SEQUENCES FOR COMPLEXES II

LARS WINTHER CHRISTENSEN

1. Introduction and Notation

This short paper elaborates on an example given in [4] to illustrate an application of sequences for complexes:

> Let R be a local ring with a dualizing complex D, and let M be a finitely generated R–module; then a sequence $x_1, \dots, x_n$ is part of a system of parameters for M if and only if it is a $\mathbf{R}\mathrm{Hom}_R(M, D)$–sequence [4, 5.10].

The final Theorem 3.9 of this paper generalizes the result above in two directions: the dualizing complex is replaced by a Cohen–Macaulay semi–dualizing complex (see [3, Sec. 2] or 3.8 below for definitions), and the finite module is replaced by a complex with finite homology.

Before we can even state, let alone prove, this generalization of [4, 5.10] we have to introduce and study *parameters* for complexes. For a finite R–module M every M–sequence is part of a system of parameters for M, so, loosely speaking, regular elements are just special parameters. For a complex X, however, parameters and regular elements are two different things, and kinship between them implies strong relations between two measures of the size of X: the *amplitude* and the *Cohen–Macaulay defect* (both defined below). This is described in 3.5, 3.6, and 3.7.

The definition of parameters for complexes is based on a notion of *anchor prime ideals*. These do for complexes what minimal prime ideals do for modules, and the quantitative relations between dimension and depth under dagger duality — studied in [3] — have a qualitative description in terms of anchor and associated prime ideals.

Throughout R denotes a commutative, Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We use the same notation as in [4], but for convenience we recall a few basic facts.

The homological position and size of a complex X is captured by the *supremum*, *infimum*, and *amplitude*:

$$\begin{aligned} \sup X &= \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \\ \inf X &= \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0\}, \quad \text{and} \\ \mathrm{amp}\, X &= \sup X - \inf X. \end{aligned}$$

By convention, $\sup X = -\infty$ and $\inf X = \infty$ if $\mathrm{H}(X) = 0$.

The *support* of a complex X is the set

$$\mathrm{Supp}_R X = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid X_\mathfrak{p} \not\simeq 0\} = \bigcup_\ell \mathrm{Supp}_R \mathrm{H}_\ell(X).$$

As usual $\mathrm{Min}_R X$ is the subset of minimal elements in the support.

The *depth* and the (*Krull*) *dimension* of an R–complex X are defined as follows:

$$\operatorname{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X)), \text{ for } X \in \mathcal{D}_-(R), \quad \text{and}$$
$$\dim_R X = \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\},$$

cf. [6, Sec. 3]. For modules these notions agree with the usual ones. It follows from the definition that

$$\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p} \tag{1.0.1}$$

for $X \in \mathcal{D}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$; and there are always inequalities:

$$-\inf X \leq \dim_R X \quad \text{for} \quad X \in \mathcal{D}_+(R); \quad \text{and} \tag{1.0.2}$$

$$-\sup X \leq \operatorname{depth}_R X \quad \text{for} \quad X \in \mathcal{D}_-(R). \tag{1.0.3}$$

A complex $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ is *Cohen–Macaulay* if and only if $\dim_R X = \operatorname{depth}_R X$, that is, if an only if the *Cohen–Macaulay defect*,

$$\operatorname{cmd}_R X = \dim_R X - \operatorname{depth}_R X,$$

is zero. For complexes in $\mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ the Cohen–Macaulay defect is always non-negative, cf. [6, Cor. 3.9].

2. Anchor Prime Ideals

In [4] we introduced associated prime ideals for complexes. The analysis of the support of a complex is continued in this section, and the aim is now to identify the prime ideals that do for complexes what the minimal ones do for modules.

Definitions 2.1. Let $X \in \mathcal{D}_+(R)$; we say that $\mathfrak{p} \in \operatorname{Spec} R$ is an *anchor prime ideal* for X if and only if $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p} > -\infty$. The set of anchor prime ideals for X is denoted by $\operatorname{Anc}_R X$; that is,

$$\operatorname{Anc}_R X = \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \inf X_\mathfrak{p} = 0\}.$$

For $n \in \mathbb{N}_0$ we set

$$\mathrm{W}_n(X) = \{\mathfrak{p} \in \operatorname{Supp}_R X \mid \dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} \leq n\}.$$

