Geometric Transformations and Wallpaper Groups

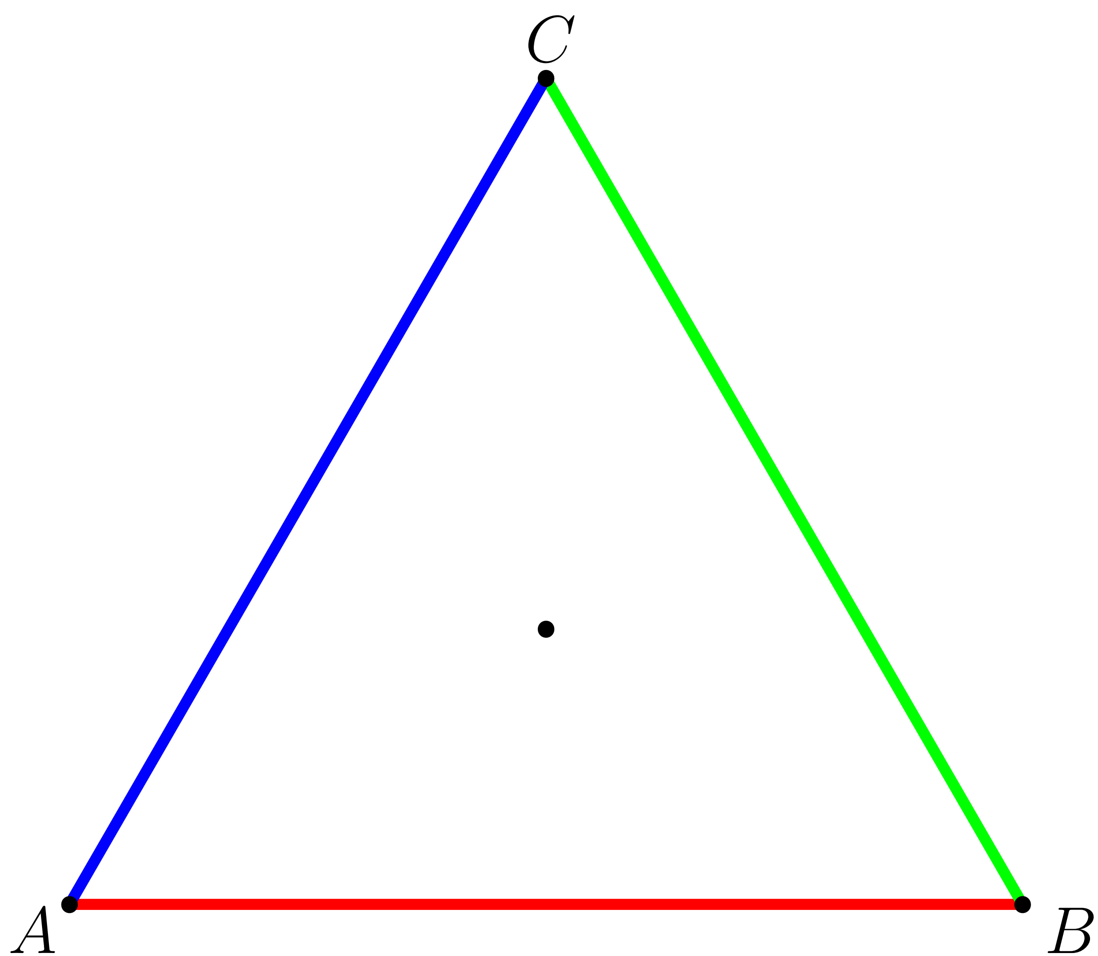
Lance Drager

Texas Tech University

Introduction to Groups of Isometrics

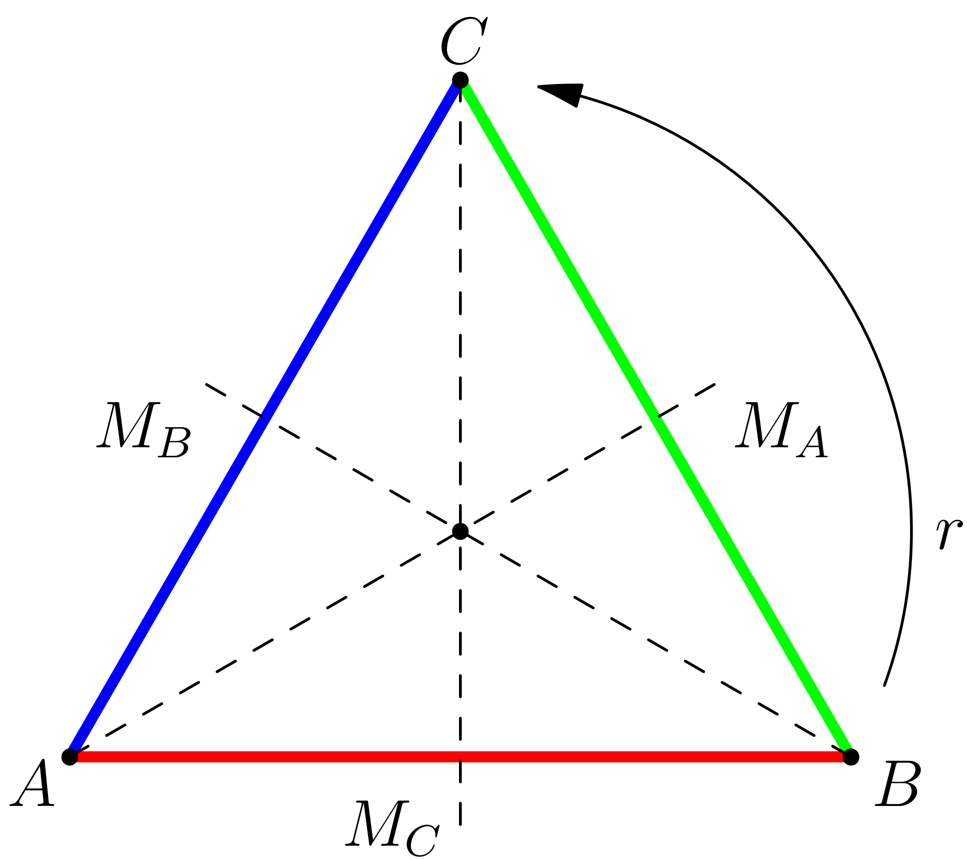
Symmetries of an Equilateral Triangle

