so by a standard application of Zorn's lemma one can choose a set U of homogeneous elements with $M/\mathfrak{J}M = \coprod_{u\in U} \Sigma^{|u|}\boldsymbol{k}u$. As every cyclic $\boldsymbol{k}$-module is a direct sum of simple modules generated by idempotents, one can assume that each u is an idempotent in $\boldsymbol{k}$. By assumption, each u now lifts to an idempotent e_u in R, so by B.41 the canonical surjection $\varkappa^u\colon Re_u \to \boldsymbol{k}u$ is a projective cover of $\boldsymbol{k}u$, and the desired morphism is $\varkappa = \coprod_{u\in U}\varkappa^u$. Finally, if M is degreewise finitely generated, then one can choose U with only finitely many elements of each degree. □

We can now link semi-perfectness of rings to existence of projective covers.

B.46 Theorem. *The following conditions are equivalent.*

(*i*) *R is semi-perfect.*
(*ii*) *Every degreewise finitely generated graded R-module has a projective cover.*
(*iii*) *Every degreewise finitely generated graded R-module is semi-perfect.*
(*iv*) *The R-module R is semi-perfect.*

Moreover, if R is semi-perfect and $P \twoheadrightarrow M$ is a projective cover of a degreewise finitely generated graded R-module, then P is degreewise finitely generated.

Proof. Let $\mathfrak{J}$ denote the Jacobson radical of R and set $\boldsymbol{k} = R/\mathfrak{J}$. The implication (*ii*) $\Rightarrow$ (*iii*) follows from the definition of semi-perfect modules B.38, and (*iv*) is a special case of (*iii*).

(*i*) $\Rightarrow$ (*ii*): Let M be a degreewise finitely generated graded R-module. By B.45 there exists a graded-projective and degreewise finitely generated R-module F and surjective morphism $\varkappa\colon F \longrightarrow M/\mathfrak{J}M$ with $\operatorname{Ker}\varkappa = \mathfrak{J}F$. Let β be the canonical morphism $M \twoheadrightarrow M/\mathfrak{J}M$; by 5.2.2 there exists a morphism $\gamma\colon F \longrightarrow M$ with $\beta\gamma = \varkappa$. Thus, one has $M = \gamma(F) + \mathfrak{J}M$. It follows from Nakayama's lemma B.32 that $\mathfrak{J}M$ is superfluous in M, so γ is surjective. To see that γ is a projective cover, it remains to verify that $\operatorname{Ker}\gamma$ is superfluous in F. This follows as $\operatorname{Ker}\gamma$ is contained in $\operatorname{Ker}\varkappa = \mathfrak{J}F$, which is a superfluous submodule by Nakayama's lemma. As a projective cover is unique up to isomorphism, see B.36, the final assertion in the theorem also follows.

(*iv*) $\Rightarrow$ (*i*): By B.40(d) the $\boldsymbol{k}$-module $\boldsymbol{k}$ is semi-simple, so $\boldsymbol{k}$ is a semi-simple ring. From B.40(b) it follows that $\mathfrak{J}$ is superfluous in R, whence the canonical map $R \twoheadrightarrow \boldsymbol{k}$ is a projective cover. Let u be an idempotent in $\boldsymbol{k}$; the goal is to lift u to R. There is an equality $\boldsymbol{k} = \boldsymbol{k}u \oplus \boldsymbol{k}(1-u)$. As R is a semi-perfect R-module, its homomorphic images $\boldsymbol{k}u$ and $\boldsymbol{k}(1-u)$ have projective covers $\pi\colon P \twoheadrightarrow \boldsymbol{k}u$ and $\pi'\colon P' \twoheadrightarrow \boldsymbol{k}(1-u)$. The morphism $\pi \oplus \pi'$ is a projective cover of $\boldsymbol{k}$. By B.35 there is an isomorphism $\gamma\colon P \oplus P' \longrightarrow R$ such that γ followed by the canonical map $R \twoheadrightarrow \boldsymbol{k}$ is $\pi \oplus \pi'$. Thus, the modules P and P' are isomorphic to ideals $\mathfrak{e}$ and $\mathfrak{e}'$ in R, and one has $R = \mathfrak{e} \oplus \mathfrak{e}'$. Choose elements $e \in \mathfrak{e}$ and $e' \in \mathfrak{e}'$ with $1 = e + e'$ in R. It is elementary to verify that e and e' are orthogonal idempotents and that e maps to u in $\boldsymbol{k}$. $\square$

Remark. Every finitely generated flat module over a semi-perfect ring is projective; see E B.21. It is a result of Kaplansky [154] that the finiteness hypothesis in the next corollary can be omitted.

B.47 Corollary. *Let R be local. A degreewise finitely generated graded-projective R-module is graded-free.*

Proof. Let P be a degreewise finitely generated and graded-projective R-module. The identity 1^P is a projective cover of P. By B.46 the graded module P is semi-perfect, so by B.42 it has the form $\coprod_{u \in U} \Sigma^{n_u} Re_u$, where each e_u is an idempotent in R. As R is local, 1 and 0 are the only idempotents in R, so P is graded-free. $\square$

Perfect Rings

B.48 Definition. A left ideal $\mathfrak{a}$ in R is called *left T-nilpotent* if for every sequence of elements $(a_i)_{i\in\mathbb{N}}$ in $\mathfrak{a}$ there is a number $n \in \mathbb{N}$ such that $a_1a_2\cdots a_n = 0$ holds.

Every nilpotent left ideal is left T-nilpotent; in particular, the Jacobson radical of a left (or right) Artinian ring is left T-nilpotent. A T-nilpotent ideal has properties similar to the Jacobson radical as captured by Nakayama's lemma.

B.49 Lemma. *For a left ideal $\mathfrak{a}$ in R, the following conditions are equivalent.*

(*i*) *The left ideal $\mathfrak{a}$ is left T-nilpotent.*

(*ii*) *For every R^{o}-module $N \neq 0$ one has $(0 :_N \mathfrak{a}) \neq 0$.*
(*iii*) *For every R-module $M \neq 0$ one has $\mathfrak{a}M \neq M$.*
(*iv*) *For every graded R-module M and every proper graded submodule $M' \subset M$ one has $M' + \mathfrak{a}M \neq M$.*
(*v*) *For every graded R-module M the submodule $\mathfrak{a}M$ is superfluous.*

Proof. (*i*) $\Rightarrow$ (*ii*): Assume that there exists an R^{o}-module $N \neq 0$ with $(0 :_N \mathfrak{a}) = 0$. For every element $x \neq 0$ in N there exists then an element $a \in \mathfrak{a}$ with $xa \neq 0$. It follows by induction that there exists a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathfrak{a}$ with $a_1a_2\cdots a_n \neq 0$ for every $n \in \mathbb{N}$. Thus, $\mathfrak{a}$ is not left T-nilpotent.

(*ii*) $\Rightarrow$ (*iii*): Let $M \neq 0$ be an R-module; consider the ideal $\mathfrak{b} = (0 :_R M)$ and the right ideal $\mathfrak{B} = (\mathfrak{b} :_R \mathfrak{a}) = \{r \in R \mid r\mathfrak{a} \subseteq \mathfrak{b}\}$. Now (*ii*) yields $\mathfrak{B}/\mathfrak{b} = (0 :_{R/\mathfrak{b}} \mathfrak{a}) \neq 0$. It follows that $\mathfrak{b}$ is strictly contained in $\mathfrak{B}$, whence $\mathfrak{B}M$ is non-zero. However, one has $\mathfrak{B}\mathfrak{a} \subseteq \mathfrak{b}$ and, therefore, $\mathfrak{B}(\mathfrak{a}M) = 0$, so $\mathfrak{a}M \neq M$ holds.

(*iii*) $\Rightarrow$ (*iv*): Apply (*iii*) to the non-zero R-module M/M'.

(*iv*) $\Rightarrow$ (*v*): This implication follows immediately from the definition, B.28, of superfluous submodules.

(*v*) $\Rightarrow$ (*i*): Let a sequence $(a_i)_{i\in\mathbb{N}}$ of elements in $\mathfrak{a}$ be given. For $j > i \geqslant 1$ let α^{ji} be the homothety given by right multiplication on R with $a_i\cdots a_{j-1}$ and set $\alpha^{ii} = 1^R$. These maps form a direct system of R-modules; let A denote its colimit. For every element a in A there is a ring element r and an integer $i \geqslant 1$ with $a = \alpha^i(r)$, where α^i is the canonical morphism $R \to A$; see 3.3.2. From the equalities $\alpha^i(r) = \alpha^{i+1}(ra_i) = ra_i\alpha^{i+1}(1)$ one gets $\mathfrak{a}A = A$. By (*v*) the submodule $\mathfrak{a}A$ is superfluous in A, so the colimit A is zero. In particular, $\alpha^1(1)$ is zero, so by 3.3.2 one has $\alpha^{j1}(1) = a_1\cdots a_{j-1} = 0$ some $j > 1$. Thus, $\mathfrak{a}$ is left T-nilpotent. $\square$

B.50 Definition. Let $\mathfrak{J}$ be the Jacobson radical of R. The ring R is called *left perfect* if $R/\mathfrak{J}$ is semi-simple and $\mathfrak{J}$ is left T-nilpotent.

B.51 Example. As a nilpotent ideal is left T-nilpotent, every left (or right) Artinian ring is left perfect.

Remark. Perfectness is not a left–right symmetric property; Bass gives an example in [29]. It is a consequence of Livitzki's theorem that a left perfect and left Noetherian ring is left Artinian; see E B.28. Björk [41] proves that a left perfect and right Noetherian ring is right Artinian. See E 5.5.22 for an example of a commutative perfect ring that is not Artinian and hence not Noetherian.

B.52. If the Jacobson radical $\mathfrak{J}$ of R is left T-nilpotent, then every element in $\mathfrak{J}$ is nilpotent, whence idempotents lift from $R/\mathfrak{J}$ to R; see B.44. Thus, every left perfect ring is semi-perfect.

B.53 Theorem. *The following conditions are equivalent.*

(*i*) *R is left perfect.*
(*ii*) *Every graded R-module has a projective cover.*
(*iii*) *Every graded R-module is semi-perfect.*

Proof. Let $\mathfrak{J}$ denote the Jacobson radical of R and set $\boldsymbol{k} = R/\mathfrak{J}$. The implication (*ii*) $\Rightarrow$ (*iii*) follows from the definition B.38 of semi-perfect modules.

$(i) \Rightarrow (ii)$: Let M be a graded R-module. Since R is semi-perfect, it follows from B.45 that there exists a graded-projective R-module F and a surjective morphism $\varkappa\colon F \to M/\mathfrak{J}M$ with $\operatorname{Ker}\varkappa = \mathfrak{J}F$. Let β be the canonical morphism $M \twoheadrightarrow M/\mathfrak{J}M$; by 5.2.2 there is a morphism $\gamma\colon F \to M$ with $\beta\gamma = \varkappa$. Thus, one has $M = \gamma(F) + \mathfrak{J}M$. It follows from B.49 that $\mathfrak{J}M$ is superfluous in M, so γ is surjective. To see that γ is a projective cover, it remains to verify that $\operatorname{Ker}\gamma$ is superfluous in F. This follows as $\operatorname{Ker}\gamma$ is contained in $\operatorname{Ker}\varkappa = \mathfrak{J}F$, which is a superfluous submodule by B.49.

$(iii) \Rightarrow (i)$: It follows from B.40(d) that $\boldsymbol{k}$ is semi-simple as a $\boldsymbol{k}$-module; i.e. it is a semi-simple ring. For every graded R-module M the submodule $\mathfrak{J}M$ is superfluous in M by B.40(b), so $\mathfrak{J}$ is left T-nilpotent by B.49. Thus, R is left perfect. ◻

Another homological characterization of perfect rings is given in 5.5.30.

Minimal Complexes of Projective Modules

For a complex of projective modules, minimality means that the subcomplex of boundaries is superfluous.

B.54 Lemma. *Let $\mathfrak{J}$ be the Jacobson radical of R and P a complex of projective R-modules. If $P^\natural$ is semi-perfect and $\partial^P(P) \subseteq \mathfrak{J}P$ holds, then P is minimal.*

Proof. Let $\varepsilon\colon P \to P$ be a morphism with $\varepsilon \sim 1^P$; it suffices by B.6 to prove that ε is an isomorphism. By assumption there exists a homomorphism $\sigma\colon P \to P$ of degree 1 such that $1^P - \varepsilon = \partial^P\sigma + \sigma\partial^P$ holds. For every $p \in P$, the element $p - \varepsilon(p)$ belongs to $\mathfrak{J}P + \sigma(\mathfrak{J}P) = \mathfrak{J}P$. It follows that the induced morphism $\bar{\varepsilon}\colon P/\mathfrak{J}P \to P/\mathfrak{J}P$ is the identity $1^{P/\mathfrak{J}P}$, so ε is an isomorphism by B.40(f). ◻

B.55 Theorem. *Let $\mathfrak{J}$ denote the Jacobson radical of R and P be a complex of projective R-modules such that $P^\natural$ is semi-perfect. There is an equality $P = P' \oplus P''$, where P' and P'' are complexes of projective R-modules, P' is minimal, and P'' is contractible. Moreover, the following assertions hold.*

(a) *The complex P' is unique in the following sense: if one has $P = F' \oplus F''$, where F' is minimal and F'' is contractible, then F' is isomorphic to P'.*
(b) *P is minimal if and only if $\mathrm{B}(P)^\natural$ is superfluous in $P^\natural$ if and only if the inclusion $\partial^P(P) \subseteq \mathfrak{J}P$ holds.*
(c) *If P is semi-projective, then P' and P'' are semi-projective.*

Proof. Set $\boldsymbol{k} = R/\mathfrak{J}$; the graded $\boldsymbol{k}$-module $(P/\mathfrak{J}P)^\natural$ is semi-simple by B.40(d). Set $B = \mathrm{B}(P/\mathfrak{J}P)$ and $H = \mathrm{H}(P/\mathfrak{J}P)$; by 4.2.17 there is a split exact sequence of $\boldsymbol{k}$-complexes,

$$(\star) \qquad 0 \longrightarrow H \longrightarrow P/\mathfrak{J}P \xrightarrow{\ \tau\ } \operatorname{Cone} 1^B \longrightarrow 0\,.$$

Because it is a homomorphic image of the semi-perfect graded R-module $P^\natural$, the graded module $B^\natural$ is semi-perfect. In particular, $B^\natural$ has a projective cover $\varkappa\colon F \twoheadrightarrow B^\natural$.

Set $P'' = \operatorname{Cone} 1^F$ and $C = \operatorname{Cone} 1^B$; both complexes are contractible by 4.3.31. The projective cover $\varkappa$ induces a surjective morphism $\chi = \varkappa \oplus \Sigma\varkappa\colon P'' \to C$. It is elementary to verify that $\chi\colon P''^{\natural} \twoheadrightarrow C^{\natural}$ is a projective cover. By $(\star)$ the graded module $C^{\natural}$ is a homomorphic image of $P^{\natural}$, so it is semi-perfect and, therefore, $P''^{\natural}$ is semi-perfect by B.40(c).

Let π denote the canonical map $P \twoheadrightarrow P/\mathfrak{J}P$. The complexes $\operatorname{Hom}_R(P, P'')$ and $\operatorname{Hom}_R(P, C)$ are contractible by 4.3.29 so $\operatorname{Hom}_R(P, \chi)$ is a quasi-isomorphism. Moreover, $\operatorname{Hom}_R(P, \chi)$ is surjective by 5.2.2 and hence surjective on cycles; see 4.2.7. Thus, there exists a morphism $\gamma\colon P \to P''$ with $\chi\gamma = \tau\pi$. Let π' be the restriction of π to the subcomplex $P' = \operatorname{Ker}\gamma$ and consider the commutative diagram

$$(\diamond) \qquad \begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "\pi'"] & P \arrow[r, "\gamma"] \arrow[d, "\pi"] & P'' \arrow[r] \arrow[d, "\chi"] & 0 \\ 0 \arrow[r] & H \arrow[r] & P/\mathfrak{J}P \arrow[r, "\tau"] & C \arrow[r] & 0\,. \end{tikzcd}$$

The bottom row is the split exact sequence $(\star)$, and by construction the top row is exact at P' and at P. To see that it is exact at P'', notice that $\bar{\chi}\colon P''/\mathfrak{J}P'' \to C/\mathfrak{J}C$ is bijective by B.37 and $\bar{\pi}$ is the identity morphism $1^{P/\mathfrak{J}P}$, while $\bar{\tau} = \tau$ is surjective. It follows that $\bar{\gamma}\colon P/\mathfrak{J}P \to P''/\mathfrak{J}P''$ is surjective, and then γ is surjective by B.40(f). As $P''^{\natural}$ is a graded-projective R-module, the top row in $(\diamond)$ is degreewise split by 5.2.2 and, therefore, split by B.1 as P'' is contractible. Thus, one has $P = P' \oplus P''$.

To see that P' is minimal, note that $(\diamond)$ now yields a commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & P'/\mathfrak{J}P' \arrow[r] \arrow[d, "\bar{\pi}'"] & P/\mathfrak{J}P \arrow[r, "\bar{\gamma}"] \arrow[d, "1^{P/\mathfrak{J}P}"] & P''/\mathfrak{J}P'' \arrow[r] \arrow[d, "\bar{\chi}", "\cong"'] & 0 \\ 0 \arrow[r] & H \arrow[r] & P/\mathfrak{J}P \arrow[r, "\tau"] & C \arrow[r] & 0\,, \end{tikzcd}$$

with exact rows. It follows from the Five Lemma that $\bar{\pi}'$ is an isomorphism. The differential on H is zero, so $\partial^{P'}(P')$ is contained in $\mathfrak{J}P'$. As the top row in $(\diamond)$ is split, the graded-projective R-module $P'^{\natural}$ is a homomorphic image of $P^{\natural}$ and hence semi-perfect. It follows from B.54 that P' is a minimal complex, which finishes the proof of the first assertion. Parts (a) and (c) follow from B.7 and 5.2.18, respectively.

(b): Since $P^{\natural}$ is graded-projective and semi-perfect, it follows from B.37 and B.40(b) that $\mathrm{B}(P)^{\natural} = (\partial^P(P))^{\natural}$ is superfluous in $P^{\natural}$ if and only if the inclusion $\partial^P(P) \subseteq \mathfrak{J}P$ holds. In view of B.54 it remains to prove that $\partial^P(P) \subseteq \mathfrak{J}P$ holds if P is minimal. Assume that P is minimal; by the arguments above one has $P = P' \oplus P''$, where P'' is contractible and $\partial^{P'}(P')$ is contained in $\mathfrak{J}P'$. The surjection $P \to P'$ is a homotopy equivalence by B.1 and hence an isomorphism by B.7. Thus $\partial^P(P)$ is contained in $\mathfrak{J}P$. □

Minimal Semi-Projective Resolutions

Every quasi-isomorphism with a semi-projective codomain has by 5.2.20/6.3.2 a homotopy right inverse. Under the additional assumption that the codomain complex is minimal, such a quasi-isomorphism has a genuine right inverse.

B.56 Proposition. *For an R-complex P, the following conditions are equivalent.*

(i) *P is semi-projective and minimal.*

(ii) *Every quasi-isomorphism $M \to P$ has a right inverse.*

Proof. Assume that P is semi-projective and minimal. If $\alpha \colon M \to P$ is a quasi-isomorphism, then there exists by 5.2.20 a morphism $\gamma \colon P \to M$ with $\alpha\gamma \sim 1^P$. Set $\varepsilon = \alpha\gamma$; by assumption, ε has an inverse, so $\gamma\varepsilon^{-1}$ is a right inverse for α.

Assume that every quasi-isomorphism $M \to P$ has a right inverse. In particular, every homotopy equivalence $P \to P$ has a right inverse and hence P is minimal by B.6. It follows from 5.2.10 that P is semi-projective. □

B.57 Definition. Let M be an R-complex. A semi-projective resolution $P \xrightarrow{\simeq} M$ is called *minimal* if the semi-projective complex P is minimal.

B.58 Theorem. *Let M be an R-complex. If $L \xrightarrow{\simeq} M$ is a minimal semi-projective resolution, then L is unique up to isomorphism, and it has the following properties.*

(a) *$L_v = 0$ holds for all $v < \inf M$.*

(b) *For every semi-projective resolution $P \xrightarrow{\simeq} M$ the complex L is a direct summand of P.*

Proof. Part (a) follows from part (b), as one by 5.2.15 can choose a semi-projective resolution $P \xrightarrow{\simeq} M$ with $P_v = 0$ for all $v < \inf M$. To prove part (b), let $P \xrightarrow{\simeq} M$ be any semi-projective resolution. It follows from 5.2.19 that there is a quasi-isomorphism $\alpha \colon P \to L$, and by B.56 it has right inverse β. Thus L is a direct summand of P. The morphism β is a quasi-isomorphism, so if P is minimal, then β has a right inverse by the same argument, and then it is an isomorphism with $\beta^{-1} = \alpha$. This proves the uniqueness statement. □

B.59 Example. If M is an acyclic R-complex, then $0 \xrightarrow{\simeq} M$ is the minimal semi-projective resolution; this morphism is only surjective if M is the zero complex.

B.60 Theorem. *If R is left perfect, then every R-complex has a minimal semi-projective resolution.*

Proof. Let M be an R-complex, by 5.2.15 there is a semi-projective resolution $\pi \colon P \xrightarrow{\simeq} M$. By B.53 the graded R-module $P^\natural$ is semi-perfect, so by B.55 one has $P = L \oplus P''$, where L is minimal and semi-projective, and P'' is contractible. Let ι be the embedding $L \rightarrowtail P$; by 4.2.6 it is a quasi-isomorphism as P'' is acyclic. Thus $\pi\iota \colon P \xrightarrow{\simeq} M$ is the desired resolution. □

Remark. Minimal projective resolutions of modules over perfect rings were treated by Eilenberg [75] as early as 1956.

B.61 Theorem. *Assume that R is left Noetherian and semi-perfect. Every R-complex M with $\mathrm{H}(M)$ bounded below and degreewise finitely generated has a minimal semi-projective resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated.*

Proof. By 5.2.16 there is a semi-projective resolution $\pi\colon P \xrightarrow{\simeq} M$ with P degreewise finitely generated. By B.46 the graded R-module $P^\natural$ is semi-perfect, so by B.55 one has $P = L \oplus P''$, where L is minimal and semi-projective, and P'' is contractible. Let ι be the embedding $L \rightarrowtail P$; by 4.2.6 it is a quasi-isomorphism as P'' is acyclic. Thus $\pi\iota\colon L \xrightarrow{\simeq} M$ is the desired resolution. □

B.62 Definition. Let M be an R-complex. A semi-free resolution $L \xrightarrow{\simeq} M$ is called *minimal* if the semi-free complex L is minimal.

B.63 Corollary. *Assume that R is left Noetherian and local. Every R-complex M with* $\mathrm{H}(M)$ *bounded below and degreewise finitely generated has a minimal semi-free resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated.*

Proof. As R is semi-perfect, see B.44, it follows from B.61 that M has a minimal semi-projective resolution $L \xrightarrow{\simeq} M$ with L degreewise finitely generated. By B.58 and B.47 the complex L is bounded below and consists of free R-modules, so L is semi-free by 5.1.3. □

Exercises

E B.1 Show that the conclusions in B.1 may fail if the exact sequence is not degreewise split.

E B.2 Assume that R is left Noetherian. Show that every injective R-module has an indecomposable direct summand.

E B.3 Let $\{M^u\}_{u\in U}$ and $\{N^u\}_{u\in U}$ be a families of graded R-modules such that N^u is a submodule of M^u for every $u \in U$. The graded submodule $\coprod_{u\in U} N^u$ of $\coprod_{u\in U} M^u$ is essential if and only if N^u is essential in M^u for every $u \in U$.

E B.4 Let R be an integral domain with field of fractions Q. Show that the embedding $R \rightarrowtail Q$ is an injective envelope of the R-module R.

E B.5 Let $\iota\colon M \rightarrowtail I$ be an injective preenvelope of an R-module. Show that ι is an injective envelope if and only if every endomorphism $\gamma\colon I \to I$ with $\gamma\iota = \iota$ is an automorphism.

E B.6 Let $\iota\colon M \rightarrowtail I$ and $\iota'\colon M' \rightarrowtail I'$ be injective envelopes of graded R-modules. Show that the direct sum $\iota \oplus \iota'\colon M \oplus M' \to I \oplus I'$ is an injective envelope.

E B.7 Assume that R is left Noetherian and let $\{\iota^n\colon M^n \rightarrowtail I^n\}_{n\in\mathbb{N}}$ be a family of injective envelopes. Show that the coproduct $\coprod_{n\in\mathbb{N}} \iota^n\colon \coprod_{n\in\mathbb{N}} M^n \to \coprod_{n\in\mathbb{N}} I^n$ is an injective envelope. *Hint:* See Xu [256, 1.4].

E B.8 Let p be a prime and consider the injective envelope $\mathbb{Z}/p\mathbb{Z} \rightarrowtail \mathbb{Z}(p^\infty)$ from B.15. Show that the product $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \longrightarrow \mathbb{Z}(p^\infty)^{\mathbb{N}}$ is not an injective envelope.

E B.9 Consider an R-complex $M = 0 \to M' \to M'' \to 0$. Show that if $\operatorname{Hom}_R(M'', M') = 0$ holds, then M is minimal.

E B.10 Show how to construct a minimal injective resolution of an R-module by taking successive injective envelopes.

E B.11 Show that in a minimal semi-injective resolution $\iota\colon M \to I$ of an R-complex the map ι need not be injective.

E B.12 Let I be a complex of injective R-modules. Show that the complexes $I_{\leqslant n}$ and $I_{\geqslant n}$ are minimal for every $n \in \mathbb{Z}$.

E B.13 Let $p \in \mathbb{Z}$ be a prime. Show that the canonical ring homomorphism $\mathbb{Z}_{p\mathbb{Z}} \to \mathbb{Z}/p\mathbb{Z}$ is a projective cover of the $\mathbb{Z}_{p\mathbb{Z}}$-module $\mathbb{Z}/p\mathbb{Z}$.

