Geometric Transformations and Wallpaper Groups

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Isometries of the Plane

Isometries

- A transformation of the plane is an **isometry** if it is one-to-one and onto and

  $$\mathrm{dist}(T(p), T(q)) = \mathrm{dist}(p, q), \qquad \text{for all points } p \text{ and } q.$$

- If S and T are isometries, so is ST, where $(ST)(p) = S(T(p))$.
- If T is an isometry, so is T^{-1}.
- Our goal is to find all the isometries.

Orthogonal Matrices

- A matrix A is **orthogonal** if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^2$.
- If A and B are orthogonal, so is AB.
- If A is orthogonal, it is invertible and A^{-1} is orthogonal.
- Rotation matrices are orthogonal.
- **Proposition**. A is orthogonal if and only if $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^2$.
- ($\Longrightarrow$) $\|Ax\|^2 = Ax \cdot Ax = x \cdot x = \|x\|^2$.
- Recall
$$2x \cdot y = \|x\|^2 + \|y\|^2 - \|x - y\|^2.$$

Orthogonal Matrices

- $(\Longleftarrow)$

$$\begin{aligned} 2Ax \cdot Ay &= \|Ax\|^2 + \|Ay\|^2 - \|Ax - Ay\|^2 \\ &= \|Ax\|^2 + \|Ay\|^2 - \|A(x-y)\|^2 \quad \text{(distributive law)} \\ &= \|x\|^2 + \|y\|^2 - \|x-y\|^2 \quad (A \text{ is orthogonal}) \\ &= 2x \cdot y, \end{aligned}$$

  so dividing by 2 gives the result.

- An orthogonal matrix preserves angles.
- **Proposition**. A matrix is orthogonal if and only is its columns are orthogonal unit vectors.

Orthogonal Matrices

- $(\Longrightarrow)$

$$\|Ae_i\| = \|e_i\| = 1.$$
$$Ae_1 \cdot Ae_2 = e_1 \cdot e_2 = 0.$$

- $(\Longleftarrow)$ Let $A = [u \mid v]$ where u and v are orthogonal unit vectors. Note $Ax = x_1 u + x_2 v$.

$$\begin{aligned}
\|Ax\|^2 &= Ax \cdot Ax \\
&= (x_1 u + x_2 v) \cdot (x_1 u + x_2 v) \\
&= x_1^2 u \cdot u + 2x_1 x_2 u \cdot v + x_2^2 v \cdot v \\
&= x_1^2 (1) + 2x_1 x_2 (0) + x_2^2 (1) \\
&= x_1^2 + x_2^2 = \|x\|^2.
\end{aligned}$$

Classifying Orthogonal Matrices

- If A is orthogonal, $\mathrm{Col}_1(A) = (\cos(\theta), \sin(\theta))$ for some θ. So the possibilities are

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta),$$

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = S(\theta).$$

In the first case $A = R(\theta)$ is a rotation.

Classifying Orthogonal Matrices

- In the second case, let $u = (\cos(\theta/2), \sin(\theta/2))$ and $v = (-\sin(\theta/2), \cos(\theta/2))$, which are orthogonal unit vectors.
-

$$S(\theta)u = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\theta/2) + \sin(\theta)\sin(\theta/2) \\ \sin(\theta)\cos(\theta/2) - \cos(\theta)\sin(\theta/2) \end{bmatrix}$$