- **Problem.** Find all the symmetries of an equilateral triangle.



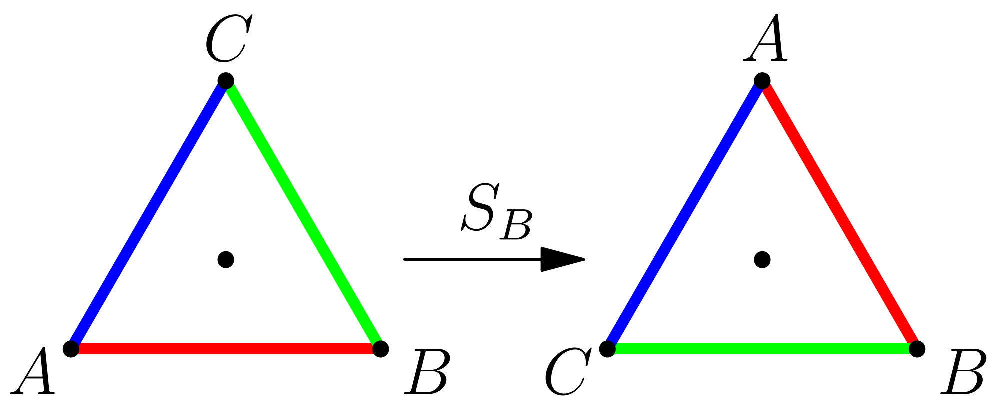
Symmetries of an Equilateral triangle

- There are six, the identity, (call it e), rotation by 120°, call it r, r^2 and the reflections S_A, S_B and $S_C = s$ in the mirror lines M_A, M_B and M_C

