LOCAL RINGS OF EMBEDDING CODEPTH 3. EXAMPLES

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Abstract. A complete local ring of embedding codepth 3 has a minimal free resolution of length 3 over a regular local ring. Such resolutions carry a differential graded algebra structure, based on which one can classify local rings of embedding codepth 3. We give examples of algebra structures that have been conjectured not to occur.

1. Introduction

A classification of commutative noetherian local rings of embedding codepth $c \leq 3$ took off more than a quarter of a century ago. Up to completion, a local ring of embedding codepth c is a quotient of regular local ring Q by an ideal I of grade c, and the classification is based on an algebra structure on $\mathrm{Tor}_*^Q(Q/I, k)$, where k is the residue field of Q. The possible isomorphism classes of these algebras were identified by Weyman [7] and by Avramov, Kustin, and Miller [2]. Significant restrictions on the invariants that describe these isomorphism classes were recently worked out by Avramov [1]. Here is a précis that will suffice for our purposes.

Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Denote by e the minimal number of generators of $\mathfrak{m}$ and by d the depth of R. The number e is called the *embedding dimension* of R, and $c = e - d$ is the *embedding codepth*. By Cohen's Structure Theorem the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has the form $\widehat{R} = Q/I$, where Q is a complete regular local ring with the same embedding dimension and residue field as R; we refer to I as the *Cohen ideal* of R.

The projective dimension of $\widehat{R}$ over Q is c, by the Auslander–Buchsbaum Formula. From now on let $c \leq 3$; the minimal free resolution F of $\widehat{R}$ over Q then carries a differential graded algebra structure. It induces a graded algebra structure on $F \otimes_Q k = \mathrm{H}(F \otimes_Q k) = \mathrm{Tor}_*^Q(\widehat{R}, k)$, which identifies R as belonging to one of six (parametrized) classes, three of which are called $\mathbf{B}$, $\mathbf{C}(c)$, and $\mathbf{G}(r)$ for $r \geq 2$. The ring R is in $\mathbf{C}(c)$ if and only if it is an embedding codepth c complete intersection. If R is Gorenstein but not a complete intersection, then it belongs to the class $\mathbf{G}(r)$ with $r = \mu(I)$, the minimal number of generators of the Cohen ideal. Work of J. Watanabe [6] shows that $\mu(I)$ is odd and at least 5. Brown [3] identified rings in $\mathbf{B}$ of type 2, and thus far no other examples of $\mathbf{B}$ rings have been known. Rings in $\mathbf{G}(r)$ that are not Gorenstein—rings in $\mathbf{G}(3)$ and $\mathbf{G}(2n)$ for $n \in \mathbb{N}$ in particular—have also been elusive; in fact, it has been conjectured [1, 3.10] that every ring in $\mathbf{G}(r)$ would be Gorenstein and, by implication, that the classes $\mathbf{G}(3)$ and $\mathbf{G}(2n)$ would be empty.

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In this paper we provide examples of some of the sorts of rings that have hitherto dodged detection; the precise statements follow in Theorems I and II below.

Theorem I. *Let $\Bbbk$ be a field, set $Q = \Bbbk[\![x,y,z]\!]$, and consider these ideals in Q:*

$$\begin{aligned}\mathfrak{g}_1 &= (xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)\\ \mathfrak{g}_2 &= \mathfrak{g}_1 + (x^3y - z^4)\\ \mathfrak{g}_3 &= \mathfrak{g}_2 + (x^2z^2)\\ \mathfrak{g}_4 &= \mathfrak{g}_3 + (x^3z)\ .\end{aligned}$$

Each algebra $Q/\mathfrak{g}_n$ has embedding codepth 3 and type 2, and $Q/\mathfrak{g}_n$ is in $\mathbf{G}(n+1)$.

The theorem provides counterexamples to the conjecture mentioned above: The classes $\mathbf{G}(2)$, $\mathbf{G}(3)$, and $\mathbf{G}(4)$ are not empty, and rings in $\mathbf{G}(5)$ need not be Gorenstein. The second theorem provides examples of rings in $\mathbf{B}$ of type different from 2.

Theorem II. *Let $\Bbbk$ be a field, set $Q = \Bbbk[\![x,y,z]\!]$, and consider these ideals in Q:*

$$\begin{aligned}\mathfrak{b}_1 &= (x^3, x^2y, yz^2, z^3)\\ \mathfrak{b}_2 &= \mathfrak{b}_1 + (xyz)\\ \mathfrak{b}_3 &= \mathfrak{b}_2 + (xy^2 - y^3)\\ \mathfrak{b}_4 &= \mathfrak{b}_3 + (y^2z)\ .\end{aligned}$$

Each algebra $Q/\mathfrak{b}_n$ has embedding codepth 3 and belongs to $\mathbf{B}$. The algebras $Q/\mathfrak{b}_1$ and $Q/\mathfrak{b}_2$ have type 1 while $Q/\mathfrak{b}_3$ and $Q/\mathfrak{b}_4$ have type 3.

The algebras $Q/\mathfrak{b}_1$ and $Q/\mathfrak{b}_2$ have embedding codimension 2, which is the largest possible value for a non-Gorenstein ring of embedding codepth 3 and type 1. The algebras $Q/\mathfrak{b}_3$ and $Q/\mathfrak{b}_4$ are artinian; that is, they have embedding codimension 3. An artinian local ring of codepth 3 and type 2 belongs to $\mathbf{B}$ only if the minimal number of generators of its Cohen ideal is odd and at least 5; see [1, 3.4]. In that light it appears noteworthy that $\mathfrak{b}_3$ and $\mathfrak{b}_4$ are minimally generated by 6 and 7 elements, respectively.