E B.14 Let M be a graded R-module with graded submodules $N \subseteq M$ and $N' \subseteq M' \subseteq M$. Show that if N' is superfluous in M' and $M' \subseteq N + N'$ holds, then M' is contained in N.

E B.15 Show that the unique maximal ideal in a local ring is both essential and superfluous.

E B.16 Set $R = \{\frac{s}{t} \in \mathbb{Q} \mid (s,t) = 1 \text{ and } t \text{ odd}\}$. Show that the Jacobson radical of R is $\mathfrak{J} = R(2)$. Set $F = R^{(\mathbb{N})}$ and let $\mu\colon F \to \mathbb{Q}$ be the map given by $(a_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} \frac{a_n}{2^n}$. Show that $\mu(\mathfrak{J}F)$ is not superfluous in $\mathbb{Q}$ and conclude that $\mathfrak{J}F$ is not superfluous in F.

E B.17 Let $\mathfrak{a}$ be an ideal in R such that $R/\mathfrak{a}$ is local. Show that idempotents lift from $R/\mathfrak{a}$ to R.

E B.18 Show that a sum of orthogonal idempotents is an idempotent.

E B.19 Let $\pi\colon P \twoheadrightarrow M$ be a projective precover of an R-module. Show that π is a projective cover if and only if every endomorphism $\gamma\colon P \to P$ with $\pi\gamma = \pi$ is an automorphism.

E B.20 Show that every flat module over a perfect ring is projective. *Hint:* 1.3.44.

E B.21 Show that every finitely generated flat module over a semi-perfect ring is projective. Give an example of a semi-perfect ring that is not left Noetherian.

E B.22 Let $\pi\colon P \twoheadrightarrow M$ and $\pi'\colon P' \twoheadrightarrow M'$ be projective covers of graded R-modules. Show that the direct sum $\pi \oplus \pi'\colon P \oplus P' \to M \oplus M'$ is a projective cover.

E B.23 Assume that R is left perfect and let $\{\pi^n\colon P^n \twoheadrightarrow M^n\}_{n\in\mathbb{N}}$ be a family of projective covers. Show that $\coprod_{n\in\mathbb{N}} \pi^n\colon \coprod_{n\in\mathbb{N}} P^n \to \coprod_{n\in\mathbb{N}} M^n$ is a projective cover. *Hint:* See Xu [256, 1.4].

E B.24 Let $p \in \mathbb{Z}$ be a prime and consider the projective cover $\mathbb{Z}_{p\mathbb{Z}} \to \mathbb{Z}/p\mathbb{Z}$ from E B.13. Show that the coproduct $(\mathbb{Z}_{p\mathbb{Z}})^{(\mathbb{N})} \to (\mathbb{Z}/p\mathbb{Z})^{(\mathbb{N})}$ is not a projective cover.

E B.25 Let P be a minimal semi-projective R-complex and M a simple R-module. Show that the complex $\operatorname{Hom}_R(P,M)$ has zero differential.

E B.26 Assume that R is left perfect. Show how to construct a minimal projective resolution of an R-module by taking successive projective covers.

E B.27 Show that in a minimal semi-projective resolution $\pi\colon P \to M$ of an R-complex the map π need not be surjective.

E B.28 Assume that R is left Noetherian. Let $\mathfrak{J}$ be the Jacobson radical of R and assume that the ring $R/\mathfrak{J}$ is left Artinian. Show that $R/\mathfrak{J}^n$ is an Artinian R-module and a left Artinian ring for every $n \in \mathbb{N}$. Use Levitzki's theorem [168, §10] to conclude that a left perfect and left Noetherian ring is left Artinian.

E B.29 Let $\Bbbk$ be a field and consider the local ring $R = \Bbbk[\![x,y]\!]$. Identify the minimal free resolution of $R/(x,y)$ as a summand of each of the resolutions

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & 0 \\ 0 & 1 \end{pmatrix}} R^3 \xrightarrow{(x\ y\ 0)} R \longrightarrow 0$$

and

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -y & y \\ x & -x \\ 0 & 1 \end{pmatrix}} R^3 \xrightarrow{(x\ y\ 0)} R \longrightarrow 0\,.$$

Appendix C
Structure of Injective Modules

SYNOPSIS. Hom vanishing; faithfully injective module; structure of injective module over Noetherian ring; injective precover; indecomposable injective module; endomorphism of ∼.

In his thesis [179] Matlis developed a structure theory for injective modules over Noetherian rings. Our goal is to expose its main points, which are that every injective module is a coproduct of indecomposable modules (C.4 below), and that these have elementary descriptions over Artinian rings (C.6) and over commutative rings (C.23).

We open with a general result that is known as the Hom Vanishing Lemma. A stronger version is available over commutative Noetherian rings; see 17.1.3.

C.1 Lemma. *Let M and N be R-modules. A necessary condition for* $\operatorname{Hom}_R(M,N)$ *to be non-zero is existence of elements m in M and $n \neq 0$ in N with* $(0 :_R m) \subseteq (0 :_R n)$. *If N is injective, then this condition is also sufficient.*

PROOF. If $\varphi\colon M \to N$ is a non-zero homomorphism, then there is an element $m \in M$ with $\varphi(m) \neq 0$, and by R-linearity of φ one has $(0 :_R m) \subseteq (0 :_R \varphi(m))$. Conversely, if m and $n \neq 0$ are elements with $(0 :_R m) \subseteq (0 :_R n)$, then the assignment $rm \mapsto rn$ defines a non-zero homomorphism from the submodule $R\langle m\rangle \subseteq M$ to N, and if N is injective, then it extends to a homomorphism $M \to N$. □

C.2 Proposition. *Every indecomposable injective R-module is isomorphic to the injective envelope* $\mathrm{E}_R(R/\mathfrak{a})$ *for some left ideal $\mathfrak{a}$ in R. In particular, the collection of isomorphism classes of indecomposable injective R-modules constitutes a set.*

PROOF. Let E be an indecomposable injective R-module. Every non-zero element in E generates a submodule isomorphic to $R/\mathfrak{a}$ for some left ideal $\mathfrak{a}$; see 1.3.1. As E is injective it follows from B.19 that the injective envelope $\mathrm{E}_R(R/\mathfrak{a})$ is a direct summand of E, so by indecomposability of E one has $\mathrm{E}_R(R/\mathfrak{a}) \cong E$. □

C.3 Proposition. *Let* Max R *denote the set of maximal left ideals of R. The R-module*

$$\prod_{\mathfrak{m}\in\operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$$

is faithfully injective.

Proof. The module $E = \coprod_{\mathfrak{m}\in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$ is injective by 1.3.27. To see that the functor $\operatorname{Hom}_R(-,E)$ is faithful, notice that for every R-module M and for every $m \neq 0$ in M one has $(0 :_R m) \subseteq \mathfrak{m}$ for some $\mathfrak{m}$ in $\operatorname{Max} R$. Now it follows from C.1 that there is a non-zero homomorphism $M \to \mathrm{E}_R(R/\mathfrak{m}) \rightarrowtail E$. $\square$

If R is left Noetherian, then also the coproduct $\coprod_{\mathfrak{m}\in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$ of the modules from C.3 is a faithfully injective R-module; cf. 8.2.20.

Left Noetherian Rings

C.4 Theorem. *If R is left Noetherian, then every injective R-module is a coproduct of indecomposable injective modules.*

Proof. We first argue that every injective R-module $I \neq 0$ contains an indecomposable injective submodule. Let $i \neq 0$ be an element in I; by B.19 there is an injective submodule E of I such that $R\langle i\rangle$ is essential in E. As R is left Noetherian, the module $R\langle i\rangle$ cannot accommodate an infinite independent family of non-zero submodules, see 1.1.24(b), and since $R\langle i\rangle$ is essential in E, neither can E. It follows that E contains an indecomposable direct summand. Indeed, if E is indecomposable, then that is the desired module. Otherwise, one has $E = M^1 \oplus E^1$ for non-zero modules M^1 and E^1. If E^1 is indecomposable, then that is the desired module, and if not then one has $E^1 = M^2 \oplus E^2$, etc. As the family $M^1, M^2, \dots$ is independent, the process terminates after finitely many iterations with an indecomposable direct summand E^n of E. In particular, E^n is an injective submodule of I.

Given an injective R-module I, consider the set of all independent families of indecomposable injective submodules of I. By the argument above, this set is non-empty, and furthermore it is inductively ordered under inclusion. By Zorn's lemma there exists a maximal such family $\{I^u\}_{u\in U}$. The submodule $\sum_{u\in U} I^u \cong \coprod_{u\in U} I^u$ is injective by 8.2.20, so one has $I = E \oplus \sum_{u\in U} I^u$ for some R-module E, which is also injective. If E were non-zero, then it would contain an indecomposable submodule E', and the family $\{I^u\}_{u\in U} \cup \{E'\}$ would be independent, which would contradict the maximality of $\{I^u\}_{u\in U}$. Thus, one has $E = 0$ and $I \cong \coprod_{u\in U} I^u$. $\square$

Remark. The decomposition of injective modules described in C.4 is unique to Noetherian rings; this is also a result of Matlis [180].

For the next result, recall that a direct sum is finite by convention.

C.5 Corollary. *Assume that R is left Noetherian and let M be a finitely generated R-module. The injective envelope $\mathrm{E}_R(M)$ is a direct sum of indecomposable injective modules.*

Proof. By C.4 the module $\mathrm{E}_R(M)$ is the sum $\sum_{u\in U} E^u$ of an independent family of indecomposable injective submodules. As M is finitely generated, it is contained in the sum $E' = E^{u_1} + \cdots + E^{u_n} \cong E^{u_1} \oplus \cdots \oplus E^{u_n}$ of finitely many of these modules. Now, since M is an essential submodule of $\mathrm{E}_R(M)$, one has $E' = \mathrm{E}_R(M)$. $\square$

It is evident that the injective envelope of a simple module is indecomposable. To fully appreciate the next result, recall that—up to isomorphism—a left Artinian ring has only finitely many simple modules.

C.6 Theorem. *Assume that R is left Artinian. The assignment $N \mapsto \mathrm{E}_R(N)$ yields a one-to-one correspondence between isomorphism classes of simple R-modules and isomorphism classes of indecomposable injective R-modules. In particular, every indecomposable injective R-module is isomorphic to $\mathrm{E}_R(R/\mathfrak{m})$ for some maximal left ideal $\mathfrak{m}$ in R.*

Proof. Let E be an indecomposable injective R-module. For every element $e \neq 0$ in E the finitely generated submodule $R\langle e \rangle$ has a composition series, so it contains a simple R-module N. Because E is indecomposable, it is an injective envelope of N by B.19. To see that non-isomorphic simple modules have non-isomorphic injective envelopes, let N and N' be simple R-modules with $E = \mathrm{E}_R(N) \cong \mathrm{E}_R(N')$. There are then injective homomorphisms $\iota \colon N \to E$ and $\iota' \colon N' \to E$ such that $\operatorname{Im}\iota$ and $\operatorname{Im}\iota'$ are essential in E. By B.11 the submodule $\operatorname{Im}\iota \cap \operatorname{Im}\iota'$ is essential in E; in particular, it is non-zero. It is also isomorphic to a submodule of N and to a submodule of N', and hence isomorphic to both N and N'. Thus, one has $N \cong N'$. A simple R-module is cyclic and hence isomorphic to $R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ in R. □

Caveat. For every maximal left ideal $\mathfrak{m}$ in R the quotient $R/\mathfrak{m}$ is a simple R-module, and every simple R-module is isomorphic to one of those. However, if R is not commutative, then distinct maximal left ideals may yield isomorphic simple modules; see also E 8.2.16 and E 12.4.1.

C.7 Definition. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An *$\mathcal{X}$-precover* of M is a homomorphism $\varphi \colon X \to M$ with $X \in \mathcal{X}$ such that for every homomorphism $\varphi' \colon X' \to M$ with $X' \in \mathcal{X}$ there is a homomorphism $\chi \colon X' \to X$ that makes the following diagram commutative,

$$\begin{tikzcd} & X' \arrow[dl, dotted, "\chi"'] \arrow[d, "\varphi'"] \\ X \arrow[r, "\varphi"'] & M \, . \end{tikzcd}$$

C.8 Example. Let $\mathcal{P}_0$ denote the class of projective R-modules. By 1.3.12 and 1.3.17 every R-module has a $\mathcal{P}_0$-precover, also called a *projective precover*.

C.9. Let $\mathcal{X}$ be a class of R-modules and M an R-module. A homomorphism $\varphi \colon X \to M$ with $X \in \mathcal{X}$ is an $\mathcal{X}$-precover if and only if $\operatorname{Hom}_R(X', \varphi)$ is surjective for every $X' \in \mathcal{X}$. Notice that if M is a homomorphic image of some module from $\mathcal{X}$, then every $\mathcal{X}$-precover $X \to M$ is surjective.

Let $\mathcal{I}_0$ denote the class of injective R-modules; an $\mathcal{I}_0$-precover is called an *injective precover;* cf. C.8.

C.10 Proposition. *If R is left Noetherian, then every R-module has an injective precover.*

PROOF. Let $\{E_u\}_{u \in U}$ be a set of representatives for the isomorphism classes of indecomposable injective R-modules, see C.2, and let M be an R-module. Consider the module $E = \coprod_{u \in U} E_u^{(\operatorname{Hom}_R(E_u,M))}$, which is injective by 8.2.20. For $u \in U$ and $\alpha \in \operatorname{Hom}_R(E_u, M)$ write $\varepsilon_{u,\alpha} \colon E_u \to E$ for the embedding. Let $\varphi \colon E \to M$ be the unique homomorphism that satisfies $\varphi\varepsilon_{u,\alpha} = \alpha$ for every u and α; cf. 1.1.20. For every $u \in U$ the homomorphism $\operatorname{Hom}_R(E_u, \varphi) \colon \operatorname{Hom}_R(E_u, E) \to \operatorname{Hom}_R(E_u, M)$ maps $\varepsilon_{u,\alpha}$ to α, and thus $\operatorname{Hom}_R(E_u, \varphi)$ is surjective. Every injective R-module E' is by C.4 a coproduct of modules E_u, so $\operatorname{Hom}_R(E', \varphi)$ is a product of homomorphisms of the form $\operatorname{Hom}_R(E_u, \varphi)$; see 3.1.27. Since each map $\operatorname{Hom}_R(E_u, \varphi)$ is surjective, so is $\operatorname{Hom}_R(E', \varphi)$. This means that every homomorphism $\varphi' \colon E' \to M$ has the form $\varphi' = \varphi\chi$ for some $\chi \colon E' \to E$. □

C.11 Proposition. *Let $R \to S$ be a ring homomorphism and M an R-module. If S is flat as an R^{o}-module and $\varphi \colon I \to M$ an injective precover, then the induced map of S-modules,* $\operatorname{Hom}_R(S, \varphi) \colon \operatorname{Hom}_R(S, I) \to \operatorname{Hom}_R(S, M)$*, is an injective precover.*

PROOF. The S-module $\operatorname{Hom}_R(S, I)$ is injective by 5.4.28(a), and for every S-module E there are by adjunction 4.4.12 and the unitor 4.4.1 isomorphisms,

$$(\star) \qquad \operatorname{Hom}_S(E, \operatorname{Hom}_R(S, \varphi)) \cong \operatorname{Hom}_R(S \otimes_S E, \varphi) \cong \operatorname{Hom}_R(E, \varphi) .$$

If E is injective, then it is also injective as an R-module, see 5.4.28(b), and it follows from C.9 that $\operatorname{Hom}_R(E, \varphi)$ is surjective. Now $(\star)$ shows that $\operatorname{Hom}_S(E, \operatorname{Hom}_R(S, \varphi))$ is surjective, whence $\operatorname{Hom}_R(S, \varphi)$ is an injective precover, again by C.9. □

COMMUTATIVE NOETHERIAN RINGS

If R is commutative and Artinian with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$, then by C.6 and C.3 the indecomposable injective R-modules are precisely the injective envelopes $\mathrm{E}_R(R/\mathfrak{m}_1), \dots, \mathrm{E}_R(R/\mathfrak{m}_n)$, and the direct sum $D = \bigoplus_{u=1}^n \mathrm{E}_R(R/\mathfrak{m}_u)$ is a faithfully injective R-module. The endomorphism ring of D is by 18.2.2 isomorphic to R, that is, every R-linear map $D \to D$ is a homothety. This is just one striking consequence of the structure of injective R-modules.

In the rest of this appendix, we use some of the standard notation and terminology from commutative algebra recalled in Sect. 12.4. An injective module over a commutative Noetherian ring is by C.4 a direct sum of indecomposable modules, and another theorem due to Matlis describes the building blocks.

C.12 Theorem. *Assume that R is commutative and Noetherian. An indecomposable injective R-module has exactly one associated prime ideal, and the assignment* $\mathfrak{p} \mapsto \mathrm{E}_R(R/\mathfrak{p})$ *yields a one-to-one correspondence between prime ideals of R and isomorphism classes of indecomposable injective R-modules.*

PROOF. If E is an indecomposable injective R-module, then E is by B.19 an injective envelope of every submodule $N \neq 0$ of E. Since R is Noetherian, the module E has an associated prime ideal $\mathfrak{p}$, that is, it contains a submodule N isomorphic to $R/\mathfrak{p}$. Suppose $\mathfrak{p}'$ is also an associated prime ideal of E, then there is a submodule

$N' \cong R/\mathfrak{p}'$ of E, and the intersection $N \cap N'$ is non-zero as N is essential in E. It follows that there is a non-zero submodule of $R/\mathfrak{p}$ that is annihilated by $\mathfrak{p}'$, whence one has $\mathfrak{p}' \subseteq \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal. By symmetry, one gets $\mathfrak{p} = \mathfrak{p}'$. Thus, $\mathfrak{p}$ is the only associated prime ideal of E, and one has $E \cong \mathrm{E}_R(R/\mathfrak{p})$.

Conversely, let $\mathfrak{p}$ be a prime ideal in R. To see that $\mathrm{E}_R(R/\mathfrak{p})$ is indecomposable, assume that one has $\mathrm{E}_R(R/\mathfrak{p}) = E' \oplus E''$. There are ideals $\mathfrak{a}' \supseteq \mathfrak{p}$ and $\mathfrak{a}'' \supseteq \mathfrak{p}$ in R with $E' \cap R/\mathfrak{p} = \mathfrak{a}'/\mathfrak{p}$ and $E'' \cap R/\mathfrak{p} = \mathfrak{a}''/\mathfrak{p}$. The intersection $\mathfrak{a}'/\mathfrak{p} \cap \mathfrak{a}''/\mathfrak{p}$ in $R/\mathfrak{p}$ is trivial, so one has $\mathfrak{a}'\mathfrak{a}'' \subseteq \mathfrak{a}' \cap \mathfrak{a}'' \subseteq \mathfrak{p}$. As $\mathfrak{p}$ is a prime ideal this implies that, say, $\mathfrak{a}'$ is contained in $\mathfrak{p}$, so $E' \cap R/\mathfrak{p}$ is trivial. As $R/\mathfrak{p}$ is essential in $\mathrm{E}_R(R/\mathfrak{p})$, this forces $E' = 0$. □

C.13 Example. It follows from C.12 and B.15 that the indecomposable injective $\mathbb{Z}$-modules are precisely $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Q}$ and $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}(p^\infty)$ where p is a prime. It is routine to verify that the $\mathbb{Z}$-modules $\mathbb{Q}/\mathbb{Z}$ and $\coprod_{p \text{ prime}} \mathbb{Z}(p^\infty)$ are isomorphic.

For a prime ideal $\mathfrak{p}$ in R, the next result establishes that $\mathrm{E}_R(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion in the sense of 11.2.6.

C.14 Proposition. *Assume that R is commutative and Noetherian and let $\mathfrak{p}$ be a prime ideal in R. Every element in $\mathrm{E}_R(R/\mathfrak{p})$ is annihilated by a power of $\mathfrak{p}$.*

Proof. Let e be a non-zero element of $E = \mathrm{E}_R(R/\mathfrak{p})$ and set $\mathfrak{a} = (0 :_R e)$; the goal is to prove that some power $\mathfrak{p}^n$ is contained in $\mathfrak{a}$. As $\mathfrak{p}$ is finitely generated, it suffices to show that every element $x \in \mathfrak{p}$ has a power in $\mathfrak{a}$. Fix an x in $\mathfrak{p}$; the ideals $\mathfrak{b}_n = (\mathfrak{a} :_R x^n)$ form an ascending chain, so $\mathfrak{b}_n = \mathfrak{b}_{n+1}$ holds for some n.

As $\mathfrak{p}$ is the only associated prime ideal of E, see C.12, the submodule $R\langle e\rangle \cong R/\mathfrak{a}$ of E has an element re with $(0 :_R re) = \mathfrak{p}$, whence there is an element y in R with $(\mathfrak{a} :_R y) = \mathfrak{p}$. One has $\mathfrak{a} = (\mathfrak{a} + Ry) \cap (\mathfrak{a} + Rx^n)$. Indeed, an element z in the intersection has the form $a + ry = z = a' + r'x^n$, so $xz = xa + rxy$ is in $\mathfrak{a}$ and hence so is $xz - xa' = r'x^{n+1}$. Thus, the element r' is in $\mathfrak{b}_{n+1} = \mathfrak{b}_n$, so $r'x^n$ and hence z is in $\mathfrak{a}$. The ideal $\mathfrak{a} + Ry$ strictly contains $\mathfrak{a}$, so it corresponds to a non-zero submodule of $R/\mathfrak{a} \cong R\langle e\rangle \subseteq E$. Every non-zero submodule of E is essential, cf. B.19, so it follows that $\mathfrak{a} + Rx^n$ corresponds to the zero submodule, i.e. x^n is in $\mathfrak{a}$. □

C.15 Proposition. *Assume that R is commutative and Noetherian and let $\mathfrak{p}$ be a prime ideal in R. The following assertions hold.*

(a) $\operatorname{Ass}_R \mathrm{E}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

(b) $\operatorname{Supp}_R \mathrm{E}_R(R/\mathfrak{p}) = \mathrm{V}(\mathfrak{p})$.

(c) *For every prime ideal $\mathfrak{q}$ the following conditions are equivalent*:

  (*i*) $\mathfrak{p} \subseteq \mathfrak{q}$.

  (*ii*) $\operatorname{Hom}_R(R/\mathfrak{p}, \mathrm{E}_R(R/\mathfrak{q})) \neq 0$.

  (*iii*) $\operatorname{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{q})) \neq 0$.

Proof. Part (a) follows from C.12.

(b): Let $e \neq 0$ be an element in $\mathrm{E}_R(R/\mathfrak{p})$. If $\mathfrak{p} \not\subseteq \mathfrak{q}$, then one can choose an element $x \in \mathfrak{p} \setminus \mathfrak{q}$. By C.14 one has $x^n e = 0$ for some $n \in \mathbb{N}$, and since $\mathfrak{q}$ is a prime ideal, x^n is not in $\mathfrak{q}$. Thus $\frac{e}{r} = \frac{x^n e}{x^n r} = 0$ holds for every $r \in R \setminus \mathfrak{q}$, so $\mathrm{E}_R(R/\mathfrak{p})_\mathfrak{q} = 0$. On the other hand, if $\mathfrak{q}$ contains $\mathfrak{p}$, then $[1]_\mathfrak{p} \neq 0$ in $\mathrm{E}_R(R/\mathfrak{p})_\mathfrak{q}$.

(c): If $\mathfrak{q}$ contains $\mathfrak{p}$, then the composite $R/\mathfrak{p} \twoheadrightarrow R/\mathfrak{q} \rightarrowtail \mathrm{E}_R(R/\mathfrak{q})$ of canonical homomorphisms is non-zero, and it extends by injectivity of $\mathrm{E}_R(R/\mathfrak{q})$ to a non-zero homomorphism $\mathrm{E}_R(R/\mathfrak{p}) \to \mathrm{E}_R(R/\mathfrak{q})$. On the other hand, existence of such a homomorphism implies by C.1 and C.14 that a power of $\mathfrak{p}$ is contained in the annihilator of a non-zero element in the module $\mathrm{E}_R(R/\mathfrak{q})$ and hence contained in its unique associated prime ideal $\mathfrak{q}$; cf. (a). Thus, $\mathfrak{q}$ contains $\mathfrak{p}$. ◻

C.16 Proposition. *Assume that R is commutative and Noetherian and let $\mathfrak{a}$ be an ideal in R. For every prime ideal $\mathfrak{p}$ in R there is an isomorphism of $R/\mathfrak{a}$-modules,*

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}_R(R/\mathfrak{p})) \cong \begin{cases} \mathrm{E}_{R/\mathfrak{a}}(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p} \\ 0 & \text{if } \mathfrak{a} \not\subseteq \mathfrak{p}\,. \end{cases}$$

Proof. For a prime ideal $\mathfrak{p}$ that does not contain $\mathfrak{a}$, the Hom Vanishing Lemma C.1 and C.15(a) yield $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}_R(R/\mathfrak{p})) = 0$. Now let $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$. The injective evelope $R/\mathfrak{p} \rightarrowtail \mathrm{E}_R(R/\mathfrak{p})$ yields an injective homomorphism,

$$(\mathfrak{b}) \qquad \mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{p}) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}_R(R/\mathfrak{p}))\,,$$

and by 5.4.28(a) the $R/\mathfrak{a}$-module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}_R(R/\mathfrak{p}))$ is injective. The map $(\mathfrak{b})$ is per 1.1.8 isomorphic to the inclusion of $R/\mathfrak{p}$ into the submodule $(0 :_{\mathrm{E}_R(R/\mathfrak{p})} \mathfrak{a})$ of $\mathrm{E}_R(R/\mathfrak{p})$, and $R/\mathfrak{p}$ being essential in $\mathrm{E}_R(R/\mathfrak{p})$ is essential $(0 :_{\mathrm{E}_R(R/\mathfrak{p})} \mathfrak{a})$. ◻

C.17 Lemma. *Assume that R is commutative and Noetherian and let $\mathfrak{p} \in \operatorname{Spec} R$. For every $x \in R \setminus \mathfrak{p}$ the homothety $x^{\mathrm{E}_R(R/\mathfrak{p})}$ is an automorphism.*

Proof. Multiplication by $x \in R \setminus \mathfrak{p}$ is injective on the essential submodule $R/\mathfrak{p}$ of $E = \mathrm{E}_R(R/\mathfrak{p})$ and hence injective on E. The submodule $xE \cong E$ of E is injective, whence it is a direct summand of E, and since E is indecomposable it is all of E. ◻

Via the next result, questions about indecomposable injective modules can often be dealt with in a local setting.

