FINITE GORENSTEIN REPRESENTATION TYPE IMPLIES SIMPLE SINGULARITY

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To Lucho Avramov on his sixtieth birthday

Abstract. Let R be a commutative noetherian local ring and consider the set of isomorphism classes of indecomposable totally reflexive R-modules. We prove that if this set is finite, then either it has exactly one element, represented by the rank 1 free module, or R is Gorenstein and an isolated singularity (if R is complete, then it is even a simple hypersurface singularity). The crux of our proof is to argue that if the residue field has a totally reflexive cover, then R is Gorenstein or every totally reflexive R-module is free.

Introduction

Remarkable connections between the module theory of a local ring and the character of its singularity emerged in the 1980s. They show how finiteness conditions on the category of maximal Cohen–Macaulay modules[1] characterize particular isolated singularities. We develop these connections in several directions.

A local ring with only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules is said to be of finite Cohen–Macaulay (CM) representation type. By work of Auslander [5], every complete Cohen–Macaulay local ring of finite CM representation type is an isolated singularity.

Specialization to Gorenstein rings opens to a finer description of the singularities; it centers on the simple hypersurface singularities identified in Arnol′d's work on germs of holomorphic functions [1]. By work of Buchweitz, Greuel, and Schreyer [12], Herzog [18], and Yoshino [32], a complete Gorenstein ring of finite CM representation type is a simple singularity in the generalized sense of [32]. Under extra assumptions on the ring, the converse holds by work of Knörrer [21] and Solberg [25].

In this introduction, R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. To avoid the *a priori* condition in [12, 18, 32] that R is Gorenstein, we replace finite CM representation type with a finiteness condition on the category $\mathcal{G}(R)$ of modules of Gorenstein dimension 0. Over a Gorenstein ring, these modules are precisely the maximal Cohen–Macaulay modules, but they are known to exist over any ring, unlike maximal Cohen–Macaulay modules.

Theorem A. *Let R be complete. If the set of isomorphism classes of non-free indecomposable modules in $\mathcal{G}(R)$ is finite and not empty, then R is a simple singularity.*

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[1]The finitely generated modules whose depth equals the Krull dimension of the ring.

The category $\mathcal{G}(R)$ was introduced by Auslander and Bridger [4, 6]. An R-module G is in $\mathcal{G}(R)$ if there is an exact complex of finitely generated free R-modules

$$\boldsymbol{F} = \cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \cdots,$$

such that G is isomorphic to $\operatorname{Coker} \partial_0$ and the complex $\operatorname{Hom}_R(\boldsymbol{F}, R)$ is exact. Every finitely generated free R-module is in $\mathcal{G}(R)$, and the modules in this category have Gorenstein dimension 0 as in [4, 6]; following [11] we call them *totally reflexive*.

The aforementioned works [12, 18, 32] show that Theorem A follows from the next result, which is proved as (4.3).

Theorem B. *If the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R)$ is finite, then R is Gorenstein or every module in $\mathcal{G}(R)$ is free.*

As this theorem does not require R to be complete, we considerably strengthen Theorem A using work of Huneke, Leuschke, and R. Wiegand [19, 22, 30]; this occurs in (4.5). Theorem B was conjectured by R. Takahashi [29], who proved it for henselian rings of depth at most two [27, 28, 29]. The class of rings over which all totally reflexive modules are free is poorly understood, but it is known to include all Golod rings [11], in particular, all Cohen–Macaulay rings of minimal multiplicity.

To prove Theorem B we use a notion of $\mathcal{G}(R)$-*approximations*, which is close kin to the CM-approximations of Auslander and Buchweitz [7]. When R is Gorenstein, a $\mathcal{G}(R)$-approximation is exactly a CM-approximation. By [7], every module over a Gorenstein ring has a CM-approximation. Our proof of Theorem B goes via the following strong converse, proved as (3.4).

Theorem C. *Let R be a local ring and assume there is a non-free module in $\mathcal{G}(R)$. If the residue field k has a $\mathcal{G}(R)$-approximation, then R is Gorenstein.*

This theorem complements recent developments in relative homological algebra. The notion of totally reflexive modules has two extensions to non-finitely generated modules; see [13] for details. One is Gorenstein projective modules, which allows arbitrary free modules in the definition above. By recent work of Jørgensen [20], every module over a complete local ring has a Gorenstein projective precover. The other extension is Gorenstein flat modules. By a result of Enochs and López-Ramos [15], every module has a Gorenstein flat precover.

Theorem C counterposes these developments; it shows that for finitely generated modules, the precovers found in [20] and [15] cannot, in general, be finitely generated. Assume that R is complete. Then a finitely generated R-module has a $\mathcal{G}(R)$-approximation if and only if it has a $\mathcal{G}(R)$-precover. Assume further that R is not Gorenstein. Theorem C shows that if $X \to \mathsf{k}$ is a Gorenstein projective/flat precover and X is not free, then X is not finitely generated.

1. Categories and covers

In this paper, rings are commutative and noetherian; modules are finitely generated (unless otherwise specified). We write $\operatorname{mod}(R)$ for the category of finitely generated modules over a ring R.

For an R-module M, we denote by M_i the ith syzygy in a free resolution. When R is local, we denote by $\Omega_i^R(M)$ the ith syzygy in the minimal free resolution of M. For an R-module M, set $M^* = \operatorname{Hom}_R(M, R)$; we refer to this module as the *algebraic dual* of M.

We only consider full subcategories of $\mathrm{mod}(R)$; this allows us to define a subcategory by specifying its objects. In the following, $\mathcal{B}$ is a subcategory of $\mathrm{mod}(R)$.

(1.1) **Closures.** Recall that the category $\mathcal{B}$ is said to be closed under extensions if for every short exact sequence $0 \to B \to X \to B' \to 0$ with B and B' in $\mathcal{B}$ also X is in $\mathcal{B}$. The closure of $\mathcal{B}$ under extensions is by definition the smallest subcategory containing $\mathcal{B}$ and closed under extensions. Recall also that $\mathcal{B}$ is closed under direct sums and direct summands when a direct sum $M \oplus N$ is in $\mathcal{B}$ if and only if both summands are in $\mathcal{B}$. The closure of $\mathcal{B}$ under addition is by definition the smallest subcategory containing $\mathcal{B}$ and closed under direct sums and direct summands; we denote it by $\mathrm{add}(\mathcal{B})$.