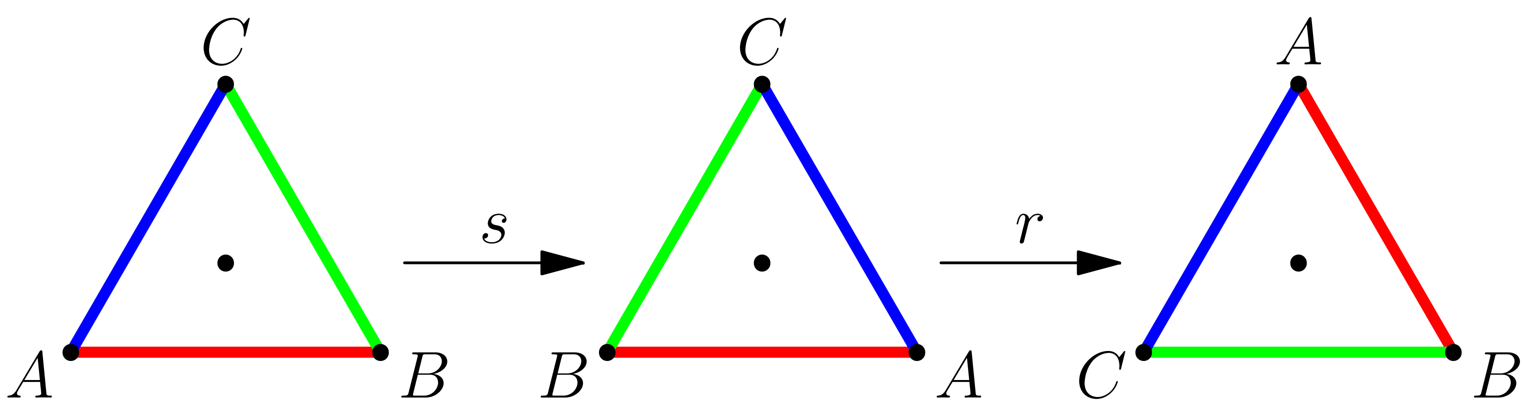


Symmetries of an Equilateral Triangle

- Here's S_B.

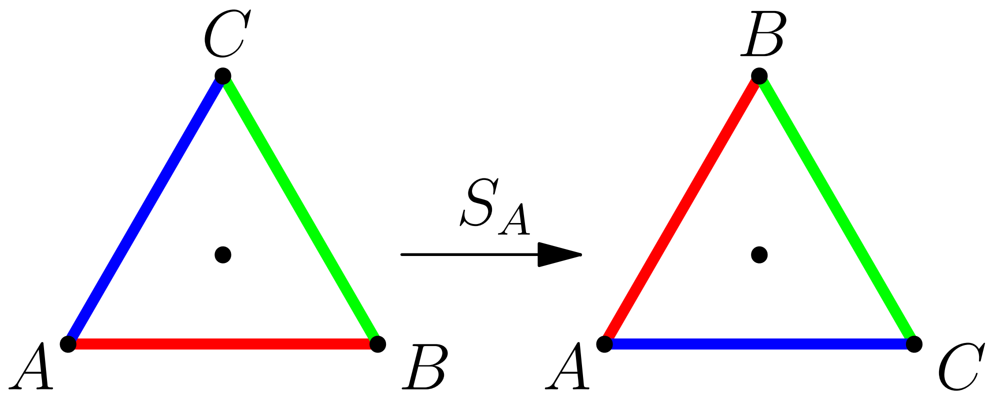


- Here's rs.

