RESTRICTED HOMOLOGICAL DIMENSIONS AND COHEN–MACAULAYNESS

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ABSTRACT. The classical homological dimensions—the projective, flat, and injective ones—are usually defined in terms of resolutions and then proved to be computable in terms of vanishing of appropriate derived functors. In this paper we define restricted homological dimensions in terms of vanishing of the same derived functors but over classes of test modules that are restricted to assure automatic finiteness over commutative Noetherian rings of finite Krull dimension. When the ring is local, we use a mixture of methods from classical commutative algebra and the theory of homological dimensions to show that vanishing of these functors reveals that the underlying ring is a Cohen–Macaulay ring—or at least close to be one.

INTRODUCTION

The first restricted dimension comes about like this: Let R be a commutative Noetherian ring; the flat dimension of an R–module M can then be computed by non-vanishing of Tor modules,

$$\operatorname{fd}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(T,M) \neq 0 \text{ for some module } T \,\},$$

and hence we define a *restricted flat dimension* as

$$\mathsf{Rfd}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(T,M) \neq 0 \text{ for some module } T \text{ with } \operatorname{fd}_R T < \infty \,\}.$$

(The flat dimension is sometimes called the *Tor–dimension* and the dimension defined above has similarly been referred to as the *restricted Tor–dimension*).

The restricted flat dimension is often finite: First, it follows from [5, Thm. 2.4] that $\mathsf{Rfd}_R M \leq \dim R$ for all R–modules M. Second, by our Theorem (2.5), for all R–modules M there is an inequality $\mathsf{Rfd}_R M \leq \operatorname{fd}_R M$ with equality if $\operatorname{fd}_R M < \infty$; we say that the restricted flat dimension is a *refinement* of the flat dimension.

Furthermore, the restricted flat dimension is also a refinement of other dimension concepts: H. Holm [24] has proved that our Rfd_R is a refinement of Enochs' and Jenda's *Gorenstein flat dimension* Gfd_R introduced in [15]. Moreover, our Theorem (2.8) shows that for finitely generated modules Rfd_R is a refinement of the *Cohen–Macaulay dimension* CM-dim$_R$ of A. Gerko in [23] (and thus of Auslander's G–dimension [2], as well as the CI–dimension by Avramov, Gasharov, and Peeva [8] and the projective dimension).

The restricted flat dimension can by our Theorem (2.4.b) always be computed by the formula

$$\mathsf{Rfd}_R M = \sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}$$

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where $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ denotes the index of the first non-vanishing $\operatorname{Ext}^m_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}, M_\mathfrak{p})$ module. We refer to this as the *local depth of M at $\mathfrak{p}$*. The equation above is an extension of Chouinard's formula [11, Cor. 1.2] where M has finite flat dimension.

Moreover, Chouinard's formula generalizes the classical Auslander–Buchsbaum formula, $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ [4, Thm. 3.7], for finite modules of finite projective dimension over a local ring. It is, therefore, natural to ask when the restricted flat dimension satisfies a formula of this type. The answer, provided in Theorem (3.4), is that $\mathsf{Rfd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ for every finitely generated R–module M if and only if R is Cohen–Macaulay.

We proceed by asking the obvious question: what happens if one tests only by finitely generated modules of finite flat (or equivalently projective) dimension? This leads to the definition of a *small* restricted flat dimension, $\mathsf{rfd}_R M$, which turns out to satisfy a Chouinard-like formula

$$\text{(I.1)} \qquad \mathsf{rfd}_R M = \sup \{ \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, M) \mid \mathfrak{p} \in \operatorname{Spec} R \},$$

where $\operatorname{depth}_R(\mathfrak{p}, M)$ is the index of the first non-vanishing $\operatorname{Ext}^m_R(R/\mathfrak{p}, M)$ module. This is called the *$\mathfrak{p}$–depth of M* (or grade of $\mathfrak{p}$ on M), and we refer to it as the *non-local depth of M at $\mathfrak{p}$*. We have always $\operatorname{depth}_R(\mathfrak{p}, M) \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$.

The next question is: when do the small and large restricted flat dimensions agree? We prove, in Theorem (3.2), that the two dimensions agree over a local ring R if and only if it is *almost Cohen–Macaulay* in the sense that $\dim R_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \leq 1$ for all prime ideals $\mathfrak{p}$. These rings are studied in detail in section 3.

Following this pattern we introduce, in section 5, four dimensions modeled on the formulas for computing projective and injective dimension by vanishing of Ext–modules. For a number of reasons these dimensions do not behave as nicely as those based on vanishing of Tor–modules.

The *small restricted injective dimension*,

$$\mathsf{rid}_R N = \sup \{ m \in \mathbb{N}_0 \mid \operatorname{Ext}^m_R(T, N) \neq 0 \text{ for some module } T \text{ with } \operatorname{pd}_R T < \infty \},$$

is a *finer* invariant than the injective dimension over any commutative ring in the sense that there is always an inequality $\mathsf{rid}_R N \leq \operatorname{id}_R N$. Furthermore, by Corollary (5.9) a local ring R is almost Cohen–Macaulay if and only if $\mathsf{rid}_R N = \operatorname{id}_R N$ for all R–modules of finite injective dimension, that is, if and only if rid_R is a refinement of id_R.

A formula dual to (I.1) is satisfied by this restricted injective dimension

$$\text{(I.2)} \qquad \mathsf{rid}_R N = \sup \{ \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{width}_R(\mathfrak{p}, N) \mid \mathfrak{p} \in \operatorname{Spec} R \}.$$

The $\mathfrak{p}$–width of N, $\operatorname{width}_R(\mathfrak{p}, N)$, is a notion dual to the $\mathfrak{p}$–depth; it is introduced and studied in section 4. For finite non-zero modules over a local ring (I.2) reduces to

$$\mathsf{rid}_R N = \operatorname{depth} R$$

and hence emerges as a generalization of Bass' celebrated formula [10, Lem. (3.3)].

The small and large restricted projective dimensions (with obvious definitions) are finer invariants than the usual projective dimension but only refinements for finitely generated modules. Still, they detect Cohen–Macaulayness of the underlying ring as proved in Theorem (5.22).

1. Prerequisites

Throughout this paper R is a non-trivial, commutative, and Noetherian ring. When R is *local*, $\mathfrak{m}$ denote its unique maximal ideal and k denotes its residue field $R/\mathfrak{m}$. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ the residue field of the local ring $R_\mathfrak{p}$ is denoted by $k(\mathfrak{p})$, i.e., $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. As usual, the set of prime ideals containing an ideal $\mathfrak{a}$ is written $\mathrm{V}(\mathfrak{a})$. By $\boldsymbol{x}$ we denote an sequence of elements from R, e.g., $\boldsymbol{x} = x_1, \dots, x_n$. Finitely generated modules are called *finite* modules.

In this paper definitions and results are formulated within the framework of the derived category $\mathcal{D}(R)$ of the category of R–modules, and although some arguments have a touch of classical commutative algebra, the proofs draw heavily on the theory of homological dimensions for complexes using the derived functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$. Throughout we use notation and results from [13] and [7]. However, in order to make the text readable we list the most needed facts below.

For an object X in $\mathcal{D}(R)$ (that is, a complex X of R–modules) the *supremum* $\sup X$ and the *infimum* $\inf X$ of $X \in \mathcal{D}(R)$ are the (possibly infinite) numbers $\sup \{ \ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0 \}$ and $\inf \{ \ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0 \}$, respectively. (Here $\sup \varnothing = -\infty$ and $\inf \varnothing = \infty$, as usual.) The *full subcategories* $\mathcal{D}_-(R)$ and $\mathcal{D}_+(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. We set $\mathcal{D}_\mathrm{b}(R) = \mathcal{D}_-(R) \cap \mathcal{D}_+(R)$. The full subcategory $\mathcal{D}_0(R)$ of $\mathcal{D}_\mathrm{b}(R)$ consists of X with $\mathrm{H}_\ell(X) = 0$ for $\ell \neq 0$. Since each R–module M can be considered as a complex concentrated in degree 0 (hence $M \in \mathcal{D}_0(R)$) and since each $X \in \mathcal{D}_0(R)$ is isomorphic (in $\mathcal{D}(R)$) to the module $\mathrm{H}_0(X)$, we identify $\mathcal{D}_0(R)$ with the category of R–modules. The full subcategory $\mathcal{D}^\mathrm{f}(R)$ of $\mathcal{D}(R)$ consists of complexes X with all the modules $\mathrm{H}_\ell(X)$ finite for $\ell \in \mathbb{Z}$. The superscript f is also used with the full subcategories; for example, $\mathcal{D}^\mathrm{f}_\mathrm{b}(R)$ consists of complexes X with $\mathrm{H}(X)$ finite in each degree and bounded.

