PROBLEM SET

Problems on Change of Basis

Math 2360, Spring 2011

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ANSWERS

Problem 1. Recall that

$$P_3 = \{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

is the space of polynomials of degree less than 3 and that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

In each part you are given a row $\mathcal{U}$ of vectors in P_3. Find a matrix A so that $\mathcal{U} = \mathcal{P}A$ and determine if $\mathcal{U}$ is an ordered basis of P_3.

A.
$$\mathcal{U} = \begin{bmatrix} 6\,x^2 + x + 5 & 12\,x^2 + 2\,x + 10 & 5\,x^2 + x + 4 \end{bmatrix}$$

Answer:

Just reading off the coefficients we have

$$\begin{bmatrix} 6\,x^2 + x + 5 & 12\,x^2 + 2\,x + 10 & 5\,x^2 + x + 4 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 6 & 12 & 5 \\ 1 & 2 & 1 \\ 5 & 10 & 4 \end{bmatrix}.$$

To check this, just carry out the matrix multiplication. Thus,

$$\mathcal{U} = \mathcal{P}A,$$

where

$$A = \begin{bmatrix} 6 & 12 & 5 \\ 1 & 2 & 1 \\ 5 & 10 & 4 \end{bmatrix}.$$

A quick check with a calculator (RREF or determinant) shows that A is *not* invertible, so $\mathcal{U}$ is not and ordered basis of P_3.

B.
$$\mathcal{U} = \begin{bmatrix} 11\,x^2 + 15\,x + 5 & 6\,x^2 + 10\,x + 3 & 2\,x^2 + 3\,x + 1 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} 11\,x^2 + 15\,x + 5 & 6\,x^2 + 10\,x + 3 & 2\,x^2 + 3\,x + 1 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 11 & 6 & 2 \\ 15 & 10 & 3 \\ 5 & 3 & 1 \end{bmatrix},$$

so we have

$$\mathcal{U} = \mathcal{P}A$$

where

$$A = \begin{bmatrix} 11 & 6 & 2 \\ 15 & 10 & 3 \\ 5 & 3 & 1 \end{bmatrix}.$$

A quick check with a calculator shows that A is invertible, so $\mathcal{U}$ is an ordered basis of P_3

Problem 2. Recall that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

Consider the following polynomials in P_3.

$$p_1(x) = 2x^2 - x + 3, \quad p_2(x) = x^2 - 1, \quad p_3(x) = 3x^2 - 2x + 2.$$

Let $\mathcal{Q}$ be the row of vectors in P_3 given by

$$\mathcal{Q} = \begin{bmatrix} p_1(x) & p_2(x) & p_3(x) \end{bmatrix}.$$

A. Show that $\mathcal{Q}$ is an ordered basis of P_3. Find the change of basis matrix $S_{\mathcal{P}\mathcal{Q}}$.

Answer:

We can easily read off the coefficients to get

$$\begin{bmatrix} 2\,x^2 - x + 3 & x^2 - 1 & 3\,x^2 - 2\,x + 2 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}.$$

so we have

$$\mathcal{Q} = \mathcal{P}A$$

where the matrix A is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}.$$

A quick check shows that A is invertible. Thus, $\mathcal{Q}$ is an ordered basis of P_3 and we have

$$\mathcal{Q} = \mathcal{P}A.$$

The defining equation of $S_{\mathcal{P}\mathcal{Q}}$ is

$$\mathcal{Q} = \mathcal{P}S_{\mathcal{P}\mathcal{Q}}$$

so we have

$$S_{\mathcal{PQ}} = A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}.$$

B. Find the change of basis matrix $S_{\mathcal{QP}}$.

Answer:

We have

$$S_{\mathcal{QP}} = (S_{\mathcal{PQ}})^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix}.$$

C. Express x^2, x and 1 as linear combinations of the basis vectors in $\mathcal{Q}$

Answer:

