EXAM

Exam 3
Takehome Exam

Math 2360, Spring 2015

November 18, 2015

- Write all of your answers on separate sheets of paper. Do not write on the exam handout. You can keep the exam questions.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **400 points total**.

Good luck!

70 pts. **Problem 1**.

Consider the space

$$P_3 = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R}\},$$

of polynomials of degree less than three. One basis of P_3 is

$$\mathcal{P} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}.$$

Define polynomials by

$$q_1(x) = 1, \quad q_2(x) = x - 3, \quad q_3(x) = (x-3)^2 = x^2 - 6x + 9,$$

and define

$$\mathcal{Q} = \begin{bmatrix} q_1(x) & q_2(x) & q_3(x) \end{bmatrix}.$$

A. Verify that $\mathcal{Q}$ is a basis and find the change of basis matrices $S_{\mathcal{PQ}}$ and $S_{\mathcal{QP}}$.

B. Let $f(x) = 2 + 3x + 5x^2$. Find the coordinates $[f(x)]_{\mathcal{P}}$.

C. Find the coordinates $[f(x)]_{\mathcal{Q}}$ of $f(x)$ with respect to the basis $\mathcal{Q}$. Write $f(x)$ as a linear combination of $q_1(x)$, $q_2(x)$ and $q_3(x)$. Expand and simplify this expression to check that it's correct.

D. Let
$$g(x) = 3 - 5(x-3) - 3(x-3)^2.$$
Find the coordinates $[g(x)]_{\mathcal{P}}$ of $g(x)$ with respect to the basis $\mathcal{P}$ by using the change of basis matrix. Check that this is correct.

Problem 2. Recall that

$$\mathcal{P} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$$

is an ordered basis of P_3.

Another ordered basis for P_3 is

$$\mathcal{Q} = \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix}$$

Let $T\colon P_3 \to P_3$ be the linear transformation defined by

$$T(p(x)) = p'(x) - 3p(x).$$

If it's not obvious to you that this is linear, check it.

A. Find the matrix of T with respect to the basis $\mathcal{P}$, i.e., find $[T]_{\mathcal{P}\mathcal{P}}$.

B. Find the matrix of T with respect to $\mathcal{Q}$, i.e., find $[T]_{\mathcal{Q}\mathcal{Q}}$

C. Let $g(x)$ be the element of P_3 with $[g(x)]_{\mathcal{Q}} = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix}^T$. Find $[T(g(x))]_{\mathcal{Q}}$. Write $g(x)$ and $T(g(x))$ as linear combinations of $\mathcal{Q}$.

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80 pts. **Problem 3**. Recall that the standard basis of $\mathbb{R}^2$ is $\mathcal{E} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix},$$

where

$$u_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

and let

$$\mathcal{V} = \begin{bmatrix} v_1 & v_2 \end{bmatrix},$$

where

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then $\mathcal{U}$ and $\mathcal{V}$ are ordered bases of $\mathbb{R}^2$ (you don't need to check that).

A. Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{E}\mathcal{V}}$.

B. Find the change of basis matrices $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$.

C. Let $w \in \mathbb{R}^2$ be the vector such that

$$[w]_{\mathcal{U}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

Find $[w]_{\mathcal{E}}$ and express w as column vector in $\mathbb{R}^2$

D. Find $[w]_{\mathcal{V}}$, the coordinate vector of w with respect to $\mathcal{V}$.

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90 pts.

Problem 4. Recall that the standard basis of $\mathbb{R}^2$ is $\mathcal{E} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix},$$

where

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

and let

$$\mathcal{V} = \begin{bmatrix} v_1 & v_2 \end{bmatrix},$$

where

$$v_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Then $\mathcal{U}$ and $\mathcal{V}$ are ordered bases of $\mathbb{R}^2$ (you don't need to check that).

Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$\begin{aligned} L(u_1) &= 2u_1 - 3u_2 \\ L(u_2) &= -3u_1 + 2u_2. \end{aligned}$$

A. Find the transition matrices $S_{\mathcal{E}\mathcal{U}}$, $S_{\mathcal{E}\mathcal{V}}$, $S_{\mathcal{U}\mathcal{E}}$, $S_{\mathcal{V}\mathcal{E}}$, $S_{\mathcal{U}\mathcal{V}}$ and $S_{\mathcal{V}\mathcal{U}}$.

B. Find $[L]_{\mathcal{U}\mathcal{U}}$, the matrix of L with respect to the basis $\mathcal{U}$.

C. Find $[L]_{\mathcal{E}\mathcal{E}}$, the matrix of L with respect to the standard basis.

D. Find $[L]_{\mathcal{V}\mathcal{V}}$, the matrix of L with respect to the basis $\mathcal{V}$.

E. Let $w \in \mathbb{R}^2$ be the vector whose coordinate vector with respect to $\mathcal{V}$ is

$$[w]_{\mathcal{V}} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

Find $[L(w)]_{\mathcal{V}}$, the coordinate vector of $L(v)$ with respect to the basis $\mathcal{V}$.

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40 pts.

Problem 5. Let

$$A = \begin{bmatrix} 4 & -1 \\ 6 & -1 \end{bmatrix}.$$

Find the characteristic polynomial and the eigenvalues of A. (Do not find any eigenvectors.)

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60 pts.

Problem 6. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 11 & 10 & -22 \\ 9 & 11 & -21 \\ 9 & 9 & -19 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

B. The matrix is

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

and the eigenvalues are -1 and 2.

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60 pts.

Problem 7. In each part, you are given a matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A and determine if A is diagonalizable. If so, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$. This problem will require the use of complex numbers.

A. The matrix is

$$\begin{bmatrix} -31 & 27 & -3 & -6 \\ -36 & 32 & -3 & -12 \\ 48 & -33 & 8 & -36 \\ -3 & 3 & 0 & -1 \end{bmatrix}$$

and the eigenvalues are $2 \pm 3i$.

B. The matrix is

$$\begin{bmatrix} -9 & 125 & -2 & -40 \\ -52 & 657 & -4 & -210 \\ 2 & 9 & 1 & -3 \\ -160 & 2018 & -12 & -645 \end{bmatrix}$$

and the eigenvalues are $1 \pm 2i$.
