A NOTE ON THE LEBESGUE COVERING LEMMA

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The purpose of this note is to give a proof of the Lebesgue Covering Lemma, which we used in class.

First, we recall a couple of definitions. Let \((X, d)\) be a metric space and let \(E \subseteq X\) be a nonempty set. We define \(\text{diam}(E)\), the diameter of \(E\) by

\[
\text{diam}(E) = \sup \{d(x, y) \mid x, y \in E\},
\]

(this might be \(\infty\), of course). The diameter of the empty set is 0.

Let \(\mathcal{U}\) be an open cover of \(X\), i.e., \(\mathcal{U}\) is a collection of open subsets of \(X\) and every point \(x \in X\) is in some element \(U\) of \(\mathcal{U}\). A number \(\lambda > 0\) is a Lebesgue Number for \(\mathcal{U}\) if every set of diameter less than \(\lambda\) is contained in some element in the cover \(\mathcal{U}\), i.e., if \(\text{diam}(E) < \lambda\), there is some \(U \in \mathcal{U}\) such that \(E \subseteq U\).

**Lebesgue Covering Lemma.** If \(X\) is a compact metric space, every open cover of \(X\) has a Lebesgue number.

The proof will occupy the remainder of this note.

Let \(\mathcal{U}\) be an open cover of our compact metric space \(X\). For every point \(x \in X\), select an element \(U(x)\) of \(\mathcal{U}\) so that \(x \in U(x)\). Since \(U(x)\) is open, we can find some radius \(r(x) > 0\) such that \(B_{r(x)}(x) \subseteq U(x)\).

The collection of open balls

\[
\mathcal{B} = \{B_{r(x)}/2(x) \mid x \in X\}
\]

is clearly an open cover of \(X\). Since \(X\) is compact, there is a finite subcover, say \(\mathcal{B}' = \{B_{r(x_i)/2(x_i)} \mid i = 1, 2, \ldots, n\}\).

Let

\[
\lambda = \min \{r(x_i)/2 \mid i = 1, 2, \ldots, n\},
\]

so \(\lambda > 0\).

Of course we want to show that \(\lambda\) is a Lebesgue number for \(\mathcal{U}\). To do this, suppose that \(E \subseteq X\) and \(\text{diam}(E) < \lambda\).

Fix a point \(p \in E\). The point \(p\) must be in some element of the cover \(\mathcal{B}'\), say \(p \in B_{r(x_i)/2(x_i)}\).

Let \(q\) be any point in \(E\). Then

\[
d(p, q) \leq \text{diam}(E) < \lambda \leq r(x_i)/2.
\]

Then

\[
d(x_i, q) \leq d(x_i, p) + d(p, q) < r(x_i)/2 + r(x_i)/2 = r(x_i).
\]

This shows that

\[
E \subseteq B_{r(x_i)}(x_i).
\]
But then
\[ E \subseteq B_{\tau(x_i)}(x_i) \subseteq U(x_i), \]
so \( E \) is contained in the element \( U(x_i) \) of \( U \). This completes the proof.

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