70 pts.

Problem 1. Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & -12 & -4 & 1 & 11 \\ 1 & 1 & -1 & 0 & 0 & 5 \\ 4 & 3 & -6 & -1 & 1 & 17 \\ 2 & -1 & -8 & -3 & 1 & 3 \\ 2 & 1 & -4 & -1 & 2 & 6 \end{bmatrix}.$$

The RREF of A is the matrix

$$R = \begin{bmatrix} 1 & 0 & -3 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A. Find a basis for the nullspace of A.

B. Find a basis for the rowspace of A.

C. Find a basis for the columnspace of A

D. What is the rank of A?

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50 pts.

Problem 2. Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 8 \\ 0 \\ 3 \end{bmatrix}.$$

A. Cut down the list of vectors above to a basis for S. What is the dimension of S?

B. For each of the following vectors, determine if the vector is in S and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

$$\mathbf{w}_1 = \begin{bmatrix} 18 \\ 12 \\ 5 \\ 7 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 15 \\ 2 \\ 9 \\ 3 \end{bmatrix}$$

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50 pts.

Problem 3. Let A be a 6×8 matrix and let B be a 7×7 matrix.

A. What is the largest possible value of the rank of A?

B. If the nullspace of A has dimension 5, what is the rank of A?

C. If the rowspace of B has dimension 4, what is the dimension of the nullspace of B?

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40 pts.

Problem 4. Let

$$A = \begin{bmatrix} -4 & 4 \\ -3 & 4 \end{bmatrix}.$$

Find the characteristic polynomial of A and the eigenvalues of A.

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60 pts.

Problem 5. In each part you are given an matrix A and its eigenvalues. Find a basis for each of the eigenspaces of A. Determine if A is diagonalizable, and if it is, find a matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

A.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \qquad \text{Eigenvalues} = 1, 2.$$

B.

$$A = \begin{bmatrix} 8 & 9 & 9 \\ 0 & 2 & 0 \\ -6 & -9 & -7 \end{bmatrix}, \qquad \text{Eigenvalues} = -1, 2.$$

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80 pts.

Problem 6.

Let $\mathcal{U} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{bmatrix}$ be the ordered basis of $\mathbb{R}^2$ where

$$\mathbf{u}_1 = \begin{bmatrix}1\\3\end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix}1\\2\end{bmatrix}.$$

A. Find the change of basis matrices $S_{\mathcal{EU}}$ and $S_{\mathcal{UE}}$.

B. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$L(\mathbf{u}_1) = 2\mathbf{u}_1 - 1\mathbf{u}_2$$
$$L(\mathbf{u}_2) = 3\mathbf{u}_1 - 5\mathbf{u}_2.$$

Find $[L]_{\mathcal{UU}}$, the matrix of L with respect to the basis $\mathcal{U}$.

C. Find the matrix of L with respect to the standard basis $\mathcal{E}$ of $\mathbb{R}^2$

D. Let $\mathbf{v}$ be the vector

$$\mathbf{v} = \begin{bmatrix}2\\3\end{bmatrix}.$$

a. Express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

b. Express $T(\mathbf{v})$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

c. Express $T(\mathbf{v})$ as a column vector.

d. Check the last part against $[L]_{\mathcal{EE}}\mathbf{v}$.

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40 pts.

Problem 7. Let S be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 18 \\ 38 \\ 11 \\ 11 \\ 2 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Let A be the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 & 1 & -3 \\ -2 & 3 & -4 & 1 & -5 & 12 \\ -8 & 15 & -22 & 3 & -25 & 56 \\ 2 & -3 & 4 & -1 & 5 & -12 \\ -8 & 14 & -20 & 3 & -24 & 53 \\ 7 & 9 & -25 & -2 & 1 & 18 \end{bmatrix}.$$

Define K by

$$K = \{\mathbf{v} \in S \mid A\mathbf{v} = \mathbf{0}\}.$$

Find a basis of K. Explain your reasoning.

EXAM

Exam # 2
Take-home Exam

Math 3351, Spring 2003

Feb. 28, 2003

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).
- This exam has 7 problems. There are **390 points total**.

Good luck!