Observation 2.2. Let S be a multiplicative system in R, and let $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p} \cap S = \emptyset$ then $S^{-1}\mathfrak{p}$ is a prime ideal in $S^{-1}R$, and for $X \in \mathcal{D}(R)$ there is an isomorphism $S^{-1}X_{S^{-1}\mathfrak{p}} \simeq X_\mathfrak{p}$ in $\mathcal{D}(R_\mathfrak{p})$. In particular, $\inf S^{-1}X_{S^{-1}\mathfrak{p}} = \inf X_\mathfrak{p}$ and $\dim_{S^{-1}R_{S^{-1}\mathfrak{p}}} S^{-1}X_{S^{-1}\mathfrak{p}} = \dim_{R_\mathfrak{p}} X_\mathfrak{p}$. Thus, the next biconditional holds for $X \in \mathcal{D}_+(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$.

$$\mathfrak{p} \in \operatorname{Anc}_R X \iff S^{-1}\mathfrak{p} \in \operatorname{Anc}_{S^{-1}R} S^{-1}X. \tag{2.2.1}$$

Theorem 2.3. *For $X \in \mathcal{D}_+(R)$ there are inclusions:*

(a) $$\operatorname{Min}_R X \subseteq \operatorname{Anc}_R X; \quad \textit{and}$$

(b) $$\mathrm{W}_0(X) \subseteq \operatorname{Anc}_R X.$$

Furthermore, if $\operatorname{amp} X = 0$, that is, if X is equivalent to a module up to a shift, then

(c) $$\operatorname{Anc}_R X = \operatorname{Min}_R X \subseteq \operatorname{Ass}_R X;$$

and if X is Cohen–Macaulay, that is, $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then

(d) $$\operatorname{Ass}_R X \subseteq \operatorname{Anc}_R X = \mathrm{W}_0(X).$$

Proof. In the following X belongs to $\mathcal{D}_+(R)$.

(a): If $\mathfrak{p}$ belongs to $\operatorname{Min}_R X$ then $\operatorname{Supp}_{R_\mathfrak{p}} X_\mathfrak{p} = \{\mathfrak{p}_\mathfrak{p}\}$, so $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, that is, $\mathfrak{p}_\mathfrak{p} \in \operatorname{Anc}_{R_\mathfrak{p}} X_\mathfrak{p}$ and hence $\mathfrak{p} \in \operatorname{Anc}_R X$ by (2.2.1).

(b): Assume that $\mathfrak{p}$ belongs to $\mathrm{W}_0(X)$, then $\dim_R X = \dim R/\mathfrak{p} - \inf X_\mathfrak{p}$, and since $\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}$ and $\dim_{R_\mathfrak{p}} X_\mathfrak{p} \geq -\inf X_\mathfrak{p}$, cf. (1.0.1) and (1.0.2), it follows that $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$, as desired.

(c): For $M \in \mathcal{D}_0(R)$ we have

$$\operatorname{Anc}_R M = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0\} = \operatorname{Min}_R M,$$

and the inclusion $\operatorname{Min}_R M \subseteq \operatorname{Ass}_R M$ is well-known.

(d): Assume that $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\dim_R X = \operatorname{depth}_R X$, then $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp}_R X$, cf. [5, (16.17)]. If $\mathfrak{p} \in \operatorname{Ass}_R X$ we have

$$\dim_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} = -\sup X_\mathfrak{p} \leq -\inf X_\mathfrak{p},$$

cf. [4, Def. 2.3], and it follows by (1.0.2) that equality must hold, so $\mathfrak{p}$ belongs to $\operatorname{Anc}_R X$.

For each $\mathfrak{p} \in \operatorname{Supp}_R X$ there is an equality

$$\dim_R X = \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p},$$

cf. [5, (17.4)(b)], so $\dim_R X - \dim R/\mathfrak{p} + \inf X_\mathfrak{p} = 0$ for $\mathfrak{p}$ with $\dim_{R_\mathfrak{p}} X_\mathfrak{p} = -\inf X_\mathfrak{p}$. This proves the inclusion $\operatorname{Anc}_R X \subseteq \mathrm{W}_0(X)$. □

Corollary 2.4. *For $X \in \mathcal{D}_\mathrm{b}(R)$ there is an inclusion:*

(a) $$\operatorname{Min}_R X \subseteq \operatorname{Ass}_R X \cap \operatorname{Anc}_R X;$$

and for $\mathfrak{p} \in \operatorname{Ass}_R X \cap \operatorname{Anc}_R X$ there is an equality:

(b) $$\operatorname{cmd}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{amp} X_\mathfrak{p}.$$

Proof. Part (a) follows by 2.3(a) and [4, Prop. 2.6]; part (b) is immediate by the definitions of associated and anchor prime ideals, cf. [4, Def. 2.3]. □