We have the basic equation

$$\mathcal{P} = \mathcal{Q} S_{\mathcal{QP}},$$

and so

$$\begin{bmatrix} x^2 & x & 1 \end{bmatrix} = \begin{bmatrix} 2\,x^2 - x + 3 & x^2 - 1 & 3\,x^2 - 2\,x + 2 \end{bmatrix} \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix}.$$

From this equation we read off

$$\begin{aligned} x^2 &= \frac{2}{5}(2x^2 - x + 3) + \frac{4}{5}(x^2 - 1) - \frac{1}{5}(3x^2 - 2x + 2) \\ x &= (2x^2 - x + 3) + (x^2 - 1) - (3x^2 - 2x + 2) \\ 1 &= \frac{2}{5}(2x^2 - x + 3) - \frac{1}{5}(x^2 - 1) - \frac{1}{5}(3x^2 - 2x + 2). \end{aligned}$$

The reader is invited to do the algebraic simplifications of the right-hand sides these equations to check they are correct.

D. Let $q(x) = -x^2 + 2x - 5$. Find the $[q(x)]_{\mathcal{Q}}$. Express $q(x)$ as a linear combination of the vectors in $\mathcal{Q}$.

Answer:

We have, course

$$q(x) = -x^2 + 2x - 5 = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}.$$

In other words,

$$[q(x)]_{\mathcal{P}} = \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix}.$$

We can then find $[q(x)]_{\mathcal{Q}}$ by using the basic equation

$$[q(x)]_{\mathcal{Q}} = S_{\mathcal{QP}}[q(x)]_{\mathcal{P}}.$$

We computed $S_{\mathcal{QP}}$ above, so

$$\begin{aligned} [[q(x)]_{\mathcal{Q}} &= S_{\mathcal{QP}}[q(x)]_{\mathcal{P}} \\ &= \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -2/5 \\ 11/5 \\ -4/5 \end{bmatrix}. \end{aligned}$$

Since

$$q(x) = \mathcal{Q}[q(x)]_{\mathcal{Q}}$$

we have

$$\begin{aligned} q(x) &= \begin{bmatrix} 2\,x^2 - x + 3 & x^2 - 1 & 3\,x^2 - 2\,x + 2 \end{bmatrix} \begin{bmatrix} -2/5 \\ 11/5 \\ -4/5 \end{bmatrix} \\ &= -\frac{2}{5}(2x^2 - x + 3) + \frac{11}{5}(x^2 - 1) - \frac{4}{5}(3x^2 - 2x + 2). \end{aligned}$$

as the reader is invited to check.

E. Suppose that

$$[g(x)]_{\mathcal{Q}} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}.$$

Find $[g(x)]_{\mathcal{P}}$ and express $g(x)$ as a linear combinations of $\mathcal{P}$ and $\mathcal{Q}$

Answer:

Use the change of basis equation

$$[g(x)]_{\mathcal{P}} = S_{\mathcal{P}\mathcal{Q}}[g(x)]_{\mathcal{Q}}.$$

We've computed $S_{\mathcal{P}\mathcal{Q}}$ above, so

$$\begin{aligned}
[g(x)]_{\mathcal{P}} &= S_{\mathcal{P}\mathcal{Q}}[g(x)]_{\mathcal{Q}} \\
&= \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 11 \\ -7 \\ -1 \end{bmatrix}.
\end{aligned}$$

Thus, we have

$$g(x) = \mathcal{P}[g(x)]_{\mathcal{P}} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 11 \\ -7 \\ -1 \end{bmatrix} = 11x^2 - 7x - 1.$$

On the other hand

$$\begin{aligned}
g(x) &= \mathcal{Q}[g(x)]_{\mathcal{Q}} \\
&= \begin{bmatrix} 2\,x^2 - x + 3 & x^2 - 1 & 3\,x^2 - 2\,x + 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} \\
&= -3(2x^2 - x + 3) + 2(x^2 - 2) + 5(3x^2 - 2x + 2)
\end{aligned}$$

The reader is invited to check that the two expressions for $g(x)$ are the same.

