FREE RESOLUTIONS OF DYNKIN FORMAT AND THE LICCI PROPERTY OF GRADE 3 PERFECT IDEALS

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Abstract. Recent work on generic free resolutions of length 3 attaches to every resolution a graph and suggests that resolutions whose associated graph is a Dynkin diagram are distinguished. We conjecture that in a regular local ring, every grade 3 perfect ideal whose minimal free resolution is distinguished in this way is in the linkage class of a complete intersection.

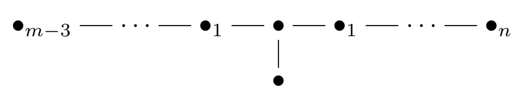
1. Introduction

Let Q be a commutative Noetherian ring. Quotient rings of Q that have projective dimension at most 3 as Q-modules have been and still are investigated vigorously. A recent development is Weyman's [14] construction of generic rings for resolutions of length 3. For a free resolution,

$$F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} F_3 \longleftarrow 0\,,$$

the *format* is the quadruple (f_0, f_1, f_2, f_3) with $f_i = \operatorname{rank} F_i$. To each resolution format Weyman associates a graph and a generic ring, and he proves that the generic ring is Noetherian exactly if the graph is a Dynkin diagram. This suggests that free resolutions of formats corresponding to Dynkin diagrams play a special role; here we explore what that role might be in the context of linkage.

Let Q be local and I in Q be a perfect ideal of grade 3. Let m be the minimal number of generators of I, the minimal free resolution of Q/I over Q then has format $\mathfrak{f} = (1, m, m+n-1, n)$ for $m \geqslant 3$ and some integer $n \geqslant 1$ which, when Q is regular, is referred to as the *type* of the ring Q/I. For convenience we refer to the tuple $\mathfrak{f}$ as the *resolution format*, or simply the *format,* of I. The graph associated to $\mathfrak{f}$ is



A complete intersection ideal has resolution format $(1,3,3,1)$, and the corresponding graph is the Dynkin diagram A_3. A Gorenstein ideal that is not complete intersection has format $(1, m, m, 1)$ for some odd $m \geqslant 5$, and the associated graph is the Dynkin diagram D_m. There is another way to obtain Dynkin diagrams

Date: 1 October 2019.

2010 *Mathematics Subject Classification.* 13C40; 13D02.

Key words and phrases. Free resolution, licci ideal, linkage, perfect ideal.

This work is part of a body of research that started during the authors' visit to MSRI in Spring 2013 and continued during a months-long visit by L.W.C. to Northeastern University; the hospitality of both institutions is acknowledged with gratitude. L.W.C. was partly supported by NSA grant H98230-14-0140 and Simons Foundation collaboration grant 428308, and J.W. was partly supported by NSF DMS grant 1400740.

of type D, namely from almost complete intersection ideals, which have formats $(1,4,n+3,n)$, corresponding to D_{n+3}, for $n \geqslant 2$. It is elementary to verify that E_6, E_7, and E_8 are the only other Dynkin diagrams that can possibly arise in this manner. In the remainder of this paper, we say that a (resolution) format is *Dynkin* if the corresponding graph is a Dynkin diagram.

If Q is regular, then every grade 2 perfect ideal I in Q is *licci*, that is, in the linkage class of a complete intersection ideal. Not every perfect ideal of grade 3 is licci, and in a certain sense licci ideals of grade 4 are rare; see Miller and Ulrich [12]. Works of Kunz [11] and Watanabe [13] show that a grade 3 perfect ideal is licci if it is Gorenstein or almost complete intersection; these results predate the term licci, but Buchsbaum and Eisenbud interpret them in the introduction to [3]. The purpose of this paper is to motivate the following:

Conjecture. Let Q be a regular local ring and $\mathfrak{f} = (1, m, m+n-1, n)$ be a resolution format realized by some grade 3 perfect ideal in Q.

- **I** If $\mathfrak{f}$ is not Dynkin, then there exists a grade 3 perfect ideal in Q of format $\mathfrak{f}$ that is not licci.
- **II** If $\mathfrak{f}$ is Dynkin, then every grade 3 perfect ideal in Q of format $\mathfrak{f}$ is licci.

The paper is organized as follows. In Section 2 we recall basics on linkage and prove a few technical results. In Section 3 we collect evidence for part I of the conjecture; in fact, we prove it for local rings that are obtained by localizing a polynomial algebra at the irrelevant maximal ideal. Some of our evidence for part II was already discussed above, and further evidence is provided in Section 4.

2. Linkage of grade 3 perfect ideals

To goal of this section, other than to recall the language of linkage, is to establish a proposition that allows some control over the resolution formats of directly linked ideals. To this end we need a folklore application of Prime Avoidance for which we did not find a citable reference.[1]

(2.1) **Lemma.** *Let Q be a commutative ring and $P_1, \dots, P_s$ be prime ideals in Q. For every ideal J and element x in Q with $J + (x) \not\subseteq P_1 \cup P_2 \cup \ldots \cup P_s$ there exists an element $v \in J$ with $v + x \notin P_1 \cup P_2 \cup \ldots \cup P_s$.*

Proof. Without loss of generality we may assume that there are no containments among the prime ideals $P_1, \dots, P_s$ and that they are ordered such that there is an r between 0 and s with $x \in P_1, \dots, P_r$ and $x \notin P_{r+1}, \dots, P_s$. Notice that if $r = 0$, then one can take $v = 0$.

Let $r > 0$. From the assumption on $J + (x)$ one gets $J \not\subseteq P_1 \cup \ldots \cup P_r$; choose an element $u \in J \setminus (P_1 \cup \ldots \cup P_r)$. If $r = s$, then one can take $v = u$. In the case $r < s$ one has $P_{r+1} \cap \ldots \cap P_s \not\subseteq P_1 \cup \ldots \cup P_r$ as, indeed, by Prime Avoidance containment would imply $P_{r+1} \cap \ldots \cap P_s \subseteq P_i$ for some i and then one would have $P_j \subseteq P_i$ for some $j \neq i$, which contradicts the assumption on $P_1, \dots, P_s$. Now choose an element $y \in P_{r+1} \cap \ldots \cap P_s \setminus P_1 \cup \ldots \cup P_r$. It is elementary to verify that $uy + x$ is not in $P_1 \cup P_2 \cup \ldots \cup P_s$. □

[1] Note added in proof: A, not widely available, reference is Kaplansky [10, Theorem 124].