(1.1) ***Depth***. If R is local the local *depth* $\operatorname{depth}_R Y$ of $Y \in \mathcal{D}_-(R)$ is the (possibly infinite) number $-\sup(\mathbf{R}\mathrm{Hom}_R(k, Y))$, cf. [19, Sec. 3]. For finite modules this agrees with the classical definition.

In the following, R is any (commutative Noetherian) ring and $\mathfrak{a}$ is an ideal.

The non-local $\mathfrak{a}$–*depth* $\operatorname{depth}_R(\mathfrak{a}, Y)$ is the number $-\sup(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y))$ when $Y \in \mathcal{D}(R)$. The $\mathfrak{a}$–depth is an extension to complexes of a well-known invariant, the grade, for (finite) modules. In particular, $\operatorname{depth}_R(\mathfrak{a}, R)$ is the maximal length of an R–sequence in $\mathfrak{a}$.

Let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite sequence of generators for $\mathfrak{a}$, and let $\mathrm{K}(\boldsymbol{a})$ be the Koszul complex. When $Y \in \mathcal{D}_-(R)$, by [25, 6.1] there is an equality

(1.1.1) $$\operatorname{depth}_R(\mathfrak{a}, Y) = t - \sup(\mathrm{K}(\boldsymbol{a}) \otimes_R Y)\,.$$

For $Y \in \mathcal{D}_-(R)$ the relation to local depth is given by [14, Prop. 4.5]

(1.1.2) $$\operatorname{depth}_R(\mathfrak{a}, Y) = \inf \{ \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \}.$$

In particular, we have

(1.1.3) $$\operatorname{depth}_R(\mathfrak{p}, Y) \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p}\,.$$

From (1.1.2) it also follows that

(1.1.4) $$\operatorname{depth}_R(\mathfrak{b}, Y) \geq \operatorname{depth}_R(\mathfrak{a}, Y) \geq -\sup Y.$$

The *Cohen–Macaulay defect* $\operatorname{cmd} R$ of a local ring R is the (always non-negative) difference $\dim R - \operatorname{depth} R$ between the Krull dimension and the depth. For a non-local ring R the Cohen–Macaulay defect is the supremum over the defects at all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$.

(1.2) ***Width***. If R is local, then the (local) *width* $\operatorname{width}_R X$ of $X \in \mathcal{D}_+(R)$ is the number $\inf(X \otimes^{\mathbf{L}}_R k)$, cf. [31, Def. 2.1], and note that if $X \in \mathcal{D}^{\mathrm{f}}_+(R)$, then [18, Lem. 2.1] and Nakayama's lemma give

$$\operatorname{width}_R X = \inf X \tag{1.2.1}$$

For any $X \in \mathcal{D}_+(R)$ [18, Lem. 2.1] give the next inequality

$$\operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} \geq \inf X_\mathfrak{p} \geq \inf X. \tag{1.2.2}$$

(1.3) ***Homological dimensions***. The *projective*, *injective*, and *flat dimensions* are abbreviated as pd, id, and fd, respectively. The full subcategories $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ of $\mathcal{D}_\mathrm{b}(R)$ consist of complexes of finite, respectively, projective, injective, and flat dimension, cf. [13, 1.4]. For example, a complex belongs to $\mathcal{F}(R)$ if and only if it is isomorphic in $\mathcal{D}(R)$ to a bounded complex of flat modules. Again we use the superscript f to denote finite homology and the subscript 0 to denote modules. For example, $\mathcal{P}^{\mathrm{f}}_0(R)$ denotes the category of finite modules of finite projective dimension.

We close this section by summing up some results on bounds for these dimensions; they will be used extensively in the rest of the text.

By [28, Thm. (3.2.6)] and [9, Prop. 5.4] we have

$$\sup\{\operatorname{pd}_R M \mid M \in \mathcal{P}_0(R)\} = \dim R\,. \tag{1.3.1}$$

The next shrewd observation due to Auslander and Buchsbaum [5] is often handy.

(1.4) **Lemma.** *If R is local and $\dim R > 0$, then there exists a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that $\operatorname{depth} R_\mathfrak{p} = \dim R - 1$.*

In particular, for any local ring we have

$$\sup\{\operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} = \begin{cases} \dim R & \textit{if } R \textit{ is Cohen–Macaulay; and} \\ \dim R - 1 & \textit{if } R \textit{ is not Cohen–Macaulay.} \end{cases} \qquad \square$$

By [5, Prop. 2.8] and [26, Thm. 1] this implies that,

$$\operatorname{fd}_R M, \operatorname{id}_R N \leq \begin{cases} \dim R & \text{if } R \text{ is Cohen–Macaulay, and} \\ \dim R - 1 & \text{if } R \text{ is not Cohen–Macaulay} \end{cases} \tag{1.4.1}$$

for $M \in \mathcal{F}_0(R)$ and $N \in \mathcal{I}_0(R)$.

The classical Auslander–Buchsbaum formula extends to complexes [20, (0.1)]

(1.5) **Auslander–Buchsbaum formula.** *If R is local and $X \in \mathcal{P}^{\mathrm{f}}(R)$, then*

$$\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth}_R X. \quad \square$$

Actually, it is a special case of the following [20, Lem. 2.1]

(1.6) **Theorem.** *Let R be local. If $X \in \mathcal{F}(R)$ and $Y \in \mathcal{D}_{\mathrm{b}}(R)$, then the next three equalities hold.*

(a) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = -\sup(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth}_R Y.$$

(b) $$\operatorname{depth}_R X = -\sup(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth} R.$$

(c) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{depth}_R X + \operatorname{depth}_R Y - \operatorname{depth} R. \quad \square$$

In [25, Thm. 4.1] it was demonstrated that it is sufficient to take Y bounded on the left, i.e., $Y \in \mathcal{D}_{-}(R)$. We treat dual versions of this theorem in section 4.

2. Tor–dimensions

This section is devoted to the Tor–dimensions: the flat dimension, the large restricted flat dimension, and the small restricted flat dimension. The second one was introduced in [21], and the proofs of the first four results can be found in [12, Chap. 5].

(2.1) **Definition.** The *large restricted flat dimension*, $\mathsf{Rfd}_R X$, of $X \in \mathcal{D}_{+}(R)$ is

$$\mathsf{Rfd}_R X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{F}_0(R)\}.$$

For an R–module M we get

$$\mathsf{Rfd}_R M = \sup\{m \in \mathbb{N}_0 \mid \operatorname{Tor}^R_m(T, M) \neq 0 \text{ for some } T \in \mathcal{F}_0(R)\}.$$

The latter expression explains the name, which is justified further by Proposition (2.2) below. The number $\mathsf{Rfd}_R X$ is sometimes referred to as the *restricted Tor–dimension*, and it is denoted $\operatorname{Td}_R X$ in [21] and [12].