* * *

In preparation for the proofs, we recall a few definitions and facts. Let Q be a regular local ring with residue field k and let I be an ideal in Q of grade 3. The quotient ring $R = Q/I$ has codepth 3 and its minimal free resolution over Q has the form

$$F \;=\; 0 \longrightarrow Q^n \longrightarrow Q^m \longrightarrow Q^{l+1} \longrightarrow Q \longrightarrow 0\ ,$$

where n is the type of R and one has $m = n+l$ and $l+1 = \mu(I)$. It has a structure of a graded-commutative differential graded algebra; this was proved by Buchsbaum and Eisenbud [4, 1.3]. While this structure is not unique, the induced graded-commutative algebra structure on $A = \mathrm{H}(F \otimes_Q k)$ is unique up to isomorphism. Given bases

$$\begin{aligned}\mathbf{e}_1,\dots,\mathbf{e}_{l+1} &\quad \text{for } A_1,\\ \mathbf{f}_1,\dots,\mathbf{f}_m &\quad \text{for } A_2, \text{ and}\\ \mathbf{g}_1,\dots,\mathbf{g}_n &\quad \text{for } A_3\end{aligned} \tag{1.1}$$

graded-commutativity yields

$$\begin{aligned} -\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i \text{ and } \mathbf{e}_i^2 = 0 \quad & \text{for all } 1 \le i,j \le l+1\,; \\ \mathbf{e}_i\mathbf{f}_j = \mathbf{f}_j\mathbf{e}_i \quad & \text{for all } \ 1 \le i \le l+1 \ \text{ and all } \ 1 \le j \le m\,. \end{aligned} \tag{1.2}$$

We recall from [2, 2.1] the definitions of the classes $\mathbf{G}(r)$ with $r \ge 2^*$ and $\mathbf{B}$.

The ring R belongs to $\mathbf{G}(r)$ if there is a basis (1.1) for $A_{\ge 1}$ such that one has

$$\mathbf{e}_i\mathbf{f}_i = \mathbf{g}_1 \qquad \text{for all } 1 \le i \le r \tag{1.3}$$

and all other products of basis elements not fixed by (1.3) via (1.2) are zero.

The ring R belongs to $\mathbf{B}$ if there is a basis (1.1) for $A_{\ge 1}$ such that one has

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{f}_3 \qquad \text{and} \qquad \begin{aligned} \mathbf{e}_1\mathbf{f}_1 = \mathbf{g}_1 \\ \mathbf{e}_2\mathbf{f}_2 = \mathbf{g}_1 \end{aligned} \tag{1.4}$$

and all other products of basis elements not fixed by (1.4) via (1.2) are zero.

The proofs of both theorems use the fact that the graded algebra A is isomorphic to the Koszul homology algebra over R. We fix notation for the Koszul complex. Let R be of embedding dimension 3 and let $\mathfrak{m} = (x,y,z)$ be its maximal ideal. We denote by K^R the Koszul complex over the canonical homomorphism $\pi\colon R^3 \twoheadrightarrow \mathfrak{m}$. It is the exterior algebra of the rank 3 free R-module with basis ε_x, ε_y, ε_z, endowed with the differential induced by π. For brevity we set

$$\varepsilon_{xy} = \varepsilon_x \wedge \varepsilon_y\,, \quad \varepsilon_{xz} = \varepsilon_x \wedge \varepsilon_z\,, \quad \varepsilon_{yz} = \varepsilon_y \wedge \varepsilon_z \quad \text{and} \quad \varepsilon_{xyz} = \varepsilon_x \wedge \varepsilon_y \wedge \varepsilon_z\,.$$

The differential is then given by,

$$\begin{array}{lll} \partial(\varepsilon_x) = x & \partial(\varepsilon_{xy}) = x\varepsilon_y - y\varepsilon_x & \\ \partial(\varepsilon_y) = y & \partial(\varepsilon_{xz}) = x\varepsilon_z - z\varepsilon_x & \partial(\varepsilon_{xyz}) = x\varepsilon_{yz} - y\varepsilon_{xz} + z\varepsilon_{xy}\,, \\ \partial(\varepsilon_z) = z & \partial(\varepsilon_{yz}) = y\varepsilon_z - z\varepsilon_y & \end{array}$$

and it makes K^R into a graded-commutative differential graded algebra. The induced algebra structure on $\mathrm{H}(\mathrm{K}^R)$ is graded-commutative, and there is an isomorphism of graded algebras $A \cong \mathrm{H}(\mathrm{K}^R)$; see [1, (1.7.1)].

Note that the homology module $\mathrm{H}_3(\mathrm{K}^R)$ is isomorphic, as a k-vector space, to the socle of R, that is, the ideal $(0:\mathfrak{m})$. To be precise, if $s_1,\ldots,s_n$ is a basis for the socle of R, then the homology classes of the cycles $s_1\varepsilon_{xyz},\ldots,s_n\varepsilon_{xyz}$ in K^R_3 form a basis for $\mathrm{H}_3(\mathrm{K}^R)$. From the isomorphism $A \cong \mathrm{H}(\mathrm{K}^R)$ one gets, in particular, $\sum_{i=0}^{3}(-1)^i \operatorname{rank}_{\Bbbk} \mathrm{H}_i(\mathrm{K}^R) = 0$, as the ranks of the Koszul homology modules equal the ranks of the free modules in F. To sum up one has,

$$\begin{aligned} \operatorname{rank}_{\Bbbk} \mathrm{H}_1(\mathrm{K}^R) &= l+1 = \mu(I) \\ \operatorname{rank}_{\Bbbk} \mathrm{H}_2(\mathrm{K}^R) &= l + \operatorname{rank}_{\Bbbk} \mathrm{H}_3(\mathrm{K}^R) \\ \operatorname{rank}_{\Bbbk} \mathrm{H}_3(\mathrm{K}^R) &= \operatorname{rank}_{\Bbbk}(0:\mathfrak{m})\,. \end{aligned} \tag{1.5}$$

Theorem I is proved in Sections 2–4 and Theorem II in Sections 5–7. For each quotient algebra $Q/\mathfrak{g}_n$ and $Q/\mathfrak{b}_n$ we shall verify that the Koszul homology algebra has the desired multiplicative structure as described in (1.3) and (1.4), and we shall determine the type of the quotient algebra. As described above, the latter means determining the socle rank, as each of these algebras has depth 0.

* One does not define $\mathbf{G}(1)$ because it would overlap with another class called $\mathbf{H}(0,1)$.

