
EXAM

Exam #2

Math 1351-004, Spring 2003

April 4, 2002

ANSWERS

80 pts.

Problem 1. In each part, find the derivative of the given function. After eliminating all the derivatives, it is not necessary to simplify your answer.

A. $f(x) = 7x^5 - 5x^4 + x^2 - 3x + 12$

Answer: $f'(x) = 35x^4 - 20x^3 + 2x - 3$

B. $f(x) = \frac{3x^2 + 2}{x - 1}$

Answer:

$$\begin{aligned} f'(x) &= \frac{(x-1)\frac{d}{dx}(3x^2+2) - (3x^2+2)\frac{d}{dx}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(6x) - (3x^2+2)}{(x-1)^2} && \text{acceptable answer} \\ &= \frac{3x^2 - 6x - 2}{(x-1)^2} \end{aligned}$$

C. $f(x) = x^2 \cos(x)$

Answer:

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx} x^2 \right] \cos(x) + x^2 \frac{d}{dx} \cos(x) \\ &= 2x \cos(x) - x^2 \sin(x) \end{aligned}$$

D. $f(x) = x^3(x^2 + 1)^{10}$

Answer:

$$\begin{aligned} f'(x) &= \left[\frac{d}{dx} x^3 \right] (x^2 + 1)^{10} + x^3 \frac{d}{dx} (x^2 + 1)^{10} \\ &= 3x^2(x^2 + 1)^{10} + x^3 10(x^2 + 1)^9(2x) \\ &= 3x^2(x^2 + 1)^{10} + 20x^4(x^2 + 1)^9 \end{aligned}$$

E. $f(x) = \sec^4(5x)$

Answer:

$$\begin{aligned} f'(x) &= 4 \sec^3(5x) \frac{d}{dx} \sec 5x \\ &= 4 \sec^3(5x) \sec(5x) \tan(5x) \frac{d}{dx} (5x) \\ &= 4 \sec^3(5x) \sec(5x) \tan(5x) (5) \\ &= 20 \sec^4(5x) \tan(5x) \end{aligned}$$

F. $f(x) = e^{x^2+5x+3}$

Answer:

$$f'(x) = e^{x^2+5x+3} = e^{x^2+5x+3} \frac{d}{dx}(x^2 + 5x + 3) = (2x + 5)e^{x^2+5x+3}$$

G. $f(x) = \frac{\ln(x)}{x}$

Answer:

$$\begin{aligned} f'(x) &= \frac{x \frac{d}{dx} \ln(x) - \ln(x) \frac{d}{dx} x}{x^2} \\ &= \frac{x(1/x) - \ln(x)}{x^2} \\ &= \frac{1 - \ln(x)}{x^2} \end{aligned}$$

H. $f(x) = \tan^{-1}(x^2)$

Answer:

$$\begin{aligned} f'(x) &= \frac{1}{1 + (x^2)^2} \frac{d}{dx} x^2 \\ &= \frac{2x}{1 + x^4} \end{aligned}$$

40 pts.

Problem 2. Find the slope of the line tangent to the curve

$$x^3 + x^2y + y^3 = 11$$

at the point $(1, 2)$.

Answer:

Differentiate both sides of the equation, remembering that y is a function of x .

$$\begin{aligned} \frac{d}{dx}(x^3 + x^2y + y^3) &= \frac{d}{dx} 11 \\ 3x^2 + \left[\frac{d}{dx} x^2 \right] y + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} &= 0 \\ 3x^2 + 2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} &= 0 \end{aligned}$$

Now, solve the last equation for dy/dx .

$$\begin{aligned}x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} &= -(3x^2 + 2xy) \\(x^2 + 3y^2) \frac{dy}{dx} &= -(3x^2 + 2xy) \\ \frac{dy}{dx} &= -\frac{3x^2 + 2xy}{x^2 + 3y^2}.\end{aligned}$$

Now substitute $x = 1$, $y = 2$ into the last expression.

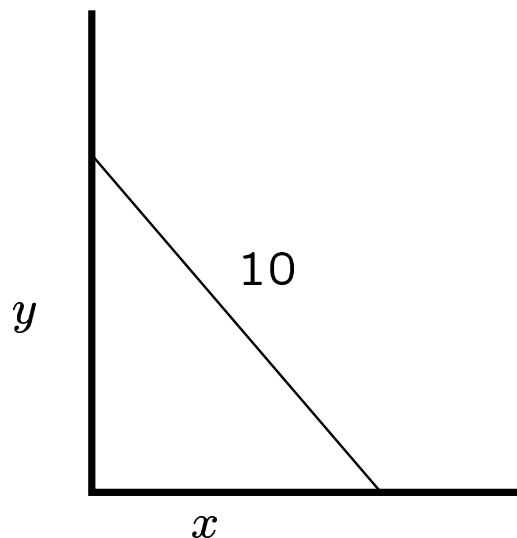
$$\begin{aligned}\left. \frac{dy}{dx} \right|_{(x,y)=(1,2)} &= -\frac{3(1)^2 + 2(1)(2)}{1^2 + 3(2)^2} \\ &= -\frac{7}{13}\end{aligned}$$

40 pts.

Problem 3. A 10 foot long ladder is leaning against the wall. The foot of the ladder is being pushed toward the wall at 3 feet per second. How fast is the top of the ladder moving when the foot of the ladder is 6 feet from the wall?