(2.2) **Proposition.** *If $X \in \mathcal{D}_{+}(R)$, then*

$$\sup X \leq \mathsf{Rfd}_R X \leq \sup X + \dim R\,.$$

In particular, $\mathsf{Rfd}_R X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rfd}_R X < \infty$ if (and only if) $X \in \mathcal{D}_{\mathrm{b}}(R)$. $\square$

(2.3) **Proposition.** *For every $\mathfrak{p} \in \operatorname{Spec} R$ and $X \in \mathcal{D}_{+}(R)$ there is an inequality*

$$\mathsf{Rfd}_{R_\mathfrak{p}} X_\mathfrak{p} \leq \mathsf{Rfd}_R X. \quad \square$$

The equation (b) below is the *Ultimate Auslander–Buchsbaum Formula.*

(2.4) **Theorem.** *If $X \in \mathcal{D}_{\mathrm{b}}(R)$, then*

(a) $$\mathsf{Rfd}_R X = \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{F}(R) \ \wedge\ \mathrm{H}(U) \neq 0\}$$

(b) $$\mathsf{Rfd}_R X = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}. \quad \square$$

The large restricted flat dimension is a refinement of the flat dimension, that is,

(2.5) **Theorem.** *For every complex $X \in \mathcal{D}_{+}(R)$ there is an inequality*

$$\mathsf{Rfd}_R X \leq \operatorname{fd}_R X,$$

and equality holds if $\operatorname{fd}_R X < \infty$. $\square$

(2.6) ***Gorenstein flat dimension***. E. Enochs and O. Jenda have in [15] introduced the *Gorenstein flat dimension* $\mathrm{Gfd}_R\, M$ of any R–module M. H. Holm has studied this concept further in [24] and proved that $\mathrm{Gfd}_R\, M$ is a refinement of $\mathrm{fd}_R\, M$ and that $\mathsf{Rfd}_R\, M$ is a refinement of $\mathrm{Gfd}_R\, M$, that is, for any R–module M there is a chain of inequalities

$$\mathsf{Rfd}_R\, M \leq \mathrm{Gfd}_R\, M \leq \mathrm{fd}_R\, M$$

with equality to the left left of any finite number. Thus, in particular, if $\mathrm{Gfd}_R\, M < \infty$ then Theorem (2.4.b) yields the formula

$$\mathrm{Gfd}_R\, M = \sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}\,.$$

(2.7) ***Cohen–Macaulay dimension***. In [23] A. Gerko has defined the *CM–dimension* $\text{CM-dim}_R\, M$ of any finite module M over a local ring R in such a way that the ring is Cohen–Macaulay if and only if $\text{CM-dim}_R\, M < \infty$ for all finite M. Furthermore, this dimension is a refinement of the Auslander G–dimension G-dim_R, and thereby of the projective one pd_R. On the other hand, the next result shows that Rfd_R is a refinement of CM-dim_R.

(2.8) **Theorem.** *If R is local and M is a finite R–module, then* $\mathsf{Rfd}_R\, M \leq \text{CM-dim}_R\, M$ *with equality if* $\text{CM-dim}_R\, M < \infty$.

Proof. It suffices to assume that $\text{CM-dim}_R\, M$ is finite. By [23, Thm. 3.8, Prop. 3.10] we have then $\text{CM-dim}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$ and $\text{CM-dim}_{R_{\mathfrak{p}}}\, M_{\mathfrak{p}} \leq \text{CM-dim}_R\, M$ for all $\mathfrak{p} \in \operatorname{Spec} R$. These results combined with Theorem (2.4.b) yield for a suitable $\mathfrak{p}$ that

$$\begin{aligned}\mathsf{Rfd}_R\, M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{CM-dim}_{R_{\mathfrak{p}}}\, M_{\mathfrak{p}} \\ &\leq \text{CM-dim}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M \leq \mathsf{Rfd}_R\, M\,. \quad \square\end{aligned}$$

When testing flat dimension by non-vanishing of Tor modules, cf. (I.1), it is sufficient to use finite, even cyclic, test modules. It is natural to ask if something similar holds for the large restricted flat dimension. In general the answer is negative, and (3.2) tells us exactly when it is positive. But testing by only finite modules of finite flat dimension gives rise to a new invariant with interesting properties of its own, e.g., see (2.11.b).

(2.9) **Definition.** The *small restricted flat dimension*, $\mathsf{rfd}_R\, X$, of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{rfd}_R\, X = \sup\{\, \sup\,(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{P}^{\mathrm{f}}_0(R)\,\}.$$

(2.10) **Observation.** Let $X \in \mathcal{D}_+(R)$. It is immediate from the definition that

(2.10.1) $$\sup X = \sup\,(R \otimes^{\mathbf{L}}_R X) \leq \mathsf{rfd}_R\, X \leq \mathsf{Rfd}_R\, X \leq \sup X + \dim R,$$

cf. (2.2). In particular, $\mathsf{rfd}_R\, X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{rfd}_R\, X < \infty$ if (and only if) $X \in \mathcal{D}_{\mathrm{b}}(R)$.

By the Ultimate Auslander–Buchsbaum Formula (2.4.b) the large restricted flat dimension is a supremum of differences of *local* depths; the next result shows that the small one is a supremum of differences of *non-local* depths.

(2.11) **Theorem.** *If* $X \in \mathcal{D}_b(R)$, *then there are the next two equalities.*

(a) $\mathsf{rfd}_R X = \sup\{ \sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge \ \mathrm{H}(U) \neq 0 \}$

(b) $\mathsf{rfd}_R X = \sup\{ \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R \}$.

Proof. As it is immediate from the definition that $\mathsf{rfd}_R X$ is less than or equal to the first supremum, it suffices to prove the next two inequalities

$$\sup\{ U \otimes^{\mathbf{L}}_R X - \sup U \} \leq \sup\{ \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \} \leq \mathsf{rfd}_R X$$

where $U \in \mathcal{P}^{\mathrm{f}}(R)$, $\mathrm{H}(U) \neq 0$, and $\mathfrak{p} \in \operatorname{Spec} R$.

First, let $U \in \mathcal{P}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$ be given; we then want to prove the existence of a prime ideal $\mathfrak{p}$ such that

$$(*) \qquad \sup(U \otimes^{\mathbf{L}}_R X) - \sup U \leq \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X).$$

We can assume that $\mathrm{H}(U \otimes^{\mathbf{L}}_R X) \neq 0$, otherwise $(*)$ holds for every $\mathfrak{p}$. Set $s = \sup(U \otimes^{\mathbf{L}}_R X)$, choose $\mathfrak{p}$ in $\operatorname{Ass}_R(\mathrm{H}_s(U \otimes^{\mathbf{L}}_R X))$, and choose by (1.1.2) a prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$, such that $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_{\mathfrak{q}}$. The first equality in the computation below follows by [18, Lem. 2.1], the second by (1.6.a) and [6, Cor. 2.10.F and Prop. 5.5], and the third by (1.5); the last inequality is by (1.1.3) and [18, Lem. 2.1].

$$\begin{aligned}
\sup(U \otimes^{\mathbf{L}}_R X) - \sup U &= -\operatorname{depth}_{R_{\mathfrak{p}}}(U_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}) - \sup U \\
&= \operatorname{pd}_{R_{\mathfrak{p}}} U_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U \\
&\leq \operatorname{pd}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U \\
&= \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \sup U \\
&= \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{q}}} U_{\mathfrak{q}} - \sup U \\
&\leq \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X).
\end{aligned}$$

Second, let $\mathfrak{p} \in \operatorname{Spec} R$ be given, and the task is to find a finite module T of finite projective dimension with

$$\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \leq \sup(T \otimes^{\mathbf{L}}_R X).$$

Set $d = \operatorname{depth}_R(\mathfrak{p}, R)$, choose a maximal R–sequence $\boldsymbol{x} = x_1, \ldots, x_d$ in $\mathfrak{p}$, and set $T = R/(\boldsymbol{x})$. Then T belongs to $\mathcal{P}^{\mathrm{f}}_0(R)$ and the Koszul complex $\mathrm{K}(\boldsymbol{x})$ is its minimal free resolution. By (1.1.1) and (1.1.4) we now have the desired

$$\begin{aligned}
\sup(T \otimes^{\mathbf{L}}_R X) = \sup(\mathrm{K}(\boldsymbol{x}) \otimes_R X) &= d - \operatorname{depth}_R((\boldsymbol{x}), X) \\
&\geq d - \operatorname{depth}_R(\mathfrak{p}, X) = \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \quad \square
\end{aligned}$$

(2.12) **Observation.** Let $X \in \mathcal{D}_b(R)$. It follows from (2.11.b) that

$$\sup\{ \operatorname{depth} R_{\mathfrak{m}} - \operatorname{depth}_{R_{\mathfrak{m}}} X_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max} R \} \leq \mathsf{rfd}_R X;$$

and in view of (1.1.2) and (1.1.4) we also have

$$\mathsf{rfd}_R X \leq \sup\{ \operatorname{depth} R_{\mathfrak{m}} \mid \mathfrak{m} \in \operatorname{Max}_R X \} + \sup X.$$

In particular: if R is local, then

$$(2.12.1) \qquad \operatorname{depth} R - \operatorname{depth}_R X \leq \mathsf{rfd}_R X \leq \operatorname{depth} R + \sup X.$$

The example below shows that the two restricted flat dimensions may differ, even for finite modules over local rings, and it shows that the small restricted flat dimension can grow under localization. The latter, unfortunate, property is reflected in the non-local nature of the formula given in (2.11.b).