In the rest of this section, we keep the setup and notation close to [1] by Avramov, Kustin, and Miller; the reader may see [1, Section 1] for more background.

Let I be a perfect ideal of grade 3 in a local ring Q. An ideal $J \subseteq Q$ is said to be *directly linked* to I if there exists a regular sequence x_1, x_2, x_3 of elements in I with $J = (x_1, x_2, x_3) : I$. The ideal J is then also a perfect ideal of grade 3. An ideal J is said to be *linked* to I if there exists a sequence of ideals $I = J_0, J_1, \ldots, J_n = J$ such that J_{i+1} is directly linked to J_i for each $i = 0, \ldots, n-1$.

(2.2) **Proposition.** *Let Q be a local ring and $I \subseteq Q$ a grade 3 perfect ideal of resolution format $(1, m, m+n-1, n)$.*

(a) *If $m \geqslant 4$ holds, then there exists a grade 3 perfect ideal of resolution format $(1, n+3, m+n, m-2)$ that is directly linked to I.*

(b) *If $m \geqslant 5$ holds, then there exists a grade 3 perfect ideal of resolution format $(1, n+3, m+n-1, m-3)$ that is directly linked to I.*

Proof. Denote by $\mathfrak{n}$ the maximal ideal of Q and by k the residue field $Q/\mathfrak{n}$. Let $F_\bullet$ be the minimal free resolution of Q/I. For a regular sequence x_1, x_2, x_3 in I the corresponding Koszul complex is denoted by $K_\bullet$. The canonical surjection $Q/(x_1, x_2, x_3) \to Q/I$ lifts to a morphism $\varphi_\bullet \colon K_\bullet \to F_\bullet$. As recalled in [1, Section 1], the dual of the mapping cone of $\varphi_\bullet$ yields a, not necessarily minimal, free resolution of Q/J. Further, the ranks of the modules in the minimal free resolution $G_\bullet$ of Q/J are given by

$$\begin{aligned} \operatorname{rank}_Q G_1 &= n + 3 - \operatorname{rank}_k(\varphi_3 \otimes k) - \operatorname{rank}_k(\varphi_2 \otimes k) \\ \operatorname{rank}_Q G_2 &= m + n + 2 - \operatorname{rank}_k(\varphi_2 \otimes k) - \operatorname{rank}_k(\varphi_1 \otimes k) \\ \operatorname{rank}_Q G_3 &= m - \operatorname{rank}_k(\varphi_1 \otimes k)\,. \end{aligned}$$

Set $A_\bullet = \operatorname{Tor}^Q_\bullet(Q/I, k)$. The morphism $\varphi_\bullet \otimes k \colon \operatorname{H}(K_\bullet) \to A_\bullet$ is a morphism of graded-commutative k-algebras.

(a): By assumption, I is not generated by a regular sequence, so by [1, Theorem 2.1] one has $(A_1)^3 = 0$ and, therefore, $\operatorname{rank}_k(\varphi_3 \otimes k) = 0$. It now suffices to show that one can choose a regular sequence x_1, x_2, x_3 such that $\operatorname{rank}_k(\varphi_1 \otimes k) = 2$ and $\operatorname{rank}_k(\varphi_2 \otimes k) = 0$ hold. By [1, Theorem 2.1] one can choose minimal generators x'_1 and x'_2 of I such that the corresponding cycles z'_1 and z'_2 in F_1 satisfy $[z'_1] \cdot [z'_2] = 0$ in $A_\bullet$. Let $P_1, \ldots, P_s$ be the associated prime ideals of Q. As I has grade 3, the ideal $\mathfrak{n}I + (x'_1)$ is not contained in the union $\bigcup_{i=1}^s P_i$. It now follows from Lemma (2.1) that there exists an element v_1 in $\mathfrak{n}I$ such that $x_1 = v_1 + x'_1$ is not in $\bigcup_{i=1}^s P_i$, i.e. not a zero-divisor. Similarly, there exists a $v_2 \in \mathfrak{n}I$ such that $x_2 = v_2 + x'_2$ is not in the union of the associated primes of the Q-module $Q/(x_1)$. Thus, x_1, x_2 form a regular sequence of minimal generators of I, and the corresponding cycles z_1 and z_2 in F_1 satisfy $[z_1] = [z'_1]$ and $[z_2] = [z'_2]$ in A_1. Finally there exists, again by Lemma (2.1), an element x_3 in $\mathfrak{n}I$ and not in the union of the associated primes of the Q-module $Q/(x_1, x_2)$. With this regular sequence x_1, x_2, x_3 one has $\operatorname{rank}_k(\varphi_1 \otimes k) = 2$, as the cycle z_3 in F_1 corresponding to x_3 satisfies $[z_3] = 0$ in A_1. In particular, one has $[z_1] \cdot [z_3] = 0 = [z_2] \cdot [z_3]$ and hence $\operatorname{rank}_k(\varphi_2 \otimes k) = 0$.

