Problem 1. Consider the matrix

\[
A = \begin{bmatrix}
4 & 0 & -12 & -4 & 1 & 11 \\
1 & 1 & -1 & 0 & 0 & 5 \\
4 & 3 & -6 & -1 & 1 & 17 \\
2 & -1 & -8 & -3 & 1 & 3 \\
2 & 1 & -4 & -1 & 2 & 6
\end{bmatrix}
\]

The RREF of \(A\) is the matrix

\[
R = \begin{bmatrix}
1 & 0 & -3 & -1 & 0 & 3 \\
0 & 1 & 2 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

A. Find a basis for the nullspace of \(A\).
B. Find a basis for the rowspace of \(A\).
C. Find a basis for the columnspace of \(A\).
D. What is the rank of \(A\)?

Problem 2. Let \(S\) be the subspace of \(\mathbb{R}^4\) spanned by the vectors

\[
v_1 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 6 \\ 8 \\ 0 \\ 3 \end{bmatrix}.
\]

A. Cut down the list of vectors above to a basis for \(S\). What is the dimension of \(S\)?
B. For each of the following vectors, determine if the vector is in \(S\) and, if so, express it as a linear combination of the basis vectors you found in the previous part of the problem.

\[
w_1 = \begin{bmatrix} 18 \\ 12 \\ 5 \\ 7 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 15 \\ 2 \\ 9 \\ 3 \end{bmatrix}.
\]
Problem 3. Let $A$ be a $6 \times 8$ matrix and let $B$ be a $7 \times 7$ matrix.

A. What is the largest possible value of the rank of $A$?

B. If the nullspace of $A$ has dimension 5, what is the rank of $A$?

C. If the rowspace of $B$ has dimension 4, what is the dimension of the nullspace of $B$?

Problem 4. Let

$$A = \begin{bmatrix} -4 & 4 \\ -3 & 4 \end{bmatrix}. $$

Find the characteristic polynomial of $A$ and the eigenvalues of $A$.

Problem 5. In each part you are given an matrix $A$ and its eigenvalues. Find a basis for each of the eigenspaces of $A$. Determine if $A$ is diagonalizable, and if it is, find a matrix $P$ and a diagonal matrix $D$ so that $P^{-1}AP = D$.

A. 

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 0 \\ -1 & 2 & 1 \end{bmatrix}, \quad \text{Eigenvalues = 1, 2.}$$

B. 

$$A = \begin{bmatrix} 8 & 9 & 9 \\ 0 & 2 & 0 \\ -6 & -9 & -7 \end{bmatrix}, \quad \text{Eigenvalues = -1, 2.}$$
Let $U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ be the ordered basis of $\mathbb{R}^2$ where

$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

A. Find the change of basis matrices $S_{EU}$ and $S_{UE}$.

B. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$L(u_1) = 2u_1 - u_2$
$L(u_2) = 3u_1 - 5u_2$.

Find $[L]_{idE}$, the matrix of $L$ with respect to the basis $U$.

C. Find the matrix of $L$ with respect to the standard basis $E$ of $\mathbb{R}^2$.

D. Let $v$ be the vector $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

a. Express $v$ as a linear combination of $u_1$ and $u_2$.

b. Express $T(v)$ as a linear combination of $u_1$ and $u_2$.

c. Express $T(v)$ as a column vector.

d. Check the last part against $[L]_{EE}v$. 


**Problem 7.** Let $S$ be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$v_1 = \begin{bmatrix} 18 \\ 38 \\ 11 \\ 11 \\ 2 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 9 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 8 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Let $A$ be the matrix

$$A = \begin{bmatrix} 0 & -1 & 2 & 0 & 1 & -3 \\ -2 & 3 & -4 & 1 & -5 & 12 \\ -8 & 15 & -22 & 3 & -25 & 56 \\ 2 & -3 & 4 & -1 & 5 & -12 \\ -8 & 14 & -20 & 3 & -24 & 53 \\ 7 & 9 & -25 & -2 & 1 & 18 \end{bmatrix}.$$ 

Define $K$ by

$$K = \{ v \in S \mid Av = 0 \}.$$ 

Find a basis of $K$. Explain your reasoning.
EXAM

Exam # 2
Take-home Exam

Math 3351, Spring 2003

Feb. 28, 2003

• Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.

• You **must** show enough work to justify your answers. Unless otherwise instructed, give exact answers, not approximations (e.g., $\sqrt{2}$, not 1.414).

• This exam has 7 problems. There are **390 points total**.

Good luck!