Classifying Orthogonal Matrices

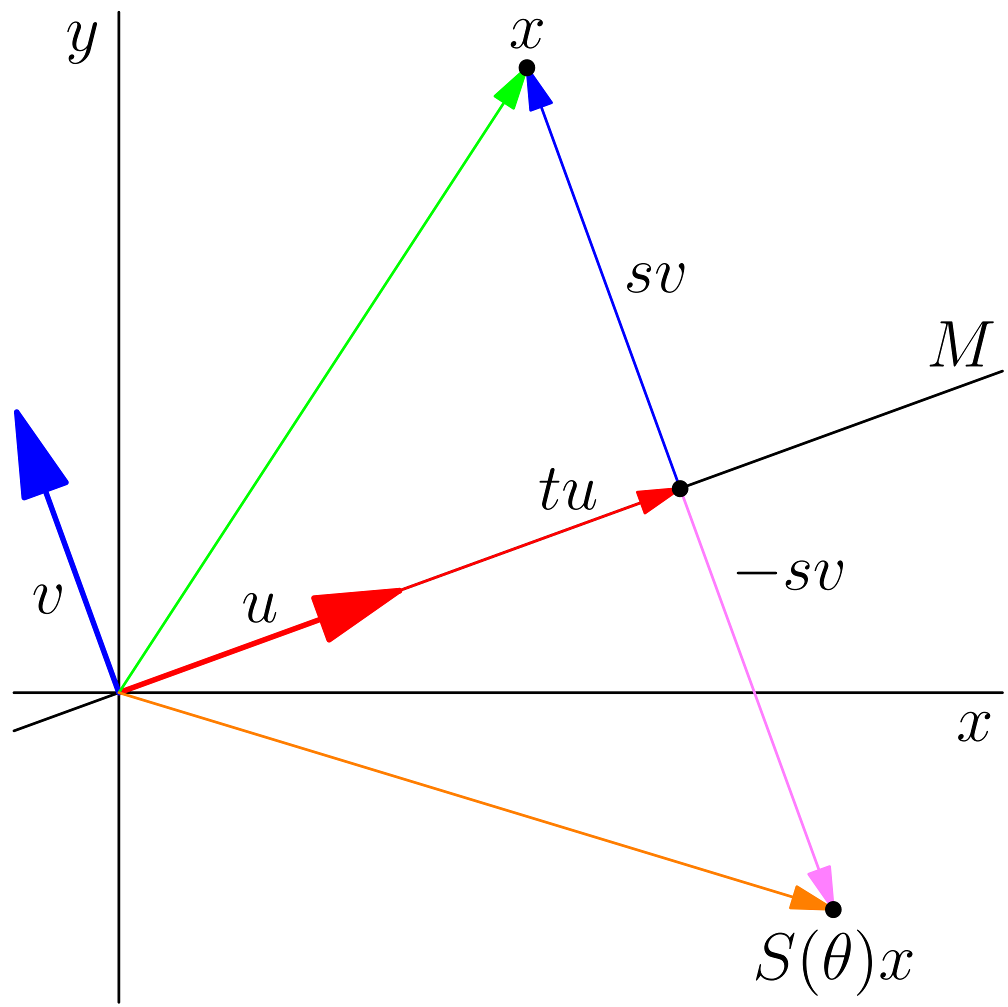
- In the second case, let $u = (\cos(\theta/2), \sin(\theta/2))$ and $v = (-\sin(\theta/2), \cos(\theta/2))$, which are orthogonal unit vectors.
- $$\begin{aligned} S(\theta)u &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta)\cos(\theta/2) + \sin(\theta)\sin(\theta/2) \\ \sin(\theta)\cos(\theta/2) - \cos(\theta)\sin(\theta/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta - \theta/2) \\ \sin(\theta - \theta/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = u. \end{aligned}$$

Classifying Orthogonal Matrices

- $$\begin{aligned} S(\theta)v &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \\ &= \begin{bmatrix} -\cos(\theta)\sin(\theta/2) + \sin(\theta)\cos(\theta/2) \\ -\sin(\theta)\sin(\theta/2) - \cos(\theta)\cos(\theta/2) \end{bmatrix} \\ &= \begin{bmatrix} \sin(\theta - \theta/2) \\ -\cos(\theta - \theta/2) \end{bmatrix} \\ &= \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2) \end{bmatrix} = -\begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} = -v. \end{aligned}$$

Classifying Orthogonal Matrices

- $S(\theta)u = u$ and $S(\theta)v = -v$. If $x = tu + sv$, then $S(\theta)x = tu - sv$.



Classifying Orthogonal matrices

- $S(\theta)$ is a **reflection** with its mirror at an angle of $\theta/2$.
- **Exercise**. Let A be an orthogonal matrix. Then A is a rotation (or the identity) if and only if $\det(A) = 1$, and A is a reflection if and only if $\det(A) = -1$.
- **Exercise**. If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ define the **transpose** of A by $A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A is orthogonal if and only if $A^{-1} = A^T$.

