Geometric Transformations and Wallpaper Groups

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Vector and Matrix Algebra

Vectors

- Our setting is the Euclidean Plane
- A **vector** is a quantity that has a magnitude and a direction.

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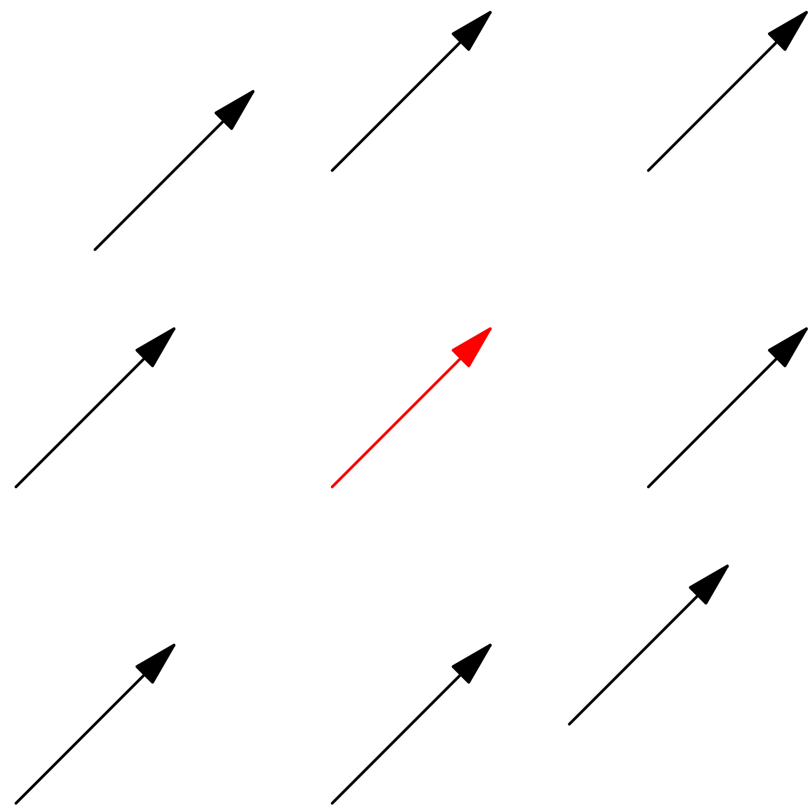


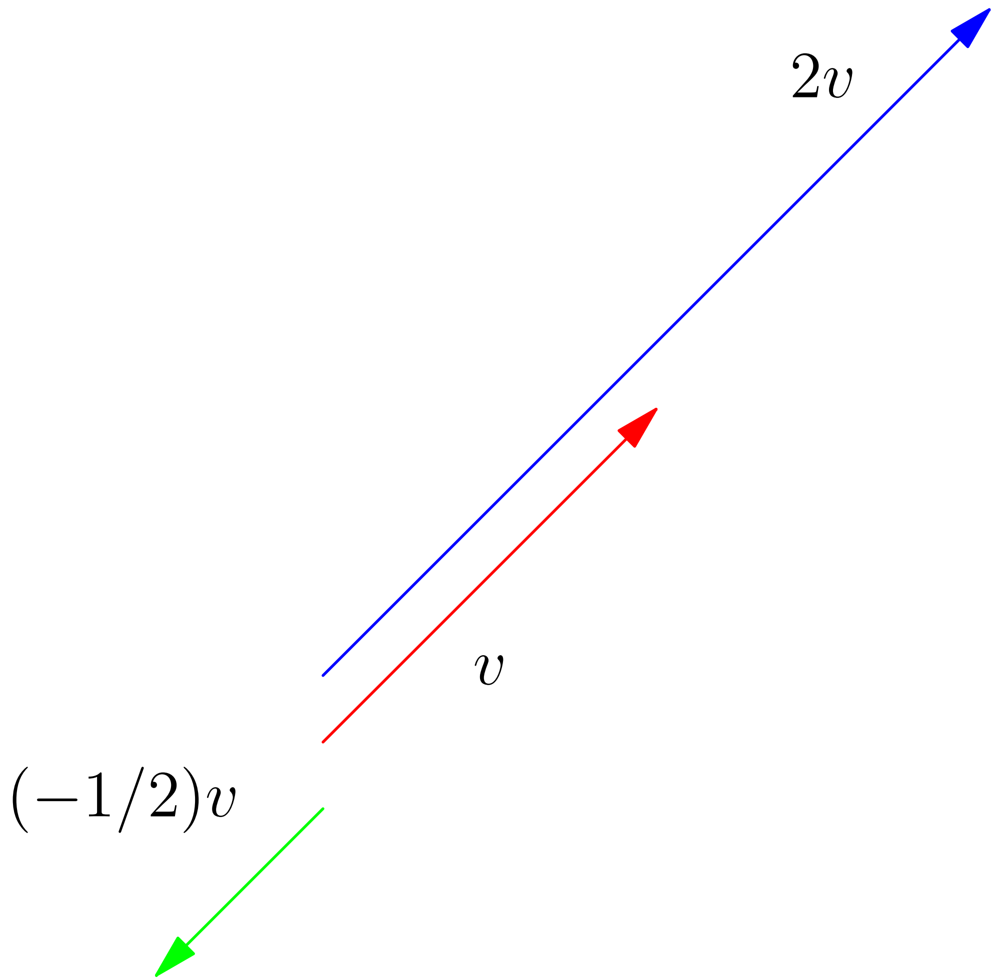
Vector Operations

- 0 is the vector with magnitude zero. Draw it as a point.
- $\|u\|$ denotes the magnitude of the vector u.
- A **scalar** is a quantity with a magnitude ($\pm$) but no direction, i.e., just a number.
- Two operations on vectors
  - Scalar Multiplication.
  - Vector Addition.