C.18 Proposition. *Assume that R is commutative and Noetherian and let $\mathfrak{q} \subseteq \mathfrak{p}$ be prime ideals in R. The R-module $\mathrm{E}_R(R/\mathfrak{q})$ has a canonical structure of an $R_\mathfrak{p}$-module, and as such it is isomorphic to $\mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p})$. In particular, there is an isomorphism $\mathrm{E}_R(R/\mathfrak{p}) \cong \mathrm{E}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}))$ of $R_\mathfrak{p}$-modules, where $\kappa(\mathfrak{p})$ is the field $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.*

Proof. It follows from C.17 that $E = \mathrm{E}_R(R/\mathfrak{q})$ is an $R_\mathfrak{p}$-module when one sets $\frac{r}{x}e = r(x^E)^{-1}(e)$ for $r \in R$, $x \in R \setminus \mathfrak{p}$, and $e \in E$. With this structure, E is an injective $R_\mathfrak{p}$-module; indeed, by 1.2.6 there are natural isomorphisms,

$$\begin{aligned} \mathrm{Hom}_{R_\mathfrak{p}}(-, E) &\cong \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p} \otimes_R -, E) \\ &\cong \mathrm{Hom}_R(-, \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E)) \\ &\cong \mathrm{Hom}_R(-, E)\,, \end{aligned}$$

of functors on $\mathcal{M}(R_\mathfrak{p})$. It follows from B.12 that $(R/\mathfrak{q})_\mathfrak{p}$ is essential in $E_\mathfrak{p} \cong E$, so E is by B.16 isomorphic to the injective envelope $\mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}_\mathfrak{p})$. ◻

C.19 Example. Let p be a prime. One has $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Q}$ and $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$ by B.15. It follows from C.18 applied with $\mathfrak{p} = p\mathbb{Z}$ and $\mathfrak{q} = 0$ that $\mathbb{Q}$ as a $\mathbb{Z}_{p\mathbb{Z}}$-module is the injective envelope of $\mathbb{Z}_{p\mathbb{Z}}$. Applied with $\mathfrak{p} = p\mathbb{Z} = \mathfrak{q}$ it shows that the Prüfer p-group $\mathbb{Z}(p^\infty)$ is the injective envelope of the residue field of the local ring $\mathbb{Z}_{p\mathbb{Z}}$.

C.20 Construction. Assume that R is commutative and Noetherian. Let $\mathfrak{p} \in \operatorname{Spec} R$, set $E = \mathrm{E}_R(R/\mathfrak{p})$, and for $n \geqslant 0$ set

$$E^n = (0 :_E \mathfrak{p}^n) .$$

By C.14 these submodules yield a filtration $0 = E^0 \subseteq E^1 \subseteq \cdots$ with $E = \bigcup_{n\geqslant 0} E^n$.

C.21 Proposition. *Assume that R is commutative Noetherian and local with unique maximal ideal $\mathfrak{m}$; set $\boldsymbol{k} = R/\mathfrak{m}$ and $E = \mathrm{E}_R(\boldsymbol{k})$.*

(a) *For a $\boldsymbol{k}$-vector space V of finite rank, one has*

$$\operatorname{Hom}_R(V, E) = \operatorname{Hom}_R(V, \boldsymbol{k}) \cong V .$$

(b) *For every $\boldsymbol{k}$-vector space V of finite rank, the biduality homomorphism from* 1.4.2, *i.e.* $\delta_E^V \colon V \to \operatorname{Hom}_R(\operatorname{Hom}_R(V, E), E)$*, is an isomorphism.*

(c) *With the notation from* C.20 *there are isomorphisms of R-modules,*

$$E^n/E^{n-1} \cong \mathfrak{m}^{n-1}/\mathfrak{m}^n \qquad \textit{and} \qquad \operatorname{Hom}_R(E^n, E) \cong R/\mathfrak{m}^n .$$

The isomorphism $R/\mathfrak{m}^n \to \operatorname{Hom}_R(E^n, E)$ maps an element $[r]_{\mathfrak{m}^n}$ in $R/\mathfrak{m}^n$ to the homomorphism $E^n \to E$ given by multiplication by r. In particular, one has $(0 :_R E^n) = \mathfrak{m}^n$.

Proof. (a): The image of a homomorphism $V \to E$ is contained in E^1, cf. C.1, and E^1 is a $\boldsymbol{k}$-vector space of rank 1, as $\boldsymbol{k}$ is essential in E. Thus one has $E^1 = \boldsymbol{k}$ and $\operatorname{Hom}_R(V, E) = \operatorname{Hom}_R(V, E^1) \cong \operatorname{Hom}_{\boldsymbol{k}}(V, \boldsymbol{k}) \cong V$.

(b): It follows from C.3 and 4.5.3 that δ_E^V is injective and hence an isomorphism, as the $\boldsymbol{k}$-vector spaces V and $\operatorname{Hom}_R(\operatorname{Hom}_R(V, E), E)$ have the same rank by (a).

(c): For every $n \geqslant 0$ one has $\operatorname{Hom}_R(R/\mathfrak{m}^n, E) \cong E^n$; see 1.1.8. Apply the exact functor $\operatorname{Hom}_R(-, E)$ to the sequence $0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \to R/\mathfrak{m}^n \to R/\mathfrak{m}^{n-1} \to 0$ to get an exact sequence

$$0 \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow \operatorname{Hom}_R(\mathfrak{m}^{n-1}/\mathfrak{m}^n, E) \longrightarrow 0 .$$

By (a) one now has $E^n/E^{n-1} \cong \operatorname{Hom}_R(\mathfrak{m}^{n-1}/\mathfrak{m}^n, E) \cong \mathfrak{m}^{n-1}/\mathfrak{m}^n$.

The module $\operatorname{Hom}_R(E^n, E)$ is isomorphic to $\operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{m}^n, E), E)$, so to prove the second isomorphism in (c), it suffices to show that biduality, $\delta_E^{R/\mathfrak{m}^n}$ is an isomorphism. Note that $\delta_E^{\mathfrak{m}^{n-1}/\mathfrak{m}^n}$ is an isomorphism by (b); in particular, $\delta_E^{R/\mathfrak{m}}$ is an isomorphism. Let $n > 1$ and assume that $\delta_E^{R/\mathfrak{m}^{n-1}}$ is an isomorphism. Set $(-)^\vee = \operatorname{Hom}_R(-, E)$. An application of the Five Lemma 1.1.2 to the commutative diagram below shows that $\delta_E^{R/\mathfrak{m}^n}$ is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^{n-1}/\mathfrak{m}^n \arrow[r] \arrow[d, "\cong"', "\delta_E^{\mathfrak{m}^{n-1}/\mathfrak{m}^n}"] & R/\mathfrak{m}^n \arrow[r] \arrow[d, "\delta_E^{R/\mathfrak{m}^n}"] & R/\mathfrak{m}^{n-1} \arrow[r] \arrow[d, "\cong"', "\delta_E^{R/\mathfrak{m}^{n-1}}"] & 0 \\
0 \arrow[r] & (\mathfrak{m}^{n-1}/\mathfrak{m}^n)^{\vee\vee} \arrow[r] & (R/\mathfrak{m}^n)^{\vee\vee} \arrow[r] & (R/\mathfrak{m}^{n-1})^{\vee\vee} \arrow[r] & 0\,.
\end{tikzcd}$$

Finally, note that the isomorphism $R/\mathfrak{m}^n \to \operatorname{Hom}_R(E^n, E)$ is the composite

$$R/\mathfrak{m}^n \xrightarrow{\delta_E^{R/\mathfrak{m}^n}} \operatorname{Hom}_R(\operatorname{Hom}_R(R/\mathfrak{m}^n, E), E) \xrightarrow{\operatorname{Hom}(\varphi, E)} \operatorname{Hom}_R(E^n, E)$$

where $\delta_E^{R/\mathfrak{m}^n}$ is the biduality homomorphism, whose action is described in 1.4.2, and $\varphi\colon E^n \to \operatorname{Hom}_R(R/\mathfrak{m}^n, E)$ is the isomorphism from 1.1.8. Consequently, the composite isomorphism maps $[r]_{\mathfrak{m}^n}$ to multiplication by r on E^n, as claimed. □

C.22 Corollary. *Assume that R is commutative and Noetherian and let $\mathfrak{m}$ be a maximal ideal in R.*

(a) *For every $n \in \mathbb{N}$ the submodule $(0 :_{\mathrm{E}_R(R/\mathfrak{m})} \mathfrak{m}^n)$ has finite length.*
(b) *Every finitely generated submodule of $\mathrm{E}_R(R/\mathfrak{m})$ has finite length.*

Proof. Set $E = \mathrm{E}_R(R/\mathfrak{m})$ and adopt the notation from C.20. Part (b) follows from part (a) as one has $E = \bigcup_{n\in\mathbb{N}} E^n$. To prove part (a) one may by C.18 assume that R is local with maximal ideal $\mathfrak{m}$, and then it follows from C.21(c) that each subquotient E^n/E^{n-1} is an $R/\mathfrak{m}$-vector space of finite rank. □

We close with Matlis' structure theorem for injective modules.

C.23 Theorem. *Assume that R is commutative and Noetherian. Let I be an injective R-module. For every $\mathfrak{p} \in \operatorname{Spec} R$ there exists by* C.4 *and* C.12 *a set $U(\mathfrak{p})$ such that there is an isomorphism,*

$$I \cong \coprod_{\mathfrak{p}\in\operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{p})^{(U(\mathfrak{p}))}\,.$$

This decomposition is unique in the sense that the cardinality of each $U(\mathfrak{p})$ is given by the rank of the vector space $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p})$ over the field $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$.

Proof. By 1.1.11, 3.1.13, and C.15(b) there are isomorphisms

$$I_\mathfrak{p} \cong \Big(\coprod_{\mathfrak{q}\in\operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{q})^{(U(\mathfrak{q}))}\Big)_\mathfrak{p} \cong \coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{E}_R(R/\mathfrak{q})_\mathfrak{p}^{(U(\mathfrak{q}))}\,.$$

Moreover, one has $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{q})_\mathfrak{p}) \cong \operatorname{Hom}_R(R/\mathfrak{p}, \mathrm{E}_R(R/\mathfrak{q}))_\mathfrak{p}$ by 12.1.21, and by C.15(c) this module is zero if $\mathfrak{p} \nsubseteq \mathfrak{q}$. This explains the 3rd isomorphisms below; the 2nd isomorphism follows from 3.1.33 and the 4th from C.18 and C.21(a).

$$\begin{aligned}
\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), I_\mathfrak{p}) &\cong \operatorname{Hom}_{R_\mathfrak{p}}\Big(\kappa(\mathfrak{p}), \coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{E}_R(R/\mathfrak{q})_\mathfrak{p}^{(U(\mathfrak{q}))}\Big) \\
&\cong \coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{q})_\mathfrak{p})^{(U(\mathfrak{q}))} \\
&\cong \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{p})_\mathfrak{p})^{(U(\mathfrak{p}))}
\end{aligned}$$

$$\cong \; \kappa(\mathfrak{p})^{(U(\mathfrak{p}))} \,. \qquad \square$$

C.24 Proposition. *Assume that R is commutative and Noetherian. Let I be an injective R-module, decomposed per* C.23 *as*

$$I \;\cong\; \coprod_{\mathfrak{q}\in \operatorname{Spec} R} \mathrm{E}_R(R/\mathfrak{q})^{(U(\mathfrak{q}))} \,.$$

For $\mathfrak{p} \in \operatorname{Spec} R$ there are isomorphisms of $R_{\mathfrak{p}}$-modules,

$$\coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{E}_R(R/\mathfrak{q})^{(U(\mathfrak{q}))} \;\cong\; I_{\mathfrak{p}} \;\cong\; \coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}})^{(U(\mathfrak{q}))} \,.$$

In particular, $I_{\mathfrak{p}}$ is injective as an R-module and as an $R_{\mathfrak{p}}$-module.

Proof. By C.15(b) one has $\operatorname{Supp}_R \mathrm{E}_R(R/\mathfrak{q}) = \mathrm{V}(\mathfrak{q})$ and localization preserves coproducts per 3.1.13; thus one has

$$I_{\mathfrak{p}} \;\cong\; \coprod_{\mathfrak{q}\subseteq\mathfrak{p}} \mathrm{E}_R(R/\mathfrak{q})_{\mathfrak{p}}^{(U(\mathfrak{q}))} \,.$$

The asserted isomorphisms now follow from C.18. $\square$

Remark. Over a commutative non-noetherian ring, the localization of an injective module need not be injective, see Everett [72].

Exercises

E C.1 Let $\mathcal{X}$ be a class of R-modules, M an R-module, and $\varphi\colon X \to M$ an $\mathcal{X}$-precover. Show that a homomorphism $\varphi'\colon X' \to M$ with $X' \in \mathcal{X}$ is an $\mathcal{X}$-precover if there exists a homomorphism $\psi\colon X \to X'$ with $\varphi'\psi = \varphi$.

E C.2 (Cf. C.13) Show that the $\mathbb{Z}$-modules $\mathbb{Q}/\mathbb{Z}$ and $\coprod_{p \text{ prime}} \mathbb{Z}(p^{\infty})$ are isomorphic.

E C.3 An R-module E is called *Σ-injective* if $E^{(U)}$ is injective for every set U. Show that the field of fractions of an integral domain is Σ-injective. *Hint:* Baer's criterion 1.3.30.

E C.4 Assume that R is commutative and Noetherian and let U be a multiplicative subset of R. Show that the indecomposable injective $U^{-1}R$-modules are the modules $\mathrm{E}_R(R/\mathfrak{p})$ with $\mathfrak{p} \cap U = \emptyset$.

Appendix D
Projective Dimension of Flat Modules

SYNOPSIS. Jensen's theorem; continuous chain; Eklof's lemma; transfinite extension of modules of finite projective dimension; countably related flat module; finitistic projective dimension of Noetherian ring; flat preenvelope.

The first goal is to prove the following result of Jensen [147]; stated in 8.5.18.

Theorem. *For every flat R-module F one has* $\operatorname{pd}_R F \leqslant \operatorname{FPD} R$.

The proof, which comes after D.12, is based on an in-depth study of kernels of free precovers of flat modules. As part of it emerges a result of independent interest, D.9, which asserts that a flat module has projective dimension at most 1 if it can be generated by a set of elements with only countably many relations among them.

CONTINUOUS CHAINS

The notion of a continuous chain plays a key role in the proof of Jensen's theorem, which is based on transfinite induction.

D.1 Definition. Let M be an R-complex and λ an ordinal. A family $\{M^\alpha\}_{\alpha<\lambda}$ of subcomplexes of M is called a *continuous chain* if the next conditions are satisfied.

(1) For $\alpha \leqslant \beta < \lambda$ one has $M^\alpha \subseteq M^\beta$.
(2) If $\beta < \lambda$ is a limit ordinal, then one has $M^\beta = \bigcup_{\alpha<\beta} M^\alpha$.

The following result is known as Eklof's lemma, as the proof originates in [79]. Here it is stated as it appears in [81] by Eklof and Trlifaj.

D.2 Lemma. *Let $\mathcal{Y}$ be a class of R-modules and set*

$$^{\perp}\mathcal{Y} = \{M \in \mathcal{M}(R) \mid \operatorname{Ext}^1_R(M,Y) = 0 \text{ for every } Y \in \mathcal{Y}\}\,.$$

If an R-module M is the union of a continuous chain $\{M^\alpha\}_{\alpha<\lambda}$ of submodules with $M^0 \in {}^{\perp}\mathcal{Y}$ and $M^{\alpha+1}/M^\alpha \in {}^{\perp}\mathcal{Y}$ for every ordinal α with $\alpha+1<\lambda$, then $M \in {}^{\perp}\mathcal{Y}$.

Proof. Without loss of generality one can assume that λ is a limit ordinal. Indeed, if $\lambda = \kappa + 1$ is a successor ordinal, then $M = \bigcup_{\alpha<\lambda} M^\alpha = M^\kappa$ holds, so one can choose a limit ordinal $\lambda_0 > \lambda$ and extend $\{M^\alpha\}_{\alpha<\lambda}$ to a continuous chain $\{M^\alpha\}_{\alpha<\lambda_0}$ with $M^\alpha = M$ for $\lambda \leqslant \alpha < \lambda_0$.

Now, set $M^\lambda = M = \bigcup_{\alpha<\lambda} M^\alpha$. We show by transfinite induction that $M^\alpha \in {}^\perp\mathcal{Y}$ holds for every $\alpha \leqslant \lambda$; in particular, $M^\lambda = M$ is in ${}^\perp\mathcal{Y}$, as desired.

By assumption one has $M_0 \in {}^\perp\mathcal{Y}$. Now let $0 < \delta \leqslant \lambda$ be an ordinal and assume that $M^\alpha \in {}^\perp\mathcal{Y}$ holds for every $\alpha < \delta$. It must be argued that one has $M^\delta \in {}^\perp\mathcal{Y}$. There are two cases: $\delta < \lambda$ is a successor ordinal or $\delta \leqslant \lambda$ is a limit ordinal.

If $\delta < \lambda$ is a successor ordinal, say, $\delta = \alpha + 1$, one considers the exact sequence,

$$0 \longrightarrow M^\alpha \longrightarrow M^{\alpha+1} \longrightarrow M^{\alpha+1}/M^\alpha \longrightarrow 0\,.$$

By the induction hypothesis one has $M^\alpha \in {}^\perp\mathcal{Y}$ and by assumption the quotient $M^{\alpha+1}/M^\alpha$ belongs to ${}^\perp\mathcal{Y}$, so it follows from 7.3.35 that $M^\delta = M^{\alpha+1}$ is in ${}^\perp\mathcal{Y}$.

Now let $\delta \leqslant \lambda$ be a limit ordinal. To show $M^\delta \in {}^\perp\mathcal{Y}$, let $Y \in \mathcal{Y}$ be given. To prove that $\operatorname{Ext}^1_R(M^\delta, Y) = 0$ holds, it suffices by 7.3.36 to see that every exact sequence,

$$(\dagger) \qquad 0 \longrightarrow Y \xrightarrow{\;\psi\;} X \xrightarrow{\;\varphi\;} M^\delta \longrightarrow 0\,,$$

is split. Let $\alpha \leqslant \delta$ be given. As $\operatorname{Im}\psi = \operatorname{Ker}\varphi = \varphi^{-1}(\{0\}) \subseteq \varphi^{-1}(M^\alpha)$ holds, ψ corestricts to a homomorphism $Y \to \varphi^{-1}(M^\alpha)$, which we also denote by ψ. Evidently, φ (co)restricts to a surjective homomorphism $\varphi^{-1}(M^\alpha) \to M^\alpha$, which we also denote by φ. Thus, for every $\alpha \leqslant \delta$ there is an exact sequence,

$$(\dagger_\alpha) \qquad 0 \longrightarrow Y \xrightarrow{\;\psi\;} \varphi^{-1}(M^\alpha) \xrightarrow{\;\varphi\;} M^\alpha \longrightarrow 0\,.$$

The sequence $(\dagger_\delta)$ coincides with $(\dagger)$, and for $\alpha < \delta$ the sequence $(\dagger_\alpha)$ is split by 7.3.36 as one has $M^\alpha \in {}^\perp\mathcal{Y}$. By transfinite induction we now construct a family of homomorphisms $\{\sigma^\alpha \colon M^\alpha \to \varphi^{-1}(M^\alpha)\}_{\alpha\leqslant\delta}$ such that for every $\alpha \leqslant \delta$ one has:

($\flat_\alpha$) For every $x \in M^\alpha$ one has $\varphi\sigma^\alpha(x) = x$.
($\sharp_\alpha$) For every $\beta \leqslant \alpha$ and $x \in M^\beta$ one has $\sigma^\alpha(x) = \sigma^\beta(x)$.

Once such a family is constructed, the homomorphism $\sigma^\delta \colon M^\delta \to \varphi^{-1}(M^\delta) = X$ is a right inverse of $\varphi\colon X \to M^\delta$ by ($\flat_\delta$), so the sequence $(\dagger)$ is split, as desired.

As the sequence $(\dagger_0)$ is split there is a homomorphism $\sigma^0 \colon M^0 \to \varphi^{-1}(M^0)$ with $\varphi\sigma^0(x) = x$ for $x \in M^0$. Thus condition ($\flat_0$) holds, and trivially so does ($\sharp_0$).

Now, let $0 < \gamma \leqslant \delta$ be given and assume that we have constructed a family $\{\sigma^\alpha \colon M^\alpha \to \varphi^{-1}(M^\alpha)\}_{\alpha<\gamma}$ such that conditions ($\flat_\alpha$) and ($\sharp_\alpha$) hold for every $\alpha < \gamma$. The goal is to construct a homomorphism $\sigma^\gamma \colon M^\gamma \to \varphi^{-1}(M^\gamma)$ satisfying conditions ($\flat_\gamma$) and ($\sharp_\gamma$). Again there are two cases to consider: $\gamma \leqslant \delta$ is a limit ordinal or $\gamma < \delta$ is a successor ordinal.

Assume first that $\gamma \leqslant \delta$ is a limit ordinal. As one has $M^\gamma = \bigcup_{\alpha<\gamma} M^\alpha$, every element $x \in M^\gamma$ is in M^α for some $\alpha < \gamma$. If x is in both M^α and M^β for some $\alpha, \beta < \gamma$, then one has $\sigma^\alpha(x) = \sigma^\beta(x)$. Indeed, as λ is well-ordered one can assume that $\beta \leqslant \alpha$ holds, so condition ($\sharp_\alpha$), which holds by the induction hypothesis, yields $\sigma^\alpha(x) = \sigma^\beta(x)$. Thus one can define a homomorphism $\sigma^\gamma \colon M^\gamma \to \varphi^{-1}(M^\gamma)$ by setting $\sigma^\gamma(x) = \sigma^\alpha(x)$ for any ordinal $\alpha < \gamma$ with $x \in M^\alpha$. Condition ($\sharp_\gamma$) holds by construction. For $\alpha < \gamma$ and $x \in M^{\alpha}$ one has $\varphi\sigma^{\gamma}(x) = \varphi\sigma^{\alpha}(x) = x$ by the definition of σ^{γ} and condition $(\flat_{\alpha})$, which holds by the induction hypothesis. Thus, condition $(\flat_{\gamma})$ holds as well.

Finally, assume that $\gamma < \delta$ is a successor ordinal, say, $\gamma = \omega + 1$. By the induction hypothesis there is a homomorphism $\sigma^{\omega}\colon M^{\omega} \to \varphi^{-1}(M^{\omega})$ with $\varphi\sigma^{\omega}(x) = x$ for $x \in M^{\omega}$, see condition $(\flat_{\omega})$. As the sequence $(\dagger_{\omega+1})$ is split there is a homomorphism $\tau\colon M^{\omega+1} \to \varphi^{-1}(M^{\omega+1})$ with $\varphi\tau(x) = x$ for $x \in M^{\omega+1}$. It follows that one has $\tau(M^{\omega}) \subseteq \varphi^{-1}(M^{\omega})$, so τ (co)restricts to a homomorphism $\tau\colon M^{\omega} \to \varphi^{-1}(M^{\omega})$. For $x \in M^{\omega}$ one has $\varphi\sigma^{\omega}(x) - \varphi\tau(x) = x - x = 0$, so the image of the homomorphism $\sigma^{\omega} - \tau\colon M^{\omega} \to \varphi^{-1}(M^{\omega})$ is contained in the kernel of $\varphi\colon \varphi^{-1}(M^{\omega}) \to M^{\omega}$. Now exactness of the sequence $(\dagger_{\omega})$ yields a (unique) homomorphism $\vartheta\colon M^{\omega} \to Y$ with $\psi\vartheta = \sigma^{\omega} - \tau$. By 7.3.35 and 7.3.27 application of $\operatorname{Hom}_R(-,Y)$ to the exact sequence $0 \to M^{\omega} \to M^{\omega+1} \to M^{\omega+1}/M^{\omega} \to 0$ yields an exact sequence,

$$\operatorname{Hom}_R(M^{\omega+1},Y) \longrightarrow \operatorname{Hom}_R(M^{\omega},Y) \longrightarrow \operatorname{Ext}^1_R(M^{\omega+1}/M^{\omega},Y) \ .$$

By assumption one has $\operatorname{Ext}^1_R(M^{\omega+1}/M^{\omega},Y) = 0$ so $\vartheta\colon M^{\omega} \to Y$ extends to a homomorphism $\tilde{\vartheta}\colon M^{\omega+1} \to Y$. Set $\sigma^{\omega+1} = \psi\tilde{\vartheta} + \tau\colon M^{\omega+1} \to \varphi^{-1}(M^{\omega+1})$. For $x \in M^{\omega+1}$ one has $\varphi\sigma^{\omega+1}(x) = \varphi\psi\tilde{\vartheta}(x) + \varphi\tau(x) = 0 + x = x$ so condition $(\flat_{\omega+1})$ is satisfied. Further, for $\beta < \gamma = \omega + 1$, i.e. for $\beta \leqslant \omega$, and $x \in M^{\beta} \subseteq M^{\omega}$ one has

$$\sigma^{\omega+1}(x) \ = \ \psi\tilde{\vartheta}(x) + \tau(x) \ = \ \psi\vartheta(x) + \tau(x) \ = \ \sigma^{\omega}(x) \ = \ \sigma^{\beta}(x) \ ,$$

where the last equality follows from condition $(\sharp_{\omega})$, which holds by the induction hypothesis. This shows that condition $(\sharp_{\omega+1})$ holds. □

The next result is due to Auslander [7]; here we derive it as an easy consequence of Eklof's lemma. The case $n = 0$ says that the class of projective modules is closed under transfinite extensions.

