The Method of Undetermined Coefficients and the Shifting Rule.

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In these notes, we will show how to use operator polynomials and the shifting rule to find a particular solution for a linear, constant coefficient, differential equation.

This method works for the same class of forcing functions (right hand sides) as the method discussed in the book. This method is computationally more efficient than the method in the book. It does not require you to solve the homogeneous equation first.

To use the method, you have to write the equation in operator form. We use D to stand for the differentiation operator, so $Dy = y'$, $D^2y = D[Dy] = y''$ and so forth. Thus we can write the equation

$$y'' + 3y' + 5y = \sin(x)$$

in the form

$$D^2y + 3Dy + 5y = \sin(x).$$

We can rewrite this in the form

$$(D^2 + 3D + 5)y = \sin(x)$$

or just

$$P(D)y = \sin(x)$$

where $P(D)$ is the differential operator

$$P(D) = D^2 + 3D + 5.$$

This operator can be thought of as a polynomial in the differential operator D. In fact, it can be thought of as the result of substituting D for λ in the polynomial

$$P(\lambda) = \lambda^2 + 3\lambda + 5,$$

which is, of course, the characteristic polynomial of the equation.

With this notation, we can state the **Shifting Rule** as follows: Let $P(D)$ be a polynomial in D and let α be a real or complex number. Then, for any sufficiently differentiable function y, we have

$$e^{\alpha x}P(D)y = P(D-\alpha)[e^{\alpha x}y].$$

Note that if we replace α by $-\alpha$, we get the alternate form

$$e^{-\alpha x}P(D)y = P(D+\alpha)[e^{-\alpha x}y]$$

of the shifting rule, which is often helpful.

We'll give some examples of how to apply the shifting rule to find a particular solution in these notes.

Polynomial Right Hand Sides. We first consider the case of an equation of the form

$$P(D)y = q(x)$$

where $q(x)$ is a polynomial of degree n. If $P(D)$ has a non-zero constant term, we look for a solution which is a polynomial of the same degree as $q(x)$.

(1) Example: $(D^2 + 2D + 1)y = x^2 + 1$.

We look for a solution which is a polynomial of degree 2. Thus, we try

$$y = Ax^2 + Bx + C.$$

For this to be a solution we must have

$$\begin{aligned}
x^2+1 &= (D^2+2D+1)y \\
&= D^2y + 2Dy + y \\
&= 2A + 2[2Ax+B] + [Ax^2+Bx+C] \\
&= 2A + 4Ax + 2B + Ax^2 + Bx + C \\
&= Ax^2 + (4A+B)x + (2A+2B+C)
\end{aligned}$$

Equating coefficients of the powers of x on both sides we get

$$\begin{aligned}
A &= 1 \\
4A+B &= 0 \\
2A+2B+C &= 1.
\end{aligned}$$

The solution of this system of equations is clearly

$$A=1, \qquad B=-4, \qquad C=7.$$

and so our particular solution is

$$y = x^2 - 4x + 7$$

The other case to consider is where $P(D)$ has no constant term. In this case, we take out as many factors of D as we can, which reduces the problem to solving a lower order equation and then integrating. The procedure is shown in the next example.

(2) Example: $(D^2+3D)y = x^2$.

We rewrite the equation as

$$(D+3)Dy = x^2.$$

Temporarily, let $z = Dy$. Then z satisfies

$$(D+3)z = x^2.$$

We can't pull out any more factors of D, so we look for a solution which is a polynomial of the same degree as the right hand side. Thus, we try

$$z = Ax^2 + Bx + C.$$

Plugging this into the equation for z yields

$$\begin{aligned}
x^2 &= Dz + 3z \\
&= 2Ax + B + 3(Ax^2+Bx+C) \\
&= 2Ax + B + 3Ax^2 + 3Bx + 3C \\
&= 3Ax^2 + (2A+3B)x + (B+3C)
\end{aligned}$$

This yields the system of equations

$$\begin{aligned}
3A &= 1 \\
2A+3B &= 0 \\
B+3C &= 0
\end{aligned}$$

which have the solution

$$A = \frac{1}{3}, \qquad B = -\frac{2}{9}, \qquad C = \frac{2}{27}.$$

Thus, we get

$$z = \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}.$$

Now, recall that $z = Dy$. Thus

$$Dy = \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}.$$

We can then find y be taking anti-derivatives:

$$y = \frac{1}{9}x^3 - \frac{1}{9}x^2 + \frac{2}{27}x$$

(we can take the constant of integration to be zero because we're looking for a particular solution, i.e., just one function that satisfies the equation).