We define the closure $\langle\mathcal{B}\rangle$ to be the smallest subcategory containing $\mathcal{B}$ and closed under direct summands and extensions. It is straightforward to verify that the closure $\langle\mathcal{B}\rangle$ is reached by countable alternating iteration, starting with $\mathcal{B}$, between closure under addition and closure under extensions.

We say that $\mathcal{B}$ is *closed under algebraic duality* if for every module B in $\mathcal{B}$ the module B^* is also in $\mathcal{B}$. Similarly, we say that $\mathcal{B}$ is *closed under syzygies* if for every module B in $\mathcal{B}$ every first syzygy B_1 is in $\mathcal{B}$; then every syzygy B_i is in $\mathcal{B}$.

(1.2) **Precovers and covers.** Let M be an R-module. A $\mathcal{B}$-*precover* of M is a homomorphism $\varphi\colon B \to M$, with $B \in \mathcal{B}$, such that every homomorphism $X \to M$ with $X \in \mathcal{B}$, factors through φ; i.e., the homomorphism

$$\mathrm{Hom}_R(X,\varphi)\colon\ \mathrm{Hom}_R(X,B) \longrightarrow \mathrm{Hom}_R(X,M)$$

is surjective for each module X in $\mathcal{B}$. A $\mathcal{B}$-precover $\varphi\colon B \to M$ is a $\mathcal{B}$-*cover* if every $\gamma \in \mathrm{Hom}_R(B,B)$ with $\varphi\gamma = \varphi$ is an automorphism.

Note that if the category $\mathcal{B}$ contains R, then every $\mathcal{B}$-precover is surjective.

(1.3) If there are only finitely many isomorphism classes of indecomposable modules in $\mathcal{B}$, then every finitely generated R-module has a $\mathcal{B}$-precover; see [2, Prop. 4.2].

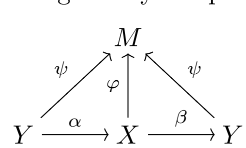
(1.4) Consider a diagram $B \xrightarrow{\varphi} M \oplus N \underset{\iota}{\overset{\pi}{\rightleftarrows}} M$, where $\pi\iota$ is the identity on M. If φ is a $\mathcal{B}$-precover, then so is $\pi\varphi\colon B \to M$.

The next two lemmas appear in Xu's book [31, 2.1.1 and 1.2.8]. We include a proof of the second one since Xu left it to the reader.

(1.5) **Wakamatsu's lemma.** Let $\mathcal{B}$ be a subcategory of $\mathrm{mod}(R)$, and let φ be a $\mathcal{B}$-cover of an R-module M. If $\mathcal{B}$ is closed under extensions, then $\mathrm{Ext}^1_R(X,\ker\varphi) = 0$ for all $X \in \mathcal{B}$.

(1.6) **Lemma.** *Let $\mathcal{B}$ be a subcategory of* $\mathrm{mod}(R)$*, and let M be an R-module. If M has a $\mathcal{B}$-cover, then a $\mathcal{B}$-precover $\varphi\colon X \to M$ is a cover if and only if* $\ker\varphi$ *contains no non-zero direct summand of X.*

Proof. Let $\psi\colon Y \to M$ be a $\mathcal{B}$-cover. For the "if" part, consider the commutative diagram below, where α and β are given by the precovering properties of φ and ψ.



Since $\psi\beta\alpha = \psi$ and ψ is a cover, the composite $\beta\alpha$ is an automorphism, so β is surjective. It also follows that X is isomorphic to $\operatorname{Ker}\beta \oplus \operatorname{Im}\alpha$. As $\operatorname{Ker}\varphi$ contains no non-zero summand of X, the inclusion $\operatorname{Ker}\beta \subseteq \operatorname{Ker}\varphi$ implies that β is also injective. Consequently, φ is a $\mathcal{B}$-cover.

For the "only if" part, consider a decomposition $X = Y \oplus Z$, and assume there is an inclusion $Z \subseteq \operatorname{Ker}\varphi$. Let π be the endomorphism of X projecting onto Y, then $\varphi\pi = \varphi$. Since φ is a cover, π is an automorphism, whence $Z = 0$. □

2. Approximations and reflexive subcategories

Stability of (pre-)covers under base change is delicate to track. To avoid this task, we develop a notion between precover and cover. The next definition is in line with that of CM-approximations [7]; for $\mathcal{G}(R)$ it broadens the notion used in [11].

(2.1) **Definitions.** Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}(R)$ and set

$$\mathcal{B}^{\perp} = \{\, L \in \operatorname{mod}(R) \mid \operatorname{Ext}^i_R(B, L) = 0 \text{ for all } B \in \mathcal{B} \text{ and all } i > 0 \,\}.$$

Let M be an R-module. A $\mathcal{B}$-*approximation* of M is a short exact sequence

$$0 \longrightarrow L \longrightarrow B \longrightarrow M \longrightarrow 0,$$

where B is in $\mathcal{B}$ and L is in $\mathcal{B}^{\perp}$.

(2.2) Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}(R)$ and M be an R-module.

(a) If $0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow B \xrightarrow{\varphi} M \longrightarrow 0$ is a $\mathcal{B}$-approximation of M, then φ is a *special* $\mathcal{B}$-precover of M; see [31, Prop. 2.1.3].

(b) If $B \xrightarrow{\varphi} M$ is a surjective $\mathcal{B}$-cover, and $\mathcal{B}$ is closed under syzygies and extensions, then the sequence $0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow B \xrightarrow{\varphi} M \longrightarrow 0$ is a $\mathcal{B}$-approximation of M by Wakamatsu's lemma.

(c) Assume $\operatorname{mod}(R)$ has the Krull–Schmidt property (e.g., R is henselian) and $\mathcal{B}$ is closed under direct summands. The module M has a $\mathcal{B}$-cover if and only if it has a $\mathcal{B}$-precover; see [29, Cor. 2.5].

The next two results study the behavior of approximations under base change.

Let $\vartheta\colon R \to S$ be a ring homomorphism. We say that ϑ is of finite flat dimension if S, viewed as an R-module through ϑ, has a bounded resolution by flat R-modules. We write $\operatorname{Tor}^R_{i>0}(S, \mathcal{B}) = 0$ if for all $B \in \mathcal{B}$, and for all $i > 0$, the modules $\operatorname{Tor}^R_i(S, B)$ vanish. We denote by $S \otimes \mathcal{B}$ the subcategory of S-modules $S \otimes_R B$ with $B \in \mathcal{B}$.

