**Problem 1.** In this problem, we will work in the vectorspace

$$P_3 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \},$$

the space of polynomials of degree less than 3. Let $\mathcal{P}$ be the ordered basis $\mathcal{P} = [x^2 \ x \ 1]$ of $P_3$ and let $\mathcal{Q}$ be the ordered basis

$$\mathcal{Q} = [2x^2 + x + 1 \ x^2 + 2x + 1, x^2 + x + 1]$$

of $P_3$.

A. Find the change of basis matrices $S_{\mathcal{P}\mathcal{Q}}$ and $S_{\mathcal{Q}\mathcal{P}}$.

*Answer:*

The defining equation of the change of basis matrix $S_{\mathcal{P}\mathcal{Q}}$ is $\mathcal{Q} = \mathcal{P} S_{\mathcal{P}\mathcal{Q}}$. Reading off the coefficients, we have

\[
\begin{bmatrix}
2x^2 + x + 1 \\ x^2 + 2x + 1, x^2 + x + 1
\end{bmatrix}_{\mathcal{Q}} =
\begin{bmatrix}
x^2 \\ x \\ 1
\end{bmatrix}_{\mathcal{P}} \cdot
\begin{bmatrix}
2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1
\end{bmatrix}
\]

(a matrix multiplication), so

\[
S_{\mathcal{P}\mathcal{Q}} =
\begin{bmatrix}
2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1
\end{bmatrix}
\]  \hspace{1cm} (1)

To find $S_{\mathcal{Q}\mathcal{P}}$, we use the fact that $S_{\mathcal{Q}\mathcal{P}} = S_{\mathcal{P}\mathcal{Q}}^{-1}$. Thus,

\[
S_{\mathcal{Q}\mathcal{P}} =
\begin{bmatrix}
1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3
\end{bmatrix}
\]

where we have used a calculator to find the inverse of the matrix in (1).

B. Let $p(x) = 2x^2 - x + 1$. Find $[p(x)]_{\mathcal{Q}}$, the coordinate vector of $p(x)$ with respect to the ordered basis $\mathcal{Q}$. Show how $p(x)$ can be written as a linear combination to the elements of $\mathcal{Q}$

*Answer:*

First we find $[p(x)]_{\mathcal{P}}$, since that’s easy. The defining equation of $[p(x)]_{\mathcal{P}}$ is $p(x) = \mathcal{P}[p(x)]_{\mathcal{P}}$. We have

\[
p(x) = 2x^2 - x + 1 =
\begin{bmatrix}
x^2 \\ x \\ 1
\end{bmatrix}_{\mathcal{P}} \cdot
\begin{bmatrix}
2 \\ -1 \\ 1
\end{bmatrix}
\]

1
and so

\[ [p(x)]_P = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}. \]

To find \([p(x)]_Q\), we use the change of coordinate equation

\[ [p(x)]_Q = S_{QP} [p(x)]_P, \]

so

\[
[p(x)]_Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix},
\]

using our value of \(S_{QP}\) from above. Thus,

\[
[p(x)]_Q = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.
\]

The defining equation of \([p(x)]_Q\) is \(p(x) = Q[p(x)]_Q\). Thus, we have

\[
p(x) = Q[p(x)]_Q
= \underbrace{[2x^2 + x + 1 \hspace{1em} x^2 + 2x + 1, x^2 + x + 1]}_Q \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}_{[p(x)]_Q}
= 1(2x^2 + x + 1) - 2(x^2 + 2x + 1) + 2(x^2 + x + 1).
\]

Thus, we can write \(p(x)\) as a linear combination of the elements of \(Q\) as follows

\[
p(x) = 1(2x^2 + x + 1) - 2(x^2 + 2x + 1) + 2(x^2 + x + 1).
\]

---

**Problem 2.** In this problem, we will work in the vectorspace

\[
P_3 = \{ ax^2 + bx + c \mid a, b, c \in \mathbb{R} \},
\]

the space of polynomials of degree less than 3. Let \(P\) be the ordered basis \(P = [x^2 \hspace{1em} x \hspace{1em} 1]\) of \(P_3\).

Let \(A\) be the matrix

\[
A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & 8 \end{bmatrix},
\]

which is invertible. Let \(Q\) be the ordered basis defined by \(Q = PA\). Let \(T: P_3 \to P_3\) be the linear transformation defined by \(T(p(x)) = p'(x) + p(x)\).
A. Find $[T]_{PP}$, the matrix of $T$ with respect to the ordered basis $P$.