Corollary 2.5. *If $X \in \mathcal{D}^{\mathrm{f}}_{+}(R)$, then*

$$\dim_R X = \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\}.$$

Proof. It is immediate by the definitions that

$$\begin{aligned}\dim_R X &= \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}\\ &\geq \sup\{\dim R/\mathfrak{p} - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\}\\ &= \sup\{\dim R/\mathfrak{p} + \dim_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Anc}_R X\};\end{aligned}$$

and the opposite inequality follows by 2.3(b). □

Proposition 2.6. *The following hold:*

(a) *If $X \in \mathcal{D}_+(R)$ and $\mathfrak{p}$ belongs to $\operatorname{Anc}_R X$, then $\dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) = 0$.*

(b) *If $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then $\operatorname{Anc}_R X$ is a finite set.*

Proof. (a): Assume that $\mathfrak{p} \in \operatorname{Anc}_R X$; by [6, Prop. 3.5] we have

$$-\inf X_\mathfrak{p} = \dim_{R_\mathfrak{p}} X_\mathfrak{p} \geq \dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) - \inf X_\mathfrak{p},$$

and hence $\dim_{R_\mathfrak{p}}(\mathrm{H}_{\inf X_\mathfrak{p}}(X_\mathfrak{p})) = 0$.

(b): By (a) every anchor prime ideal for X is minimal for one of the homology modules of X, and when $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ each of the finitely many homology modules has a finite number of minimal prime ideals. □

Observation 2.7. By Nakayama's lemma it follows that

$$\inf \mathrm{K}(x_1, \dots, x_n; Y) = \inf Y,$$

for $Y \in \mathcal{D}_+^{\mathrm{f}}(R)$ and elements $x_1, \dots, x_n \in \mathfrak{m}$.

Proposition 2.8 (Dimension of Koszul Complexes)**.** *The following hold for a complex $Y \in \mathcal{D}_+^{\mathrm{f}}(R)$ and elements $x_1, \dots, x_n \in \mathfrak{m}$:*

(a) $$\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \sup \{\dim R/\mathfrak{p} - \inf Y_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)\}; \quad \textit{and}$$

(b) $$\dim_R Y - n \leq \dim_R \mathrm{K}(x_1, \dots, x_n; Y) \leq \dim_R Y.$$

Furthermore:

(c) *The elements $x_1, \dots, x_n$ are contained in a prime ideal $\mathfrak{p} \in \mathrm{W}_n(Y)$; and*

(d) $\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y$ *if and only if* $x_1, \dots, x_n \in \mathfrak{p}$ *for some* $\mathfrak{p} \in \mathrm{W}_0(Y)$.

Proof. Since $\mathrm{Supp}_R \mathrm{K}(x_1, \dots, x_n; Y) = \mathrm{Supp}_R Y \cap \mathrm{V}(x_1, \dots, x_n)$ (see [6, p. 157] and [4, 3.2]) (a) follows by the definition of Krull dimension and 2.7. In (b) the second inequality follows from (a); the first one is established through four steps:

1° $Y = R$: The second equality below follows from the definition of Krull dimension as $\mathrm{Supp}_R \mathrm{K}(x_1, \dots, x_n) = \mathrm{Supp}_R \mathrm{H}_0(\mathrm{K}(x_1, \dots, x_n)) = \mathrm{V}(x_1, \dots, x_n)$, cf. [4, 3.2]; the inequality is a consequence of Krull's Principal Ideal Theorem, see for example [8, Thm. 13.6].

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_n)\\ &= \sup \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(x_1, \dots, x_n)\}\\ &= \dim R/(x_1, \dots, x_n)\\ &\geq \dim R - n\\ &= \dim_R Y - n.\end{aligned}$$

2° $Y = B$, a cyclic module: By $\bar{x}_1, \dots, \bar{x}_n$ we denote the residue classes in B of the elements $x_1, \dots, x_n$; the inequality below is by 1°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(\bar{x}_1, \dots, \bar{x}_n)\\ &= \dim_B \mathrm{K}(\bar{x}_1, \dots, \bar{x}_n)\\ &\geq \dim B - n\\ &= \dim_R Y - n.\end{aligned}$$

3° $Y = H \in \mathcal{D}_0^{\mathrm{f}}(R)$: We set $B = R/\mathrm{Ann}_R H$; the first equality below follows by [6, Prop. 3.11] and the inequality by 2°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_n; B)\\ &\geq \dim_R B - n\\ &= \dim_R Y - n.\end{aligned}$$

4° $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$: The first equality below follows by [6, Prop. 3.12] and the last by [6, Prop. 3.5]; the inequality is by 3°.

$$\begin{aligned}\dim_R \mathrm{K}(x_1,\dots,x_n;Y) &= \sup\{\dim_R \mathrm{K}(x_1,\dots,x_n;\mathrm{H}_\ell(Y)) - \ell \mid \ell \in \mathbb{Z}\}\\ &\geq \sup\{\dim_R \mathrm{H}_\ell(Y) - n - \ell \mid \ell \in \mathbb{Z}\}\\ &= \dim_R Y - n.\end{aligned}$$

This proves (b).

