EXAM

Exam 3

Math 2360–102, Summer I, 2015

June 26, 2015

- Write all of your answers on separate sheets of paper. Do not write on the exam handout. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **430 points total**.

Good luck!

70 pts. **Problem 1**. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 & 5 \\ 1 & 1 & 1 & 0 & 4 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 & 5 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A

D. What is the rank of A.

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50 pts. **Problem 2**. Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -7 \\ -10 \\ 12 \\ -16 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} -3 \\ -5 \\ 5 \\ -8 \end{bmatrix}.$$

Let $S \subseteq \mathbb{R}^4$ be defined by

$$S = \operatorname{span}(v_1, v_2, v_3, v_4, v_5).$$

A. Find a basis for S. What is the dimension of S?

B. Consider the vectors

$$w_1 = \begin{bmatrix} 4 \\ 5 \\ -7 \\ 9 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 5 \\ 2 \\ -8 \\ 9 \end{bmatrix}.$$

Determine if each of these vectors is in S. If the vector is in S, write it as a linear combination of the basis vectors for S you found in the first part.

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50 pts.

Problem 3. In each part, determine if the given vectors in $\mathbb{R}^5$ are linearly independent. Justify your answer. If the vectors are linearly dependent, find scalars c_1, c_2, c_3 and c_4, not all zero, so that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0.$$

A.

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

B.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

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60 pts.

Problem 4. In this problem, we're working in the vector space

$$P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\},$$

the space of polynomials of degree less than three. Let $\mathcal{U}$ be the basis of P_3 given by

$$\mathcal{U} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix},$$

and let $\mathcal{V}$ be the basis of P_3 given by

$$\mathcal{V} = \begin{bmatrix} 3x^2 + 2x + 1 & 2x^2 + x + 1 & 2x^2 + 1 \end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$.

B. Let $p(x) = -x^2 + 2x + 5$. Find $[p(x)]_{\mathcal{V}}$, the coordinates of $p(x)$ with respect to $\mathcal{V}$. Write $p(x)$ as a linear combination of the elements of $\mathcal{V}$

C. Suppose that

$$[q(x)]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Find $[q(x)]_{\mathcal{U}}$. Find $q(x)$ in the form $ax^2 + bx + c$.

100 pts.

Problem 5. Let $\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ be the basis of $\mathbb{R}^3$ given by

$$u_1 = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let $\mathcal{V}$ be the basis of ^{3} given by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -5 \\ -1 \end{bmatrix}.$$

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$\begin{aligned} T(u_1) &= u_1 + u_2 + u_3 \\ T(u_2) &= u_2 - u_3 \\ T(u_3) &= 2u_1 + 3u_2 - u_3 \end{aligned}$$

A. Find the transition matrices $S_{\mathcal{E}\mathcal{U}}$, $S_{\mathcal{E}\mathcal{V}}$, $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$. Recall that $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$.

B. Find the matrix $[T]_{\mathcal{U}\mathcal{U}}$ of T with respect to $\mathcal{U}$.

C. Find the matrix $[T]_{\mathcal{V}\mathcal{V}}$ of T with respect to $\mathcal{V}$.

D. Let $w \in \mathbb{R}^3$ be the vector with

$$[w]_{\mathcal{V}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Find $[T(w)]_{\mathcal{V}}$.

E. Find $[Tw]_{\mathcal{U}}$.

40 pts.

Problem 6. Let

$$A = \begin{bmatrix} -4 & 3 \\ -2 & 3 \end{bmatrix}$$

Find the characteristic polynomial and the eigenvalues of A. (Do not find any eigenvectors.)

60 pts.

Problem 7. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 11 & 10 & -22 \\ 9 & 11 & -21 \\ 9 & 9 & -19 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

B. The matrix is

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

and the eigenvalues are -1 and 2.