**Answer:**

The defining equation of $[T]_{PP}$ is

$$T(P) = P[T]_{PP}. \quad (2)$$

and $T(P) = [T(x^2) \ T(x) \ T(1)]$. Since $T(p(x)) = p'(x) + p(x)$ we have

$$T(x^2) = (x^2)' + x^2 = 2x + x^2$$
$$T(x) = (x)' + x = 1 + x$$
$$T(1) = (1)' + 1 = 0 + 1 = 1,$$

so $T(P) = [x^2 + 2x \ x + 1 \ 1]$. Reading off the coefficients, we have

$$[x^2 + 2x \ x + 1 \ 1]_{T(P)} = [x^2 \ x \ 1]_P \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

(matrix multiplication). Comparing this with equation (2), we see that

$$[T]_{PP} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

B. Find $[T]_{QQ}$, the matrix of $T$ with respect to the basis $Q$.

**Answer:**

We will use the change of basis equation for linear transformations, i.e.,

$$[T]_{QQ} = S_{QP}[T]_{PP}S_{QP}^{-1}. \quad (3)$$

Since $S_{QP} = S_{QP}^{-1}$, we can rewrite this equation as

$$[T]_{QQ} = S_{QP}^{-1}[T]_{PP}S_{QP}. \quad (3)$$

The defining equation of $S_{QP}$ is

$$Q = PS_{QP}. \quad (4)$$

Since the basis $Q$ is defined by $Q = P A$, where $A$ is given in the problem, comparison with equation (4) shows that $S_{QP} = A$. Hence, plugging into (3) shows that

$$[T]_{QQ} = A^{-1}[T]_{PP} A.$$
Using the given value of $A$ and the value of $[T]_{PP}$ from the previous part of the problem, we have

$$
[T]_{QQ} = \begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & 1 \\
2 & 2 & 8 \\
\end{bmatrix}^{-1} \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 1 & 3 \\
0 & 1 & 1 \\
2 & 2 & 8 \\
\end{bmatrix}.
$$

Using a calculator for the matrix computations, we find

$$
[T]_{QQ} = \begin{bmatrix}
-1 & -3 & 7 \\
2 & 5/2 & 11/2 \\
0 & 1/2 & 3/2 \\
\end{bmatrix}
$$

---

**Problem 3.** Let $U = [u_1 \ u_2]$ be the ordered basis of $\mathbb{R}^2$ where

$$u_1 = \begin{bmatrix}
2 \\
1 \\
\end{bmatrix}, \quad u_2 = \begin{bmatrix}
2 \\
2 \\
\end{bmatrix}.$$

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose matrix with respect to the standard basis $\mathcal{E}$ is

$$A = \begin{bmatrix}
1 & 2 \\
1 & 0 \\
\end{bmatrix},$$

in other words, $T(x) = Ax$.

A. Find the change of basis matrices $S_{EU}$ and $S_{UE}$.

*Answer:*

The defining equation of $S_{EU}$ is $U = E S_{EU}$. The corresponding matrix equation is

$$\text{mat}(U) = \text{mat}(E) S_{EU} = I S_{EU} = S_{EU}$$

Thus, we have

$$S_{EU} = \text{mat}(U) = \begin{bmatrix}
2 & 2 \\
1 & 2 \\
\end{bmatrix}.$$

Thus,

$$S_{EU} = \begin{bmatrix}
2 & 2 \\
1 & 2 \\
\end{bmatrix}.$$

We know that $S_{UE} = S_{EU}^{-1}$. Using a calculator for the computation of the inverse, we have

$$S_{UE} = \begin{bmatrix}
1 & -1 \\
-1/2 & 1 \\
\end{bmatrix}.$$
B. Find $[T]_{UU}$, the matrix of $T$ with respect to $U$.

Answer:
We are given the matrix of $T$ with respect to the standard basis, so

$$[T]_{EE} = A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$  

To find $[T]_{UU}$, we use the change of coordinate equation for linear transformations, i.e.,

$$[T]_{UU} = S_{EU}[T]_{EE}S_{EU}$$

$$= \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}.$$  

Thus, the answer to this part of the problem is

$$[T]_{UU} = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}.$$  

C. Find the scalars $c_1$ and $c_2$ such that

$$T(2u_1 - 3u_2) = c_1u_1 + c_2u_2.$$  

Answer:
Let $v = 2u_1 - 3u_2$. Then, of course,

$$v = \begin{bmatrix} u_1 & u_2 \\ -3 \end{bmatrix}.$$  

The defining equation of $[v]_U$ is $v = U[v]_U$, so we have

$$[v]_U = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$  

The equation for the action of the linear transformation $T$ in the $U$-coordinates is

$$[T(v)]_U = [T]_{UU}[v]_U.$$  

Thus, we have

$$[T(v)]_U = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}.$$
The defining equation of \( [T(v)]_{ul} \) is
\[
T(v) = U[T(v)]_{ul}
\]
So we have
\[
T(2u_1 - 3u_2) = T(v) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} = -8u_1 + 3u_2,
\]
So the answer is \( c_1 = -8, \; c_2 = 3 \).