Classifying Orthogonal Matrices

- **Exercise**. Verify the following identities.

$$R(\theta)S(\varphi) = S(\theta + \varphi),$$
$$S(\varphi)R(\theta) = S(\varphi - \theta),$$
$$S(\theta)S(\varphi) = R(\theta - \varphi).$$

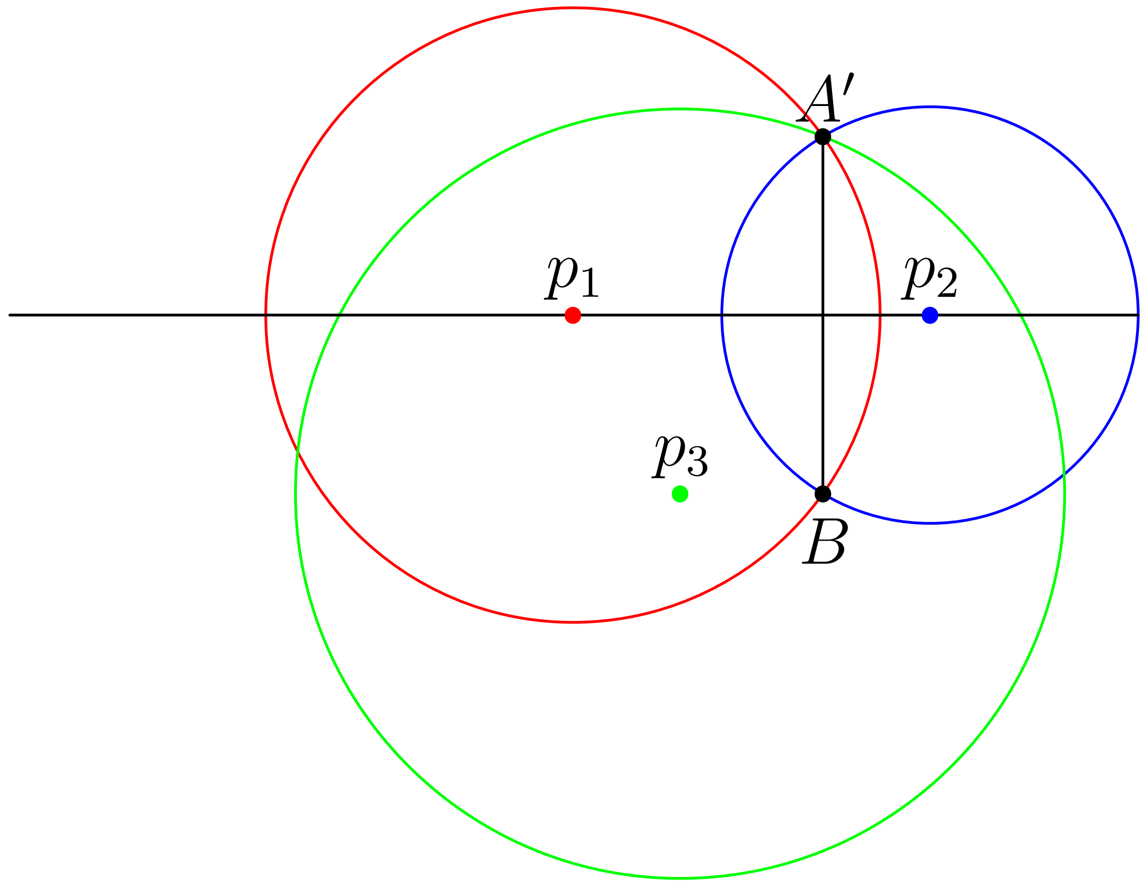
Translations

- For $v \in \mathbb{R}^2$, define $T_v \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T_v(x) = x + v$. We say T_v is **translation by v**.
- $T_u T_v = T_{u+v}$ and $T_v^{-1} = T_{-v}$.
- Translations are isometries

$$
\begin{aligned}
\operatorname{dist}(T_v(x), T_v(y)) &= \|T_v(x) - T_v(y)\| \\
&= \|(x + v) - (y + v)\| \\
&= \|x + v - y - v\| \\
&= \|x - y\| \\
&= \operatorname{dist}(x, y).
\end{aligned}
$$

Classifying Isometries

- Let p_1, p_2 and p_3 be noncollinear points. A point x is uniquely determined by the three numbers $r_1 = \mathrm{dist}(x, p_1)$, $r_2 = \mathrm{dist}(x, p_2)$ and $r_3 = \mathrm{dist}(x, p_3)$.



