Vectors

- A vector in the plane is a quantity that has both a magnitude and a direction. We can represent it by an arrow.

![Arrow Representation](image)

- Arrows that have the same direction and length represent the same vector.

![Arrow Comparison](image)

- A special case is the zero vector which has zero magnitude and no particular direction (we just draw it as a dot).

- We'll usually denote a vector by a lower case letter. If \( u \) is a vector, \( \|u\| \) denotes the magnitude of \( u \).

- A scalar is a quantity that has magnitude but no direction, i.e., just a
• If $v$ is a vector, $-v$ denotes the vector that has the same length but the opposite direction. Thus, $\|v\| = \|v\|$.

• If $v$ is a vector and $s$ is a scalar we define the vector $sv$ as follows.
  - If $s = 0$, $sv$ is the zero vector.
  - If $s > 0$, $sv$ has the same direction as $v$ and $\|sv\| = s\|v\|$.
  - If $s < 0$, then $sv = |s|(-v)$, i.e., it points in the direction opposite to $v$ and $\|sv\| = |s|\|v\|$.

• If $u$ and $v$ are vectors, as in the picture, we define $u + v$ by the parallelogram law, as in the following diagram.
The diagram shows that vector addition is commutative, i.e., \( u + v = v + u \).

- Subtraction \( u - v \) is defined as \( u + ( -v ) \), but the diagram is worth drawing.
• A choice of a coordinate system sets up a 1-1 correspondence between points in the plane and pairs of real numbers. It also gives a 1-1 correspondence between vectors and pairs of real numbers.

\[ u = (u_1, u_2) \]

• The magnitude of the vector is given by

\[ \|u\| = \sqrt{u_1^2 + u_2^2}. \]

• The set of ordered pairs of real numbers is denoted \( \mathbb{R}^2 \).

• How do scalar multiplication and vector addition look in coordinates? Consider the case shown in the diagram

\[ \text{so } s > 0. \] 

Let the coordinates of the tip of \( sv \) be \( (a, b) \). From similar triangles we have

\[ \frac{v_1}{\|v\|} = \frac{a}{\|sv\|} = \frac{a}{s\|v\|}. \]

from which we conclude that

\[ a = sv_1. \]
Similarly, we have
\[ \frac{v_2}{\|v\|} = \frac{b}{s\|v\|} \]
and so
\[ b = sv_2. \]
Thus, the components of \( sv \) are \((sv_1, sv_2)\). In other words,
\[ s(v_1, v_2) = (sv_1, sv_2). \]
For vector addition, consider the diagram

We conclude
\[ (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2). \]

- The vector operations satisfy the following rules
  1. \((u + v) + w = u + (v + w)\) (Associative Law)
  2. \(u + v = v + u\). (Commutative Law)
  3. \(0 + v = v\). (Additive identity)
  4. \(v + (-v) = 0\). (Additive inverse)
  5. \(s(tv) = (st)v\). (Associative Law)
  6. \((s + t)v = sv + tv\). (Distributive Law)
  7. \(s(u + v) = su + sv\). (Distributive Law)
  8. \(1v = v\).
• We can define the angle between two vectors.

![Diagram of vectors](image)

• The *dot product* of two vectors is defined by

\[ u \cdot v = ||u|| \ ||v|| \cos(\theta). \]

(This is zero if \( u \) or \( v \) is zero.) Observe

\[ u \cdot u = ||u||^2. \]

Recall the law of cosines:

\[ c^2 = a^2 + b^2 - 2ab \cos(\theta). \]

Apply this to the diagram

![Diagram of triangle](image)

We get

\[ ||u - v||^2 = ||u||^2 + ||v||^2 - 2||u|| \ ||v|| \cos(\theta), \]

which we recognize as

\[ ||u - v||^2 = ||u||^2 + ||v||^2 - 2u \cdot v. \]
Thus, we get
\[ 2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2 \]
\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 \]
\[ - [(u_1 - v_1)^2 + (u_2 - v_2)^2] \]
\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 \]
\[ - (u_1^2 - 2u_1v_1 + v_1^2) - (u_2^2 - 2u_2v_2 + v_2^2) \]
\[ = u_1^2 + u_2^2 + v_1^2 + v_2^2 \]
\[ - u_1^2 + 2u_1v_1 - v_1^2 - u_2^2 + 2u_2v_2 - v_2^2 \]
\[ = 2u_1v_2 + 2u_2v_2. \]

Thus, we have the formula
\[ u \cdot v = u_1v_1 + u_2v_2. \]

- We can use this to compute angles between vectors in terms of the components
\[ \cos(\theta) = \frac{u \cdot v}{\|u\| \|v\|}. \]

- Properties of the dot product.
  1. \( u \cdot v = v \cdot u. \)
  2. \((su) \cdot v = s(u \cdot v). \)
  3. \((u + v) \cdot w = u \cdot w + v \cdot w. \)
Matrices

- An $m \times n$ matrix is a rectangular array of numbers with $m$ rows and $n$ columns. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In this notation, $a_{ij}$ denotes the entry in row $i$ and column $j$.

- To multiply a matrix by a scalar, multiply each entry by the scalar.

- Matrices of the same size can be added by adding corresponding entries, like vectors.

- An $m \times n$ matrix $A$ and an $n \times p$ matrix $B$ can be multiplied to give an $m \times p$ matrix $AB$.