D.3 Corollary. *Let M be an R-module and $n \geqslant 0$ an integer. If M is the union of a continuous chain $\{M^{\alpha}\}_{\alpha<\lambda}$ of submodules with*

$$\operatorname{pd}_R M^0 \ \leqslant \ n \qquad \textit{and} \qquad \operatorname{pd}_R(M^{\alpha+1}/M^{\alpha}) \ \leqslant \ n$$

for every ordinal α with $\alpha + 1 < \lambda$, then one has $\operatorname{pd}_R M \leqslant n$.

Proof. Choose for each R-module N an injective replacement I^N and notice from 8.2.6 that one has $\operatorname{Ext}^{n+1}_R(M',N) \cong \operatorname{Ext}^1_R(M',\mathrm{Z}_{-n}(I^N))$ for every R-module M'. Thus, it follows from 8.1.8 that with $\mathcal{Z}_n = \{\mathrm{Z}_{-n}(I^N) \mid N \in \mathcal{M}(R)\}$ the class

$$^{\perp}\mathcal{Z}_n \ = \ \{M' \in \mathcal{M}(R) \mid \operatorname{Ext}^1_R(M',Z) = 0 \text{ for every } Z \in \mathcal{Z}_n\} \ .$$

coincides with the class of R-modules M' with $\operatorname{pd}_R M' \leqslant n$. Now apply D.2. □

D.4 Lemma. *Let $\varkappa\colon M \to N$ be a homomorphism of R-modules and assume that M is the union of a continuous chain $\{M_{\alpha}\}_{\alpha<\lambda}$ of submodules. If $\varkappa|_{M_{\alpha}}\colon M_{\alpha} \to N$ is a pure monomorphism for every $\alpha < \lambda$, then $\varkappa$ is a pure monomorphism.*

Proof. For $\alpha < \beta < \lambda$ there is a commutative diagram with exact rows,

$$\begin{tikzcd}
\eta_\alpha \; = \; 0 \arrow[r] & M_\alpha \arrow[r, "\varkappa|_{M_\alpha}"] \arrow[d, tail] & N \arrow[r] \arrow[d, equal] & \operatorname{Coker}(\varkappa|_{M_\alpha}) \arrow[r] \arrow[d] & 0 \\
\eta_\beta \; = \; 0 \arrow[r] & M_\beta \arrow[r, "\varkappa|_{M_\beta}"] & N \arrow[r] & \operatorname{Coker}(\varkappa|_{M_\beta}) \arrow[r] & 0 \, .
\end{tikzcd}$$

These diagrams yield a filtered direct systems of R-complexes with

$$\eta \;=\; \operatorname*{colim}_{\alpha<\lambda} \eta_\alpha \;=\; 0 \longrightarrow M \xrightarrow{\;\varkappa\;} N \longrightarrow \operatorname{Coker}\varkappa \longrightarrow 0\,,$$

see 3.3.3, and η is exact by 3.3.16. Let K be an R^{o}-module; by 3.2.23 one has

$$K \otimes_R \eta \;=\; K \otimes_R \big(\operatorname*{colim}_{\alpha<\lambda} \eta_\alpha\big) \;\cong\; \operatorname*{colim}_{\alpha<\lambda} (K \otimes_R \eta_\alpha)\,.$$

Each sequence $K \otimes_R \eta_\alpha$ is exact as $\varkappa|_{M_\alpha}$ is a pure monomorphism, see 5.5.15, so it follows, again by 3.3.16, that $K \otimes_R \eta$ is exact. Thus, $K \otimes_R \eta$ is exact for every R^{o}-module K, whence η is pure exact; that is, $\varkappa$ is a pure monomorphism. $\square$

Countably Related Flat Modules

D.5 Definition. Let $\aleph$ be an infinite cardinal. An R-module is called $\aleph$-*generated* if it has a set E of generators with $\operatorname{card} E \leqslant \aleph$, and it is called $\aleph^<$-*generated* if it has a set E of generators with $\operatorname{card} E < \aleph$. For an ordinal α the notation $\alpha < \aleph$ means $\alpha < \omega_\aleph$, where $\omega_\aleph$ is the first ordinal of cardinality $\aleph$. For an $\aleph$-generated module this convention allows us to write a set of generators on the form $E = \{e_\alpha\}_{\alpha<\aleph}$.

An $\aleph_0$-generated module is called *countably generated*; notice that a module is $\aleph_0^<$-generated if and only if it is finitely generated.

D.6. Let L be a free R-module with basis $\{e_u\}_{u\in U}$. Every element l in L has a representation $l = \sum_{u\in U_l} r_u e_u$ for a uniquely determined finite subset U_l of U and uniquely determined elements $r_u \neq 0$ in R. Note that U_0 is the empty set.

Given a submodule K of L, set $U_K = \bigcup_{k\in K} U_k$; now K is contained in the free submodule $L' = R\langle U_K\rangle$ of L, and the quotient L/L' is free with basis $U \setminus U_K$. Notice that if K is generated by $k_1, \dots, k_n$, then one has $U_K = U_{k_1} \cup \dots \cup U_{k_n}$; that is, U_K is finite. Similarly, if $\aleph$ is an infinite cardinal and K is $\aleph$-generated, then $\operatorname{card} U_K \leqslant \aleph$ holds.

D.7 Proposition. *For an exact sequence $\eta = 0 \longrightarrow K \xrightarrow{\;\varkappa\;} L \xrightarrow{\;\pi\;} F \longrightarrow 0$ of R-modules where L is free, the following conditions are equivalent.*

(*i*) *F is flat.*

(*ii*) *The sequence η is pure.*

(*iii*) *For every finitely generated submodule H of K there is a homomorphism $\varrho\colon L \to K$ that satisfies $\varrho\varkappa|_H = 1^H$.*

Proof. The equivalence of conditions (*i*) and (*ii*) is a special case of 5.5.18.

$(ii) \Rightarrow (iii)$: Let H be a finitely generated submodule of K. Choose by 1.3.12 a surjective homomorphism $\pi' : L' \to H$ with L' finitely generated and free. As $\operatorname{Im}(\varkappa\pi') = \varkappa(H)$ is a finitely generated submodule of L, it is contained in a finitely generated direct summand L'' of L; cf. D.6. Now consider the commutative diagram,

$$\begin{tikzcd} L' \arrow[r, "\varkappa\pi'"] \arrow[d, "\pi'"'] & L'' \arrow[d, tail, "\varepsilon"] \\ K \arrow[r, "\varkappa"] & L\,. \end{tikzcd}$$

By (ii) the sequence η is pure exact, so 5.5.14 yields a homomorphism $\varrho' : L'' \to K$ with $\varrho'(\varkappa\pi') = \pi'$. As L'' is a direct summand of L, there is a projection $\varpi : L \twoheadrightarrow L''$ with $\varpi\varepsilon = 1^{L''}$. Set $\varrho = \varrho'\varpi$ and note that one has $\varrho\varkappa\pi' = \varrho'\varpi\varepsilon\varkappa\pi' = \varrho'\varkappa\pi' = \pi'$. As $\operatorname{Im}\pi'$ is H it follows that $\varrho\varkappa|_H = 1^H$ holds, as desired.

$(iii) \Rightarrow (ii)$: Consider a commutative diagram,

$$\begin{tikzcd} L' \arrow[r, "\varkappa'"] \arrow[d, "\varphi'"'] & L'' \arrow[d, "\varphi''"] \\ K \arrow[r, "\varkappa"] & L\,, \end{tikzcd}$$

where also L' and L'' are finitely generated free R-modules. To show that η is pure, it is by 5.5.14 sufficient to show the existence of a homomorphism $\varrho' : L'' \to K$ with $\varrho'\varkappa' = \varphi'$. As $\operatorname{Im}\varphi'$ is a finitely generated submodule of K, it follows from (iii) that there is a homomorphism $\varrho : L \to K$ with $\varrho\varkappa\varphi' = \varphi'$. Now set $\varrho' = \varrho\varphi''$, and note that one has $\varrho'\varkappa' = \varrho\varphi''\varkappa' = \varrho\varkappa\varphi' = \varphi'$, as desired. □

D.8. Consider a short exact sequence of R-modules, $0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\pi} F \longrightarrow 0$ where L is free with basis $\{e_u\}_{u \in U}$ and F is flat. Let K' be a submodule of K and L' a free submodule of L with $\varkappa(K') \subseteq L'$; cf. D.6. Let $\varkappa' : K' \to L'$ be the restriction of $\varkappa$, set $F' = \operatorname{Coker}\varkappa'$, and let $\pi' : L' \to F'$ be the canonical homomorphism. If K' is a pure submodule of K, then F' is flat. Indeed, there is a commutative diagram

(D.8.1)
$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r, "\varkappa'"] \arrow[d, tail, "\iota"] & L' \arrow[r, "\pi'"] \arrow[d, tail] & F' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r, "\varkappa"] & L \arrow[r, "\pi"] & F \arrow[r] & 0\,. \end{tikzcd}$$

It is elementary to verify that a composite $\alpha\beta$ of monomorphisms is pure if both α and β are pure and only if β is pure. By D.7 the monomorphism $\varkappa$ is pure. Thus, if ι is pure, then $\varkappa\iota$ is pure, whence $\varkappa'$ is pure by commutativity of (D.8.1), and F' is flat.

An R-module M is called *countably related* if there exists a short exact sequence $0 \to K \to L \to M \to 0$ where L is free and K is countably generated. The next theorem asserts: A countably related flat module has projective dimension at most 1.

D.9 Theorem. *Let $0 \to K \to L \to F \to 0$ be an exact sequence of R-modules where L is free and F is flat. If K is countably generated, then K is projective.*

Proof. By 8.1.20 the module K is projective if and only if $\operatorname{pd}_R F \leqslant 1$ holds. As in D.8 denote the homomorphisms in the exact sequence by $\varkappa$ and π. Since K is countably generated, there is a diagram as (D.8.1) with L' countably generated and $L'' = L/L'$ free; see D.6. As one has $F = F' \oplus L''$, one can assume that L is countably generated in the first place. To prove the inequality $\operatorname{pd}_R F \leqslant 1$, we write K as the union of a chain $K^1 \subseteq K^2 \subseteq \cdots$ of finitely generated submodules and construct endomorphisms $\varphi^u \colon L \to L$ with the following properties.

(a) $\pi\varphi^u = \pi$ for all u in $\mathbb{N}$.
(b) $\varphi^u\varkappa|_{K^u} = 0$ for all u in $\mathbb{N}$.
(c) $\varphi^w\varphi^v = \varphi^w$ for all $w > v$ in $\mathbb{N}$.

From these data we construct an exact sequence,

$$(\diamond) \qquad 0 \longrightarrow L^{(\mathbb{N})} \xrightarrow{\;\Phi\;} L^{(\mathbb{N})} \xrightarrow{\;\Pi\;} F \longrightarrow 0\,,$$

showing that $\operatorname{pd}_R F \leqslant 1$ holds as desired.

Assume for the moment that the submodules K^u and the endomorphisms φ^u have been constructed. The sequence $(\diamond)$ is then obtained by setting

$$\Phi((l^u)_{u\in\mathbb{N}}) \;=\; (l^u - \varphi^{u-1}(l^{u-1}))_{u\in\mathbb{N}} \qquad\text{and}\qquad \Pi((l^u)_{u\in\mathbb{N}}) \;=\; \pi\big(\sum_{u\in\mathbb{N}} l^u\big)\,,$$

with the ingenuous convention $\varphi^0(l^0) = 0$. It is evident that Φ is injective and that Π is surjective; furthermore, (a) yields

$$\Pi\Phi((l^u)_{u\in\mathbb{N}}) \;=\; \pi\big(\sum_{u\in\mathbb{N}}(l^u - \varphi^{u-1}(l^{u-1}))\big) \;=\; \pi\big(\sum_{u\in\mathbb{N}} l^u\big) - \pi\big(\sum_{u\in\mathbb{N}}\varphi^u(l^u)\big) \;=\; 0\,.$$

To show that $(\diamond)$ is exact, it remains to see that $\operatorname{Ker}\Pi$ is contained in $\operatorname{Im}\Phi$. Let $l = (l^u)_{u\in\mathbb{N}}$ be an element in $\operatorname{Ker}\Pi$, then $\sum_{u\in\mathbb{N}} l^u$ is in $\operatorname{Ker}\pi = \operatorname{Im}\varkappa$, so there is an element k in $K = \bigcup_{u\in\mathbb{N}} K^u$ with $\varkappa(k) = \sum_{u\in\mathbb{N}} l^u$. As the submodules K^u form an ascending chain, one has $k \in K^u$ for all $u \gg 0$. Choose v such that k is in K^v and such that $l^u = 0$ holds for all $u > v$. Define $x = (x^u)_{u\in\mathbb{N}}$ in $L^{(\mathbb{N})}$ by setting $x^1 = l^1$,

$$x^u \;=\; l^u + \varphi^{u-1}\big(\sum_{i=1}^{u-1} l^i\big) \quad\text{for}\quad 1 < u \leqslant v \qquad\text{and}\qquad x^u \;=\; 0 \quad\text{for}\quad u > v\,.$$

One has $\Phi(x) = l$; that is, $x^u - \varphi^{u-1}(x^{u-1}) = l^u$ holds for all $u \in \mathbb{N}$. It is evident for $u = 1$, and for $u = 2$ it is a short computation, $x^2 - \varphi^1(x^1) = l^2 + \varphi^1(l^1) - \varphi^1(l^1) = l^2$. For $2 < u \leqslant v$ one has $\varphi^{u-1}\varphi^{u-2} = \varphi^{u-1}$ by (c) and, consequently,

$$x^u - \varphi^{u-1}(x^{u-1}) \;=\; l^u + \varphi^{u-1}\big(\sum_{i=1}^{u-1} l^i\big) - \varphi^{u-1}\big(l^{u-1} + \varphi^{u-2}\big(\sum_{i=1}^{u-2} l^i\big)\big) \;=\; l^u\,.$$

For $u > v+1$ one has $x^u - \varphi^{u-1}(x^{u-1}) = 0 = l^u$ and, finally, (c) and (b) yield

$$x^{v+1} - \varphi^v(x^v) \;=\; 0 - \varphi^v\big(l^v + \varphi^{v-1}\big(\sum_{i=1}^{v-1} l^i\big)\big) \;=\; -\varphi^v\big(\sum_{i=1}^{v} l^i\big) \;=\; \varphi^v\varkappa(k) \;=\; 0 \;=\; l^{v+1}\,.$$

It remains to construct the submodules K^u and the endomorphisms φ^u. Recall that L is countably generated and let $\{e_u\}_{u\in\mathbb{N}}$ be a basis; we proceed with a construction that applies to every finitely generated submodule H of K. Assume that H is generated by elements $h_1, \dots, h_n$. By D.7 there is a homomorphism $\varrho\colon L \to K$ with $\varrho\varkappa|_H = 1^H$. Choose $v \gg 0$ such that $\varkappa(H)$ is contained in $R\langle e_1, \dots, e_v\rangle$, and denote by $\widetilde{H}$ the submodule of K generated by $\varrho(e_1), \dots, \varrho(e_v)$. The assignments

$$e_u \mapsto e_u - \varkappa\varrho(e_u) \quad \text{for} \quad 1 \leqslant u \leqslant v \qquad \text{and} \qquad e_u \mapsto e_u \quad \text{for} \quad u > v$$

define an endomorphism $\varphi^H : L \to L$ with the following properties,

(1) For every $u \in \mathbb{N}$ one has $\varphi^H(e_u) = e_u + x_u$ for some $x_u \in \varkappa(\widetilde{H})$.
(2) $\pi\varphi^H = \pi$.
(3) $\varphi^H\varkappa|_H = 0$.
(4) $H \subseteq \widetilde{H}$.

Property (1) is immediate from the definitions of φ^H and $\widetilde{H}$, and then (2) holds as the composite $\pi\varkappa$ is the zero morphism. To establish (3) and (4) write

$$\varkappa(h_\jmath) = \sum_{u=1}^{v} r_{\jmath u} e_u \quad \text{for} \quad \jmath \in \{1, \dots, n\}\,.$$

For every $\jmath \in \{1, \dots, n\}$ one has $h_\jmath = \varrho\varkappa(h_\jmath) = \sum_{u=1}^{v} r_{\jmath u}\varrho(e_u)$, which proves (4). The next computation establishes (3); the penultimate equality uses the identity $\varrho\varkappa|_H = 1^H$.

$$\begin{aligned} \varphi^H\varkappa(h_\jmath) &= \varphi^H\Big(\sum_{u=1}^{v} r_{\jmath u} e_u\Big) \\ &= \sum_{u=1}^{v} r_{\jmath u}(e_u - \varkappa\varrho(e_u)) \\ &= \varkappa(h_\jmath) - \varkappa\varrho\varkappa(h_\jmath) \\ &= \varkappa(h_\jmath - \varrho\varkappa(h_\jmath)) \\ &= 0 \end{aligned}$$

Being countably generated, K is the union of an ascending chain $H^1 \subseteq H^2 \subseteq \cdots$ of finitely generated submodules. Set $K^1 = H^1$ and $K^u = \widetilde{K}^{u-1} + H^u$ for $u > 1$. By these definitions and (4) one has $K = \bigcup_{u\in\mathbb{N}} K^u$ and

$$K^1 \subseteq \widetilde{K}^1 \subseteq K^2 \subseteq \widetilde{K}^2 \subseteq K^3 \subseteq \cdots\,.$$

For every $u \in \mathbb{N}$ set $\varphi^u = \varphi^{K^u}$; the properties (a) and (b) follow immediately from (2) and (3). To prove (c), let $w > v$ and apply (1) to write $\varphi^{K^v}(e_u) = e_u + x_u$ for some $x_u \in \varkappa(\widetilde{K}^v) \subseteq \varkappa(K^w)$. As $\varphi^{K^w}\varkappa|_{K^w} = 0$ holds by (b) one has $\varphi^{K^w}(x_u) = 0$ and, therefore, $\varphi^{K^w}\varphi^{K^v}(e_u) = \varphi^{K^w}(e_u + x_u) = \varphi^{K^w}(e_u)$. □

Proof of Jensen's Theorem

Theorem D.9 and the next result furnish the base cases for the induction arugments that establish Jensen's theorem.

D.10 Lemma. *Let* $0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\pi} F \longrightarrow 0$ *be an exact sequence of* R*-modules where* L *is free and* F *is flat. For every countably generated submodule* H *of* K *there is a countably generated submodule* H' *of* K *such that* H *is contained in* H' *and* $\varkappa|_{H'}$ *is a pure monomorphism.*

Proof. Let $\{e_u\}_{u\in U}$ be a basis for L and $\{h_n\}_{n\in\mathbb{N}}$ a set of generators for H. Set $H^0 = 0$; we construct a family of modules $\{H^n\}_{n\in\mathbb{N}}$ and a family of homomorphisms $\{\varrho^n : L \to H^n\}_{n\in\mathbb{N}}$ with the following properties.

(a) H^n is a finitely generated submodule of K with $h_n \in H^n$ and $H^{n-1} \subseteq H^n$.

(b) $\varrho^n\varkappa|_{H^{n-1}} = 1^{H^{n-1}}$.

Assuming that such families have been constructed, set $H' = \bigcup_{n\in\mathbb{N}} H^n$. By construction, H' is a submodule of K and countably generated. Moreover, H' contains H as each generator h_n is in H' by (a). To see that the monomorphism $\varkappa|_{H'}$ is pure, it suffices by D.7 to argue that for every finitely generated submodule H'' of H' there is a homomorphism $\varrho : L \to H'$ with $\varrho\varkappa|_{H''} = 1^{H''}$. Thus, let H'' be a finitely generated submodule of H' and choose by (a) an m in $\mathbb{N}$ such that H'' is contained in H^m. Since $\varrho^{m+1} : L \to H^{m+1} \subseteq H'$ satisfies $\varrho^{m+1}\varkappa|_{H^m} = 1^{H^m}$ by (b), the homomorphism $\varrho = \varrho^{m+1}$ has the sought after property.

To construct the submodules H^n and the homomorphisms ϱ^n, set $\varrho^0 = 0$ and proceed recursively. Assuming that H^{n-1} and ϱ^{n-1} have been constructed, the submodule $K^n = H^{n-1} + R\langle h_n\rangle$ of K is finitely generated, so there exists by D.7 a homomorphism $\varrho' : L \to K$ with $\varrho'\varkappa|_{K^n} = 1^{K^n}$. Let $U_{\varkappa(K^n)}$ be the set constructed in D.6. The assignments

$$e_u \mapsto \varrho'(e_u) \quad \text{for} \quad u \in U_{\varkappa(K^n)} \qquad \text{and} \qquad e_u \mapsto 0 \quad \text{for} \quad u \notin U_{\varkappa(K^n)}$$

define a homomorphism $\varrho^n : L \to K$, and with $H^n = \operatorname{Im}\varrho^n$ both (a) and (b) hold. Indeed, the set $U_{\varkappa(K^n)}$ is finite, and thus H^n is finitely generated. For $h \in K^n$ one has $\varkappa(h) = \sum_{u\in U_{\varkappa(h)}} r_u e_u$, and since $U_{\varkappa(h)}$ is contained in $U_{\varkappa(K^n)}$, one obtains

$$(\dagger) \qquad h = \varrho'\varkappa(h) = \sum_{u\in U_{\varkappa(h)}} r_i\varrho'(e_u) = \sum_{u\in U_{\varkappa(h)}} r_i\varrho^n(e_u) = \varrho^n\varkappa(h)\,;$$

in particular, h belongs to H^n. With K^n contained in H^n, one has both $h_n \in H^n$ and $H^{n-1} \subseteq H^n$, which establishes (a). Furthermore, as H^{n-1} is contained in K^n one has $\varrho^n\varkappa|_{H^{n-1}} = 1^{H^{n-1}}$, also by (†), whence (b) holds as well. □

The next result is inspired by Osofsky [198].

D.11 Proposition. *Let* $0 \longrightarrow K \xrightarrow{\varkappa} L \xrightarrow{\pi} F \longrightarrow 0$ *be an exact sequence of* R*-modules where* L *is free and* F *is flat. Let* $\aleph$ *be an infinite cardinal and* H *an* $\aleph$*-generated submodule of* K.

(a) *There is an* $\aleph$*-generated submodule* H' *of* K *such that* H *is contained in* H' *and* $\varkappa|_{H'}$ *is a pure monomorphism.*

(b) *If* $\aleph$ *is uncountable, then there is a continuous chain* $\{H^\alpha\}_{\alpha<\aleph}$ *of submodules in* K*, such that each module* H^α *is* $\max\{\operatorname{card}\alpha, \aleph_0\}$*-generated, each monomorphism* $\varkappa|_{H^\alpha}$ *is pure, and one has* $H \subseteq \bigcup_{\alpha<\aleph} H^\alpha$.

Proof. We proceed by induction on $\aleph$. For $\aleph = \aleph_0$ part (a) holds by D.10. Assume now that $\aleph$ is uncountable and that (a) holds for all $\aleph^<$-generated submodules of K. The goal is to prove (a) and (b) for all $\aleph$-generated submodules H of K; we start by noting that (a) follows from (b). Indeed, assume (b) holds and set $H' = \bigcup_{\alpha<\aleph} H^\alpha$.

This module contains H, and it follows from D.4 that $\varkappa|_{H'}$ is a pure monomorphism. Since each module H^α has a set E_α of generators with $\operatorname{card} E_\alpha \leqslant \aleph$ (actually, $\operatorname{card} E_\alpha < \aleph$) the set $E' = \bigcup_{\alpha<\aleph} E_\alpha$ of generators for H' has $\operatorname{card} E' \leqslant \aleph^2 = \aleph$.

To prove (b), let H be an $\aleph$-generated submodule of K and $\{h_\alpha\}_{\alpha<\aleph}$ a set of generators for H. We construct a continuous chain $\{H^\alpha\}_{\alpha<\aleph}$ of $\max\{\operatorname{card}\alpha, \aleph_0\}$-generated submodules of K such that each monomorphism $\varkappa|_{H^\alpha}$ is pure, and one has $h_\alpha \in H^\beta$ for all $\alpha < \beta$. The module $H' = \bigcup_{\alpha<\aleph} H^\alpha$ then contains the family $\{h_\alpha\}_{\alpha<\aleph}$ and hence it contains H.

Set $H^0 = 0$; it is generated by $\emptyset$, it is a submodule of every other submodule of K, and the zero morphism is pure. Now, let $\gamma < \aleph$ and assume that there is a continuous chain $\{H^\alpha\}_{\alpha<\gamma}$ of $\max\{\operatorname{card}\alpha, \aleph_0\}$-generated submodules of K such that each monomorphism $\varkappa|_{H^\alpha}$ is pure, and one has $h_\alpha \in H^\beta$ for all $\alpha < \beta < \gamma$.

If γ is a limit ordinal, then $H^\gamma = \bigcup_{\alpha<\gamma} H^\alpha$ contains h_α for all $\alpha < \gamma$, the family $\{H^\alpha\}_{\alpha<\gamma+1}$ is a continuous chain of submodules, and $\varkappa|_{H^\gamma}$ is pure by D.4. To see that H^γ is $\max\{\operatorname{card}\gamma, \aleph_0\}$-generated, note that each module H^α for $\alpha < \gamma$ has a set of generators E_α with $\operatorname{card} E_\alpha \leqslant \max\{\operatorname{card}\alpha, \aleph_0\} \leqslant \operatorname{card}\gamma$. Thus, the set $E_\gamma = \bigcup_{\alpha<\gamma} E_\alpha$ of generators for H^γ has $\operatorname{card} E_\gamma \leqslant (\operatorname{card}\gamma)^2 = \operatorname{card}\gamma = \max\{\operatorname{card}\gamma, \aleph_0\}$.