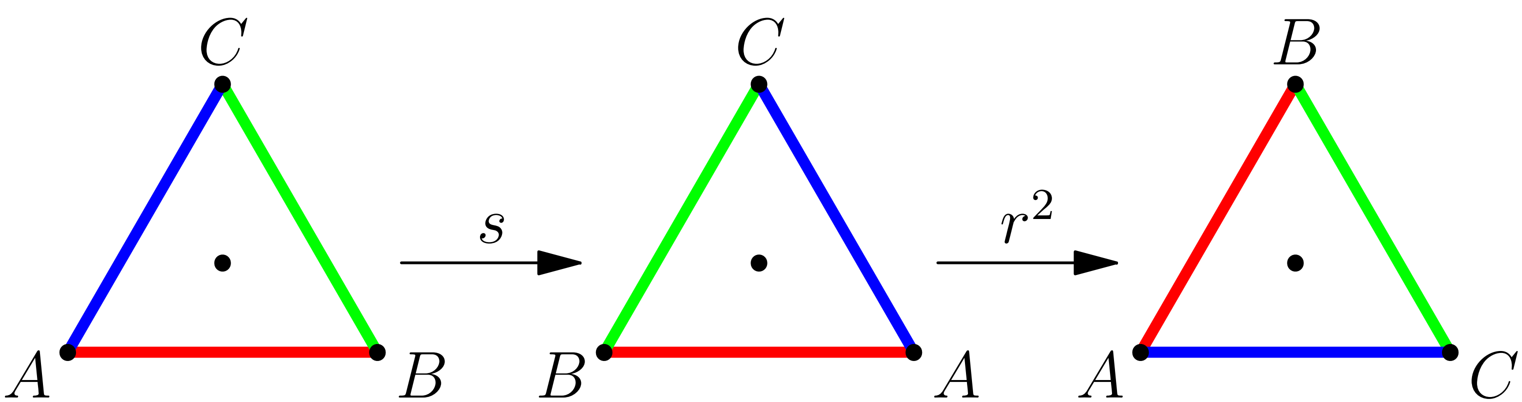


Symmetries of an Equilateral Triangle

- Here's S_A.

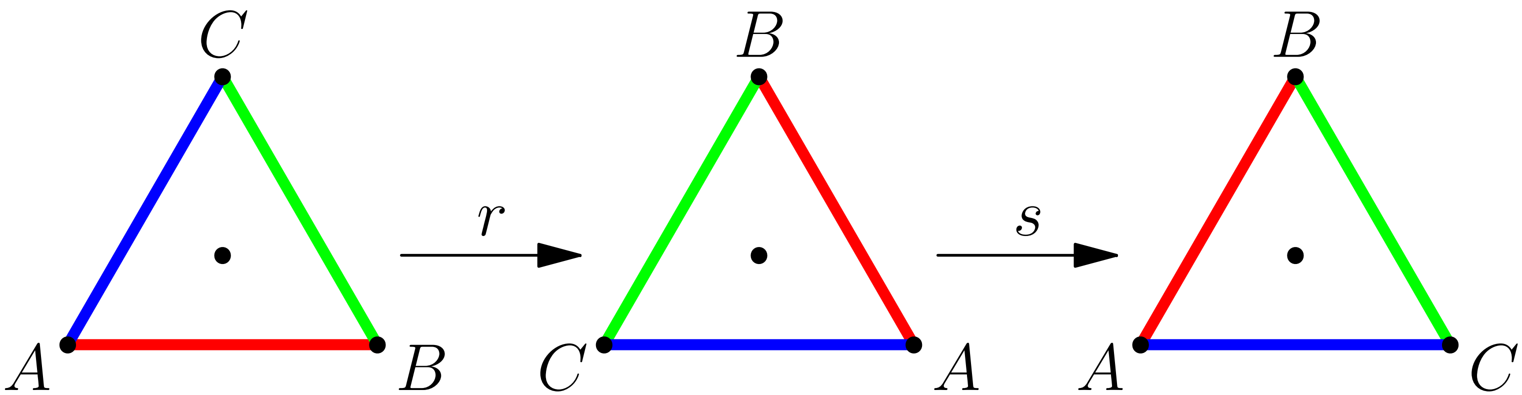


- Here's $r^2 s$.