For the product of an $1 \times n$ matrix (row vector) and a $n \times 1$ matrix (column vector) the product should be $1 \times 1$, i.e., just a number. The definition is

$$\begin{bmatrix} a_1 \\ b_1 \\ b_2 \\ \vdots \\ a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n.$$

If $A$ is $m \times n$ and $B$ is $n \times p$, $C = AB$ is $m \times p$ and is given by

$$c_{ij} = \text{Row}_i(A) \text{Col}_j(B).$$
Here are the two cases that will be most important to us. First a $2 \times 2$ matrix $A$ times a two entry column vector $u$

$$Au = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 \\ a_{21}u_1 + a_{22}u_2 \end{bmatrix},$$

yielding a two entry column vector.

Second, the product of two $2 \times 2$ matrices $A$ and $B$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$ 

Matrix multiplication is not commutative!

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$ 

Exercise: Let $A$ and $B$ be $2 \times 2$ matrices. Write $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ where $b_1$ and $b_2$ are the columns of $B$ considered as column vectors. Show $AB = \begin{bmatrix} Ab_1 \\ Ab_2 \end{bmatrix}$. Similarly, write

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where $a_1$ and $a_2$ are the rows of $A$ considered as row vectors. Show that $AB = \begin{bmatrix} a_1B \\ a_2B \end{bmatrix}$. 
• Rules of Matrix Algebra. Matrix addition and scalar multiplication obey the same rules as addition and scalar multiplication of vectors. In addition, we have the following rules (The matrices are assumed to have compatible sizes)

1. \(A(BC) = (AB)C\). (Multiplication is Associative)

2. \(A(B + C) = AB + AC\). (Left Distributive Law)

3. \((A + B)C = AC + BC\). (Right Distributive Law).

4. \((sA)B = s(AB) = A(sB)\)

• The 2 \times 2 identity matrix is

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

If \(A\) and \(B\) have compatible sizes, \(IA = A\) and \(BI = B\).

• A (2 \times 2) matrix \(A\) is invertible if there is a (2 \times 2) matrix \(B\) so that

\[ AB = BA = I. \]

It’s easy to show there is at most one matrix \(B\) with this property. If \(B\) exists, we write \(B = A^{-1}\). Note that if \(A^{-1}\) exists, \((A^{-1})^{-1} = A\).

• The determinant of the 2 \times 2 matrix

\[ A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \]

is defined by

\[ \det(A) = ad - bc. \]
• **Theorem** Let $A$ be a $2 \times 2$ matrix. Then, the following conditions are equivalent.

1. $A$ is invertible.
2. The only vector $v$ such that $Av = 0$ is $v = 0$.
3. $\det(A) \neq 0$.

To show that $(1) \implies (2)$, suppose $(1)$ holds and that $Av = 0$. Then we have $A^{-1}Av = A^{-1}0 = 0$. Then $0 = A^{-1}Av = Iv = v$, so $v = 0$. Thus, $(2)$ holds.

To show that $(2) \implies (3)$, it will suffice to show that $\neg(3) \implies \neg(2)$. So assume $\det(A) = ad - bc = 0$. If $A = 0$ then $Av = 0$ for all $v$ so $(2)$ fails. So, suppose $A \neq 0$.

Notice that
\[
\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} d \\ -b \end{bmatrix} = \begin{bmatrix} ad - bc \\ bd - bc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0
\]
\[
\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} -c \\ a \end{bmatrix} = \begin{bmatrix} -ac + ac \\ -bc + ad \end{bmatrix} = 0.
\]

If $A \neq 0$, one of these vectors is nonzero, so $(2)$ fails.

Finally, to show that $(3) \implies (1)$, assume that $\det(A) \neq 0$. Just check that the following formula works
\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.
\]

• **Exercise:** Define the *trace* of $A$, denoted $\text{tr}(A)$, by
\[
\text{tr} \left( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) = a + d,
\]
(i.e., the sum of the diagonal entries).
1. Show that 
\[ \text{tr}(AB) = \text{tr}(BA). \]
(Use brute force.)

2. Use the first part to show 
\[ \text{tr}(P^{-1}AP) = \text{tr}(A). \]

**Exercise:** \( \det(A) = 0 \) if and only if one of the columns of \( A \) is a multiple of the other.

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**Linear Transformations**

- Let \( e_1 \) and \( e_2 \) be the vectors
  \[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
  \]

  Note that
  \[
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_1e_2 + u_2e_2.
  \]

- If \( A \) is a \( 2 \times 2 \) matrix then \( Ae_1 = \text{Col}_1(A) \) and \( Ae_2 = \text{Col}_2(A) \).

- A transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is **linear** if
  \[ T(\alpha u + \beta v) = \alpha T(u) + \beta T(v) \]
  for all vectors \( u \) and \( v \) and scalars \( \alpha \) and \( \beta \).

- An example is the transformation
  \[ T(v) = Av \]
  for a \( 2 \times 2 \) matrix \( A \).
**Theorem** If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear, there is a unique matrix $A$ such that $T(v) = Av$. In fact,

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}.$$ 

To see this, suppose $T$ is linear and $v$ is a vector. Then,

$$T(v) = T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = T(v_1 e_1 + v_2 e_2) = v_1 T(e_1) + v_2 T(e_2) = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$