Right Hand Sides of the form $q(x)e^{\alpha x}$. Suppose we have an equation of the form

$$P(D)y = q(x)e^{\alpha x}.$$

where $q(x)$ is a polynomial. To solve this equation, we rewrite it as

$$e^{-\alpha x}P(D)y = q(x).$$

By the shifting rule, this is the same as

$$P(D+\alpha)[e^{-\alpha x}y] = q(x).$$

If we let $Q(D) = P(D+\alpha)$ and $z = e^{-\alpha x}y$, then z satisfies the equation

$$Q(D)z = q(x).$$

Since the right hand side is a polynomial, we know how to solve this equation. Once we've found z, we can solve $z = e^{-\alpha x}y$ for y, namely $y = e^{\alpha x}z$.

(3) Example: $(D^2 + 2D + 1)y = xe^{2x}$.

We write the equation in the form

$$P(D)y = xe^{2x} \tag{1}$$

where

$$P(D) = D^2 + 2D + 1. \tag{2}$$

We rewrite equation (1) in the form

$$e^{-2x}P(D)y = x.$$

By the shifting rule, this is equivalent to

$$P(D+2)[e^{-2x}y] = x.$$

Temporarily, let $z = e^{-2x}y$, so

$$P(D+2)z = x.$$

Now, we calculate $P(D+2)$ by substituting $D+2$ for D in equation (2). This gives

$$\begin{aligned} P(D+2) &= (D+2)^2 + 2(D+2) + 1 \\ &= D^2 + 4D + 4 + 2D + 4 + 1 \\ &= D^2 + 6D + 9. \end{aligned}$$

Thus, z satisfies the equation

(3) $$(D^2 + 6D + 9)z = x.$$

For z, we try a polynomial of the same degree as the right hand side, so we try

$$z = Ax + B.$$

Plugging this into (3) yields

$$\begin{aligned} x &= D^2 z + 6Dz + 9z \\ &= 0 + 6(A) + 9(Ax + B) \\ &= 6A + 9Ax + 9B \end{aligned}$$

This yields the equations

$$9A = 1, \qquad 6A + 9B = 0,$$

and so we have the solution

$$A = \frac{1}{9}, \qquad B = -\frac{2}{27}.$$

This gives us

$$z = \frac{1}{9}x - \frac{2}{27}.$$

Recalling that $z = e^{-2x}y$, we get $y = ze^{2x}$. Thus, finally, we get the particular solution

$$y = (\frac{1}{9}x - \frac{2}{27})e^{2x}.$$

(4) Example: $(D^2 - 4D + 3)y = 2e^x$.

For the solution, write the equation as

$$P(D)y = 2e^x, \qquad P(D) = D^2 - 4D + 3.$$

Move the exponential over to the left hand side and write the equation as

$$e^{-x}P(D)y = 2.$$

By the shifting rule, this is the same as

$$P(D+1)[e^{-x}y] = 2.$$

Let $z = e^{-x}y$, so we want to solve the equation

$$P(D+1)z = 2.$$

Now calculate $P(D+1)$ as follows:

$$\begin{aligned} P(D+1) &= (D+1)^2 - 4(D+1) + 3 \\ &= D^2 + 2D + 1 - 4D - 4 + 3 \\ &= D^2 - 2D \end{aligned}$$

Thus, the equation for z becomes

$$(D^2 - 2D)z = 2.$$

Since we can factor out a D, we do so and rewrite the equation as

$$(D-2)Dz = 2.$$

Let $w = Dz$, so we have the equation

$$(D-2)w = 2$$

for w. Our trial solution should be a polynomial of the same degree as the right hand side. In this case, the right hand side has degree zero, so we want a polynomial of degree zero, i.e., a constant. Thus, we try

$$w = A.$$

Plugging into the equation yields

$$\begin{aligned} 2 &= Dw - 2w \\ &= 0 - 2A \\ &= -2A \end{aligned}$$

so, clearly, $A = -1$. Thus we have $w = -1$. Since $w = Dz$, we have

$$Dz = -1.$$

Anti-differentiation yields

$$z = -x.$$

Recalling that $z = e^{-x}y$, we finally have

$$y = ze^x = -xe^x.$$

Right Hand Sides of the Form $q(x)e^{\alpha x}\cos(\beta x)$ or $q(x)e^{\alpha x}\sin(\beta x)$.
Consider the equations

$$\begin{aligned} P(D)y &= q(x)e^{\alpha x}\cos(\beta x) \\ P(D)y &= q(x)e^{\alpha x}\sin(\beta x) \end{aligned}$$

where we assume that α and β are real numbers, $q(x)$ is a polynomial with real coefficients, and that the coefficients of $P(D)$ are real.