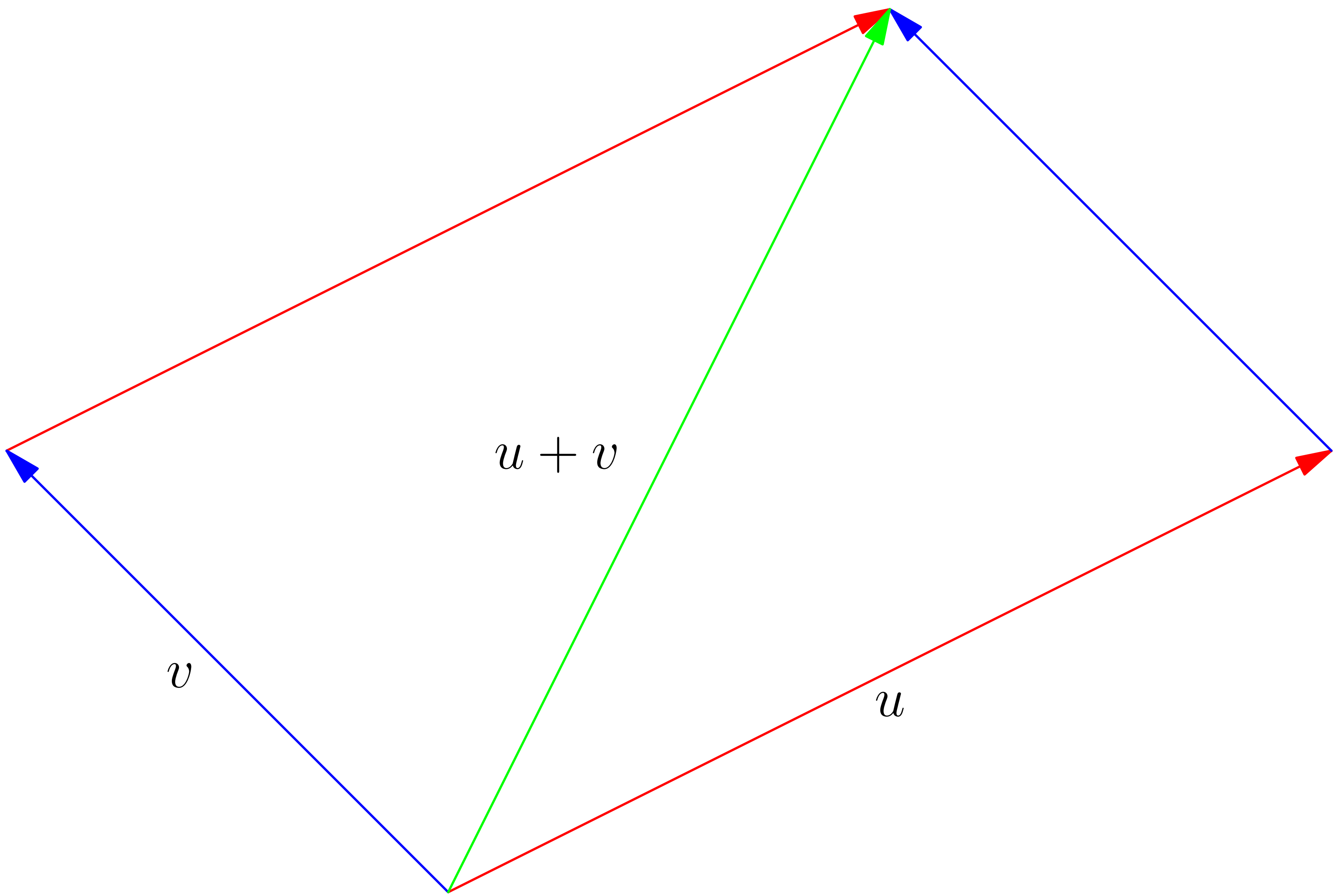
Scalar Multiplication

- Scalar Multiplication sv.
  - If $s = 0$, then sv is the zero vector.
  - If $s > 0$, then sv is the vector whose magnitude is $s\|v\|$ and whose direction in the same as the direction of v.
  - If $s < 0$, then sv is the vector whose magnitude is $|s|\,\|v\|$ and whose direction is opposite to the direction of v.
- $\|sv\| = |s|\,\|v\|$.