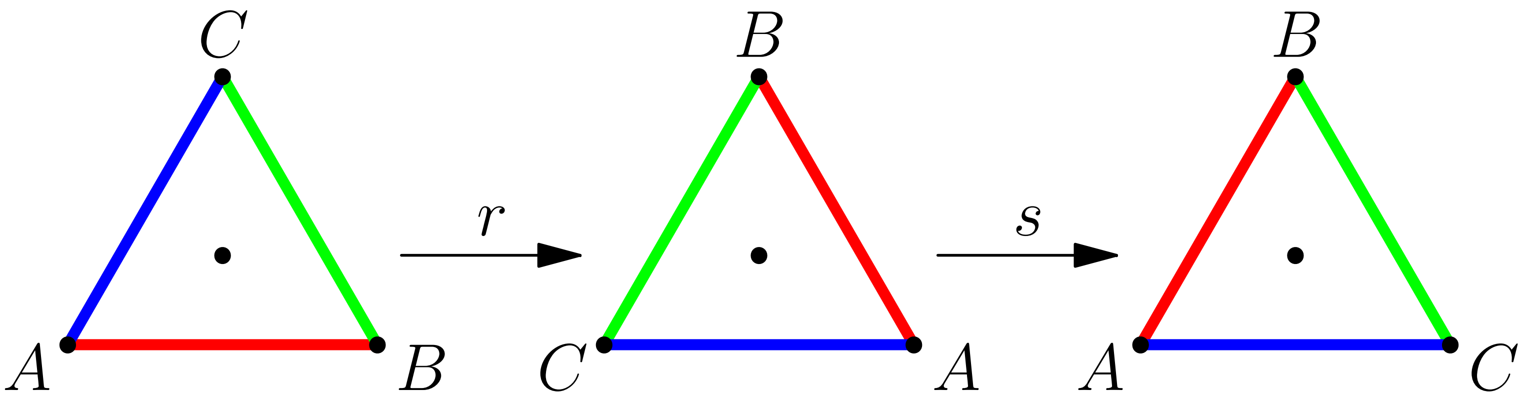
Symmetries of an Equilateral Triangle

- What is sr?

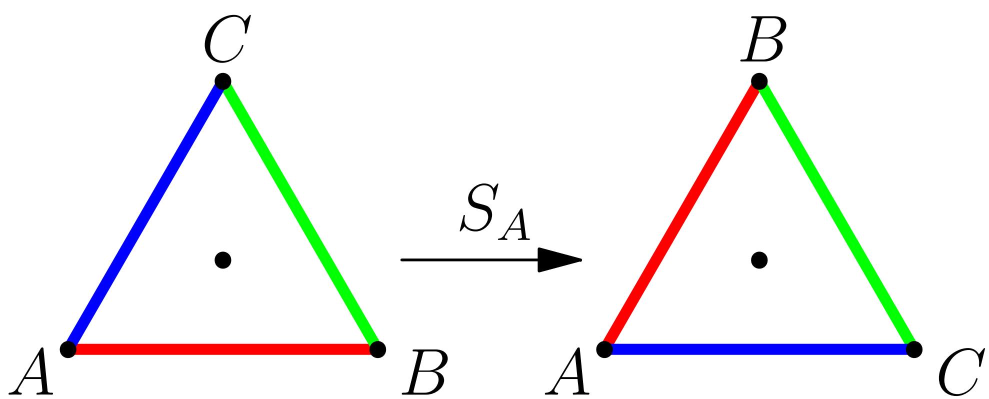


Symmetries of an Equilateral Triangle

- What is sr?