To deal with these problems, recall that if $\sigma = \alpha + i\beta$ is the complex number with real part α and imaginary part β, then

$$e^{\sigma x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x).$$

Of course, the independent variable x is **always** a real number, so this formula is the expression of $e^{\sigma x}$ is terms of its real and imaginary parts. Thus,

$$q(x)e^{\sigma x} = q(x)e^{\alpha x}\cos(\beta x) + iq(x)e^{\alpha x}\sin(\beta x).$$

Since $q(x)$ is always real, we conclude that

$$q(x)e^{\alpha x}\cos(\beta x) = \mathrm{Re}[q(x)e^{\sigma x}]$$
$$q(x)e^{\alpha x}\sin(\beta x) = \mathrm{Im}[q(x)e^{\sigma x}]$$

Now consider the equation

$$P(D)y = q(x)e^{\alpha x}\cos(\beta x) = \mathrm{Re}[q(x)e^{\sigma x}]. \tag{4}$$

To solve this equation, we find a **complex** solution of

$$P(D)y = q(x)e^{\sigma x}. \tag{5}$$

Once we've found the complex solution, we can get a solution of (4) by taking its real part. Similarly, the imaginary part of the solution of (5) is a solution of

$$P(D)y = \mathrm{Im}[q(x)e^{\sigma x}] = q(x)e^{\alpha x}\sin(\beta x). \tag{6}$$

Thus, by solving the complex equation (5) we solve the two real equations (4) and (6) at the same time.

We can solve (5) by exactly the same method we used above for exponential right hand sides; the only difference is that we now have to deal with complex coefficients.

(5) Example: $(D^2 + 2D + 1)y = x\sin(2x)$

We want to solve

$$(D^2 + 2D + 1)y = x\sin(2x) \tag{7}$$

The right hand side if of the form $q(x)e^{\alpha x}\sin(\beta x)$ where $\alpha = 0$ and $\beta = 2$. Thus,

$$x\sin(2x) = \mathrm{Im}[xe^{2ix}].$$

The complex equation we want to solve is

$$(D^2 + 2D + 1)y = xe^{2ix}. \tag{8}$$

The solution of (7) will be the imaginary part of the solution of equation (8).

We write equation (8) as

$$P(D)y = xe^{2ix}, \qquad P(D) = D^2 + 2D + 1.$$

Moving the exponential over gives

$$e^{-2ix}P(D)y = x.$$

By the shifting rule, this can be rewritten as

$$P(D + 2i)[e^{-2ix}y] = x.$$

Let $z = e^{-2ix}y$, so we want to solve

$$P(D + 2i)z = x.$$

Next, we calculate

$$\begin{aligned} P(D + 2i) &= (D + 2i)^2 + 2(D + 2i) + 1 \\ &= D^2 + 4iD + 4i^2 + 2D + 4i + 1 \\ &= D^2 + 4iD - 4 + 2D + 4i + 1 \\ &= D^2 + (2 + 4i)D + (-3 + 4i) \end{aligned}$$

Thus, the equation for z is

$$[D^2 + (2+4i)D + (-3+4i)]z = x.$$

Our trial solution is

$$z = Ax + B.$$

Plugging this into the equation yields

$$\begin{aligned} x &= D^2 z + (2+4i)Dz + (-3+4i)z \\ &= 0 + (2+4i)A + (-3+4i)(Ax+B) \\ &= [(-3+4i)A]x + [(2+4i)A + (-3+4i)B]. \end{aligned}$$

Equating coefficients gives the equations

$$(-3+4i)A = 1 \tag{9}$$
$$(2+4i)A + (-3+4i)B = 0 \tag{10}$$

From equation (9) we have

$$A = \frac{1}{-3+4i} = \frac{1}{-3+4i}\frac{-3-4i}{-3-4i} = \frac{-3-4i}{(-3)^2+(4)^2} = -3/25 - 4i/25.$$