2. Proof that $Q/\mathfrak{g}_1$ is a type 2 algebra in $\mathbf{G}(2)$

The ideal $\mathfrak{g}_1 = (xy^2, xyz, yz^2, x^4 - y^3z, xz^3 - y^4)$ in $Q = \Bbbk[\![x,y,z]\!]$ is generated by homogeneous elements, so $R = Q/\mathfrak{g}_1$ is a graded $\Bbbk$-algebra. For $n \geq 0$ denote by R_n the subspace of R of homogeneous polynomials of degree n. It is simple to verify that the elements in the second column below form bases for the subspaces R_n; for convenience, the third column lists the relations among non-zero monomials.

$$
\begin{array}{lll}
R_0 & 1 & \\
R_1 & x,\ y,\ z & \\
R_2 & x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2 & \\
R_3 & x^3,\ x^2y,\ x^2z,\ xz^2,\ y^3,\ y^2z,\ z^3 & \\
R_4 & x^4,\ x^3y,\ x^3z,\ x^2z^2,\ xz^3,\ z^4 & y^4 = xz^3,\ y^3z = x^4 \\
R_5 & x^4y,\ x^3z^2,\ z^5 & xz^4 = y^4z = x^4y \\
R_{n\geq 6} & z^n &
\end{array}
\tag{2.1}
$$

Set $A = \mathrm{H}(\mathrm{K}^R)$; we shall verify that A has the multiplicative structure described in (1.3) with $r = 2$, and that R has socle rank 2.

A basis for A_3. From (2.1) it is straightforward to verify that the socle of R is generated by x^4y and x^3z^2, so it has rank 2 and the homology classes of the cycles

$$
g_1 = x^4y\varepsilon_{xyz} \qquad \text{and} \qquad g_2 = x^3z^2\varepsilon_{xyz}
\tag{2.2}
$$

form a basis for A_3. As there are no non-zero boundaries in K_3^R, the homology classes $\bar{g}_1$ and $\bar{g}_2$ contain only g_1 and g_2, respectively, and the bar merely signals that we consider the cycles as elements in A rather than K^R.

As the ideal $\mathfrak{g}_1$ is minimally generated by 5 elements, one has $\operatorname{rank}_\Bbbk A_1 = 5$, and hence $\operatorname{rank}_\Bbbk A_2 = 6$ by (1.5).

A basis for A_2. It is elementary to verify that the next elements in K_2^R are cycles.

$$
\begin{aligned}
f_1 &= -yz\varepsilon_{xy} + y^2\varepsilon_{xz} \\
f_2 &= yz\varepsilon_{xz} \\
f_3 &= x^3z\varepsilon_{xy} \\
f_4 &= xz^3\varepsilon_{xy} \\
f_5 &= (x^3y - z^4)\varepsilon_{xy} \\
f_6 &= x^3z^2\varepsilon_{xz}
\end{aligned}
\tag{2.3}
$$

To see that their homology classes form a basis for A_2 it suffices, as A_2 is a $\Bbbk$-vector space of rank 6, to verify that they are linearly independent modulo boundaries. A boundary in K_2^R has the form

$$
\partial(a\varepsilon_{xyz}) = ax\varepsilon_{yz} - ay\varepsilon_{xz} + az\varepsilon_{xy}\ ,
\tag{2.4}
$$

for some a in R. As the differential is graded, one needs to verify that f_1 and f_2 are linearly independent modulo boundaries, that f_3, f_4, and f_5 are linearly independent modulo boundaries, and that f_6 is not a boundary.

If u and v are elements in $\Bbbk$ such that $uf_1 + vf_2 = -uyz\varepsilon_{xy} + (uy^2 + vyz)\varepsilon_{xz}$ is a boundary, then it has the form (2.4) for some a in R_1. In particular, one has $ax = 0$, and that forces $a = 0$; see (2.1). With this one has $uyz = 0$ and $uy^2 + vyz = 0$, whence $u = 0 = v$. Thus, f_1 and f_2 are linearly independent modulo boundaries.

If u, v, and w in $\Bbbk$ are such that $uf_3+vf_4+wf_5 = (wx^3y+ux^3z+vxz^3-wz^4)\varepsilon_{xy}$ is a boundary, then it has the form (2.4) for some element

$$a = a_1x^3 + a_2x^2y + a_3x^2z + a_4xz^2 + a_5y^3 + a_6y^2z + a_7z^3$$

in R_3. From $az = wx^3y + ux^3z + vxz^3 - wz^4$ one gets $w = 0$, $u = a_1$, and $v = a_4$; see (2.1). From $ax = 0$ one gets $a_1 = 0 = a_4$, that is, $u = 0 = v$.

Finally, f_6 is not a boundary as no element a in R_4 satisfies $-ay = x^3z^2$.

A basis for A_1. The following elements are cycles in K_1^R.

$$\begin{aligned} e_1 &= x^3\varepsilon_x - y^2z\varepsilon_y \\ e_2 &= z^3\varepsilon_x - y^3\varepsilon_y \\ e_3 &= yz\varepsilon_x \\ e_4 &= z^2\varepsilon_y \\ e_5 &= y^2\varepsilon_x \end{aligned} \tag{2.5}$$

The vector space A_1 has rank 5, so as above the task is to show that $e_1, \dots, e_5$ are linearly independent modulo boundaries. A boundary in K_1^R has the form

$$\partial(a\varepsilon_{xy} + b\varepsilon_{xz} + c\varepsilon_{yz}) = -(ay+bz)\varepsilon_x + (ax-cz)\varepsilon_y + (bx+cy)\varepsilon_z\ , \tag{2.6}$$

for a, b, and c in R. As above, the fact that the differential is graded allows us to treat cycles with coefficients in R_2 and R_3 independently.

If u and v are elements in $\Bbbk$ such that $ue_1+ve_2 = (ux^3+vz^3)\varepsilon_x - (uy^2z+vy^3)\varepsilon_y$ is a boundary in K_1^R, then it has the form (2.6) for elements

$$\begin{aligned} a &= a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2\ , \\ b &= b_1x^2 + b_2xy + b_3xz + b_4y^2 + b_5yz + b_6z^2\ , \quad \text{and} \\ c &= c_1x^2 + c_2xy + c_3xz + c_4y^2 + c_5yz + c_6z^2 \end{aligned}$$

in R_2. The equality $cz - ax = uy^2z + vy^3$ yields $v = 0$ and $u = c_4$, and from $bx + cy = 0$ one gets $c_4 = 0$.