(2.13) **Example.** Let R be a local ring with $\dim R = 2$ and $\operatorname{depth} R = 0$. By (1.4) choose $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{q} = 1$, choose $x \in \mathfrak{q}$ such that the fraction ${}^{x}/_{1}$ is $R_\mathfrak{q}$–regular, and set $M = R/(x)$. It follows by (2.12.1) that $\mathsf{rfd}_R\, M = 0$, but

$$\mathsf{Rfd}_R\, M \geq \mathsf{Rfd}_{R_\mathfrak{q}}\, M_\mathfrak{q} \geq \mathsf{rfd}_{R_\mathfrak{q}}\, M_\mathfrak{q} \geq \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} = 1 - 0 > \mathsf{rfd}_R\, M$$

by (2.3), (2.10.1), and (2.12.1).

The ring considered above is of Cohen–Macaulay defect two, so in a sense — to be made clear by (3.2) — the example is a minimal one.

(2.14) **Remark.** It is straightforward to see that the restricted flat dimensions of modules can be described in terms of resolutions: If M is any R–module, then $\mathsf{Rfd}_R\, M$ [respectively, $\mathsf{rfd}_R\, M$] is less than or equal to a non-negative integer g if and only if there is an exact sequence of modules

$$0 \to T_g \to \cdots \to T_n \to \cdots \to T_0 \to M \to 0$$

such that $\operatorname{Tor}_i^R(T, T_n) = 0$ for all $i > 0$, all $T \in \mathcal{F}_0(R)$ [respectively, all $T \in \mathcal{P}_0^{\mathrm{f}}(R)$], and all n, $0 \leq n \leq g$.

3. Almost Cohen–Macaulay Rings

In this section we characterize the rings over which the small and large restricted flat dimensions agree for for all complexes. It is evident from (2.4.b), (2.11.b), and (1.1.2) that the two dimensions will agree if $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, and in (3.2) we show that this condition is also necessary. These rings are called *almost Cohen–Macaulay,* and Lemma (3.1) explains why.

Next we consider the question: when do the restricted flat dimensions satisfy an Auslander–Buchsbaum equality? The answer, provided by (3.4), is that it happens if and only if the ring is Cohen–Macaulay.

(3.1) **Lemma.** *The following are equivalent.*

(i) $\operatorname{cmd} R \leq 1$.

(ii) $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.

(iii) $\mathfrak{p} \in \operatorname{Ass}_R \big(R/(\boldsymbol{x})\big)$ *whenever* $\mathfrak{p} \in \operatorname{Spec} R$ *and* $\boldsymbol{x}$ *is a maximal R–sequence in* $\mathfrak{p}$.

(iv) *For every* $\mathfrak{p} \in \operatorname{Spec} R$ *there exists* $M \in \mathcal{P}_0^{\mathrm{f}}(R)$ *with* $\mathfrak{p} \in \operatorname{Ass}_R M$.

(v) *For every* $\mathfrak{p} \in \operatorname{Spec} R$ *there exists* $X \in \mathcal{P}^{\mathrm{f}}(R)$ *with* $\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$.

Proof. Conditions (i) through (iii) are the equivalent conditions (3), (4), and (5) in [17, Prop. 3.3], and the implications (iii) ⇒ (iv) and (iv) ⇒ (v) are obvious.

(v) ⇒ (i): It suffices to assume that R is local. If $\dim R = 0$ there is nothing to prove, so we assume that $\dim R > 0$ and choose by (1.4) a prime ideal $\mathfrak{p}$, such that $\operatorname{depth} R_\mathfrak{p} = \dim R - 1$. For $X \in \mathcal{P}^{\mathrm{f}}(R)$ with $\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$ (1.5) and [18, Lem. 2.1] yield

$$\begin{aligned}\operatorname{depth} R = \operatorname{pd}_R X + \operatorname{depth}_R X &\geq \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup X \\ = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup X &= \dim R - 1\,. \quad \square\end{aligned}$$

(3.2) **Theorem.** *If* R *is local, then the following are equivalent.*

(i) $\operatorname{cmd} R \leq 1$.

(ii) $\mathsf{rfd}_R X = \mathsf{Rfd}_R X$ *for all complexes* $X \in \mathcal{D}_+(R)$.

(iii) $\mathsf{rfd}_R M = \mathsf{Rfd}_R M$ *for all finite* R*–modules* M.

Proof. The second condition is, clearly, stronger than the third, so there are two implications to prove.

(i) $\Rightarrow$ (ii): If X is not bounded, then $\infty = \mathsf{rfd}_R X = \mathsf{Rfd}_R X$, cf. (2.10.1). Suppose $X \in \mathcal{D}_\mathrm{b}(R)$; the (in)equalities in the next computation follow by, respectively, (2.11.b), (3.1), (1.1.2), and (2.4.b).

$$\begin{aligned}\mathsf{rfd}_R X &= \sup\{\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &\geq \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \mathsf{Rfd}_R X.\end{aligned}$$

The opposite inequality always holds, cf. (2.10.1), whence equality holds.

(iii) $\Rightarrow$ (i): We can assume that $\dim R > 0$ and choose a prime ideal $\mathfrak{q}$ such that $\operatorname{depth} R_\mathfrak{q} = \dim R - 1$, cf. (1.4). Set $M = R/\mathfrak{q}$ and apply (2.12.1) and (2.4.b) to get

$$\operatorname{depth} R \geq \mathsf{rfd}_R M = \mathsf{Rfd}_R M \geq \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} = \dim R - 1\,. \quad \square$$

Over almost Cohen–Macaulay rings it is, actually, sufficient to use special cyclic modules for testing the restricted flat dimensions:

(3.3) **Corollary.** *If* $\operatorname{cmd} R \leq 1$ *and* $X \in \mathcal{D}_+(R)$, *then*

$$\mathsf{rfd}_R X = \mathsf{Rfd}_R X = \sup\{\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\}.$$

Proof. If X is not bounded, then

$$\mathsf{rfd}_R X = \mathsf{Rfd}_R X = \infty = \sup(R \otimes^{\mathbf{L}}_R X).$$

For $X \in \mathcal{D}_\mathrm{b}(R)$ we have $\mathsf{rfd}_R X = \mathsf{Rfd}_R X$ by (3.2). The definition yields

$$\sup\{\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\} \leq \mathsf{rfd}_R X.$$

By (2.4.b) it is sufficient for each $\mathfrak{p} \in \operatorname{Spec} R$ to find an R–sequence $\boldsymbol{x}$ such that

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \leq \sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X).$$

This is easy: let $\boldsymbol{x}$ be any maximal R–sequence in $\mathfrak{p}$, then, by (3.1), $\mathfrak{p}$ is associated to $R/(\boldsymbol{x})$, in particular, $\operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}))_\mathfrak{p} = 0$, so by [18, Lem. 2.1] and (1.6.c) we have

$$\begin{aligned}\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) &\geq -\operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X)_\mathfrak{p} \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}}(R/(\boldsymbol{x}))_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \\ &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}. \quad \square\end{aligned}$$

For finite modules the large restricted flat dimension is a refinement of the projective dimension, and over a local ring we, therefore, have $\mathsf{Rfd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$ for $M \in \mathcal{P}_0^{\mathrm{f}}(R)$. Now we ask when such a formula holds for all finite modules:

(3.4) **Theorem.** *If R is local, then the following are equivalent.*

(*i*) *R is Cohen–Macaulay.*

(*ii*) $\mathsf{Rfd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X$ *for all complexes* $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.