If $\gamma = \alpha + 1$ is a successor ordinal, consider the submodule $K' = H^\alpha + R\langle h_\alpha\rangle$ of K. By assumption, H^α and, therefore, K' has a set of generators of cardinality at most $\max\{\operatorname{card}\alpha, \aleph_0\} = \max\{\operatorname{card}\gamma, \aleph_0\} < \aleph$. By the assumption that (a) holds for all $\aleph^<$-generated submodules of K, there is a $\max\{\operatorname{card}\gamma, \aleph_0\}$-generated submodule H^γ of K such that $\varkappa|_{H^\gamma}$ is pure and K' is contained in H^γ. This ensures that h_α is in H^γ for all $\alpha < \gamma$ and that $\{H^\alpha\}_{\alpha<\gamma+1}$ is a continuous chain of submodules. □

The next result is essentially a restatement of Jensen's theorem.

D.12 Theorem. *If* FPD R *is finite, then every pure submodule of a free R-module has finite projective dimension.*

Proof. Let L be a free R-module, $\aleph$ an infinite cardinal, and H an $\aleph$-generated pure submodule of L. By transfinite induction we prove that H has finite projective dimension. Note that L/H is a flat module by D.7.

If H is $\aleph_0$-generated, then it is projective by D.9. Now let $\aleph > \aleph_0$ and assume that all $\aleph^<$-generated pure submodules of L have finite projective dimension. An application of D.11 to the exact sequence $0 \to H \to L \to L/H \to 0$ shows that H is the union of a continuous chain $\{H^\alpha\}_{\alpha<\aleph}$ of $\aleph^<$-generated pure submodules of L. Set $p = \operatorname{FPD} R$; by the induction hypothesis the modules in the chain have $\operatorname{pd}_R H^\alpha \leqslant p$. For every α with $\alpha + 1 < \aleph$ it follows from 8.1.9 applied to the sequence $0 \to H^\alpha \to H^{\alpha+1} \to H^{\alpha+1}/H^\alpha \to 0$ that also $\operatorname{pd}_R(H^{\alpha+1}/H^\alpha) \leqslant p$ holds. The desired conclusion, $\operatorname{pd}_R H \leqslant p$, is now immediate from D.3. □

Proof of 8.5.18. One can assume that FPD R is finite. Let F be a flat R-module and choose by 1.3.12 a surjective homomorphism $\pi\colon L \to F$ with L free. The exact sequence $0 \to \operatorname{Ker}\pi \to L \to F \to 0$ is pure by D.7, so $\operatorname{Ker}\pi$ has finite projective dimension by D.12. It follows from 8.1.9 that F has finite projective dimension as well, whence $\operatorname{pd}_R F \leqslant \operatorname{FPD} R$ holds. □

MODULES OF FINITE PROJECTIVE DIMENSION OVER NOETHERIAN RINGS

The next goal is to prove a companion result to 8.5.18, which was noticed by Jensen in [148, §5] and stated here in 8.5.24:

Theorem. *Assume that R is left Noetherian and let M be an R-module. If M has finite projective dimension, then* $\mathrm{pd}_R\, M \leqslant \mathrm{FFD}\, R + 1$ *holds.*

They key ingredient in the proof, which comes after D.15, is D.14; we prove it here with an argument learned from Raynaud and Gruson [207].

D.13 Lemma. *Assume that R is left Noetherian and let M be an R-module. If M is countably generated, then every submodule of M is countably generated.*

PROOF. Let M be generated by elements $\{x_n \mid n \in \mathbb{N}\}$; in particular, M is the union $\bigcup_{k\in\mathbb{N}} R\langle x_1,\ldots,x_k\rangle$. A submodule N of M is the union $\bigcup_{k\in\mathbb{N}}(R\langle x_1,\ldots,x_k\rangle \cap N)$, and each submodule $R\langle x_1,\ldots,x_k\rangle \cap N$ is finitely generated, as R is left Noetherian. Thus N is countably generated. □

REMARK. A ring with the property that every ideal is countably generated, equivalently, every submodule of a countably generated module is countably generated, is called left $\aleph_0$-Noetherian. See Jensen [146] and Osofsky [198].

D.14 Proposition. *Assume that R is left Noetherian and let M be an R-module. If M has finite projective dimension, then it is the union of a continuous chain $\{M^\alpha\}_{\alpha<\lambda}$ of submodules such that M^0 and $M^{\alpha+1}/M^\alpha$ for every ordinal α with $\alpha+1<\lambda$ are countably generated modules of finite projective dimension.*

PROOF. Let M be an R-module of finite projective dimension and $L' \xrightarrow{\simeq} M$ a free resolution, see 5.1.16. By 8.1.8 the module $\mathrm{C}_p(L')$ is projective for some $p>0$, and by Eilenberg's swindle 1.3.20 there is a free R-module L'' with $\mathrm{C}_p(L') \oplus L'' \cong L''$. The complex $L = \mathrm{D}^p(L'') \oplus L'_{\subseteq p}$ is now a semi-free replacement of M with $L_v = 0$ for $v > p$ and $v < 0$. Let $\{e_u \mid u \in U\}$ be a graded basis for the graded-free module $L^\natural$. We proceed to write L as the union of a continuous chain $\{L^\alpha\}_{\alpha<\lambda}$ of subcomplexes in such a way that for every $\alpha < \lambda$ one has:

(1) The graded submodule $(L^\alpha)^\natural$ has the form $\coprod_{u\in U_\alpha} Re_u$ where U_α is a subset of U, and $U_\beta \subseteq U_\alpha$ holds for every $\beta \leqslant \alpha$.

(2) The embedding $L^\alpha \to L$ induces an injective morphism $\mathrm{H}(L^\alpha) \to \mathrm{H}(L)$.

Notice that if L^α is a subcomplex of L that satisfies requirements (1) and (2), then $(L^\alpha)^\natural$ is graded-free with graded basis $\{e_u \mid u \in U_\alpha\}$, one has $\mathrm{H}_v(L^\alpha) = 0$ for $v \neq 0$, and $\mathrm{H}_0(L^\alpha)$ is a submodule of $\mathrm{H}_0(L) \cong M$. Further, if L^β for $\beta \leqslant \alpha$ is another subcomplex that satisfies (1) and (2), then $L^\beta \rightarrowtail L$ factors through L^α, per the inclusion $U_\beta \subseteq U_\alpha$. It follows that the injective morphism $\mathrm{H}(L^\beta) \to \mathrm{H}(L)$ factors through $\mathrm{H}(L^\alpha)$, so $L^\beta \rightarrowtail L^\alpha$ induces an injective morphism $\mathrm{H}(L^\beta) \to \mathrm{H}(L^\alpha)$.

Set $L^0 = 0$; with $U_0 = \varnothing$ it has the form specified in (1) and condition (2) is trivial. Let α be an ordinal and assume that L^β has been constructed for all ordinals $\beta < \alpha$.

In the case α is a limit ordinal, set $L^\alpha = \bigcup_{\beta<\alpha} L^\beta$ and notice from 3.3.3 and 3.3.15(d) that one has

$$(\dagger) \qquad \mathrm{H}(L^{\alpha}) \;\cong\; \mathrm{H}\big(\operatorname*{colim}_{\beta<\alpha} L^{\beta}\big) \;\cong\; \operatorname*{colim}_{\beta<\alpha} \mathrm{H}(L^{\beta}) \;\cong\; \bigcup_{\beta<\alpha} \mathrm{H}(L^{\beta})\,.$$

The complex L^{α} evidently satisfies condition (2). To see that $(L^{\alpha})^{\natural}$ has the form specified in (1), set $U_{\alpha} = \bigcup_{\beta<\alpha} U_{\beta}$ and notice that there are equalities,

$$(L^{\alpha})^{\natural} \;=\; \big(\bigcup_{\beta<\alpha} L^{\beta}\big)^{\natural} \;=\; \bigcup_{\beta<\alpha} (L^{\beta})^{\natural} \;=\; \bigcup_{\beta<\alpha} \big(\coprod_{u\in U_{\beta}} Re_u\big) \;=\; \coprod_{u\in U_{\alpha}} Re_u\,.$$

Now assume that $\alpha = \beta + 1$ is a successor ordinal. If $U_{\beta} = U$ the construction is complete. Otherwise, let ∂ denote the differential on the quotient complex $\overline{L} = L/L^{\beta}$ and for $u \in U$ let $\bar{e}_u$ denote the coset of e_u in $\overline{L}$. The graded module $(\overline{L})^{\natural}$ is graded-free with graded basis $\{\bar{e}_u \mid u \in U \setminus U_{\beta}\}$. Let $I \subseteq U \setminus U_{\beta}$ be a countable set. The submodule $\partial(\coprod_{u\in I} R\bar{e}_u)$ is countably generated and, since R is left Noetherian, so is $\mathrm{B}(\overline{L}) \cap \coprod_{u\in I} R\bar{e}_u$, see D.13. Thus, adding countably many elements from $U \setminus U_{\beta}$ to I one gets a countable subset I' with

$$\partial\big(\coprod_{u\in I} R\bar{e}_u\big) \;\subseteq\; \coprod_{u\in I'} R\bar{e}_u \qquad \text{and} \qquad \mathrm{B}(\overline{L}) \cap \coprod_{u\in I} R\bar{e}_u \;\subseteq\; \partial\big(\coprod_{u\in I'} R\bar{e}_u\big)\,.$$

Starting with $J_0 = \{\bar{e}_u\}$ for some $u \in U \setminus U_{\beta}$ the construction above yields a sequence $J_0 \subseteq J_1 \subseteq \cdots$ of countable subsets of $U \setminus U_{\beta}$, where $J_{n+1} = J_n'$, such that one has

$$\partial\big(\coprod_{u\in J_n} R\bar{e}_u\big) \;\subseteq\; \coprod_{u\in J_{n+1}} R\bar{e}_u \qquad \text{and} \qquad \mathrm{B}(\overline{L}) \cap \coprod_{u\in J_n} R\bar{e}_u \;\subseteq\; \partial\big(\coprod_{u\in J_{n+1}} R\bar{e}_u\big)\,.$$

Set $J = \bigcup_{n\geqslant 0} J_n$ and $F = \coprod_{u\in J} R\bar{e}_u$. By construction one has $\mathrm{B}(\overline{L}) \cap F = \partial(F)$; it follows that F defines a subcomplex of $\overline{L}$ and that the map $\mathrm{H}(F) \to \mathrm{H}(\overline{L})$ induced by the embedding $F \to \overline{L}$ is injective. Let $L^{\beta+1}$ be the unique subcomplex of L with $L^{\beta+1}/L^{\beta} = F$. Evidently, one has

$$(L^{\beta+1})^{\natural} \;=\; \coprod_{u\in U_{\beta+1}} Re_u \qquad \text{where} \qquad U_{\beta+1} \;=\; J \cup U_{\beta}\,.$$

The complex $L^{\beta+1}$ satisfies condition (1). To see that it satiesfies (2), notice first that the exact sequence $0 \to L^{\beta} \to L \to \overline{L} \to 0$ per 2.2.21 induces an exact sequence

$$\cdots \longrightarrow \mathrm{H}_1(L) \longrightarrow \mathrm{H}_1(\overline{L}) \longrightarrow \mathrm{H}_0(L^{\beta}) \longrightarrow \mathrm{H}_0(L) \longrightarrow \mathrm{H}_0(\overline{L}) \longrightarrow 0\,.$$

As L^{β} satisfies condition (2), one has $\mathrm{H}_v(L^{\beta}) = 0 = \mathrm{H}_v(L)$ for $v \geqslant 1$, which implies $\mathrm{H}_v(\overline{L}) = 0$ for $v \geqslant 2$. Subsequently, injectivity of the map $\mathrm{H}_0(L^{\beta}) \to \mathrm{H}_0(L)$ forces $\mathrm{H}_1(\overline{L}) = 0$. As the morphism $\mathrm{H}(F) \to \mathrm{H}(\overline{L})$ is injective, it now follows that $\mathrm{H}(F)$ is concentrated in degree 0. Applying the argument above to the exact sequence of homology modules associated to the exact sequence $0 \to F \to \overline{L} \to L/L^{\beta+1} \to 0$ one now gets that $\mathrm{H}(L/L^{\beta+1})$ is concentrated in degree 0. Finally, associated to the exact sequence $0 \to L^{\beta+1} \to L \to L/L^{\beta+1} \to 0$ one has the exact sequence,

$$\cdots \longrightarrow \mathrm{H}_1(L/L^{\beta+1}) \longrightarrow \mathrm{H}_0(L^{\beta+1}) \longrightarrow \mathrm{H}_0(L) \longrightarrow \mathrm{H}_0(L/L^{\beta+1}) \longrightarrow 0\,.$$

It shows that the morphism $\mathrm{H}(L^{\beta+1}) \to \mathrm{H}(L)$ is injective.

For every successor ordinal $\beta+1$ with $U_{\beta} \subset U$ one has $U_{\beta} \subset U_{\beta+1}$ so the process described above terminates; as desired, L is now the union of a continuous chain $\{L^{\alpha}\}_{\alpha<\lambda}$ where each subcomplex L^{α} satisfies (1) and (2).

It follows from (†) and the consequences of (2) discussed before the construction of the continuous chain $\{L^\alpha\}_{\alpha<\lambda}$ that $\{\mathrm{H}_0(L^\alpha)\}_{\alpha<\lambda}$ is a continuous chain with union $\mathrm{H}_0(L) \cong M$. Further, for every ordinal β with $\beta + 1 < \lambda$, the short exact sequence $0 \to L^\beta \to L^{\beta+1} \to L^{\beta+1}/L^\beta \to 0$ induces an exact sequence,

$$0 \longrightarrow \mathrm{H}_0(L^\beta) \longrightarrow \mathrm{H}_0(L^{\beta+1}) \longrightarrow \mathrm{H}_0(L^{\beta+1}/L^\beta) \longrightarrow 0\,,$$

which shows that $\mathrm{H}_0(L^{\beta+1})/\mathrm{H}_0(L^\beta)$ is isomorphic to $\mathrm{H}_0(L^{\beta+1}/L^\beta) = \mathrm{H}_0(F)$. In particular, the quotient $\mathrm{H}_0(L^{\beta+1})/\mathrm{H}_0(L^\beta)$ is countably generated and has projective dimension at most p; indeed, F is degreewise countably generated and a semi-free replacement of $\mathrm{H}_0(F)$. □

Remark. The proof of D.14 shows that for a module M of projective dimension $p > 0$, the projective dimension of M^0 and each quotient $M^{\alpha+1}/M^\alpha$ is at most p. For our applications of D.14 it makes no difference, but the same is true for a module of projective dimension $p = 0$. If fact, Kaplansky [154] proves an even stronger result: A projective module is a coproduct of countably generated projective modules. This results is also a key ingredient in the proof of freeness of projective modules over local rings, cf. the Remarks after 1.3.21 and B.46.

D.15 Corollary. *If R is left Noetherian, then there is an equality,*

$$\operatorname{FPD} R \;=\; \sup\{\operatorname{pd}_R M \mid M \text{ is a countably generated } R\text{-module with } \operatorname{pd}_R M < \infty\}\,.$$

Proof. Per D.3 the assertion follows immediately from D.14. □

Proof of 8.5.24. Set $n = \operatorname{FFD} R$ and assume that it is finite, otherwise there is nothing to prove. Let M be an R-module of finite projective dimension; per D.14 and D.3 one can assume that it is countably generated. Choose a surjective homomorphism $\pi\colon L_0 \to M$ where L_0 is a countably generated free R-module, see the proof of 1.3.12. As R is left Noetherian, the submodule $\operatorname{Ker}\pi$ is countably generated, see D.13, so recursively one builds a degreewise countably generated free resolution $L \xrightarrow{\simeq} M$. By 8.3.6 and 8.3.11 the cokernel $\mathrm{C}_n(L)$ is flat, so it follows from D.9 applied to the exact sequence $0 \to \mathrm{C}_{n+1}(L) \to L_n \to \mathrm{C}_n(L) \to 0$ that $\mathrm{C}_{n+1}(L)$ is projective. Thus, one has $\operatorname{pd}_R M \leqslant n + 1$. □

Flat Preenvelopes

D.16. Let α be an infinite cardinal and $\{M^u\}_{u\in U}$ a family of R-modules with $\operatorname{card} M^u \leqslant \alpha$ for every $u \in U$. For the coproduct one has

$$\operatorname{card}\Big(\coprod_{u\in U} M^u\Big) \;\leqslant\; \max\{\alpha, \operatorname{card} U\}\,.$$

If each M^u is a submodule of a common R-module M, then one can consider the sum $\sum_{u\in U} M^u$. There is a canonical surjective homomorphism $\coprod_{u\in U} M^u \to \sum_{u\in U} M^u$, see 3.1.4, and hence one also has $\operatorname{card}(\sum_{u\in U} M^u) \leqslant \max\{\alpha, \operatorname{card} U\}$.

D.17 Lemma. *Given a cardinal α there is a cardinal $\widetilde{\alpha}$, depending on α and $\operatorname{card} R$, with the following property: For every R-module M and every submodule $N \subseteq M$ with $\operatorname{card} N \leqslant \alpha$, there exists a pure submodule $\widetilde{N} \subseteq M$ with $\operatorname{card}\widetilde{N} \leqslant \widetilde{\alpha}$ and $N \subseteq \widetilde{N}$.*

Proof. Let Δ be the set of all triples $(m,n,\varkappa)$ where $m,n \in \mathbb{N}$ and $\varkappa\colon R^m \to R^n$ is a homomorphism. Let $\kappa\colon F \to F'$ be the coproduct over Δ of the homomorphisms $\varkappa$. That is, one has $F = \coprod_{(m,n,\varkappa)\in\Delta} R^m$ and $F' = \coprod_{(m,n,\varkappa)\in\Delta} R^n$, and κ is uniquely determined by commutativity of the diagrams,

$$\begin{tikzcd} R^m \arrow[r, "\varkappa"] \arrow[d, "\varepsilon_{(m,n,\varkappa)}"'] & R^n \arrow[d, "\varepsilon'_{(m,n,\varkappa)}"] \\ F \arrow[r, "\kappa"] & F' , \end{tikzcd}$$

where ε and ε' denote the canonical embeddings.

We now construct a sequence $N = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$ of submodules of M. Set $N_0 = N$. Given N_u we construct N_{u+1} as follows. Let Ψ_u be the set of all homomorphisms $\psi\colon F \to N_u$ for which there exists some commutative diagram,

$$(\diamond) \qquad \begin{tikzcd} F \arrow[r, "\kappa"] \arrow[d, "\psi"'] & F' \arrow[d, dotted] \\ N_u \arrow[r, tail, "\nu_u"] & M , \end{tikzcd}$$

where ν_u is the embedding. Choose for every $\psi \in \Psi_u$ a homomorphism $\psi'\colon F' \to M$, that makes the diagram $(\diamond)$ commutative and set $N_{u+1} = N_u + \sum_{\psi\in\Psi_u} \operatorname{Im}\psi'$. We show that $\widetilde{N} = \bigcup_{u=0}^{\infty} N_u$ has the desired properties. Notice that each ν_u factors as a composite of embeddings $\widetilde{\nu}_u\colon N_u \rightarrowtail \widetilde{N}$ and $\widetilde{\nu}\colon \widetilde{N} \rightarrowtail M$.

Evidently one has $N \subseteq \widetilde{N} \subseteq M$. To see that $\widetilde{N}$ is a pure submodule of M it suffices, by 5.5.14, to show that for every commutative diagram,

$$(\star) \qquad \begin{tikzcd} R^m \arrow[r, "\varkappa"] \arrow[d, "\varphi"'] & R^n \arrow[d, "\varphi'"] \\ \widetilde{N} \arrow[r, tail, "\widetilde{\nu}"] & M , \end{tikzcd}$$

there exists a homomorphism $\varrho\colon R^n \to \widetilde{N}$ with $\varrho\varkappa = \varphi$. Let $(\star)$ be given and notice that $(m,n,\varkappa)$ is an element in Δ. As R^m is finitely generated, φ factors through some N_u, that is, φ can be written as a composite

$$R^m \xrightarrow{\bar{\varphi}} N_u \xrightarrow{\widetilde{\nu}_u} \widetilde{N} .$$

Let $\psi\colon F \to N_u$ and $\xi\colon F' \to M$ be the homomorphisms given by

$$\psi\varepsilon_{(\grave{m},\grave{n},\grave{\varkappa})} = \begin{cases} \bar{\varphi} & \text{for } (\grave{m},\grave{n},\grave{\varkappa}) = (m,n,\varkappa) \\ 0 & \text{for } (\grave{m},\grave{n},\grave{\varkappa}) \neq (m,n,\varkappa) \end{cases}$$

and

$$\xi\varepsilon'_{(\grave{m},\grave{n},\grave{\varkappa})} = \begin{cases} \varphi' & \text{for } (\grave{m},\grave{n},\grave{\varkappa}) = (m,n,\varkappa) \\ 0 & \text{for } (\grave{m},\grave{n},\grave{\varkappa}) \neq (m,n,\varkappa) \end{cases}$$

for $(\grave{m},\grave{n},\grave{\varkappa}) \in \Delta$. Now $\nu_u\psi = \xi\kappa$ holds, that is, the diagram $(\diamond)$ with ξ as the dotted arrow is commutative. Indeed, one has

$$\nu_u\psi\varepsilon_{(m,n,\varkappa)} \;=\; \nu_u\bar{\varphi} \;=\; \widetilde{\nu}\widetilde{\nu}_u\bar{\varphi} \;=\; \widetilde{\nu}\varphi \;=\; \varphi'\varkappa \;=\; \xi\varepsilon'_{(m,n,\varkappa)}\varkappa \;=\; \xi\kappa\varepsilon_{(m,n,\varkappa)}$$

and $\nu_u\psi\varepsilon_{(\mathring{m},\mathring{n},\mathring{\varkappa})} = 0 = \xi\varepsilon'_{(\mathring{m},\mathring{n},\mathring{\varkappa})}\varkappa = \xi\kappa\varepsilon_{(\mathring{m},\mathring{n},\mathring{\varkappa})}$ for all $(\mathring{m},\mathring{n},\mathring{\varkappa}) \neq (m,n,\varkappa)$ in Δ. This means that the homomorphism $\psi\colon F \to N_u$ belongs to the set Ψ_u. By construction of N_{u+1} there exists a homomorphism ψ' that makes the diagram below commutative,

$$\begin{tikzcd} F \arrow[r, "\kappa"] \arrow[d, "\psi"'] & F' \arrow[d, dotted, "\psi'"] \\ N_u \arrow[r, tail, "\nu_{u,u+1}"] & N_{u+1} \end{tikzcd} ;$$

here $\nu_{u,u+1}$ denotes the embedding. We argue that $\varrho = \widetilde{\nu}_{u+1}\psi'\varepsilon'_{(m,n,\varkappa)}\colon R^n \to \widetilde{N}$ satisfies $\varrho\varkappa = \varphi$ as required. As $\widetilde{\nu}$ is injective, it suffices to show that $\widetilde{\nu}\varrho\varkappa = \widetilde{\nu}\varphi$ holds. To verify this is a computation based on commutativity of the diagrams above:

$$\begin{aligned} \widetilde{\nu}\varrho\varkappa \;&=\; \widetilde{\nu}\widetilde{\nu}_{u+1}\psi'\varepsilon'_{(m,n,\varkappa)}\varkappa \\ &=\; \nu_{u+1}\psi'\varepsilon'_{(m,n,\varkappa)}\varkappa \\ &=\; \nu_{u+1}\psi'\kappa\varepsilon_{(m,n,\varkappa)} \\ &=\; \nu_{u+1}\nu_{u,u+1}\psi\varepsilon_{(m,n,\varkappa)} \\ &=\; \nu_u\psi\varepsilon_{(m,n,\varkappa)} \\ &=\; \nu_u\bar{\varphi} \\ &=\; \widetilde{\nu}\widetilde{\nu}_u\bar{\varphi} \\ &=\; \widetilde{\nu}\varphi\,. \end{aligned}$$

It remains to prove the assertion about cardinality. Without loss of generality one can assume that α is infinite. Set $\lambda = \max\{\aleph_0, \operatorname{card} R\}$ and $\widetilde{\alpha} = \alpha^{\lambda}$. We show that if $\operatorname{card} N \leqslant \alpha$ holds, then one has $\operatorname{card}\widetilde{N} \leqslant \widetilde{\alpha}$. As $\widetilde{N}$ is the countable union $\bigcup_{u=0}^{\infty} N_u$, it suffices to argue that $\operatorname{card} N_u \leqslant \widetilde{\alpha}$ holds for every u. We prove this by induction. For $u = 0$ we have $N_0 = N$ and hence $\operatorname{card} N_0 \leqslant \alpha \leqslant \widetilde{\alpha}$. Now assume that $\operatorname{card} N_u \leqslant \widetilde{\alpha}$ holds for some u. Note that $\operatorname{card}\Delta = \lambda$ whence the free modules F and F' have bases of cardinality λ. It follows that

$$\operatorname{Hom}_R(F, N_u) \;\cong\; \operatorname{Hom}_R(R^{(\lambda)}, N_u) \;\cong\; N_u^{\lambda}\,,$$

and consequently, $\operatorname{card}(\operatorname{Hom}_R(F,N_u)) \leqslant \widetilde{\alpha}^{\lambda} = (\alpha^{\lambda})^{\lambda} = \alpha^{\lambda\cdot\lambda} = \alpha^{\lambda} = \widetilde{\alpha}$, where the inequality comes from the assumption $\operatorname{card} N_u \leqslant \widetilde{\alpha}$. As Ψ_u is a subset of $\operatorname{Hom}_R(F, N_u)$ one also has $\operatorname{card}\Psi_u \leqslant \widetilde{\alpha}$. Recall that $N_{u+1} = N_u + \sum_{\psi\in\Psi_u} \operatorname{Im}\psi'$ where each ψ' is a homomorphism $F' \to X$. In particular, one has

$$\operatorname{card}(\operatorname{Im}\psi') \;\leqslant\; \operatorname{card} F' = \operatorname{card} R^{(\lambda)} \;\leqslant\; \max\{\lambda, \operatorname{card} R\} \;=\; \lambda \;\leqslant\; \widetilde{\alpha}\,.$$

Thus N_{u+1} is a sum, over a set of cardinality at most $\widetilde{\alpha}$, of modules of cardinality at most $\widetilde{\alpha}$. It follows that $\operatorname{card} N_{u+1} \leqslant \widetilde{\alpha}$; see D.16. □

D.18 Definition. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An $\mathcal{X}$-*preenvelope* of M is a homomorphism $\varphi\colon M \to X$ with $X \in \mathcal{X}$ such that for every homomorphism $\varphi' \colon M \to X'$ with $X' \in \mathcal{X}$ there is a homomorphism $\chi \colon X \to X'$ that makes the following diagram commutative,

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & X \arrow[dl, dotted, "\chi"] \\ X' . & \end{tikzcd}$$

D.19 Example. Let $\mathcal{I}_0$ denote the class of injective R-modules. By 5.3.30 and 5.3.27 every R-module has an $\mathcal{I}_0$-preenvelope, also called an *injective preenvelope*.