If u, v, and w are elements in $\Bbbk$ such that $ue_3+ve_4+we_5 = (uyz+wy^2)\varepsilon_x+vz^2\varepsilon_y$ is a boundary, then it has the form (2.6) for $a = a_1x+a_2y+a_3z$, $b = b_1x+b_2y+b_3z$, and $c = c_1x + c_2y + c_3z$ in R_1. This yields equations:

$$\begin{aligned} -a_1xy - a_2y^2 - (a_3+b_2)yz - b_1xz - b_3z^2 &= uyz + wy^2\ , \\ a_1x^2 + a_2xy + (a_3-c_1)xz - c_2yz - c_3z^2 &= vz^2\ , \quad \text{and} \\ b_1x^2 + (b_2+c_1)xy + b_3xz + c_2y^2 + c_3yz &= 0\ . \end{aligned}$$

From the last equation one gets, in particular, $c_3 = 0$ and $b_2 + c_1 = 0$. The second one now yields $v = 0$, $a_2 = 0$, and $a_3 = c_1$. With these equalities, the first equation yields $w = 0$ and $u = 0$.

The product $A_1 \cdot A_1$. To determine the multiplication table $A_1 \times A_1$ it is by (1.2) sufficient to compute the products e_ie_j for $1 \le i < j \le 5$. The product e_3e_5 is zero by graded-commutativity of K^R, and the following products are zero because the coefficients vanish in R; cf. (2.1).

$$
\begin{aligned}
e_1e_2 &= (-x^3y^3 + y^2z^4)\varepsilon_{xy} & e_3e_4 &= yz^3\varepsilon_{xy}\\
e_1e_3 &= y^3z^2\varepsilon_{xy} & e_4e_5 &= -y^2z^2\varepsilon_{xy}\\
e_2e_5 &= y^5\varepsilon_{xy}
\end{aligned}
$$

Finally, the computations

$$
\begin{aligned}
e_1e_4 &= x^3z^2\varepsilon_{xy} = \partial(x^3z\varepsilon_{xyz})\ ,\\
e_1e_5 &= y^4z\varepsilon_{xy} = xz^4\varepsilon_{xy} = \partial(xz^3\varepsilon_{xyz})\ ,\\
e_2e_3 &= y^4z\varepsilon_{xy} = xz^4\varepsilon_{xy} = \partial(xz^3\varepsilon_{xyz})\ , \quad \text{and}\\
e_2e_4 &= z^5\varepsilon_{xy} = \partial((z^4 - x^3y)\varepsilon_{xyz})
\end{aligned}
$$

show that also the products $\bar{e}_1\bar{e}_4$, $\bar{e}_1\bar{e}_5$, $\bar{e}_2\bar{e}_3$, and $\bar{e}_2\bar{e}_4$ in homology are zero. Thus, one has $A_1 \cdot A_1 = 0$.

The product $A_1 \cdot A_2$**.** Among the products e_if_j for $1 \le i \le 5$ and $1 \le j \le 6$ several are zero by graded-commutativity of K^R:

$$
e_3f_j \ \text{ and } \ e_5f_j \ \text{ for } 1 \le j \le 6
$$
$$
e_1f_3\ ,\ e_1f_4\ ,\ e_1f_5\ ,\ e_2f_3\ ,\ e_2f_4\ ,\ e_2f_5\ ,\ e_4f_3\ ,\ e_4f_4\ ,\ \text{and}\ \ e_4f_5\ .
$$

The following products are zero because the coefficients vanish in R.

$$
\begin{aligned}
e_1f_2 &= y^3z^2\varepsilon_{xyz} & e_4f_1 &= -y^2z^2\varepsilon_{xyz}\\
e_1f_6 &= x^3y^2z^3\varepsilon_{xyz} & e_4f_2 &= -yz^3\varepsilon_{xyz}\\
e_2f_1 &= y^5\varepsilon_{xyz} & e_4f_6 &= -x^3z^4\varepsilon_{xyz}\\
e_2f_6 &= x^3y^3z^2\varepsilon_{xyz}
\end{aligned}
$$

This leaves two products to compute, namely $e_1f_1 = y^4z\varepsilon_{xyz} = e_2f_2$.

The computations above show that in terms of the $\Bbbk$-basis $\bar{e}_1,\ldots,\bar{e}_5$, $\bar{f}_1,\ldots,\bar{f}_6$, $\bar{g}_1,\bar{g}_2$ for $A_{\ge 1}$ the non-zero products of basis vectors are

$$
\bar{e}_1\bar{f}_1 = \bar{e}_2\bar{f}_2 = \bar{g}_1\ , \tag{2.7}
$$

whence R belongs to $\mathbf{G}(2)$.

3. Proof that $Q/\mathfrak{g}_2$ is a type 2 algebra in $\mathbf{G}(3)$

Set $R = Q/\mathfrak{g}_2$; as one has $\mathfrak{g}_2 = \mathfrak{g}_1 + (x^3y - z^4)$ it follows from (2.1) that the elements listed below form bases for the subspaces R_n. As in (2.1) the third column records the relations among non-zero monomials.

$$
\begin{array}{lll}
R_0 & 1 & \\
R_1 & x,\ y,\ z & \\
R_2 & x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2 & \\
R_3 & x^3,\ x^2y,\ x^2z,\ xz^2,\ y^3,\ y^2z,\ z^3 & \\
R_4 & x^4,\ x^3y,\ x^3z,\ x^2z^2,\ xz^3 & y^4 = xz^3,\ y^3z = x^4,\ z^4 = x^3y\\
R_5 & x^4y,\ x^3z^2 & xz^4 = y^4z = x^4y
\end{array} \tag{3.1}
$$

Set $A = \mathrm{H}(\mathrm{K}^R)$; we shall verify that A has the multiplicative structure described in (1.3) with $r = 3$, and that R has socle rank 2.