(*iii*) $\mathsf{rfd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$ *for all finite R–modules M.*

Proof. (*i*) $\Rightarrow$ (*ii*): Let $\mathfrak{p}$ be a prime ideal. It follows by a complex version of [10, Lem. (3.1)], cf. [16, Chp. 13], that $\operatorname{depth}_R X \le \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \dim R/\mathfrak{p}$. Thus we get the first inequality in the next chain.

$$\begin{aligned} \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} &\le \operatorname{depth} R_\mathfrak{p} - (\operatorname{depth}_R X - \dim R/\mathfrak{p}) \\ &\le \dim R_\mathfrak{p} + \dim R/\mathfrak{p} - \operatorname{depth}_R X \\ &\le \dim R - \operatorname{depth}_R X \\ &= \operatorname{depth} R - \operatorname{depth}_R X. \end{aligned}$$

The desired equality now follows by (2.4.b).

(*ii*) $\Rightarrow$ (*iii*): Immediate as

$$\operatorname{depth} R - \operatorname{depth}_R M \le \mathsf{rfd}_R\, M \le \mathsf{Rfd}_R\, M$$

by (2.12.1) and (2.10.1).

(*iii*) $\Rightarrow$ (*i*): We assume that R is not Cohen–Macaulay and seek a contradiction. Set $d = \operatorname{depth} R$ and let $\boldsymbol{x} = x_1, \ldots, x_d$ be a maximal R–sequence. Since R is not Cohen–Macaulay, the ideal generated by the sequence is not $\mathfrak{m}$–primary; that is, there exists a prime ideal $\mathfrak{p}$ such that $(\boldsymbol{x}) \subseteq \mathfrak{p} \subset \mathfrak{m}$. Set $M = R/\mathfrak{p}$, then $\operatorname{depth}_R M > 0$, but $\operatorname{depth}_R(\mathfrak{p}, M) = 0$ and $\operatorname{depth}_R(\mathfrak{p}, R) = d$, so by (2.11.b) we have

$$\operatorname{depth} R - \operatorname{depth}_R M < d = \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, M) \le \mathsf{rfd}_R\, M,$$

and the desired contradiction has been obtained. □

(3.5) **Corollary.** *If R is a Cohen–Macaulay local ring and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\mathsf{rfd}_R\, X = \mathsf{Rfd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X.$$

Proof. Immediate by (3.4) and (3.2). □

4. Width of Complexes

To study dual notions of the restricted flat dimensions we need to learn more about width of complexes; in particular, we need a non-local concept of width. Inspired by Iyengar's [25] approach to depth, we will introduce the non-local width by way of Koszul complexes. When $\boldsymbol{x}$ is a sequence of elements in R the Koszul complex $\mathrm{K}(\boldsymbol{x})$ consists of free modules, so the functors $-\otimes_R \mathrm{K}(\boldsymbol{x})$ and $-\otimes_R^{\mathbf{L}} \mathrm{K}(\boldsymbol{x})$ are naturally isomorphic, and we will not distinguish between them. The next result follows from the discussion [25, 1.1–1.3].

(4.1) **Lemma.** *Let $X \in \mathcal{D}(R)$ and let $\mathfrak{a}$ be an ideal in R. If $\boldsymbol{a} = a_1, \dots, a_t$ and $\boldsymbol{x} = x_1, \dots, x_u$ are two finite sequences of generators for $\mathfrak{a}$, then*

$$\inf (X \otimes_R \mathrm{K}(\boldsymbol{a})) = \inf (X \otimes_R \mathrm{K}(\boldsymbol{x})). \quad \square$$

The lemma shows that the next definition of a non-local width makes sense.

(4.2) **Definition.** Let $\mathfrak{a}$ be an ideal in R, and let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite sequence of generators for $\mathfrak{a}$. For $X \in \mathcal{D}(R)$ we define the $\mathfrak{a}$–*width*, $\mathrm{width}_R(\mathfrak{a}, X)$, as

$$\mathrm{width}_R(\mathfrak{a}, X) = \inf (X \otimes_R \mathrm{K}(\boldsymbol{a})).$$

(4.3) **Observation.** A Koszul complex has infimum at least zero, so by [18, Lem. 2.1] there is always an inequality

$$\mathrm{width}_R(\mathfrak{a}, X) \geq \inf X. \tag{4.3.1}$$

If $\boldsymbol{a} = a_1, \dots, a_t$ generates a proper ideal $\mathfrak{a}$ in R, then $\inf \mathrm{K}(\boldsymbol{a}) = 0$ as $\mathrm{H}_0(\mathrm{K}(\boldsymbol{a})) \cong R/\mathfrak{a}$. For $X \in \mathcal{D}_+^{\mathrm{f}}(R)$ with $\mathrm{H}(X) \neq 0$ it then follows by [18, Lem. 2.1] that equality holds in (4.3.1) if and only if $\mathrm{Supp}_R(\mathrm{H}_{\inf X}(X)) \cap \mathrm{V}(\mathfrak{a}) \neq \emptyset$. In particular: if R is local, then

$$\mathrm{width}_R(\mathfrak{a}, X) = \inf X \tag{4.3.2}$$

for all $X \in \mathcal{D}_+^{\mathrm{f}}(R)$ and all proper ideals $\mathfrak{a}$.

(4.4) **Observation.** Let $X \in \mathcal{D}(R)$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R and $\mathfrak{b} \supseteq \mathfrak{a}$, then it follows easily by the definition that $\mathrm{width}_R(\mathfrak{b}, X) \geq \mathrm{width}_R(\mathfrak{a}, X)$.

(4.5) ***Matlis Duality***. If E be a faithfully injective R–module and $X \in \mathcal{D}(R)$ then

$$\sup (\mathbf{R}\mathrm{Hom}_R(X, E)) = -\inf (X) \quad \text{and} \quad \inf (\mathbf{R}\mathrm{Hom}_R(X, E)) = -\sup (X)\,.$$

Every ring R admits a faithfully injective module E, e.g., $E = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. For any faithfully injective module E we use the notation $-^{\vee} = \mathbf{R}\mathrm{Hom}_R(-, E)$. If R is local, then $-^{\vee} = \mathrm{Hom}_R(-, \mathrm{E}_R(R/\mathfrak{m}))$ is known as the *Matlis duality functor*. Here we tacitly use that $\mathrm{Hom}_R(-, E)$ and $\mathbf{R}\mathrm{Hom}_R(-, E)$ are naturally isomorphic in $\mathcal{D}(R)$ and we do not distinguish between them.

The next results will spell out the expected relations between $\mathfrak{a}$–width and width over local rings as well as the behavior of $\mathfrak{a}$-width and $\mathfrak{a}$-depth under duality with respect to faithfully injective modules. But first we establish a useful lemma and a remark

(4.6) **Lemma.** *If $\boldsymbol{a} = a_1, \dots, a_t$ generates the ideal $\mathfrak{a}$ in R and $X \in \mathcal{D}(R)$ then*

$$\mathrm{width}_R(\mathfrak{a}, X) = t + \inf (\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)).$$

Proof. When Σ denotes the shift functor we have $X \otimes_R \mathrm{K}(\boldsymbol{a}) \cong \Sigma^t \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)$, and the desired equality follows

$$\mathrm{width}_R(\mathfrak{a}, X) = \inf (\Sigma^t \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)) = t + \inf (\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)). \quad \square$$

(4.7) **Remark.** For $Z \in \mathcal{D}_+^{\mathrm{f}}(R)$ and $W \in \mathcal{D}(R)$ we have the following special case of the Hom-evaluation morphism [1, Thm. 1, p. 27]

$$Z \otimes_R^{\mathbf{L}} W^{\vee} \simeq \mathbf{R}\mathrm{Hom}_R(Z, W)^{\vee}$$

provided either $\operatorname{pd}_R Z < \infty$ or $W \in \mathcal{D}_-(R)$.