D.20. Let $\mathcal{X}$ be a class of R-modules and M an R-module. A homomorphism $\varphi \colon M \to X$ with $X \in \mathcal{X}$ is an $\mathcal{X}$-preenvelope if and only if $\operatorname{Hom}_R(\varphi, X')$ is surjective for every $X' \in \mathcal{X}$. Notice that if M can be embedded in some module from $\mathcal{X}$, then every $\mathcal{X}$-preenvelope $M \to X$ is injective.

D.21 Theorem. *Let $\mathcal{X}$ be a class of R-modules. If the next two conditions are satisfied, then every R-module has an $\mathcal{X}$-preenvelope.*

(1) *Every pure submodule of a module in $\mathcal{X}$ belongs to $\mathcal{X}$.*
(2) *For every family $\{X_u\}_{u \in U}$ of modules in $\mathcal{X}$ the product $\prod_{u \in U} X_u$ is in $\mathcal{X}$.*

Proof. Let M be an R-module. Set $\alpha = \operatorname{card} M$ and let $\widetilde{\alpha}$ be as in D.17. The collection of isomorphism classes of modules in $\mathcal{X}$ of cardinality $\leqslant \widetilde{\alpha}$ constitutes a set; let $\{X_u\}_{u \in U}$ be a set of representatives for these classes. That is, every $X \in \mathcal{X}$ with $\operatorname{card} X \leqslant \widetilde{\alpha}$ is isomorphic to some X_u. By assumption the product

$$X = \prod_{u \in U} X_u^{\operatorname{Hom}_R(M, X_u)}$$

belongs to $\mathcal{X}$. For every $u \in U$ and $\beta \in \operatorname{Hom}_R(M, X_u)$ let $\pi_{u,\beta} \colon X \to X_u$ be the projection. Now, let $\varphi \colon M \to X$ be the unique homomorphism that satisfies $\pi_{u,\beta}\varphi = \beta$ for every u and β; see 1.1.19. To see that φ is an $\mathcal{X}$-preenvelope, let $\varphi' \colon M \to X'$ be a homomorphism with $X' \in \mathcal{X}$. Note that the submodule $\operatorname{Im} \varphi' \subseteq X'$ has $\operatorname{card}(\operatorname{Im} \varphi') \leqslant \operatorname{card} M = \alpha$, so D.17 yields a pure submodule $Y \subseteq X'$ with $\operatorname{Im} \varphi' \subseteq Y$ and $\operatorname{card} Y \leqslant \widetilde{\alpha}$. By assumption, Y belongs to $\mathcal{X}$. Hence there is an isomorphism $Y \cong X_u$ for some u. As φ' factors through $\operatorname{Im} \varphi'$, it also factors through $Y \cong X_u$. Thus there exist homomorphisms $\beta \colon M \to X_u$ and $\gamma \colon X_u \to X'$ with $\gamma\beta = \varphi'$. Therefore the homomorphism $\chi = \gamma\pi_{u,\beta} \colon X \to X'$ satisfies $\chi\varphi = \gamma\pi_{u,\beta}\varphi = \gamma\beta = \varphi'$. □

Let $\mathcal{F}_0$ denote the class of flat R-modules; an $\mathcal{F}_0$-preenvelope is called a *flat preenvelope*; cf. D.19.

D.22 Corollary. *Assume that R is right Noetherian and let $n \geqslant 0$ be an integer. Set*

$$\mathcal{F}_n = \{M \in \mathcal{M}(R) \mid \operatorname{fd}_R M \leqslant n\} .$$

Every R-module has an $\mathcal{F}_n$-preenvelope; in particular, every R-module has a flat preenvelope.

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence of R-modules. By 5.5.14 there is a split exact sequence of R^{o}-modules,

$$0 \longrightarrow \mathrm{Hom}_{\Bbbk}(M'', \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(M, \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(M', \mathbb{E}) \longrightarrow 0\,,$$

so 8.3.17 and 8.2.12 show that one has $\mathrm{fd}_R M \;=\; \max\{\mathrm{fd}_R M', \mathrm{fd}_R M''\}$. In particular, $\mathrm{fd}_R M' \leqslant \mathrm{fd}_R M$ holds, so the class $\mathcal{F}_n$ is closed under pure submodules. By 8.3.27 the class $\mathcal{F}_n$ is closed under products, so the assertion follows from D.21. $\square$

As noted in D.19 above, every module has an injective preenvelope; in fact, every module has an injective envelope by B.17. Here is a related result:

D.23 Corollary. *Assume that R is left Noetherian and let $n \geqslant 0$ be an integer. Set*

$$\mathcal{I}_n \;=\; \{M \in \mathcal{M}(R) \mid \mathrm{id}_R M \leqslant n\}\,.$$

Every R-module has an $\mathcal{I}_n$-preenvelope.

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a pure exact sequence of R-modules. By 5.5.14 there is a split exact sequence of R^{o}-modules,

$$0 \longrightarrow \mathrm{Hom}_{\Bbbk}(M'', \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(M, \mathbb{E}) \longrightarrow \mathrm{Hom}_{\Bbbk}(M', \mathbb{E}) \longrightarrow 0\,,$$

so 8.3.18 and 8.3.13 show that one has $\mathrm{id}_R M \;=\; \max\{\mathrm{id}_R M', \mathrm{id}_R M''\}$. In particular, $\mathrm{id}_R M' \leqslant \mathrm{id}_R M$ holds, so the class $\mathcal{I}_n$ is closed under pure submodules. By 8.2.12 the class $\mathcal{I}_n$ is closed under products, so the assertion follows from D.21. $\square$

D.24 Proposition. *Let $R \to S$ be a ring homomorphism and M an R-module. If S is flat as an R-module and $\varphi\colon M \to F$ a flat preenvelope, then the induced homomorphism of S-modules, $S \otimes_R \varphi\colon S \otimes_R M \to S \otimes_R F$, is a flat preenvelope.*

Proof. The S-module $S \otimes_R F$ is flat by 5.4.24(a), and for every S-module G there are by adjunction 4.4.12 and the counitor 4.4.2 ismorphisms,

$$(\diamond) \qquad \mathrm{Hom}_S(S \otimes_R \varphi, G) \;\cong\; \mathrm{Hom}_R(\varphi, \mathrm{Hom}_S(S, G)) \;\cong\; \mathrm{Hom}_R(\varphi, G)\,.$$

If G is flat, then it is also flat as an R-module, see 5.4.24(b), and it follows from D.20 that $\mathrm{Hom}_R(\varphi, G)$ is surjective. Now $(\diamond)$ shows that $\mathrm{Hom}_S(S \otimes_R \varphi, G)$ is surjective, whence $S \otimes_R \varphi$ is a flat preenvelope by another application of D.20. $\square$

Exercises

E D.1 Let $\mathcal{X}$ be a class of R-modules, M an R-module, and $\varphi\colon M \to X$ an $\mathcal{X}$-preenvelope. Show that a homomorphism $\varphi'\colon M \to X'$ with $X' \in \mathcal{X}$ is an $\mathcal{X}$-preenvelope if there exists a homomorphism $\psi\colon X' \to X$ with $\psi\varphi' = \varphi$.

E D.2 Let $R \to S$ be a ring homomorphism and M an R-module. If S is flat as an R-module and $\varphi\colon M \to F$ an $\mathcal{F}_n$-preenvelope, then the induced homomorphism of S-modules, $S \otimes_R \varphi\colon S \otimes_R M \to S \otimes_R F$, is an $\mathcal{F}_n$-preenvelope.

Appendix E
Triangulated Categories

SYNOPSIS. Axioms for triangulated category; triangulated functor; triangulated subcategory; Five Lemma; split triangle.

In this appendix $\mathcal{T}$ is an additive category equipped with an additive automorphism Σ. We describe the conditions for $(\mathcal{T}, \Sigma)$ to form a triangulated category; the first step is to settle on a collection of triangles.

E.1 Definition. A *candidate triangle* in $\mathcal{T}$ is a diagram

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M\,,$$

such that the composites $\beta\alpha$, $\gamma\beta$, and $(\Sigma\alpha)\gamma$ are all zero. An *morphism* (φ, ψ, χ) of candidate triangles is a commutative diagram in $\mathcal{T}$,

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\
M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M'
\end{tikzcd}\;;$$

it is called an *isomorphism* if φ, ψ, and χ are isomorphisms in $\mathcal{T}$.

E.2. For a collection $\triangle$ of candidate triangles in $\mathcal{T}$, consider the next conditions.

(TR0) For every M in $\mathcal{T}$, the candidate triangle

$$M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M$$

is in $\triangle$. Every candidate triangle that is isomorphic to one from $\triangle$ is in $\triangle$.

(TR1) Every morphism $\alpha\colon M \to N$ in $\mathcal{T}$ fits in a candidate triangle from $\triangle$,

$$M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M\,.$$

(TR2) For every candidate triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\triangle$, the following two candidate triangles belong to $\triangle$ as well,

$$N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N \quad \text{and} \quad \Sigma^{-1} X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X\,.$$

(TR2′) Consider two candidate triangles,

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M \quad \text{and} \quad N \xrightarrow{-\beta} X \xrightarrow{-\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N\,.$$

If one belongs to $\triangle$ then so does the other.

(TR3) For every commutative diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\ M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' \end{tikzcd} \tag{E.2.1}$$

where the rows are candidate triangles in $\triangle$, there exists a (not necessarily unique) morphism $\chi\colon X \to X'$, such that (φ, ψ, χ) is a morphism of candidate triangles.

(TR4′) For every commutative diagram (E.2.1), where the rows are candidate triangles in $\triangle$, there exists a (not necessarily unique) morphism $\chi\colon X \to X'$ such that (φ, ψ, χ) is a morphism of candidate triangles, and such that the following candidate triangle belongs to $\triangle$,

$$\begin{matrix} M' \\ \oplus \\ N \end{matrix} \xrightarrow{\begin{pmatrix} \alpha' & \psi \\ 0 & -\beta \end{pmatrix}} \begin{matrix} N' \\ \oplus \\ X \end{matrix} \xrightarrow{\begin{pmatrix} \beta' & \chi \\ 0 & -\gamma \end{pmatrix}} \begin{matrix} X' \\ \oplus \\ \Sigma M \end{matrix} \xrightarrow{\begin{pmatrix} \gamma' & \Sigma\varphi \\ 0 & -\Sigma\alpha \end{pmatrix}} \begin{matrix} \Sigma M' \\ \oplus \\ \Sigma N \end{matrix}\,. \tag{E.2.2}$$

The candidate triangle (E.2.2) is called the *mapping cone* of (φ, ψ, χ).

Condition (TR4′) is evidently stronger than (TR3), and it is proved in E.5 below that (TR2) and (TR2′) are equivalent under assumption of (TR0). The conditions in E.2 supply the axioms for a triangulated category.

E.3 Definition. A *triangulated category* is an additive category $\mathcal{T}$ equipped with an additive automorphism Σ and a collection $\triangle$ of candidate triangles, called *distinguished triangles*, such that (TR0), (TR1), (TR2′), and (TR4′) are satisfied.

In a triangulated category $(\mathcal{T}, \Sigma)$, the functor Σ is usually called the *translation*. Axiom (TR4′) is perhaps the less intuitive of the lot; here is a simple application.

E.4 Example. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. For X and M in $\mathcal{T}$ it follows from (TR0) and (TR2′) that there are distinguished triangles

$$\Delta^X = \Sigma^{-1}X \longrightarrow 0 \longrightarrow X \xrightarrow{-1^X} X \quad \text{and} \quad \Delta^M = M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma M\,.$$

The only morphism from Δ^X to Δ^M is the zero morphism, and its mapping cone,

$$M \xrightarrow{\begin{pmatrix} 1^M \\ 0 \end{pmatrix}} \begin{matrix} M \\ \oplus \\ X \end{matrix} \xrightarrow{(0\ 1^X)} X \xrightarrow{0} \Sigma M\,,$$

is a distinguished triangle by (TR4′).

Remark. If the collection $\triangle$ in $(\mathcal{T},\Sigma)$ satisfies only (TR0), (TR1), (TR2′), and (TR3), then $\mathcal{T}$ is called *pretriangulated*. It can be proved that for a pretriangulated category the so-called Octahedral Axiom, which is usually denoted (TR4), is equivalent to (TR4′); see Neeman [190]. That is, a triangulated category is a pretriangulated category that satisfies the Octahedral Axiom. This perspective goes back to Verdier's thesis on derived categories [247] from the mid 1960s—it was published 30 years late and only after Verdier's passing. Indeed, triangulated categories in algebra were originally defined through axiomatization of the properties of derived categories; the axioms being (TR0), (TR1), (TR2), (TR3), and (TR4); usually with (TR0) included in (TR1). The contemporary formulation of the definition in E.3 follows Neeman's monograph [191].

E.5 Lemma. *Let $\triangle$ be a collection of candidate triangles in* $(\mathcal{T},\Sigma)$ *such that* (TR0) *is satisfied. Condition* (TR2) *is then satisfied if and only if* (TR2′) *is satisfied.*

Proof. Assume that (TR2′) is satisfied. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a candidate triangle in $\triangle$. Consider the following isomorphism of candidate triangles,

$$(\diamond)\qquad \begin{tikzcd} N \arrow[r, "\beta"] \arrow[d, equal] & X \arrow[r, "\gamma"] \arrow[d, "\cong"', "-1^X"] & \Sigma M \arrow[r, "-\Sigma\alpha"] \arrow[d, equal] & \Sigma N \arrow[d, equal] \\ N \arrow[r, "-\beta"] & X \arrow[r, "-\gamma"] & \Sigma M \arrow[r, "-\Sigma\alpha"] & \Sigma N\,. \end{tikzcd}$$

By (TR2′) the lower row in $(\diamond)$ belongs to $\triangle$, and hence so does the upper row by (TR0). To show that the candidate triangle

$$\Sigma^{-1}X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X$$

is in $\triangle$, is by (TR2′) equivalent to showing that $M \xrightarrow{-\alpha} N \xrightarrow{-\beta} X \xrightarrow{\gamma} \Sigma M$ is in $\triangle$, and that follows from (TR0) and the next isomorphism of candidate triangles,

$$\begin{tikzcd} M \arrow[r, "-\alpha"] \arrow[d, equal] & N \arrow[r, "-\beta"] \arrow[d, "\cong"', "-1^N"] & X \arrow[r, "\gamma"] \arrow[d, equal] & \Sigma M \arrow[d, equal] \\ M \arrow[r, "\alpha"] & N \arrow[r, "\beta"] & X \arrow[r, "\gamma"] & \Sigma M\,. \end{tikzcd}$$

Similar arguments show that (TR2) implies (TR2′). $\square$

Although triangulated categories are seldom Abelian, they are close enough to Abelian categories that one can do homological algebra in much the same fashion. The object X in a distinguished triangle $M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M$ is a pseudo-cokernel—and $\Sigma^{-1}X$ is a pseudo-kernel—of α. In this perspective, the distinguished triangles correspond to short exact sequences.

E.6 Proposition. *Let* $(\mathcal{T},\Sigma)$ *be a triangulated category. The opposite category* $(\mathcal{T}^{\mathrm{op}},\Sigma^{-1})$ *is triangulated in the following canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1}M$ in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the corresponding diagram $\Sigma^{-1}M \to X \to N \to M$ is a distinguished triangle in $\mathcal{T}$.*

Proof. It is evident that a diagram in $\mathcal{T}^{\mathrm{op}}$ is a candidate triangle if and only if the corresponding diagram in $\mathcal{T}$ is a candidate triangle. Let $\triangle$ be the collection of distinguished triangles in $\mathcal{T}$. It is elementary to verify that the collection of diagrams $M \to N \to X \to \Sigma^{-1}M$ in $\mathcal{T}^{\mathrm{op}}$ such that the corresponding diagram in $\mathcal{T}$ belongs to $\triangle$ satisfies the axioms in E.3. As an example, we provide the details for (TR0).

Let M be an object in $\mathcal{T}^{\mathrm{op}}$ and hence in $\mathcal{T}$. The candidate triangle

$$(\diamond) \qquad M \xrightarrow{1^M} M \longrightarrow 0 \longrightarrow \Sigma^{-1}M$$

in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the corresponding candidate triangle in $\mathcal{T}$,

$$(\flat) \qquad \Sigma^{-1}M \longrightarrow 0 \longrightarrow M \xrightarrow{1^M} M\,,$$

belongs to $\triangle$. By (TR2′), applied twice, ($\flat$) is in $\triangle$ if and only if the following candidate triangle is in $\triangle$,

$$(\dagger) \qquad M \xrightarrow{1^M} M \longrightarrow \Sigma 0 \longrightarrow \Sigma M\,.$$

There is an isomorphism $\Sigma 0 \cong 0$ in $\mathcal{T}$, so by (TR0) the triangle ($\dagger$) is in $\triangle$, whence ($\diamond$) is distinguished in $\mathcal{T}^{\mathrm{op}}$. Next, let

$$(\ddagger) \qquad \begin{array}{ccccccc} M' & \xrightarrow{\alpha'} & N' & \xrightarrow{\beta'} & X' & \xrightarrow{\gamma'} & \Sigma^{-1}M' \\ \cong\big\downarrow\varphi & & \cong\big\downarrow\psi & & \cong\big\downarrow\chi & & \cong\big\downarrow\Sigma^{-1}\varphi \\ M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & X & \xrightarrow{\gamma} & \Sigma^{-1}M \end{array}$$

be an isomorphism of candidate triangles in $\mathcal{T}^{\mathrm{op}}$, and assume that the bottom row is distinguished. In the corresponding diagram in $\mathcal{T}$,

$$\begin{array}{ccccccc} \Sigma^{-1}M & \xrightarrow{\gamma} & X & \xrightarrow{\beta} & N & \xrightarrow{\alpha} & M \\ \cong\big\downarrow\Sigma^{-1}\varphi & & \cong\big\downarrow\chi & & \cong\big\downarrow\psi & & \cong\big\downarrow\varphi \\ \Sigma^{-1}M' & \xrightarrow{\gamma'} & X' & \xrightarrow{\beta'} & N' & \xrightarrow{\alpha'} & M'\,, \end{array}$$

the top row belongs to $\triangle$, and by (TR0) so does the bottom row. Hence, the top row in ($\ddagger$) is a distinguished triangle in $\mathcal{T}^{\mathrm{op}}$. □

Triangulated Functors and Triangulated Subcategories

E.7 Definition. Consider triangulated categories $(\mathcal{T}, \Sigma_{\mathcal{T}})$ and $(\mathcal{U}, \Sigma_{\mathcal{U}})$. A functor $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ is called *triangulated* if it is additive and there is a natural isomorphism $\phi\colon \mathrm{F}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ such that for every distinguished triangle in $\mathcal{T}$,

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma_{\mathcal{T}} M\,,$$

the induced candidate triangle in $\mathcal{U}$,

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\phi^M \mathrm{F}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}(M)\,,$$

is distinguished. Although a triangulated functor is a pair (F, ϕ), it is customary to suppress ϕ and when needed refer to it as the natural isomorphism associated to F.

In the suggested analogy with Abelian categories, triangulated functors correspond to exact functors. The translation functor, for example, is triangulated.

E.8 Example. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. Given a distinguished triangle

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$$

in $\mathcal{T}$, successive applications of (TR2′) yield distinguished triangles

$$N \xrightarrow{-\beta} X \xrightarrow{-\gamma} \Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N\,,$$
$$X \xrightarrow{\gamma} \Sigma M \xrightarrow{\Sigma\alpha} \Sigma N \xrightarrow{\Sigma\beta} \Sigma X\,, \quad \text{and}$$
$$\Sigma M \xrightarrow{-\Sigma\alpha} \Sigma N \xrightarrow{-\Sigma\beta} \Sigma X \xrightarrow{-\Sigma\gamma} \Sigma^2 M\,.$$

Now it follows from the isomorphism of candidate triangles,

$$\begin{tikzcd}
\Sigma M \arrow[r, "-\Sigma\alpha"] \arrow[d, equal] & \Sigma N \arrow[r, "-\Sigma\beta"] \arrow[d, "-1^N", "\cong"'] & \Sigma X \arrow[r, "-\Sigma\gamma"] \arrow[d, equal] & \Sigma^2 M \arrow[d, equal] \\
\Sigma M \arrow[r, "\Sigma\alpha"] & \Sigma N \arrow[r, "\Sigma\beta"] & \Sigma X \arrow[r, "-\Sigma\gamma"] & \Sigma^2 M\,,
\end{tikzcd}$$

that $\Sigma M \xrightarrow{\Sigma\alpha} \Sigma N \xrightarrow{\Sigma\beta} \Sigma X \xrightarrow{-\Sigma\gamma} \Sigma^2 M$ is distinguished, so the functor $\Sigma\colon \mathcal{T} \to \mathcal{T}$ is triangulated with $-1^{\Sigma^2 M}$ for the natural isomorphism $\Sigma(\Sigma M) \to \Sigma(\Sigma M)$.

E.9 Proposition. *Let $(\mathcal{T}, \Sigma_{\mathcal{T}})$, $(\mathcal{U}, \Sigma_{\mathcal{U}})$, and $(\mathcal{V}, \Sigma_{\mathcal{V}})$ be triangulated categories and $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ and $\mathrm{G}\colon \mathcal{U} \to \mathcal{V}$ be functors. If F and G are triangulated with associated natural isomorphisms $\phi\colon \mathrm{F}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ and $\psi\colon \mathrm{G}\Sigma_{\mathcal{U}} \to \Sigma_{\mathcal{V}}\mathrm{G}$, then GF is triangulated with associated natural isomorphism $\psi\mathrm{F} \circ \mathrm{G}\phi\colon \mathrm{GF}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{V}}\mathrm{GF}$.*

Proof. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a distinguished triangle in $\mathcal{T}$. As F is triangulated, the diagram

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\phi^M \mathrm{F}(\gamma)} \Sigma_{\mathcal{U}}\mathrm{F}(M)$$

is a distinguished triangle in $\mathcal{U}$. It yields, as G is triangulated, a distinguished triangle

$$\mathrm{GF}(M) \xrightarrow{\mathrm{GF}(\alpha)} \mathrm{GF}(N) \xrightarrow{\mathrm{GF}(\beta)} \mathrm{GF}(X) \xrightarrow{\psi^{\mathrm{F}(M)}\mathrm{G}(\phi^M \mathrm{F}(\gamma))} \Sigma_{\mathcal{U}}\mathrm{GF}(M)$$

in $\mathcal{V}$. It remains to note that one has $\psi^{\mathrm{F}(M)}\mathrm{G}(\phi^M\mathrm{F}(\gamma)) = \psi^{\mathrm{F}(M)}\mathrm{G}(\phi^M)\,\mathrm{GF}(\gamma)$. □

E.10. Let $(\mathcal{T}, \Sigma_{\mathcal{T}})$ and $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be triangulated categories. It is straightforward to verify that if $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ is a triangulated functor with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}}\mathrm{F}$, then the opposite functor $\mathrm{F}^{\mathrm{op}}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{U}^{\mathrm{op}}$ is triangulated with associated natural isomorphism $(\Sigma_{\mathcal{U}}^{-1}\,\phi\,\Sigma_{\mathcal{T}}^{-1})^{\mathrm{op}}\colon \Sigma_{\mathcal{U}}^{-1}\,\mathrm{F}^{\mathrm{op}} \to \mathrm{F}^{\mathrm{op}}\,\Sigma_{\mathcal{T}}^{-1}$.

E.11 Definition. For triangulated categories $\mathcal{T}$ and $\mathcal{U}$, an equivalence $\mathcal{T} \rightleftarrows \mathcal{U}$ is called an *equivalence of triangulated categories* if the functors are triangulated.

It is not automatic that a natural transformations between triangulated functors commutes with translation; hence one makes the following definition.

E.12 Definition. Let $(\mathcal{T}, \Sigma_{\mathcal{T}})$ and $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be triangulated categories and F and G be triangulated functors $\mathcal{T} \to \mathcal{U}$ with associated natural isomorphisms $\phi\colon \mathrm{F}\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}}\mathrm{F}$ and $\psi\colon \mathrm{G}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{G}$. A natural transformation $\tau\colon \mathrm{F} \to \mathrm{G}$ is called *triangulated* if the following diagram is commutative for every X in $\mathcal{T}$,

$$\begin{tikzcd}
\mathrm{F}(\Sigma_{\mathcal{T}} X) \arrow[r, "\phi^X", "\cong"'] \arrow[d, "\tau^{\Sigma X}"'] & \Sigma_{\mathcal{U}}\mathrm{F}(X) \arrow[d, "\Sigma\tau^X"] \\
\mathrm{G}(\Sigma_{\mathcal{T}} X) \arrow[r, "\psi^X", "\cong"'] & \Sigma_{\mathcal{U}}\mathrm{G}(X)\,.
\end{tikzcd}$$

That is, $\tau^{\Sigma_{\mathcal{T}} X}$ and $\Sigma_{\mathcal{U}}\tau^X$ are isomorphic, and the isomorphism is natural in X.