Problem 3. Recall that

$$\mathcal{P} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix}$$

is an ordered basis of P_3.

Another ordered basis for P_3 is

$$\mathcal{Q} = \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix}$$

Let $T\colon P_3 \to P_3$ be the linear transformation defined by

$$T(p(x)) = p'(x) + 2p(x).$$

If it's not obvious to you that this is linear, check it.

A. Find the matrix of T with respect to the basis $\mathcal{P}$, i.e., find $[T]_{\mathcal{P}\mathcal{P}}$.

Answer:

We can easily calculate what the linear map T does to the elements of $\mathcal{P}$. We have

$$\begin{aligned} T(x^2) &= (x^2)' + 2x^2 = 2x^2 + 2x \\ T(x) &= (x)' + 2x = 2x + 1 \\ T(1) &= (1)' + 2(1) = 0 + 2 = 2. \end{aligned}$$

Thus,

$$T(\mathcal{P}) = \begin{bmatrix} T(x^2) & T(x) & T(1) \end{bmatrix} = \begin{bmatrix} 2x^2 + 2x & 2x + 1 & 2 \end{bmatrix}.$$

The defining equation for $[T]_{\mathcal{P}\mathcal{P}}$ is

$$T(\mathcal{P}) = \mathcal{P}[T]_{\mathcal{P}\mathcal{P}}.$$

We can now just read off the coefficents

$$T(\mathcal{P}) = \begin{bmatrix} 2x^2 + 2x & 2x + 1 & 2 \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

This means

$$[T]_{\mathcal{P}\mathcal{P}} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

B. Find the matrix of T with respect to the basis $\mathcal{Q}$, i.e., find $[T]_{\mathcal{Q}\mathcal{Q}}$.

Answer:

We need the change of basis matrices $S_{\mathcal{P}\mathcal{Q}}$ and $S_{\mathcal{Q}\mathcal{P}}$. We did these in a previous problem and got

$$S_{\mathcal{P}\mathcal{Q}} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix}$$

$$S_{\mathcal{Q}\mathcal{P}} = (S_{\mathcal{P}\mathcal{Q}})^{-1} = \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix}$$

We can then use the basic equation

$$[T]_{\mathcal{Q}\mathcal{Q}} = S_{\mathcal{Q}\mathcal{P}}[T]_{\mathcal{P}\mathcal{P}}S_{\mathcal{P}\mathcal{Q}}.$$

Carying out the computation, we have

$$\begin{aligned}[T]_{\mathcal{Q}\mathcal{Q}} &= S_{\mathcal{Q}\mathcal{P}}[T]_{\mathcal{P}\mathcal{P}}S_{\mathcal{P}\mathcal{Q}} \\ &= \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix} \begin{bmatrix} 2/5 & 1 & 2/5 \\ 4/5 & 1 & -1/5 \\ -1/5 & -1 & -1/5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ 3 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{28}{5} & 2 & \frac{26}{5} \\ \frac{21}{5} & 4 & \frac{32}{5} \\ -\frac{19}{5} & -2 & -\frac{18}{5} \end{bmatrix}\end{aligned}$$

C. Let $g(x)$ be the element of P_3 with $[g(x)]_{\mathcal{Q}} = \begin{bmatrix}-2 & 1 & 3\end{bmatrix}^T$. Find $[T(g(x))]_{\mathcal{Q}}$. Write $g(x)$ and $T(g(x))$ as linear combinations of $\mathcal{Q}$.

