Geometric Transformations and Wallpaper Groups

Lance Drager

Texas Tech University
Isometries

A transformation of the plane is an **isometry** if it is one-to-one and onto and

\[
\text{dist}(T(p), T(q)) = \text{dist}(p, q), \quad \text{for all points } p \text{ and } q.
\]

If \( S \) and \( T \) are isometries, so is \( ST \), where \((ST)(p) = S(T(p))\).

If \( T \) is an isometry, so is \( T^{-1} \).

Our goal is to find all the isometries.
Orthogonal Matrices

- A matrix $A$ is **orthogonal** if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^2$.
- If $A$ and $B$ are orthogonal, so is $AB$.
- If $A$ is orthogonal, it is invertible and $A^{-1}$ is orthogonal.
- Rotation matrices are orthogonal.

**Proposition.** $A$ is orthogonal if and only if $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^2$.

$(\Rightarrow)$ $\|Ax\|^2 = Ax \cdot Ax = x \cdot x = \|x\|^2$.

Recall

$$2x \cdot y = \|x\|^2 + \|y\|^2 - \|x - y\|^2.$$
Orthogonal Matrices

\[
2Ax \cdot Ay = \|Ax\|^2 + \|Ay\|^2 - \|Ax - Ay\|^2
\]

\[
= \|Ax\|^2 + \|Ay\|^2 - \|A(x - y)\|^2 \quad \text{(distributive law)}
\]

\[
= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \quad \text{\(A \) is orthogonal}
\]

\[
= 2x \cdot y,
\]

so dividing by 2 gives the result.

An orthogonal matrix preserves angles.

**Proposition.** A matrix is orthogonal if and only if its columns are orthogonal unit vectors.
Orthogonal Matrices

$(\implies)$

\[\|A\mathbf{e}_i\| = \|\mathbf{e}_i\| = 1.\]
\[A\mathbf{e}_1 \cdot A\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 = 0.\]

$(\iff)$ Let $A = [u \mid v]$ where $u$ and $v$ are orthogonal unit vectors. Note $A\mathbf{x} = x_1u + x_2v$.

\[\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x}\]
\[= (x_1u + x_2v) \cdot (x_1u + x_2v)\]
\[= x_1^2u \cdot u + 2x_1x_2u \cdot v + x_2^2v \cdot v\]
\[= x_1^2(1) + 2x_1x_2(0) + x_2^2(1)\]
\[= x_1^2 + x_2^2 = \|\mathbf{x}\|^2.\]
Classifying Orthogonal Matrices

If $A$ is orthogonal, $\text{Col}_1(A) = (\cos(\theta), \sin(\theta))$ for some $\theta$. So the possibilities are

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta),$$

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = S(\theta).$$

In the first case $A = R(\theta)$ is a rotation.
In the second case, let \( u = (\cos(\theta/2), \sin(\theta/2)) \) and \( v = (-\sin(\theta/2), \cos(\theta/2)) \), which are orthogonal unit vectors.

\[
S(\theta)u = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
\sin(\theta) & -\cos(\theta)
\end{bmatrix}
\begin{bmatrix}
\cos(\theta/2) \\
\sin(\theta/2)
\end{bmatrix}
= \begin{bmatrix}
\cos(\theta) \cos(\theta/2) + \sin(\theta) \sin(\theta/2) \\
\sin(\theta) \cos(\theta/2) - \cos(\theta) \sin(\theta/2)
\end{bmatrix}
\]
Classifying Orthogonal Matrices

In the second case, let \( u = (\cos(\theta/2), \sin(\theta/2)) \) and \( v = (-\sin(\theta/2), \cos(\theta/2)) \), which are orthogonal unit vectors.

\[
S(\theta)u = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} \\
= \begin{bmatrix} \cos(\theta) \cos(\theta/2) + \sin(\theta) \sin(\theta/2) \\ \sin(\theta) \cos(\theta/2) - \cos(\theta) \sin(\theta/2) \end{bmatrix} \\
= \begin{bmatrix} \cos(\theta - \theta/2) \\ \sin(\theta - \theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = u.
\]
Classifying Orthogonal Matrices

\[ S(\theta)v = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} = \begin{bmatrix} -\cos(\theta)\sin(\theta/2) + \sin(\theta)\cos(\theta/2) \\ -\sin(\theta)\sin(\theta/2) - \cos(\theta)\cos(\theta/2) \end{bmatrix} = \begin{bmatrix} \sin(\theta - \theta/2) \\ -\cos(\theta - \theta/2) \end{bmatrix} = \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{bmatrix} = -v. \]
Classifying Orthogonal Matrices

$S(\theta)u = u$ and $S(\theta)v = -v$. If $x = tu + sv$, then $S(\theta)x = tu - sv$. 