(b): By assumption, I is not generated by a regular sequence, so by [1, Theorem 2.1] one has $(A_1)^3 = 0$ and, therefore, $\operatorname{rank}_k(\varphi_3 \otimes k) = 0$. It now suffices to show that one can choose a regular sequence x_1, x_2, x_3 such that $\operatorname{rank}_k(\varphi_1 \otimes k) = 3$ and

$\operatorname{rank}_k(\varphi_2 \otimes k) = 0$ hold. By [1, Theorem 2.1] one can choose minimal generators x'_1, x'_2, and x'_3 of I such that the corresponding cycles $z'_1, z'_2, z'_3 \in F_1$ satisfy $[z'_i]\cdot[z'_j] = 0$ in $A_\bullet$. (Notice that the assumption $m \geqslant 5$ is needed in order to choose these elements in case the multiplicative structure is TE; for the other possible structures, B, G, and H, such elements can be chosen as long as m is at least 4.) As in the proof of part (a) there exist elements $v_1, v_2, v_3 \in \mathfrak{n}I$ such that $v_1 + x'_1, v_2 + x'_2, v_3 + x'_3$ is a regular sequence in I with the desired properties. □

(2.3) **Corollary.** *Let Q be a local ring and $I \subseteq Q$ a grade 3 perfect ideal of resolution format $(1, m, m+n-1, n)$.*

(a) *If $m \geqslant 4$ holds, then there exists a grade 3 perfect ideal of resolution format $(1, m+1, m+n+1, n+1)$ that is linked to I.*

(b) *If $m \geqslant 5$ holds, then there exists a grade 3 perfect ideal of resolution format $(1, m, m+n, n+1)$ that is linked to I.*

(c) *If $m \geqslant 4$ and $n \geqslant 2$ holds, then there exists a grade 3 perfect ideal of resolution format $(1, m+1, m+n, n)$ that is linked to I.*

Proof. Part (a) follows from two consecutive applications of Proposition (2.2)(a). Part (b) follows from an application of (2.2)(b) followed an application of (2.2)(a). Finally, part (c) follows by application of (2.2)(a) and (2.2)(b) in that order. □

Let $I \subseteq Q$ be a perfect ideal of grade 3. For every regular sequence x_1, x_2, x_3 in I one has $(x_1, x_2, x_3) : ((x_1, x_2, x_3) : I) = I$; see Golod [7]. Thus if an ideal J is directly linked to I, then I is also directly linked to J. It follows that "being linked" is an equivalence relation. The ideal I is called *licci* if it is in the linkage class of a complete intersection ideal.

3. Evidence for Part I

By a Dynkin diagram we always mean a simply laced Dynkin diagram, i.e. a diagram from the ADE classification. Recall from the Introduction that

$$\begin{array}{ccccccccccccc} \bullet_{m-3} & — & \cdots & — & \bullet_1 & — & \bullet & — & \bullet_1 & — & \cdots & — & \bullet_n \\ & & & & & & | & & & & & & \\ & & & & & & \bullet & & & & & & \end{array}$$

is the graph associated to the resolution format $(1, m, m+n-1, n)$. The Dynkin formats that can be realized by grade 3 perfect ideals are

$$\begin{array}{rcl} \mathrm{A}_3 & \text{corresponding to} & (1,3,3,1) \\ \mathrm{D}_m & — & (1,m,m,1) \text{ for odd } m \geqslant 5 \\ \mathrm{D}_{n+3} & — & (1,4,n+3,n) \text{ for } n \geqslant 2 \\ \mathrm{E}_6 & — & (1,5,6,2) \\ \mathrm{E}_7 & — & (1,6,7,2) \text{ and } (1,5,7,3) \\ \mathrm{E}_8 & — & (1,7,8,2) \text{ and } (1,5,8,4) \end{array} \tag{3.1}$$

This is straightforward to verify when one recalls that

- $(1,3,3,1)$ is the only possible format with $m = 3$; this excludes the formats $(1,3,n+2,n)$ corresponding to A_{n+2} for $n \geqslant 2$.
- The minimal number of generators of grade 3 Gorenstein ideal is odd, see [3]. This excludes the formats $(1,m,m,1)$ corresponding to D_m for even $m \geqslant 4$.

(3.2) **Theorem.** *Let $\Bbbk$ a field and $e \geqslant 3$; set $Q = \Bbbk[X_1, \ldots, X_e]_{(X_1,\ldots,X_e)}$. For every resolution format $\mathfrak{f} = (1, m, m+n-1, n)$ with $m \geqslant 3$ and $n \geqslant 1$ that is not Dynkin there exists a grade 3 perfect ideal that has resolution format $\mathfrak{f}$ and is not licci.*

It is implicit in this statement that every format $(1, m, m+n-1, n)$ that is not Dynkin is realized by a degree 3 perfect ideal in the ring Q. A proof was sketched in the last section of [14], we provide the details in the Appendix.

The proof of Theorem (3.2) takes up the bulk of this section; indeed the Theorem follows from Propositions (3.3), (3.5), and (3.7). It proceeds in two steps: First we prove that it is enough to exhibit non-licci ideals for two specific formats, $(1, 6, 8, 3)$ and $(1, 8, 9, 2)$, which are not Dynkin. Second we produce such examples, in fact entire families of them.

(3.3) **Proposition.** *Let Q be a local ring. If there is a resolution format $\mathfrak{f}$ such that*

(1) *there exists a grade 3 perfect ideal in Q of format $\mathfrak{f}$,*
(2) *$\mathfrak{f}$ is not Dynkin, and*
(3) *every perfect ideal in Q of format $\mathfrak{f}$ is licci*

then $(1, 6, 8, 3)$ or $(1, 8, 9, 2)$ is such a format.

Proof. Assume that there exists at least one resolution format that satisfies conditions (1)–(3). Let $\mathfrak{f} = (1, m, m+n-1, n)$ be minimal among all such formats with regard to the total rank $\beta(\mathfrak{f}) = 2(m+n)$ of the modules in the minimal free resolution. Necessarily, one has $m \geqslant 4$ as the Dynkin format $(1, 3, 3, 1)$ is the only format with $m = 3$. It follows that the format $\mathfrak{f}' = (1, n+2, m+n-2, m-3)$ is Dynkin. Indeed, one has $\beta(\mathfrak{f}') = \beta(\mathfrak{f}) - 2$ and every perfect ideal of format $\mathfrak{f}'$ is by Proposition (2.2)(a) is directly linked to a perfect ideal of format $\mathfrak{f}$ and hence licci.