(2.3) **Lemma.** *Let $R \to S$ be a ring homomorphism of finite flat dimension. Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}(R)$ such that $\operatorname{Tor}^R_{i>0}(S, \mathcal{B}) = 0$. If $L \in \mathcal{B}^{\perp}$ and $\operatorname{Tor}^R_{i>0}(S, L) = 0$, then for every $m \in \mathbb{Z}$ and every $B \in \mathcal{B}$ there is an isomorphism*

$$\operatorname{Ext}^m_S(S \otimes_R B, S \otimes_R L) \cong \operatorname{Tor}^R_{-m}(S, \operatorname{Hom}_R(B, L)).$$

In particular, there are isomorphisms $\operatorname{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \operatorname{Hom}_R(B, L)$, and $S \otimes_R L$ is in $\langle S \otimes \mathcal{B} \rangle^{\perp}$.

Proof. Fix $B \in \mathcal{B}$. Take a free resolution $\boldsymbol{E} \to B$ and a bounded flat resolution $\boldsymbol{F} \to S$ over R. By the vanishing of (co)homology, the induced morphisms

$$S \otimes_R \boldsymbol{E} \to S \otimes_R B, \qquad \boldsymbol{F} \otimes_R L \to S \otimes_R L, \quad \text{and} \quad \operatorname{Hom}_R(B, L) \to \operatorname{Hom}_R(\boldsymbol{E}, L)$$

are homology isomorphisms. In particular, the first one is a free resolution of the S-module $S \otimes_R B$. The functors $\operatorname{Hom}_R(\boldsymbol{E}, -)$ and $\boldsymbol{F} \otimes_R -$ preserves homology isomorphisms. This explains the first, third, and fifth isomorphisms below.

$$\begin{aligned}\operatorname{Ext}_S^m(S \otimes_R B, S \otimes_R L) &\cong \mathrm{H}^m(\operatorname{Hom}_S(S \otimes_R \boldsymbol{E}, S \otimes_R L))\\ &\cong \mathrm{H}^m(\operatorname{Hom}_R(\boldsymbol{E}, S \otimes_R L))\\ &\cong \mathrm{H}^m(\operatorname{Hom}_R(\boldsymbol{E}, \boldsymbol{F} \otimes_R L))\\ &\cong \mathrm{H}^m(\boldsymbol{F} \otimes_R \operatorname{Hom}_R(\boldsymbol{E}, L))\\ &\cong \mathrm{H}^m(\boldsymbol{F} \otimes_R \operatorname{Hom}_R(B, L))\\ &\cong \operatorname{Tor}^R_{-m}(S, \operatorname{Hom}_R(B, L))\end{aligned}$$

The second isomorphism follows from Hom-tensor adjointness, and the fourth is tensor evaluation; see [17, Prop. II.5.14]. For $m = 0$ the composite isomorphism reads $\operatorname{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \operatorname{Hom}_R(B, L)$. That $S \otimes_R L$ is in $\langle S \otimes \mathcal{B}\rangle^{\perp}$ follows as Tor_i^R is zero for $i < 0$. □

(2.4) **Proposition.** *Let $R \to S$ be a ring homomorphism and $\mathcal{B}$ be a subcategory of* $\operatorname{mod}(R)$. *Let M be an R-module with a $\mathcal{B}$-approximation $0 \to L \to B \to M \to 0$. If $\operatorname{Tor}^R_{i>0}(S, \mathcal{B}) = 0$ and $\operatorname{Tor}^R_{i>0}(S, M) = 0$, then*

$$0 \longrightarrow S \otimes_R L \longrightarrow S \otimes_R B \longrightarrow S \otimes_R M \longrightarrow 0$$

is an $\langle S \otimes \mathcal{B}\rangle$-approximation.

Proof. By the assumptions on $\mathcal{B}$ and M, application of the functor $S \otimes_R -$ to the $\mathcal{B}$-approximation of M yields the desired short exact sequence and also equalities $\operatorname{Tor}^R_{i>0}(S, L) = 0$. Now Lemma (2.3) gives that $S \otimes_R L$ is in $\langle S \otimes \mathcal{B}\rangle^{\perp}$. □

(2.5) Let $\mathcal{B}$ be a subcategory of $\operatorname{mod}(R)$ with $R \in \mathcal{B}^{\perp}$. For every $B \in \mathcal{B}$ and every R-module N, dimension shifting yields

$$\operatorname{Ext}^i_R(B, N) \cong \operatorname{Ext}^{i+h}_R(B, N_h) \quad \text{for } i > 0 \text{ and } h \geqslant 0.$$

Moreover, for $h \geqslant 0$ the algebraic dual B^* is a hth syzygy of $(B_h)^*$, so

$$\operatorname{Ext}^i_R(B^*, N) \cong \operatorname{Ext}^{i+h}_R((B_h)^*, N) \quad \text{for } i > 0 \text{ and } h \geqslant 0.$$

If, furthermore, $\mathcal{B}$ is closed under syzygies and algebraic duality, then these isomorphisms combine to yield

(2.5.1) $$\operatorname{Ext}^i_R(B^*, N_j) \cong \operatorname{Ext}^i_R((B_h)^*, N_{j-h}) \quad \text{for } i > 0 \text{ and } j \geqslant h \geqslant 0.$$

In particular, (2.5.1) holds when $\mathcal{B}$ is a category satisfying the next definition.

(2.6) **Definition.** A subcategory $\mathcal{B}$ of $\operatorname{mod}(R)$ is *reflexive* if R is in $\mathcal{B} \cap \mathcal{B}^{\perp}$ and $\mathcal{B}$ is closed under

(1) direct sums and direct summands,
(2) syzygies, and
(3) algebraic duality.

It is standard that the category $\mathcal{G}(R)$ of totally reflexive R-modules is a reflexive subcategory of $\operatorname{mod}(R)$. Moreover, using the characterization of $\mathcal{G}(R)$ provided by [13, (1.1.2) and (4.1.4)], it is straightforward to verify that every reflexive subcategory of $\operatorname{mod}(R)$ is, in fact, a subcategory of $\mathcal{G}(R)$.

(2.7) In the rest of the paper, $\mathcal{F}(R)$ denotes the category of finitely generated free R-modules. Let $\mathcal{B}$ be a reflexive subcategory of $\mathrm{mod}(R)$. There are containments

$$\mathcal{F}(R) \subseteq \mathcal{B} \subseteq \mathcal{G}(R).$$

Further, let $R \to S$ be a ring homomorphism of finite flat dimension, then

$$\mathrm{Tor}^R_{i>0}(S, \mathcal{B}) = 0,$$

as every module in $\mathcal{B}$ is an infinite syzygy.