(4.8) **Proposition.** *If $X \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, then*

$$\operatorname{depth}_R(\mathfrak{a}, X^\vee) = \operatorname{width}_R(\mathfrak{a}, X) \quad \text{and} \quad \operatorname{width}_R(\mathfrak{a}, Y^\vee) = \operatorname{depth}_R(\mathfrak{a}, Y)\,.$$

Proof. By (1.1.1) and (4.6), the equalities follow from (4.7) and adjointness, respectively. □

(4.9) **Proposition.** *If M is a finite R–module with support $\mathrm{V}(\mathfrak{a})$ and $X \in \mathcal{D}_+(R)$, then*

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf\,(X \otimes^{\mathbf{L}}_R R/\mathfrak{a}) = \inf\,(X \otimes^{\mathbf{L}}_R M)\,.$$

Proof. By [14, Prop. 4.5] we obtain

$$\operatorname{depth}_R(\mathfrak{a}, X^\vee) = -\sup\,(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X^\vee)) = -\sup\,(\mathbf{R}\mathrm{Hom}_R(M, X^\vee))\,.$$

Hence (4.8), the adjointness isomorphism, and (4.5) yield the desired assertions. □

(4.10) **Corollary.** *If $\mathfrak{q} \subseteq \mathfrak{p}$ are prime ideals in R and $X \in \mathcal{D}_+(R)$, then*

$$\operatorname{width}_R(\mathfrak{q}, X) \le \operatorname{width}_{R_\mathfrak{p}}(\mathfrak{q}_\mathfrak{p}, X_\mathfrak{p})\,. \quad \square$$

(4.11) **Corollary.** *If R is local and $X \in \mathcal{D}_+(R)$, then* $\operatorname{width}_R(\mathfrak{m}, X) = \operatorname{width}_R X$. □

(4.12) **Corollary.** *For $\mathfrak{p} \in \operatorname{Spec} R$ and $X \in \mathcal{D}_+(R)$ there is an inequality*

$$\operatorname{width}_R(\mathfrak{p}, X) \le \operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p}. \quad \square$$

Finally, we want to dualize (1.6). The first result in this direction is [31, Lem. 2.6] which is stated just below (cf. also [1, Thm. 1(2), p. 27]).

(4.13) **Theorem.** *If R is local, $X \in \mathcal{D}_-(R)$, and $Y \in \mathcal{I}(R)$, then the following hold.*

(a) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \inf\,(\mathbf{R}\mathrm{Hom}_R(k, Y)) + \operatorname{depth}_R X.$$

(b) $$\operatorname{width}_R Y = \inf\,(\mathbf{R}\mathrm{Hom}_R(k, Y)) + \operatorname{depth} R.$$

(c) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \operatorname{width}_R Y + \operatorname{depth}_R X - \operatorname{depth} R. \quad \square$$

If the hypothesis $(X, Y) \in \mathcal{D}_-(R) \times \mathcal{I}(R)$ above is replaced by the hypothesis $(X, Y) \in \mathcal{P}(R) \times \mathcal{D}_+(R)$ we get a corresponding result which is stated below. (By the Auslander–Buchsbaum formula (1.5) part (b) below can be written exactly like (c) above.)

(4.14) **Theorem.** *Let R be local and $Y \in \mathcal{D}_+(R)$. If $X \in \mathcal{P}(R)$, then*

(a) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \operatorname{width}_R Y - \sup\,(X \otimes^{\mathbf{L}}_R k).$$

In particular: if $X \in \mathcal{P}^{\mathrm{f}}(R)$, then

(b) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \operatorname{width}_R Y - \operatorname{pd}_R X.$$

Proof. Let P be a bounded projective resolution of X, and let K be the Koszul complex on a sequence of generators for the maximal ideal $\mathfrak{m}$. It is straightforward to check that

$$(*) \qquad \mathrm{Hom}_R(P,Y)\otimes_R K \cong \mathrm{Hom}_R(P, Y\otimes_R K);$$

it uses that K is a bounded complex of finite free modules, and that tensoring by finite modules commutes with direct products. We can assume that $Y_\ell = 0$ for $\ell \ll 0$, the same then holds for $Y\otimes_R K$ and $\mathrm{Hom}_R(P, Y\otimes_R K)$. By $(*)$ the spectral sequence corresponding to the double complex $\mathrm{Hom}_R(P, Y\otimes_R K)$ converges to $\mathrm{H}(\mathrm{Hom}_R(P,Y)\otimes_R K)$. Filtrating by columns, cf. [30, Def. 5.6.1] we may write

$$E^2_{pq} = \mathrm{H}_p(\mathrm{Hom}_R(P, \mathrm{H}_q(Y\otimes_R K))),$$

as P is a complex of projectives. Evoking the fact that the homology module $\mathrm{H}_q(Y\otimes_R K)$ is a vector space over k, we can compute E^2_{pq} as follows

$$(**) \qquad \begin{aligned} E^2_{pq} &= \mathrm{H}_p(\mathrm{Hom}_R(P, \mathrm{Hom}_k(k, \mathrm{H}_q(Y\otimes_R K)))) \\ &= \mathrm{H}_p(\mathrm{Hom}_k(P\otimes_R k, \mathrm{H}_q(Y\otimes_R K))) \\ &= \mathrm{Hom}_k(\mathrm{H}_{-p}(P\otimes_R k), \mathrm{H}_q(Y\otimes_R K)). \end{aligned}$$

In $\mathcal{D}(R)$ there are isomorphisms

$$(***) \qquad P\otimes_R k \simeq X\otimes^{\mathbf{L}}_R k \quad \text{and} \quad \mathrm{Hom}_R(P,Y)\otimes_R K \simeq \mathbf{R}\mathrm{Hom}_R(X,Y)\otimes_R K.$$

If $\mathrm{H}(P\otimes_R k) = 0$ or $\mathrm{H}(Y\otimes_R K) = 0$ (i.e., $\sup(X\otimes^{\mathbf{L}}_R k) = -\infty$ or $\mathrm{width}_R Y = \infty$), then $E^2_{pq} = 0$ for all p and q, so also $\mathrm{H}(\mathrm{Hom}_R(P,Y)\otimes_R K)$ vanishes making $\mathrm{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \infty$. Otherwise, it is easy to see from $(**)$ that

$$\begin{aligned} &E^2_{pq} = 0 \quad \text{for } -p > \sup(P\otimes_R k) \quad \text{or} \quad q < \inf(Y\otimes_R K); \quad \text{and} \\ &E^2_{pq} \neq 0 \quad \text{for } -p = \sup(P\otimes_R k) \quad \text{and} \quad q = \inf(Y\otimes_R K). \end{aligned}$$

A standard "corner" argument now shows that

$$\inf(\mathrm{Hom}_R(P,Y)\otimes_R K) = \inf(Y\otimes_R K) - \sup(P\otimes_R k);$$

and by $(***)$, (4.2), and (4.12) this is the desired equality (a).

Part (b) follows from (a) in view of [6, Cor. 2.10.F and Prop. 5.5]. □

The difference between the proofs of the last two theorems is, basically, that between k and K: In [31] Yassemi uses the k–structure of the complex $\mathbf{R}\mathrm{Hom}_R(k,X)$, and to get in a position to do so he needs the power of evaluation-morphisms, cf. [6, Lem. 4.4]. The same procedure could be applied to produce (4.14.b), but not part (a). The proof of (4.14) uses only the k–structure of the homology modules $\mathrm{H}_\ell(Y\otimes_R K)$, where K is the Koszul complex on a generating sequence for $\mathfrak{m}$, and the simple structure of the Koszul complex allows us to avoid evaluation-morphisms, cf. $(*)$ in the proof.

5. Ext–dimensions

Restricted injective dimensions are analogues to the restricted flat ones. When it comes to generalizing Bass' formula [10, Lem. (3.3)] the small restricted injective dimension is the more interesting, and we start with that one.

Furthermore, we study restricted projective dimensions, and for all four Ext–dimensions we examine their ability to detect (almost) Cohen–Macaulayness of the underlying ring.

(5.1) **Definition.** The *small restricted injective dimension*, $\mathsf{rid}_R\, Y$, of $Y \in \mathcal{D}_-(R)$ is

$$\mathsf{rid}_R\, Y = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) \mid T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}.$$

For an R–module N the definition reads

$$\mathsf{rid}_R\, N = \sup\{\, m \in \mathbb{N}_0 \mid \ \mathrm{Ext}_R^m(T,N) \neq 0 \text{ for some } T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}.$$

(5.2) **Observation.** Let $Y \in \mathcal{D}_-(R)$. It is immediate from the definition that

$$-\inf Y = -\inf(\mathbf{R}\mathrm{Hom}_R(R,Y)) \leq \mathsf{rid}_R\, Y;$$

and for $T \in \mathcal{P}_0(R)$ we have

$$-\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) \leq -\inf Y + \mathrm{pd}_R\, T,$$

cf. [13, (1.4.3)], so by (1.3.1) there are always inequalities

(5.2.1) $$-\inf Y \leq \mathsf{rid}_R\, Y \leq -\inf Y + \dim R.$$

In particular, $\mathsf{rid}_R\, Y > -\infty$ if (and only if) $\mathrm{H}(Y) \neq 0$; and if $\dim R$ is finite, then $\mathsf{rid}_R\, Y < \infty$ if (and only if) $Y \in \mathcal{D}_\mathrm{b}(R)$.