Given the Dynkin formats (3.1), the potential minimal formats $\mathfrak{f}$ are as follows:

Dynkin format $\mathfrak{f}'$	Potential minimal format $\mathfrak{f}$
$(1, 2l+1, 2l+1, 1)$ for $l \geqslant 1$	$(1, 4, 2l+2, 2l-1)$
$(1, 4, l+3, l)$ for $l \geqslant 2$	$(1, l+3, l+4, 2)$
$(1, 5, 6, 2)$	$(1, 5, 7, 3)$
$(1, 5, 7, 3)$	$(1, 6, 8, 3)$
$(1, 5, 8, 4)$	$(1, 7, 9, 3)$
$(1, 6, 7, 2)$	$(1, 5, 8, 4)$
$(1, 7, 8, 2)$	$(1, 5, 9, 5)$

Among the potential minimal formats, $(1, 4, 2l+2, 2l-1)$ does not exist for $l = 1$, and for $l \geqslant 2$ the format is Dynkin corresponding to D_{2l+2}. The format $\mathfrak{f}_l = (1, l+3, l+4, 2)$ is for $l \in \{2, 3, 4\}$ Dynkin corresponding to E_6, E_7, and E_8. Further, the formats $(1, 5, 7, 3)$ and $(1, 5, 8, 4)$ are Dynkin corresponding to E_7 and E_8. Thus, possible minimal formats that are not Dynkin and such that every ideal of the format is licci are $(1, 5, 9, 5)$, $(1, 6, 8, 3)$, $(1, 7, 9, 3)$ and $\mathfrak{f}_l$ for $l \geqslant 5$.

An ideal of format $(1, 5, 9, 5)$ is by Proposition (2.2)(b) linked to an ideal of format $\mathfrak{f}_5 = (1, 8, 9, 2)$. An ideal of format $(1, 6, 8, 3)$ is by Corollary (2.3)(c) linked one of format $(1, 7, 9, 3)$. Similarly, an ideal of format $\mathfrak{f}_l$ is linked to one of format f_{l+1}. Thus, if there exists a format that is not Dynkin with the property that every ideal of that format is licci, then $(1, 6, 8, 3)$ or $(1, 8, 9, 2)$ has that property. □

To provide examples of ideals of formats $(1, 6, 8, 3)$ and $(1, 8, 9, 2)$ that are not licci, we rely on numerical obstructions found by Huneke and Ulrich [8].

(3.4) Let $\Bbbk$ be a field and $e \geqslant 3$; set $\mathcal{Q} = \Bbbk[X_1, \dots, X_e]$. Let $\mathcal{I}$ be a homogeneous perfect ideal in $\mathcal{Q}$ of grade 3 with minimal free resolution

$$\mathcal{Q} \longleftarrow \bigoplus_{i=1}^{m} \mathcal{Q}(-d_{1,i}) \longleftarrow \bigoplus_{i=1}^{m+n-1} \mathcal{Q}(-d_{2,i}) \longleftarrow \bigoplus_{i=1}^{n} \mathcal{Q}(-d_{3,i}) \longleftarrow 0\,.$$

Set $Q = \mathcal{Q}_{(X_1,\dots,X_e)}$ and $I = \mathcal{I}_{(X_1,\dots,X_e)}$. Assuming that one has $d_{1,1} \leqslant d_{1,2} \leqslant \cdots \leqslant d_{1,m}$ and $d_{3,1} \leqslant \cdots \leqslant d_{3,n}$ it follows from [8, Corollary 5.13] that if the inequality

$$d_{3,n} \leqslant 2d_{1,1}$$

holds, then the ideal I is not licci.

Notice that if Q/I is Artinian, i.e. $e = 3$, then one has $Q/I \cong \mathcal{Q}/\mathcal{I}$ and the inequality says that 3 plus the socle degree of $\mathcal{Q}/\mathcal{I}$ does not exceed 2 times the initial degree of $\mathcal{I}$.

Resolution format $(1, 6, 8, 3)$**.**

(3.5) **Proposition.** *Adopt the notation from* (3.4). *Let* Φ *be a* 2×4 *matrix of linear forms in* $\mathcal{Q}$ *or a* 3×3 *symmetric matrix of linear forms, and let* $\mathcal{I}$ *be the ideal of* 2×2 *minors of* Φ. *The ideal* I *has resolution format* $(1, 6, 8, 3)$ *and it is not licci.*

Proof. The graded minimal free resolution of $\mathcal{Q}/\mathcal{I}$ has the form

$$\mathcal{Q} \longleftarrow \mathcal{Q}(-2)^6 \longleftarrow \mathcal{Q}(-3)^8 \longleftarrow \mathcal{Q}(-4)^3 \longleftarrow 0\,;$$

see Eagon and Nothcott [5, Theorem 2] and Józefiak [9, Theorem 3.1] or Weyman [15, Proposition (6.1.7) and thm. (6.3.1)]. It follows from (3.4) that I is not licci. □

The next example illustrates Proposition (3.5) and the linkage argument in the proof of Proposition (3.3). Recall that an Artinian local ring is called *level* if the socle is exactly the highest nonvanishing power of the maximal ideal.

(3.6) **Example.** Let $\Bbbk$ be a field; set $\mathcal{Q} = \Bbbk[X, Y, Z]$ and $\mathcal{N} = (X, Y, Z)$. The ideal $\mathcal{N}^2$ is generated by the six quadratic monomials, and the quotient $\mathcal{Q}/\mathcal{N}^2$ is an Artinian local level algebra of socle degree 1 and type 3, so the minimal free resolution over $\mathcal{Q}$ is

$$\mathcal{Q} \longleftarrow \mathcal{Q}(-2)^6 \longleftarrow \mathcal{Q}(-3)^8 \longleftarrow \mathcal{Q}(-4)^3 \longleftarrow 0\,;$$

Notice that $\mathcal{N}^2$ is the ideal of 2×2 minors of either matrix

$$\begin{pmatrix} X & Y & Z & 0 \\ 0 & X & Y & Z \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} X & Y & Z \\ Y & 0 & X \\ Z & X & Y \end{pmatrix}\,.$$

