PROBLEM SET

Problems on Change of Basis

Math 2360, Spring 2011

March 27, 2011

- Write all of your answers on separate sheets of paper. You can keep the question sheet.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This problem et has 7 problems. There are **0 points total**.

Good luck!

Problem 1. Recall that

$$P_3 = \{a_2 x^2 + a_1 x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

is the space of polynomials of degree less than 3 and that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

In each part you are given a row $\mathcal{U}$ of vectors in P_3. Find a matrix A so that $\mathcal{U} = \mathcal{P}A$ and determine if $\mathcal{U}$ is an ordered basis of P_3.

A.
$$\mathcal{U} = \begin{bmatrix} 6\,x^2 + x + 5 & 12\,x^2 + 2\,x + 10 & 5\,x^2 + x + 4 \end{bmatrix}$$

B.
$$\mathcal{U} = \begin{bmatrix} 11\,x^2 + 15\,x + 5 & 6\,x^2 + 10\,x + 3 & 2\,x^2 + 3\,x + 1 \end{bmatrix}.$$

Problem 2. Recall that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

Consider the following polynomials in P_3.

$$p_1(x) = 2x^2 - x + 3, \quad p_2(x) = x^2 - 1, \quad p_3(x) = 3x^2 - 2x + 2.$$

Let $\mathcal{Q}$ be the row of vectors in P_3 given by

$$\mathcal{Q} = \begin{bmatrix} p_1(x) & p_2(x) & p_3(x) \end{bmatrix}.$$

A. Show that $\mathcal{Q}$ is an ordered basis of P_3. Find the change of basis matrix $S_{\mathcal{PQ}}$.

B. Find the change of basis matrix $S_{\mathcal{QP}}$.

C. Express x^2, x and 1 as linear combinations of the basis vectors in $\mathcal{Q}$

D. Let $q(x) = -x^2 + 2x - 5$. Find the $[q(x)]_{\mathcal{Q}}$. Express $q(x)$ as a linear combination of the vectors in $\mathcal{Q}$.

E. Suppose that

$$[g(x)]_{\mathcal{Q}} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}.$$

Find $[g(x)]_{\mathcal{P}}$ and express $g(x)$ as a linear combinations of $\mathcal{P}$ and $\mathcal{Q}$

Problem 3. Recall that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

Another ordered basis for P_3 is

$$\mathcal{Q} = \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix}$$

Let $T\colon P_3 \to P_3$ be the linear transformation defined by

$$T(p(x)) = p'(x) + 2p(x).$$

If it's not obvious to you that this is linear, check it.

A. Find the matrix of T with respect to the basis $\mathcal{P}$, i.e., find $[T]_{\mathcal{P}\mathcal{P}}$.

B. Find the matrix of T with respect to the basis $\mathcal{Q}$, i.e., find $[T]_{\mathcal{Q}\mathcal{Q}}$.

C. Let $g(x)$ be the element of P_3 with $[g(x)]_{\mathcal{Q}} = \begin{bmatrix} -2 & 1 & 3 \end{bmatrix}^T$. Find $[T(g(x))]_{\mathcal{Q}}$. Write $g(x)$ and $T(g(x))$ as linear combinations of $\mathcal{Q}$.

Problem 4. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$ (why?). Recall that

$$\mathcal{E} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

is the standard basis of $\mathbb{R}^2$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

B Let v be the vector

$$v = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find $[v]_{\mathcal{E}}$ and $[v]_{\mathcal{U}}$. Express v as a linear combination of u_1 and u_2.

To express v as a linear combination of $\mathcal{U}$, we have

$$\begin{aligned} v &= \mathcal{U}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -18 \\ 13 \end{bmatrix} \\ &= -18u_1 + 13u_2 \\ &= -18 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 13 \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned}$$

we invite the reader to do the simplification to see if this is correct.

Problem 5. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

is a basis of $\mathbb{R}^2$. The row of vectors

$$\mathcal{W} = \begin{bmatrix} w_1 & w_2 \end{bmatrix},$$

where

$$w_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is also an ordered basis of $\mathbb{R}^2$ (why?).

Let $v \in \mathbb{R}^2$ be the vector such that $[v]_{\mathcal{U}} = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$.

A. Find the change of basis matrices $S_{\mathcal{U}\mathcal{W}}$ and $S_{\mathcal{W}\mathcal{U}}$.

B. Express v as a linear combination of $\mathcal{U}$.

C. Find v as an element of $\mathbb{R}^2$, equivalently, find $v = [v]_{\mathcal{E}}$.
   You should check this by comparing with the previous part of the problem.

D. Find $[v]_{\mathcal{W}}$. Express v as a linear combination of $\mathcal{W}$.

Problem 6. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$.

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix}.$$

A. Find the matrix of T with respect to the standard basis, i.e., find $[T]_{\mathcal{E}\mathcal{E}}$. Another way to say it is that we're looking for the matrix A so that $T(x) = Ax$

B. Find the matrix of T with respect to the basis $\mathcal{U}$.

C. Let v be the vector such that

$$[v]_{\mathcal{U}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Find $[T(v)]_{\mathcal{U}}$. Express v and $T(v)$ as linear combinations of $\mathcal{U}$.

Problem 7. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$.

Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$T(u_1) = u_1 + u_2$$
$$T(u_2) = 2u_1 - u_2.$$

A. Find the matrix of T with respect to the basis $\mathcal{U}$, i.e., find $[T]_{\mathcal{U}\mathcal{U}}$.

B. Find the matrix of T with respect to the standard basis $\mathcal{E}$, i.e., find $[T]_{\mathcal{E}\mathcal{E}}$.

C. If v is the vector in $\mathbb{R}^2$ given by

$$v = \begin{bmatrix} 3 \\ -5 \end{bmatrix},$$

find $T(v)$.