Classifying Isometries

- If an isometry fixes three noncollinear points p_1, p_2 and p_3, it is the identity.

$$\operatorname{dist}(x, p_i) = \operatorname{dist}(T(x), T(p_i)) = \operatorname{dist}(T(x), p_i), \qquad i = 1, 2, 3,$$
$$\implies x = T(x).$$

- Isometries preserve angles.
- **Theorem**. Every isometry T can be written as $T(x) = Ax + v$, where A is an orthogonal matrix, i.e. T is an orthogonal matrix followed by a translation.

Classifying isometries

- Proof: The points $0, e_1, e_2$ are noncollinear.

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- Let $u = -T(0)$. Then $T_u T(0) = 0$.
- $\mathrm{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix R so that $R T_u T(e_1) = e_1$.

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- $R T_u T(e_2) = \pm e_2$.
- If $R T_u T(e_2) = -e_2$, let S be reflection through the x-axis, otherwise $S = I$.

Classifying isometries

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- Let $u = -T(0)$. Then $T_u T(0) = 0$.
- $\text{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix R so that $RT_u T(e_1) = e_1$.
- $RT_u T(e_2) = \pm e_2$.
- If $RT_u T(e_2) = -e_2$, let S be reflection through the x-axis, otherwise $S = I$.
- $SRT_u T(0) = 0$, $SRT_u T(e_1) = e_1$ and $SRT_u T(e_2) = e_2$.

Classifying isometries

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- $\mathrm{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix R so that $R T_u T(e_1) = e_1$.
- $R T_u T(e_2) = \pm e_2$.
- If $R T_u T(e_2) = -e_2$, let S be reflection through the x-axis, otherwise $S = I$.
- $S R T_u T(0) = 0$, $S R T_u T(e_1) = e_1$ and $S R T_u T(e_2) = e_2$.
- $S R T_u T = I$, so $T = T_{-u} R^{-1} S^{-1}$.

Classifying isometries

- Proof: The points $0, e_1, e_2$ are noncollinear.
- Let $u = -T(0)$. Then $T_u T(0) = 0$.
- $\mathrm{dist}(T_u T(e_1), 0) = 1$, so we can find a rotation matrix R so that $RT_u T(e_1) = e_1$.
- $RT_u T(e_2) = \pm e_2$.
- If $RT_u T(e_2) = -e_2$, let S be reflection through the x-axis, otherwise $S = I$.
- $SRT_u T(0) = 0$, $SRT_u T(e_1) = e_1$ and $SRT_u T(e_2) = e_2$.
- $SRT_u T = I$, so $T = T_{-u} R^{-1} S^{-1}$.
- Let $A = R^{-1} S^{-1}$ and $v = -u$. Then $T(x) = Ax + v$.

Classifying Isometries

- Abbreviate $T(x) = Ax + v$ by $(A \,|\, v)$.
- $$\begin{aligned}(A \,|\, u)(B \,|\, v)x &= (A \,|\, u)(Bx + v) \\ &= A(Bx + v) + u \\ &= ABx + Av + u \\ &= (AB \,|\, u + Av)x.\end{aligned}$$
- $(A \,|\, u)(B \,|\, v) = (AB \,|\, u + Av)$.
- The identity transformation is $(I \,|\, 0)$. Translation by v is $(I \,|\, v)$.
- $(A \,|\, u)^{-1} = (A^{-1} \,|\, -A^{-1}u)$.