[Diagram showing vectors $u$, $v$, $tu$, $sv$, $x$, $S(\theta)x$, and line $M$.]
Classifying Orthogonal matrices

- $S(\theta)$ is a **reflection** with its mirror at an angle of $\theta/2$.

- **Exercise.** Let $A$ be an orthogonal matrix. Then $A$ is a rotation (or the identity) if and only if $\det(A) = 1$, and $A$ is a reflection if and only if $\det(A) = -1$.

- **Exercise.** If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ define the transpose of $A$ by

  $$A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$  

  Show that $A$ is orthogonal if and only if $A^{-1} = A^T$. 

Exercise. Verify the following identities.

\[ R(\theta)S(\varphi) = S(\theta + \varphi), \]
\[ S(\varphi)R(\theta) = S(\varphi - \theta), \]
\[ S(\theta)S(\varphi) = R(\theta - \varphi). \]
For $v \in \mathbb{R}^2$, define $T_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T_v(x) = x + v$. We say $T_v$ is translation by $v$.

$T_uT_v = T_{u+v}$ and $T_v^{-1} = T_{-v}$.

Translations are isometries

$$
\text{dist}(T_v(x), T_v(y)) = \|T_v(x) - T_v(y)\|
= \| (x + v) - (y + v) \|
= \| x + v - y - v \|
= \| x - y \|
= \text{dist}(x, y).
$$
Let \( p_1, p_2 \) and \( p_3 \) be noncollinear points. A point \( x \) is uniquely determined by the three numbers \( r_1 = \text{dist}(x, p_1), \ r_2 = \text{dist}(x, p_2) \) and \( r_3 = \text{dist}(x, p_3). \)
If an isometry fixes three noncollinear points \( p_1, p_2 \) and \( p_3 \), it is the identity.

\[
dist(x, p_i) = dist(T(x), T(p_i)) = dist(T(x), p_i), \quad i = 1, 2, 3,
\]

\[
\implies x = T(x).
\]

Isometries preserve angles.

**Theorem.** Every isometry \( T \) can be written as 
\[
T(x) = Ax + v,
\]
where \( A \) is an orthogonal matrix, i.e. \( T \) is an orthogonal matrix followed by a translation.
Classifying isometries

Proof: The points 0, e_1, e_2 are noncollinear.
Classifying isometries

Proof: The points 0, \( e_1, e_2 \) are noncollinear.

Let \( u = -T(0) \). Then \( T_u T(0) = 0 \).
Classifying isometries

- Proof: The points $0, e_1, e_2$ are noncollinear.
- Let $u = -T(0)$. Then $T_u T(0) = 0$.
- $\text{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix $R$ so that $R T_u T(e_1) = e_1$. 
Classifying isometries

- Proof: The points $0, e_1, e_2$ are noncollinear.
- Let $u = -T(0)$. Then $T_u T(0) = 0$.
- $\text{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix $R$ so that $RT_u T(e_1) = e_1$.
- $RT_u T(e_2) = \pm e_2$. 
Classifying isometries

Proof: The points $0, e_1, e_2$ are noncollinear. Let $u = -T(0)$. Then $T_u T(0) = 0$. \[ \text{dist}(T_u T(e_1), 0) = 1, \text{ so we can find a rotation matrix } R \text{ so that } RT_u T(e_1) = e_1. \]

$RT_u T(e_2) = \pm e_2$. If $RT_u T(e_2) = -e_2$, let $S$ be reflection through the $x$-axis, otherwise $S = I$. 


Classifying isometries

Proof: The points $0, e_1, e_2$ are noncollinear.

Let $u = -T(0)$. Then $T_{u}T(0) = 0$.

$\text{dist}(T_{u}T(e_1), 0) = 1$, so we can find a rotation matrix $R$ so that $RT_{u}T(e_1) = e_1$.

$RT_{u}T(e_2) = \pm e_2$.

If $RT_{u}T(e_2) = -e_2$, let $S$ be reflection through the $x$-axis, otherwise $S = I$.

$SRT_{u}T(0) = 0$, $SRT_{u}T(e_1) = e_1$ and $SRT_{u}T(e_2) = e_2$. 
Classifying isometries

Proof: The points 0, e₁, e₂ are noncollinear.

Let \( u = -T(0) \). Then \( T_u T(0) = 0 \).

\( \text{dist}(T_u T(e_1), 0) = 1 \), so we can find a rotation matrix \( R \) so that \( RT_u T(e_1) = e_1 \).

\( RT_u T(e_2) = \pm e_2 \).

If \( RT_u T(e_2) = -e_2 \), let \( S \) be reflection through the \( x \)-axis, otherwise \( S = I \).

\( SRT_u T(0) = 0, SRT_u T(e_1) = e_1 \) and \( SRT_u T(e_2) = e_2 \).

