EXAM

Practice Questions for Exam #2

Math 3350, Spring 2004

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ANSWERS

Problem 1. Find the general solution.

A. $D^3(D-2)(D-3)^2y=0$.

Answer:

The characteristic polynomial is $\lambda^3(\lambda-2)(\lambda-3)^2$. Thus, $\lambda=0$ is a root of multiplicity 3, so it contributes the basic solutions $e^{0x}, xe^{0x}, x^2e^{0x}$, i.e., $1, x, x^2$.

We have $\lambda=2$ as a root of multiplicity 1, so it contributes the basic solution e^{2x}.

Finally, we have $\lambda=3$ as a root of multiplicity 2, so it contributes the basic solutions e^{3x}, xe^{3x}.

The general solution of the equation is a linear combination, with arbitrary coefficients, of the basic solutions, so the general solution is

$$y=C_1+C_2x+C_3x^2+C_4e^{2x}+C_5e^{3x}+C_6xe^{3x}.$$

B. $(D-1)(D-2)(D^2-4D+13)^2y=0$.

Answer:

We have $\lambda=1$ and $\lambda=2$ as roots of multiplicity 1, so they contribute basic solutions e^x and e^{2x}. The roots of the quadratic $\lambda^2-4\lambda+13$ are $\lambda=2\pm3i$ and these conjugate roots both have multiplicity 2. Thus, this pair of conjugate roots contributes the basic solutions $e^{2x}\cos(3x)$, $e^{2x}\sin(3x)$, $xe^{2x}\cos(3x)$ and $xe^{2x}\sin(3x)$. Thus, the general solution is

$$\begin{aligned}(1.1)\quad y=C_1e^x+C_2e^{2x}+C_3e^{2x}\cos(3x)+\\ C_4e^{2x}\sin(3x)+C_5xe^{2x}\cos 3x+C_6xe^{2x}\sin(3x).\end{aligned}$$

Problem 2. Find the general solution **by the method of undetermined coefficients**.

A. $(D^2-2D+1)y=x^2+3x+1$.

Answer:

Since the right-hand side is a polynomial of degree 2, the trial solution should be a polynomial of degree 2, say $y=Ax^2+Bx+C$. Now plug this into the left-hand side and solve for the coefficients. The answer is

$$y=x^2+7x+13+C_1e^x+C_2xe^x.$$

B. $(D^3 + 2D^2)y = x$.

Answer:

Factor out as many D's as possible and write the equation as

$$(D + 2)[D^2 y] = x.$$

Let $z = D^2 y$ so the equation is

$$(D + 2)z = x.$$

Since the right-hand side is a polynomial of degree 1, the trial solution should be a polynomial of degree 1, say $z = Ax + B$. Plugging into the equation gives

$$A + 2Ax + 2B = x.$$

By equating coefficients of powers of x, we get the equations

$$2A = 1, \qquad A + 2B = 0.$$

The solution is $A = 1/2$, $B = -1/4$. Thus, we have

$$z = \frac{1}{2}x - \frac{1}{4}.$$

Since $z = D^2 y$, we have

$$D^2 y = \frac{1}{2}x - \frac{1}{4}.$$

Integrating once gives

$$Dy = \frac{1}{4}x^2 - \frac{1}{4}x$$

and integrating again gives

$$y = \frac{1}{12}x^3 - \frac{1}{8}x^2.$$

The solution of the homogeneous equation is

$$y_h = C_1 + C_2 x + C_3 e^{2x},$$

so the general solution of the inhomogeneous equation is

$$y = \frac{1}{12}x^3 - \frac{1}{8}x^2 + C_1 + C_2 x + C_3 e^{2x},$$

C. $(D^2 - 2D + 1)y = e^{2x}$.

Answer:

Write the equation as $P(D)y = e^{2x}$, where $P(D) = D^2 - 2D + 1$. Move the exponential to the right-hand side, and write the equation as

$$e^{-2x} P(D)y = 1.$$

By the shifting rule, this is the same as

$$P(D+2)[e^{-2x}y] = 1.$$

Let $z = e^{-2x}y$ and calculate

$$P(D+2) = (D+2)^2 - 2(D+2) + 1 = D^2 + 4D + 1.$$

Thus, the equation we have to solve for z is

$$(D^2 + 4D + 1)z = 1.$$

Since the right-hand side is a polynomial of degree zero, we try a polynomial of degree zero for z, i.e., $z = A$, where A is an undetermined constant. Plugging into the equation yields the equation $A = 1$, so $z = 1$. Since $1 = z = e^{-2x}y$, we have $y = e^{2x}$.

