Answers to Review problems
for Exam 2
Math 2360 - Dol, Fall 2017

(1)

$$A = \begin{bmatrix} 1 & 1 & -3 & 0 & 1 \\ 1 & 0 & -1 & 0 & -2 \\ 1 & 1 & -3 & 1 & 6 \\ 1 & 0 & -1 & 1 & 3 \\ -3 & 0 & 3 & -1 & 1 \end{bmatrix}$$

The RREF of A is

$$R = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(A) Find the nullspace of A

A and R have the same nullspace, so use R

We're solving $Ax=0$, which has the same solutions as $Rx = 0$. Call the unknowns x_1, x_2, x_3, x_4, x_5. The leading entries are in columns 1, 2 and 4 of R, so x_1, x_2, x_4 are leading variables and x_3 and x_5 are free variables

Set $x_3 = \alpha$, $x_5 = \beta$, solve eqns bottom up:

Row 5: $0 = 0$

Row 4: $0 = 0$

Row 3: $x_4 + 5x_5 = 0$

$$x_4 = -5x_5 = -5\beta$$

Row 2: $x_2 - 2x_3 + 3x_5 = 0$

$$x_2 = 2x_3 - 3x_5 = 2\alpha - 3\beta$$

Row 1: $x_1 - x_3 - 2x_5 = 0$

$$x_1 = x_3 + 2x_5 = \alpha + 2\beta$$

So the solutions are parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ 2\alpha - 3\beta \\ \alpha \\ -5\beta \\ \beta \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

So a basis of the nullspace is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

(B) Find a basis of the rowspace of A.

R has the same rowspace as A

A basis of this space is obtained by taking the non zero rows from R.

So a basis of the rowspace is

$$[1 \ 0 \ -1 \ 0 \ -2], \ [0 \ 1 \ -2 \ 0 \ 3], \ [0 \ 0 \ 0 \ 1 \ 5].$$

(C) Find a basis of the comumn space of A.

R and A do <u>not</u> have the same column space.

Find the columns of R that contain the leading entries.

These are $Col_1(R)$, $Col_2(R)$ and $Col_4(R)$.
A basis of the column space of A is given by the corresponding columns of A, ie $Col_1(A)$, $Col_2(A)$ and $Col_4(A)$. Hence, a basis of the column space of A is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

(D) What is the rank of A?

rank(A) = dimension of Row space of A = dimension of column space of A = $\underline{\underline{3}}$

(2) Solution (see problem sheet).

Form the matrix

$A = [v_1 \mid v_2 \mid v_3 \mid v_4 \mid v_5]$

and the augmented matrix

$B = [v_1 \mid v_2 \mid v_3 \mid v_4 \mid v_5 \,\|\, w_1 \mid w_2]$

One calculation gives

$$R = \operatorname{Rref}(B) = \begin{array}{c} \text{Columns} \\ \begin{array}{ccccccc} \boxed{1} & \boxed{2} & 3 & \boxed{4} & 5 & 6 & 7 \end{array} \\ \underbrace{\left[\begin{array}{ccccc|cc} \boxed{1} & 0 & -2 & 0 & 1 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{C = RREF(A)} \end{array}$$

(A) A basis of the column space of A $= S'$ is $\operatorname{Col}_1(A)$, $\operatorname{Col}_2(A)$, $\operatorname{Col}_4(A)$

$$\operatorname{Col}_1(A) = V_1, \quad \operatorname{Col}_2(A) = V_2, \quad \operatorname{Col}_4(A) = V_4.$$

So V_1, V_2, V_4 are a basis for S',

So Dimension of $S' = 3$.

(B) V_3 and V_5 are not in our basis.

We have

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\mathrm{Col}_3(G) = \mathrm{Col}_1(G)\cdot(-2) + \mathrm{Col}_2(G)$

A and G have the same nullspace, so the columns of A and the columns of G satisfy the same linear relations, thus

$$\underset{V_3}{\underset{\parallel}{\mathrm{Col}_3(A)}} = -2\,\underset{V_1}{\underset{\parallel}{\mathrm{Col}_1(A)}} + \underset{V_2}{\underset{\parallel}{\mathrm{Col}_2(A)}}$$

so $V_3 = -2V_1 + V_2 + 0V_4$ $\quad (V_1, V_2, V_4) =$ basis.

By the same reasoning

$$V_5 = V_1 - 3V_2 + 2V_4$$

(check!)