Answer:

We use the basic equation

$$[T(g(x))]_{\mathcal{Q}} = [T]_{\mathcal{Q}\mathcal{Q}}[g(x)]_{\mathcal{Q}}.$$

Carrying out the computation, we have

$$\begin{aligned}[T(g(x))]_{\mathcal{Q}} &= [T]_{\mathcal{Q}\mathcal{Q}}[g(x)]_{\mathcal{Q}} \\ &= \begin{bmatrix} \frac{28}{5} & 2 & \frac{26}{5} \\ \frac{21}{5} & 4 & \frac{32}{5} \\ -\frac{19}{5} & -2 & -\frac{18}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{32}{5} \\ \frac{74}{5} \\ -\frac{26}{5} \end{bmatrix}.\end{aligned}$$

We have

$$\begin{aligned} g(x) &= \mathcal{Q}[g(x)]_{\mathcal{Q}} \\ &= \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \\ (*) \qquad &= -2(2x^2 - x + 3) + (x^2 - 1) + 3(3x^2 - 2x + 2). \end{aligned}$$

and we have

$$
\begin{aligned}
T(g(x)) &= \mathcal{Q}[T(g(x))]_{\mathcal{Q}} \\
&= \begin{bmatrix} 2x^2 - x + 3 & x^2 - 1 & 3x^2 - 2x + 2 \end{bmatrix} \begin{bmatrix} \frac{32}{5} \\ \frac{74}{5} \\ -\frac{26}{5} \end{bmatrix} \\
(**) \qquad &= \frac{32}{5}(2x^2 - x + 3) + \frac{74}{5}(x^2 - 1) - \frac{26}{5}(3x^2 - 2x + 2).
\end{aligned}
$$

Carry out the following check: Simplify $(*)$ and apply the linear transformation T to the simplification. Show that that agrees with the simplification of $(**)$.

Problem 4. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$ (why?). Recall that

$$\mathcal{E} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

is the standard basis of $\mathbb{R}^2$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A Find the change of basis matrices $S_{\mathcal{E}\mathcal{U}}$ and $S_{\mathcal{U}\mathcal{E}}$.

Answer:

The defining equation of $S_{\mathcal{E}\mathcal{U}}$ is

$$\mathcal{U} = \mathcal{E} S_{\mathcal{E}\mathcal{U}}.$$

This is equivalent to the matrix equation

$$\mathrm{mat}(\mathcal{U}) = \mathrm{mat}(\mathcal{E}) S_{\mathcal{E}\mathcal{U}}$$

where

$$
\begin{aligned}
\mathrm{mat}(\mathcal{U}) &= [u_1 \mid u_2] = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\
\mathrm{mat}(\mathcal{E}) &= [e_1 \mid e_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\end{aligned}
$$

Thus,

$$S_{\mathcal{E}\mathcal{U}} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

We then have

$$S_{\mathcal{U}\mathcal{E}} = (S_{\mathcal{E}\mathcal{U}})^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

B Let v be the vector

$$v = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Find $[v]_{\mathcal{E}}$ and $[v]_{\mathcal{U}}$. Express v as a linear combination of u_1 and u_2.

Answer:

The defining equation of $[v]_{\mathcal{E}}$ is

$$v = \mathcal{E}[v]_{\mathcal{E}}.$$

We have, of course,

$$v = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3e_1 - 5e_2 = \begin{bmatrix} e_1 & e_2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \mathcal{E}v$$

so

$$[v]_{\mathcal{E}} = v. \tag{1}$$

You'll recall that (1) holds for any vector $v \in \mathbb{R}^2$. That's what's so special about the standard basis.

To find $[v]_{\mathcal{U}}$, we use the change of coordinates equation

$$[v]_{\mathcal{U}} = S_{\mathcal{U}\mathcal{E}}[v]_{\mathcal{E}}$$

Thus, we can calculate

$$\begin{aligned} [v]_{\mathcal{U}} &= S_{\mathcal{U}\mathcal{E}}[v]_{\mathcal{E}} \\ &= S_{\mathcal{U}\mathcal{E}}v \\ &= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -18 \\ 13 \end{bmatrix}. \end{aligned}$$

To express v as a linear combination of $\mathcal{U}$, we have

$$\begin{aligned} v &= \mathcal{U}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} -18 \\ 13 \end{bmatrix} \\ &= -18u_1 + 13u_2 \\ &= -18\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 13\begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned}$$

we invite the reader to do the simplification to see if this is correct.