The next observation is crucial for our proofs of the main theorems.

(2.8) Assume $\mathrm{mod}(R)$ has the Krull–Schmidt property (e.g., R is henselian) and let $\mathcal{B}$ be a reflexive subcategory of $\mathrm{mod}(R)$ closed under extensions. We claim that an R-module M has a $\mathcal{B}$-precover if and only if it has a $\mathcal{B}$-approximation. Indeed, let $\varphi\colon B \to M$ be a $\mathcal{B}$-precover; by (2.2)(c) the module M also has a $\mathcal{B}$-cover. Decompose B as $B' \oplus B''$, where B'' is the largest direct summand of B contained in $\mathrm{Ker}\,\varphi$. By Lemma (1.6) the factorization $\varphi'\colon B' \to M$ is a cover, and by (2.2)(b) the sequence $0 \to \mathrm{Ker}\,\varphi' \to B' \to M \to 0$ is a $\mathcal{B}$-approximation.

(2.9) **Lemma.** *Let $\mathcal{B}$ be a reflexive subcategory of $\mathrm{mod}(R)$ and M be an R-module. If M has a $\mathcal{B}$-approximation, then every syzygy of M has a $\mathcal{B}$-approximation.*

Proof. Let $0 \to L \to B \to M \to 0$ be a $\mathcal{B}$-approximation. It is sufficient to prove that every first syzygy M_1 has a $\mathcal{B}$-approximation. By the horseshoe construction, there is a short exact sequence $0 \to L_1 \to B_1 \to M_1 \to 0$, and the syzygy B_1 is in $\mathcal{B}$ by assumption. Let X be in $\mathcal{B}$. Since $\mathcal{B}$ is reflexive, there is an isomorphism $X \cong X^{**}$, and also the module $((X^*)_1)^*$ is in $\mathcal{B}$. Now (2.5.1) yields the second isomorphism in the chain

$$\mathrm{Ext}^i_R(X, L_1) \cong \mathrm{Ext}^i_R(X^{**}, L_1) \cong \mathrm{Ext}^i_R(\,((X^*)_1)^*, L\,) = 0. \qquad \square$$

(2.10) **Proposition.** *Let $R \to S$ be a ring homomorphism of finite flat dimension. If $\mathcal{B}$ is a reflexive subcategory of $\mathrm{mod}(R)$, then $\langle S \otimes \mathcal{B} \rangle$ is a reflexive subcategory of $\mathrm{mod}(S)$. In particular, $\langle S \otimes \mathcal{G}(R) \rangle$ is reflexive.*

Proof. The ring S is in $\langle S \otimes \mathcal{B} \rangle$. As $R \in \mathcal{B}^\perp$, it follows from (2.7) and Lemma (2.3) that S is in $\langle S \otimes \mathcal{B} \rangle^\perp$. By definition, $\langle S \otimes \mathcal{B} \rangle$ is closed under direct sums and direct summands; this leaves (2) and (3) in Definition (2.6) to verify.

First we prove closure under syzygies. Take $B \in \mathcal{B}$ and consider a short exact sequence $0 \to B_1 \to F \to B \to 0$, where F is a free R-module. By assumption, the syzygy B_1 is in $\mathcal{B}$. By (2.7) the sequence

$$0 \longrightarrow S \otimes_R B_1 \longrightarrow S \otimes_R F \longrightarrow S \otimes_R B \longrightarrow 0$$

is exact. It shows that the syzygy $S \otimes_R B_1$ of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, it follows that any summand of $S \otimes_R B$ has a first syzygy in $\mathrm{add}(S \otimes \mathcal{B})$, in particular, in $\langle S \otimes \mathcal{B} \rangle$. By Schanuel's lemma, a module in $\langle S \otimes \mathcal{B} \rangle$ with some first syzygy in $\langle S \otimes \mathcal{B} \rangle$ has every first syzygy in $\langle S \otimes \mathcal{B} \rangle$. Finally, given a short exact sequence $0 \to M \to X \to N \to 0$, where M, N, and their first syzygies are in $\langle S \otimes \mathcal{B} \rangle$, we claim that also a first syzygy of X is in $\langle S \otimes \mathcal{B} \rangle$. Indeed, take presentations of M and N. Since $\langle S \otimes \mathcal{B} \rangle$ is closed under extensions, it follows from the horseshoe construction that a first syzygy of X is in $\langle S \otimes \mathcal{B} \rangle$.

Next we prove closure under algebraic duality. Take $B \in \mathcal{B}$ and note that by (2.7), Lemma (2.3) applies (with $L = R$) to yield the isomorphism

$$\mathrm{Hom}_S(S \otimes_R B, S) \cong S \otimes_R \mathrm{Hom}_R(B, R).$$

Thus, the algebraic dual of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, the algebraic dual of any summand of $S \otimes_R B$ is in $\mathrm{add}(S \otimes \mathcal{B})$, in particular, in $\langle S \otimes \mathcal{B} \rangle$. It is now sufficient to prove that for every short exact sequence $0 \to M \to X \to N \to 0$, where M, N, and the duals M^* and N^* are in $\langle S \otimes \mathcal{B} \rangle$, also the dual X^* is in $\langle S \otimes \mathcal{B} \rangle$. Since $\langle S \otimes \mathcal{B} \rangle$ is closed under extensions, this is immediate from the exact sequence

$$0 \to N^* \to X^* \to M^* \to \mathrm{Ext}^1_S(N, S),$$

where $\mathrm{Ext}^1_S(N, S) = 0$ as S is in $\langle S \otimes \mathcal{B} \rangle^{\perp}$. □

3. Approximations detect the Gorenstein property

The main result of this section is Theorem C from the introduction. Lemma (3.2) furnishes the base case; for that we study a standard homomorphism.

(3.1) For modules X and N over a ring S there is a natural map

$$\theta_{XN} \colon X \otimes_S N \longrightarrow \mathrm{Hom}_S(X^*, N),$$

given by evaluation $\theta(x \otimes n)(\zeta) = \zeta(x)n$. Auslander computed the kernel and cokernel of this map in [3, Prop. 6.3]. Because the map is pivotal for our proof of the next lemma, we include a computation for the case where X is totally reflexive.