The next two results are parallel to (2.11.b) and (3.3); the key is the duality expressed by the first equation in (5.3), and it essentially hinges on (4.7).

(5.3) **Proposition.** *If* $Y \in \mathcal{D}_\mathrm{b}(R)$, *then*

(a) $\mathsf{rid}_R\, Y = \mathsf{rfd}_R\, Y^\vee$

(b) $\mathsf{rid}_R\, Y = \sup\{\, -\sup U - \inf(\mathbf{R}\mathrm{Hom}_R(U,Y)) \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge \ \mathrm{H}(U) \neq 0\,\}$

(c) $\mathsf{rid}_R\, Y = \sup\{\, \mathrm{depth}_R(\mathfrak{p},R) - \mathrm{width}_R(\mathfrak{p},Y) \mid \mathfrak{p} \in \mathrm{Spec}\, R\,\}$.

Proof. It follows by (4.7) that

$$\sup(T \otimes_R^{\mathbf{L}} Y^\vee) = \sup(\mathbf{R}\mathrm{Hom}_R(T,Y)^\vee) = -\inf(\mathbf{R}\mathrm{Hom}_R(T,Y))$$

for $T \in \mathcal{P}_0^{\mathrm{f}}(R)$, and (a) is proved. Now, (b) follows by (2.11.a), and (c) is a consequence of (4.8) and (2.11.b). □

(5.4) **Proposition.** *If* $\mathrm{cmd}\, R \leq 1$ *and* $Y \in \mathcal{D}_\mathrm{b}(R)$, *then*

$$\mathsf{rid}_R\, Y = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(R/(\boldsymbol{x}),Y)) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\,\}.$$

Proof. Use (4.7), (5.3.a), and (3.3). □

(5.5) **Corollary.** *If* R *is local,* $Y \in \mathcal{D}_-^{\mathrm{f}}(R)$, *and* $N \neq 0$ *is an* R*–modules, then*

(a) $$\mathsf{rid}_R\, Y = \mathrm{depth}\, R - \inf Y$$

(b) $$\mathsf{rid}_R\, N = \mathrm{depth}\, R\,.$$

Proof. Use also (5.3.c), (4.3.2), and (1.1.4). □

(5.6) **Corollary.** *If* $\mathrm{cmd}\, R \leq 1$ *then* $\mathsf{rid}_{R_\mathfrak{p}}\, Y_\mathfrak{p} \leq \mathsf{rid}_R\, Y$ *for* $Y \in \mathcal{D}_-(R)$ *and* $\mathfrak{p} \in \mathrm{Spec}\, R$.

Proof. Apply also (3.1) and (4.10). □

(5.7) **Remark.** Just like the small restricted flat dimension, the small restricted injective dimension can grow under localization: Let R and $\mathfrak{q}$ be as in (2.13); for every finite R–module N with $\mathfrak{q} \in \operatorname{Supp}_R N$ (e.g., $N = R/\mathfrak{q}$) we then have

$$\mathsf{rid}_{R_\mathfrak{q}} N_\mathfrak{q} = \operatorname{depth} R_\mathfrak{q} = 1 > 0 = \operatorname{depth} R = \mathsf{rid}_R N.$$

It is well-known that a local ring must be Cohen–Macaulay in order to allow a non-zero finite module of finite injective dimension (this is the Bass conjecture proved by Peskine and Szpiro [27] and Roberts [29]), so it follows by the next result that the original Bass formula (I.4) is contained in (5.5.b).

(5.8) **Proposition.** *For every complex $Y \in \mathcal{D}_-(R)$ there is an inequality*

$$\mathsf{rid}_R Y \leq \operatorname{id}_R Y,$$

and equality holds if $\operatorname{id}_R Y < \infty$ *and* $\operatorname{cmd} R \leq 1$.

Proof. Since the inequality is immediate and equality holds if $Y \simeq 0$, we assume that $\operatorname{id}_R Y = n \in \mathbb{Z}$. By [6, Prop. 5.3.I] there exists then $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), Y_\mathfrak{p}))$ is non-trivial. Choose next, by (3.1), a module $T \in \mathcal{P}_0^{\mathrm{f}}(R)$ with $\mathfrak{p} \in \operatorname{Ass}_R T$. The short exact sequence

$$0 \to R/\mathfrak{p} \to T \to C \to 0,$$

induces a long exact homology sequence which shows that $\mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(T, Y)) \neq 0$. Thus

$$\mathsf{rid}_R Y \geq -\inf(\mathbf{R}\mathrm{Hom}_R(T, Y)) \geq n\,. \quad \square$$

(5.9) **Corollary.** *For a local ring R the next three conditions are equivalent.*

(*i*) $\operatorname{cmd} R \leq 1$.
(*ii*) $\mathsf{rid}_R Y = \operatorname{id}_R Y$ *for all complexes* $Y \in \mathcal{I}(R)$.
(*iii*) $\mathsf{rid}_R M = \operatorname{id}_R M$ *for all R–modules of finite injective dimension.*

Proof. By the proposition, (*i*) implies (*ii*) which is stronger than (*iii*). To see that (*iii*) implies (*i*) we may assume that $\dim R > 0$. Choose an R–module M with $\operatorname{id}_R M = \dim R - 1$ and a finite R–module T of finite projective dimension such that $\mathsf{rid}_R M = -\inf(\mathbf{R}\mathrm{Hom}_R(T, M))$. Now [6, Thm. 2.4.P] and (1.5) yield

$$\dim R - 1 = -\inf(\mathbf{R}\mathrm{Hom}_R(T, M)) \leq \operatorname{pd}_R T \leq \operatorname{depth} R\,. \quad \square$$

(5.10) **Definition.** The *large restricted injective dimension*, $\mathsf{Rid}_R Y$, of $Y \in \mathcal{D}_-(R)$ is

$$\mathsf{Rid}_R Y = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(T, Y)) \mid T \in \mathcal{P}_0(R)\,\}.$$

For an R–module M the definition reads

(5.10.1) $$\mathsf{Rid}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(T, M) \neq 0 \text{ for some } T \in \mathcal{P}_0(R)\,\}.$$

(5.11) **Observation.** Let $Y \in \mathcal{D}_-(R)$. As in (5.2) we see that there are inequalities

(5.11.1) $$-\inf Y \leq \mathsf{rid}_R Y \leq \mathsf{Rid}_R Y \leq -\inf Y + \dim R.$$

In particular, $\mathsf{Rid}_R Y > -\infty$ if (and only if) $\mathrm{H}(Y) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rid}_R Y < \infty$ if (and only if) $Y \in \mathcal{D}_\mathrm{b}(R)$.

(5.12) **Remark.** Below we show that the large injective dimension is a refinement of the injective dimension, at least over almost Cohen–Macaulay rings. For a complex Y of finite injective dimension over such a ring we, therefore, have

$$\mathsf{Rid}_R\, Y = \sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\},$$

by the extension to complexes [31, Thm. 2.10] of Chouinard's formula (I.3). It is, however, easy to see that this formula fails in general. Let R be local and not Cohen–Macaulay, and let T be a module with $\operatorname{pd}_R T = \dim R$, cf. [9, Prop. 5.4]. By [6, 2.4.P] there is then an R–module N with $-\inf(\mathbf{R}\mathrm{Hom}_R(T,N)) = \dim R$, so $\mathsf{Rid}_R\, N \geq \dim R$ but by (4.3.1) and (1.4) we have

$$\sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\} \leq \sup\{\, \operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\} = \dim R - 1\,.$$

(5.13) **Proposition.** *For every complex $Y \in \mathcal{D}_-(R)$ there is an inequality*

$$\mathsf{Rid}_R\, Y \leq \operatorname{id}_R Y,$$

and equality holds if $\operatorname{id}_R Y < \infty$ *and* $\operatorname{cmd} R \leq 1$.