Classifying Isometries

- Consider $(R \,|\, v)$ where $R = R(\theta) \neq I$ is a rotation.
- Look for a point p so that $(R \,|\, v)p = p$.

$$Rp + v = p$$
$$\implies \ p - Rp = v$$
$$\implies \ (I - R)p = v.$$

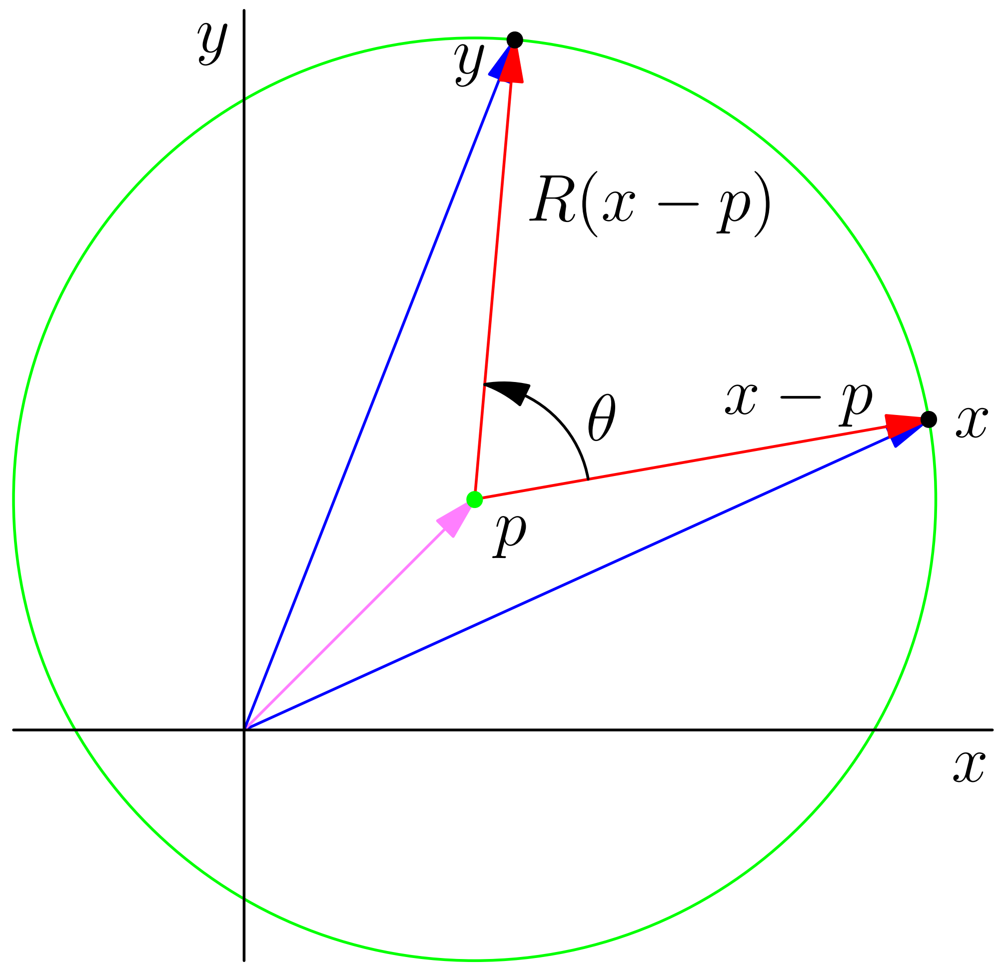
If $\det(I - R) \neq 0$ there is a unique solution p.

Classifying Isometries

- $$\det(I - R) = \begin{vmatrix} 1-\cos(\theta) & \sin(\theta) \\ -\sin(\theta) & 1-\cos(\theta) \end{vmatrix}$$
  $$= (1-\cos(\theta))^2 + \sin^2(\theta)$$
  $$= 1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta)$$
  $$= 2(1-\cos(\theta)) \neq 0$$
- There is a unique fixed point p. $(R \,|\, v) = (R \,|\, p - Rp)$.
- $(R \,|\, p - Rp)x = Rx + p - Rp = p + R(x-p)$.

Classifying Isometries

- $y = (R \,|\, p - Rp)x = p + R(x - p)$ says that this isometry is rotation though angle θ around the point p.