The general solution of the homogeneous equation is $y_h = C_1e^x + C_2xe^x$, so the general solution of the inhomogeneous equation is

$$y = e^{2x} + C_1e^x + C_2xe^x.$$

D. $(D^2 - 3D + 2)y = x^2e^{2x}$.

Answer:

Write the equation as

$$P(D)y = x^2e^{2x},$$

where $P(D) = D^2 - 3D + 2$. Moving the exponential to the other side, this equation becomes

$$e^{-2x}P(D)y = x^2.$$

By the shifting rule, this is equivalent to

$$P(D+2)[e^{-2x}y] = x^2.$$

Set $z = e^{-2x}y$ and calculate

$$P(D+2) = (D+2)^2 - 3(D+2) + 2 = D^2 + 4D + 4 - 3D - 6 + 2 = D^2 - D.$$

Thus, the equation for z is

$$(D^2 - D)z = x^2.$$

Since we can factor out a D, we do so and write the equation as

$$(D-1)[Dz] = x^2.$$

Set $w = Dz$, so the equation for w is

$$(D-1)w = x^2.$$

Since the right-hand side is a polynomial of degree 2, our trial solution is a polynomial of degree 2, say $w = Ax^2 + Bx + C$. Plugging this into the equation gives

$$2Ax + B - Ax^2 - Bx - C = x^2.$$

Equating coefficients of powers of x gives

$$-A = 1, \quad 2A - B = 0, \quad B - C = 0.$$

Thus, $A = -1$, $B = -2$ and $C = -2$. Plugging this in gives

$$w = -x^2 - 2x - 2$$

Since $w = Dz$, we get z by integrating once. Thus,

$$z = -\frac{1}{3}x^3 - x^2 - 2x.$$

Since $z = e^{-2x}y$, we finally get

$$y = -\left(\frac{1}{3}x^3 + x^2 + 2x\right)e^{2x}.$$

The general solution of the homogeneous equation is $y_h = C_1e^x + C_2e^{2x}$, so the general solution of the inhomogeneous equation is

$$y = -\left(\frac{1}{3}x^3 + x^2 + 2x\right)e^{2x} + C_1e^x + C_2e^{2x}$$

E. $(D^2 - 3D + 2)y = x^2\sin(2x)$.

Answer:

Write the equation as

$$(*) \qquad P(D)y = x^2\sin(2x),$$

where $P(D) = D^2 - 3D + 2$. Note that $x^2\sin(2x)$ is the imaginary part of x^2e^{2ix}. Thus, to solve $(*)$, we want to solve the complex equation

$$(\heartsuit) \qquad P(D)y = x^2e^{2ix},$$

and then take the imaginary part of the solution. To solve equation $(\heartsuit)$, we move the exponential to the left side are write the equation as

$$e^{-2ix}P(D)y = x^2.$$

By the shifting rule, this is equivalent to

$$P(D + 2i)[e^{-2ix}y] = x^2.$$

We set $z = e^{2ix}y$ and calculate

$$\begin{aligned} P(D+2i) &= (D+2i)^2 - 3(D+2i) + 2 \\ &= D^2 + 4iD - 4 - 3D - 6i + 2 \\ &= D^2 + (-3+4i)D + (-2-6i). \end{aligned}$$

Thus, the equation for z is

$$\Big(D^2 + (-3+4i)D + (-2-6i)\Big)z = x^2$$

Since the right-hand side is a polynomial of degree 2, we try a polynomial of degree 2, $z = Ax^2 + Bx + C$. Plugging into the equations and collecting coefficients yields

$$(-2-6i)Ax^2 + ((-6+8i)A + (-2-6i)B)x + (2A + (-3+4i)B + (-2-6i)C) = x^2,$$

so we get the equations

$$\begin{gathered} (-2-6i)A = 1 \\ (-6+8i)A + (-2-6i)B = 0 \\ 2A + (-3+4i)B + (-2-6i)C = 0 \end{gathered}$$