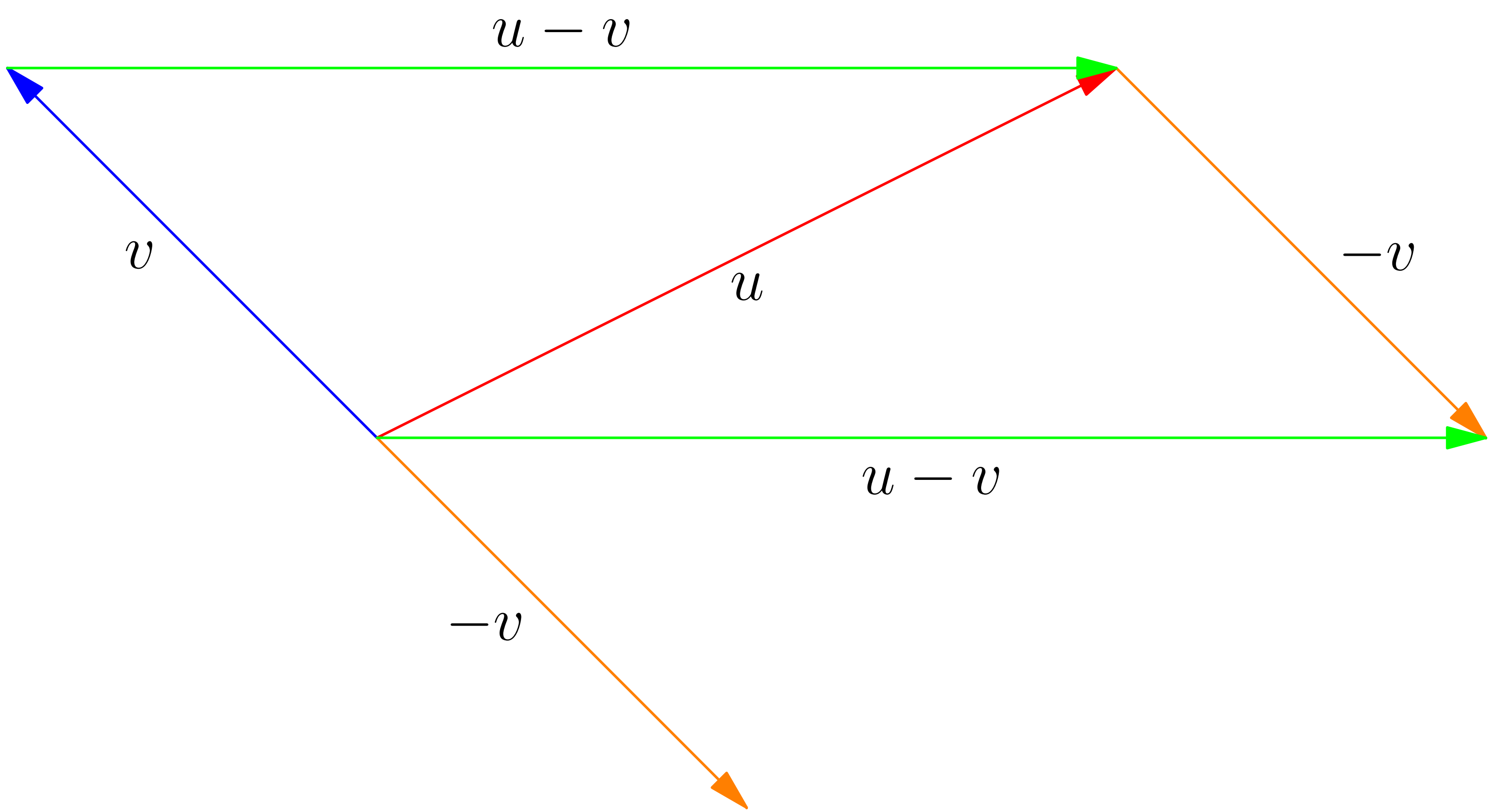
Scalar Multiplication



Vector Addition

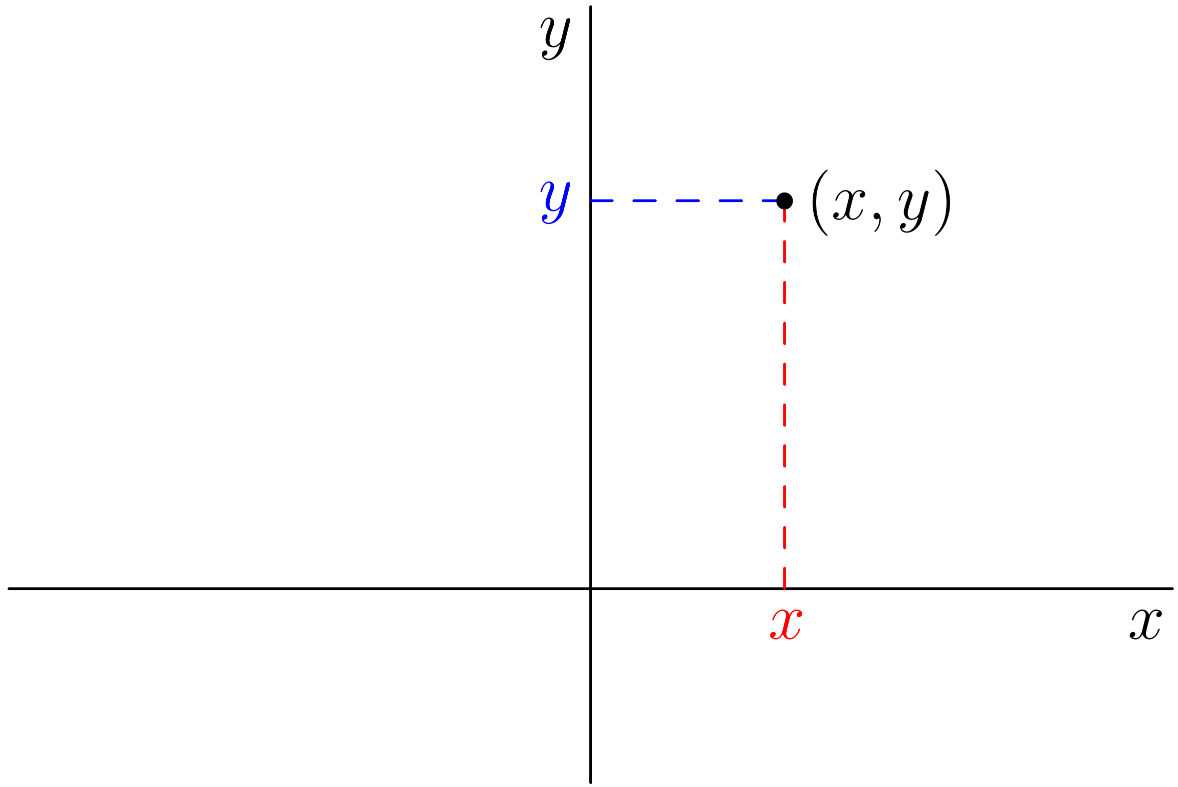


Vector Subtraction



Vectors in Coordinates

- We can put a coordinate system on the plane.

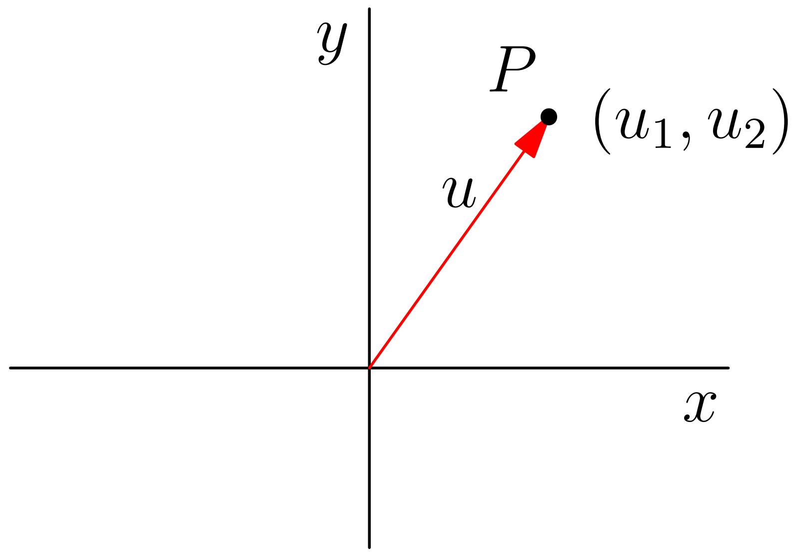


- Distance Formula: $\mathrm{dist}((0,0),(x,y)) = \sqrt{x^2 + y^2}$

Vectors in Coordinates

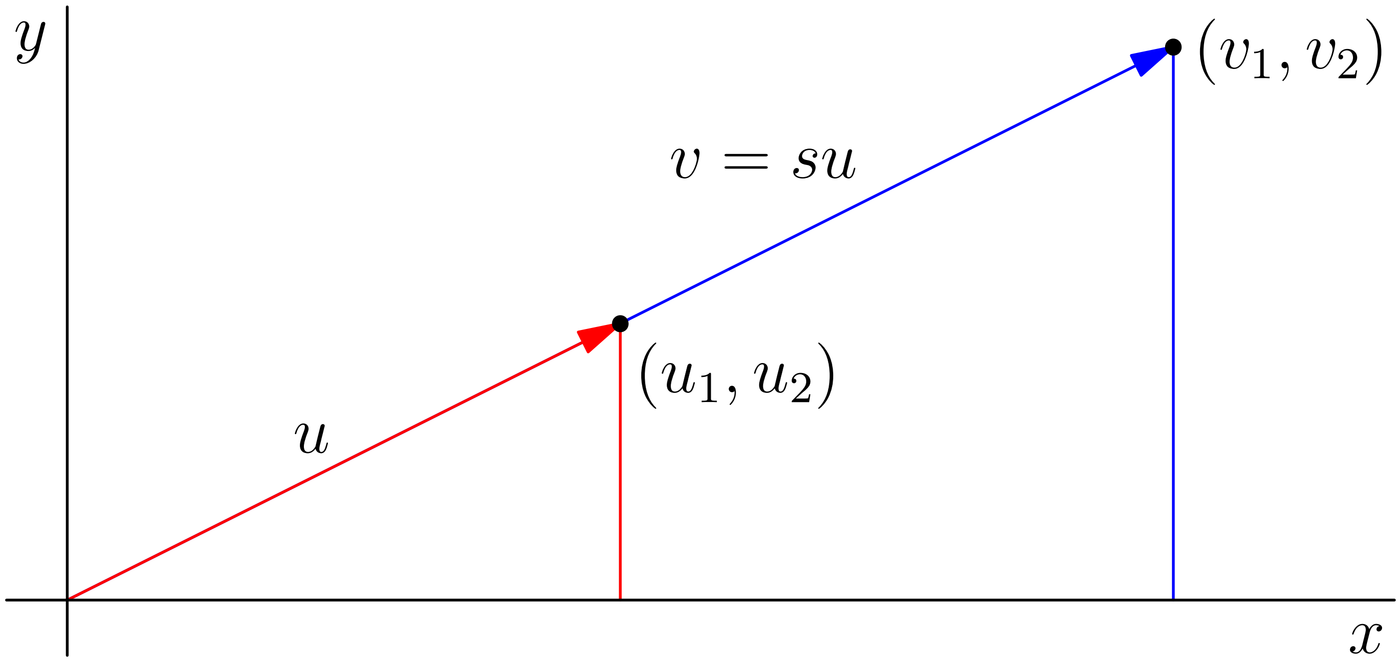
- Coordinates give one-to-one correspondences

$$\text{Vectors} \leftrightarrow \text{Points} \leftrightarrow (u_1, u_2) \in \mathbb{R}^2.$$



- Write a vector as $u = (u_1, u_2)$. We call u_1 and u_2 the **components** of u. $\|u\| = \sqrt{u_1^2 + u_2^2}$.