Classifying Isometries

- Consider $(S \,|\, w)$, where $S = S(\theta)$ is a reflection matrix.
- There are orthogonal unit vectors u and v so that $Su = u$ and $Sv = -v$.
- Write $w = \alpha u + \beta v$.
- **Case 1**: $\alpha = 0$.
- The point $p = \beta v/2$ is fixed

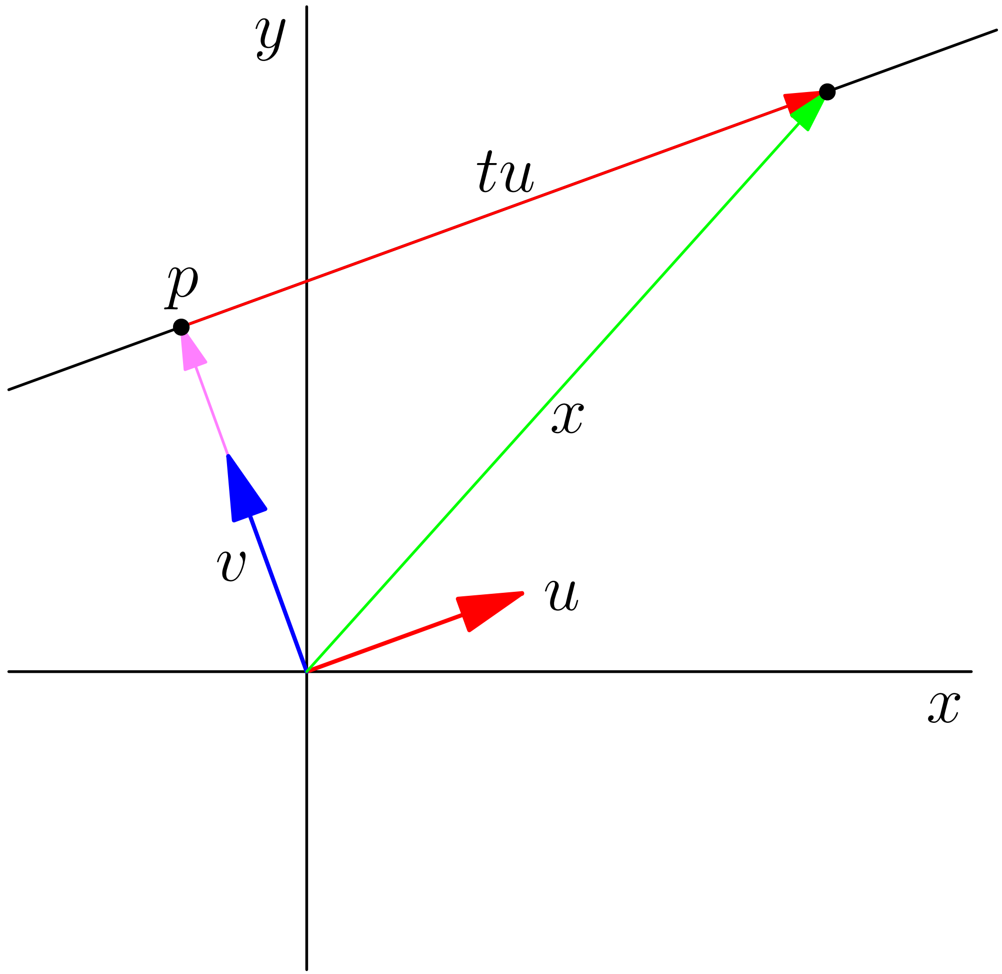
$$(S \,|\, \beta v)p = S(\beta v/2) + \beta v = -\beta v/2 + \beta v = \beta v/2 = p.$$

- The fixed points are exactly $x = p + tu$ for $t \in \mathbb{R}$.