Then, from equation (10), we have

$$\begin{aligned} (-3+4i)B &= -(2+4i)A = (-2-4i)(-3/25-4i/25) = 6/25 - 16/25 + 12i/25 + 8i/25 = -2/5 + 4i/5 \\ B &= (-2/5+4i/5)/(-3+4i) = (-2/5+4i/5)(-3/25-4i/25) = 22/125 - 4i/125. \end{aligned}$$

Thus, we get

$$z = (-3/25 - 4i/25)x + (22/125 - 4i/125) = [(-3/25)x + 22/125] - i[(4/25)x + 4/125]$$

Recalling that $z = e^{-2ix}y$, we get

$$\begin{aligned} y &= ze^{2ix} \\ &= \{[(-3/25)x + 22/125] - i[(4/25)x + 4/125]\}\{\cos(2x) + i\sin(2x)\} \\ &= \{[(-3/25)x + 22/125]\cos(2x) + [(4/25)x + 4/125]\sin(2x)\} \\ &\quad + i\{[(-3/25)x + 22/125]\sin(2x) - [(4/25)x + 4/125]\cos(2x)\} \end{aligned}$$

This is a particular solution of the complex equation (8). Taking the imaginary part of this solution, we finally get the particular solution

$$y = [(-3/25)x + 22/125]\sin(2x) - [(4/25)x + 4/125]\cos(2x)$$

for the original equation (7).

Notice that the real part

$$[(-3/25)x + 22/125]\cos(2x) + [(4/25)x + 4/125]\sin(2x)$$

of the solution of (8) is a particular solution of the equation

$$(D^2 + 2D + 1)y = \mathrm{Re}[xe^{2ix}] = x\cos(2x).$$

(6) Example: $(D^2 - 2D + 2)y = e^x \cos(x)$.

We want to solve

$$(D^2 - 2D + 2)y = e^x \cos(x) = \mathrm{Re}[e^{(1+i)x}], \tag{11}$$

so we solve the complex equation

$$(D^2 - 2D + 2)y = e^{(1+i)x} \tag{12}$$

and then take the real part. We can rewrite the complex equation as

$$P(D)y = e^{(1+i)x}, \qquad P(D) = D^2 - 2D + 2.$$

Moving the exponential to the left side gives

$$e^{-(1+i)x} P(D)y = 1.$$

By the shifting rule, this is the same as

$$P(D + (1+i))[e^{-(1+i)x} y] = 1.$$

Let $z = e^{-(1+i)x} y$, so we want to solve

$$P(D + (1+i))z = 1.$$

Now, we calculate

$$\begin{aligned} P(D + (1+i)) &= [D + (1+i)]^2 - 2[D + (1+i)] + 2 \\ &= D^2 + 2(1+i)D + (1+i)^2 - 2D - 2 - 2i + 2 \\ &= D^2 + (2+2i)D + 2i - 2D - 2 - 2i + 2 \\ &= D^2 + 2iD \end{aligned}$$

Thus, the equation for z is

$$(D^2 + 2iD)z = 1.$$

We factor out a D and write this as

$$(D + 2i)Dz = 1.$$

Thus, $w = Dz$ satisfies

$$(D + 2i)w = 1.$$

Since the right hand side is a polynomial of degree zero, we try a polynomial of degree zero for w, so let $w = A$. Plugging into the equation gives just

$$2iA = 1.$$

Thus,

$$w = A = \frac{1}{2i} = -\frac{1}{2}i.$$

Since, $Dz = w$, anti-differentiation gives

$$z = -\frac{1}{2}ix.$$

Since $z = e^{-(1+i)x} y$, we have

$$\begin{aligned} y &= -\frac{1}{2}ixe^{(1+i)x} \\ &= -\frac{1}{2}ix[e^x \cos(x) + ie^x \sin(x)] \\ &= \frac{1}{2}xe^x \sin(x) - i[\frac{1}{2}xe^x \cos(x)]. \end{aligned}$$

This is a particular solution of the complex equation (12). The real part of this is a particular solution to the original equation, so our final answer (for a particular solution of equation (11)) is

$$y = \frac{1}{2}xe^x \sin(x).$$