The next remark will also be used in later sections; loosely speaking, it allows us to recycle the computations from Section 2 in the analysis of A.

(3.2) **Remark.** Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals in Q. The canonical epimorphism $Q/\mathfrak{a} \twoheadrightarrow Q/\mathfrak{b}$ yields a morphism of complexes $\mathrm{K}^{Q/\mathfrak{a}} \to \mathrm{K}^{Q/\mathfrak{b}}$. It maps cycles to cycles and boundaries to boundaries. To be explicit, let $E = \{1, \varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{xyz}\}$ be the standard basis for either Koszul complex; if $\sum_{\varepsilon \in E}(q_\varepsilon + \mathfrak{a})\varepsilon$ is a cycle (boundary) in $\mathrm{K}^{Q/\mathfrak{a}}$, then $\sum_{\varepsilon \in E}(q_\varepsilon + \mathfrak{b})\varepsilon$ is a cycle (boundary) in $\mathrm{K}^{Q/\mathfrak{b}}$. By habitual abuse of notation, we write x, y, and z for the cosets of x, y, and z in any quotient algebra of Q, and as such we make no notational distinction between an element in $\mathrm{K}^{Q/\mathfrak{a}}$ and its image in $\mathrm{K}^{Q/\mathfrak{b}}$.

A basis for $A_{\geq 1}$. From (3.1) it is straightforward to verify that the socle of R is R_5, so it has rank 2, and the homology classes of the cycles g_1 and g_2 from (2.2) form a basis for A_3. The ideal $\mathfrak{g}_2$ is minimally generated by 6 elements, so one has $\operatorname{rank}_\Bbbk A_1 = 6$ and $\operatorname{rank}_\Bbbk A_2 = 7$; see (1.5). Proceeding as in Section 2 it is straightforward to verify that the homology classes $e_1, \dots, e_5$ from (2.5) together with the homology class of the cycle

$$e_6 = x^2 y\varepsilon_x - z^3\varepsilon_z$$

form a basis for A_1. Similarly, one verifies that the homology classes of f_1, f_2, f_4, and f_6 from (2.3) together with those of the cycles

$$\begin{aligned} f_3 &= x^3\varepsilon_{xy} - z^3\varepsilon_{xz} + y^3\varepsilon_{yz}\ , \\ f_5 &= z^4\varepsilon_{xz} - y^3 z\varepsilon_{yz}\ , \text{ and} \\ f_7 &= x^2 y\varepsilon_{xy} + z^3\varepsilon_{yz} \end{aligned}$$

make up a basis for A_2.

The product $A_1 \cdot A_1$. It follows from the computations in the previous section that one has $\bar{e}_i\bar{e}_j = 0$ for $1 \leq i, j \leq 5$. To complete the multiplication table $A_1 \times A_1$ it is by (1.2) sufficient to compute the products $e_i e_6$ for $1 \leq i \leq 5$. The products e_1e_6 and e_2e_6 are zero as one has $R_3 \cdot R_3 = 0$. The remaining products involving e_6,

$$\begin{aligned} e_3e_6 &= -yz^4\varepsilon_{xz}\ , \\ e_4e_6 &= -x^2yz^2\varepsilon_{xy} - z^5\varepsilon_{yz}\ , \text{ and} \\ e_5e_6 &= -y^2z^3\varepsilon_{xz} \end{aligned}$$

are zero because the coefficients vanish in R. Thus, one has $A_1 \cdot A_1 = 0$.

The product $A_1 \cdot A_2$. It follows from the computations in the previous section that the only non-zero products $\bar{e}_i\bar{f}_j$ for $1 \leq i \leq 5$ and $j \in \{1, 2, 4, 6\}$ are the ones listed in (2.7). The products $e_i f_5$ for $1 \leq i \leq 6$ are zero as one has $R_{\geq 2} \cdot R_4 = 0$; similarly, the products

$$e_1f_3\ ,\ e_1f_7\ ,\ e_2f_3\ ,\ e_2f_7\ ,\ e_6f_3\ ,\ e_6f_4\ ,\ \text{and}\ \ e_6f_7$$

are zero as one has $R_3 \cdot R_{\geq 3} = 0$. Among the remaining products, e_4f_7 and e_6f_2 are zero by graded-commutativity, and the following are zero because the coefficients vanish in R.

$$e_3f_7 = yz^4\varepsilon_{xyz} \qquad e_5f_7 = y^2z^3\varepsilon_{xyz}$$
$$e_4f_3 = z^5\varepsilon_{xyz} \qquad e_6f_1 = yz^4\varepsilon_{xyz}$$
$$e_5f_3 = y^5\varepsilon_{xyz}$$

The one remainging product is $e_3f_3 = y^4z\varepsilon_{xyz} = g_1$.

In terms of the basis $\bar{e}_1,\dots,\bar{e}_6,\bar{f}_1,\dots,\bar{f}_7,\bar{g}_1,\bar{g}_2$ for $A_{\geq 1}$ the only non-zero products of basis vectors are

$$\bar{e}_i\bar{f}_i = \bar{g}_1 \quad \text{for } 1 \leq i \leq 3\ ,$$

whence R belongs to $\mathbf{G}(3)$.

4. Proof that $Q/\mathfrak{g}_3$ and $Q/\mathfrak{g}_4$ are type 2 algebras in $\mathbf{G}(4)$ and $\mathbf{G}(5)$

The arguments that show that $Q/\mathfrak{g}_3$ and $Q/\mathfrak{g}_4$ are $\mathbf{G}$ algebras follow the argument in Section 3 closely; we summarize them below.

(4.1) **The quotient by $\mathfrak{g}_3$.** Set $R = Q/\mathfrak{g}_3$; as one has $\mathfrak{g}_3 = \mathfrak{g}_2 + (x^2z^2)$ it follows from (3.1) that the elements listed in the second column below form bases for the subspaces R_n.