Consider a short exact sequence $0 \to N_1 \to F \to N \to 0$, where F is a free S-module. For any totally reflexive S-module X, the evaluation homomorphism θ_{XF} is an isomorphism, and the commutative diagram

$$\begin{array}{ccccccccc} & & X \otimes_S N_1 & \longrightarrow & X \otimes_S F & \longrightarrow & X \otimes_S N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_{XN_1}} & & \cong \downarrow{\scriptstyle \theta_{XF}} & & \downarrow{\scriptstyle \theta_{XN}} & & \\ 0 & \longrightarrow & \mathrm{Hom}_S(X^*, N_1) & \longrightarrow & \mathrm{Hom}_S(X^*, F) & \longrightarrow & \mathrm{Hom}_S(X^*, N) & \longrightarrow & \mathrm{Ext}^1_S(X^*, N_1) \longrightarrow 0 \end{array}$$

shows that there is an isomorphism $\mathrm{Coker}\, \theta_{XN} \cong \mathrm{Ext}^1_S(X^*, N_1)$. The snake lemma applies to yield $\mathrm{Ker}\, \theta_{XN} \cong \mathrm{Coker}\, \theta_{XN_1} \cong \mathrm{Ext}^1_S(X^*, N_2)$, and then (2.5.1) gives

(3.1.1) $$\mathrm{Ker}\, \theta_{XN} \cong \mathrm{Ext}^1_S((X_2)^*, N) \quad \text{and} \quad \mathrm{Coker}\, \theta_{XN} \cong \mathrm{Ext}^1_S((X_1)^*, N).$$

(3.2) **Lemma.** *Let $(S, \mathfrak{n}, \ell)$ be a complete local ring of depth 0. Let $\mathcal{C}$ be a reflexive subcategory of $\mathrm{mod}(S)$. If ℓ has a $\mathcal{C}$-approximation and ℓ is not in $\mathcal{C}$, then $\mathcal{C} = \mathcal{F}(S)$.*

Proof. Consider a $\mathcal{C}$-approximation $0 \longrightarrow L \xrightarrow{\alpha} C \longrightarrow \ell \longrightarrow 0$, and dualize to get $0 \longrightarrow \ell^* \longrightarrow C^* \xrightarrow{\alpha^*} L^*$. Let I be the image of α^*, and let φ be the factorization of α^* through the inclusion $I \hookrightarrow L^*$.

First we prove that the surjection φ is a $\langle \mathcal{C} \rangle$-precover of I. Let X be a module in $\langle \mathcal{C} \rangle$. If X is a free S-module, then any homomorphism $X \to I$ lifts through φ. We may now assume that X is indecomposable and not free. Because $\mathrm{Hom}_S(X, I)$ is a submodule of $\mathrm{Hom}_S(X, L^*)$, it suffices to prove surjectivity of

$$\mathrm{Hom}_S(X, \alpha^*) \colon \mathrm{Hom}_S(X, C^*) \longrightarrow \mathrm{Hom}_S(X, L^*),$$

which we do next.

The vertical maps in the commutative diagram below are evaluation homomorphisms, see (3.1).

$$\begin{array}{ccccccc} & & X \otimes_R L & \xrightarrow{\iota} & X \otimes_R C & \longrightarrow & X \otimes_R \ell & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \theta_{XL}} & & \big\downarrow{\scriptstyle \theta_{XC}} & & \big\downarrow{\scriptstyle \theta_{X\ell}} & & \\ 0 & \longrightarrow & \mathrm{Hom}_S(X^*, L) & \longrightarrow & \mathrm{Hom}_S(X^*, C) & \longrightarrow & \mathrm{Hom}_S(X^*, \ell) & \longrightarrow & \mathrm{Ext}^1_S(X^*, L) \end{array}$$

First we argue that the rows of this diagram are short exact sequences. The module X is in $\langle \mathcal{C} \rangle$ and hence in $\mathcal{G}(S)$, see (2.7), so $\mathrm{Ext}^1_S(X^*, L) = 0$. Moreover, θ_{XL} is an isomorphism by (3.1.1), hence ι is injective. Next note that for every $\zeta \in X^*$ the image of $\zeta \colon X \to S$ is in $\mathfrak{n}$ as X is indecomposable and not free. Thus, for all $x \in X$ and $u \in \ell$, we have $\theta_{X\ell}(x \otimes u)(\zeta) = \zeta(x)u = 0$. Finally, apply $\mathrm{Hom}_S(-, S)$ to the diagram above and use Hom-tensor adjointness to get

$$\begin{array}{ccccccc} \mathrm{Hom}_S(X^*, \ell)^* & \longrightarrow & \mathrm{Hom}_S(X^*, C)^* & \longrightarrow & \mathrm{Hom}_S(X^*, L)^* & \longrightarrow & \mathrm{Ext}^1_S(\mathrm{Hom}_S(X^*, \ell), S) \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle \theta^*_{XC}} & & \big\downarrow{\scriptstyle \theta^*_{XL}}\,\cong & & \big\downarrow{\scriptstyle 0} \\ \mathrm{Hom}_S(X, \ell^*) & \longrightarrow & \mathrm{Hom}_S(X, C^*) & \xrightarrow{\mathrm{Hom}_S(X, \alpha^*)} & \mathrm{Hom}_S(X, L^*) & \longrightarrow & \mathrm{Ext}^1_S(X \otimes_R \ell, S). \end{array}$$

The diagram shows that $\mathrm{Hom}_S(X, \alpha^*)$ is surjective, as desired.

Now $\varphi \colon C^* \to I$ is a $\langle \mathcal{C} \rangle$-precover, so by completeness of S, the module I has a $\langle \mathcal{C} \rangle$-cover; see (2.2)(c). The ring has depth 0, so ℓ^* is a non-zero ℓ-vector space. By the assumptions on $\mathcal{C}$, the residue field ℓ cannot be a direct summand of C^*. As $\mathrm{Ker}\,\varphi = \ell^*$, it follows from Lemma (1.6) that φ is a $\langle \mathcal{C} \rangle$-cover. For every $X \in \langle \mathcal{C} \rangle$ Wakamatsu's lemma gives $\mathrm{Ext}^1_S(X, \ell^*) = 0$. Consequently, every module in $\mathcal{C}$ is projective and hence free, since S is local. □

(3.3) Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring and denote by $\mathcal{M}(R)$ the category of maximal Cohen–Macaulay R-modules.

(a) If R is Cohen–Macaulay, then $\mathcal{G}(R) \subseteq \mathcal{M}(R)$ by the Auslander–Bridger formula [4, §3.2 Prop. 3]. Conversely, if $\mathcal{G}(R) \subseteq \mathcal{M}(R)$, then R is Cohen–Macaulay.