Scalar Multiplication in Coordinates

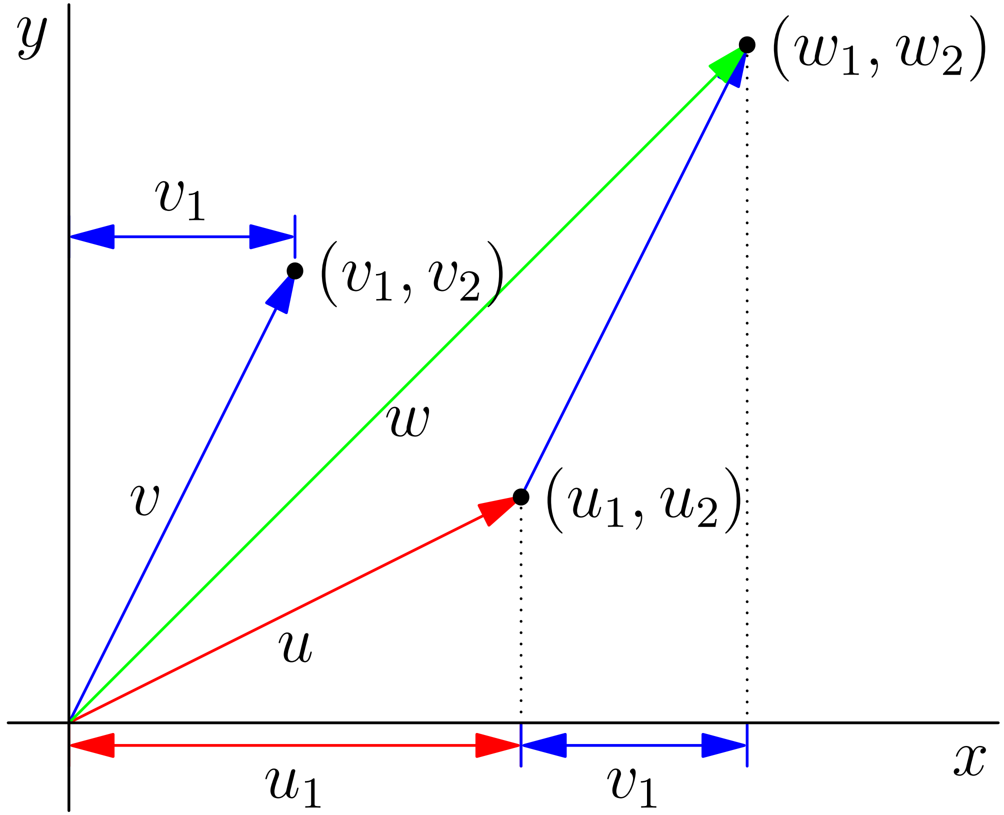


$$\frac{u_2}{\|u\|} = \frac{v_2}{\|v\|} = \frac{v_2}{s\|u\|}$$

$$v_2 = su_2$$

$$s(u_1, u_2) = (su_1, su_2)$$

Vector Addition in Coordinates



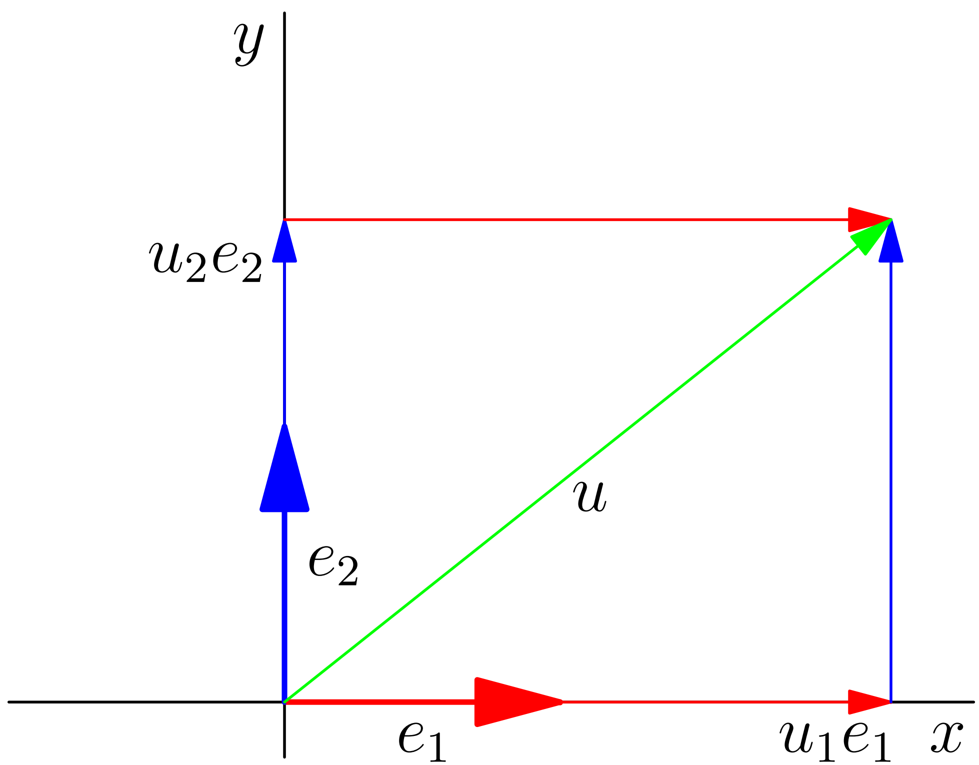
$$w_1 = u_1 + v_1$$

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

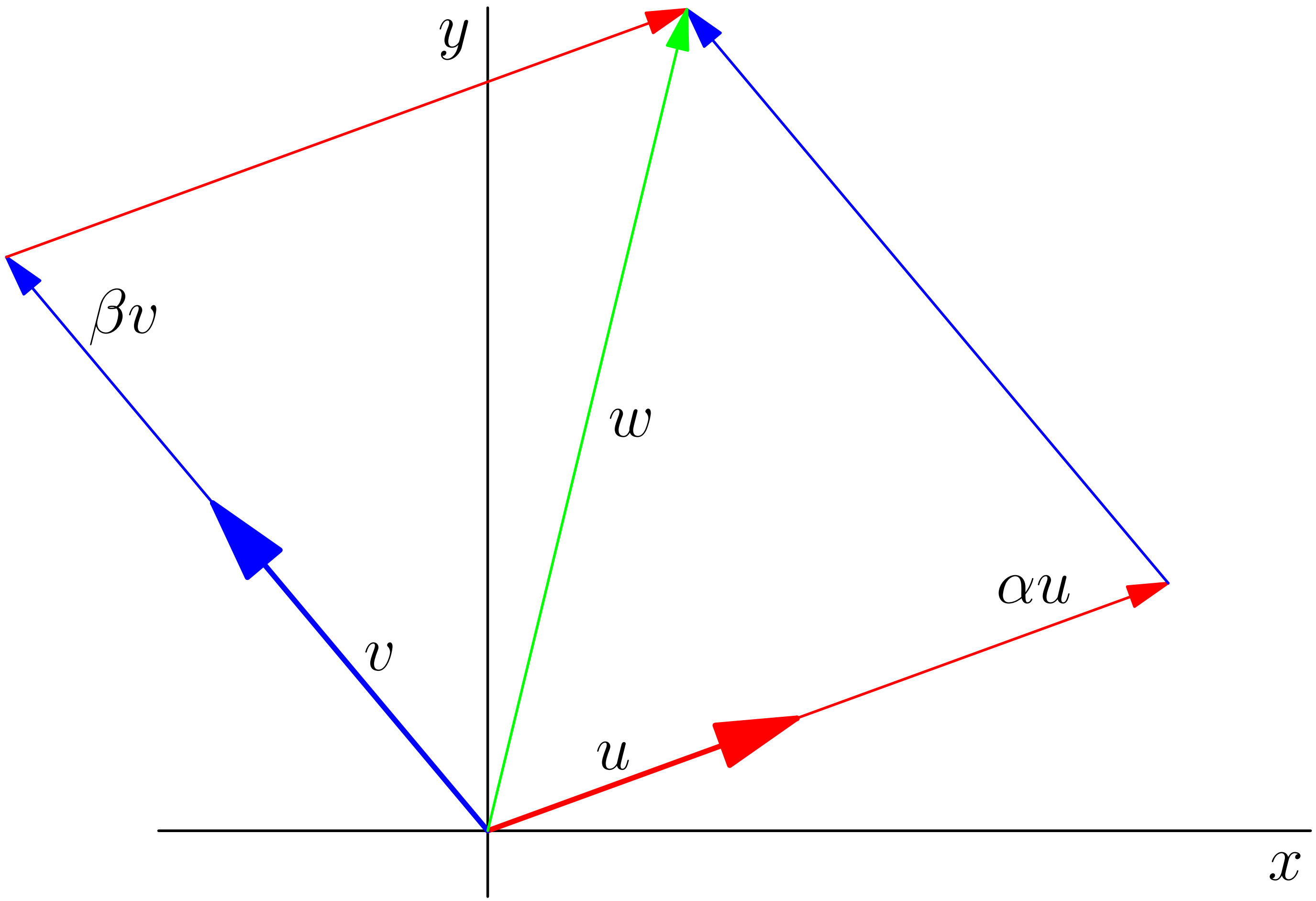
Standard Basis Vectors

$$(u_1, u_2) = u_1(1,0) + u_2(0,1) = u_1e_1 + u_2e_2$$

$$e_1 = (1,0), \quad e_2 = (0,1)$$



A Nonstandard Basis



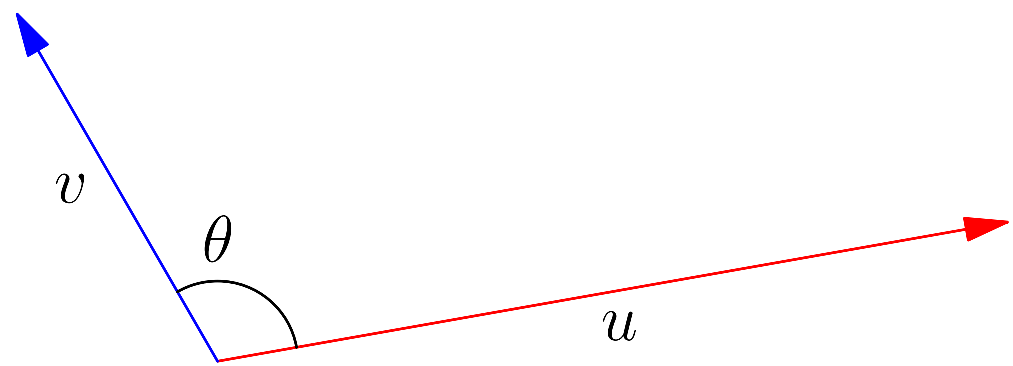
$$w = \alpha u + \beta v$$