R_0	1	
R_1	$x,\ y,\ z$	
R_2	$x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2$	
R_3	$x^3,\ x^2y,\ x^2z,\ xz^2,\ y^3,\ y^2z,\ z^3$	
R_4	$x^4,\ x^3y,\ x^3z,\ xz^3$	$y^4 = xz^3,\ y^3z = x^4,\ z^4 = x^3y$
R_5	x^4y	$xz^4 = y^4z = x^4y$

It is straightforward to verify that the socle of R is generated by the elements x^4y and x^3z, so it has rank 2. Set $A = \mathrm{H}(\mathrm{K}^R)$; the homology classes of cycles

$$g_1 = x^4y\varepsilon_{xyz} \quad \text{and} \quad g_2 = x^3z\varepsilon_{xyz}$$

form a basis for A_3. One readily verifies that the homology classes of $e_1,\dots,e_6$ and f_1,f_2,f_3,f_5,f_7 from Section 3, see also (3.2), together with those of the cycles

$$e_7 = xz^2\varepsilon_x\ , \quad f_4 = y^3\varepsilon_{xy} - xz^2\varepsilon_{xz}\ , \quad f_6 = xz^2\varepsilon_{xy}\ , \quad \text{and} \quad f_8 = x^3z\varepsilon_{xz}$$

form bases for A_1 and A_2.

The products e_1e_7, e_2e_7, and e_6e_7 vanish as one has $R_3 \cdot R_3 = 0$, while e_3e_7 and e_5e_7 are zero by graded-commutativity. Finally, one has $e_4e_7 = -xz^4\varepsilon_{xy} = -\partial(xz^3\varepsilon_{xyz})$, so also the product $\bar{e}_4\bar{e}_7$ is zero. Together with the computations from the previous section this shows that $A_1 \cdot A_1$ is zero.

The products e_7f_j vanish for $3 \leq j \leq 8$ as one has $R_3 \cdot R_{\geq 3} = 0$, and for the same reason any one of the elements e_1, e_2, and e_6 multiplied by either f_4 or f_6 is zero. All products e_if_8 vanish as one has $R_{\geq 2} \cdot R_4 = 0$. Among the remaining products involving e_7, f_4, or f_6 all but one are zero by graded-commutativity, the non-zero product is $e_4f_4 = xz^4\varepsilon_{xyz} = g_1$. It follows that in terms of the $\Bbbk$-basis $\bar{e}_1,\dots,\bar{e}_7$, $\bar{f}_1,\dots,\bar{f}_8$, $\bar{g}_1,\bar{g}_2$ for $A_{\geq 1}$ the only non-zero products of basis-elements are

$$\bar{e}_i\bar{f}_i = \bar{g}_1 \quad \text{for } 1 \leq i \leq 4\ ,$$

whence R belongs to $\mathbf{G}(4)$.

(4.2) **The quotient by $\mathfrak{g}_4$.** Set $R = Q/\mathfrak{g}_4$; as one has $\mathfrak{g}_4 = \mathfrak{g}_3 + (x^3z)$ it follows from (4.1) that the elements listed in the second column below form bases for the subspaces R_n.

R_0	1	
R_1	$x,\ y,\ z$	
R_2	$x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2$	
R_3	$x^3,\ x^2y,\ x^2z,\ xz^2,\ y^3,\ y^2z,\ z^3$	
R_4	$x^4,\ x^3y,\ xz^3$	$y^4 = xz^3,\ y^3z = x^4,\ z^4 = x^3y$
R_5	x^4y	$xz^4 = y^4z = x^4y$

It is straightforward to check that the socle of R is generated by the elements x^4y and x^2z, so it has rank 2. Set $A = \mathrm{H}(\mathrm{K}^R)$; the homology classes of the cycles

$$g_1 = x^4y\varepsilon_{xyz} \quad \text{and} \quad g_2 = x^2z\varepsilon_{xyz}$$

form a basis for A_3. One readily verifies that the homology classes of $e_1, \dots, e_7$ and $f_1, f_2, f_3, f_4, f_6, f_7$ from (4.1), see also (3.2), together with those of

$$e_8 = x^2z\varepsilon_x\,, \quad f_5 = -x^3\varepsilon_{xz} + y^2z\varepsilon_{yz}\,, \quad f_8 = x^2z\varepsilon_{xy}\,, \quad \text{and} \quad f_9 = x^2z\varepsilon_{xz}$$

form bases for A_1 and A_2.

All the products e_ie_8 are zero as x^2z is in the socle of R. Together with the computations from (4.1) this shows that $A_1 \cdot A_1$ is zero.

All products e_8f_j, e_if_8, and e_if_9 vanish as x^2z is in the socle of R. Any one of the elements e_1, e_2, e_6, and e_7 multiplied by f_5 is zero for as one has $R_3 \cdot R_3 = 0$. The remaining products involving f_5 are

$$e_3f_5 = y^3z^2\varepsilon_{xyz} = 0\,, \quad e_4f_5 = x^3z^2\varepsilon_{xyz} = 0\,, \quad \text{and} \quad e_5f_5 = y^4z\varepsilon_{xyz} = g_1\,.$$

It follows that in terms of the $\Bbbk$-basis $\bar{e}_1, \dots, \bar{e}_8$, $\bar{f}_1, \dots, \bar{f}_9$, $\bar{g}_1, \bar{g}_2$ for $A_{\geq 1}$ the only non-zero products of basis vectors are

$$\bar{e}_i\bar{f}_i = \bar{g}_1 \quad \text{for } 1 \leq i \leq 5\,,$$

whence R belongs to $\mathbf{G}(5)$.

5. Proof that $Q/\mathfrak{b}_1$ is a type 1 algebra in $\mathbf{B}$

The ideal $\mathfrak{b}_1 = (x^3, x^2y, yz^2, z^3)$ in $Q = \Bbbk[\![x,y,z]\!]$ is generated by homogeneous elements, so $R = Q/\mathfrak{b}_1$ is a graded $\Bbbk$-algebra. It is straightforward to verify that the elements listed below form bases for the subspaces R_n.