Problem 5. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

is a basis of $\mathbb{R}^2$. The row of vectors

$$\mathcal{W} = \begin{bmatrix} w_1 & w_2 \end{bmatrix},$$

where

$$w_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is also an ordered basis of $\mathbb{R}^2$ (why?).

Let $v \in \mathbb{R}^2$ be the vector such that $[v]_{\mathcal{U}} = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$.

A. Find the change of basis matrices $S_{\mathcal{U}\mathcal{W}}$ and $S_{\mathcal{W}\mathcal{U}}$.

Answer:

It's easiest to go through the standand basis. We have

$$\begin{aligned} S_{\mathcal{E}\mathcal{U}} &= \operatorname{mat}(\mathcal{U}) = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ S_{\mathcal{E}\mathcal{W}} &= \operatorname{mat}(\mathcal{W}) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

Then we have

$$\begin{aligned} S_{\mathcal{U}\mathcal{W}} &= S_{\mathcal{U}\mathcal{E}} S_{\mathcal{E}\mathcal{W}} \\ &= (S_{\mathcal{E}\mathcal{U}})^{-1} S_{\mathcal{E}\mathcal{W}} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

and we also have

$$S_{\mathcal{W}\mathcal{U}} = (S_{\mathcal{U}\mathcal{W}})^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

B. Express v as a linear combination of $\mathcal{U}$.

Answer:

$$\begin{aligned} v &= \mathcal{U}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= 2u_1 - u_2 \\ &= 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

C. Find v as an element of $\mathbb{R}^2$, equivalently, find $v = [v]_{\mathcal{E}}$.

Answer:

$$\begin{aligned} v &= [v]_{\mathcal{E}} \\ &= S_{\mathcal{E}\mathcal{U}}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

You should check this by comparing with the previous part of the problem.

D. Find $[v]_{\mathcal{W}}$. Express v as a linear combination of $\mathcal{W}$.

Answer:

We have

$$\begin{aligned} [v]_{\mathcal{W}} &= S_{\mathcal{W}\mathcal{U}}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} v &= \mathcal{W}[v]_{\mathcal{W}} \\ &= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= -w_1 + 3w_2 \\ &= -\begin{bmatrix} 5 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Simplify and compare with the previous part of the problem.

Problem 6. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$.

Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix}.$$

A. Find the matrix of T with respect to the standard basis, i.e., find $[T]_{\mathcal{E}\mathcal{E}}$. Another way to say it is that we're looking for the matrix A so that $T(x) = Ax$

Answer:

One way to do it is to recall that

$$A = [T(e_1) \mid T(e_2)].$$

But we can easily calculate that

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

and so

$$[T]_{\mathcal{E}\mathcal{E}} = A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

Another way to look at it is to read off coefficents

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

B. Find the matrix of T with respect to the basis $\mathcal{U}$.

Answer:

Use the basic equation

$$[T]_{\mathcal{U}\mathcal{U}} = S_{\mathcal{U}\mathcal{E}}[T]_{\mathcal{E}\mathcal{E}}S_{\mathcal{E}\mathcal{U}}$$

We have

$$S_{\mathcal{E}\mathcal{U}} = \mathrm{mat}(\mathcal{U}) = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$S_{\mathcal{U}\mathcal{E}} = (S_{\mathcal{E}\mathcal{U}})^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

Thus, our calculation is

$$\begin{aligned} [T]_{\mathcal{U}\mathcal{U}} &= S_{\mathcal{U}\mathcal{E}}[T]_{\mathcal{E}\mathcal{E}}S_{\mathcal{E}\mathcal{U}} \\ &= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -3 \\ 5 & 5 \end{bmatrix} \end{aligned}$$

C. Let v be the vector such that

$$[v]_{\mathcal{U}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Find $[T(v)]_{\mathcal{U}}$. Express v and $T(v)$ as linear combinations of $\mathcal{U}$.