(b) If R is Gorenstein, then the categories $\mathcal{G}(R)$ and $\mathcal{M}(R)$ coincide by [4, §3.2 Thm. 3] and the Auslander–Bridger formula. Conversely, if $\mathcal{G}(R) = \mathcal{M}(R)$, then R is Gorenstein. Indeed, R is Cohen–Macaulay by (a), so $\Omega^R_{\dim R}(\mathsf{k})$ is in $\mathcal{M}(R)$, hence in $\mathcal{G}(R)$, and therefore R is Gorenstein by [4, §3.2, Rmk. after Thm. 3].

(c) If R is Gorenstein, then a short exact sequence $0 \to L \to G \to M \to 0$ is a CM-approximation if and only if it is a $\mathcal{G}(R)$-approximation. This follows from (b) and the fact that L is in $\mathcal{M}(R)^\perp$ if and only if L has finite injective dimension.

If R is Gorenstein, then every R-module has a CM-approximation by [7, Thm. A]. In view of (3.3)(c) the next result contains a converse, cf. Theorem C.

(3.4) **Theorem.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring and $\mathcal{B}$ be a reflexive subcategory of* $\mathrm{mod}(R)$. *If k has a $\mathcal{B}$-approximation, then R is Gorenstein or $\mathcal{B} = \mathcal{F}(R)$.*

In our proof of this theorem we use the next lemma. We do not know a reference giving a direct argument, so one is supplied here.

(3.5) **Lemma.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring, and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence in $\mathfrak{m} \setminus \mathfrak{m}^2$. If $\boldsymbol{x}$ is linearly independent modulo $\mathfrak{m}^2$, then k is a direct summand of the module $\Omega^R_n(\mathsf{k})/\boldsymbol{x}\Omega^R_n(\mathsf{k})$.*

Proof. Let $(\mathrm{K}(\boldsymbol{x}),\mathrm{d})$ be the Koszul complex on $\boldsymbol{x}$. If necessary, supplement $\boldsymbol{x}$ to a minimal generating sequence $\boldsymbol{x},\boldsymbol{y}$ for $\mathfrak{m}$. Let $(\boldsymbol{F},\partial)$ be a minimal free resolution of k. The identification $R/(\boldsymbol{x},\boldsymbol{y})=\mathsf{k}$ lifts to a morphism of complexes $\sigma\colon \mathrm{K}(\boldsymbol{x},\boldsymbol{y})\to\boldsymbol{F}$. Serre proves in [24, Appendix I.2] that σ is injective and degreewise split. The natural inclusion $\iota\colon \mathrm{K}(\boldsymbol{x})\hookrightarrow\mathrm{K}(\boldsymbol{x},\boldsymbol{y})$ is also degreewise split, so the composite $\rho=\sigma\iota$ is an injective morphism of complexes and degreewise split.

From the short exact sequence $0\longrightarrow\Omega^R_n(\mathsf{k})\xrightarrow{\iota}F_{n-1}\longrightarrow\Omega^R_{n-1}(\mathsf{k})\longrightarrow 0$, we get an exact sequence in homology that reads in part

$$(*)\qquad \mathrm{Tor}^R_1(R/(\boldsymbol{x}),\Omega^R_{n-1}(\mathsf{k}))\longrightarrow R/(\boldsymbol{x})\otimes_R\Omega^R_n(\mathsf{k})\xrightarrow{R/(\boldsymbol{x})\otimes_R\iota}R/(\boldsymbol{x})\otimes_R F_{n-1}.$$

The module $\mathrm{Tor}^R_1(R/(\boldsymbol{x}),\Omega^R_{n-1}(\mathsf{k}))\cong\mathrm{Tor}^R_n(R/(\boldsymbol{x}),\mathsf{k})$ is annihilated by $\mathfrak{m}$.

Let e be a generator of $\mathrm{K}(\boldsymbol{x})_n$. The image $\rho_n(e)$ in F_n is a minimal generator as ρ_n is split. Set $\varepsilon=\partial_n\rho_n(e)\in\Omega^R_n(\mathsf{k})$; since $\boldsymbol{F}$ is minimal, ε is a minimal generator of the syzygy $\Omega^R_n(\mathsf{k})$. The minimal generator $1\otimes\varepsilon$ of $R/(\boldsymbol{x})\otimes_R\Omega^R_n(\mathsf{k})$ is in the kernel of $(R/(\boldsymbol{x})\otimes_R\iota)$, as the element $\varepsilon=\partial_n\rho_n(e)=\rho_{n-1}\mathrm{d}_n(e)$ is in $\boldsymbol{x}F_{n-1}$. By exactness of $(*)$ the element $1\otimes\varepsilon$ is annihilated by $\mathfrak{m}$, hence it generates a 1-dimensional k-vector space that is a direct summand of $\Omega^R_n(\mathsf{k})/\boldsymbol{x}\Omega^R_n(\mathsf{k})$. □

Proof of (3.4). We aim to apply Lemma (3.2). By Propositions (2.4) and (2.10), and by faithful flatness of $\widehat{R}$, we may assume R is complete. Set $d=\operatorname{depth}R$; by Lemma (2.9) the dth syzygy $\Omega^R_d(\mathsf{k})$ has a $\mathcal{B}$-approximation:

$$0\to L\to B\to\Omega^R_d(\mathsf{k})\to 0.$$

Let $\boldsymbol{x}=x_1,\dots,x_d$ be an R-regular sequence in $\mathfrak{m}\setminus\mathfrak{m}^2$ linearly independent modulo $\mathfrak{m}^2$. The Koszul homology modules

$$\mathrm{H}_i(\mathrm{K}(\boldsymbol{x})\otimes_R\Omega^R_d(\mathsf{k}))\cong\mathrm{Tor}^R_i(R/(\boldsymbol{x}),\Omega^R_d(\mathsf{k}))\cong\mathrm{Tor}^R_{i+d}(R/(\boldsymbol{x}),\mathsf{k})$$

vanish for $i>0$, so $\boldsymbol{x}$ is also $\Omega^R_d(\mathsf{k})$-regular.

Set $S=R/(\boldsymbol{x})$; by (2.7) and Proposition (2.4) the sequence

$$0\longrightarrow S\otimes_R L\longrightarrow S\otimes_R B\xrightarrow{\varphi}S\otimes_R\Omega^R_d(\mathsf{k})\longrightarrow 0$$

is a $\langle S\otimes\mathcal{B}\rangle$-approximation. Moreover, the category $\langle S\otimes\mathcal{B}\rangle$ is reflexive by Proposition (2.10). By Lemma (3.5) the residue field k is a direct summand of $S\otimes_R\Omega^R_d(\mathsf{k})$, so by (1.4) there is an $\langle S\otimes\mathcal{B}\rangle$-precover of k. Since S is complete, it follows from (2.8) that k has a $\langle S\otimes\mathcal{B}\rangle$-approximation.