Remark. Another name for triangulated natural transformation is 'graded' natural transformation; see Bondal and Orlov [42]. It is proved ibid. that the unit and counit in an adjunction of triangulated functors are automatically triangulated.

E.13 Example. Let $(\mathcal{T}, \Sigma_{\mathcal{T}})$ and $(\mathcal{U}, \Sigma_{\mathcal{U}})$ be triangulated categories and $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ a triangulated functor with associated natural isomorphism $\phi\colon \mathrm{F}\Sigma \to \Sigma_{\mathcal{U}}\mathrm{F}$. It follows from E.8 and E.9 that the functors $\mathrm{F}\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{U}}\mathrm{F}$ are triangulated with associated natural isomorphisms

$$-\phi\,\Sigma_{\mathcal{T}}\colon\ \mathrm{F}\Sigma_{\mathcal{T}}^2 \longrightarrow \Sigma_{\mathcal{U}}\mathrm{F}\Sigma_{\mathcal{T}} \qquad \text{and} \qquad -\Sigma_{\mathcal{U}}\,\phi\colon\ \Sigma_{\mathcal{U}}\mathrm{F}\Sigma_{\mathcal{T}} \longrightarrow \Sigma_{\mathcal{U}}^2\mathrm{F}\,.$$

Thus, the natural isomorphism ϕ is triangulated.

E.14 Definition. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. A *triangulated subcategory* of $\mathcal{T}$ is a full additive subcategory $\mathcal{S}$ that satisfies the following conditions.

(1) If N and N' are isomorphic objects in $\mathcal{T}$, then N belongs to $\mathcal{S}$ if and only if N' belongs to $\mathcal{S}$.
(2) An object N in $\mathcal{T}$ belongs to $\mathcal{S}$ if and only if ΣN belongs to $\mathcal{S}$.
(3) For every distinguished triangle $M \to N \to X \to \Sigma M$ in $\mathcal{T}$, such that the objects M and N belong to $\mathcal{S}$, also X belongs to $\mathcal{S}$.

Conveniently, the conditions (1)–(3) above are expressed as: $\mathcal{S}$ is closed under isomorphisms, shifts, and distinguished triangles in $\mathcal{T}$.

Notice that if $\mathcal{S}$ is a triangulated subcategory of $(\mathcal{T}, \Sigma)$, then $(\mathcal{S}, \Sigma)$ is on its own a triangulated category.

The Five Lemma

E.15 Definition. Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $\mathcal{E}$ an Abelian category. An additive functor $\mathrm{F}\colon \mathcal{T} \to \mathcal{E}$ is called *homological* if the sequence

$$\mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X)$$

in $\mathcal{E}$ is exact for every distinguished triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \longrightarrow \Sigma M$ in $\mathcal{T}$.

E.16. Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $\mathcal{E}$ an Abelian category. It follows from (TR2′) that a homological functor $\mathrm{F}\colon \mathcal{T} \to \mathcal{E}$ induces an exact sequence in $\mathcal{E}$,

$$\cdots \longrightarrow \mathrm{F}(\Sigma^{-1}X) \xrightarrow{\mathrm{F}(\Sigma^{-1}\gamma)} \mathrm{F}(M) \xrightarrow{\mathrm{F}(\alpha)} \mathrm{F}(N) \xrightarrow{\mathrm{F}(\beta)} \mathrm{F}(X) \xrightarrow{\mathrm{F}(\gamma)} \mathrm{F}(\Sigma M) \xrightarrow{\mathrm{F}(\Sigma\alpha)} \mathrm{F}(\Sigma N) \longrightarrow \cdots ,$$

for every distinguished triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\mathcal{T}$.

E.17 Lemma. *Let $(\mathcal{T}, \Sigma)$ be a triangulated category. For every object Y in $\mathcal{T}$, the functor $\mathcal{T}(Y, -)\colon \mathcal{T} \to \mathcal{M}(\mathbb{Z})$ is homological.*

PROOF. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ be a distinguished triangle in $\mathcal{T}$. It must be shown that the sequence

$$\mathcal{T}(Y, M) \xrightarrow{\mathcal{T}(Y,\alpha)} \mathcal{T}(Y, N) \xrightarrow{\mathcal{T}(Y,\beta)} \mathcal{T}(Y, X)$$

is exact. As one has $\beta\alpha = 0$, it follows that the inclusion $\operatorname{Im}\mathcal{T}(Y, \alpha) \subseteq \operatorname{Ker}\mathcal{T}(Y, \beta)$ holds. Conversely, assume that $\vartheta\colon Y \to N$ is in $\operatorname{Ker}\mathcal{T}(Y, \beta)$, that is, one has $\beta\vartheta = 0$. It follows that the next diagram is commutative,

$$\begin{tikzcd}
Y \arrow[r, "1^Y"] & Y \arrow[r] \arrow[d, "\vartheta"] & 0 \arrow[r] \arrow[d] & \Sigma Y \\
M \arrow[r, "\alpha"] & N \arrow[r, "\beta"] & X \arrow[r, "\gamma"] & \Sigma M .
\end{tikzcd}$$

The rows are distinguished triangles by (TR1) and by assumption. By (TR2′) and (TR4′) there is a morphism $\upsilon\colon Y \to M$ with $\alpha\upsilon = \vartheta$, whence ϑ is in $\operatorname{Im}\mathcal{T}(Y, \alpha)$. $\square$

REMARK. Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $\mathcal{E}$ an Abelian category. An additive functor $\mathrm{G}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{E}$ is called *cohomological* if the opposite functor $\mathrm{G}^{\mathrm{op}}\colon \mathcal{T} \to \mathcal{E}^{\mathrm{op}}$ is homological. Dually to E.17 one can show that the functor $\mathcal{T}(-, Y)\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{M}(\mathbb{Z})$ is cohomological for every Y in $\mathcal{T}$.

The next result is the triangulated analogue of the Five Lemma.

E.18 Lemma. *Let $(\mathcal{T}, \Sigma)$ be a triangulated category and consider a morphism,*

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\varphi"] & N \arrow[r, "\beta"] \arrow[d, "\psi"] & X \arrow[r, "\gamma"] \arrow[d, "\chi"] & \Sigma M \arrow[d, "\Sigma\varphi"] \\
M' \arrow[r, "\alpha'"] & N' \arrow[r, "\beta'"] & X' \arrow[r, "\gamma'"] & \Sigma M' ,
\end{tikzcd}$$

of distinguished triangles. If two of the morphisms φ, ψ, and χ are isomorphisms, then so is the third.

Proof. By axiom (TR2′) it suffices to argue that if φ and ψ are isomorphisms, then so is χ. To this end, it is enough to show that for every object Y in $\mathcal{T}$, the induced homomorphism $\mathcal{T}(Y,\chi)\colon \mathcal{T}(Y,X) \to \mathcal{T}(Y,X')$ of Abelian groups is an isomorphism. Indeed, by surjectivity of $\mathcal{T}(X',\chi)$ there is a morphism $\xi\colon X' \to X$ with $\chi\xi = 1^{X'}$. Now the equalities $\chi(\xi\chi) = 1^{X'}\chi = \chi 1^X$ and injectivity of $\mathcal{T}(X,\chi)$ yield $\xi\chi = 1^X$, whence ξ is an inverse of χ. To prove that $\mathcal{T}(Y,\chi)$ is an isomorphism, consider the following commutative diagram,

$$\begin{tikzcd}
\mathcal{T}(Y,M) \arrow[r, "\mathcal{T}(Y,\alpha)"] \arrow[d, "\mathcal{T}(Y,\varphi)"] & \mathcal{T}(Y,N) \arrow[r, "\mathcal{T}(Y,\beta)"] \arrow[d, "\mathcal{T}(Y,\psi)"] & \mathcal{T}(Y,X) \arrow[r, "\mathcal{T}(Y,\gamma)"] \arrow[d, "\mathcal{T}(Y,\chi)"] & \mathcal{T}(Y,\Sigma M) \arrow[r, "\mathcal{T}(Y,\Sigma\alpha)"] \arrow[d, "\mathcal{T}(Y,\Sigma\varphi)"] & \mathcal{T}(Y,\Sigma N) \arrow[d, "\mathcal{T}(Y,\Sigma\psi)"] \\
\mathcal{T}(Y,M') \arrow[r, "\mathcal{T}(Y,\alpha')"'] & \mathcal{T}(Y,N') \arrow[r, "\mathcal{T}(Y,\beta')"'] & \mathcal{T}(Y,X') \arrow[r, "\mathcal{T}(Y,\gamma')"'] & \mathcal{T}(Y,\Sigma M') \arrow[r, "\mathcal{T}(Y,\Sigma\alpha')"'] & \mathcal{T}(Y,\Sigma N')\,,
\end{tikzcd}$$

whose rows are exact by E.16 and E.17. As φ and ψ are isomorphisms, the Five Lemma 1.1.2 yields the desired conclusion. □

E.19 Proposition. *Let* $\mathrm{F},\mathrm{G}\colon \mathcal{T} \to \mathcal{U}$ *be triangulated functors between triangulated catgories and* $\tau\colon \mathrm{F} \to \mathrm{G}$ *a triangulated natural transformation. The class*

$$\{M \in \mathcal{T} \mid \tau^M \text{ is an isomorphism}\}$$

is a triangulated subcategory of $\mathcal{T}$.

Proof. Write $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{U}}$ for the translation functors on $\mathcal{T}$ and $\mathcal{U}$. If M and N are isomorphic objects in $\mathcal{T}$, then evidently τ^M is an isomorphism if and only if τ^N is so. As τ is triangulated, see E.12, one has $\tau^{\Sigma_{\mathcal{T}} M} \cong \Sigma_{\mathcal{U}}\tau^M$ for every object M in $\mathcal{T}$, and hence τ^M is an isomorphism if and only if $\tau^{\Sigma_{\mathcal{T}} M}$ is so. Let $M \to N \to X \to \Sigma_{\mathcal{T}} M$ be a distinguished triangle in $\mathcal{T}$ such that τ^M and τ^N are isomorphisms. Application of E.18 to the commutative diagram below now shows that τ^X is an isomorphism.

$$\begin{tikzcd}
\mathrm{F}(M) \arrow[r] \arrow[d, "\tau^M", "\cong"'] & \mathrm{F}(N) \arrow[r] \arrow[d, "\tau^N", "\cong"'] & \mathrm{F}(X) \arrow[r] \arrow[d, "\tau^X"] & \Sigma_{\mathcal{U}}\mathrm{F}(M) \arrow[d, "\Sigma_{\mathcal{U}}\tau^M", "\cong"'] \\
\mathrm{G}(M) \arrow[r] & \mathrm{G}(N) \arrow[r] & \mathrm{G}(X) \arrow[r] & \Sigma_{\mathcal{U}}\mathrm{G}(M)\,.
\end{tikzcd}$$

Thus the asserted class is a triangulated subcategory of $\mathcal{T}$ by E.14. □

E.20 Proposition. *Let* $\mathrm{F}\colon \mathcal{T} \to \mathcal{U}$ *be a triangulated functor and* $\mathcal{V}$ *a triangulated subcatgory of* $\mathcal{U}$. *The preimage of* $\mathcal{V}$, *that is,*

$$\mathrm{F}^{-1}(\mathcal{V}) = \{M \in \mathcal{T} \mid \mathrm{F}(M) \in \mathcal{V}\}\,,$$

is a triangulated subcategory of $\mathcal{T}$.

Proof. Write $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{U}}$ for the translation functors on $\mathcal{T}$ and $\mathcal{U}$. If M and N are isomorphic objects in $\mathcal{T}$, then $\mathrm{F}(M)$ and $\mathrm{F}(N)$ are isomorphic objects in $\mathcal{U}$, so it follows that one has $M \in \mathrm{F}^{-1}(\mathcal{V})$ if and only if $N \in \mathrm{F}^{-1}(\mathcal{V})$. For every object M in $\mathcal{T}$ one has $\mathrm{F}(\Sigma_{\mathcal{T}} M) \cong \Sigma_{\mathcal{U}}\mathrm{F}(M)$, so it follows that one has $M \in \mathrm{F}^{-1}(\mathcal{V})$ if and only if $\Sigma_{\mathcal{T}}M \in \mathrm{F}^{-1}(\mathcal{V})$. Finally, let $M \to N \to X \to \Sigma_{\mathcal{T}}M$ be a distinguished triangle in $\mathcal{T}$ with M and N in $\mathrm{F}^{-1}(\mathcal{V})$. Since $\mathrm{F}(M) \to \mathrm{F}(N) \to \mathrm{F}(X) \to \Sigma_{\mathcal{U}}\mathrm{F}(M)$ is a distinguished triangle in $\mathcal{U}$ with $\mathrm{F}(M)$ and $\mathrm{F}(N)$ in $\mathcal{V}$, it follows that $\mathrm{F}(X)$ is in $\mathcal{V}$, i.e. X belongs to $\mathrm{F}^{-1}(\mathcal{V})$. Thus $\mathrm{F}^{-1}(\mathcal{V})$ is a triangulated subcategory of $\mathcal{T}$ by E.14. □

In the suggested analogy with Abelian categories, distinguished triangles of the form $M \longrightarrow N \longrightarrow X \xrightarrow{0} \Sigma M$ correspond to split exact sequences.

E.21 Definition. Let $(\mathcal{T}, \Sigma)$ be a triangulated category. A distinguished triangle $M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\mathcal{T}$ is called *split* if there exist morphisms $\varrho\colon N \to M$ and $\sigma\colon X \to N$ such that the following hold,

$$\varrho\alpha = 1^M, \quad \alpha\varrho + \sigma\beta = 1^N, \quad \text{and} \quad \beta\sigma = 1^X.$$

E.22 Proposition. *Let $(\mathcal{T}, \Sigma)$ be a triangulated category. For a distinguished triangle $\Delta = M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\mathcal{T}$, the following conditions are equivalent.*

(*i*) *The distinguished triangle Δ is split.*

(*ii*) *There exists a morphism $\varrho\colon N \to M$ such that $\varrho\alpha = 1^M$.*

(*iii*) *There exists a morphism $\sigma\colon X \to N$ such that $\beta\sigma = 1^X$.*

(*iv*) *Δ is isomorphic to the distinguished triangle $M \xrightarrow{\varepsilon} M \oplus X \xrightarrow{\varpi} X \xrightarrow{0} \Sigma M$, where ε and ϖ are the injection and the projection, respectively.*

(*v*) *The morphism γ is zero.*

Moreover, if Δ is split, then the diagram $X \xrightarrow{\sigma} N \xrightarrow{\varrho} M \xrightarrow{0} \Sigma X$, where ϱ and σ are as in E.21*, is a split distinguished triangle in $\mathcal{T}$.*

Proof. Conditions (*ii*) and (*iii*) follow from (*i*). Condition (*v*) follows from (*iii*) as one has $\gamma\beta = 0$, and similarly (*v*) follows from (*ii*), as $(\Sigma\alpha)\gamma = 0$ holds.

(*v*) ⇒ (*iv*): Consider the commutative diagram

$$(\dagger) \qquad \begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, equal] & N \arrow[r, "\beta"] & X \arrow[r, "\gamma"] \arrow[d, equal] & \Sigma M \arrow[d, equal] \\ M \arrow[r, "\varepsilon^M"] & M \oplus X \arrow[r, "\varpi^X"] & X \arrow[r, "0"] & \Sigma M\,. \end{tikzcd}$$

The lower row in (†) is a distinguished triangle by E.4. By (TR4′) there exists a morphism $\psi\colon N \to M \oplus X$ such that the resulting diagram is commutative, and it follows from the Five Lemma E.18 that ψ is an isomorphism.

(*iv*) ⇒ (*i*): By assumption there is an isomorphism of distinguished triangles,

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\varphi", "\simeq"'] & N \arrow[r, "\beta"] \arrow[d, "\psi", "\simeq"'] & X \arrow[r, "\gamma"] \arrow[d, "\chi", "\simeq"'] & \Sigma M \arrow[d, "\Sigma\varphi", "\simeq"'] \\ M \arrow[r, "\varepsilon^M"] & M \oplus X \arrow[r, "\varpi^X"] & X \arrow[r, "0"] & \Sigma M\,. \end{tikzcd}$$

With $\varrho = \varphi^{-1}\varpi^M\psi$ and $\sigma = \psi^{-1}\varepsilon^X\chi$, one has $\varrho\alpha = \varphi^{-1}\varpi^M\psi\alpha = 1^M$ and, similarly, $\beta\sigma = \beta\psi^{-1}\varepsilon^X\chi = 1^X$. Finally, one has

$$\alpha\varrho + \sigma\beta \;=\; \alpha\varphi^{-1}\varpi^{M}\psi + \psi^{-1}\varepsilon^{X}\chi\beta \;=\; \psi^{-1}(\varepsilon^{M}\varpi^{M} + \varepsilon^{X}\varpi^{X})\psi \;=\; 1^{N}\;.$$

Now assume that Δ is split. The diagram $\Delta' = X \xrightarrow{\sigma} N \xrightarrow{\varrho} M \xrightarrow{0} \Sigma X$ is a candidate triangle; indeed, pre- and postcomposing the identity $\alpha\varrho + \sigma\beta = 1^{N}$ by σ and ϱ one gets $\varrho\sigma = 0$. If Δ' is distinguished, then it is split because $M \to \Sigma X$ is the zero morphism. In the commutative diagram

$$(\ddagger)\qquad \begin{tikzcd}
X \arrow[r, "\sigma"] \arrow[d, equal] & N \arrow[r, "\varrho"] & M \arrow[r, "0"] \arrow[d, equal] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\varepsilon^{X}"] & X \oplus M \arrow[r, "\varpi^{M}"] & M \arrow[r, "0"] & \Sigma M\;,
\end{tikzcd}$$

the lower row is a distinguished triangle by E.4. Since Δ is split, the morphism $\psi = (\beta\ \varrho)^{\mathrm{T}}$ from N to $X \oplus M$ is an isomorphism with inverse $\psi^{-1} = (\sigma\ \alpha)$. It yields an isomorphism between Δ' and the lower row in ($\ddagger$), whence Δ' is distinguished. $\square$

The next result facilitates verification of axiom (TR4$'$).

E.23 Proposition. *Let $\mathcal{T}$ be an additive category equipped with an additive automorphism Σ. Consider a commutative diagram in $\mathcal{T}$,*

$$\begin{tikzcd}[column sep=small]
& \widetilde{M}^1 \arrow[rr, "\widetilde{\alpha}^1"] \arrow[dd, "\widetilde{\varphi}"' near start] & & \widetilde{N}^1 \arrow[rr, "\widetilde{\beta}^1"] \arrow[dd, "\widetilde{\psi}"' near start] & & \widetilde{X}^1 \arrow[rr, "\widetilde{\gamma}^1"] \arrow[dd, "\widetilde{\chi}"' near start] & & \Sigma\widetilde{M}^1 \arrow[dd, "\Sigma\widetilde{\varphi}"] \\
M^1 \arrow[ur, "\mu^1", "\cong"'] \arrow[rr, "\alpha^1"] \arrow[dd, "\varphi"'] & & N^1 \arrow[ur, "\nu^1", "\cong"'] \arrow[rr, "\beta^1"] \arrow[dd, "\psi"'] & & X^1 \arrow[ur, "\kappa^1", "\cong"'] \arrow[rr, "\gamma^1"] \arrow[dd, "\chi"'] & & \Sigma M^1 \arrow[ur, "\Sigma\mu^1", "\cong"'] \arrow[dd, "\Sigma\varphi"'] & \\
& \widetilde{M}^2 \arrow[rr, "\widetilde{\alpha}^2"] & & \widetilde{N}^2 \arrow[rr, "\widetilde{\beta}^2"] & & \widetilde{X}^2 \arrow[rr, "\widetilde{\gamma}^2"] & & \Sigma\widetilde{M}^2\;, \\
M^2 \arrow[ur, "\cong", "\mu^2"'] \arrow[rr, "\alpha^2"] & & N^2 \arrow[ur, "\cong", "\nu^2"'] \arrow[rr, "\beta^2"] & & X^2 \arrow[ur, "\cong", "\kappa^2"'] \arrow[rr, "\gamma^2"] & & \Sigma M^2 \arrow[ur, "\cong", "\Sigma\mu^2"'] &
\end{tikzcd}$$

where (φ, ψ, χ) and $(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\chi})$ are morphisms of candidate triangles, and (μ^1, ν^1, κ^1) and (μ^2, ν^2, κ^2) are isomorphisms of candidate triangles. The mapping cone candidate triangles of (φ, ψ, χ) and $(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\chi})$ are isomorphic.

Proof. The commutative diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
\begin{matrix} M^2 \\ \oplus \\ N^1 \end{matrix} \arrow[r, "\begin{pmatrix} \alpha^2 & \psi \\ 0 & -\beta^1 \end{pmatrix}"] \arrow[d, "\cong"', "\begin{pmatrix} \mu^2 & 0 \\ 0 & \nu^1 \end{pmatrix}"] & \begin{matrix} N^2 \\ \oplus \\ X^1 \end{matrix} \arrow[r, "\begin{pmatrix} \beta^2 & \chi \\ 0 & -\gamma^1 \end{pmatrix}"] \arrow[d, "\cong"', "\begin{pmatrix} \nu^2 & 0 \\ 0 & \kappa^1 \end{pmatrix}"] & \begin{matrix} X^2 \\ \oplus \\ \Sigma M^1 \end{matrix} \arrow[r, "\begin{pmatrix} \gamma^2 & \Sigma\varphi \\ 0 & -\Sigma\alpha^1 \end{pmatrix}"] \arrow[d, "\cong"', "\begin{pmatrix} \kappa^2 & 0 \\ 0 & \Sigma\mu^1 \end{pmatrix}"] & \begin{matrix} \Sigma M^2 \\ \oplus \\ \Sigma N^1 \end{matrix} \arrow[d, "\cong"', "\begin{pmatrix} \Sigma\mu^2 & 0 \\ 0 & \Sigma\nu^1 \end{pmatrix}"] \\
\begin{matrix} \widetilde{M}^2 \\ \oplus \\ \widetilde{N}^1 \end{matrix} \arrow[r, "\begin{pmatrix} \widetilde{\alpha}^2 & \widetilde{\psi} \\ 0 & -\widetilde{\beta}^1 \end{pmatrix}"] & \begin{matrix} \widetilde{N}^2 \\ \oplus \\ \widetilde{X}^1 \end{matrix} \arrow[r, "\begin{pmatrix} \widetilde{\beta}^2 & \widetilde{\chi} \\ 0 & -\widetilde{\gamma}^1 \end{pmatrix}"] & \begin{matrix} \widetilde{X}^2 \\ \oplus \\ \Sigma\widetilde{M}^1 \end{matrix} \arrow[r, "\begin{pmatrix} \widetilde{\gamma}^2 & \Sigma\widetilde{\varphi} \\ 0 & -\Sigma\widetilde{\alpha}^1 \end{pmatrix}"] & \begin{matrix} \Sigma\widetilde{M}^2 \\ \oplus \\ \Sigma\widetilde{N}^1 \end{matrix}
\end{tikzcd}$$

is an isomorphism from the mapping cone candidate triangle of (φ, ψ, χ) to the mapping cone candidate triangle of $(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{\chi})$. □

Recall that in any category, a morphism is called a *split monomorphism* or *split epimorphism* if it has a left inverse, respectively, a right inverse. Evidently, a split monomorphism is a monomorphism and a split epimorphism is an epimorphism. The converse rarely holds in an Abelian category, but it does hold in a triangulated category. Thus, every object in a triangulated category is injective and projective.

E.24 Proposition. *In a triangulated category, every monomorphism is a split monomorphism and every epimorphism is a split epimorphism.*

Proof. Let $(\mathcal{T}, \Sigma)$ be a triangulated category and $\alpha\colon M \to N$ a monomorphism in $\mathcal{T}$. By (TR1) there is a distinguished triangle $\Delta = M \xrightarrow{\alpha} N \xrightarrow{\beta} X \xrightarrow{\gamma} \Sigma M$ in $\mathcal{T}$ which by (TR2) yields a distinguished triangle,

$$\Sigma^{-1}X \xrightarrow{-\Sigma^{-1}\gamma} M \xrightarrow{\alpha} N \xrightarrow{\beta} X\,.$$

As one has $\alpha \circ \Sigma^{-1}\gamma = 0$ and α is monomorphism, it follows that $\gamma = 0$. Thus Δ is split and α has a left inverse; see E.22. Similarly one shows that every epimorphism in $\mathcal{T}$ has a right inverse. □

Exercises

E E.1 Let $\alpha\colon M \to N$ be a morphism in a triangulated category $(\mathcal{T}, \Sigma)$. Show that the complex X in a distinguished triangle $M \xrightarrow{\alpha} N \longrightarrow X \longrightarrow \Sigma M$ in $\mathcal{T}$ is unique up to isomorphism.

E E.2 (Cf. E.6) Let $(\mathcal{T}, \Sigma)$ be a triangulated category. Show that $(\mathcal{T}^{\mathrm{op}}, \Sigma^{-1})$ is triangulated in the canonical way: A candidate triangle $M \to N \to X \to \Sigma^{-1}M$ in $\mathcal{T}^{\mathrm{op}}$ is distinguished if and only if the candidate triangle $\Sigma^{-1}M \to X \to N \to M$ is distinguished in $\mathcal{T}$.

E E.3 Two homomorphisms of R-modules $\alpha, \beta\colon M \to N$ are called stably equivalent if $\alpha - \beta$ factors through a projective R-module. The *stable module category* $\underline{\mathcal{M}}(R)$ has as objects all R-modules. The hom-set $\underline{\mathcal{M}}(R)(M,N)$, often written as $\underline{\mathrm{Hom}}_R(M,N)$, is the set of classes of stably equivalent homomorphisms $M \to N$. (a) Show that $\underline{\mathcal{M}}(R)$ is a $\Bbbk$-linear category with coproducts. (b) For an R-module M, let $\Omega(M)$ be the kernel of any projective precover $P \twoheadrightarrow M$. Show that Ω is a well-defined $\Bbbk$-linear endofunctor on $\underline{\mathcal{M}}(R)$. (c) Show that the category $\underline{\mathcal{M}}(R)$ is triangulated if R is quasi-Frobenius.