**Problem 4.** Let
\[
A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.
\]
Find the characteristic polynomial of \( A \) and the eigenvalues of \( A \).

**Answer:**
We have
\[
A - \lambda I = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 3 \\ 2 & -\lambda \end{bmatrix}.
\]
Thus, the characteristic polynomial \( p(\lambda) \) of \( A \) is
\[
p(\lambda) = \det(A - \lambda I)
\]
\[
= \begin{vmatrix} 1 - \lambda & 3 \\ 2 & -\lambda \end{vmatrix}
\]
\[
= (1 - \lambda)(-\lambda) - 6
\]
\[
= -\lambda^2 - 6
\]
\[
= \lambda^2 - \lambda - 6
\]
Thus, we have
\[
p(\lambda) = \lambda^2 - \lambda - 6
\]
The eigenvalues of \( A \) are the roots of the characteristic polynomial. The characteristic polynomial factors as \( p(\lambda) = (\lambda - 3)(\lambda + 2) \), so the roots are 3 and -2. Thus,
\[
\text{Eigenvalues of } A = -2, 3.
\]

**Problem 5.** In each part you are given a matrix \( A \) and its eigenvalues. Find a basis for each of the eigenspaces of \( A \). Determine if \( A \) is diagonalizable, and if it is, find a matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).
A.

\[ A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \text{Eigenvalues} = 1, 2. \]

**Answer:**

Consider first the eigenvalue \( \lambda = 1 \). We have

\[ A - \lambda I = A - I = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}. \]

The RREF of this matrix is (by calculator)

\[ R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \]

The eigenspace \( E_1 \) is the nullspace of \( A - I \), which is the same as the nullspace of \( R \), i.e., the solution space of \( Rx = 0 \). Calling the variables \( x_1, x_2 \) and \( x_3 \), we see that \( x_3 \) is a free variable, say \( x_3 = \alpha \). The second row of \( R \) gives the equation \( x_2 - \frac{x_3}{2} = 0 \), so \( x_2 = \alpha \) and the first row gives the equation \( x_1 - x_3 = 0 \), so \( x_1 = \alpha \). Thus, the vectors in the nullspace of \( R \) are

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

Thus, a basis of the eigenspace \( E_1 \) is

\[ \text{Basis of } E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

so \( E_1 \) has dimension one.

Next, consider the eigenvalue \( \lambda = 2 \). We have

\[ A - \lambda I = A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \]

which has the RREF

\[ R = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}. \]
The eigenspace $E_2$ is the nullspace of $A - 2I$, which is the same as the nullspace of $R$. Abbreviating the computation of the nullspace of $R$, we have

$$x_3 = \alpha$$

$$x_2 - \frac{3}{2}x_3 = 0 \implies x_2 = \frac{3}{2}\alpha$$

$$x_1 - \frac{1}{2}x_3 = 0 \implies x_1 = \frac{1}{2}\alpha$$

so the nullspace is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\alpha \\ \frac{3}{2}\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Thus, we have

$$\text{Basis of } E_2 = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Alternatively, we could multiply this vector by 2 and say

$$\text{Basis of } E_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and so avoid the agony of dealing with fractions.

We have only produced 2 independent eigenvectors, and there are no additional eigenvectors that are independent of these two. Thus, $A$ does not have a basis of eigenvectors, we we conclude

$$A \text{ is not diagonalizable}.$$

B.

$$A = \begin{bmatrix} 10 & -3 & -6 \\ 12 & -5 & -6 \\ 12 & -3 & -8 \end{bmatrix}, \quad \text{Eigenvalues} = -2, 1.$$

Answer:

First consider the eigenvalue $\lambda = -2$. Then we have

$$A - \lambda I = A + 2I = \begin{bmatrix} 12 & -3 & -6 \\ 12 & -3 & -6 \\ 12 & -3 & -6 \end{bmatrix},$$
which has the RREF
\[ R = \begin{bmatrix} 1 & -1/4 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

For the abbreviated computation of the nullspace we have
\[
x_2 = \alpha \\
x_3 = \beta \\
x_1 - \frac{1}{4}x_2 - \frac{1}{2}x_3 = 0 \implies x_1 = \frac{1}{4}\alpha + \frac{1}{2}\beta.
\]

The parametrization of the nullspace is
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}.
\]

Thus, we get
\[
\text{Basis of } E_{-2} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 0 \\ 0 \end{bmatrix}. \tag{5}
\]

As matter of personal taste, I’ll get rid of the fractions by multiplying the first vector by 4 and the second by 2. That gives me
\[
\text{Basis of } E_{-2} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}. \tag{6}
\]

We see that $E_{-2}$ has dimension 2.