\( SRT_u T = I \), so \( T = T_u R^{-1} S^{-1} \).
Classifying isometries

Proof: The points 0, e₁, e₂ are noncollinear.

Let \( u = -T(0) \). Then \( T_u T(0) = 0 \).

\[ \text{dist}(T_u T(e_1), 0) = 1, \] so we can find a rotation matrix \( R \) so that \( R T_u T(e_1) = e_1 \).

\( R T_u T(e_2) = \pm e_2 \).

If \( R T_u T(e_2) = -e_2 \), let \( S \) be reflection through the \( x \)-axis, otherwise \( S = I \).

\[ S R T_u T(0) = 0, \ S R T_u T(e_1) = e_1 \ \text{and} \ S R T_u T(e_2) = e_2. \]

\( S R T_u T = I \), so \( T = T_{-u} R^{-1} S^{-1} \).

Let \( A = R^{-1} S^{-1} \) and \( v = -u \). Then \( T(x) = Ax + v \).
Abbreviate $T(x) = Ax + v$ by $(A \mid v)$.

\[
(A \mid u)(B \mid v)x = (A \mid u)(Bx + v)
= A(Bx + v) + u
= ABx + Av + u
= (AB \mid u + Av)x.
\]

$(A \mid u)(B \mid v) = (AB \mid u + Av)$.

The identity transformation is $(I \mid 0)$. Translation by $v$ is $(I \mid v)$.

$(A \mid u)^{-1} = (A^{-1} \mid -A^{-1}u)$. 
Classifying Isometries

Consider \((R \mid v)\) where \(R = R(\theta) \neq I\) is a rotation.

Look for a point \(p\) so that \((R \mid v)p = p\).

\[
Rp + v = p
\]

\[
\implies p - Rp = v
\]

\[
\implies (I - R)p = v.
\]

If \(\det(I - R) \neq 0\) there is a unique solution \(p\).
Classifying Isometries

\[
\det(I - R) = \begin{vmatrix} 1 - \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & 1 - \cos(\theta) \end{vmatrix}
\]

\[
= (1 - \cos(\theta))^2 + \sin^2(\theta)
\]

\[
= 1 - 2 \cos(\theta) + \cos^2(\theta) + \sin^2(\theta)
\]

\[
= 2(1 - \cos(\theta)) \neq 0
\]

There is a unique fixed point \( p \). \((R \mid v) = (R \mid p - Rp)\).

\((R \mid p - Rp)x = Rx + p - Rp = p + R(x - p)\).
$y = (R | p - Rp)x = p + R(x - p)$ says that this isometry is rotation though angle $\theta$ around the point $p$. 
Consider \((S \mid w)\), where \(S = S(\theta)\) is a reflection matrix.

There are orthogonal unit vectors \(u\) and \(v\) so that \(Su = u\) and \(Sv = -v\).

Write \(w = \alpha u + \beta v\).

**Case 1:** \(\alpha = 0\).

The point \(p = \beta v/2\) is fixed

\[
(S \mid \beta v)p = S(\beta v/2) + \beta v = -\beta v/2 + \beta v = \beta v/2 = p.
\]

The fixed points are exactly \(x = p + tu\) for \(t \in \mathbb{R}\).

\[
(S \mid 2p)(p + tu) = Sp + tSu + 2p = -p + tu + 2p = p + tu.
\]
The line through \( p \) parallel to \( u \) is parametrized by \( x = p + tu \), for \( t \in \mathbb{R} \).
We can write a point \( x \) as \( x = p + tu + sv \). Then \( y = (S | 2p)x = p + tu - sv \).
Classifying Isometries

Thus, $(S \mid \beta v) = (S \mid 2p)$ is a reflection through the mirror $M$.

**Case 2:** $(S \mid \alpha + u + \beta v)$ with $\alpha \neq 0$.

We have

$$(I \mid \alpha u)(S \mid \beta v) = (S \mid \alpha u + \beta v).$$

Thus, $(S \mid \alpha u + \beta v)$ is a reflection $(S \mid \beta v)$ followed by translation in a direction that is parallel to the mirror of the reflection. This is called a glide reflection or just a glide. The mirror line of the reflection is called the glide line.

A glide has no fixed points.
Classifying Isometries

A glide with a horizontal glide line, and the translation vector show in blue.
Classifying Isometries

Theorem. Every isometry of the plane falls into one of the following five mutually exclusive classes.
1. The identity.
2. A translation (not the identity).
3. Rotation about some point (not the identity).
4. A reflection about some line.
5. A glide along some line.

Exercise. Consider the cases for the product $T_1T_2$ of two isometries $T_1$ and $T_2$. In these cases, when do the isometries commute, i.e., when does $T_1T_2 = T_2T_1$?