Limiting value of higher Mahler measure

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Abstract

We consider the k-higher Mahler measure $m_k(P)$ of a Laurent polynomial P as the integral of $\log^k |P|$ over the complex unit circle. In this paper we derive an explicit formula for the value of $|m_k(P)|/k!$ as $k \to \infty$.

Keywords: Mahler measure, higher Mahler measure

1. Introduction

For a non-zero Laurent polynomial $P(z) \in \mathbb{C}[z, z^{-1}]$, the k-higher Mahler measure of P is defined [4] as

$$m_k(P) = \int_0^1 \log^k \left| P\left(e^{2\pi i t}\right) \right| \, \mathrm{d}t.$$

For $k = 1$ this coincides with the classical (log) Mahler measure defined as

$$m(P) = \log|a| + \sum_{j=1}^{n} \log\left(\max\{1, |r_j|\}\right), \text{ for } P(z) = a\prod_{j=1}^{n}(z - r_j),$$

since by Jensen's formula $m(P) = m_1(P)$ [3].

Though classical Mahler measure was studied extensively, higher Mahler measure was introduced and studied very recently by Kurokawa, Lalin and Ochiai [4] and Akatsuka [1]. It is very difficult to evaluate k-higher Mahler measure for polynomials except few specific examples shown in [1] and [4] , but it is relatively easy to find their limiting values.

In [5] Lalin and Sinha answered Lehmer's question [3] for higher Mahler measure by finding non-trivial lower bounds for m_k on $\mathbb{Z}[z]$ for $k \geq 2$.

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In [2] it has been shown using Akatsuka's zeta function of [4] that for $|a| = 1$, $|m_k(z + a)|/k! \to 1/\pi$ as $k \to \infty$. In this paper we generalize this result by computing the same limit for an arbitrary Laurent polynomial $P(z) \in \mathbb{C}[z, z^{-1}]$ using a different technique.

Theorem 1.1. *Let $P(z) \in \mathbb{C}\left[z, z^{-1}\right]$ be a Laurent polynomial, possibly with repeated roots. Let $z_1, \ldots, z_n$ be the distinct roots of P. Then*

$$\lim_{k\to\infty} \frac{|m_k(P)|}{k!} = \frac{1}{\pi} \sum_{z_j \in S^1} \frac{1}{|P'(z_j)|},$$

where S^1 is the complex unit circle $|z| = 1$, and the right-hand side is taken as ∞ if $P'(z_j) = 0$ for some $z_j \in S^1$, i.e., if P has a repeated root on S^1.

2. Proof of the theorem

We first prove several lemmas which essentially show that the integrand may be linearly approximated near the roots of P on S^1.

Lemma 2.1. *Let $P(z) \in \mathbb{C}\left[z, z^{-1}\right]$ be a Laurent polynomial and $A \subseteq [0, 1]$ be a closed set such that $P\left(e^{2\pi i t}\right) \neq 0$ for all $t \in A$. Then*

$$\lim_{k\to\infty} \frac{1}{k!} \int_A \log^k \left|P\left(e^{2\pi i t}\right)\right| \mathrm{d}t = 0$$

Proof. Since A is closed, due to the periodicity of $e^{2\pi i t}$ and continuity of $P(e^{2\pi i t})$ there exist constants b and B such that $0 < b \leq \left|P\left(e^{2\pi i t}\right)\right| \leq B$ on A. Then for each positive integer k, $(\log^k \left|P\left(e^{2\pi i t}\right)\right|)/k!$ is bounded between $(\log^k b)/k!$ and $(\log^k B)/k!$, and therefore $(1/k!) \int_A \log^k \left|P\left(e^{2\pi i t}\right)\right| \mathrm{d}t$ is bounded between $(\mu A \log^k b)/k!$ and $(\mu A \log^k B)/k!$, where μA is the Lebesgue measure of A. The result follows by letting k tend to infinity. □

Lemma 2.2. *Let $P(z) \in \mathbb{C}\left[z, z^{-1}\right]$ be a Laurent polynomial with a root of order one at $z_0 = e^{2\pi i t_0}$, and $P'(z)$ be its derivative with respect to z. Then for each $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies*

$$\left|2\pi(1 - \varepsilon)(t - t_0)P'\left(e^{2\pi i t_0}\right)\right| \leq \left|P\left(e^{2\pi i t}\right)\right| \leq \left|2\pi(1 + \varepsilon)(t - t_0)P'\left(e^{2\pi i t_0}\right)\right|.$$

Proof. Set $f(t) = P\left(e^{2\pi i t}\right)$. Then $f'(t_0) = 2\pi i P'\left(e^{2\pi i t_0}\right) \neq 0$ and