(use the calculator!) The solutions are

$$\begin{aligned} A &= -1/20 + \frac{3}{20}i \\ B &= -\frac{6}{25} + \frac{7}{100}i \\ C &= -\frac{227}{2000} - \frac{189}{2000}i. \end{aligned}$$

Plugging this in yields

$$z = \left(-1/20 + \frac{3}{20}i\right)x^2 - \left(\frac{6}{25} - \frac{7}{100}i\right)x - \frac{227}{2000} - \frac{189}{2000}i$$

Since $z = e^{-2ix}y$, we have

$$y = \left[\left(-1/20 + \frac{3}{20}i\right)x^2 - \left(\frac{6}{25} - \frac{7}{100}i\right)x - \frac{227}{2000} - \frac{189}{2000}i\right]e^{2ix}$$

This is a particular solution of equation ($\heartsuit$). To find the solution equation ($*$) we must find the imaginary part of this. Putting in $e^{2ix} = \cos(2x) + i\sin(2x)$ and multiplying out gives

$$\begin{aligned} y = {} & \cos(2\,x)\left(-1/20\,x^2 - \frac{6}{25}\,x - \frac{227}{2000}\right) - \sin(2\,x)\left(\frac{3}{20}\,x^2 + \frac{7}{100}\,x - \frac{189}{2000}\right) \\ & + i\left[\sin(2\,x)\left(-1/20\,x^2 - \frac{6}{25}\,x - \frac{227}{2000}\right) + \cos(2\,x)\left(\frac{3}{20}\,x^2 + \frac{7}{100}\,x - \frac{189}{2000}\right)\right] \end{aligned}$$

Taking the imaginary part of this give

$$y = \sin(2\,x)\left(-1/20\,x^2 - \frac{6}{25}\,x - \frac{227}{2000}\right) + \cos(2\,x)\left(\frac{3}{20}\,x^2 + \frac{7}{100}\,x - \frac{189}{2000}\right)$$

as a particular solution of equation $(*)$. The general solution of the homogeneous equation is $y_h = C_1e^x + C_2e^{2x}$, so the general solution of equation $(*)$ is

$$\begin{aligned}(2.1)\quad y = \sin(2\,x)&\left(-1/20\,x^2 - \frac{6}{25}\,x - \frac{227}{2000}\right)\\ &+ \cos(2\,x)\left(\frac{3}{20}\,x^2 + \frac{7}{100}\,x - \frac{189}{2000}\right) + C_1e^x + C_2e^{2x}.\end{aligned}$$

F. $(D^2 - 4D + 5)y = e^{2x}\cos(x)$.

Answer:

Write the equation as

$$(*) \qquad P(D)y = e^{2x}\cos(x),$$

where $P(D) = D^2 - 4D + 5$. Note that $e^{2x}\cos(x)$ is the real part of $e^{(2+i)x}$, so we can solve $(*)$ by solving the complex equation

$$(\heartsuit) \qquad P(D)y = e^{(2+i)x}$$

and then taking the real part. To solve equation $(\heartsuit)$, move the exponential to the left side and rewrite the equation as

$$e^{-(2+i)x}P(D)y = 1.$$

By the shifting rule, this is equivalent to

$$P(D + (2+i))[e^{-(2+i)x}y] = 1.$$

Set $z = e^{-(2+i)}x$ and calculate

$$\begin{aligned}P(D + (2+i)) &= (D + (2+i))^2 - 4(D + (2+i)) + 5\\ &= D^2 + 2(2+i)D + (2+i)^2 - 4(D + 2 + i) + 5\\ &= D^2 + (4+2i)D + 3 + 4i - 4d - 8 - 4i + 5\\ &= D^2 + 2iD.\end{aligned}$$