Answer:

Use the basic equation

$$[T(v)]_{\mathcal{U}} = [T]_{\mathcal{U}\mathcal{U}}[v]_{\mathcal{U}}.$$

From our previous work,

$$\begin{aligned} [T(v)]_{\mathcal{U}} &= [T]_{\mathcal{U}\mathcal{U}}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} -4 & -3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -7 \\ 15 \end{bmatrix} \end{aligned}$$

We have

$$\begin{aligned} v &= \mathcal{U}[v]_{\mathcal{U}} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= -2u_1 + 5u_2 \\ &= -2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 1 \end{bmatrix}. \end{aligned}$$

and we have

$$\begin{aligned} T(v) &= \mathcal{U}[T(v)]_{\mathcal{U}} \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -7 \\ 15 \end{bmatrix} \\ &= -7u_1 + 15u_2 \\ &= -7 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 15 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

A good exercise is to simplify the last parts of the last two calcuations and check that we got the right answer for $T(v)$, using the original formula for $T(x)$.

Problem 7. Let

$$\mathcal{U} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

where

$$u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then $\mathcal{U}$ is a basis of $\mathbb{R}^2$.

Suppose that $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that

$$\begin{aligned} T(u_1) &= u_1 + u_2 \\ T(u_2) &= 2u_1 - u_2. \end{aligned}$$

A. Find the matrix of T with respect to the basis $\mathcal{U}$, i.e., find $[T]_{\mathcal{U}\mathcal{U}}$.

Answer:

The defining equation for $[T]_{\mathcal{U}\mathcal{U}}$ is

$$T(\mathcal{U}) = \mathcal{U}[T]_{\mathcal{U}\mathcal{U}}.$$

But, we have

$$T(\mathcal{U}) = \begin{bmatrix} T(u_1) & T(u_2) \end{bmatrix} = \begin{bmatrix} u_1 + u_2 & 2u_1 - u_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix},$$

from which we conclude that

$$[T]_{\mathcal{U}\mathcal{U}} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

B. Find the matrix of T with respect to the standard basis $\mathcal{E}$, i.e., find $[T]_{\mathcal{E}\mathcal{E}}$.

Answer:

Use the basic equation

$$[T]_{\mathcal{E}\mathcal{E}} = S_{\mathcal{E}\mathcal{U}}[T]_{\mathcal{U}\mathcal{U}}S_{\mathcal{U}\mathcal{E}}.$$

The change of basis matrices are

$$S_{\mathcal{E}\mathcal{U}} = \text{mat}(\mathcal{U}) = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$
$$S_{\mathcal{U}\mathcal{E}} = (S_{\mathcal{E}\mathcal{U}})^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}.$$

This gives us

$$\begin{aligned} [T]_{\mathcal{E}\mathcal{E}} &= S_{\mathcal{E}\mathcal{U}}[T]_{\mathcal{U}\mathcal{U}}S_{\mathcal{U}\mathcal{E}} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 13 \\ -1 & 4 \end{bmatrix} \end{aligned}$$

C. If v is the vector in $\mathbb{R}^2$ given by

$$v = \begin{bmatrix} 3 \\ -5 \end{bmatrix},$$

find $T(v)$.

Answer:

Use the basic equation

$$T(v) = [T(v)]_{\mathcal{E}} = [T]_{\mathcal{E}\mathcal{E}}[v]_{\mathcal{E}} = [T]_{\mathcal{E}\mathcal{E}}v.$$

Thus, we calculate that

$$\begin{aligned} T(v) &= [T]_{\mathcal{E}\mathcal{E}}v \\ &== \begin{bmatrix} -4 & 13 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -77 \\ -23 \end{bmatrix}. \end{aligned}$$