Proof. The inequality is immediate; apply (5.8) and (5.11.1) to complete the proof. □

(5.14) **Definition.** The *large restricted projective dimension*, $\mathsf{Rpd}_R\, X$ of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{Rpd}_R\, X = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{I}_0(R)\,\}.$$

For an R–module M the definition reads

$$\mathsf{Rpd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{I}_0(R)\,\}.$$

(5.15) **Observation.** For $X \in \mathcal{D}_+(R)$ there are inequalities

(5.15.1) $$\sup X \leq \mathsf{Rpd}_R\, X \leq \sup X + \dim R$$

by (4.5), cf. [13, (1.4.2)], and (1.4.1). In particular, $\mathsf{Rpd}_R\, X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rpd}_R\, X < \infty$ if (and only if) $X \in \mathcal{D}_\mathrm{b}(R)$.

(5.16) **Lemma.** *If $X \in \mathcal{D}_\mathrm{b}(R)$ then $\mathsf{Rfd}_R\, X \leq \mathsf{Rpd}_R\, X$ with equality when $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$.*

Proof. For an R–module T of finite flat dimension, the Matlis dual $T^\vee$ is a module of finite injective dimension. By adjointness $\sup(T \otimes_R^{\mathbf{L}} X)$ equals $-\inf(\mathbf{R}\mathrm{Hom}_R(X,T^\vee))$ and (a) follows.

To show (b) let next T denote an R–module of finite injective dimension. Then $T^\vee$ is a module of finite flat dimension, and the desired equality follows since $\inf(\mathbf{R}\mathrm{Hom}_R(X,T))$ equals $-\sup(X \otimes_R^{\mathbf{L}} T^\vee)$ by (4.7). □

(5.17) **Observation.** The inequality $\mathsf{Rpd}_R\, X \leq \operatorname{pd}_R X$ holds for every $X \in \mathcal{D}_+(R)$. If R is local and $X \in \mathcal{P}^\mathrm{f}(R)$, then (2.5), [13, (1.4.4)], and (5.16) yield

$$\operatorname{pd}_R X = \operatorname{fd}_R X = \mathsf{Rfd}_R\, X = \mathsf{Rpd}_R\, X\,.$$

(5.18) **Remark.** One can prove that Rpd_R is a refinement of pd_R when R is a Cohen–Macaulay local ring with a dualizing module. If R is not Cohen–Macaulay, then $\mathsf{Rpd}_R\, M \le \dim R - 1$ for every R–module M by (1.4.1), but there exists an R–module M with $\mathrm{pd}_R\, M = \dim R$ (by [9, Prop. 5.4]).

A small restricted projective dimension based on the expression

$$\sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{I}_0^{\mathrm{f}}(R)\,\}$$

would be trivial over non-Cohen–Macaulay rings as they do not allow non-zero finite modules of finite injective dimension. Inspired by (2.11.a) and (5.3.b) we instead make the following:

(5.19) **Definition.** The *small restricted projective dimension* $\mathsf{rpd}_R\, X$ of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{rpd}_R\, X = \sup\{\, \inf U - \inf(\mathbf{R}\mathrm{Hom}_R(X,U)) \mid U \in \mathcal{I}^{\mathrm{f}}(R) \ \wedge\ \mathrm{H}(U) \neq 0\,\}.$$

It should be noted that the supremum above is of a non-empty set as any (commutative Noetherian) ring admits a $U \in \mathcal{I}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$; for example, $U = \mathrm{K}(\boldsymbol{x})^{\vee}$ when $\boldsymbol{x}$ is a sequence of generators of a maximal ideal.

(5.20) **Lemma.** *If R is local and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\inf U - \inf(\mathbf{R}\mathrm{Hom}_R(X,U)) = \operatorname{depth} R - \operatorname{depth}_R X$$

for every $U \in \mathcal{I}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$. In particular,

$$\mathsf{rpd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X.$$

Proof. For X and U as above $\mathbf{R}\mathrm{Hom}_R(X,U)$ is in $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, and the first equality follows by applying (1.2.1) twice and then (4.13.c). □

(5.21) **Observation.** If $X \in \mathcal{D}_+(R)$ then $\mathsf{rpd}_R\, X \le \mathrm{pd}_R\, X$, and if R is local and $X \in \mathcal{P}^{\mathrm{f}}(R)$ then equality holds by (5.20).

(5.22) **Theorem.** *If R is local, then the following are equivalent.*

(*i*) *R is Cohen–Macaulay.*

(*ii*) *$\mathsf{rpd}_R\, X = \mathsf{Rpd}_R\, X$ for all complexes $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.*

(*iii*) *$\mathsf{Rpd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$ for all finite R–modules M.*

Proof. (*i*) ⇒ (*ii*) follows from (5.16), (3.4), and (5.20). (*ii*) ⇒ (*iii*) is immediate by (5.20), while (*iii*) ⇒ (*i*) results from (5.16), and (3.4). □

(5.23) **Corollary.** *If R is a Cohen–Macaulay local ring and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\mathsf{rpd}_R\, X = \mathsf{Rpd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X. \quad \square$$

(5.24) ***Other Ext–dimensions***. The restrictions on the test modules T in the definitions of the restricted Ext–dimension have been made such that (among other things) these dimensions are always finite for non-zero modules over (e.g.) local rings. In the case of modules, we also consider the next two alternative projective dimensions

$$\mathsf{Apd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \ \mathrm{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{P}_0(R)\,\}$$

$$\mathsf{apd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \ \mathrm{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}\,.$$

Even when the ring is local these two numbers are not always finite (the ring is Gorenstein if and only if they are always finite). However, it is easy to verify that the dimension Apd_R is always a refinement of pd_R and that apd_R is a refinement of pd_R over finite modules. Actually, it is proved in [24] that Apd_R is a refinement of the Gorenstein projective dimension Gpd_R, and in [3, Thm. 4.13] that apd_R is, for finite modules, a refinement of Auslander's G–dimension G-dim_R.

Moreover, if R is a complete local ring and $M \in \mathcal{P}_0(R)$, then it is proved in [22] that the Auslander–Buchsbaum Formula holds (without finiteness condition on M), that is,

$$\mathsf{apd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

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References

1. Dmitri Apassov, *Complexes and differential graded modules*, Ph.D. thesis, Lund University, 1999.
2. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.
3. Maurice Auslander and Mark Bridger, *Stable module theory*, American Mathematical Society, Providence, R.I., 1969, Memoirs of the American Mathematical Society, No. 94.
4. Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.
5. ______, *Homological dimension in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), 194–206.
6. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
7. ______, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.
8. Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 67–114 (1998).
9. Hyman Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.
10. ______, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
11. Leo G. Chouinard, II, *On finite weak and injective dimension*, Proc. Amer. Math. Soc. **60** (1976), 57–60 (1977).
12. Lars Winther Christensen, *Gorenstein Dimensions*, Springer-Verlag, Berlin, 2000, Lecture Notes in Mathematics 1747.
13. ______, *Semi–dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), 1839–1883.
14. ______, *Sequences for complexes*, Math. Scandinavica **89** (2001), 1–20.
15. Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and flat dimensions*, Math. Japon. **44** (1996), no. 2, 261–268.
16. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
17. ______, *n-Gorenstein rings*, Proc. Amer. Math. Soc. **42** (1974), 67–72.
18. ______, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
19. ______, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
20. ______, *Homological dimensions of complexes of modules*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, pp. 360–368.

21. ______, *Auslander–Buchsbaum equalities*, Amer. Math. Soc. 904th Meeting, Kent State Univ., Amer. Math. Soc., Providence, RI, 1995, pp. 759–760.
22. Anders Frankild, *Vanishing of local homology*, in preparation.
23. A. A. Gerko, *On homological dimensions*, in preparation, in Russian.
24. Henrik Holm, *Gorenstein homological dimensions*, manuscript, 2000.
25. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.
26. Eben Matlis, *Applications of duality*, Proc. Amer. Math. Soc. **10** (1959), 659–662.
27. Christian Peskine and Lucien Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.
28. Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.
29. Paul Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 7, 177–180.
30. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.
31. Siamak Yassemi, *Width of complexes of modules*, Acta Math. Vietnam. **23** (1998), no. 1, 161–169.

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