Problem 1. Consider the matrix

\[
A = \begin{bmatrix}
-1 & 3 & 11 & 0 & 5 & 2 & 18 \\
-1 & 5 & 17 & 1 & 10 & 2 & 22 \\
1 & -4 & -14 & -1 & -8 & -2 & -19 \\
-1 & 1 & 5 & 0 & 1 & 1 & 7 \\
3 & 4 & 6 & 2 & 13 & 0 & 11 \\
1 & 3 & 7 & 0 & 7 & 1 & 15
\end{bmatrix}
\]

The RREF of \( A \) is the matrix

\[
R = \begin{bmatrix}
1 & 0 & -2 & 0 & 1 & 0 & 1 \\
0 & 1 & 3 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

A. Find a basis for the nullspace of \( A \).

B. Find a basis for the rowspace of \( A \).

C. Find a basis for the columnspace of \( A \).

D. What is the rank of \( A \)?
Problem 2.

A. Consider the vectors

\[ v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \]

in \( \mathbb{R}^4 \). Determine if these vectors are linearly independent or linearly dependent. If they are dependent, find scalars \( c_1, c_2 \) and \( c_3 \), not all zero, so that \( c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \).

B. Consider the vectors

\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \]

in \( \mathbb{R}^4 \). Determine if these vectors are linearly independent or linearly dependent. If they are dependent, find scalars \( c_1, c_2 \) and \( c_3 \), not all zero, so that \( c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \).

Problem 3. Let \( S = \text{span}(v_1, v_2, v_3, v_4, v_5) \) be the subspace of \( \mathbb{R}^5 \) spanned by the vectors

\[
\begin{align*}
v_1 &= \begin{bmatrix} 2 \\ -1 \\ 1 \\ -3 \\ 5 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \\ 5 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 2 \\ -1 \\ -3 \\ -3 \\ 5 \end{bmatrix}, \\
v_4 &= \begin{bmatrix} -2 \\ 1 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \\
v_5 &= \begin{bmatrix} -2 \\ 1 \\ -9 \\ -2 \\ 12 \end{bmatrix}.
\end{align*}
\]

A. Find a basis for \( S \). What is the dimension of \( S \)?

B. Consider the vectors

\[ u = \begin{bmatrix} 13 \\ -3 \\ 1 \\ -13 \\ 25 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -9 \\ 9 \end{bmatrix}. \]

Determine if each of these vector is in \( S \) and, if so, express it as a linear combination of the basis vectors of \( S \) you found in the first part.
Problem 4. Consider the vectors
\[ u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]
in \( \mathbb{R}^2 \).

A. Show that \( \mathcal{U} = [u_1 \quad u_2] \) is an ordered basis of \( \mathbb{R}^2 \). Find the change of basis matrices \( S_{\mathcal{E} \mathcal{U}} \) and \( S_{\mathcal{U} \mathcal{E}} \), where \( \mathcal{E} \) is the standard basis of \( \mathbb{R}^2 \).

B. Let \( v \in \mathbb{R}^2 \) be the vector
\[ v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]
Find \( [v]_{\mathcal{U}} \), the coordinate vector of \( v \) with respect to the basis \( \mathcal{U} \).

C. Let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation whose matrix with respect to the standard basis is
\[ [T]_{\mathcal{E} \mathcal{E}} = \begin{bmatrix} 9 & -14 \\ 7 & -12 \end{bmatrix}. \]
Find \( [T]_{\mathcal{U} \mathcal{U}} \), the matrix of \( T \) with respect to the basis \( \mathcal{U} \).

D. Find \( [T(v)]_{\mathcal{U}} \), the coordinates of \( T(v) \) with respect to \( \mathcal{U} \), where \( v \) is the vector in part B.

Problem 5. Let
\[ A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}. \]
Find the characteristic polynomial and the eigenvalues of \( A \). (Do not find any eigenvectors.)
Problem 6. In each part, you are given a matrix $A$ and its eigenvalues. Find a basis for each of the eigenspaces of $A$ and determine if $A$ is diagonalizable. If so, find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$.

A. The matrix is

$$A = \begin{bmatrix} 8 & -6 & -6 \\ 6 & -4 & -6 \\ 3 & -3 & -1 \end{bmatrix}$$

and the eigenvalues are $-1$ and $2$.

B. The matrix is

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

and the eigenvalue is $2$. 

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4
EXAM

Exam #2
Math 2360, Spring 2006
April 4, 2006

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You must show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., \(\sqrt{2}\), not 1.414).
- This exam has 6 problems. There are 325 points total.

Good luck!