In view of (a) it now follows that

$$\dim_R Y - n \leq \dim R/\mathfrak{p} - \inf Y_\mathfrak{p}$$

for some $\mathfrak{p} \in \mathrm{Supp}_R Y \cap \mathrm{V}(x_1,\dots,x_n)$. That is, the elements $x_1,\dots,x_n$ are contained in a prime ideal $\mathfrak{p} \in \mathrm{Supp}_R Y$ with

$$\dim_R Y - \dim R/\mathfrak{p} + \inf Y_\mathfrak{p} \leq n,$$

and this proves (c).

Finally, it is immediate by the definitions that

$$\dim_R Y = \sup\{\dim R/\mathfrak{p} - \inf Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R Y \cap \mathrm{V}(x_1,\dots,x_n)\}$$

if and only if $\mathrm{W}_0(Y) \cap \mathrm{V}(x_1,\dots,x_n) \neq \emptyset$. This proves (d). □

Theorem 2.9. *If $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then the next two numbers are equal.*

$$\begin{aligned}&\mathrm{d}(Y) = \dim_R Y + \inf Y; \quad \textit{and}\\ &\mathrm{s}(Y) = \inf\{s \in \mathbb{N}_0 \mid \exists\, x_1,\dots,x_s : \mathfrak{m} \in \mathrm{Anc}_R \mathrm{K}(x_1,\dots,x_s;Y)\}.\end{aligned}$$

Proof. There are two inequalities to prove.

$\mathrm{d}(Y) \leq \mathrm{s}(Y)$: Let $x_1,\dots,x_s \in \mathfrak{m}$ be such that $\mathfrak{m} \in \mathrm{Anc}_R \mathrm{K}(x_1,\dots,x_s;Y)$; by 2.8(b) and 2.7 we then have

$$\dim_R Y - s \leq \dim_R \mathrm{K}(x_1,\dots,x_s;Y) = -\inf \mathrm{K}(x_1,\dots,x_s;Y) = -\inf Y,$$

so $\mathrm{d}(Y) \leq s$, and the desired inequality follows.

$\mathrm{s}(Y) \leq \mathrm{d}(Y)$: We proceed by induction on $\mathrm{d}(Y)$. If $\mathrm{d}(Y) = 0$ then $\mathfrak{m} \in \mathrm{Anc}_R Y$ so $\mathrm{s}(Y) = 0$. If $\mathrm{d}(Y) > 0$ then $\mathfrak{m} \notin \mathrm{Anc}_R Y$, and since $\mathrm{Anc}_R Y$ is a finite set, by 2.6(b), we can choose an element $x \in \mathfrak{m} - \cup_{\mathfrak{p} \in \mathrm{Anc}_R Y}\, \mathfrak{p}$. We set $K = \mathrm{K}(x;Y)$; it is cleat that $\mathrm{s}(Y) \leq \mathrm{s}(K) + 1$. Furthermore, it follows by 2.8(a) and 2.3(b) that $\dim_R K < \dim_R Y$ and thereby $\mathrm{d}(K) < \mathrm{d}(Y)$, cf. 2.7. Thus, by the induction hypothesis we have

$$\mathrm{s}(Y) \leq \mathrm{s}(K) + 1 \leq \mathrm{d}(K) + 1 \leq \mathrm{d}(Y);$$

as desired. □

3. Parameters

By 2.9 the next definitions extend the classical notions of systems and sequences of parameters for finite modules (e.g., see [8, § 14] and the appendix in [2]).

Definitions 3.1. Let Y belong to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and set $d = \dim_R Y + \inf Y$. A set of elements $x_1,\dots,x_d \in \mathfrak{m}$ are said to be a *system of parameters* for Y if and only if $\mathfrak{m} \in \mathrm{Anc}_R \mathrm{K}(x_1,\dots,x_d;Y)$.

A sequence $\boldsymbol{x} = x_1,\dots,x_n$ is said to be a *Y–parameter sequence* if and only if it is part of a system of parameters for Y.

Lemma 3.2. *Let Y belong to $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and set $d = \dim_R Y + \inf Y$. The next two conditions are equivalent for elements $x_1, \dots, x_d \in \mathfrak{m}$.*

(*i*) *$x_1, \dots, x_d$ is a system of parameters for Y.*

(*ii*) *For every $j \in \{0, \dots, d\}$ there is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$.