Rules of Vector Algebra

- $(u + v) + w = u + (v + w)$. (Associative Law)
- $u + v = v + u$. (Commutative Law)
- $0 + v = v$. (Additive Identity)
- $v + (-v) = 0$. (Additive Inverse)
- $s(u + v) = su + sv$. (Distributive Law)
- $(s + t)u = su + tu$. (Distributive Law)
- $1u = u$.

Dot Product

- The angle θ between two vectors.



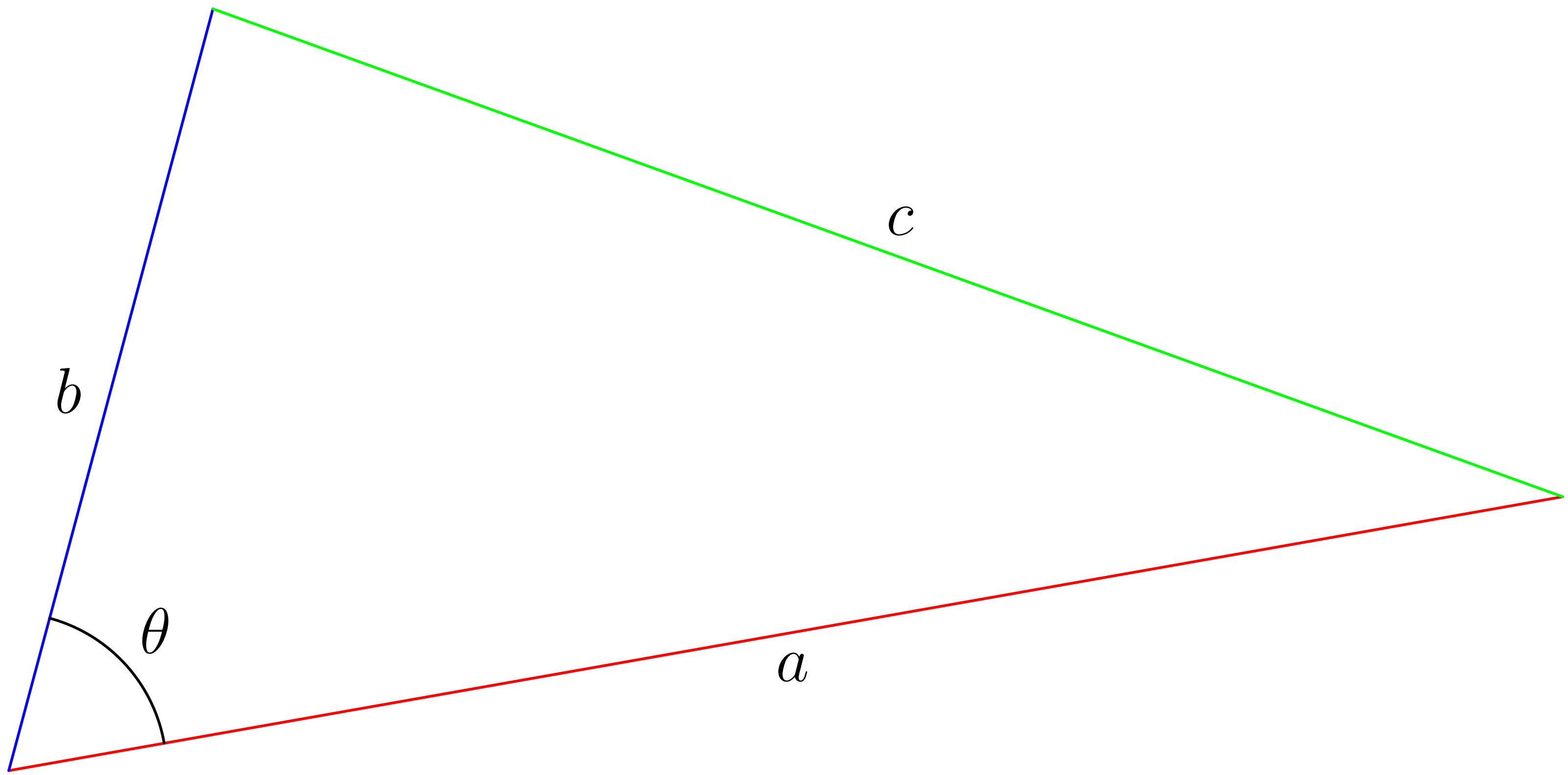
- Definition of dot product

$$u \cdot v = \|u\| \, \|v\| \, \cos(\theta).$$

$$0 \cdot v = 0.$$

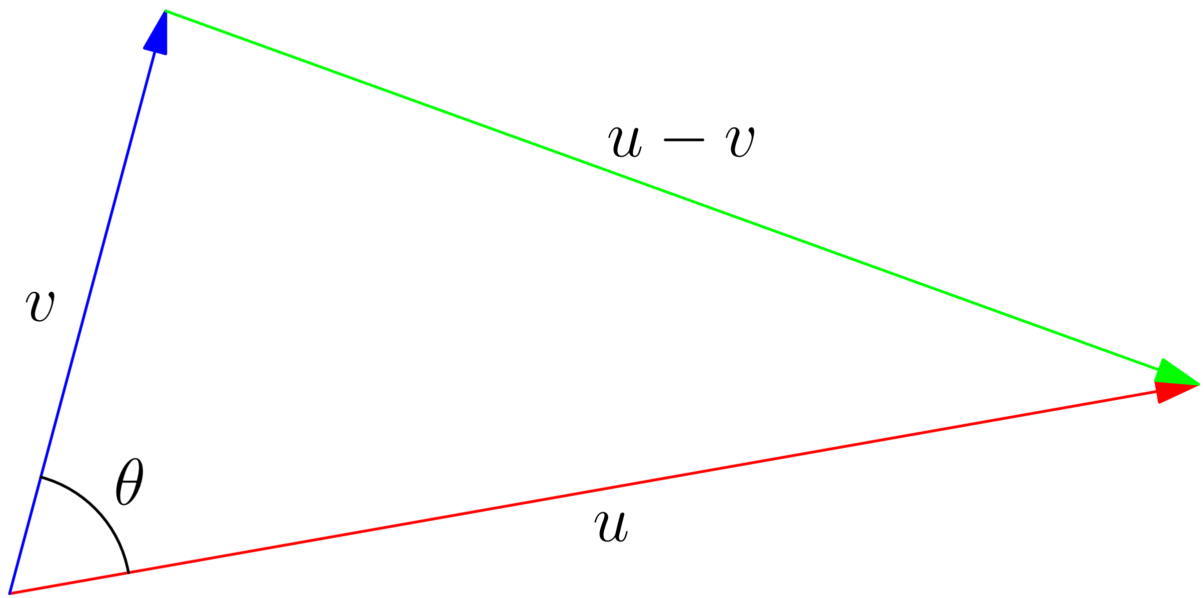
- $u \cdot u = \|u\|^2$.
- $u \cdot v = 0 \iff u \perp v$.

Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Formula for Dot Product



$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos(\theta)$$
$$= \|u\|^2 + \|v\|^2 - 2u \cdot v$$
$$2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2.$$

Dot Product in Components

- $$\begin{aligned} 2u \cdot v &= \|u\|^2 + \|v\|^2 - \|u - v\|^2 \\ &= \|(u_1, u_2)\|^2 + \|(v_1, v_2)\|^2 - \|(u_1 - v_1, u_2 - v_2)\|^2 \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - [(u_1 - v_1)^2 + (u_2 - v_2)^2] \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2 \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1^2 - 2u_1v_1 + v_1^2) - (u_2^2 - 2u_2v_2 + v_2^2) \\ &= u_1^2 + u_2^2 + v_1^2 + v_2^2 - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2 \\ &= 2u_1v_1 + 2u_2v_2. \end{aligned}$$

- $$u \cdot v = u_1v_1 + u_2v_2.$$

Properties of Dot Product

- Rules of Algebra for Dot Product
  - $u \cdot u = \|u\|^2$.
  - $u \cdot v = v \cdot u$.
  - $(su) \cdot v = s(u \cdot v) = u \cdot (sv)$
  - $u \cdot (v + w) = u \cdot v + u \cdot w$.
- Compute the angle in terms of components

$$\theta = \arccos\left(\frac{u \cdot v}{\|u\| \, \|v\|}\right).$$

- **Exercise:** If u and v are orthogonal unit vectors (i.e., $\|u\| = 1$) and $w = \alpha u + \beta v$, show $\alpha = w \cdot u$ and $\beta = w \cdot v$.

Matrices

- An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.
- Example:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}, \qquad 2 \times 3.$$

- a_{ij} denotes the entry in row i and column j.

Matrices

- A general matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, \qquad m \times n.$$

- Matrix addition and scalar multiplication are defined slotwise, like vectors

Matrix Multiplication

- Size condition for multiplication:

$$(m \times n) \cdot (n \times p) = m \times p.$$

- $(1 \times n) \cdot (n \times 1) = 1 \times 1$

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n.$$

Matrix Multiplication

- Entries of $C = AB$

$$c_{ij} = \mathrm{Row}_i(A)\,\mathrm{Col}_j(B).$$

- Important Cases:

$$Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}. \end{aligned}$$

Multiplication is not Commutative!

- If AB and BA are the same size, they must both be $n \times n$ for some n.
- Example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix}.$$

Ways of Looking at Multiplication

- Consider Ax

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2$$
$$= x_1 \,\mathrm{Col}_1(A) + x_2 \,\mathrm{Col}_2(A).$$

- $Ae_1 = \mathrm{Col}_1(A)$ and $Ae_2 = \mathrm{Col}_2(A)$.
- Consider yB

$$\begin{bmatrix} y_1 & y_1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} y_1 b_{11} + y_2 b_{21} & y_1 b_{12} + y_2 b_{22} \end{bmatrix}$$
$$= y_1 \,\mathrm{Row}_1(B) + y_2 \,\mathrm{Row}_2(B).$$

Ways of Looking at Multiplication

- In terms of rows and columns

$$A[b_1 \mid b_2] = [Ab_1 \mid Ab_2].$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \end{bmatrix}.$$

- Rules of Algebra for Matrix Multiplication
  - $(AB)C = A(BC)$. (Assocative Law)
  - $A(B + C) = AB + AC$. (Left Distributive Law)
  - $(A + B)C = AC + BC$. (Right Distributive Law)
  - $(sA)B = s(AB) = A(sB)$.
  - $A0 = 0$ and $0B = 0$.

Invertible Matrices

- I_n denotes the $n \times n$ identity matrix. It has ones on the diagonal and zeros elsewhere.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- If A is $m \times n$

$$I_m A = A = A I_n.$$

- A is **invertible** or **nonsingular** if there is a matrix B such that

$$AB = BA = I$$

Invertible Matrices

- Only square matrices can be invertible.
- B is unique, if it exists, and is denoted A^{-1}.

$$A^{-1}A = AA^{-1} = I.$$

- $(A^{-1})^{-1} = A$.
- The **Determinant** of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is defined by

$$\det(A) = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Invertible Matrices

- **Theorem** TFCAE.
  1. A is invertible.
  2. The only column vector x such that $Ax = 0$ is $x = 0$.
  3. $\det(A) \neq 0$.
- $(1) \implies (2)$. Multiply both sides of $Ax = 0$ by A^{-1}.
- $(2) \implies (3)$. Instead, do Not $(3) \implies$ Not (2). Assume $ad - bc = 0$. Of course, $0x = 0$ for all x. Suppose $A \neq 0$. Then at least one of the vectors

$$\begin{bmatrix} d \\ -b \end{bmatrix}, \quad \begin{bmatrix} -c \\ a \end{bmatrix}$$

is not zero.

Invertible Matrices

- Continuing, compute

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} d \\ -b \end{bmatrix} = \begin{bmatrix} ad - bc \\ bd - bd \end{bmatrix} = 0,$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} -c \\ a \end{bmatrix} = \begin{bmatrix} -ac + ac \\ -bc + ad \end{bmatrix} = 0,$$

  so (2) fails.

- $(3) \implies (1)$. Just check that this forumla works:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Exercises on Invertiblity

- If A and B are invertible, so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.
- A is invertible if and only if the equation $Ax = b$ has a unique solution x for every column vector b.
- Show by brute force computation (for 2×2 matrices) that $\det(AB) = \det(A)\det(B)$.
- Show that if $\det(A) = 0$, one of the columns of A is multiple of the other.
- Show for 2×2 matrices that if $AB = I$, then $B = A^{-1}$ and if $BA = I$ then $B = A^{-1}$.

Linear Transformations

- A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is **linear** if

$$T(u+v) = T(u) + T(v); T(su) = sT(u),$$

or, equivalently,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

- If A is a matrix, $T(x) = Ax$ is linear.
- **Proposition**. If T is linear, there is a unique matrix A so that $T(x) = Ax$. In fact, $A = [T(e_1) \mid T(e_2)]$.

