EXAM

Exam 3
Final Exam

Math 2360, Summer I, 2016

July 9, 2016

- Write all of your answers on separate sheets of paper. Do not write on the exam handout. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **410 points total**.

Good luck!

40 pts.

Problem 1. In each part, solve the linear system using the Gauss-Jordan method (i.e., reduce the coefficent matrix to Reduced Row Echelon Form). Show the augmented matrix you start with and the augmented matrix you finish with. It's not necessary to show individual row operations, you can just find the Reduced Row Echelon Form with your calculator.

A.

$$\begin{array}{rcrcrcl} 4x & + & 2y & + & z & = & 9 \\ 2x & + & 2y & & & = & 2 \\ 2x & + & 2y & + & z & = & 5 \end{array}$$

B.

$$\begin{array}{rcrcrcl} 2x & + & y & - & z & = & 4 \\ x & + & y & + & z & = & 3 \\ 3x & + & 2y & & & = & 7 \end{array}$$

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40 pts.

Problem 2. In each part, **Use row operations** to determine if the matrix A is invertible and, if so, to find the inverse. Show the indivdual row operations, one by one. You can use a calculator to do the row operations if you wish. Give the matrix entries in fractional form. Be sure to label your answer for the inverse. Sorry, no credit for using a different method!

A.

$$A = \begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}.$$

B.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

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40 pts.

Problem 3. Consider the matrix

$$A = \begin{bmatrix} -2 & 5 & 7 \\ 3 & 9 & 1 \\ 2 & 10 & 6 \end{bmatrix}$$

(a.) Find the cofactors A_{12},A_{22} and A_{33}.

(b.) Show how to compute $\det(A)$ by using a cofactor expansion along some row or column.

70 pts.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 15 & 29 & -30 & 43 & 4 \\ 9 & 7 & 5 & -14 & 3 & 39 \\ 3 & 2 & 1 & -4 & 0 & 13 \\ 0 & 5 & 10 & -10 & 15 & 0 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A.

D. What is the rank of A?

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80 pts.

Problem 5. Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ -5 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -2 \\ 7 \\ 6 \\ -7 \\ 1 \end{bmatrix}.$$

Let $S \subseteq \mathbb{R}^5$ be defined by

$$S = \operatorname{span}(v_1, v_2, v_3, v_4, v_5).$$

A. Find a basis for S. What is the dimension of S?

B. For each of the vectors v_1, v_2, v_3, v_4, v_5 which is not in the basis, express that vector as linear combination of the basis vectors.

C. Consider the vectors

$$w_1 = \begin{bmatrix} 9 \\ 6 \\ 4 \\ 1 \\ 8 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \\ 2 \end{bmatrix}.$$

Determine if each of these vectors is in S. If the vector is in S, write it as a linear combination of the basis vectors for S you found in the first part.

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100 pts.

Problem 6. Let $\mathcal{U} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ be the basis of $\mathbb{R}^3$ given by

$$u_1 = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let $\mathcal{V}$ be the basis of ^{3} given by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}.$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$\begin{aligned} T(u_1) &= u_1 + u_2 + u_3 \\ T(u_2) &= u_2 - u_3 \\ T(u_3) &= 2u_1 + 3u_2 - u_3 \end{aligned}$$

A. Find the transition matrices $S_{\mathcal{E}\mathcal{U}}$, $S_{\mathcal{E}\mathcal{V}}$, $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$. Recall that $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$.

B. Find the matrix $[T]_{\mathcal{U}\mathcal{U}}$ of T with respect to $\mathcal{U}$.

C. Find the matrix $[T]_{\mathcal{V}\mathcal{V}}$ of T with respect to $\mathcal{V}$.

D. Let $w \in \mathbb{R}^3$ be the vector with

$$[w]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Find $[T(w)]_{\mathcal{V}}$.

E. Find $[T(w)]_{\mathcal{U}}$.

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40 pts.

Problem 7. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 11 & -3 & -3 \\ 9 & -1 & -3 \\ 27 & -9 & -7 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

B. The matrix is

$$A = \begin{bmatrix} -4 & -1 & 4 \\ -6 & -2 & 7 \\ -6 & -1 & 6 \end{bmatrix}$$

and the eigenvalues are -1 and 2.