Assume R is not Gorenstein. Then S is not Gorenstein, so the residue field k is not in $\mathcal{G}(S)$ and hence not in $\langle S\otimes\mathcal{B}\rangle$; see [4, §3.2, Rmk. after Thm. 3] or [13, Thm. (1.4.9)]. By Lemma (3.2) every module in $\langle S\otimes\mathcal{B}\rangle$ is now free, so for every $B\in\mathcal{B}$ the module $S\otimes_R B$ is free over S. By (2.7) the sequence $\boldsymbol{x}$ is B-regular; therefore, B is a free R-module by Nakayama's lemma. □

An approximation of a module M is *minimal* if the map onto M is a cover. When R is Gorenstein, every R-module has a minimal CM-approximation by unpublished work of Auslander; see [8, Sec. 4] and [14, Thm. 5.5]. Hence we have

(3.6) **Corollary.** *Let $(R,\mathfrak{m},\mathsf{k})$ be a local ring and assume there is a non-free module in $\mathcal{G}(R)$. The following are then equivalent:*

(*i*) *R is Gorenstein.*

(*ii*) *k has a $\mathcal{G}(R)$-approximation.*

(*iii*) *Every finitely generated R-module has a minimal $\mathcal{G}(R)$-approximation.* □

(3.7) If R has a dualizing complex, cf. [17, V.§2], then k has a Gorenstein projective precover $X \to \mathsf{k}$ by [20, Thm. 2.11]. Assume X is finitely generated, i.e., X is in $\mathcal{G}(R)$ and, further, that R is henselian. If X is free, then it follows from (2.8) that k has a $\mathcal{G}(R)$-approximation $0 \to L \to X' \to \mathsf{k} \to 0$, where X' is free. Hence, k is in $\mathcal{G}(R)^{\perp}$ and then $\mathcal{G}(R) = \mathcal{F}(R)$. If X is not free, then R is Gorenstein by (3.6).

(3.8) **Questions.** Let $(R, \mathfrak{m}, \mathsf{k})$ be a local ring. If k has a $\mathcal{G}(R)$-precover, is then $\mathcal{G}(R)$ precovering? If $\mathcal{G}(R)$ is precovering and contains a non-free module, is then R Gorenstein?

4. On the number of totally reflexive modules

In this section we prove Theorems A and B. Note that by (1.3) the latter would follow immediately from a positive answer to the second question in (3.8).

(4.1) **Lemma.** *Let R be a local ring and M and N be finitely generated R-modules. If only finitely many isomorphism classes of R-modules X can fit in a short exact sequence $0 \to N \to X \to M \to 0$, then the R-module $\operatorname{Ext}^1_R(M, N)$ has finite length.*

Proof. Given an R-module X, we denote by $[X]$ the subset of $\operatorname{Ext}^1_R(M, N)$ whose elements have representatives of the form $0 \to N \to Y \to M \to 0$, where $Y \cong X$. By assumption, there exist non-isomorphic R-modules $X_0, \dots, X_n$ such that $\operatorname{Ext}^1_R(M, N)$ is the disjoint union of the sets $[X_i]$. We may take $X_0 = M \oplus N$, so $[X_0]$ is the zero submodule of $\operatorname{Ext}^1_R(M, N)$. We must prove that there is an integer $q > 0$ such that $\mathfrak{m}^q \operatorname{Ext}^1_R(M, N)$ is contained in $[X_0]$.

By [16, Cor. 1] there are integers p_i such that if $M/\mathfrak{m}^p M \oplus N/\mathfrak{m}^p N \cong X_i/\mathfrak{m}^p X_i$ for some $p \geqslant p_i$, then $X_i \cong M \oplus N$. Set $q = \max\{ p_1, \dots, p_n \}$. Take a short exact sequence ξ in $\mathfrak{m}^q \operatorname{Ext}^1_R(M, N)$; it belongs to some set $[X_i]$. By [26, Thm. 1.1] the sequence $\xi \otimes_R R/\mathfrak{m}^q$ splits, so $M/\mathfrak{m}^q M \oplus N/\mathfrak{m}^q N \cong X_i/\mathfrak{m}^q X_i$. By the choice of q this implies $X_i \cong M \oplus N$, so $i = 0$, i.e. ξ is in the zero submodule $[X_0]$. □

Let $R \to S$ be a flat ring homomorphism. It does not follow from the natural isomorphism $S \otimes_R \operatorname{Ext}^1_R(M, N) \cong \operatorname{Ext}^1_S(S \otimes_R M, S \otimes_R N)$ that every extension of the S-modules $S \otimes_R N$ and $S \otimes_R M$ has the form $S \otimes_R X$ for some R-module X. In a seminar, Roger Wiegand alerted us to the next result.

(4.2) **Lemma.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat ring homomorphism with $\mathfrak{m}S = \mathfrak{n}$ and $R/\mathfrak{m} \cong S/\mathfrak{n}$. Let M and N be finitely generated R-modules and ξ be an element of the S-module $\operatorname{Ext}^1_S(S \otimes_R M, S \otimes_R N)$. If the R-module $\operatorname{Ext}^1_R(M, N)$ has finite length, then there is an element χ in $\operatorname{Ext}^1_R(M, N)$ such that $\xi = S \otimes_R \chi$.*

Proof. The functor $S \otimes_R -$ from the category $\operatorname{mod}(R)$ to itself induces a natural isomorphism $K \to S \otimes_R K$ on R-modules of finite length. Applied to $\operatorname{Ext}^1_R(M, N)$ this yields the first isomorphism below

$$\operatorname{Ext}^1_R(M, N) \xrightarrow{\cong} S \otimes_R \operatorname{Ext}^1_R(M, N) \xrightarrow{\cong} \operatorname{Ext}^1_S(S \otimes_R M, S \otimes_R N).$$

The composite sends an exact sequence χ to $S \otimes_R \chi$. □

The next result is Theorem B from the introduction.