(5.1)

R_0	1
R_1	$x,\ y,\ z$
R_2	$x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2$
R_3	$x^2z,\ xy^2,\ xyz,\ xz^2,\ y^3,\ y^2z$
R_4	$x^2z^2,\ xy^3,\ xy^2z,\ y^4,\ y^3z$
$R_{n\geq 5}$	$xy^{n-1},\ xy^{n-2}z,\ y^n,\ y^{n-1}z$

Set $A = \mathrm{H}(\mathrm{K}^R)$; we shall verify that A has the multiplicative structure described in (1.4) and that R has socle rank 1.

A basis for $A_{\geq 1}$. From (5.1) it is straightforward to verify that the socle of R is generated by x^2z^2, so it has rank 1 and the homology class of the cycle

$$g_1 = x^2z^2\varepsilon_{xyz} \tag{5.2}$$

is a basis for A_3. The ideal $\mathfrak{b}_1$ is minimally generated by 4 elements, whence one has $\operatorname{rank}_{\Bbbk} A_1 = 4 = \operatorname{rank}_{\Bbbk} A_2$; see (1.5). Proceeding as in Section 2 it is straightforward to verify that the elements $e_1, \ldots, e_4$ and $f_1, \ldots, f_4$ listed below are cycles in K_1^R and K_2^R whose homology classes form bases for the $\Bbbk$-vector spaces A_1 and A_2.

$$\begin{aligned} e_1 &= x^2\varepsilon_x & f_1 &= z^2\varepsilon_{yz} \\ e_2 &= z^2\varepsilon_z & f_2 &= x^2\varepsilon_{xy} \\ e_3 &= xy\varepsilon_x & f_3 &= x^2z^2\varepsilon_{xz} \\ e_4 &= z^2\varepsilon_y & f_4 &= xyz\varepsilon_{xz} \end{aligned} \tag{5.3}$$

The product $A_1 \cdot A_1$. To determine the multiplication table $A_1 \times A_1$ it is sufficient to compute the products e_ie_j for $1 \le i < j \le 4$; see (1.2). The product e_1e_3 is zero by graded-commutativity of K^R, and the following products are zero because the coefficients vanish in R.

$$\begin{aligned} e_2e_3 &= -xyz^2\varepsilon_{xz} \\ e_2e_4 &= -z^4\varepsilon_{yz} \\ e_3e_4 &= xyz^2\varepsilon_{xy} \end{aligned}$$

The remaining products are

$$e_1e_2 = x^2z^2\varepsilon_{xz} = f_3 \qquad \text{and} \qquad e_1e_4 = x^2z^2\varepsilon_{xy} = \partial(x^2z\varepsilon_{xyz})\ .$$

It follows that the product $\bar{e}_1\bar{e}_4$ in homology is zero, leaving us with a single non-zero product of basis vectors in A_1, namely $\bar{e}_1\bar{e}_2 = \bar{f}_3$.

The product $A_1 \cdot A_2$. One has

$$\begin{aligned} e_1f_1 &= x^2z^2\varepsilon_{xyz} = g_1 & e_4f_3 &= -x^2z^4\varepsilon_{xyz} = 0 \\ e_2f_2 &= x^2z^2\varepsilon_{xyz} = g_1 & e_4f_4 &= -xyz^3\varepsilon_{xyz} = 0 \\ & & e_3f_1 &= xyz^2\varepsilon_{xyz} = 0\ . \end{aligned}$$

The remaining products are zero by graded-commutativity.

In terms of the $\Bbbk$-basis $\bar{e}_1, \ldots, \bar{e}_4, \bar{f}_1, \ldots, \bar{f}_4, \bar{g}_1$ for $A_{\ge 1}$ the only non-zero products of basis vectors are

$$\bar{e}_1\bar{e}_2 = \bar{f}_3 \qquad \text{and} \qquad \begin{aligned} \bar{e}_1\bar{f}_1 &= \bar{g}_1 \\ \bar{e}_2\bar{f}_2 &= \bar{g}_1\ . \end{aligned} \tag{5.4}$$

It follows that R belongs to the class **B**.

6. Proof that $Q/\mathfrak{b}_2$ is a type 1 algebra in **B**

Set $R = Q/\mathfrak{b}_2$; as one has $\mathfrak{b}_2 = \mathfrak{b}_1 + (xyz)$ it follows from (5.1) that the elements listed below form bases for the subspaces R_n.

$$\begin{array}{ll} R_0 & 1 \\ R_1 & x,\ y,\ z \\ R_2 & x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2 \\ R_3 & x^2z,\ xy^2,\ xz^2,\ y^3,\ y^2z \\ R_4 & x^2z^2,\ xy^3,\ y^4,\ y^3z \\ R_{n\ge 5} & xy^{n-1},\ y^n,\ y^{n-1}z \end{array} \tag{6.1}$$

Set $A = \mathrm{H}(\mathrm{K}^R)$; we shall verify that A has the multiplicative structure described in (1.4) and that R has socle rank 1.

A basis for $A_{\geq 1}$. From (6.1) it is straightforward to verify that the socle of R is generated by x^2z^2, so it has rank 1, and the homology class of the cycle g_1 from (5.2) is a basis for A_3; cf. (3.2). The ideal $\mathfrak{b}_2$ is minimally generated by 5 elements, so one has $\operatorname{rank}_{\Bbbk} A_1 = 5 = \operatorname{rank}_{\Bbbk} A_2$; see (1.5). Proceeding as in Section 2 it is straightforward to verify that the homology classes of $e_1, \dots, e_4$ from (5.3) together with the class of the cycle

$$e_5 = yz\varepsilon_x$$

form a basis for A_1. Similarly, one verifies that the homology classes of f_1, f_2, f_3 from (5.3) together with those of the cycles

$$f_4 = xy\varepsilon_{xz} \qquad \text{and} \qquad f_5 = yz\varepsilon_{xz}$$

make up a basis for A_2.

The product $A_1 \cdot A_1$. It follows from (5.4) that $\bar{e}_1\bar{e}_2 = \bar{f}_3$ is the only non-zero product $\bar{e}_i\bar{e}_j$ for $1 \leq i < j \leq 4$. To complete the multiplication table $A_1 \times A_1$ it is by (1.2) sufficient to compute the products e_ie_5 for $1 \leq i \leq 4$. The products e_1e_5 and e_3e_5 are zero by graded-commutativity of K^R, and the remaining products involving e_5,

$$e_2e_5 = -yz^3\varepsilon_{xz} \qquad \text{and} \qquad e_4e_5 = -yz^3\varepsilon_{xy}\ ,$$

are zero because the coefficients vanish in R.