$$(S \,|\, 2p)(p + tu) = Sp + tSu + 2p = -p + tu + 2p = p + tu.$$

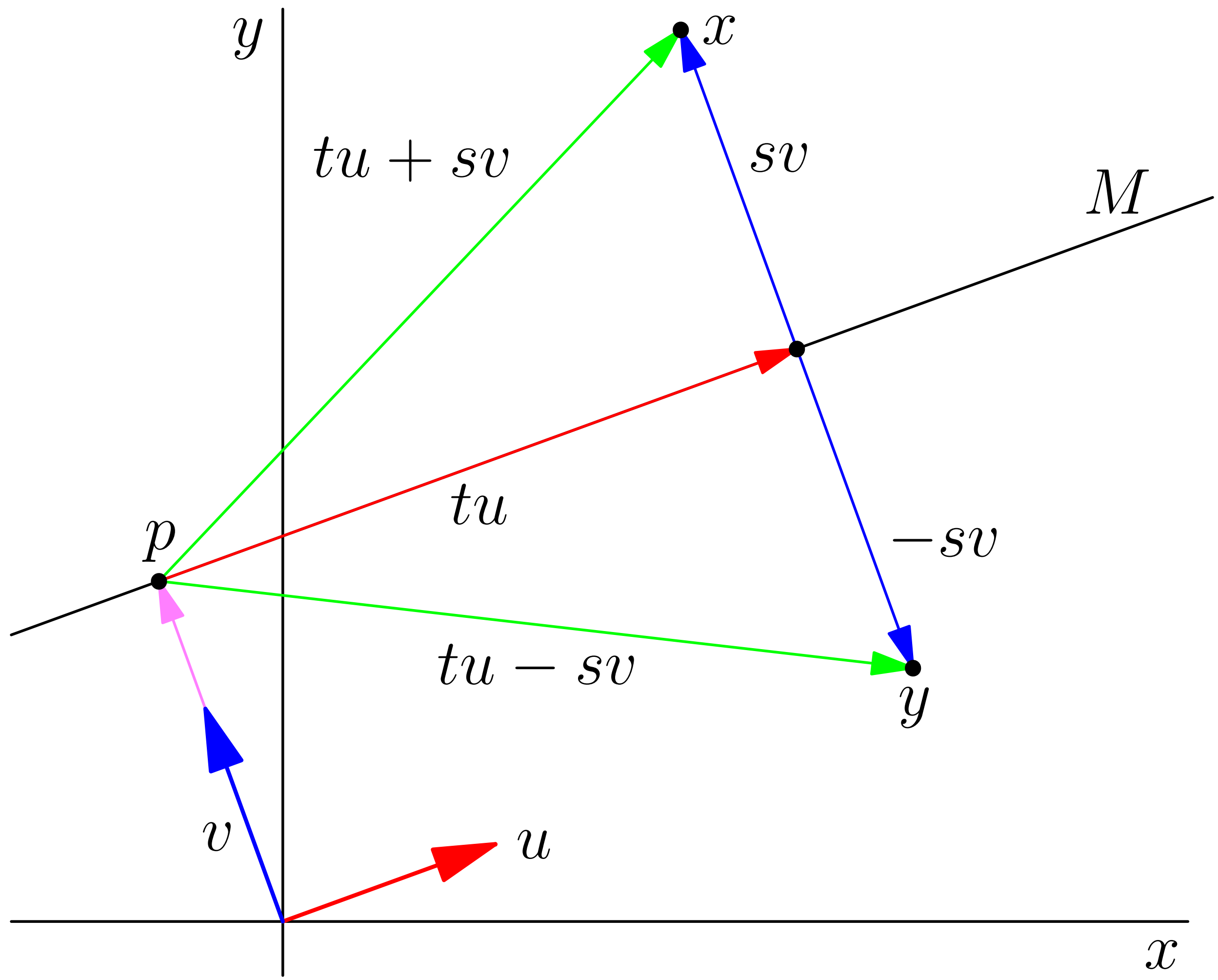
Classifying Isometries

- The line through p parallel to u is parametrized by $x = p + tu$, for $t \in \mathbb{R}$.



Classifying Isometries

- We can write a point x as $x = p + tu + sv$. Then $y = (S \mid 2p)x = p + tu - sv$.