Proof. (*i*) ⇒ (*ii*): Assume that $x_1, \dots, x_d$ is a system of parameters for Y, then

$$\begin{aligned} -\inf \mathrm{K}(x_1, \dots, x_d; Y) &= \dim_R \mathrm{K}(x_1, \dots, x_d; Y) \\ &= \dim_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)) \\ &\geq \dim_R \mathrm{K}(x_1, \dots, x_j; Y) - (d - j) && \text{by 2.8(b)} \\ &\geq \dim_R Y - j - (d - j) && \text{by 2.8(b)} \\ &= \dim_R Y - d \\ &= -\inf Y. \end{aligned}$$

By 2.7 it now follows that $-\inf Y = \dim_R \mathrm{K}(x_1, \dots, x_j; Y) - (d - j)$, so

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = d - j - \inf Y = \dim_R Y - j,$$

as desired. It also follows that $\mathrm{d}(\mathrm{K}(x_1, \dots, x_j; Y)) = d - j$, and since

$$\mathfrak{m} \in \mathrm{Anc}_R \mathrm{K}(x_1, \dots, x_d; Y) = \mathrm{Anc}_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)),$$

we conclude that $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$.

(*ii*) ⇒ (*i*): If $\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j$ then $\mathrm{d}(\mathrm{K}(x_1, \dots, x_j; Y)) = d - j$; and if $x_{j+1}, \dots, x_d$ is a system of parameters for $\mathrm{K}(x_1, \dots, x_j; Y)$ then $\mathfrak{m}$ belongs to $\mathrm{Anc}_R \mathrm{K}(x_{j+1}, \dots, x_d; \mathrm{K}(x_1, \dots, x_j; Y)) = \mathrm{Anc}_R \mathrm{K}(x_1, \dots, x_d; Y)$, so $x_1, \dots, x_d$ must be a system of parameters for Y. □

Proposition 3.3. *Let $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$.*

(*i*) *$\boldsymbol{x}$ is a Y–parameter sequence.*

(*ii*) *For each $j \in \{0, \dots, n\}$ there is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_j; Y) = \dim_R Y - j;$$

and $x_{j+1}, \dots, x_n$ is a $\mathrm{K}(x_1, \dots, x_j; Y)$–parameter sequence.

(*iii*) *There is an equality:*

$$\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y - n.$$

Proof. It follows by 3.2 that (*i*) implies (*ii*), and (*iii*) follows from (*ii*). Now, set $K = \mathrm{K}(\boldsymbol{x}; Y)$ and assume that $\dim_R K = \dim_R Y - n$. Choose, by 2.9, $s = \mathrm{s}(K) = \dim_R K + \inf K$ elements $w_1, \dots, w_s$ in $\mathfrak{m}$ such that $\mathfrak{m}$ belongs to $\mathrm{Anc}_R \mathrm{K}(w_1, \dots, w_s; K) = \mathrm{Anc}_R \mathrm{K}(x_1, \dots, x_n, w_1, \dots, w_s; Y)$. Then, by 2.7, we have

$$n + s = (\dim_R Y - \dim_R K) + (\dim_R K + \inf K) = \dim_R Y + \inf Y = d,$$

so $x_1, \dots, x_n, w_1, \dots, w_s$ is a system of parameters for Y, whence $x_1, \dots, x_n$ is a Y–parameter sequence. □

We now recover a classical result (e.g., see [2, Prop. A.4]):

Corollary 3.4. *Let M be an R–module. The following conditions are equivalent for a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{m}$.*

(*i*) *$\boldsymbol{x}$ is an M–parameter sequence.*

(*ii*) *For each $j \in \{0, \ldots, n\}$ there is an equality:*

$$\dim_R M/(x_1, \ldots, x_j)M = \dim_R M - j;$$

and $x_{j+1}, \ldots, x_n$ is an $M/(x_1, \ldots, x_j)M$–parameter sequence.

(*iii*) *There is an equality:*

$$\dim_R M/(x_1, \ldots, x_n)M = \dim_R M - n.$$

Proof. By [6, Prop. 3.12] and [5, (16.22)] we have

$$\begin{aligned}
\dim_R \mathrm{K}(x_1, \ldots, x_j; M) &= \sup\{\dim_R(M \otimes^{\mathbf{L}}_R \mathrm{H}_\ell(\mathrm{K}(x_1, \ldots, x_j))) - \ell \mid \ell \in \mathbb{Z}\} \\
&= \sup\{\dim_R(M \otimes_R \mathrm{H}_\ell(\mathrm{K}(x_1, \ldots, x_j))) - \ell \mid \ell \in \mathbb{Z}\} \\
&= \dim_R(M \otimes_R R/(x_1, \ldots, x_j)).
\end{aligned}$$

□

Theorem 3.5. *Let $Y \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. The following hold for a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{m}$.*

(a) *There is an inequality:*

$$\operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y) \geq \operatorname{amp} Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y–sequence.