E E.4 Show that the stable module category $\underline{\mathcal{M}}(\mathbb{Q}[x]/(x^2))$ is not Abelian.

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Glossary

Here we recapitulate, briefly, the definitions of several key notions. For other standard concepts that we use but do not define, we refer to the following textbooks.

- "Categories for the Working Mathematician" [175] by MacLane for notions in category theory,
- "Lectures on Modules and Rings" [167] and "A First Course in Noncommutative Rings" [168] by Lam for notions in ring theory, and
- "Commutative Ring Theory" [182] by Matsumura for notions in commutative algebra.

Abelian category. An additive category in which every morphism has a kernel and a cokernel, every monomorphism is the kernel of a morphism, and every epimorphism is the cokernel of a morphism. The opposite category of an Abelian category is Abelian. See [175, VIII.3].

In the Abelian categories $\mathcal{M}(R)$, $\mathcal{M}_{\mathrm{gr}}(R)$, and $\mathcal{C}(R)$, a kernel is identified with its domain and a cokernel is identified with its codomain.

Adjoint functors. Functors $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ and $\mathrm{G}\colon \mathcal{V} \to \mathcal{U}$ are adjoint if for all objects M in $\mathcal{U}$ and N in $\mathcal{V}$ there are isomorphisms of hom-sets $\Phi^{M,N}\colon \mathcal{V}(\mathrm{F}(M), N) \to \mathcal{U}(M, \mathrm{G}(N))$ which are natural in M and N. The functor F is then said to be a *left adjoint* for G, and G is said to be a *right adjoint* for F. In a diagram of adjoint functors,

$$\mathcal{U} \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathcal{V}\,,$$

it is standard to place left adjoint above the right adjoint.

For objects M in $\mathcal{U}$ the images of $1^{\mathrm{F}(M)}$ under $\Phi^{M,\mathrm{F}(M)}$ are natural morphisms $M \to \mathrm{GF}(M)$, whence there is a natural transformation $\alpha\colon \mathrm{Id}_{\mathcal{U}} \to \mathrm{GF}$, called the *unit of adjunction.* Similarly, for objects N in $\mathcal{V}$ the images of $1^{\mathrm{G}(N)}$ under $(\Phi^{\mathrm{G}(N),N})^{-1}$ yield a natural transformation $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{V}}$, called the *counit of adjunction.* Moreover, the composites below are the identity transformations; they are called the *zigzag identities*.

$$\mathrm{F} \xrightarrow{\mathrm{F}\alpha} \mathrm{FGF} \xrightarrow{\beta\mathrm{F}} \mathrm{F} \qquad \text{and} \qquad \mathrm{G} \xrightarrow{\alpha\mathrm{G}} \mathrm{GFG} \xrightarrow{\mathrm{G}\beta} \mathrm{G}\,.$$

Conversely, the existence of natural transformations $\alpha\colon \mathrm{Id}_{\mathcal{U}} \to \mathrm{GF}$ and $\beta\colon \mathrm{FG} \to \mathrm{Id}_{\mathcal{V}}$ such that the zigzag identities hold implies that F and G are adjoint and that α and β are the unit and counit of the adjunction. See [175, IV.1].

$\Bbbk$-*algebra*. A unital associative ring A and a homomorphism of unital rings $\Bbbk \to A$, called the *structure map*, that maps $\Bbbk$ to the center of A.

Artinian module. A module that satisfies the *Descending Chain Condition*. That is, every chain of submodules, $M^0 \supseteq M^1 \supseteq M^2 \supseteq \cdots$, stabilizes in the sense that there is an $n \in \mathbb{N}$ such that

$M^n = M^u$ holds for all $u \geqslant n$. The Artinian R-modules constitute a Serre subcategory of $\mathcal{M}(R)$; in particular, submodules and quotients of Artinian modules are Artinian.

Artinian ring. A ring that is both left and right Artinian. A ring is left Artinian if it is Artinian as a module over itself, i.e. it satisfies the Descending Chain Condition on left ideals. A ring R is right Artinian if R^{o} is left Artinian. A left Artinian ring is left Noetherian, and the quotient of a left Artinian ring by its Jacobson radical is a semi-simple ring. See [168, §§1,4].

Center of ring. The subring R^{c} of a ring R consisting of the elements that commute with all elements of R. Elements in the center are called *central*.

Coherent ring. A ring that is both left and right coherent. A ring R is left coherent if every finitely generated left ideal is finitely presented, equivalently, every finitely generated submodule of a finitely presented R-module is finitely presented. A ring R is right coherent if R^{o} is left coherent. See [167, §4G].

Colon submodule. Let M be an R-module and $M' \subseteq M$ a submodule. For a subset $\mathfrak{a} \subseteq R$ the set

$$(M' :_M \mathfrak{a}) = \{m \in M \mid \mathfrak{a}m \subseteq M'\}$$

is a submodule of the $\Bbbk$-module M; if $\mathfrak{a}$ is a right ideal, then $(M' :_M \mathfrak{a})$ is an R-submodule of M.

For a subset $X \subseteq M$, the set

$$(M' :_R X) = \{r \in R \mid rX \subseteq M'\}$$

is a left ideal, and if X is a submodule, then $(M' :_R X)$ is an ideal. Notice that for $M = R$ the symbol $(M' :_R X)$ could be interpreted, differently, according to the first display, but it should always be interpreted as in the second display.

For a subset $X \subseteq M$ the left ideal $(0 :_R X)$ is called the *annihilator* of X. If X consists of a single element m, then the annihilator is written $(0 :_R m)$ and called the annihilator of m. If R is commutative, then one has $(0 :_R m) = (0 :_R R\langle m\rangle)$, and we use the former, simpler, symbol.

Conservative functor. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ is said to *reflect* mono-/epi-/isomorphisms if given a morphism α in $\mathcal{U}$ such that $\mathrm{F}(\alpha)$ is a mono-/epi-/isomorphism in $\mathcal{V}$, then α is a mono-/epi-/isomorphism in $\mathcal{U}$. A conservative functor is one that reflects isomorphisms.

Coproduct. Let $\mathcal{V}$ be a category. The coproduct of a set-indexed family $\{M^u\}_{u\in U}$ of objects in $\mathcal{V}$ is an object $M \in \mathcal{V}$ together with morphisms $M^u \to M$, called *injections*, with the following universal property: For every family of morphisms $\{\alpha^u\colon M^u \to N\}_{u\in U}$ there is a unique morphism from M to N that for every $u \in U$ makes the next diagram commutative

$$\begin{tikzcd} M^u \arrow[r] \arrow[d, "\alpha^u"'] & M \arrow[dl, dotted] \\ N\,. & \end{tikzcd}$$

Such a coproduct M is unique up to isomorphism in $\mathcal{V}$ and usually denoted $\coprod_{u\in U} M^u$.

Let $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ be a family of morphisms in $\mathcal{V}$. If $\{M^u \to M\}_{u\in U}$ is a coproduct of $\{M^u\}_{u\in U}$ and $\{N^u \to N\}_{u\in U}$ is a coproduct of $\{N^u\}_{u\in U}$, then the unique morphism from M to N that makes the diagram

$$\begin{tikzcd} M^u \arrow[r] \arrow[d, "\alpha^u"'] & M \arrow[d, dotted] \\ N^u \arrow[r] & N \end{tikzcd}$$

commutative for every $u \in U$ is called the coproduct of $\{\alpha^u\colon M^u \to N^u\}_{u\in U}$ and denoted $\coprod_{u\in U} \alpha^u$.

One says that $\mathcal{V}$ *has coproducts* if every set-indexed family of objects in $\mathcal{V}$ has a coproduct.

A functor $\mathrm{F}\colon \mathcal{V} \to \mathcal{W}$ between categories that have coproducts is said to *preserve coproducts* if given any family $\{M^u\}_{u\in U}$ of objects in $\mathcal{V}$ the unique morphism that makes the diagram

$$\begin{tikzcd}
\mathrm{F}(M^u) \arrow[r] \arrow[d] & \coprod_{u\in U} \mathrm{F}(M^u) \arrow[dl, dotted] \\
\mathrm{F}(\coprod_{u\in U} M^u) &
\end{tikzcd}$$

in $\mathcal{W}$ commutative for every $u \in U$, is an isomorphism. Here the horizontal morphism is the injection while the vertical morphism is the image of the injection $M^u \to \coprod_{u\in U} M^u$ under F.

Let $\mathrm{F}\colon \mathcal{V} \to \mathcal{W}$ be a functor between categories that have coproducts. For every family of morphisms $\{\alpha^u : M^u \to N^u\}_{u\in U}$ in $\mathcal{V}$ there is a commutative diagram in $\mathcal{W}$,

$$\begin{tikzcd}
\coprod_{u\in U} \mathrm{F}(M^u) \arrow[r, "\coprod_{u\in U} \mathrm{F}(\alpha^u)"] \arrow[d] & \coprod_{u\in U} \mathrm{F}(N^u) \arrow[d] \\
\mathrm{F}(\coprod_{u\in U} M^u) \arrow[r, "\mathrm{F}(\coprod_{u\in U} \alpha^u)"] & \mathrm{F}(\coprod_{u\in U} N^u)\,,
\end{tikzcd}$$

so if F preserves coproducts, then the morphisms $\coprod_{u\in U} \mathrm{F}(\alpha^u)$ and $\mathrm{F}(\coprod_{u\in U} \alpha^u)$ are isomorphic.

Dedekind domain. An integral domain in which every non-zero ideal, equivalently every fractional ideal, is invertible. A Dedekind domain is Noetherian. See [167, §2].

Division ring. A ring in which every non-zero element has a multiplicative inverse. See [168, §13].

Domain. A ring R with the property that for elements $r, r' \in R$ the product rr' is zero (if and) only if one has $r = 0$ or $r' = 0$. A commutative domain is called an *integral domain.*

Endomorphism ring. Let $\mathcal{U}$ be an preadditive category and M an object in $\mathcal{U}$. A morphism $M \to M$ is called an *endomorphism* of M. The hom-set $\mathcal{U}(M, M)$ is an abelian group under addition and a ring with multiplication given by composition of endomorphisms.

Equivalence of categories. A pair of functors $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ and $\mathrm{G}\colon \mathcal{V} \to \mathcal{U}$ such that there are natural isomorphisms $\mathrm{GF} \to \mathrm{Id}_{\mathcal{U}}$ and $\mathrm{FG} \to \mathrm{Id}_{\mathcal{V}}$. An equivalence of categories $\mathcal{U}$ and $\mathcal{V}^{\mathrm{op}}$ is also called a *duality* of $\mathcal{U}$ and $\mathcal{V}$ and an equivalence of $\mathcal{U}$ and $\mathcal{U}^{\mathrm{op}}$ simply a duality on $\mathcal{U}$. See [175, I.4].

Faithful functor. A functor that is injective on hom-sets. See [175, I.3].

Filtered set. A preordered set $(U, \leqslant)$ with the property that for all elements u and v in U there is a $w \in U$ with $u \leqslant w$ and $v \leqslant w$. See [175, IX.1]. A filtered set is also called a *directed set.*

Filtration of module. Let M be an R-module. A filtration of M is a finite sequence

$$0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^n = M$$

of submodules. The number of non-zero quotients M^i/M^{i-1} is called the *length* of the filtration. A *composition series* is a filtration in which all the quotients M^i/M^{i-1} are simple modules. If M has a composition series, then all such series have the same length, and that number, written $\mathrm{length}_R\, M$, is also called the *length* of M. If M does not have a composition series, then one sets $\mathrm{length}_R\, M = \infty$. Submodules and quotients of modules of finite length have finite length, and for an exact seqeunce $0 \to M' \to M \to M'' \to 0$ one has $\mathrm{length}_R\, M = \mathrm{length}_R\, M' + \mathrm{length}_R\, M''$. See [168, §1].

Full functor. A functor that is surjective on hom-sets. See [175, I.3].

Hereditary ring. A ring that is both left and right hereditary. A ring R is left hereditary if every left ideal is projective as an R-module; R is right hereditary if R^{o} is left hereditary. See [167, §2E].

Idempotent in ring. A ring element x with $x^2 = x$. Idempotents x and y are called *orthogonal* if $xy = 0 = yx$ holds.

Inductively ordered set. A (partially) ordered set such that every totally ordered subset has an upper bound. Zorn's lemma states that a nonempty inductively ordered set has a maximal element.

Invariant Basis Number (IBN). A left–right symmetric property: R has (left) IBN if for all numbers m and n in $\mathbb{N}$ one has $R^n \cong R^m$ as R-modules if and only if $m = n$. See [167, §1A].

Indecomposable module. A non-zero module M with no other direct summands than 0 and M. See [168, §7].

Isomorphism of categories. A functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$ that has an inverse, that is, there exists a functor $\mathrm{G}\colon \mathcal{V} \to \mathcal{U}$ such that $\mathrm{GF} = \mathrm{Id}_{\mathcal{U}}$ and $\mathrm{FG} = \mathrm{Id}_{\mathcal{V}}$ hold. See [175, I.3].

Integral domain. A commutative domain.

Jacobson radical of ring. The intersection of all maximal left ideals or, equivalently, all maximal right ideals. In particular, the Jacobson radical is an ideal. See [168, §4].

Local ring. A ring with a unique maximal left ideal or, equivalently, a unique (the same) maximal right ideal. The set of units in a local ring is the complement of the maximal ideal. See [168, §19].

Localization of module or ring. Assume that R is commutative and let U be a multiplicative subset of R. The localization $U^{-1}M$ of an R-module M at U is the R-module of fractions $\frac{m}{u}$ with $m \in M$ and $u \in U$. The R-module $U^{-1}R$ is an R-algebra, and $U^{-1}M$ is a module over $U^{-1}R$. Localization, U^{-1}, is an exact functor and idempotent, i.e. $U^{-1}(U^{-1}M) \cong U^{-1}M$. See [182, §4].

Middle R-linear map. A map $\varphi\colon M \times N \to X$, where M is an R°-module, N is an R-module, and X is a $\Bbbk$-module, such that $\varphi(mr, n) = \varphi(m, rn)$ holds for all $m \in M$, $n \in N$, and $r \in R$.

Monomial. A monic polynomial with only one term.

Natural transformation. For functors $\mathrm{F}, \mathrm{G}\colon \mathcal{U} \to \mathcal{V}$ a natural transformation $\tau\colon \mathrm{F} \to \mathrm{G}$ assigns to each object M in $\mathcal{U}$ a morphism $\tau^M\colon \mathrm{F}(M) \to \mathrm{G}(M)$ in $\mathcal{V}$ which is natural in M in the sense that $\tau^N \mathrm{F}(\alpha) = \mathrm{G}(\alpha)\tau^M$ holds for every morphism $\alpha\colon M \to N$ in $\mathcal{U}$. If each morphism τ^M is an isomorphism, then τ is called a *natural isomorphism.* See [175, I.4].

Nilpotent element/ideal in ring. An element x (an ideal $\mathfrak{a}$) with $x^n = 0$ ($\mathfrak{a}^n = 0$) for some $n \in \mathbb{N}$. The same terminology is used for left and right ideals. Every element of a nilpotent (left/right) ideal is nilpotent. A (left/right) ideal is called *nil* if all its elements are nilpotent. See [168, §4].

Noetherian module. A module that satisfies the *Ascending Chain Condition*. That is, every chain of submodules, $M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots$, stabilizes in the sense that there is an $n \in \mathbb{N}$ such that $M^n = M^u$ holds for all $u \geqslant n$. The Noetherian R-modules constitute a Serre subcategory of $\mathcal{M}(R)$; in particular, submodules and quotients of Noetherian modules are Noetherian. Further, if R is commutative and $U \subseteq R$ is a multiplicative subset, then $U^{-1}M$ is a Noetherian $U^{-1}R$-module. See [168, §1] and [182, §4].

Noetherian ring. A ring that is both left and right Noetherian. A ring R is left Noetherian if it is Noetherian as an R-module, i.e. it satisfies the Ascending Chain Condition on left ideals, equivalently, every submodule of a finitely generated R-module is finitely generated. A ring R is right Noetherian if R° is left Noetherian. If R is left Noetherian and $\mathfrak{a}$ an ideal in R, then $R/\mathfrak{a}$ is left Noetherian. If R is commutative and Noetherian and $U \subseteq R$ is a multiplicative subset, then $U^{-1}R$ is Noetherian. See [168, §1] and [182, §4].

Opposite category. For a category $\mathcal{U}$, the opposite category $\mathcal{U}^{\mathrm{op}}$ has the same objects as $\mathcal{U}$. A hom-set $\mathcal{U}^{\mathrm{op}}(M, N)$ is identified with $\mathcal{U}(N, M)$ under interchange of domains and codomains. Composition $\mathcal{U}^{\mathrm{op}}(M, N) \times \mathcal{U}^{\mathrm{op}}(L, M) \to \mathcal{U}^{\mathrm{op}}(L, N)$ is given by $\mathcal{U}(M, L) \times \mathcal{U}(N, M) \to \mathcal{U}(N, L)$; that is, for $\alpha \in \mathcal{U}^{\mathrm{op}}(M, N)$ and $\beta \in \mathcal{U}^{\mathrm{op}}(L, M)$ the composite $\alpha \circ \beta$ in $\mathcal{U}^{\mathrm{op}}(L, N)$ is $\beta\alpha \in \mathcal{U}(N, L)$ with domain and codomain interchanged. See [175, II.2].

Opposite functor. For a functor $\mathrm{F}\colon \mathcal{U} \to \mathcal{V}$, the opposite functor $\mathrm{F}^{\mathrm{op}}\colon \mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ is defined on objects by $\mathrm{F}^{\mathrm{op}}(M) = \mathrm{F}(M)$. For every morphism α in $\mathcal{U}^{\mathrm{op}}(M, N)$ the morphism $\mathrm{F}^{\mathrm{op}}(\alpha)$ in $\mathcal{V}^{\mathrm{op}}(\mathrm{F}^{\mathrm{op}}(M), \mathrm{F}^{\mathrm{op}}(N))$ is $\mathrm{F}(\alpha)$ in $\mathcal{V}(\mathrm{F}(N), \mathrm{F}(M))$ with domain and codomain interchanged. For a natural transformation $\tau\colon \mathrm{F} \to \mathrm{G}$ of functors $\mathcal{U} \to \mathcal{V}$, the opposite natural transformation $\tau^{\mathrm{op}}\colon \mathrm{G}^{\mathrm{op}} \to \mathrm{F}^{\mathrm{op}}$ of the opposite functors $\mathcal{U}^{\mathrm{op}} \to \mathcal{V}^{\mathrm{op}}$ assigns to each object M the morphism in

$\mathcal{V}^{\mathrm{op}}(\mathrm{G}^{\mathrm{op}}(M), \mathrm{F}^{\mathrm{op}}(M))$ obtained from τ^M in $\mathcal{V}(\mathrm{F}(M), \mathrm{G}(M))$ by interchanging the domain and codomain. See [175, II.2].

Opposite ring. The opposite ring R^{o} of a ring R has the same underlying additive group while multiplication is given by $ab = b \cdot a$, where '$\cdot$' denotes multiplication in R. There is no distinction between a commutative ring and its opposite ring. See [168, §1].

Partially ordered set. A set endowed with a binary relation that is reflexive, transitive, and anti-symmetric. A partially ordered set is, in particular, a preordered set. See [175, I.2].

Preordered set. A set endowed with a binary relation that is reflexive and transitive. See [175, I.2]. In a preordered set $(U, \leqslant)$, an element w is called a *maximal element* if $w \leqslant u$ implies $u = w$, and it is called a *greatest element* if $u \leqslant w$ holds for all $u \in U$. Analogously one defines a *minimal element* and a *least element*.

Prime Avoidance. Assume that R is commutative and Noetherian. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be ideals in R, at most two of which are not prime. If an ideal $\mathfrak{a}$ is contained in the union $\bigcup_{i=1}^n \mathfrak{p}_i$, then $\mathfrak{a}$ is contained in one of the ideals $\mathfrak{p}_i$.

Principal (left/right) ideal domain. A domain R is a principal left ideal domain if every left ideal in R is principal, i.e. of the form Rx for some $x \in R$. A domain R is a principal right ideal domain if R^{o} is principal left ideal domain. A commutative principal left (= right) ideal domain is called a *principal ideal domain.*

Product. Let $\mathcal{V}$ be a category. The product of a set-indexed family $\{N^u\}_{u \in U}$ of objects in $\mathcal{V}$ is an object $N \in \mathcal{V}$ together with morphisms $N \to N^u$, called *projections*, with the following universal property: For every family of morphisms $\{\alpha^u : M \to N^u\}_{u \in U}$ there is a unique morphism from M to N that for every $u \in U$ makes the next diagram commutative

$$\begin{tikzcd} & N \arrow[d] \\ M \arrow[ur, dotted] \arrow[r, "\alpha^u"] & N^u . \end{tikzcd}$$

Such a product N is unique up to isomorphism in $\mathcal{V}$ and usually denoted $\prod_{u \in U} N^u$.

Let $\{\alpha^u : M^u \to N^u\}_{u \in U}$ be a family of morphisms in $\mathcal{V}$. If $\{M \to M^u\}_{u \in U}$ is a product of $\{M^u\}_{u \in U}$ and $\{N \to N^u\}_{u \in U}$ is a product of $\{N^u\}_{u \in U}$, then the unique morphism from M to N that makes the diagram

$$\begin{tikzcd} M \arrow[r, dotted] \arrow[d] & N \arrow[d] \\ M^u \arrow[r, "\alpha^u"] & N^u \end{tikzcd}$$

commutative for every $u \in U$ is called the product of $\{\alpha^u : M^u \to N^u\}_{u \in U}$ and denoted $\prod_{u \in U} \alpha^u$.

One says that $\mathcal{V}$ *has products* if every set-indexed family of objects in $\mathcal{V}$ has a product.

A functor $\mathrm{F} : \mathcal{V} \to \mathcal{W}$ between categories that have products is said to *preserve products* if given any family $\{N^u\}_{u \in U}$ of objects in $\mathcal{V}$ the unique morphism that makes the diagram

$$\begin{tikzcd} & \prod_{u \in U} \mathrm{F}(N^u) \arrow[d] \\ \mathrm{F}(\prod_{u \in U} N^u) \arrow[ur, dotted] \arrow[r] & \mathrm{F}(N^u) \end{tikzcd}$$

in $\mathcal{W}$ commutative for every $u \in U$, is an isomorphism. Here the vertical morphism is the projection while the horizontal morphism is the image of the projection $\prod_{u \in U} N^u \to N^u$ under F.

Let $\mathrm{F} : \mathcal{V} \to \mathcal{W}$ be a functor between categories that have products. For every family of morphisms $\{\alpha^u : M^u \to N^u\}_{u \in U}$ in $\mathcal{V}$ there is a commutative diagram in $\mathcal{W}$,

$$
\begin{tikzcd}
\mathrm{F}(\prod_{u\in U} M^u) \arrow[r, "\mathrm{F}(\prod_{u\in U}\alpha^u)"] \arrow[d] & \mathrm{F}(\prod_{u\in U} N^u) \arrow[d] \\
\prod_{u\in U}\mathrm{F}(M^u) \arrow[r, "\prod_{u\in U}\mathrm{F}(\alpha^u)"] & \prod_{u\in U}\mathrm{F}(N^u)\ ,
\end{tikzcd}
$$

so if F preserves products, then the morphisms $\mathrm{F}(\prod_{u\in U}\alpha^u)$ and $\prod_{u\in U}\mathrm{F}(\alpha^u)$ are isomorphic.

Quasi-Frobenius ring. A ring R such that an R-module (equivalently, an R°-module) is projective if and only if it is injective. Such a ring is Artinian and hence Noetherian. See [167, §§15A–15B].

Self-injective ring. A ring that is both left and right self-injective. A ring R is left self-injective if it is injective as a module over itself. A ring R is right self-injective if R° is left self-injective. A left (or right) Noetherian ring that is left or right self-injective is quasi-Frobenius. See [167, §15A].

Semi-simple module. A module whose every submodule is a direct summand. A module is semi-simple if and only if it is a coproduct of simple modules. See [168, §2].

Semi-simple ring. A ring R such that every R-module (equivalently: every R°-module, the R-module R, or the R°-module R) is semi-simple. A cyclic module over a semi-simple ring is isomorphic to a direct sum of simple ideals generated by idempotents. A semi-simple ring is Artinian, in particular Noetherian. See [168, §§2–3].

Serre subcategory. A full subcategory $\mathcal{U}$ of an Abelian category $\mathcal{V}$, such that 0 belongs to $\mathcal{U}$ and for every exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{V}$ the object M belongs to $\mathcal{U}$ if and only if M' and M'' belong to $\mathcal{U}$.

Simple module. A module $M \neq 0$ with no other submodules than 0 and M. See [168, §2].

Simple ring. A non-zero ring R with no other ideals than 0 and R. See [168, §1].

Torsion. Assume that R is commutative. An element m of an R-module M is torsion if $xm = 0$ holds for some non-zerodivisor x in R. The torsion elements in M form a submodule M_{T} of M called the *torsion submodule.* If one has $M_{\mathrm{T}} = M$, then M is *torsion,* and M is *torsion-free* if $M_{\mathrm{T}} = 0$ holds. See [167, §4B].

von Neumann regular ring. A ring R such that for every $x \in R$ there is an $r \in R$ with $x = xrx$; equivalently, every finitely generated left ideal (equivalently, every finitely generated right ideal) in R is generated by an idempotent. See [168, §4].

Zerodivisor. Assume that R is commutative. An element $z \in R$ is a zerodivisor if $zr = 0$ holds for some $r \neq 0$ in R. An element that is not a zerodivisor is called a *non-zerodivisor.* See [168, §1].

List of Symbols

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