The linked ideal

$$\mathcal{I} = (X^2, Y^2, Z^3) : \mathcal{N}^2 = (X^2, Y^2, XYZ, XZ^2, YZ^2, Z^3)$$

defines a local Artinian $\Bbbk$-algebra with graded basis

$$1 \quad x, y, z \quad xy, xz, yz, z^2\,.$$

Evidently, there are no linear forms in the socle of $\mathcal{Q}/\mathcal{I}$, so it is a level algebra of type 4, whence $\mathcal{I}$ has resolution format $(1, 6, 9, 4)$. Next, the ideal

$$\begin{aligned}\mathcal{J} &= (X^3 - YZ^2, Y^3 - XZ^2, Z^3) : \mathcal{I} \\ &= (X^3 - YZ^2, Y^3 - XZ^2, Z^3, X^2Y^2, X^2YZ, XY^2Z, XYZ^2)\end{aligned}$$

defines a local Artinian $\Bbbk$-algebra with graded basis

$$1 \quad x,y,z \quad x^2,xy,xz,y^2,yz,z^2 \quad x^2y,x^2z,xy^2,xyz,xz^2,y^2z,yz^2 \quad x^2z^2,y^2z^2\,.$$

It is straightforward to verify that the socle of $\mathcal{Q}/\mathcal{J}$ is generated by xyz, x^2z^2, and y^2z^2. It follows that $\mathcal{J}$ has resolution format $(1,7,9,3)$.

Resolution format $(1,8,9,2)$**.** Recall that an Artinian $\Bbbk$-algebra is called *compressed* if it has maximal length among all algebras with the same socle polynomial; see (4.1). This notion has a natural generalization to Cohen–Macaulay algebras, see Fröberg and Laksov [6], and that is the one we use in the proposition below.

(3.7) **Proposition.** *Adopt the notation from* (3.4). *Assume that the graded* $\Bbbk$-*algebra* $\mathcal{Q}/\mathcal{I}$ *is compressed with with socle polynomial* $2t^3$*; that is,*

$$\dim_{\Bbbk}(\operatorname{Tor}_3^{\mathcal{Q}}(\mathcal{Q}/\mathcal{I},\Bbbk))_i = \begin{cases} 2 & \text{for } i=6 \\ 0 & \text{for } i\neq 6\,. \end{cases}$$

The ideal I has resolution format $(1,8,9,2)$ *and it is not licci.*

Proof. By [6, Proposition 6] the Hilbert series of $\mathcal{Q}/\mathcal{I}$ is

$$\frac{1+3t+6t^2+2t^3}{(1-t)^{e-3}}\,.$$

Further, it follows from the equality $\binom{3}{2}=2\binom{3}{1}$ that $\mathcal{Q}/\mathcal{I}$ is extremely compressed, see [6, p. 133]. Now it follows from [6, Proposition 16] that the graded free resolution of $\mathcal{Q}/\mathcal{I}$ has the form

$$\mathcal{Q} \longleftarrow \mathcal{Q}(-3)^8 \longleftarrow \mathcal{Q}(-4)^9 \longleftarrow \mathcal{Q}(-6)^2 \longleftarrow 0\,;$$

The conclusion now follows from (3.4). $\square$

By work of Boij and Laksov [2, Theorem (3.4)], Proposition (3.7) applies to generic Artinian level algebras of embedding dimension 3, socle degree 3, and type 2.

The next example illustrates Proposition (3.7) and the linkage argument in the proof of Proposition (3.3).

(3.8) **Example.** Let $\Bbbk$ be a field. The ideal

$$\mathcal{I} = (X^3, X^2Y+YZ^2, X^2Z+XYZ, XY^2+XYZ, XZ^2, Y^3, Y^2Z, Z^3)$$

in $\Bbbk[X,Y,Z]$ defines an Artinian local $\Bbbk$-algebra with graded basis

$$1 \quad x,y,z \quad x^2,xy,xz,y^2,yz,z^2 \quad xyz,yz^2\,.$$

It is elementary to check that there are no quadratic forms in the socle of $\mathcal{Q}/\mathcal{I}$, so it is a compressed level algebra of type 2, and it follows that $\mathcal{I}$ has format $(1,8,9,2)$.

The linked ideal

$$\mathcal{J} = (X^3,Y^3,Z^3):\mathcal{I} = (X^3, X^2Y-YZ^2, XYZ-XZ^2-Y^2Z, Y^3, Z^3)$$

defines an Artinian local $\Bbbk$-algebra with graded basis

$$1 \quad x,y,z \quad x^2,xy,xz,y^2,yz,z^2 \quad x^2z,xy^2,xz^2,y^2z,yz^2\,.$$

Again it is elementary to verify that there are no quadratic forms in the socle, so it is a level algebra of type 5; in particular, the resolution format of $\mathcal{J}$ is $(1,5,9,5)$.

Finally, notice that under the flat extension $\Bbbk[X,Y,Z] \to \Bbbk[X,Y,Z,X_4,\ldots,X_e]$ the ideal $\mathcal{I}$ extends to an ideal that satisfies the hypothesis of Proposition (3.7).

4. Evidence for Part II

As discussed in the Introduction, it is known that grade 3 perfect ideals of resolution format $(1,4,n+3,n)$ for $n \geqslant 2$—almost complete intersection ideals—and ideals of format $(1,m,m,1)$ for odd $m \geqslant 3$—Gorenstein ideals—are licci. The remaining Dynkin formats are $(1,5,6,2)$ corresponding to E_6, $(1,6,7,2)$ and $(1,5,7,3)$ corresponding to E_7, and $(1,7,8,2)$ and $(1,5,8,4)$ corresponding to E_8; see (3.1).

In the previous section, we proved the existence of ideals that are not licci by invoking the numerical obstructions in (3.4). In this section, we prove that homogeneous grade 3 perfect ideals in $\Bbbk[X_1,X_2,X_3]$ of resolution formats corresponding to E_6, E_7, or E_8 avoid this obstruction.