- Answer: $S_A = r^2 s$



Symmetries of an Equilateral Triangle

- The set of symmetries is $D_3 = \{e, r, r^2s, rs, r^2s\}$ and we have the relations $r^3 = e$, $s^2 = e$ and $sr = r^2s$.
- Multiplication table: (Row label) $\times$ (Column label)

	e	r	r^2	s	rs	r^2s
e	e	r	r^2	s	rs	r^2s
r	r	r^2	e	rs	r^2s	s
r^2	r^2	e	r	r^2s	s	rs
s	s	r^2s	rs	e	r^2	r
rs	rs	s	r^2s	r	e	r^2
r^2s	r^2s	rs	s	r^2	r	e

Symmetries of an Equilateral Triangle

- Relations: $r^3 = e$, $s^2 = e$ and $sr = r^2 s$.
- Examples from the Multiplication Table
  - Example: $(r^2 s)(rs) = r^2 srs = r^2(sr)s = r^2(r^2 s)s = r^4 s^2 = r^4 e = rr^3 = re = r$.
  - Example: $(rs)(r^2 s) = rsr^2 s = r(sr)rs = r(r^2 s)rs = r^3 srs = srs = (sr)s = (r^2 s)s = r^2 s^2 = r^2$.

A Little Group Theory

- D_3 is a **group** of isometries: it contains the identity, it is closed under taking products, and it is closed under taking inverses.
- Other groups of isometries.
  - The group $\mathbb{E}$ of all isometries.
  - The group $\mathbb{O}$ of all orthogonal matrices.
  - D_n is the group of symmetries of a regular n-gon. It is generated by r and s with relations $r^n = e$, $s^2 = e$ and $sr = r^{n-1}s$.

A Little Group Theory

- **Definition**. A group G is a set G equipped with a binary operation $G \times G \to G \colon (g, h) \mapsto gh$, such that the following axioms are satisfied
  - **Associative Law**. $g(hk) = (gh)k$ for all $g, h, k \in G$.
  - **Existence of an identity**. There is an element $e \in G$ such that $ge = eg = g$ for all $g \in G$. (This element turns out to be unique.)
  - **Existence of Inverses**. For every $g \in G$ there is an element $h \in G$ so that $gh = hg = e$. This element h turns out to be unique and is denoted by g^{-1}.
- In general, the group operation need not be commutative!

A Little Group Theory

- If the group operation in G is commutative, we say that G is a **commutative group** or an **abelian group**.
- If G is abelian, we sometime choose to write the group operation as $+$, in which case the identity is 0 and the inverse of g is $-g$. Example: the group of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ with the operation of addition.
- If G is a group, a subset $H \subseteq G$ is said to be **a subgroup of** G if it a group in it's own right with the same operation as G, i.e.,
  - $e \in H$.
  - $h_1, h_2 \in H \implies h_1 h_2 \in H$.
  - $h \in H \implies h^{-1} \in H$.

A Little Group Theory

- If G is a group $\{e\} \subseteq G$ and $G \subseteq G$ are always subgroups of G.
- If $g \in G$ we can define the powers of g by
  - $g^0 = e$.
  - If n is a positive integer $g^n = \underbrace{gg \cdots g}_{n \text{ factors}}$.
  - If n is a negative integer, say $n = -p$, $p > 0$, then $g^n = (g^{-1})^p$.
- If $g \in G$ the set $\langle g \rangle$ of powers of g is a subgroup of G called the **cyclic subgroup of G generated by g.**

Group Maps

- Let G and H be groups. A transformation $\varphi\colon G \to H$ is a **Group Homomorphism** or **Group Mapping** if it satisfies
  - $\varphi(e) = e$
  - $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

  It follows that $\varphi(g^{-1}) = \varphi(g)^{-1}$.
- if $a \in G$ is fixed the mapping $\psi_a(g) = aga^{-1}$ is a group map $G \to G$ called **conjugation by** a.
  - $\psi_a(e) = aea^{-1} = aa^{-1} = e$.
  - $\psi_a(gh) = agha^{-1} = aga^{-1}aha^{-1} = (aga^{-1})(aha^{-1}) = \psi_a(g)\psi_a(h)$.

Group Maps and Subgroups

- **Theorem**. Let $\varphi\colon G \to H$ be a group mapping. Let $A \subseteq G$ be a subgroup of G. Let

$$\varphi(A) = \{h \in H \mid h = \varphi(a) \text{ for some } a \in A\}\,.$$

  Then $\varphi(A)$ is a subgroup of H. In particular $\varphi(G) \subseteq H$ is a subgroup of H.

