A SURFACE AREA PROBLEM

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We begin working in three dimensional space with the usual coordinates (x, y, z). We want to find the area of the part of the surface

$$\frac{x^2 + y^2}{4} + z^2 = 1,$$

that lies above the xy-plane, which we'll denote by S.

The surface S is rotationally symmetric around the z-axis,so it is convenient to introduce cylindrical coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $z = z$. The r coordinate is the distance from the z-axis, $r = \sqrt{x^2 + y^2}$. We can then write the equation of the surface as

$$\frac{r^2}{4} + z^2 = 1. \tag{1}$$

The most satisfying approach would be to start with the general framework for evaluating surface integrals by parametrizing the surface. Since our surface is a surface of revolution, we can look in an earlier section of the calculus book for the section on area of a surface of revolution.

For the general setup there, we can take the independent variable along the axis of rotation to be z and take the dependent coordinate to be r, the distance from the axis of rotation. The general setup is shown in figure . The formula we get for the area generated by rotating the curve $r = f(z)$ around the z-axis for $z = a$ to $z = b$ is

$$A := 2\pi \int_a^b f(z)\sqrt{1 + [f'(z)]^2}\, dz.$$

In our case, we can solve (1) for z to get

$$r = f(z) = 2\sqrt{1 - z^2}.$$

The positive square root gives us a profile curve with the region $r > 0$. Since we want the part of the surface given by (1) in the positive z region, the z range is $0 \le z \le 1$.

Applying our calculus training, we have

$$f(z) = 2(1 - z^2)^{1/2}$$

and so

$$f'(z) = 2\frac{1}{2}(1 - z^2)^{-1/2}(-2z) = -\frac{2z}{\sqrt{1 - z^2}}$$

Thus, we have

$$\begin{aligned}1 + [f'(z)]^2 &= 1 + \frac{4z^2}{1 - z^2} \\ &= \frac{1 - z^2}{1 - z^2} + \frac{4z^2}{1 - z^2} \\ &= \frac{1 + 3z^2}{1 - z^2},\end{aligned}$$

so, finally,

$$\sqrt{1 + [f'(z)]^2} = \frac{\sqrt{1 + 3z^2}}{\sqrt{1 - z^2}}.$$

We can now write down the integral for the area of S as

$$\begin{aligned}A &= 2\pi \int_0^1 f(z)\sqrt{1 + [f'(z)]^2}\, dz \\ &= 2\pi \int_0^1 2\sqrt{1 - z^2}\ \ \frac{\sqrt{1 + 3z^2}}{\sqrt{1 - z^2}}\, dz \\ &= 4\pi \int_0^1 \sqrt{1 + 3z^2}\, dz\end{aligned}$$

I wouldn't blame anyone in an advanced course for typing that into Maple or Mathematica, but in the calculus spirit, let's evaluate it.

To simplify a bit, consider the integral

$$J = \int_0^1 \sqrt{1 + 3z^2}\, dz. \tag{2}$$

To solve this, make the trigonometric substitution

$$\sqrt{3}\ \ z = \tan(\theta) \tag{3}$$

or, in other words, $\theta = \arctan(\sqrt{3}z)$.

From (3),

$$\sqrt{3}\ dz = \sec^2(\theta)\, d\theta$$

and

$$1 + 3z^2 = 1 + (\sqrt{3}z)^2 = 1 + \tan^2(\theta) = \sec^2(\theta),$$

from which

$$\sqrt{1 + 3z^2} = \sec(\theta)$$

Substituting in (2), we have

$$J = \frac{1}{\sqrt{3}} \int_{\theta_1}^{\theta_2} \sec^3(\theta)\, d\theta. \tag{4}$$

We have

$$\theta_0 = \arctan(\sqrt{3}(0)) = 0$$

at the lower limit and

$$\theta_1 = \arctan(\sqrt{3}(1)) = \frac{\pi}{3},$$

at the upper limit. So, we finally have

$$J = \frac{1}{\sqrt{3}} \int_0^{\pi/3} sec^3(\theta)\, d\theta.$$

For simplicity again, let

$$K = \int_0^{\pi/3} \sec^3(\theta)\, d\theta.$$

Looking at a good table of integrals, there is a reduction formula for powers of secant (which is derived by integration by parts). In this case we get

$$\int \sec^3(\theta)\, d\theta = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C.$$

Then

$$K = \left[\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)|\right]_0^{\pi/3}.$$

The values of the trig functions are

$$\tan(0) = 0, \qquad \sec(0) = 1$$
$$\tan(\pi/3) = \sqrt{3}, \qquad \sec(\pi/3) = 2.$$

Plugging all this in give

$$K = \sqrt{3} + \frac{1}{2}\ln(2 + \sqrt{3}).$$

Plugging this in, we get

$$J = \frac{1}{\sqrt{3}}K = 1 + \frac{\sqrt{3}\ln(2+\sqrt{3})}{6}$$

Finally, the area of S is

$$A = 4\pi J = 4\pi\left(1 + \frac{\sqrt{3}\ln(2+\sqrt{3})}{6}\right).$$

Evaluating this in Maple, I get

$$A \approx 17.34376541$$

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