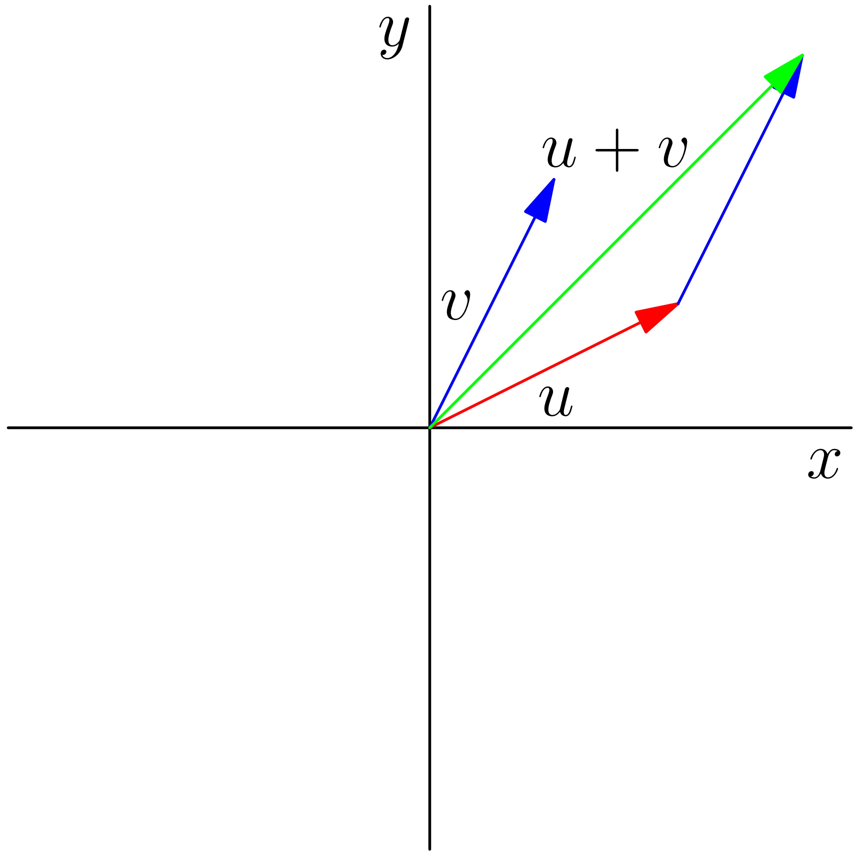
Linear Transformations

- Proof:

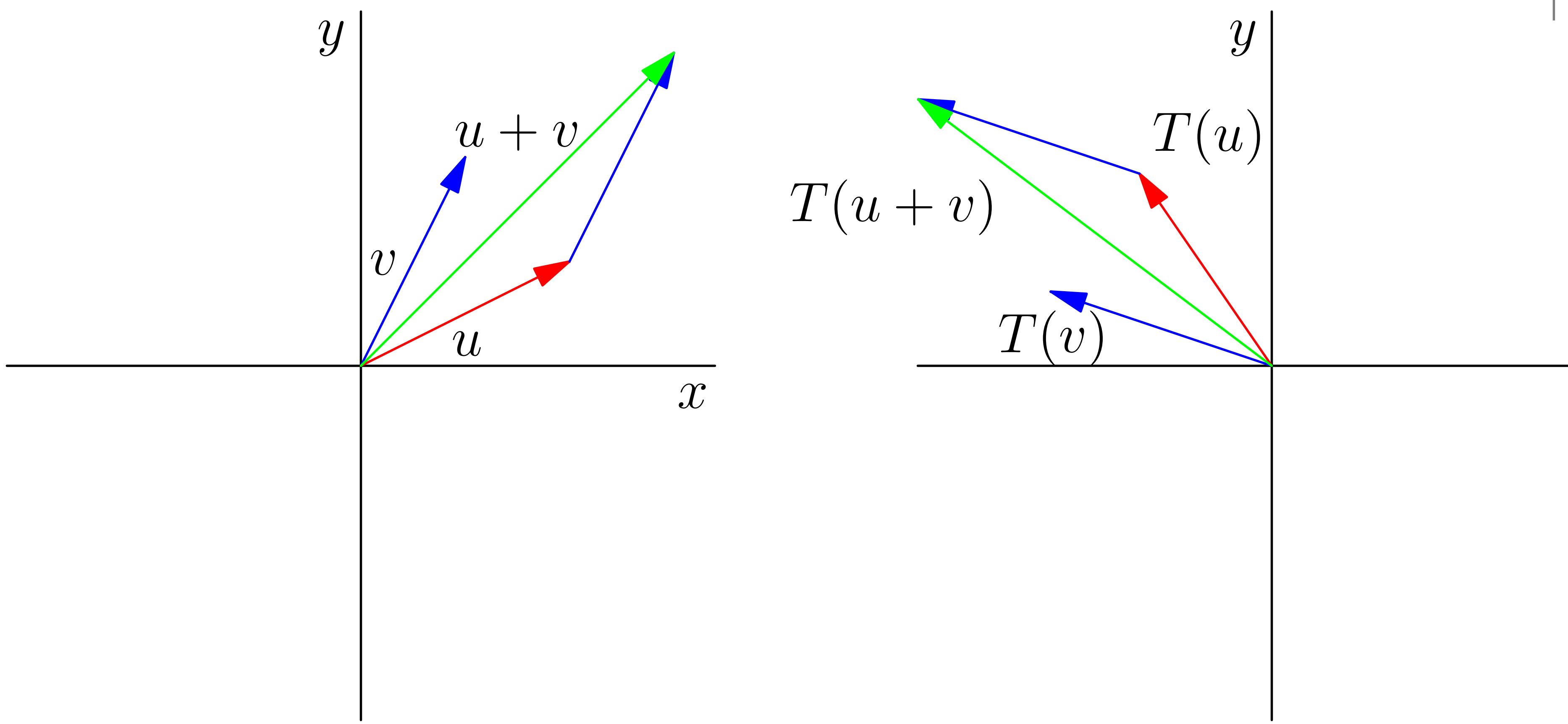
$$\begin{aligned} T(x) &= T(x_1e_1 + x_2e_2) \\ &= x_1T(e_1) + x_2T(e_2) \\ &= [T(e_1) \mid T(e_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

- Let T be rotation by angle θ around the origin. Why is this linear?