Answer:

Let x be the distance of the foot of the ladder from the wall, and let y be the height of the top of the ladder. The picture is here



Since this is a right triangle, we have

$$x^2 + y^2 = 10^2. \quad (1)$$

Differentiate both sides of this equation, remembering that x and y are functions of time t .

$$\begin{aligned} \frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}10^2 \\ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\ x \frac{dx}{dt} + y \frac{dy}{dt} &= 0. \end{aligned}$$

Thus, the relationship between the rates is

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0 \quad (2)$$

At the **specific instant** we are interested in, we are given that $x = 6$ and $dx/dt = -3$ (negative, since x is decreasing). We want to find dy/dt . In order to use (2), we need to find y , which we can do by using (1). We have $6^2 + y^2 = 10^2$, so $y = \sqrt{100 - 36} = 8$. Plugging into (2) gives

$$6(-3) + 8 \frac{dy}{dt} = 0,$$

from which we get $dy/dt = 9/4$.

40 pts.

Problem 4. The side of a cube is measured to be 10 feet. If there is a possible error of 1% in the measurement of the side, estimate the possible percentage error in the calculation of the volume of the cube.

Answer:

Let x be the length of one side of the cube. Then the volume of the cube is $V = x^3$. Differentiating this, we have

$$\frac{dV}{dx} = 3x^2$$

from which we get

$$dV = 3x^2 dx. \quad (3)$$

We have a possible error of 1% in measuring the length of the side. So the possible error is 1% of 10, which is $10/100 = 0.1$ foot. In (3) we set $dx = 0.1$ and $x = 10$ to get

$$dV = 3(10)^2(0.1) = 30$$

so our estimate of the possible error in the calculation of the volume is 30 cubic feet. To express this as a percentage error, we use

$$\text{Percentage Error} = 100\% \frac{dV}{V} = 100\% \frac{30}{1000} = 3\%.$$

40 pts.

Problem 5. In each part, use logarithmic differentiation to find dy/dx .

1. $y = \frac{x^4}{e^x(x-1)^3}$.

Answer:

First, take natural log of both sides and simplify.

$$\begin{aligned}\ln(y) &= \ln\left(\frac{x^4}{e^x(x-1)^3}\right) \\ &= \ln(x^4) - \ln(e^x) - \ln((x-1)^3) \\ &= 4\ln(x) - x - 3\ln(x-1).\end{aligned}$$

The final result is

$$\ln(y) = 4\ln(x) - x - 3\ln(x-1).$$

Now differentiate both sides of this equation. Remember that y is a function of x . This gives

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{4}{x} - 1 - \frac{3}{x-1} \frac{d}{dx}(x-1) \\ &= \frac{4}{x} - 1 - \frac{3}{x-1}.\end{aligned}$$

Solving for dy/dx gives

$$\begin{aligned}\frac{dy}{dx} &= y \left[\frac{4}{x} - 1 - \frac{3}{x-1} \right] && \text{(acceptable)} \\ &= \frac{x^4}{e^x(x-1)^3} \left[\frac{4}{x} - 1 - \frac{3}{x-1} \right]\end{aligned}$$

2. $y = x^{\sin(x)}$.

Answer:

Actually, logarithmic differentiation is the only way we can do this. Taking natural log of both sides gives

$$\ln(y) = \ln(x^{\sin(x)}) = \sin(x) \ln(x).$$

Differentiating both sides of

$$\ln(y) = \sin(x) \ln(x)$$

with respect to x gives

$$\frac{1}{y} \frac{dy}{dx} = \left[\frac{d}{dx} \sin(x) \right] \ln(x) + \sin(x) \frac{d}{dx} \ln(x) = \cos(x) \ln(x) + \frac{\sin(x)}{x}.$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= y[\cos(x) \ln(x) + \sin(x)/x] && \text{(acceptable)} \\ &= x^{\sin(x)}[\cos(x) \ln(x) + \sin(x)/x]. \end{aligned}$$

60 pts.

Problem 6. Alice throws a ball upward with a velocity of 64 feet per second, while standing on a platform 80 feet above the ground. Ignoring air resistance, answer the following questions.

1. How far above the ground is the ball, at its highest point?

Answer:

The general equation for a projectile (with no air resistance) is

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0,$$

where $s(t)$ is the height above the ground at time t , v_0 is the velocity at $t = 0$ and s_0 is the height at $t = 0$, and g is the acceleration due to gravity.

In the present problem, we are given $v_0 = 64$ and $s_0 = 80$. The experiment is taking place on earth so $g = 32$ feet per second squared. Thus, we have

$$s(t) = -16t^2 + 64t + 80.$$

The velocity is the derivative of $s(t)$, so we have

$$v(t) = s'(t) = -32t + 64.$$

To find how high the ball goes, note that when it reaches its highest point, $v = 0$. Thus, we solve for the time when the velocity is zero,

$$-32t + 64 = 0 \implies t = 2.$$

Thus, the ball reaches its highest point at $t = 2$. It's height at this time is

$$s(2) = -16(2)^2 + 64(2) + 80 = 144.$$

So, the greatest height above the ground reached by the ball is 144 feet.

2. What is the total distance traveled by the ball before it hits the ground?

Answer:

In traveling up from the platform to the highest point, the ball travels $144 - 80 = 64$ feet. In traveling from the highest point to the ground, the ball travels 144 feet. Thus, the total distance traveled is $64 + 144 = 208$.

3. How fast is the ball traveling when it hits the ground?

Answer:

To find the time when the ball hits the ground, set $s(t) = 0$ and solve for t . Thus, we have

$$-16t^2 + 64t + 80 = 0$$

$$t^2 - 4t - 5 = 0$$

$$(t - 5)(t + 1) = 0$$

So this equation has two solutions, $t = 5$ and $t = -1$. The solution $t = -1$ would be before the ball was thrown at $t = 0$, and so is not relevant to the problem. We conclude that the ball hits the ground at $t = 5$. To find the velocity with which it hits the ground, we calculate $v(5)$,

$$v(5) = -32(5) + 64 = -96.$$

Thus, when the ball hits the ground, it is traveling downward at 96 feet per second.