(4.3) **Theorem.** *Let R be a local ring. If the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R)$ is finite, then R is Gorenstein or $\mathcal{G}(R) = \mathcal{F}(R)$.*

Proof. Assume there are only finitely many isomorphism classes of indecomposable modules in $\mathcal{G}(R)$. By (1.3) the residue field k then has a $\mathcal{G}(R)$-precover $\varphi\colon B \to \mathsf{k}$. We claim that $\widehat{R} \otimes_R \varphi$ is a $\langle \widehat{R} \otimes \mathcal{G}(R)\rangle$-precover of k. Since $\widehat{R}$ is complete, this implies the existence of a $\langle \widehat{R} \otimes \mathcal{G}(R)\rangle$-approximation of k, see (2.8), and the desired conclusion follows from Theorem (3.4) and faithful flatness of $\widehat{R}$.

To prove the claim, we must show that

$$\mathrm{Hom}_{\widehat{R}}(H', \widehat{R} \otimes_R \varphi)\colon\ \mathrm{Hom}_{\widehat{R}}(H', \widehat{R} \otimes_R B) \longrightarrow \mathrm{Hom}_{\widehat{R}}(H', \mathsf{k})$$

is surjective for every module $H' \in \langle \widehat{R} \otimes \mathcal{G}(R)\rangle$. By flatness of $\widehat{R}$, surjectivity holds for modules in $\widehat{R} \otimes \mathcal{G}(R)$ and hence for every module in $\mathrm{add}(\widehat{R} \otimes \mathcal{G}(R))$. It is now sufficient to prove that the category $\mathrm{add}(\widehat{R} \otimes \mathcal{G}(R))$ is closed under extensions, because then $\langle \widehat{R} \otimes \mathcal{G}(R)\rangle$ is $\mathrm{add}(\widehat{R} \otimes \mathcal{G}(R))$.

First we show that $\widehat{R} \otimes \mathcal{G}(R)$ is closed under extensions. Fix modules G and K in $\mathcal{G}(R)$, and consider short exact sequences $0 \to G \to H \to K \to 0$. Each H is in $\mathcal{G}(R)$, and the minimal number of generators of each H is bounded by the sum of the numbers of minimal generators for G and K. Since the number of indecomposable modules in $\mathcal{G}(R)$ is finite, there are, up to isomorphism, only finitely many such modules H. By Lemma (4.1) the module $\mathrm{Ext}^1_R(K, G)$ has finite length, and by (4.2) every element of $\mathrm{Ext}^1_{\widehat{R}}(\widehat{R} \otimes_R K, \widehat{R} \otimes_R G)$ is extended from $\mathrm{Ext}^1_R(K, G)$.

To prove that $\mathrm{add}(\widehat{R} \otimes \mathcal{G}(R))$ is closed under extensions, let G' and K' be summands of extended modules, i.e., $G' \oplus G'' \cong \widehat{R} \otimes_R G$ and $K' \oplus K'' \cong \widehat{R} \otimes_R K$ for modules $G, K \in \mathcal{G}(R)$. Consider a short exact sequence $0 \to G' \to H' \to K' \to 0$. Then a sequence

$$0 \longrightarrow G' \oplus G'' \longrightarrow H' \oplus G'' \oplus K'' \longrightarrow K' \oplus K'' \longrightarrow 0,$$

is exact, so by what has already been proved, the middle term $H' \oplus G'' \oplus K''$ is in $\widehat{R} \otimes \mathcal{G}(R)$; whence H' is in $\mathrm{add}(\widehat{R} \otimes \mathcal{G}(R))$. □

In view of (3.3)(a) we have

(4.4) **Corollary.** *Let R be a Cohen–Macaulay local ring. If R is of finite CM representation type, then R is Gorenstein or $\mathcal{G}(R) = \mathcal{F}(R)$.* □

The next result contains Theorem A from the introduction.

(4.5) **Theorem.** *Let R be a local ring and assume the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R) \setminus \mathcal{F}(R)$ is finite and not empty. Then R is Gorenstein and an isolated singularity. Further, $\widehat{R}$ is a hypersurface singularity; if finite CM representation type ascends from R to $\widehat{R}$, then $\widehat{R}$ is even a simple singularity.*

Proof. By Theorem (4.3) the ring R is Gorenstein. From (3.3)(b) it follows that R is of finite CM representation type and hence an isolated singularity by [19, Cor. 2]. By [18, Satz 1.2] the completion $\widehat{R}$ is a hypersurface singularity and, assuming that also $\widehat{R}$ is of finite CM representation type, it follows from [32, Cor. (8.16)] that $\widehat{R}$ is a simple singularity. □

(4.6) **Remark.** In [23] Schreyer conjectured that a Cohen–Macaulay local k-algebra R is of finite CM representation type if and only if $\widehat{R}$ is of finite CM representation type. In [30] R. Wiegand proved descent of finite CM representation type from $\widehat{R}$ to R for any local ring R. Ascent is verified in [30] when R is Cohen–Macaulay and either $\widehat{R}$ is an isolated singularity or $\dim R \leqslant 1$. Ascent also holds for excellent Cohen–Macaulay local rings by work of Leuschke and R. Wiegand [22].

(4.7) **Remarks.** Constructing rings with infinitely many totally reflexive modules is easy using Theorem (4.3). Indeed, let Q be a local ring of positive dimension and set $R = Q[\![X]\!]/(X^2)$. As R is not reduced, it is not an isolated singularity. The R-module $R/(X)$ is in $\mathcal{G}(R)$ and is not free, cf. [13, exa. (4.1.5)], so by (4.3) there are infinitely many non-isomorphic indecomposable modules in $\mathcal{G}(R)$.

More generally, Avramov, Gasharov, and Peeva [9] construct a non-free totally reflexive module[2] G over any ring of the form $R \cong Q/(\boldsymbol{x})$, where $(Q, \mathfrak{q})$ is local and $\boldsymbol{x} \in \mathfrak{q}^2$ is a Q-regular sequence. Such a ring R is said to have an embedded deformation of codimension c, where c is the length of $\boldsymbol{x}$. Again (4.3) implies the existence of infinitely many non-isomorphic indecomposable modules in $\mathcal{G}(R)$. If $\widehat{R}$ has an embedded deformation of codimension $c \geqslant 2$, a recent argument of Avramov and Iyengar builds from G an infinite family of non-isomorphic indecomposable modules in $\mathcal{G}(R)$; see [10, Thm. 7.8 and proof of 7.4.(1)]. For such R, this gives a constructive proof of the abundance of modules in $\mathcal{G}(R)$.

(4.8) **Question.** Let R be a local ring that is not Gorenstein. Given an indecomposable totally reflexive R-module $G \ncong R$, are there constructions that produce infinite families of non-isomorphic indecomposable modules in $\mathcal{G}(R)$?

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