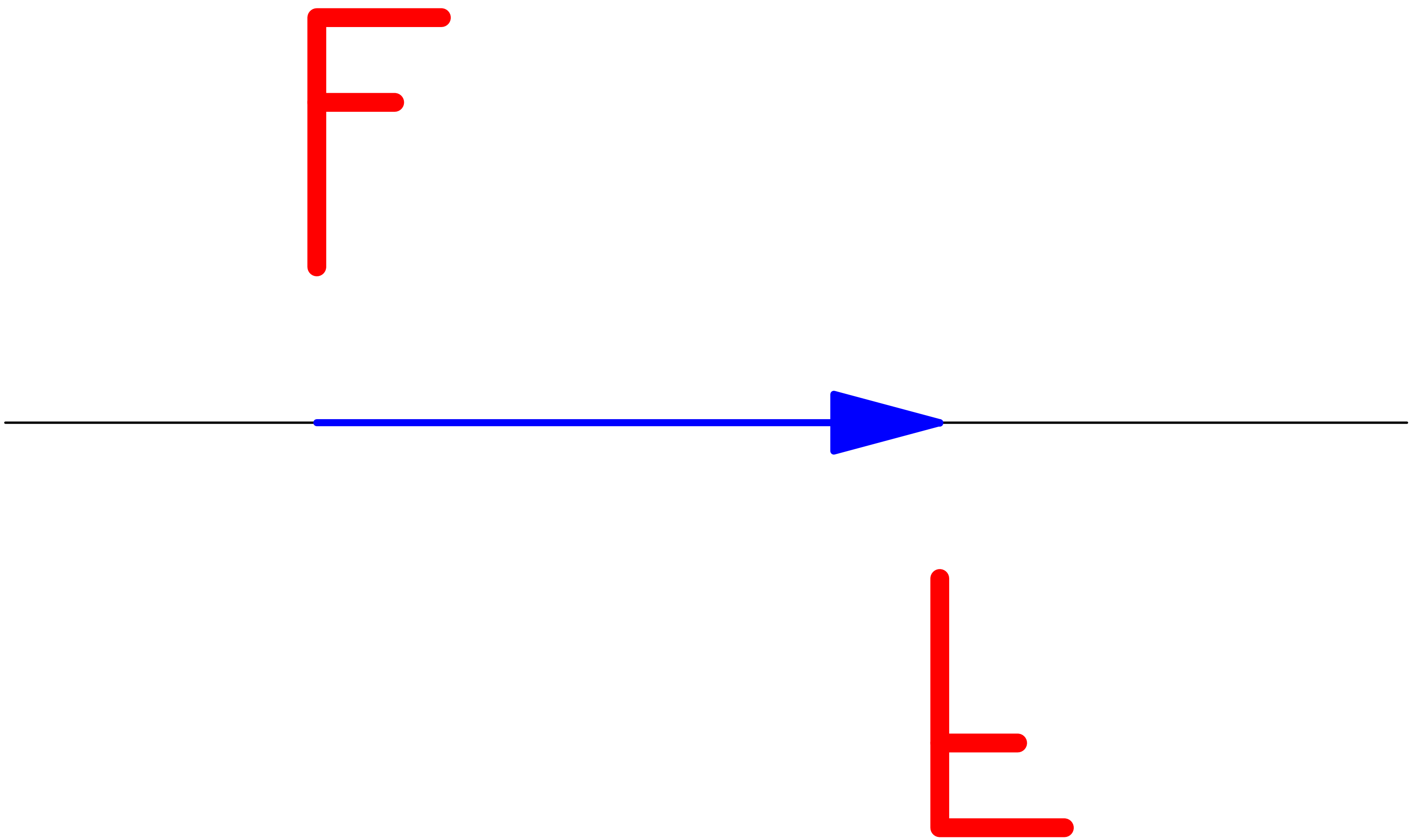


Classifying Isometries

- Thus, $(S \mid \beta v) = (S \mid 2p)$ is a reflection through the mirror M.
- **Case 2**: $(S \mid \alpha + u + \beta v)$ with $\alpha \neq 0$.
- We have
$$(I \mid \alpha u)(S \mid \beta v) = (S \mid \alpha u + \beta v).$$
- Thus, $(S \mid \alpha u + \beta v)$ is a reflection $(S \mid \beta v)$ followed by translation in a direction that is parallel to the mirror of the reflection. This is called a **glide reflection** or just a **glide**. The mirror line of the reflection is called the **glide line**.
- A glide has no fixed points.

Classifying Isometries

- A glide with a horizontal glide line, and the translation vector show in blue.



Classifying Isometries

- **Theorem**. Every isometry of the plane falls into one of the following five mutually exclusive classes.
  1. The identity.
  2. A translation (not the identity).
  3. Rotation about some point (not the identity).
  4. A reflection about some line.
  5. A glide along some line.
- **Exercise**. Consider the cases for the product $T_l T_2$ of two isometries T_1 and T_2. In these cases, when do the isometries commute, i.e, when does $T_1 T_2 = T_2 T_1$?