(4.1) Let $\mathcal{Q}$ be a polynomial algebra of embedding dimension e. For a finitely generated graded $\mathcal{Q}$-module M, let $h_M(\cdot)$ denote the Hilbert function. Let $\mathcal{I}$ be a homogeneous ideal in $\mathcal{Q}$ such that $\mathcal{Q}/\mathcal{I}$ is Artinian with socle degree s; its socle polynomial $\sum_{j=1}^{s} c_j t^j$ is the Hilbert series of the socle of $\mathcal{Q}/\mathcal{I}$. For every $i \geqslant 0$ there is an inequality

$$\begin{aligned} h_{\mathcal{Q}/\mathcal{I}}(i) &\leqslant \min\{h_{\mathcal{Q}}(i), \sum_{j=i}^{s} c_j h_{\mathcal{Q}}(j-i)\} \\ &= \min\left\{\binom{e-1+i}{e-1}, \sum_{j=i}^{s} c_j\binom{e-1+j-i}{e-1}\right\}. \end{aligned}$$

If equality holds for every i, then $\mathcal{Q}/\mathcal{I}$ is called *compressed*, see [6, Section 3].

The number $n = \sum_{j=1}^{s} c_j$ is the type of $\mathcal{Q}/\mathcal{I}$ and, evidently, one has

$$\begin{aligned} h_{\mathcal{Q}/\mathcal{I}}(i) &\leqslant \min\{h_{\mathcal{Q}}(i), n h_{\mathcal{Q}}(s-i)\} \\ &= \min\left\{\binom{e-1+i}{e-1}, n\binom{e-1+s-i}{e-1}\right\}. \end{aligned} \tag{4.1.1}$$

(4.2) **Proposition.** *Adopt the notation from* (3.4) *and let* $e=3$. *If the ideal* $\mathcal{I}$ *has initial degree* $d_{1,1} \geqslant 2$ *and resolution format* $(1,m,m+1,2)$ *for* $4 \leqslant m \leqslant 7$, *then one has* $d_{3,2} > d_{1,1} + d_{1,2}$.

As one has $d_{1,1} + d_{1,2} \geqslant 2d_{1,1}$ it follows that ideals $\mathcal{I}$ of format $(1,5,6,2)$, $(1,6,7,2)$, and $(1,7,8,2)$ avoid the numerical obstruction to being licci from (3.4).

Proof. To simplify the notation from (3.4), set

$$R = \mathcal{Q}/\mathcal{I}\,, \quad d = d_{1,1}\,, \quad d' = d_{1,2}\,, \quad \text{and} \quad s = d_{3,2} - 3\,.$$

Notice that R is Artinian of socle degree s. The inequalities $2 \leqslant d_{1,1}$ and $d_{1,1} + 2 \leqslant d_{3,2}$ yield

$$1 \leqslant s \quad \text{and} \quad d \leqslant s+1\,.$$

We assume that $s+3 \leqslant d+d'$ holds and aim for a contradiction. To this end it is by (4.1.1) sufficient to prove that there is a $u \leqslant s$ such that the inequality $h_R(u) > 2h_{\mathcal{Q}}(s-u)$ holds.

Case 1. Assume that $d < \frac{s+3}{2}$ holds. One then has $d' \geqslant s+3-d > d$, and hence

$$\begin{aligned} h_R(i) &\geqslant h_Q(i) - h_Q(i-d) - (m-1)h_Q(i-d') \\ &\geqslant h_Q(i) - h_Q(i-d) - (m-1)h_Q(i-s-3+d) \\ &\geqslant h_Q(i) - h_Q(i-d) - 6h_Q(i-s-3+d)\,. \end{aligned}$$

For $d = 2$ one has

$$h_R(s) \geqslant h_Q(s) - h_Q(s-2) = \binom{s+2}{2} - \binom{s}{2} = 2s+1 > 2 = 2h_Q(0)\,.$$

Now let $d \geqslant 3$ and set $u = s+2-\lceil \frac{d}{2} \rceil$; notice that $u \leqslant s$. Straightforward computations yield

$$2h_Q(s-u) = \begin{cases} \frac{1}{4}d(d-2) & \text{for } d \text{ even} \\ \frac{1}{4}(d+1)(d-1) & \text{for } d \text{ odd} \end{cases}$$

and

$$h_Q(u) - h_Q(u-d) - 6h_Q(u-s-3+d) = \begin{cases} \frac{1}{4}d(4s-7d+8) & \text{for } d \text{ even} \\ \frac{1}{4}(d(4s-7d+12)+3) & \text{for } d \text{ odd.} \end{cases}$$

For d even one now has

$$\begin{aligned} h_R(u) - 2h_Q(s-u) &\geqslant \frac{d(4s-7d+8)}{4} - \frac{d(d-2)}{4} \\ &= \frac{d(2(s-2d+3)-1)}{2} \\ &\geqslant \frac{d}{2}\,, \end{aligned}$$

and for d odd

$$\begin{aligned} h_R(u) - 2h_Q(s-u) &\geqslant \frac{d(4s-7d+12)+3}{4} - \frac{(d+1)(d-1)}{4} \\ &= d(s-2d+3)+1 \\ &\geqslant d+1\,. \end{aligned}$$

That is, independent of the parity of d one has $h_R(u) > 2h_Q(s-u)$.