(b) *There is an inequality:*

$$\operatorname{cmd}_R \mathrm{K}(\boldsymbol{x}; Y) \geq \operatorname{cmd}_R Y;$$

and equality holds if and only if $\boldsymbol{x}$ is a Y–parameter sequence.

(c) *If $\boldsymbol{x}$ is a maximal Y–sequence, then*

$$\operatorname{amp} Y \leq \operatorname{cmd}_R \mathrm{K}(\boldsymbol{x}; Y).$$

(d) *If $\boldsymbol{x}$ is a system of parameters for Y, then*

$$\operatorname{cmd}_R Y \leq \operatorname{amp} \mathrm{K}(\boldsymbol{x}; Y).$$

Proof. In the following K denotes the Koszul complex $\mathrm{K}(\boldsymbol{x}; Y)$.

(a): Immediate by 2.7 and [4, Prop. 5.1].

(b): By [4, Thm. 4.7(a)] and 2.8(b) we have

$$\operatorname{cmd}_R K = \dim_R K - \operatorname{depth}_R K = \dim_R K + n - \operatorname{depth}_R Y \geq \operatorname{cmd}_R Y,$$

and by 3.3 equality holds if and only if $\boldsymbol{x}$ is a Y–parameter sequence.

(c): Suppose $\boldsymbol{x}$ is a maximal Y–sequence, then

$$\begin{aligned}
\operatorname{amp} Y &= \sup Y - \inf K && \text{by 2.7} \\
&= -\operatorname{depth}_R K - \inf K && \text{by [4, Thm. 5.4]} \\
&\leq \operatorname{cmd}_R K && \text{by (1.0.2).}
\end{aligned}$$

(d): Suppose $\boldsymbol{x}$ is system of parameters for Y, then

$$\begin{aligned}
\operatorname{amp} K &= \sup K + \dim_R K \\
&\geq \dim_R K - \operatorname{depth}_R K && \text{by (1.0.3)} \\
&= \operatorname{cmd}_R Y && \text{by (b).}
\end{aligned}$$

□

Theorem 3.6. *The following hold for* $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.

(a) *The next four conditions are equivalent.*

- (*i*) *There is a maximal* Y*–sequence which is also a* Y*–parameter sequence.*
- (*ii*) $\operatorname{depth}_R Y + \sup Y \leq \dim_R Y + \inf Y$.
- (*ii'*) $\operatorname{amp} Y \leq \operatorname{cmd}_R Y$.
- (*iii*) *There is a maximal strong* Y*–sequence which is also a* Y*–parameter sequence.*

(b) *The next four conditions are equivalent.*

- (*i*) *There is a system of parameters for* Y *which is also a* Y*–sequence.*
- (*ii*) $\dim_R Y + \inf Y \leq \operatorname{depth}_R Y + \sup Y$.
- (*ii'*) $\operatorname{cmd}_R Y \leq \operatorname{amp} Y$.
- (*iii*) *There is a system of parameters for* Y *which is also a strong* Y*–sequence.*

(c) *The next four conditions are equivalent.*

- (*i*) *There is a system of parameters for* Y *which is also a maximal* Y*–sequence.*
- (*ii*) $\dim_R Y + \inf Y = \operatorname{depth}_R Y + \sup Y$.
- (*ii'*) $\operatorname{cmd}_R Y = \operatorname{amp} Y$.
- (*iii*) *There is a system of parameters for* Y *which is also a maximal strong* Y*–sequence.*

Proof. Let $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, set $\mathrm{n}(Y) = \operatorname{depth}_R Y + \sup Y$ and $\mathrm{d}(Y) = \dim_R Y + \inf Y$.

(a): A maximal Y–sequence is of length $\mathrm{n}(Y)$, cf. [4, Cor. 5.5], and the length of a Y–parameter sequence is at most $\mathrm{d}(Y)$. Thus, (*i*) implies (*ii*) which in turn is equivalent to (*ii'*). Furthermore, a maximal strong Y–sequence is, in particular, a maximal Y–sequence, cf. [4, Cor. 5.7], so (*iii*) is stronger than (*i*). It is now sufficient to prove the implication (*ii*)⇒(*iii*): We proceed by induction. If $\mathrm{n}(Y) = 0$ then the empty sequence is a maximal strong Y–sequence and a Y–parameter sequence. Let $\mathrm{n}(Y) > 0$; the two sets $\operatorname{Ass}_R Y$ and $\mathrm{W}_0(Y)$ are both finite, and since $0 < \mathrm{n}(Y) \leq \mathrm{d}(Y)$ none of them contain $\mathfrak{m}$. We can, therefore, choose an element $x \in \mathfrak{m} - \cup_{\operatorname{Ass}_R Y \cup \mathrm{W}_0(Y)} \mathfrak{p}$, and x is then a strong Y–sequence, cf. [4, Def. 3.3], and a Y–parameter sequence, cf. 3.3 and 2.8. Set $K = \mathrm{K}(x;Y)$, by [4, Thm. 4.7 and Prop. 5.1], respectively, 2.8 and 2.7 we have