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

Since $f'(t_0) \neq 0$, it follows that for each $\varepsilon \in (0, 1)$ there exists $\delta > 0$ such that $0 < |t - t_0| < \delta$ implies

$$1 - \varepsilon < \left| \frac{f(t) - f(t_0)}{(t - t_0)} \cdot \frac{1}{f'(t_0)} \right| < 1 + \varepsilon,$$

which proves the lemma since $f(t_0) = P(z_0) = 0$. □

Lemma 2.3. *Let $c \neq 0$, and $t_0 \in \mathbb{R}$. Then for all $\varepsilon > 0$,*

$$\lim_{k \to \infty} \frac{1}{k!} \left| \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \log^k |c(t - t_0)| \, \mathrm{d}t \right| = \frac{2}{|c|}.$$

Proof. For $k \geq 1$ and $x > 0$, it follows from integration by parts and induction that

$$\int_0^x \log^k u \, \mathrm{d}u = x \log^k x + x \sum_{j=1}^{k} \frac{(-1)^j \, k! \, \log^{k-j} x}{(k - j)!}.$$

Using the even symmetry of the integrand and substituting $u = |c(t - t_0)|$, we have

$$\frac{1}{k!} \left| \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \log^k |c(t - t_0)| \, \mathrm{d}t \right| = \frac{2}{|c| \, k!} \left| \int_0^{|c\varepsilon|} \log^k u \, \mathrm{d}u \right|,$$

and it follows that

$$\begin{aligned}
\lim_{k \to \infty} \frac{1}{k!} \left| \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \log^k |c(t - t_0)| \, \mathrm{d}t \right| &= \lim_{k \to \infty} \frac{2}{|c| \, k!} \left| \int_0^{|c\varepsilon|} \log^k u \, \mathrm{d}u \right| \\
&= 2\varepsilon \lim_{k \to \infty} \left| \frac{\log^k |c\varepsilon|}{k!} + \sum_{j=1}^{k} \frac{(-1)^j \log^{k-j} |c\varepsilon|}{(k - j)!} \right| \\
&= 2\varepsilon \left| \sum_{n=0}^{\infty} \frac{(-1)^n \log^n |c\varepsilon|}{n!} \right| \\
&= 2\varepsilon e^{-\log |c\varepsilon|} = 2/|c|.
\end{aligned}$$

□

Lemma 2.4. *Let $P(z) \in \mathbb{C}\left[z, z^{-1}\right]$ be a Laurent polynomial with a root of order one at $z_0 = e^{2\pi i t_0}$. Then for all sufficiently small $\delta > 0$,*

$$\lim_{k \to \infty} \frac{1}{k!} \left| \int_{t_0 - \delta}^{t_0 + \delta} \log^k \left| P\left(e^{2\pi i t}\right) \right| \, \mathrm{d}t \right| = \frac{1}{\pi \left| P'\left(e^{2\pi i t_0}\right) \right|}.$$

Proof. First notice that since z_0 has order one, it cannot be a root of $P'(z)$. Now let $\varepsilon \in (0,1)$. By Lemma 2.2 there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies

$$\left|2\pi(1-\varepsilon)(t-t_0)P'\left(e^{2\pi i t_0}\right)\right| \leq \left|P\left(e^{2\pi i t}\right)\right| \leq \left|2\pi(1+\varepsilon)(t-t_0)P'\left(e^{2\pi i t_0}\right)\right| \leq 1.$$

Setting $c = 2\pi(1-\varepsilon)P'\left(e^{2\pi i t_0}\right)$ and $d = 2\pi(1+\varepsilon)P'\left(e^{2\pi i t_0}\right)$ it follows that for $0 < |t - t_0| < \delta$,

$$\log|c(t-t_0)| \leq \log\left|P\left(e^{2\pi i t}\right)\right| \leq \log|d(t-t_0)| \leq 0,$$

and hence

$$\left|\log^k|c(t-t_0)|\right| \geq \left|\log^k\left|P\left(e^{2\pi i t}\right)\right|\right| \geq \left|\log^k|d(t-t_0)|\right| \geq 0,$$

for all $k \in \mathbb{N}$. Therefore,

$$\int_{t_0-\delta}^{t_0+\delta} \left|\log^k|c(t-t_0)|\right| \,\mathrm{d}t \geq \int_{t_0-\delta}^{t_0+\delta} \left|\log^k\left|P\left(e^{2\pi i t}\right)\right|\right| \,\mathrm{d}t \geq \int_{t_0-\delta}^{t_0+\delta} \left|\log^k|d(t-t_0)|\right| \,\mathrm{d}t \geq 0.$$

But on $(t_0 - \delta, t_0 + \delta)$, for each fixed k, either all three functions $\log^k|c(t-t_0)|$, $\log^k\left|P\left(e^{2\pi i t}\right)\right|$ and $\log^k|d(t-t_0)|$ are negative (if k is odd), or positive (if k is even). So the integrals of their absolute values are equal to the absolute values of their integrals and therefore we have

$$\left|\int_{t_0-\delta}^{t_0+\delta} \log^k|c(t-t_0)| \,\mathrm{d}t\right| \geq \left|\int_{t_0-\delta}^{t_0+\delta} \log^k\left|P\left(e^{2\pi i t}\right)\right| \,\mathrm{d}t\right| \geq \left|\int_{t_0-\delta}^{t_0+\delta} \log^k|d(t-t_0)| \,\mathrm{d}t\right|.$$