- Proof:
  - $e = \varphi(e)$ and $e \in A$, so $e \in \varphi(A)$.
  - If $h_1, h_2 \in \varphi(A)$ then $h_1 = \varphi(a_1)$ and $h_2 = \varphi(a_2)$ for some $a_1, a_2 \in A$. But then $h_1 h_2 = \varphi(a_1)\varphi(a_2) = \varphi(a_1 a_2)$. Since $a_1 a_2 \in A$, we have $h_1 h_2 \in \varphi(A)$.

Group Maps and Subgroups

- Proof Continued.
  - If $h \in \varphi(A)$, then $h = \varphi(a)$ for some $a \in A$. Then $h^{-1} = \varphi(a)^{-1} = \varphi(a^{-1})$. We have $a^{-1} \in A$, so $h^{-1} \in \varphi(A)$.
- **Theorem**. Let $\varphi \colon G \to H$ be a group map. Define

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e\} .$$

  Then $\ker(\varphi) \subseteq G$ is a subgroup of G.
- Proof:
  - $\varphi(e) = e$ so $e \in \ker(\varphi)$.
  - If $a, b \in \ker(\varphi)$ then $\varphi(a) = e$ and $\varphi(b) = e$. But then $\varphi(ab) = \varphi(a)\varphi(b) = ee = e$, so $ab \in \ker(\varphi)$.

Group Maps and Subgroups

- Proof Continued.
  - if $a \in \ker(\varphi)$ then $\varphi(a) = e$, so $\varphi(a^{-1}) = \varphi(a)^{-1} = e^{-1} = e$, hence $a^{-1} \in \ker(\varphi)$.
- **Definition**. Let G be a group and let N be a subgroup of G. We say that N is a **normal subgroup of** G if $gNg^{-1} \subseteq N$ for all $g \in G$. In other words

$$g \in G \text{ and } n \in N \implies gng^{-1} \in N.$$

- If $\varphi\colon G \to H$ is a group map, $\ker(\varphi)$ is a normal subgroup of G.

Group Maps and Subgroups

- Proof: Suppose $k \in \ker(\varphi)$ and $g \in G$. Then

$$\begin{aligned}\varphi(gkg^{-1}) &= \varphi(g)\varphi(k)\varphi(g^{-1}) \\ &= \varphi(g)e\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = e,\end{aligned}$$

  hence $gkg^{-1} \in \ker(\varphi)$.
- If G is abelian, all subgroups are normal. There are lots of subgroups of nonabelian groups that are not normal.

Exercises on Groups

- **Exercise**. Find some subgroups of D_3. Find some normal subgroups of D_3 and some non-normal subgroups.
- **Exercise** Verify that the group of symmetries of a square is the group D_4 described above.
- **Exercise** Let $\varphi\colon G \to H$ be a group map which is one-to-one and onto, so the inverse transformation $\varphi^{-1}\colon H \to G$ exists. Show that φ^{-1} is automatically a group map. We say that φ is a **group isomorphism** from G to H. Two groups G and H are **isomorphic** if there is an isomorphism between them. This means they can be put in one-to-one correspondence so that the group operations match up.

Exercises on Groups

- **Exercise**. Suppose that N is a normal subgroup of G. Suppose $n \in N$ and $g \in G$. Then $ng = gn'$ for some $n' \in N$ and $gn = n''g$ for some $n'' \in N$.
- **Exercise**. Let $\varphi\colon G \to H$ be a group map and suppose that $B \subseteq H$ is a subgroup of H. Define

$$\varphi^{-1}(B) = \{g \in G \mid \varphi(g) \in B\}\,.$$

  Show that $\varphi^{-1}(B)$ is a subgroup of G.
- **Exercise**. If $a \in G$ the conjugation $\psi_a\colon G \to G\colon g \mapsto aga^{-1}$ is an isomorphism with $\psi_a^{-1} = \psi_{a^{-1}}$.