$$\begin{aligned}\operatorname{depth}_R K + \sup K &= \mathrm{n}(Y) - 1 \\ &\leq \mathrm{d}(Y) - 1 = \dim_R K + \inf K.\end{aligned}$$

By the induction hypothesis there exists a maximal strong K–sequence $w_1, \ldots, w_{n-1}$ which is also a K–parameter sequence, and it follows by [4, 3.5] and 3.3 that $x, w_1, \ldots, w_{n-1}$ is a strong Y–sequence and a Y–parameter sequence, as wanted.

The proof of (b) i similar to the proof of (a), and (c) follows immediately by (a) and (b). □

Theorem 3.7. *The following hold for* $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$*:*

(a) *If* $\operatorname{amp} Y = 0$*, then any* Y*–sequence is a* Y*–parameter sequence.*

(b) *If* $\operatorname{cmd}_R Y = 0$*, then any* Y*–parameter sequence is a strong* Y*–sequence.*

Proof. The empty sequence is a Y–parameter sequence as well as a strong Y–sequence, this founds the base for a proof by induction on the length n of the sequence $\boldsymbol{x} = x_1, \dots, x_n$. Let $n > 0$ and set $K = \mathrm{K}(x_1, \dots, x_{n-1}; Y)$; by 2.8(a) we have

$$(*) \qquad \begin{aligned} \dim_R \mathrm{K}(x_1, \dots, x_n; Y) &= \dim_R \mathrm{K}(x_n; K) \\ &= \sup \{\dim R/\mathfrak{p} - \inf K_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R K \cap \mathrm{V}(x_n)\}. \end{aligned}$$

Assume that $\operatorname{amp} Y = 0$. If $\boldsymbol{x}$ is a Y–sequence, then $\operatorname{amp} K = 0$ by 3.5(a) and $x_n \notin \mathrm{z}_R K$, cf. [4, Def. 3.3]. As $\mathrm{z}_R K = \cup_{\mathfrak{p} \in \operatorname{Ass}_R K} \mathfrak{p}$, cf. [4, 2.5], it follows by (b) and (c) in 2.3 that x_n is not contained in any prime ideal $\mathfrak{p} \in \mathrm{W}_0(K)$; so from $(*)$ we conclude that $\dim_R \mathrm{K}(x_n; K) < \dim_R K$, and it follows by 2.8(b) that $\dim_R \mathrm{K}(x_n; K) = \dim_R K - 1$. By the induction hypothesis $\dim_R K = \dim_R Y - (n-1)$, so $\dim_R \mathrm{K}(x_1, \dots, x_n; Y) = \dim_R Y - n$ and it follows by 3.3 that $\boldsymbol{x}$ is a Y–parameter sequence. This proves (a).

We now assume that $\operatorname{cmd}_R Y = 0$. If $\boldsymbol{x}$ is a Y–parameter sequence then, by the induction hypothesis, $x_1, \dots, x_{n-1}$ is a strong Y–sequence, so it is sufficient to prove that $x_n \notin \mathrm{Z}_R K$, cf. [4, 3.5]. By 3.3 it follows that x_n is a K–parameter sequence, so $\dim_R \mathrm{K}(x_n; K) = \dim_R K - 1$ and we conclude from $(*)$ that $x_n \notin \cup_{\mathfrak{p} \in \mathrm{W}_0(K)} \mathfrak{p}$. Now, by 3.5(b) we have $\operatorname{cmd}_R K = 0$, so it follows from 2.3(d) that $x_n \notin \cup_{\mathfrak{p} \in \operatorname{Ass}_R K} \mathfrak{p} = \mathrm{Z}_R K$. This proves (b). $\square$

Semi-dualizing Complexes 3.8. We recall two basic definitions from [3]:

A complex $C \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *semi-dualizing* for R if and only if the homothety morphism $\chi_C^R \colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism [3, (2.1)].

Let C be a semi–dualizing complex for R. A complex $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ is said to be *C–reflexive* if and only if the *dagger dual* $Y^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(Y, C)$ belongs to $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ and the biduality morphism $\delta_Y^C : Y \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(Y, C), C)$ is invertible in $\mathcal{D}(R)$ [3, (2.7)].