Rotations



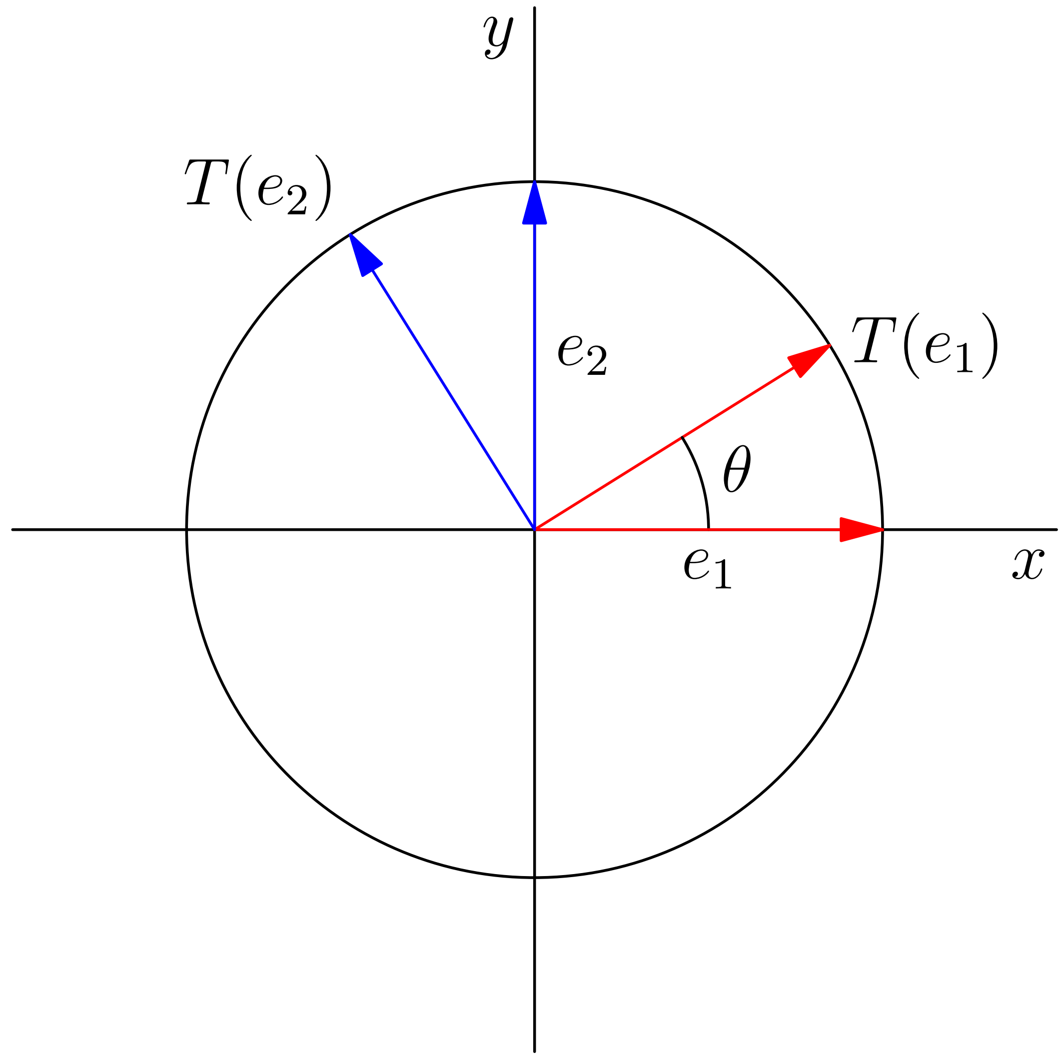
Rotations



$$T(u+v) = T(u) + T(v)$$

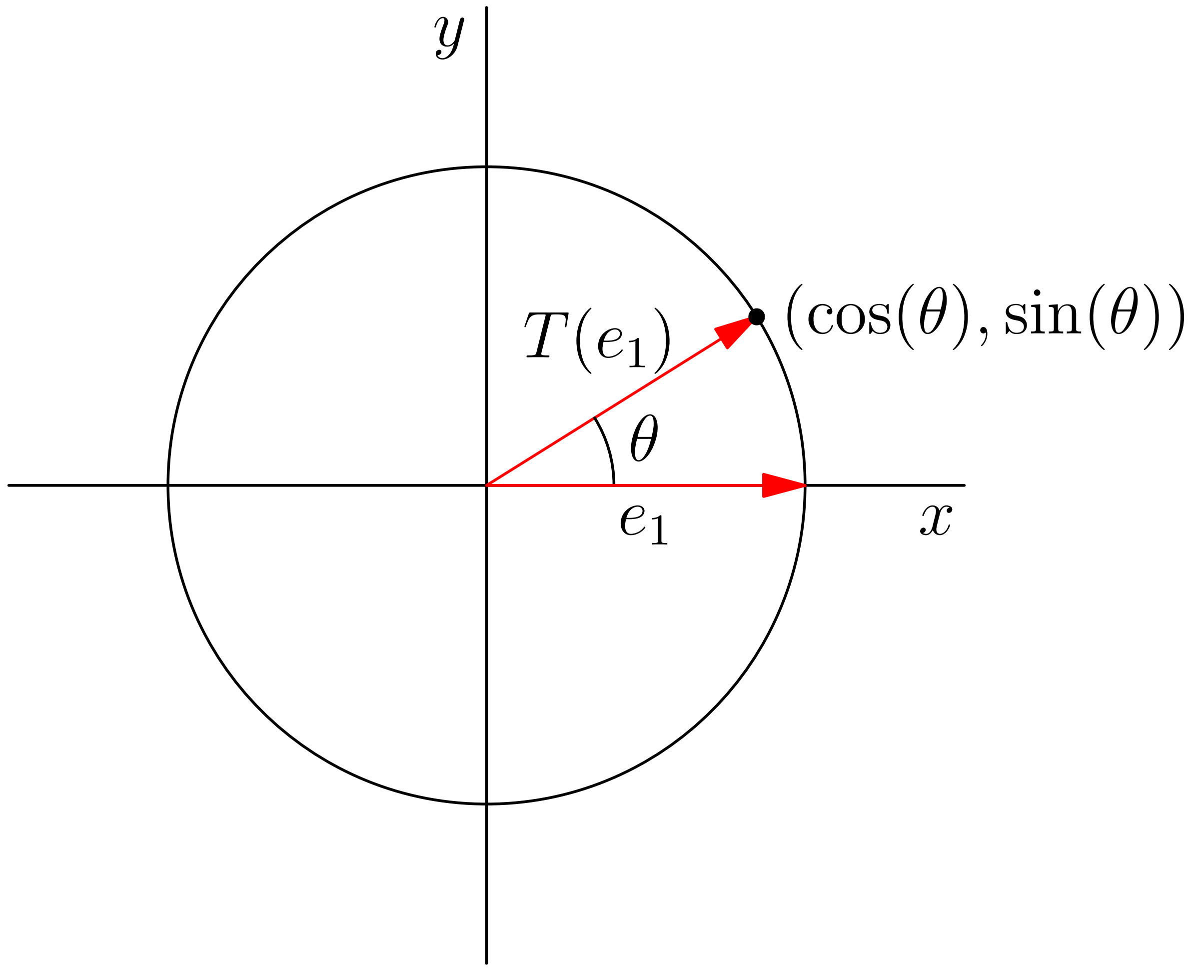
Matrix of Rotation

- We need to find $T(e_1)$ and $T(e_2)$.



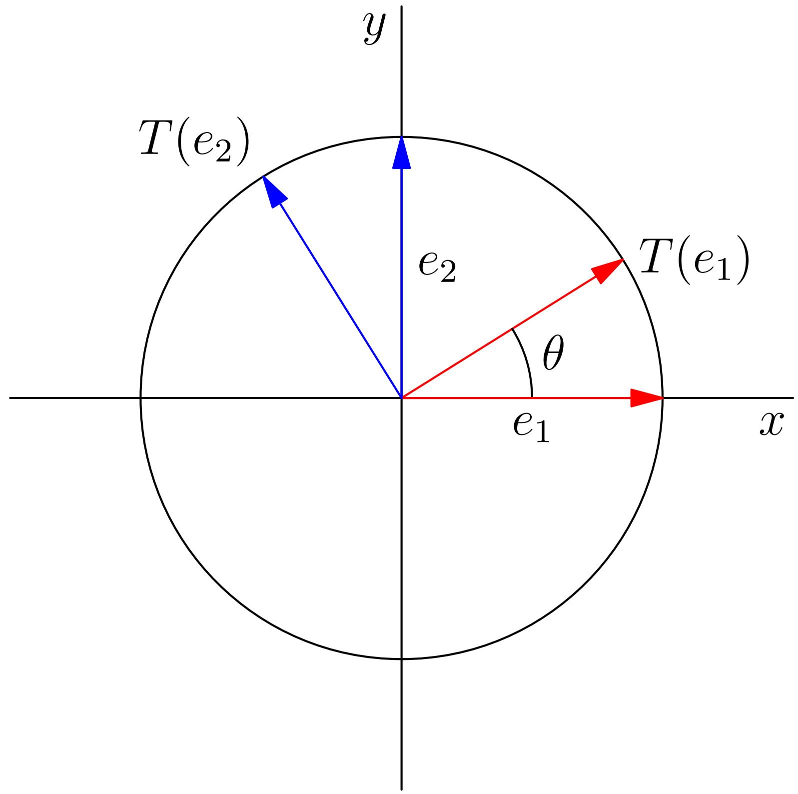
Matrix of Rotation

- We know $T(e_1)$ from trig:



Matrix of Rotation

- $T(e_2)$ must be $(-\sin(\theta), \cos(\theta))$ or $(\sin(\theta), -\cos(\theta))$. Check the signs.



- $T(e_2) = (-\sin(\theta), \cos(\theta))$.

Matrix of Rotation

- The matrix of rotation through angle θ is

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

- $R(\theta)R(\varphi) = R(\theta + \varphi) = R(\varphi)R(\theta)$.
- Compare enteries. Use $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. We get the **Additon Laws for sine and cosine**

$$\cos(\theta \pm \varphi) = \cos(\theta)\cos(\varphi) \mp \sin(\theta)\sin(\varphi),$$
$$\sin(\theta \pm \varphi) = \sin(\theta)\cos(\varphi) \pm \cos(\theta)\sin(\varphi).$$