Thus, the equation for z is

$$(D^2 + 2iD)z = 1.$$

Factor out a D and write this as

$$(D + 2i)[Dz] = 1.$$

Set $w = Dz$, so the equation for w is

$$(D + 2i)w = 1.$$

Since the right-hand side is a polynomial of degree 0, we try a polynomial of degree 0, i.e., $w = A$, where A is a constant. Plugging into the equation gives $2iA = 1$, so

$$w = \frac{1}{2i} = -\frac{1}{2}i.$$

Since $Dz = w$, integrating gives us

$$z = -\frac{1}{2}ix.$$

Since $z = e^{-(2+i)x}y$, we get

$$y = -\frac{1}{2}ixe^{(2+i)x}.$$

This is a particular solution of equation (♡). To find the solution of equation $(*)$, we need to find the real part of y. We have

$$\begin{aligned} y &= -\frac{1}{2}ixe^{(2+i)x} \\ &= -\frac{1}{2}ix(e^{2x}\cos(x) + ie^{2x}\sin(2x)) \\ &= -\frac{1}{2}ixe^{2x}\cos(x) + \frac{1}{2}xe^{2x}\sin(2x). \end{aligned}$$

Taking the real part of this gives us

$$y = \frac{1}{2}xe^{2x}\sin(x)$$

as a particular solution of $(*)$.

To find the general solution of the homogeneous equation $(D^2 - 4D + 5)y = 0$, use the quadratic formula to find the roots of the characteristic polynomial $\lambda^2 - 4\lambda + 5$. The roots are $2 \pm i$, so the general solution of the homogeneous equation is

$$y_h = C_1e^{2x}\cos(x) + C_2e^{2x}\sin(2x).$$

Thus, the general solution to $(*)$ is

$$y = \frac{1}{2}xe^{2x}\sin(x) + C_1e^{2x}\cos(x) + C_2e^{2x}\sin(2x)$$

Problem 3. Find the general solution **by the method of variation of parameters.**.

A. $y'' + 4y = \tan(2x)$.

Answer:

The basic solutions of the homogeneous equation $y'' + 4y = 0$ are $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. We are trying to find a particular solution of the inhomogeneous equation of the form $y = u(x)y_1(x) + v(x)y_2(x)$. In general the system of equations for u and v is

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= r(x), \end{aligned}$$

where $r(x)$ is the right-hand side of the inhomogeneous equation.

In our case, these equations become

$$\begin{aligned} u'\cos(2x) + v'\sin(2x) &= 0 \\ -2u'\sin(2x) + 2v'\cos(2x) &= \tan(2x). \end{aligned}$$

Use your favorite method for solving this system. If we use Cramer's rule, the determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2.$$

We then have

$$\begin{aligned} u' &= \frac{1}{\Delta}\begin{vmatrix} 0 & \sin(2x) \\ \tan(2x) & 2\cos(2x) \end{vmatrix} \\ &= -\frac{1}{2}\sin(2x)\tan(2x) = -\frac{1}{2}\frac{\sin^2(2x)}{\cos(2x)} \\ &= -\frac{1}{2}\frac{1-\cos^2(2x)}{\cos(2x)} \\ &= -\frac{1}{2}(\sec(2x) - \cos(2x)) \end{aligned}$$

and

$$v' = \frac{1}{\Delta}\begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & \tan(2x) \end{vmatrix} = \frac{1}{2}\sin(2x).$$

Integrating gives use

$$\begin{aligned} u &= -\frac{1}{2}\int(\sec(2x) - \cos(2x))\,dx = -\frac{1}{4}\ln|\sec(2x) + \tan(2x)| + \frac{1}{4}\sin(2x) \\ v &= \frac{1}{2}\int \sin(2x)\,dx = -\frac{1}{4}\cos(2x) \end{aligned}$$

(we don't need the constants of integration).

Plugging this into the formula $y = u(x)y_1(x) + v(x)y_2(x)$ for our particular solution, we get

$$\begin{aligned} y &= -\frac{1}{4}\cos(2x)\ln|\sec(2x)+\tan(2x)| + \frac{1}{4}\cos(2x)\sin(2x) - \frac{1}{4}\cos(2x)\sin(2x) \\ &= -\frac{1}{4}\cos(2x)\ln|\sec(2x)+\tan(2x)|. \end{aligned}$$

Adding on the general solution of the homogeneous equation, we find that the general solution of the inhomogeneous equation is

$$y = -\frac{1}{4}\cos(2x)\ln|\sec(2x)+\tan(2x)| + C_1\cos(2x) + C_2\sin(2x).$$

B. $x^2y'' - 4xy' + 6y = x$ (Be sure to express the equation in standard form before setting up variation of parameters.)