7

(C) Note $B = [A \mid w_1 \mid w_2]$

In $R = \text{rref}(B)$, we have

$$\text{Col}_6(R) = 2\,\text{Col}_1(R) + \text{Col}_2(R) + 2\,\text{Col}_4(R)$$

So
$$\underset{\substack{\| \\ w_1}}{\text{Col}_6(B)} = 2\,\underset{\substack{\| \\ v_1}}{\text{Col}_1(B)} + \underset{\substack{\| \\ v_2}}{\text{Col}_2(B)} + 2\,\underset{\substack{\| \\ v_4}}{\text{Col}_4(B)}$$

So $\underline{w_1 = 2v_1 + v_2 + 2v_4}$

So $\underline{w_1 \text{ is in } S'}$

For w_2, the 1 is the 4th Row of R is matched by 0's in the other columns. In B, this means w_2 is not in $\text{Span}(v_1, v_2, v_3, v_4, v_5) = S'$

③ Ⓐ Determine is vectors are linearly independent

$$v_1 = \begin{bmatrix} -4 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 10 \\ -2 \\ -1 \\ -1 \end{bmatrix}$$

Form $A = [v_1 | v_2 | v_3]$

then $R = RREF(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $Col_3(R) = -Col_1(R) - 2\,Col_2(R)$

This implies

$$\begin{array}{ccc} Col_3(A) & = -Col_1(A) & -2\,Cos l_2(A) \\ \parallel & \parallel & \parallel \\ V_3 & -V_1 & -2\;V_2 \end{array}$$

So $V_1 + 2V_2 + V_3 = 0$. Since not all Coeffs are 0, V_1, V_2, V_3 are

<u>linearly dependent.</u>

9

(3B) $V_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 4 \\ 7 \\ 13 \\ 21 \end{bmatrix}$, $V_3 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 26 \end{bmatrix}$

Let $A = [V_1 \mid V_2 \mid V_3]$.

then $R = RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The columns of R are linearly independent (they are part of the standard basis of $\mathbb{R}^4$) (why?)

So the columns of A are independent, ie V_1, V_2, V_3 are linearly independent

(4A)

$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

$v_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_5 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Let $S = \text{span}(v_1, v_2, v_3, v_4, v_5)$.

Q: Is $S = \mathbb{R}^3$?

Set $A = [v_1 \mid v_2 \mid v_3 \mid v_4 \mid v_5]$

$$= \begin{bmatrix} 2 & 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

then $R = RREF(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Noting which columns of R contain the leading entries, the corresponding cols of A, $\text{col}_1(A)$, $\text{col}_2(A)$, $\text{col}_4(A)$ form a basis of $\text{colsp}(A) = S$.

Since $S \subseteq \mathbb{R}^3$ has dimension 3, $S = \mathbb{R}^3$. The basis we get for $\mathbb{R}^3$ is $Col_1(A) = V_1$, $Col_2(A) = V_2$ and $Col_4(A) = V_4$.

(4B) In this case

$$A = [V_1 \mid V_2 \mid V_3 \mid V_4 \mid V_5]$$

$$= \begin{bmatrix} 1 & 2 & 1 & 0 & 2 \\ 4 & 8 & 3 & -1 & 7 \\ 2 & 4 & 1 & -1 & 3 \end{bmatrix}$$

$$R = RREF(A) = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, the dimension of $span(V_1, V_2, V_3, V_4, V_5) = Colsp(A)$, is only 2, so $S \subsetneq \mathbb{R}^3$, the vectors V_1, V_2, V_3, V_4, V_5 do not span $\mathbb{R}^3$

Problem 5 Add on standard basis vectors to the list

$$V_1 = \begin{bmatrix} 2 \\ 7 \\ 6 \\ -3 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 9 \\ 32 \\ 26 \\ -13 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$ to get a basis of $\mathbb{R}^4$.

Sol'n: Let $A = [V_1 | V_2 | V_3]$

Form the augmented matrix B by adding on the standard basis vectors of $\mathbb{R}^4$, which are the columns of the identity matrix, to get

$$B = [A \mid I]$$

$$B = \begin{array}{c} \left[\begin{array}{ccc|cccc} 2 & 9 & -1 & 1 & 0 & 0 & 0 \\ 7 & 32 & -3 & 0 & 1 & 0 & 0 \\ 6 & 26 & -2 & 0 & 0 & 1 & 0 \\ -3 & -13 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{ccccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \end{array} \end{array}$$

$$R = RREF(B)$$

$$= \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 7 & -4 & 0 & -5 \\ 0 & 1 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

So a basis of the column space of B is

$$\underset{v_1}{\underset{\parallel}{Col_1(B)}},\ \underset{v_2}{\underset{\parallel}{Col_2(B)}},\ \underset{v_3}{\underset{\parallel}{Col_3(B)}},\ \underset{e_3}{\underset{\parallel}{Col_6(B)}}.$$

So $\underbrace{v_1, v_2, v_3}_{\text{original}}$ can be made into a basis of $\mathbb{R}^4$ by adding e_3

v_1, v_2, v_3, e_3.