Relations between dimension and depth for C–reflexive complexes are studied in [3, sec. 3], and the next result is an immediate consequence of [3, (3.1) and (2.10)].

> Let C be a semi–dualizing complex for R and let Z be a C–reflexive complex. The following holds for $\mathfrak{p} \in \operatorname{Spec} R$: If $\mathfrak{p} \in \operatorname{Anc}_R Z$ then $\mathfrak{p} \in \operatorname{Ass}_R Z^{\dagger_C}$, and the converse holds in C is Cohen–Macaulay.

A *dualizing complex*, cf. [7], is a semi–dualizing complex of finite injective dimension, in particular, it is Cohen–Macaulay, cf. [3, (3.5)]. If D is a dualizing complex for R, then, by [7, Prop. V.2.1], all complexes $Y \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$ are D–reflexive; in particular, all finite R–modules are D–reflexive and, therefore, [4, 5.10] is a special case of the following:

Theorem 3.9. *Let C be a Cohen–Macaulay semi–dualizing complex for R, and let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence in $\mathfrak{m}$. If Y is C–reflexive, then $\boldsymbol{x}$ is a Y–parameter sequence if and only if it is a $\mathbf{R}\mathrm{Hom}_R(Y, C)$–sequence; that is*

$$\boldsymbol{x} \text{ is a } Y\text{–parameter sequence} \quad \Longleftrightarrow \quad \boldsymbol{x} \text{ is a } \mathbf{R}\mathrm{Hom}_R(Y, C)\text{–sequence}.$$

Proof. We assume that C is a Cohen–Macaulay semi–dualizing complex for R and that Y is C–reflexive, cf. 3.8. The desired biconditional follows by the next chain, and each step is explained below (we use the notation $-^{\dagger_C}$ introduced in 3.8).

$$\begin{aligned} \boldsymbol{x} \text{ is a } Y\text{–parameter sequence} &\iff \operatorname{cmd}_R \mathrm{K}(\boldsymbol{x};Y) = \operatorname{cmd}_R Y \\ &\iff \operatorname{amp} \mathrm{K}(\boldsymbol{x};Y)^{\dagger_C} = \operatorname{amp} Y^{\dagger_C} \\ &\iff \operatorname{amp} \mathrm{K}(\boldsymbol{x};Y^{\dagger_C}) = \operatorname{amp} Y^{\dagger_C} \\ &\iff \boldsymbol{x} \text{ is a } Y^{\dagger_C}\text{–sequence.} \end{aligned}$$

The first biconditional follows by 3.5(b) and the last by 3.5(a). Since $\mathrm{K}(\boldsymbol{x})$ is a bounded complex of free modules (hence of finite projective dimension), it follows from [3, Thm. (3.17)] that also $\mathrm{K}(\boldsymbol{x};Y)$ is C–reflexive, and the second biconditional is then immediate by the CMD–formula [3, Cor. (3.8)]. The third one is established as follows:

$$\begin{aligned} \mathrm{K}(\boldsymbol{x};Y)^{\dagger_C} &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}) \otimes^{\mathbf{L}}_R Y, C) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), Y^{\dagger_C}) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R \otimes^{\mathbf{L}}_R Y^{\dagger_C}) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}), R) \otimes^{\mathbf{L}}_R Y^{\dagger_C} \\ &\sim \mathrm{K}(\boldsymbol{x}) \otimes^{\mathbf{L}}_R Y^{\dagger_C} \\ &\simeq \mathrm{K}(\boldsymbol{x};Y^{\dagger_C}), \end{aligned}$$

where the second isomorphism is by adjointness and the fourth by, so-called, tensor-evaluation, cf. [1, (1.4.2)]. It is straightforward to check that $\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{x}),R)$ is isomorphic to the Koszul complex $\mathrm{K}(\boldsymbol{x})$ shifted n degrees to the right, and the symbol $\sim$ denotes isomorphism up to shift. □

If C is a semi–dualizing complex for R, then both C and R are C–reflexive complexes, cf. [3, (2.8)], so we have an immediate corollary to the theorem:

Corollary 3.10. *If C is a Cohen–Macaulay semi–dualizing complex for R, then the followiing hold for a sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{m}$.*

(a) *$\boldsymbol{x}$ is a C–parameter sequence if and only if it is an R–sequence.*

(b) *$\boldsymbol{x}$ is an R–parameter sequence if and only if it is a C–sequence.*

4. Acknowledgements

The author would like to thank Srikanth Iyengar and professor Hans-Bjørn Foxby for taking the time to discuss the material presented here.

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Matematisk Afdeling, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.
E-mail address: `winther@math.ku.dk`