Answer:

First we have to solve the homogeneous equation

$$x^2y'' - 4xy' + 6y = 0. \tag{3.1}$$

This is an Euler-Cauchy equation, so we look for solutions of the form $y = x^m$. The characteristic equation for m is

$$m(m-1) - 4m + 6 = 0.$$

Note $m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3)$, so the roots are $m = 2$ and $m = 3$. Thus, we have basic solutions $y_1 = x^2$ and $y_2 = x^3$ for the homogeneous equation (3.1).

To solve the equation in the problem, we put it in standard form by dividing both sides by x^2. This gives us

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = \frac{1}{x}.$$

We look for a particular solution of this equation of the form $y = u(x)y_1(x) + v(x)y_2(x)$. The equations for u and v are

$$\begin{aligned} u'x^2 + v'x^3 &= 0 \\ 2u'x + 3v'x^2 &= \frac{1}{x}. \end{aligned}$$

If we multiply the top equation by -2 and the bottom equation by x, the system becomes

$$\begin{aligned} -2u'x^2 - 2v'x^3 &= 0 \\ 2u'x^2 + 3v'x^3 &= 1. \end{aligned}$$

Adding these equations gives $v'x^3 = 1$, so $v' = 1/x^3$. Substituting this is the first equation gives $u' = -1/x^2$. Easy integrations then give

$$\begin{aligned} u &= \frac{1}{x} \\ v &= -\frac{1}{2x^2} \end{aligned}$$

Plugging into the formula $y = u(x)y_1(x) + v(x)y_2(x)$ for our particular solution gives

$$y = \frac{1}{x}x^2 - \frac{1}{2x^2}x^3 = \frac{1}{2}x.$$

Adding on the general solution of the homogeneous equation, we find

$$y = \frac{1}{2}x + C_1x^2 + C_2x^3$$

for the general solution of the inhomogeneous equation.

Problem 4. In each part, find the inverse Laplace transform of the given function.

A. In this part, find the partial fractions decomposition **by hand**.

$$\frac{4}{s^2(s-2)^2}$$

Answer:

The form of the partial fraction decomposition is

$$\frac{4}{s^2(s-2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}. \tag{4.1}$$

Clearing the fractions yields the equation

$$4 = As(s-2)^2 + B(s-2)^2 + Cs^2(s-2) + Ds^2. \tag{4.2}$$

Setting $s = 0$ in this equation gives $4 = 4B$, so $B = 1$. Setting $s = 2$ in (4.2) gives $4 = 4D$, so $D = 1$. Putting these values back into (4.2) and moving the known terms to the left-hand side gives

$$4 - (s-2)^2 - s^2 = As(s-2)^2 + Cs^2(s-2).$$

Differentiating this equation, we get

$$-2(s-2) - 2s = A(s-2)^2 + 2As(s-2) + 2Cs(s-2) + Cs^2. \tag{4.3}$$

Plugging $s = 0$ into this equation yields $4 = 4A$, so $A = 1$. Plugging $s = 2$ into(4.3) gives $-4 = 4C$, so $C = -1$.

Plugging these values of A,B,C and D back into (4.1) gives

$$\frac{4}{s^2(s-2)^2} = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-2} + \frac{1}{(s-2)^2}.$$

Thus, we have

$$\mathscr{L}^{-1}\left[\frac{4}{s^2(s-2)^2}\right] = \mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{1}{s^2}\right] - \mathscr{L}^{-1}\left[\frac{1}{s-2}\right] + \mathscr{L}^{-1}\left[\frac{1}{(s-2)^2}\right] \tag{4.4}$$

From the table on page 254 of the textbook, we have

$$\mathscr{L}[1] = \frac{1}{s} \tag{4.5}$$

$$\mathscr{L}[t] = \frac{1}{s^2} \tag{4.6}$$

$$\mathscr{L}[e^{2t}] = \frac{1}{s-2} \tag{4.7}$$

To find the inverse transform of $1/(s-2)^2$, we use (4.6) and the shifting rule

$$\mathscr{L}[e^{ta}f(t)] = F(s-a)$$

(see page 253 and page 255). Using these formulas we find

$$\mathscr{L}^{-1}\left[\frac{4}{s^2(s-2)^2}\right] = 1 + t - e^{2t} + te^{2t}$$

B. In this part, you can find the partial fractions decomposition with the calculator, if you wish.

$$\frac{2s^2 - 3s + 6}{(s-2)(s^2+4)}.$$

Answer:

By machine, we find the partial fraction decomposition

$$\frac{2s^2 - 3s + 6}{(s-2)(s^2+4))} = \frac{1}{2}\frac{1}{s-2} + \frac{1}{2}\frac{s-4}{s^2+4}.$$

Rewriting this slightly, we have

$$\frac{2s^2 - 3s + 6}{(s-2)(s^2+4))} = \frac{1}{2}\frac{1}{s-2} + \frac{1}{2}\frac{s}{s^2+4} - \frac{2}{s^2+4}.$$

From the table on page 254 of the text,

$$\begin{aligned}\mathscr{L}[e^{2t}] &= \frac{1}{s-2}\\ \mathscr{L}[\cos(2t)] &= \frac{s}{s^2+4}\\ \mathscr{L}[\sin(2t)] &= \frac{2}{s^2+4}.\end{aligned}$$

Thus, we get

$$\mathscr{L}^{-1}\left[\frac{2s^2-3s+6}{(s-2)(s^2+4))}\right] = \frac{1}{2}e^{2t} + \frac{1}{2}\cos(2t) - \sin(2t).$$

Problem 5. In each part, find the solution of the initial value problem **by the method of Laplace transforms**.

A. $y'' - 6y' + 9y = t$, $y(0) = 1$, $y'(0) = 0$.

Answer:

Take the transform of both sides of the equation, so we have

$$\mathscr{L}[y''] - 6\mathscr{L}[y'] + 9\mathscr{L}[y] = \mathscr{L}[t].$$

Using the table on page 254 of the book and Theorem 2 on page 259, this equation becomes

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{1}{s^2}$$

(where $Y(s) = \mathscr{L}[y(t)]$ as usual). Putting in the given initial conditions, this equation becomes

$$s^2Y(s) - s - 6sY(s) + 6 + 9Y(s) = \frac{1}{s^2}.$$

Simplifying gives

$$(s^2 - 6s + 9)Y(s) = s - 6 + \frac{1}{s^2} = \frac{s^3 - 6s^2 + 1}{s^2}.$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{s^3 - 6s^2 + 1}{(s^2 - 6s + 9)s^2} = \frac{s^3 - 6s^2 + 1}{s^2(s-3)^2}.$$

Using a machine to find the partial fraction decomposition, we have

$$Y(s) = \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} + \frac{25}{27}\frac{1}{s-3} - \frac{26}{9}\frac{1}{(s-3)^2}.$$

Taking the inverse transform of this equation (as in the previous problem) gives

$$y(t) = \frac{2}{27} + \frac{1}{9}t + \frac{25}{27}e^{3t} - \frac{26}{9}te^{3t}.$$

B. $y'' - 6y' + 9y = e^{3x}$, $y(0) = 0$, $y'(0) = 1$.

Answer:

Transforming both sides of the equation (as in the first part of this problem) gives

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{1}{s-3}.$$

Plugging in the initial conditions gives

$$s^2Y(s) - 1 - 6sY(s) + 9Y(s) = \frac{1}{s-3}.$$

Simplifying gives

$$(s^2 - 6s + 9)Y(s) = 1 + \frac{1}{s-3} = \frac{s-2}{s-3}.$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{s-2}{(s^2 - 6s + 9)(s-3)} = \frac{s-2}{(s-3)^3}$$

By machine, the partial fraction decomposition is

$$Y(s) = \frac{1}{s-3} + \frac{1}{(s-3)^3}.$$

From the table on page 254, we have

$$\mathscr{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}.$$

From the same table, we have

$$\mathscr{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2}t^2$$

Using the shifting rule

$$\mathscr{L}[e^{at}f(t)] = F(t-a),$$

we get

$$\mathscr{L}^{-1}\left[\frac{1}{(s-3)^3}\right] = \frac{1}{2}t^2e^{3t}.$$

Thus, finally,

$$y(t) = e^{3t} + \frac{1}{2}t^2e^{3t}$$