Case 2. Assume that $d \geqslant \frac{s+3}{2}$ holds; one has

$$h_R(i) \geqslant h_Q(i) - mh_Q(i-d) \geqslant h_Q(i) - 7h_Q(i-d)\,.$$

Set $u = \lfloor \frac{s+d}{2} \rfloor$ and $\sigma = s-d+3$. Notice that one has

$$1 \leqslant \sigma \leqslant d \leqslant s+1 \quad \text{and} \quad u \leqslant s\,.$$

Straightforward computations yield

$$2h_Q(s-u) = \begin{cases} \frac{1}{4}(\sigma^2-1) & \text{for } s+d \text{ even} \\ \frac{1}{4}\sigma(\sigma+2) & \text{for } s+d \text{ odd} \end{cases}$$

and

$$h_Q(u) - 7h_Q(u-d) = \begin{cases} \frac{1}{4}(2d(d+\sigma) - 3(\sigma^2-1)) & \text{for } s+d \text{ even} \\ \frac{1}{4}(2d(d+\sigma-1) - 3\sigma(\sigma-2)) & \text{for } s+d \text{ odd.} \end{cases}$$

The inequality $d \geqslant \sigma$ explains the second inequality in each of the computation below. For $s+d$ even one has

$$\begin{aligned} h_R(u) - 2h_{\mathcal{Q}}(s-u) &\geqslant \frac{2d(d+\sigma) - 3(\sigma^2-1)}{4} - \frac{\sigma^2-1}{4} \\ &\geqslant \frac{4\sigma^2 - 4(\sigma^2-1)}{4} \\ &= 1\,, \end{aligned}$$

and for $s+d$ odd

$$\begin{aligned} h_R(u) - 2h_{\mathcal{Q}}(s-u) &\geqslant \frac{2d(d+\sigma-1) - 3\sigma(\sigma-2)}{4} - \frac{\sigma(\sigma+2)}{4} \\ &\geqslant \frac{2\sigma(2\sigma-1) - 4\sigma(\sigma-1)}{4} \\ &= \frac{\sigma}{2}\,. \end{aligned}$$

That is, independent of the parity of d one has $h_R(u) > 2h_{\mathcal{Q}}(s-u)$. □

(4.3) **Proposition.** *Adopt the notation from* (3.4) *and let* $e=3$. *If the ideal* $\mathcal{I}$ *has initial degree* $d_{1,1} \geqslant 2$ *and resolution format* $(1,5,n+4,n)$ *for* $1 \leqslant n \leqslant 4$, *then one has* $d_{3,2} > d_{1,1} + d_{1,2}$.

As one has $d_{1,1} + d_{1,2} \geqslant 2d_{1,1}$ it follows, in particular, that ideals $\mathcal{I}$ of format $(1,5,7,3)$ and $(1,5,8,4)$ avoid the numerical obstruction to being licci from (3.4).

Proof. As in the proof of Proposition (4.2), set $R = \mathcal{Q}/\mathcal{I}$, $d = d_{1,1}$, $d' = d_{1,2}$, and $s = d_{3,2} - 3$. Note that R is Artinian of socle degree s, and that one has $1 \leqslant s$ and $d \leqslant s+1$. We assume that $s+3 \leqslant d+d'$ holds and aim for a contradiction. To this end it is by (4.1.1) sufficient to prove that there is a $u \leqslant s$ such that the inequality $h_R(u) > nh_{\mathcal{Q}}(s-u)$ holds.

Case 1. Assume that $d \leqslant \frac{s+3}{2}$ holds. One then has $d' \geqslant s+3-d \geqslant d$, and hence

$$\begin{aligned} h_R(i) &\geqslant h_{\mathcal{Q}}(i) - h_{\mathcal{Q}}(i-d) - 4h_{\mathcal{Q}}(i-d') \\ &\geqslant h_{\mathcal{Q}}(i) - h_{\mathcal{Q}}(i-d) - 4h_{\mathcal{Q}}(i-s-3+d)\,. \end{aligned}$$

For $d=2$ one has $s \geqslant 2$ and $h_R(s) \geqslant h_{\mathcal{Q}}(s) - h_{\mathcal{Q}}(s-2) = 2s+1 > 4 \geqslant nh_{\mathcal{Q}}(0)$ as computed in the proof of Proposition (4.2).

Now let $d \geqslant 3$ and set $u = s+2-\lceil \frac{d}{2} \rceil$; notice that $u \leqslant s$. Straightforward computations yield

$$nh_{\mathcal{Q}}(s-u) \leqslant 4h_{\mathcal{Q}}(s-u) = \begin{cases} \frac{1}{2}d(d-2) & \text{for } d \text{ even} \\ \frac{1}{2}(d+1)(d-1) & \text{for } d \text{ odd} \end{cases}$$

and

$$h_{\mathcal{Q}}(u) - h_{\mathcal{Q}}(u-d) - 4h_{\mathcal{Q}}(u-s-3+d) = \begin{cases} \frac{1}{2}d(2s-3d+5) & \text{for } d \text{ even} \\ \frac{1}{2}(d(2s-3d+6)+1) & \text{for } d \text{ odd.} \end{cases}$$

For d even one now has

$$\begin{aligned} h_R(u) - nh_{\mathcal{Q}}(s-u) &\geqslant \frac{d(2s-3d+5)}{2} - \frac{d(d-2)}{2} \\ &= \frac{d(2(s-2d+3)+1)}{2} \\ &\geqslant \frac{3d}{2}\,, \end{aligned}$$

and for d odd

$$\begin{aligned} h_R(u) - nh_{\mathcal{Q}}(s-u) &\geqslant \frac{d(2s-3d+6)+1}{2} - \frac{(d+1)(d-1)}{2} \\ &= \frac{2d(s-2d+3)+2}{2} \\ &\geqslant d+1\,. \end{aligned}$$

That is, independent of the parity of d one has $h_R(u) > nh_{\mathcal{Q}}(s-u)$.