The product $A_1 \cdot A_2$. It follows from (5.4) that the only non-zero products $\bar{e}_i\bar{f}_j$ for $1 \leq i \leq 4$ and $1 \leq j \leq 3$ are $\bar{e}_1\bar{f}_1 = \bar{e}_2\bar{f}_2 = \bar{g}_1$. To complete the multiplication table $A_1 \times A_2$ one has to compute the products e_5f_j for $1 \leq j \leq 5$ and e_if_4 and e_if_5 for $1 \leq i \leq 5$. The next products are zero because the coefficients vanish in R,

$$\begin{aligned} e_4f_4 &= -xyz^2\varepsilon_{xyz} \\ e_4f_5 &= -yz^3\varepsilon_{xyz} \\ e_5f_1 &= yz^3\varepsilon_{xyz}\ , \end{aligned}$$

and the remaining are zero by graded-commutativity.

In terms of the $\Bbbk$-basis $\bar{e}_1, \dots, \bar{e}_5, \bar{f}_1, \dots, \bar{f}_5, \bar{g}_1$ for $A_{\geq 1}$ the only non-zero products of basis vectors are the ones listed in (5.4), so R belongs to $\mathbf{B}$.

7. Proof that $Q/\mathfrak{b}_3$ and $Q/\mathfrak{b}_4$ are type 3 algebras in $\mathbf{B}$

The arguments that show that $Q/\mathfrak{b}_3$ and $Q/\mathfrak{b}_4$ are $\mathbf{B}$ algebras follow the argument in Section 6 closely; we summarize them below.

(7.1) **The quotient by $\mathfrak{b}_3$.** Set $R = Q/\mathfrak{b}_3$; as one has $\mathfrak{b}_3 = \mathfrak{b}_2 + (xy^2 - y^3)$ it follows from (6.1) that the elements listed below form bases for the subspaces R_n.

R_0	1	
R_1	$x,\ y,\ z$	
R_2	$x^2,\ xy,\ xz,\ y^2,\ yz,\ z^2$	
R_3	$x^2z,\ xy^2,\ xz^2,\ y^2z$	$y^3 = xy^2$
R_4	x^2z^2	

It is straightforward to verify that the socle of R is generated by the elements x^2z^2, xy^2, and y^2z, so it has rank 3. Set $A = \mathrm{H}(\mathrm{K}^R)$; the homology classes of the cycles

$$g_1 = x^2z^2\varepsilon_{xyz}\,, \quad g_2 = xy^2\varepsilon_{xyz}\,, \quad \text{and} \quad g_3 = y^2z\varepsilon_{xyz}$$

form a basis for A_3. One readily verifies that the homology classes of $e_1, \dots, e_5$ and $f_1, \dots, f_5$ from the previous section, see also (3.2), together with those of the cycles

$$e_6 = y^2\varepsilon_x - y^2\varepsilon_y\,, \quad f_6 = xy^2\varepsilon_{xy}\,, \quad f_7 = y^2z\varepsilon_{xy}\,, \quad \text{and} \quad f_8 = y^2z\varepsilon_{yz}$$

form bases for A_1 and A_2.

The products e_ie_6, e_if_6, e_if_7, and e_if_8 for $1 \le i \le 5$ as well as the products e_6f_j for $1 \le j \le 8$ vanish as one has $y^2R_{\ge 2} = 0$. It follows that in terms of the $\Bbbk$-basis $\bar{e}_1, \dots, \bar{e}_6, \bar{f}_1, \dots, \bar{f}_8, \bar{g}_1, \bar{g}_2, \bar{g}_3$ for $A_{\ge 1}$ the only non-zero products of basis vectors are the ones listed in (5.4), so R belongs to $\mathbf{B}$.

(7.2) **The quotient by $\mathfrak{b}_4$.** Set $R = Q/\mathfrak{b}_4$; as one has $\mathfrak{b}_4 = \mathfrak{b}_3 + (y^2z)$ it follows from (7.1) that the elements listed below form bases for the subspaces R_n.

R_0	1	
R_1	x, y, z	
R_2	x^2, xy, xz, y^2, yz, z^2	
R_3	x^2z, xy^2, xz^2	$y^3 = xy^2$
R_4	x^2z^2	

It is straightforward to check that the socle of R is generated by the elements x^2z^2, xy^2, and yz, so it has rank 3. Set $A = \mathrm{H}(\mathrm{K}^R)$; the homology classes of the cycles

$$g_1 = x^2z^2\varepsilon_{xyz}\,, \quad g_2 = xy^2\varepsilon_{xyz}\,, \quad \text{and} \quad g_3 = yz\varepsilon_{xyz}$$

form a basis for A_3. One readily verifies that the homology classes of $e_1, \dots, e_6$ and $f_1, \dots, f_6$ from (7.1) together with those of

$$e_7 = yz\varepsilon_y \quad f_7 = yz\varepsilon_{xy}\,, \quad f_8 = yz\varepsilon_{yz}\,, \quad \text{and} \quad f_9 = y^2\varepsilon_{xz} - y^2\varepsilon_{yz}$$

form bases for A_1 and A_2.

The products e_ie_7 for $1 \le i \le 6$ and e_7f_j for $1 \le j \le 9$ vanish as yz is in the socle of R, and for $1 \le i \le 6$ the products e_if_7 and e_if_8 vanish for the same reason. Finally, all products e_if_9 vanish as one has $y^2R_{\ge 2} = 0$. Thus, in terms of the $\Bbbk$-basis $\bar{e}_1, \dots, \bar{e}_7, \bar{f}_1, \dots, \bar{f}_9, \bar{g}_1, \bar{g}_2, \bar{g}_3$ for $A_{\ge 1}$ the only non-zero products of basis vectors are the ones listed in (5.4), so R belongs to $\mathbf{B}$.

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