Exercises on Groups (Harder)

- **Exercise**. What are the subgroups of $\mathbb{Z}$?
- **Exercise**. Find, up to isomorphism, all groups with 2, 3 and 4 elements.

Conjugation in $\mathbb{E}$

- We need some facts about conjugation in the group of isometries. I'll write down the facts, do some of the important ones, and leave the rest as exercises (good for the soul).
- In general,

$$\begin{aligned}(A\,|\,a)(B\,|\,b)(A\,|\,a)^{-1} &= (A\,|\,a)(B\,|\,b)(A^{-1}\,|\,-A^{-1}a)\\ &= (A\,|\,a)(BA^{-1}\,|\,b - BA^{-1}a)\\ &= (ABA^{-1}\,|\,a + Ab - ABA^{-1}a).\end{aligned}$$

- If we conjugate a translation $(I\,|\,b)$ by $(A\,|\,a)$ we get

$$(A\,|\,a)(I\,|\,b)(A\,|\,a)^{-1} = (I\,|\,Ab).$$

Conjugations in $\mathbb{E}$

- Conjugation by a translation $(I \mid a)$
  - Conjugating $(I \mid b)$ gives $(I \mid b)$ back.
  - Conjugating a rotation translates the rotocenter by a.
  - Conjugating a Reflection [glide] by $(S \mid \alpha u + \beta v)$, where $Su = u$ and $Sv = -v$ by, $(I \mid a) = (I \mid su + tv)$ gives $(S \mid \alpha u + \beta v + 2tv)$. This again a reflection [glide, same translation] with the mirror [glide] line shifted by tv in the perpendicular direction.
- Conjugation by a Rotation $(R \mid a)$
  - Conjugating $(I \mid b)$ gives $(I \mid Rb)$.
  - Conjugating $(R' \mid b)$ gives $(R' \mid c)$ for some c, i.e. it changes the rotocenter, but not the angle of rotation.

Conjugation in $\mathbb{E}$

- Note in general that $(A \,|\, a) = (I \,|\, a)(A \,|\, 0)$ and so

$$(A \,|\, a)(B \,|\, b)(A \,|\, b)^{-1} = (I \,|\, a)(A \,|\, 0)(B \,|\, b)(A \,|\, 0)^{-1}(I \,|\, a)^{-1}.$$

- Conjugation by Rotation, continued.
  - Let S be a reflection matrix with $Su = u$ and $Sv = -v$. Set $S' = RSR^{-1}$, $u' = Ru$ and $v' = Rv$. Then $S'u' = u'$ and $S'v' = -v'$. Thus, S' is a reflection matrix with the mirror line rotated by R. If we have a reflection or glide $(S \,|\, \alpha u + \beta v)$ then conjugating by $(R \,|\, 0)$ gives $(RSR^{-1} \,|\, \alpha Ru + \beta Rv) = (S' \,|\, \alpha u' + \beta v')$, which is of the same type with the mirror or glide line rotated around the origin.

Conjugation in $\mathbb{E}$

- Conjugation by a reflection or guide.
  - If we conjugate $(I \mid b)$ by $(S \mid a)$ we get $(I \mid Sb)$.
  - If R is a rotation matrix $SRS = R^{-1}$. Hence conjugating $(R \mid b)$ by $(S \mid 0)$ gives $(R^{-1} \mid Sb)$ another rotation with the opposite angle.
  - Let T be a reflection matrix with $Tx = x$ and $Ty = -y$, let S be a reflection matrix, and let $T' = STS$, $x' = Sx$ and $y' = Sy$. Then $T'x' = x'$ and $T'y' = -y'$. Thus T' is a reflection with the mirror transformed by S. If we conjugate $(T \mid \alpha x + \beta y)$ by $(S \mid 0)$, we get $(STS \mid \alpha Sx + \beta Sy) = (T' \mid \alpha x' + \beta y')$, which is of the same type.
- A conjugation map $\psi \colon \mathbb{E} \to \mathbb{E}$ preserves the type of isometries.