Next, consider the eigenvalue $\lambda = 1$. We have
\[
A - \lambda I = A - I = \begin{bmatrix} 9 & -3 & -6 \\ 12 & -3 & -9 \\ 12 & -3 & -9 \end{bmatrix},
\]

which has has the RREF
\[
R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Note that we have already computed the nullspace of this matrix $R$ in the first part of the problem. Thus, we know
\[
\text{Basis of } E_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{7}
\]
Since we’ve found three linearly independent eigenvectors, \( A \) has a basis of eigenvectors and \( A \) is diagonalizable.

Using the basis (6) of \( E_{-2} \) as a matter of taste (I could equally well use (5)), and the basis (7) of \( E_1 \), set

\[
P = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

then \( P \) is invertible, \( D \) is a diagonal matrix and \( P^{-1}AP = D \) (check it if you don’t believe me).

**Problem 6.** In each part, determine if the given transformation \( T \) is linear.

If it is linear, find a matrix so that \( T(x) = Ax \). If \( T \) is not linear, justify your answer.

A. \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) is given by

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1x_2 \\ x_1^2 + x_2^2 \end{bmatrix}
\]

*Answer:*

For \( T \) to be linear, it must preserve scalar multiplication, i.e., it must be true that \( T(\alpha x) = \alpha T(x) \) for all scalars \( \alpha \) and all \( x \in \mathbb{R}^2 \). We have

\[
T \left( \alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = T \left( \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \right) = \begin{bmatrix} (\alpha x_1)(\alpha x_2) \\ (\alpha x_1)^2 + (\alpha x_2)^2 \end{bmatrix} = \begin{bmatrix} \alpha^2 x_1x_2 \\ \alpha^2(x_1^2 + x_2^2) \end{bmatrix} \quad (8)
\]

\[
\alpha T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \alpha \begin{bmatrix} x_1x_2 \\ x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} \alpha x_1x_2 \\ \alpha(x_1^2 + x_2^2) \end{bmatrix}. \quad (9)
\]

If \( T \) was linear, the right hand sides of equations (8) and (9) would have to be equal for all values of \( \alpha, x_1 \) and \( x_2 \). In particular, we would have to have

\[
\alpha^2 x_1x_2 = \alpha x_1x_2
\]

for all values of \( \alpha, x_1 \) and \( x_2 \). This is plainly not the case (e.g., \( \alpha = 2, x_1 = x_2 = 1 \)), so

\( T \) is not linear.
B. $T: \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 5x_3 \\ x_1 - x_3 \end{bmatrix}$$

**Answer:**

We can rewrite the right hand side of the definition of $T$ as a matrix multiplication

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, where $A$ is the matrix

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 1 & 0 & -1 \end{bmatrix}.$$

Since $T$ is given by matrix multiplication, $T$ must be linear.

$T$ is linear.

---

**Problem 7.** Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

A. Cut down the list of vectors above to a basis for $S$. What is the dimension of $S$?

B. For each of the following vectors, determine if the vector is in $S$ and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} -3 \\ 5 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 5 \\ 3 \\ 4 \\ 0 \end{bmatrix}.$$
Answer:
We can do all the computation in one RREF calculation on the calculator. Make up a matrix containing all these vectors

\[
A = \begin{bmatrix}
  v_1 & v_2 & v_3 & v_4 & w_1 & w_2 \\
  2 & -1 & 8 & 1 & -3 & 5 \\
  2 & 3 & 0 & 1 & 5 & 3 \\
  1 & -1 & 5 & 1 & -2 & 4 \\
  1 & 3 & -3 & 0 & -1 & 0
\end{bmatrix}
\]

The RREF of this is

\[
R = \begin{bmatrix}
  1 & 0 & 3 & 0 & -1 & 0 \\
  0 & 1 & -2 & 0 & 2 & 0 \\
  0 & 0 & 0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Columns 1, 2 and 4 of \( R \) form a basis for the span of the columns left of the bar, so the same is true of \( A \). Thus,

\[
\text{v}_1, \text{v}_2, \text{v}_4 \text{ are a basis of } S.
\]

Since \( S \) has a basis with three elements,

\[
\text{The dimension of } S \text{ is } 3.
\]

In \( R \), we have \( \text{col}_5(R) = -\text{col}_1(R) + 2\text{col}_2(R) + \text{col}_4(R) \). The same relation must hold among the columns of \( A \), so we have

\[
\text{w}_1 = -\text{v}_1 + 2\text{v}_2 + \text{v}_4.
\]

Since \( \text{w}_1 \) is a linear combination of vectors in \( S \),

\[
\text{w}_1 \in S.
\]

The last column of \( R \) is not a linear combination of the columns to the left of the bar (because of the last row), so the same is true of \( A \). Thus,

\[
\text{w}_2 \notin S.
\]