Case 2. Assume that $d > \frac{s+3}{2}$ holds; one has

$$h_R(i) \geqslant h_{\mathcal{Q}}(i) - 5h_{\mathcal{Q}}(i-d)\,.$$

Set $u = \lfloor \frac{s+d}{2} \rfloor$ and $\sigma = s - d + 3$. Notice that one has

$$2 \leqslant \sigma + 1 \leqslant d \leqslant s+1 \quad \text{and} \quad u \leqslant s\,.$$

Straightforward computations yield

$$nh_{\mathcal{Q}}(s-u) \leqslant 4h_{\mathcal{Q}}(s-u) = \begin{cases} \frac{1}{2}(\sigma^2-1) & \text{for } s+d \text{ even} \\ \frac{1}{2}\sigma(\sigma+2) & \text{for } s+d \text{ odd} \end{cases}$$

and

$$h_{\mathcal{Q}}(u) - 5h_{\mathcal{Q}}(u-d) = \begin{cases} \frac{1}{2}(d(d+\sigma) - (\sigma^2-1)) & \text{for } s+d \text{ even} \\ \frac{1}{2}(d(d+\sigma-1) - \sigma(\sigma-2)) & \text{for } s+d \text{ odd.} \end{cases}$$

The inequality $d \geqslant \sigma+1$ explains the second inequality in each of the computation below. For $s+d$ even one has

$$\begin{aligned} h_R(u) - nh_{\mathcal{Q}}(s-u) &\geqslant \frac{d(d+\sigma) - (\sigma^2-1)}{2} - \frac{\sigma^2-1}{2} \\ &\geqslant \frac{(\sigma+1)(2\sigma+1) - 2(\sigma^2-1)}{2} \\ &= \frac{3(\sigma+1)}{2}\,, \end{aligned}$$

and for $s+d$ odd

$$\begin{aligned} h_R(u) - nh_{\mathcal{Q}}(s-u) &\geqslant \frac{d(d+\sigma-1) - \sigma(\sigma-2)}{2} - \frac{\sigma(\sigma+2)}{2} \\ &\geqslant \frac{(\sigma+1)2\sigma - 2(\sigma^2-\sigma)}{2} \\ &= 2\sigma\,. \end{aligned}$$

That is, independent of the parity of d one has $h_R(u) > nh_{\mathcal{Q}}(s-u)$. □

Appendix A. Realizability of resolution formats

In the power series ring $\Bbbk[\![X,Y,Z]\!]$ there exists by [3, Proposition 6.2] an ideal of resolution format $(1,m,m,1)$, i.e. a Gorenstein ideal of grade 3, for every odd $m \geqslant 3$. We start by recording that this is the case in any local ring of sufficient depth.

(A.1) **Lemma.** *Let Q be a local ring of depth at least 3. For every odd $m \geqslant 3$ there exists a grade 3 perfect ideal of resolution format $(1,m,m,1)$.*

Proof. Denote by $\mathfrak{n}$ the maximal ideal of Q and let x,y,z be a regular sequence in $\mathfrak{n}$. Let l be an integer with $m = 2l+1$ and define a skew symmetric matrix V_l as in [4, Section 3]. Let I be the ideal generated by the sub-maximal Pfaffians of V_l. It follows from Proposition 3.3 in *loc. cit.* that the radical of I contains x, y, and z, so I has grade (at least) 3, whence by [3, Theorem 2.1] it is a Gorenstein ideal of format $(1,m,m,1)$. □

(A.2) **Theorem.** *Let $(Q,\mathfrak{n})$ be a local ring of depth at least 3. For every resolution format $\mathfrak{f} = (1,m,m+n-1,n)$ with $m \geqslant 3$ and $n \geqslant 1$ that is not $(1,m,m,1)$ with m even and not $(1,3,n+2,n)$ with $n \geqslant 2$ there exists a grade 3 perfect ideal I in Q of resolution format $\mathfrak{f}$.*

Proof. By Lemma (A.1) there exists for every odd $g \geqslant 3$ a grade 3 perfect ideal with resolution format $(1,g,g,1)$. Fix a format $\mathfrak{f} = (1,m,m+n-1,n)$ with $m \geqslant 3$ and $n \geqslant 2$. If $m = 4$ and n is even, then it follows from Proposition (2.2)(b) that there exists an ideal of format $\mathfrak{f}$ that is linked to one of format $(1,n+3,n+3,1)$. If $m = 4$ and n is odd, then it follows from Proposition (2.2)(a) that there exists an ideal of format $\mathfrak{f}$ that is linked to one of format $(1,n+2,n+2,1)$. If $m \geqslant 5$ is odd, then it follows from $n-1$ applications of Corollary (2.3)(b) that there is an ideal of format $\mathfrak{f}$ that is linked to one of format $(1,m,m,1)$. If $m \geqslant 6$ is even, then it follows from Corollary (2.3)(a) and $n-2$ applications of Corollary (2.3)(b) that there exists an ideal of format $\mathfrak{f}$ that is linked to one of format $(1,m-1,m-1,1)$. □

References

[1] Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334

[2] Mats Boij and Dan Laksov, *Nonunimodality of graded Gorenstein Artin algebras*, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1083–1092. MR1227512

[3] David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723

[4] Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Trimming a Gorenstein ideal*, J. Commut. Algebra **11** (2019), 325–339.

[5] Jack A. Eagon and Douglas G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. Ser. A **269** (1962), 188–204. MR0142592

[6] Ralf Fröberg and Dan Laksov, *Compressed algebras*, Complete intersections (Acireale, 1983), Lecture Notes in Math., vol. 1092, Springer, Berlin, 1984, pp. 121–151. MR0775880

[7] Evgeniy S. Golod, *A note on perfect ideals*, Algebra (A.I. Kostrikin, Ed.), Moscow State Universit Press, 1980, pp. 37–39.

[8] Craig Huneke and Bernd Ulrich, *The structure of linkage*, Ann. of Math. (2) **126** (1987), no. 2, 277–334. MR0908149

[9] Tadeusz Józefiak, *Ideals generated by minors of a symmetric matrix*, Comment. Math. Helv. **53** (1978), no. 4, 595–607. MR0511849

[10] Irving Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, Chicago, Ill.-London, 1974. MR0345945

[11] Ernst Kunz, *Almost complete intersections are not Gorenstein rings*, J. Algebra **28** (1974), 111–115. MR0330158

[12] Matthew Miller and Bernd Ulrich, *Linkage and compressed algebras*, Proceedings of the conference on algebraic geometry (Berlin, 1985), Teubner-Texte Math., vol. 92, Teubner, Leipzig, 1986, pp. 267–275. MR0922918

[13] Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. **50** (1973), 227–232. MR0319985

[14] Jerzy Weyman, *Generic free resolutions and root systems*, Ann. Inst. Fourier, to appear. Preprint arXiv:1609.02083 [math.AC].

[15] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003. MR1988690

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