By Lemma 2.3 it follows that

$$\frac{2}{|c|} \geq \lim_{k\to\infty} \frac{1}{k!}\left|\int_{t_0-\delta}^{t_0+\delta} \log^k\left|P\left(e^{2\pi i t}\right)\right|\right| \geq \frac{2}{|d|}.$$

Since $c = 2\pi(1-\varepsilon)P'\left(e^{2\pi i t_0}\right)$ and $d = 2\pi(1+\varepsilon)P'\left(e^{2\pi i t_0}\right)$ and $\varepsilon > 0$ is arbitrary, we are done. $\square$

With these lemmas, we now proceed to prove the main theorem.

Proof of Theorem 1.1. First notice that

$$\frac{m_k(P)}{k!} = \frac{1}{k!}\int_0^1 \log^k\left|P\left(e^{2\pi i t}\right)\right| \,\mathrm{d}t.$$

If $P(z)$ does not have any roots on S^1 then choosing $A = [0,1]$ and applying Lemma 2.1 we see that $|m_k(P)|/k! \to 0$ as $k \to \infty$ and the theorem holds in this case.

Now let $t_1, \ldots, t_m \in [0,1]$ such that $e^{2\pi i t_1}, \ldots, e^{2\pi i t_m}$ are the distinct roots of P on S^1. Let $\delta > 0$ be sufficiently small so that $|P(e^{2\pi i t_j})| < 1$ on each interval $(t_j - \delta, t_j + \delta)$, $j = 1, \ldots, m$, and these intervals are disjoint and define

$$A = [0,1] \setminus \bigcup_{j=1}^{m} (t_j - \delta, t_j + \delta).$$

Using Lemma 2.1, and the fact that $\log|P(e^{2\pi i t})| < 0$ on $[0,1] \setminus A$, we find that

$$\begin{aligned} \lim_{k\to\infty} \frac{|m_k(P)|}{k!} &= \lim_{k\to\infty} \frac{1}{k!} \left| \int_A \log^k |P(e^{2\pi i t})| \, \mathrm{d}t + \int_{[0,1]\setminus A} \log^k |P(e^{2\pi i t})| \, \mathrm{d}t \right| \\ &= \lim_{k\to\infty} \sum_{j=1}^{m} \frac{1}{k!} \left| \int_{t_j-\delta}^{t_j+\delta} \log^k |P(e^{2\pi i t})| \, \mathrm{d}t \right| \qquad (2.5) \end{aligned}$$

If P has no repeated roots on S^1, then by Lemma 2.4, this final sum is equal $\pi^{-1} \sum_{j=1}^{m} |P'(e^{2\pi i t_j})|^{-1}$, and so the theorem is proven in this case.

Finally, if P has a repeated root on S^1, we may assume without loss of generality that $P(z_1) = P'(z_1) = 0$ where $z_1 = e^{2\pi i t_1}$. With $f(t) = P(e^{2\pi i t})$, we have that $f(t_1) = f'(t_1) = 0$. Then for each $\varepsilon \in (0,1)$ there is a $\delta_\varepsilon \in (0,1)$ such that

$$\left| \frac{f(t)}{t - t_1} \right| = \left| \frac{f(t) - f(t_1)}{t - t_1} \right| \le \varepsilon, \qquad \text{for all } 0 < |t - t_1| < \delta_\varepsilon.$$

It follows that $\log|f(t)| \le \log|\varepsilon(t - t_1)| < 0$ for all $0 < |t - t_1| < \delta_\varepsilon$, and so

$$\left| \log^k |f(t)| \right| \ge \left| \log^k |\varepsilon(t - t_1)| \right|, \qquad \text{for all } 0 < |t - t_1| < \delta_\varepsilon.$$

We may assume that $\delta_\varepsilon < \delta$, and using (2.5) and Lemma 2.3 deduce that

$$\begin{aligned} \lim_{k\to\infty} \frac{|m_k(P)|}{k!} &\ge \lim_{k\to\infty} \left| \int_{t_1-\delta}^{t_1+\delta} \log^k |P(e^{2\pi i t})| \, \mathrm{d}t \right| \\ &= \lim_{k\to\infty} \int_{t_1-\delta}^{t_1+\delta} \left| \log^k |P(e^{2\pi i t})| \right| \, \mathrm{d}t \\ &\ge \lim_{k\to\infty} \int_{t_1-\delta_\varepsilon}^{t_1+\delta_\varepsilon} \left| \log^k |\varepsilon(t - t_0)| \right| \, \mathrm{d}t \\ &= \frac{2}{|\varepsilon|}. \end{aligned}$$

Since $\varepsilon \in (0, 1)$ was arbitrary, the limit in question diverges to ∞ and the theorem is proven. □

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